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Seminar Paper

Relaxations for probabilistically constrained stochastic programming problems

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List of abbreviations and symbols

Abbreviation/Symbol	Meaning
EMS	Emergency Medical Services
ESDP	Emergent Specimen Delivery Problem
MILP	Mixed-integer linear problem
MINLP	Mixed-integer non-linear problem
CDF	Cumulative Distribution Function
PDF	Probability Density Function
$\boldsymbol{F}$	Multivariate cumulative distribution function
$\boldsymbol{F_i}$	Univariate (marginal) cumulative distribution function
$\boldsymbol{F_i^{-1}}$	Marginal quantile function

Introduction and motivation

There are many optimization problems in practice requiring taking decisions under uncertainty. For example, the project "KiMoNo" initiated in Passau by the Federal Ministry of Digital and Transport in Germany considers the following problem. An emergency, where a human specimen needs to be collected and analysed, occurs at a doctor's office. A drone 1) flies from the potential drone base location to the doctor's office, 2) picks up the specimen, 3) delivers it to the laboratory, where 4) a battery swap is conducted, and 5) comes back to the location. The trip from the base to the doctor's office should not exceed a predefined service time. The drone's battery capacity should be enough for the trips. The questions are where to locate drone bases and how many drones should be bought. This emergent specimen delivery problem (ESDP) (Petrenko, 2024) plays an essential role in saving human lives and revolutionizes the design of emergency medical services.

The main challenge of ESDP is how to incorporate the uncertainty of emergent demand. One of the common approaches is to use stochastic programming with chance constraints. Stochastic programming paradigm relies on a strong assumption that the distributions of the uncertain components are known. Chance- (probabilistic-) constrained programming is a useful and simple method in stochastic programming which allows to model the system performance to a prescribed level of reliability (probability): namely, the constraints should be satisfied for sufficiently many realizations of the random variables.

Let $\boldsymbol{x}$ be a vector of decision variables and $\boldsymbol{\xi}$ – a vector of random variables. The following sample problem is an underlying problem for static stochastic programming model formulations (the dimensions of matrices are given below):

$$min \quad \underset{(1\times n)}{c^T} \cdot \underset{(n\times 1)}{x} \tag{1a}$$

$$s.t. \quad \underset{(r\times n)}{T} \cdot \underset{(n\times 1)}{x} \geq \underset{(r\times 1)}{\xi} \tag{1b}$$

$$\underset{(m\times n)}{A} \cdot \underset{(n\times 1)}{x} \geq \underset{(m\times 1)}{b} \tag{1c}$$

$$\underset{(n\times 1)}{x} \geq 0 \tag{1d}$$

Let us introduce chance constraints for this problem. We distinguish between two types of chance constraints: *individual* (2) and *joint* (3) ones, where $\boldsymbol{p} \in (\mathbf{0}, \mathbf{1})$ is the reliability level.

$$\mathbb{P}(T_i x \geq \xi_i) \geq p_i, i = 1, \dots, r \tag{2}$$

$$\mathbb{P}(Tx \geq \xi) \geq p, \tag{3}$$

Imagine that, as in Petrenko (2024), the components $\boldsymbol{\xi_i}$ of the random vector $\boldsymbol{\xi}$ are demands on specimens at doctor's offices with unique probability distributions and $\boldsymbol{x}$ are the number of drones on each location to cover the demand at a certain doctor's office. Individual chance constraints (2) are imposed on *each* doctor's office. In words, each constraint for each doctor's office requires that the probability, that the drones reserved for this particular doctor cover the demand at this office, is at least, for example, 97% (or any other value for $\boldsymbol{p}$ between 0 and 1). In contrast to individual chance constraints, joint chance constraints are imposed on the *entire* area. The only one constraint says: the probability that the reserved drones cover the demand for each doctor's office simultaneously is at least $\boldsymbol{p}$**.** The reader should pay attention that individual and joint chance constraints are not the same. Namely, joint chance constraints imply individual ones, so the condition that the service level at each doctor's office should be at least $\boldsymbol{p}$ is *necessary, but not sufficient* for the whole system to be reliable on the level $\boldsymbol{p}$ (Lejeune and Prékopa, 2018). Moreover, for the case of individual chance constraints, the independency between demand points is assumed. But what if there is some dependency structure between the components of the random vector?

In this seminar paper, we will focus on solving the ESDP with joint chance constraints where the dependency between the demand points is present and the demand is continuous. The dependency between emergencies at doctor's offices can be explained through the cases of catastrophes, large car accidents or epidemies. The continuity of the demand can be expressed as the weight of a specimen (in grams) to be delivered. To ensure the positiveness of demand, we assume that the weight of emergent specimens at each doctor's office follows a half-normal distribution with $\boldsymbol{\mu = 0}$ and $\boldsymbol{\sigma = \sigma_i}$ – that is an absolute value of a normal distribution (Figure 1). It models the emergent demand quite well: mostly the demand is equal to 0 or slightly deviates from zero for 1-2 specimens, larger demand is less likely, but still possible.

To solve the ESDP, the ideas expressed in our basic paper of Lejeune and Prékopa (2018) will be used. We will go into the details of it later.

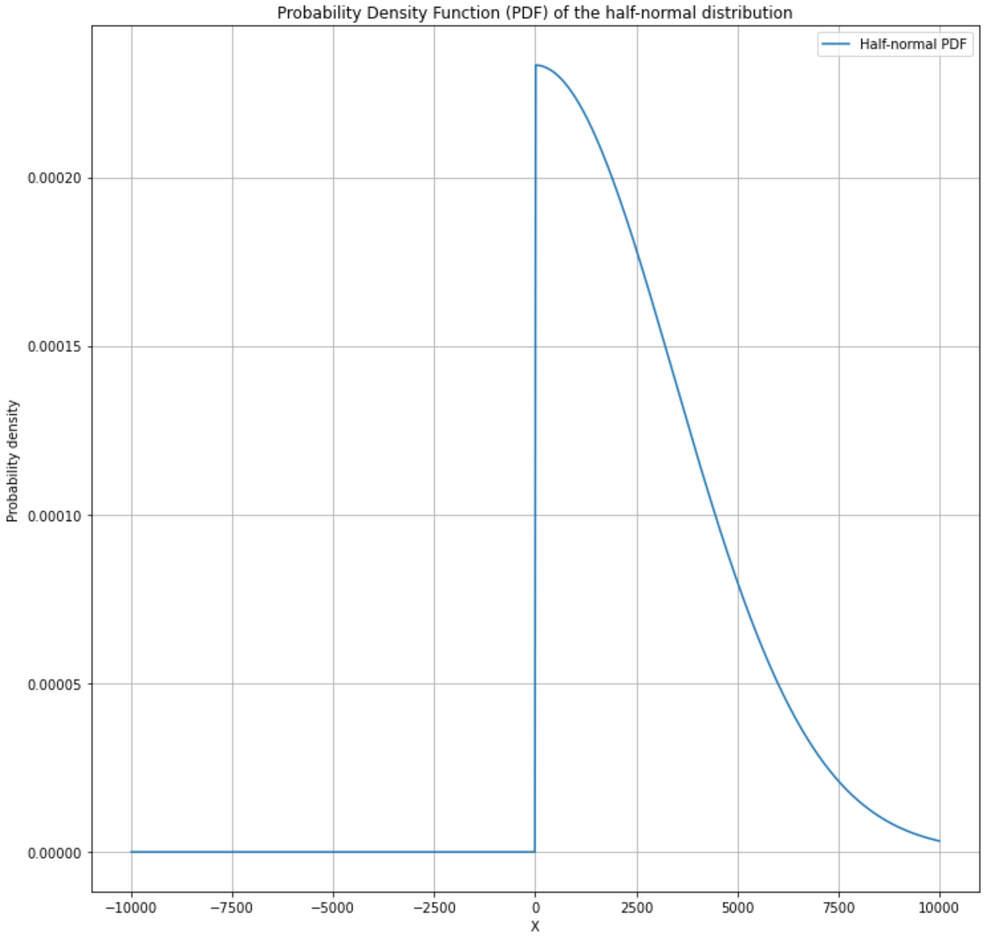

Figure 1. A probability distribution function of the half-normal distribution

To begin with, we introduce the sample program SP1 (see Lejeune and Prékopa, 2018), which is a general problem with joint chance constraints. It will serve as a basis for the later considerations. If $\boldsymbol{F}$ is a multivariate cumulative distribution function (CDF), the above joint probabilistic constraint (3) can be reformulated as follows:

$$F(Tx) \geq p \tag{4}$$

Then the problem SP1 can be formulated in this way:

SP1:

$$\min \quad c^T x \tag{5a}$$

$$s.t. \quad F(Tx) \geq p \tag{5b}$$

$$Ax \geq b \tag{5c}$$

$$x \geq 0 \tag{5d}$$

ESDP: Model formulation

Deterministic model (DM)

Petrenko (2024) formulates the underlying deterministic model for the ESDP assuming that the demand is known a priori and is derived from a given discrete probability distribution. Analogue to it, we introduce the deterministic model (DM) for the case with continuous demand using the similar notation below.

Table 1. Notation for the deterministic model without relaxation

Notation	Type	Description
$I;\ i = 1, \dots, \lvert I \rvert$	Set	A finite set of doctor's offices
$K; k = 1, \dots, \lvert K \rvert, K \neq I,$ $\lvert K \rvert < \lvert I \rvert$	Set	A finite set of medical laboratories
$J;\ j = 1, \dots, \lvert J \rvert$ $I, K \subseteq J$	Set	A finite set of potential locations for drone bases
$d_{ji}, d_{ji} \geq 0$	Parameter	Travel time (Euclidean distance) from a potential drone base j to a doctor's office i
$d_{ik}, d_{ik} \geq 0$	Parameter	Travel time (Euclidean distance) from a doctor's office i to a laboratory k
$d_{kj}, d_{kj} \geq 0$	Parameter	Travel time (Euclidean distance) from a laboratory k to a potential drone base j
$B, B > 0$	Parameter	Maximal allowed overall travel time of the drone due to a battery capacity
$S, S > 0$	Parameter	Maximal allowed service time from accepting a request to arriving to doctor's office
$W, W > 0$	Parameter	Maximal allowed weight of the freight in grams transported by the drone
$L = \{(k, j): d_{kj} \leq B\},$ $L \subseteq K \times J$	Set	A set of paths from a drone base location j to a laboratory k that are not larger than a battery capacity
$P = \{(i, k): d_{ik} \leq B\},$ $P \subseteq I \times K$	Set	A set of paths from a doctor's office i to a laboratory k that last not longer than the battery capacity
$M_{(k,j)} = \{i \in I,$ $(k, j) = 1, \dots, \lvert L \rvert :$ $\begin{cases} d_{ji} \leq S \\ d_{ji} + d_{ik} \leq B \end{cases}\}$	Set	A set of doctor's offices that can be covered by a path (k, j). A doctor's office i is covered by a path (k, j) if 1) the distance "start base – doctor's office" is not larger than a predefined service time,

		2) the distance "start base – doctor's office – laboratory" is not larger than a drone's battery capacity and 3) the distance "laboratory – start base" is not larger than a drone's battery capacity
$N_i = \{(k,j) \in L,$ $i = 1, \dots, \lvert I \rvert :$ $\begin{cases} d_{ji} \le S \\ d_{ji} + d_{ik} \le B \end{cases}\}$	Set	A set of all paths that can cover a doctor's office i. A path (k,j) can cover a doctor's office i if 1) the distance "start base – doctor's office" is not larger than a predefined service time, 2) the distance "start base – doctor's office – laboratory" is not larger than the drone's battery capacity and 3) the distance "laboratory – start base" is not larger than the drone's battery capacity
$O_j = \{(i,k) \in P,$ $j = 1, \dots, \lvert J \rvert :$ $\begin{cases} d_{ji} \le S \\ d_{ji} + d_{ik} \le B \end{cases}\}$	Set	A set of all paths (i,k) that can be covered by a location j. A path (i,k) can be covered by a location j if 1) the distance "start base – doctor's office" is not larger than a predefined service time and 2) the distance "start base – doctor's office – laboratory" is not longer than the drone's battery capacity
$KJ_{feas} = \{(k,j) : M_{(k,j)} \neq \emptyset\}$	Set	Set of paths (k,j), which cover at least one doctor's office i
$I_{feas} = \{i : N_i \neq \emptyset\},$ $i_{feas} = 1, \dots, \lvert I_{feas} \rvert$	Set	Set of doctor's offices i, which are covered at least by one path (k,j)
$J_{feas} = \{j : O_j \neq \emptyset\},$ $j_{feas} = 1, \dots, \lvert J_{feas} \rvert$	Set	Set of locations j, which cover at least one path (i,k)
$x_{ikj}, x_{ikj} \in \mathbb{Z}_0^+, i \in I, (k,j) \in L$	Decision variable	A number of drones located at j which are used to cover the service requests at the doctor's office i using a laboratory k
$y_j = \begin{cases} 1 \\ 0 \end{cases}, j = 1, \dots, \lvert J \rvert$	Decision variable	1, if the location j is open, 0 otherwise
α	Parameter	Costs for buying and maintaining one drone
β_j	Parameter	Fixed costs for opening and maintaining a drone base location j
γ	Parameter	Variable costs per kilometre
$q_j, q_j \in \mathbb{Z}_0^+, j = 1, \dots, \lvert J \rvert$	Parameter	A number of drones which can be placed at the location j (a drone base capacity)
$\theta_i, \theta_i \in \mathbb{R}_0^+, i = 1, \dots, \lvert I \rvert$	Parameter	A realization of a service request (weight to transport) generated at the doctor's office i

$p, p \in [0; 1]$	Parameter	A reliability level, i. e. a probability with which random service requests are covered

The DM can be formulated as follows:

$minimize$	$\sum_{i \in I_{feas}} \sum_{(k,j) \in KJ_{feas}} (a + \gamma \cdot (d_{ji} + d_{ik} + d_{kj})) \cdot x_{ikj} + \beta_j \sum_{j \in J_{feas}} y_j$	Minimize total costs: 1) Costs for buying and maintaining of drones, 2) variable costs per kilometre and 3) fixed costs for opening and maintaining a base location *j*	(6a)
$s.t.$	$\sum_{(k,j) \in N_i \neq \emptyset} W \cdot x_{ikj} \geq \theta_i,$ $i = 1, \dots, \lvert I_{feas} \rvert$	Covering constraints ensuring that the maximum transporting capacity of all drones is larger than the demand	(6b)
	$\sum_{(i,k) \in O_j \neq \emptyset} x_{ikj} \leq q_j y_j,$ $j = 1, \dots, \lvert J_{feas} \rvert$	Capacity constraints limiting the number of drones which can be located at each candidate station *j*	(6c)
	$d_{ji} x_{ikj} \leq B \cdot x_{ikj},$ $(k, j) \in KJ_{feas}, i \in I_{feas}$	Distance constraints limiting the overall distance according to drones' battery capacity	(6d)
	$d_{ji} x_{ikj} \leq S \cdot x_{ikj},$ $(k, j) \in KJ_{feas}, i \in I_{feas}$	Distance constraints limiting the service time according to a threshold	(6e)

$y_{j \in K} = 1$	Creating a drone base in laboratories	(6f)
$x_{ikj} \in \mathbb{Z}_0^+$ $y_j \in \{0,1\}, \forall i = 1, \ldots, \lvert I_{feas} \rvert,$ $j = 1, \ldots, \lvert J_{feas} \rvert, k = 1, \ldots, \lvert K_{feas} \rvert$	Non-negative integer and binary variables	(6g)

Probabilistic model (PM)

For the probabilistic model, we assume that the demand at each doctor's office is a random variable and replace the demand covering constraints by joint chance constraints in the deterministic model.

Table 2. Notation for the deterministic model without relaxation

Notation	Type	Description
$\theta_i, \theta_i \in \mathbb{R}_0^+, i = 1, \ldots, \lvert I \rvert$	Random variable	A service request (weight to transport) generated at the doctor's office i

The probabilistic model (PM) is formulated as follows:

$minimize$ (6a)

$s.t.$	$\mathbb{P}(\sum_{(k,j) \in N_i \neq \emptyset} W \cdot x_{ikj} \geq \theta_i,$ $i = 1, \ldots, \lvert I_{feas} \rvert) \geq p$	Covering constraints ensuring that the maximum transporting capacity of all drones is larger than the demand with the prescribed level of probability	(7)
	(6c) – (6g)		

If we look at the problem attentively, we can see that PM is a special case of SP1, so the later considerations applied to SP1 are also applicable to PM. However, from now on we will refer to the general problem SP1, providing certain numerical examples with PM.

Solving problems of type SP1

Background knowledge

Despite the possibility of modern solvers to solve such non-linear problems, they still require computing large multivariate probabilities and their gradients for joint continuous distributions, which is computationally very intensive.

Our basic paper of Lejeune and Prékopa (2018) offers several "relaxations" for SP1. The authors understand under "relaxations" approximations of SP1, where only univariate (first-order relaxations) or bivariate probabilities (second-order relaxations) are computed. In case of ESDP, cumulative distribution functions only for one or two doctor's offices' demands must be calculated.

There are inner and outer approximations (Figure 2). The feasible set of the original problem contains the feasible set of an inner approximation problem, and the feasible set of an outer approximation problem contains the feasible set of the original problem. For minimization problems, inner approximations provide an upper bound on the optimal objective value and outer approximations provide a lower bound on the optimal objective value. For maximization problems – vice versa, as we see it in the Figure 2. Lejeune and Prékopa (2018) list different possible bounds on the probabilities to formulate the approximations from them.

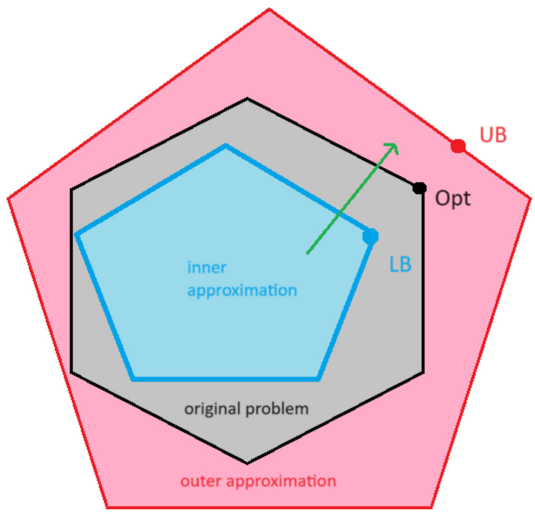

Figure 2. Illustration of feasible sets of an inner and outer approximation of the original optimization problem.

The authors also study the convexity of SP1 approximations: under what conditions the feasible sets of these relaxations are convex? Convex sets are sets in which when taking any two points of this set, the line segment between these points lies within this set. Convex feasible sets are useful in optimization, providing the opportunity to find the unique optimal objective value.

To answer the above question about convexity, we rely on Theorem 1 (Prékopa, 1995).

Theorem 1. *If* $g_i(x,\xi) \geq 0, i = 1,\dots,r$ *are concave functions (in case of SP1,* $g_i(x,\xi) = T_i x - \xi_i, i = 1,\dots,r$*) are linear functions, which are concave) and* ξ *has a logarithmically concave probability density function, then* $\mathbb{P}(g_i(x,\xi) \geq 0, i = 1,\dots,r)$ *is a logarithmically concave function of* x.

Theorem 1 implies (Prékopa, 1995) that the feasible set of SP1 is convex. Log-concave and concave functions are important in stochastic programming and convexity analysis. Log-concave function is a function whose logarithm is a concave function, a concave function is a function with a negative or zero second derivative. Concavity implies log-concavity: a concave function is also log-concave.

Let us check whether the multivariate non-degenerate half-normal distribution function is log-concave to ensure that the feasible set of SP1 is convex. Non-degenerate (half-)normal distribution means that the covariance matrix is positive-definite, so if it defines an inner product and does not turn the denominator of the PDF into zero (so the PDF exists).

Proof. For the non-degenerate multivariate half-normal distribution, we have $\boldsymbol{x} = |\boldsymbol{Z}|$, where $\boldsymbol{Z} \sim N_r(\mathbf{0}, \boldsymbol{\Sigma})$. As soon as we have that $x_i = -z_i$ or $x_i = z_i$, we have to "double" the probability density function for each dimension for $\boldsymbol{x} \geq 0$, that is why the PDF of the multivariate normal distribution is multiplied by 2^r (8). $\boldsymbol{x}$ is non-negative because of the constraint (5d). For the non-degenerate normal distribution (), it was proven that its PDF is log-concave (Lejeune and Prékopa, 2018). Products of two log-concave functions (as we can consider a constant a log-concave function) are log-concave. That is why the PDF of the non-degenerate half-normal distribution is concave and problem SP1 is convex. ■

$$f(x_1, \dots, x_r) = \frac{2^r}{\sqrt{2\pi^r |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{x}\right) \tag{8}$$

In the following sections, we will discuss the approximations considered in our basic paper Lejeune and Prékopa (2018), relying on the introduction information above. The approximations are based on different bounding schemes. Indeed, if we know a lower or an upper bound on the value of the multivariate probability in SP1 (left-hand side of (5b)), we can take this bound and replace the multivariate probability with it in the constraint (5b). The optimal objective value of this replacement will be the value that should be "at least" or "at most" reached by solving the original problem – in other words, it provides an upper or lower bound on the objective value of SP1. In case of an inner approximation, the replacement provides a feasible solution to the original problem.

First-order relaxations

Relaxations with Boole's inequality

Let event A_i be the event that the demand on the doctor's office i is smaller than the capacity of reserved for this office drones (the demand is covered), mathematically: $A_i : \xi_i \leq T_i x, i = 1, \dots, r$, and $\overline{A_\iota}$ it's complement (the demand is not covered). We will try to find the bound on the intersection of events $\bigcap_{i=1}^{r} A_i$ because in our constraints these events must occur simultaneously as we have a joint chance constraint. We will try to derive the bounds using probabilities of marginal events $\mathbb{P}(A_i)$ to reduce dimensionality for program solving.

If we take Boole's inequality to $\overline{A_\iota}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{r} A_i\right) \leq \sum_{i=1}^{r} \mathbb{P}(A_i) \tag{9}$$

we get:

$$\mathbb{P}\left(\bigcup_{i=1}^{r}\overline{A_i}\right) \leq \sum_{i=1}^{r}\mathbb{P}\left(\overline{A_i}\right) \Leftrightarrow 1-\mathbb{P}\left(\bigcup_{i=1}^{r}\overline{A_i}\right) \geq 1-\sum_{i=1}^{r}\mathbb{P}\left(\overline{A_i}\right) \Leftrightarrow$$
$$\Leftrightarrow \mathbb{P}\overline{\left(\bigcup_{i=1}^{r}\overline{A_i}\right)} \geq 1-\sum_{i=1}^{r}\mathbb{P}\left(\overline{A_i}\right) \tag{10}$$

Applying De Morgan's inequality to the above result,

$$\bigcap_{i=1}^{r} A_i = \overline{\bigcup_{\iota=1}^{r}\bar{A}_\iota} \Leftrightarrow \mathbb{P}\left(\bigcap_{i=1}^{r} A_i\right) = 1-\mathbb{P}\left(\bigcup_{i=1}^{r}\bar{A}_\iota\right) \tag{11}$$

we get:

$$\mathbb{P}\overline{\left(\bigcup_{i=1}^{r}\overline{A_i}\right)} \geq 1-\sum_{i=1}^{r}\mathbb{P}\left(\overline{A_i}\right) \Leftrightarrow \mathbb{P}\left(\bigcap_{i=1}^{r} A_i\right) \geq 1-\sum_{i=1}^{r}\mathbb{P}\left(\overline{A_i}\right) =$$
$$= 1-\mathbb{P}\left(\overline{A_1}\right)-\cdots-\mathbb{P}\left(\overline{A_i}\right) =$$
$$= 1-\left(1-\mathbb{P}(A_1)\right)-\ldots-\left(1-\mathbb{P}(A_i)\right) = 1\underbrace{-1+\mathbb{P}(A_1)-\ldots-1+\mathbb{P}(A_i)}_{r\ times} =$$
$$= \sum_{i=1}^{r}\mathbb{P}(A_i)-r+1. \tag{12}$$

The above lower bound can be reformulated as:

$$\mathbb{P}\left(\bigcap_{i=1}^{r} A_i\right) \geq \sum_{i=1}^{r}\mathbb{P}(A_i)-r+1 \Leftrightarrow$$
$$\Leftrightarrow F(Tx) \geq \sum_{i=1}^{r}F_i(T_i x)-r+1. \tag{13}$$

As we see, the joint multivariate probability should be at least as large as the sum of marginal probabilities (cumulative probabilities that there are enough drones for each doctor's office) minus the total number of doctor's offices plus one. If the right-hand side is at least p, then the left-hand side is also at least p, however, it shortens the feasible set. Therefore, we can derive an inner approximation BO1 of SP1 (14a – 14c), replacing the probabilities by the notation of a cumulative distribution function (4). The optimal objective value of BO1 provides an upper bound on the objective value of SP1:

BO1:

$$min \quad c^T x \tag{14a}$$

$$s.t. \quad \sum_{i=1}^{r} F_i(T_i x) - r + 1 \geq p \iff \iff \sum_{i=1}^{r} F_i(T_i x) \geq r - 1 + p \tag{14b}$$

(5c) – (5d)

We can also introduce "implied" individual constraints to BO1 (14c).

$$s.t. \quad F_i(T_i x) \geq p \tag{14c}$$

(14c) is a necessary, but not sufficient condition. Indeed, (14b) implies (14c) because the sum of marginals should be at least $r - 1 + p$, where each term can be maximum equal to 1. It means that $(r - 1)$ terms must be equal to 1 and the last term must be at least p. In the example of ESDP, the overall performance of the emergency medical system on a reliability level p requires that at each doctor's office the service level must be at least p.

Let us show that BO1 is convex.

Proof. The multivariate half-normal distribution is log-concave. According to properties of log-concavity, log-concavity preserves for the marginals F_i of F. However, the summation in (14b) does not preserve log-concavity.

It can be proven for the univariate half-normal distribution that it is concave, which is even stronger than log-concavity and is preserved in the summation. The probability density function is the first derivative of the cumulative distribution function. Therefore, the first derivative of the PDF is the second derivative of the CDF. Applying it to the PDF of the half-normal distribution:

$$f'^{(x_i,\sigma)} = \frac{d}{dx}\left[\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\exp\left(-\frac{x_i^2}{2\sigma^2}\right)\right] =$$

$$= \underbrace{\frac{\sqrt{2}}{\sigma\sqrt{\pi}} \cdot \exp\left(-\frac{x_i^2}{2\sigma^2}\right)}_{>0} \cdot \left(-\frac{\overbrace{x_i}^{>0\ (constraint)}}{\underbrace{\sigma^2}_{>0}}\right) \leq 0 \tag{15}$$

So, we see that that the second derivative of the CDF is negative, that is why the CDF is concave with $x_i > 0$ (Figure 3). Therefore, the feasible set of each (14d) constraint is convex since it requires a concave function to be at least equal to a constant (it borders the concave function below making a "cave" out of it). Moreover, the same applies to (14b), because the sum of concave functions is concave, and it should be also at least equal to a constant $p + r - 1$. ■

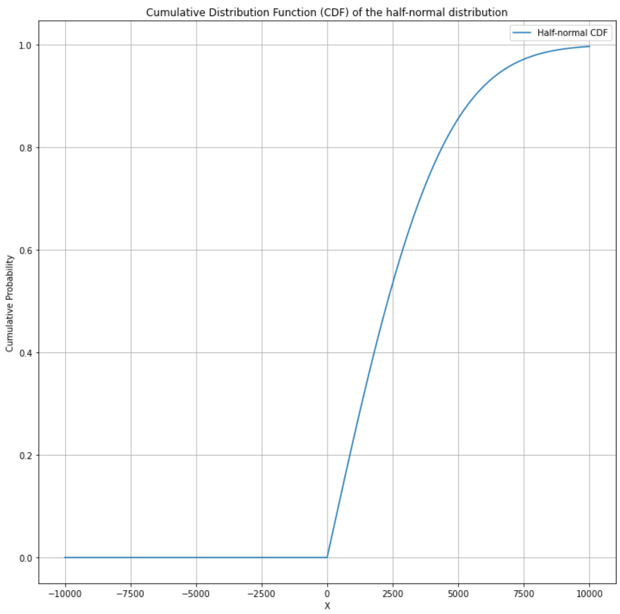

Figure 3. Concave cumulative distribution function of a half-normal distribution (example).

We can derive an inner approximation BO2 (Prékopa 1995), equivalent to BO1, by simply lifting the decisional and constraint spaces. Note that both models are actually the same and produce the same results. Let $q_1, \dots, q_r$ be *auxiliary non-negative decision variables*, with which we rewrite the constraint (14b). We get:

BO2:

$$min \quad c^T x \tag{16a}$$

$$s.t. \quad F_i(T_i x) \geq q_i, i = 1, \dots, r \tag{16b}$$

$$1 - p \geq \sum_{i=1}^{r} (1 - q_i) \tag{16c}$$

$$p \leq q_i \leq 1, i = 1, \dots, r \tag{16d}$$

(5c) – (5d) (16e)

Note that constraints (16d) are implied by (16c) and just help to speed up the solution. Let us look at (16c): opening the summation sign we get:

$$1 - p \geq r - q_1 - \dots - q_r \Leftrightarrow$$

$$p \leq \underbrace{1 - r}_{\substack{\leq 0 \; as \; the \; number \; of \\ doctor's \; offices \\ is \; at \; least \; 1}} + q_1 + \dots + q_r \tag{17}$$

So $q_1 + \dots + q_r$ (17) should be at least $r - 1 + p$, which is possible, as in BO1 (14b), if $p \leq q_i$. Moreover, $q_i \leq 1$ is implied by (16b) since the probability is not larger than 1.

The constraints (16b) – (16c) are equivalent to (14b). Taking (16c) and following from it (17), we can rewrite (16c) as:

$$r - 1 + p \leq \sum_{i=1}^{r} q_i \tag{18}$$

Summing individual constraints (16b) we get:

$$\begin{cases} F_1(T_1 x) \geq q_1 \\ \dots \\ F_r(T_r x) \geq q_r \end{cases} \Leftrightarrow \sum_{i=1}^{r} F_i(T_i x) \geq \sum_{i=1}^{r} q_i \tag{19}$$

Combining (18) and (19) we get the constraint (14b) (20). (16d) are proven to be implied.

$$\sum_{i=1}^{r} F_i(T_i x) \geq \sum_{i=1}^{r} q_i \geq r - 1 + p \tag{20}$$

The proof of convexity under the half-normal distribution is similar to BO1: marginal probabilities are concave, therefore individual constraints (16b) are convex. ■

Let us denote F_i^{-1} as a quantile function (inverse of a marginal probability distribution function). Chen et al. (2010) has derived the following approximation:

BO3:

$$min \quad c^T x \tag{21a}$$

$$s.t. \quad T_i x \geq F_i^{-1}\left(1 - \frac{1-p}{r}\right), i = 1, \dots, r \tag{21b}$$

(5c) – (5d)

Note that $F_i^{-1}\left(1 - \frac{1-p}{r}\right)$ contains no variables and equals to a constant. Therefore, (21b) is a linear constraint and BO3 is a linear program. For example, for the service level 97% and 4 doctor's offices we get: $1 - \frac{1-p}{r} = 1 - \frac{1-0.95}{4} = 0{,}9875$-quantile of a half-normal distribution with a defined standard deviation and zero mean. Let us show that this problem approximates BO1 and BO2.

Proof. Consider a feasible solution $\bar{x}$ of BO3 where the constraints (21b) are binding (22) (the inequality sign is changed to the equality sign), so each marginal probability equals to the constant $1 - \frac{1-p}{r}$. We remind the reader that F_i^{-1} is the inverse of F_i. Replacing the summands (23), we get the constraint of BO1 (14b), so this solution and, therefore, all feasible solutions of BO3 satisfy the constraint (14b) and are feasible for BO1 and for the equivalent BO2. To sum up, the feasible set of BO1/BO2 includes the feasible set of BO3. The opposite, however, is not true. ■

$$F_i(T_i \bar{x}) = 1 - \frac{1-p}{r} \tag{22}$$

$$\sum_{i=1}^{r} F_i(T_i \bar{x}) = \left(1 - \frac{1-p}{r}\right) r = r - 1 + p \tag{23}$$

Being linear, problem BO3 is computationally much faster, however, it is rather conservative in decisions.

Relaxations with Slepian's product-type inequality

Slepian's inequality (Slepian 1962) was first stated as follows. If R and R' are correlation matrices such that $R \geq R'$ holding componentwise, then, for any standard normally distributed $z_1, \dots, z_i$ we get: $F(z; R) \geq F(z; R')$, where F is the r-variate standard normal probability distribution function. If ξ is distributed normally, we can standardize it as:

$$F\left(\frac{T_i x - \mu_i}{\sigma_i},\ i = 1,\ \dots, r; R\right) \tag{24}$$

For $R \geq R'$, we get the Slepian's inequality:

$$F\left(\frac{T_i x - \mu_i}{\sigma_i},\ i = 1,\ \dots, r; R\right) \geq F\left(\frac{T_i x - \mu_i}{\sigma_i},\ i = 1,\ \dots, r; R'\right) \tag{25}$$

Clearly, similar to the logic of BO1, we can impose the probabilistic requirement on the right-hand side of (25) and derive an inner approximation S1 of SP1, shortening the feasible set of SP1:

S1:

$$min \quad c^T x \tag{26a}$$

$$s.t. \quad F\left(\frac{T_i x - \mu_i}{\sigma_i},\ i = 1,\ \dots, r; R'\right) \geq p \tag{26b}$$

(5c) – (5d)

If the correlations are all non-negative or non-positive, the inequality can provide sharp bounds. Using the second correlation matrix R' as an identity matrix (no correlations between doctor's offices) and any, for example, positive R, we get (27), since probabilities of independent events are equal to their product:

$$F\left(\frac{T_i x-\mu_i}{\sigma_i}, i=1,\dots,r; R'\right)=\prod_{i=1}^{r} F_i\left(\frac{T_i x-\mu_i}{\sigma_i}\right)\geq p \tag{27}$$

and, taking a logarithm of both sides, (28).

$$\ln\left(F\left(\frac{T_i x-\mu_i}{\sigma_i}, i=1,\dots,r; R'\right)\right)=\sum_{i=1}^{r}\ln\left(F_i\left(\frac{T_i x-\mu_i}{\sigma_i}\right)\right)\geq \ln p \tag{28}$$

For ξ with a log-normal density function we derive the following convex approximation S2:

S2:

$$min \quad c^T x \tag{29a}$$

$$s.t. \quad \sum_{i=1}^{r}\ln\left(F_i\left(\frac{T_i x-\mu_i}{\sigma_i}\right)\right)\geq \ln p \tag{29b}$$

(5c) – (5d)

Proof. Convexity results from Theorem 1: with a log-concave density function, each term in the left-hand side of (29b) is a concave function. Concavity is preserved in summations, that is why the problem is convex.

Second- and higher order relaxations

Setwise bounding schemes dependence bounding problems

We will introduce a generalization of Slepian's inequality bounds on the probability of intersection of events A_i, as in the chapter "Relaxations with Boole's inequality". The bounds rely on the concept of set dependence (Chhetry et al., 1989; Costigan, 1996). As Slepians's inequality, setwise bounds require positive dependence conditions. However, these conditions are weaker.

Let us divide the random vector to r_k-dimensional partitions (subvectors) $\tilde{\xi}_k, k=1,\dots,v$: $\xi=\left(\tilde{\xi}_1,\dots,\tilde{\xi}_v\right)$, such that $\mathbb{P}(\xi\leq Tx)=\mathbb{P}(\tilde{\xi}_k\in L_k, k=1,\dots,v)$, where $L_k,, k=1,\dots,v$ is a collection of independent events. It means that the overall multivariate probability that the demand of the EMS system is covered equals to the multivariate probability that the demand is covered at each partition. We will come later to an example.

Chhetry (1989) proved the following: let $B_k = \{i : \xi_i \in \tilde{\xi}_k,\ i = 1, \dots, r\},\ k = 1, \dots, v$ (B_k is the collection of offices in one partition $\tilde{\xi}_k$) and $\rho_{ij} = corr(\xi_i, \xi_j)$ – correlation between two doctor's offices belonging to different partitions. If the partitions are normally distributed, then:

$$\mathbb{P}(\xi \leq Tx) \geq \prod_{k=1}^{v} \mathbb{P}(\tilde{\xi}_k \in L_k) \tag{30}$$

$$\text{if } \rho_{ij} > 0 \; for\ each\ i \in B_k\ and\ j \in B_{k'}, k \neq k', \tag{31}$$

So (31) it means that positive dependencies should be between the elements of different partitions. This is a less strict requirement than in Slepian's inequality. Partitions $\tilde{\xi}_k$ are in this case positive lower orthant dependent (it means that (30) holds). If the correlations are negative, the inequality sign changes to $\leq$.

Example. Assume that we have 5 doctor's offices. The correlation between the coverage of demand at each two locations is described with the matrix and the partition to subvectors $\tilde{\xi}_k$ is shown with the colour (Table 3): $\tilde{\xi}_1 = [\text{Office 1, Office 2, Office 3}]$ and $\tilde{\xi}_2 = [\text{Office 4, Office 5}]$. Note that the events of demand coverage within each group of doctor's offices are independent: for example, the events that the demand is covered both at Office 1, Office 2 and Office 3 cannot happen at the same time.

Table 3. Correlation matrix for partitions.

Corr	**1**	**2**	**3**	**4**	**5**
1	1				
2	<0	1			
3	<0	<0	1		

4	>0	>0	>0	1	
5	>0	>0	>0	<0	1

Then the following holds:

$$\mathbb{P}(\xi_i \leq T_i x,\ i = 1, \dots, 5) \geq$$
$$\geq \mathbb{P}(\xi_1 \leq T_1 x, \xi_2 \leq T_2 x, \xi_3 \leq T_3 x)\mathbb{P}(\xi_4 \leq T_4 x, \xi_5 \leq T_5 x) \tag{32}$$

In words, the multivariate probability that the demand is covered is larger than the product of multivariate probabilities in each group of doctor's offices if the offices demands within the group are independent.

As we see, in contrast to Slepian's inequality, the setwise bounds require positive dependence only for partitions, but within the partitions correlations can be negative.

Let $\tilde{\xi}_1, \dots, \tilde{\xi}_v$ be such that (31) holds. As it follows from (30), non-linear problem BS1 is an inner approximation of SP1 providing an upper bound on the optimal objective value and for negative correlations $\rho_{ij} < 0\ for\ each\ i \in B_k\ and\ j \in B_{k'}, k \neq k'$, BS1 is an outer approximation of SP1 providing a lower bound on the optimal value.

<u>BS1:</u>

$$min \quad c^T x \tag{33a}$$

$$s.t. \quad \prod_{k=1}^{v} F(T_k x) \geq p \tag{33b}$$

(5c) – (5d)

For the lognormal density function, (33) becomes a linear and convex problem (since we can take the logarithm of both sides). The proof is similar to S2: convexity results from Theorem 1: with a log-concave density function, each term in the left-hand side of (33b) is a concave function. Concavity is preserved in summations, that is why the problem is convex. ■

Note that the partitions can be not unique and is the task is not trivial. From the computational point of view, it is better to have small partititions of, for instance, 1-3 components, to easier compute the multivariate probabilities. The authors Lejeune and Prékopa (2018) develop 1) two MILPs which define the maximal cardinality (size) partition of ξ and 2) an MILP which defines a partition in which the size of the largest subvector is minimal. However, to reduce the scope of this seminar paper, we will not describe them in detail.

Overview of other higher-order bounding schemes

Again, to reduce the scope of the seminar paper, we will not describe several bounding schemes mentioned in the basic paper in detail. However, we will provide an overview of the approaches.

We remind the reader that the event A_i is the event that the demand on the doctor's office i is smaller than the capacity of reserved for this office drones (the demand is covered), mathematically: $A_i : \xi_i \leq T_i x, i = 1, \dots, r$, and $\overline{A_\iota}$ is it's complement (the demand is not covered).

The first second-order approach is based on **a Boole-Bonferroni's inequality** (33), providing the bound on the union of events that can be reformulated to the intersection of events by (11). The bound was subsequently tightened, especially with a graph theory (spanning trees), and approximations of SP1 were derived, which are convex under certain conditions, proven for the normal distribution.

$$\sum_{i=1}^{r} \mathbb{P}(A_i) - \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \mathbb{P}(A_i, A_j) \leq \mathbb{P}\left(\bigcup_{i=1}^{r} A_i\right) \leq \sum_{i=1}^{r} \mathbb{P}(A_i) \tag{33}$$

The second bounding scheme is based on **binomial moments** and the respective binomial moment problem (Prékopa, 1999). Denoting by ν the number of events that occur out of $A_1, \dots, A_r$, the binomial moment S_k is defined as:

$$S_k = \sum_{1<i_1 \dots <i_k<r} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}), k = 1, \dots, r \tag{34}$$

Example. We have 4 doctor's offices and 4 events that the demand at these locations is covered: $A_1, \dots, A_4$. We want to compute the second moment S_2. We choose possible 2-

combinations (i_1, i_2) such that $1 < i_1 < i_2 < 4$: (1,2), (1,3), (1,4), (2,3), (2,4), (3,4). The moment is computed as follows

$$S_2 = \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap A_3) + \cdots + \mathbb{P}(A_3 \cap A_4) \tag{35}$$

The concept relies on the fact that the first m binomial moments $S_1, \dots, S_m$ (for example, the first and the second with maximum bivariate probabilities) can be computed conveniently.

The last second-order approach is bounding with the **sum of disjoint products** and is based on the addition law of probabilities. It relies on the idea that if two or more events have nothing in common, the probability that at least one of them will occur is the sum of the probabilities of the separate events. The union of r complemented events $\overline{A_\iota}$ can be written as:

$$\bigcup_{i=1}^{r} \overline{A_\iota} = \underbrace{(\overline{A_1} A_2 \dots A_r)}_{V_1} \cup \underbrace{(\overline{A_2} A_3 \dots A_r)}_{V_2} \cup \dots \cup \underbrace{(\overline{A_{r-1}} A_r)}_{V_{r-1}} \cup \underbrace{\overline{A_r}}_{V_r} \tag{34}$$

All the sets $V_i, i = 1, \dots, r$ are disjoint and constructed as the product of binary variables. Their disjointedness implies that the probability of their union is equal to the sum of their individual probabilities (34), where we can apply (11) and derive the probability of the intersection of events.

$$\mathbb{P}\left(\bigcup_{i=1}^{r} \overline{A_i}\right) = \sum_{i=1}^{r} \mathbb{P}(V_i) \tag{34}$$

The resulting bound is reduced to the second order (only bivariate probabilities). However, the resulting approximation is not convex.

Computational experiments

Approach

In this section we will apply the described approximations of the probabilistically constraint program SP1 to the case of ESDP. The basic problem and problem approximations are mixed-integer non-linear (MINLP) programs. However, for the sake of simplicity, we will not apply to non-traditional MINLP solvers and will try to use Gurobi for these purposes.

As we do not want to compute gradients for deriving second- and higher-order probabilities, we will stop on the approximations with marginal probabilities and approximations which are convex under the assumption of the half-normal distribution. These are inner approximations BO1, BO2 and BO3. The deterministic model DM was used as a benchmark.

To avoid non-linear integer programming, piecewise linear approximations of half-normal CDFs for each doctor's office were used. Linear approximations were conducted with the step 0.005, approximating the probabilities after 0.995 as multiple 1-s to avoid numerical issues.

Instance generation

For the first step of computational experiments, several instances with following parameters we created (Table 4).

Table 4. Parameter values for computational experiments

Parameter	Basic analysis	Robustness analysis
Random seed	108448	[0, 49]
$\lvert I \rvert$	Randomly (uniformly) placed, {5; 10; 15; 20; 25}	7
$\lvert J \rvert$	Randomly (uniformly) placed, {100; 200; 300; 400; 500}	100
$\lvert K \rvert$	Randomly (uniformly) placed, 1	1

p	For all instances 0.95, for I2: {0.95, 0.97, 0.99}	0.9
σ_i	Randomly (uniformly): from 2000 g to 5000 g	Randomly (uniformly): from 10000 g to 70000 g
B	Speed of a drone 17 m/s × (90 × 60 s of travel) = 91 800 m	
S	Speed of a drone 17 m/s × 60 s × 1 min = 1 020 m	
W	2500 g per drone	
α	Purchasing cost of a drone 14 760 € + transport box 500 € + spare battery 640 € = 15 900 €	
$\beta_{j\in J\setminus\{I\cup K\}}$	Rent with extra costs 15 €/m^2 × free space 100 m^2 × 12 months + salary of 3 drone technicians 55 000 € × 3 + equipment 20 000 € = 203 000 €	
$\beta_{j\in\{I\cup K\}}$	Rent with extra costs 8 €/m^2 × free space 20 m^2 × 12 months + salary of a drone technician 55 000 € + equipment 20 000 € = 76 920 €	
γ	0.0045 €/km / 1000	
$\beta_{j\in J\setminus\{I\cup K\}}$	3 drones/m^2 (using shelves) × shelf space (100 – 15) m^2 = 255 drones	
$\beta_{j\in\{I\cup K\}}$	3 drones/m^2 (using shelves) × shelf space (20 – 5) m^2 = 45 drones	
Computational time limit	200 seconds	

Results

Unfortunately, we cannot compute the original problem SP1 with Gurobi to see the precision of the approximations. However, the results of the basic paper show that the difference in the objective values is minimal.

Performance results are presented in the table below:

Table 1. Gap and computational time

Instance	Gap				Computational time in sec (rounded)			
	DM	BO1	BO2	BO3	DM	BO1	BO2	BO3
I1	0.0000%	0.0000%	0.0000%	0.0000%	0	0	0	0
I2	0.0000%	0.0043%	0.0043%	0.0000%	0	12	12	0
I3	0.0000%	0.2923%	0.5033%	0.0000%	0	200	200	0
I4	0.0000%	0.0892%	0.0829%	0.0000%	0	200	200	0
I5	0.0000%	5.9196%	6.0754%	0.0000%	0	200	200	0

BO3 is very fast to compute, which supports the authors' considerations. The approximations BO1 and BO2 become harder to solve to optimality with the growing size of instances, even when they are relatively small.

With the growing p, the number of drones increases (Figure 4). The number of drones for the probabilistic models is much larger than of the deterministic one. The approximation BO3 is quite conservative and overestimates the number of drones, which is in line with the basic paper. Moreover, it is less sensible to the changing level of reliability. BO1 and BO2 are more precise in terms of changing the reliability.

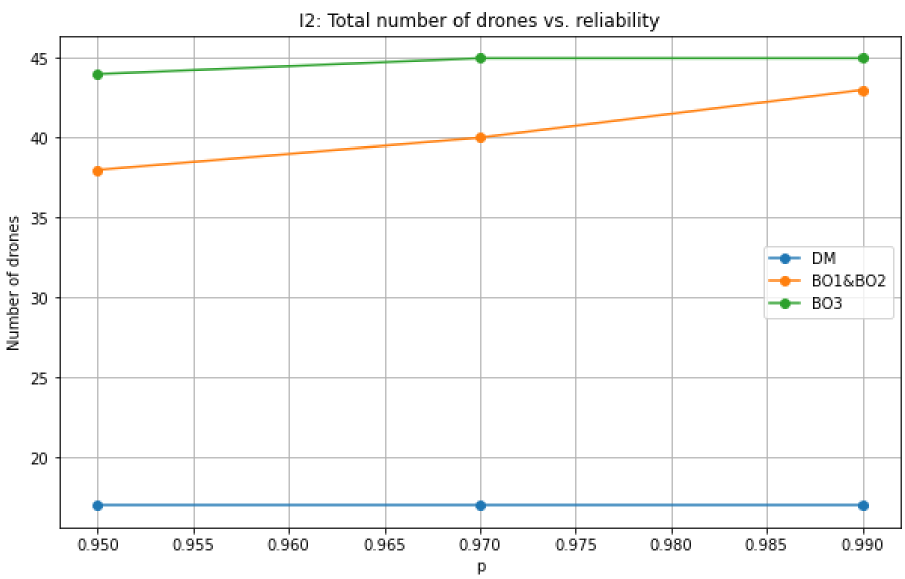

Figure 4. Total number of drones vs. reliability for the second instance.

The dynamics of the objective value are presented below. The growth of the objective value for the probabilistic models with the growing instance size is faster. There are many factors which can influence the results: a sample structure of the realizations of the demand, the growing error in demand estimation and, therefore, the need to build new drone base locations.

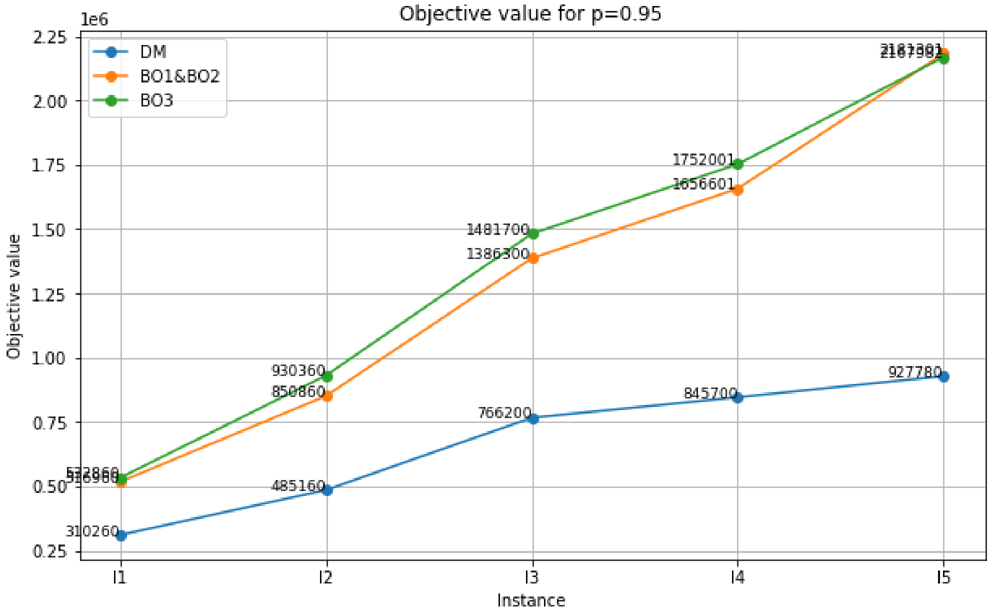

Figure 5. Objective value for different instances

Despite of being relatively conservative, the advantage of probabilistic models is their robustness. Let us calculate the service level for different demand realizations to measure the model's performance under uncertainty. The service level is calculated as:

$$SL = 1 - \frac{\sum_{i=1}^{r} missing\ drones\ at\ doctor's\ office\ i}{Actual\ needed\ number\ of\ drones} \tag{35}$$

The results of simulation are presented in the Figure 6. All three models are robust to different demand realizations except for two instances for BO1 and BO2 and one instance for BO3. This can be also explained by the methodology of service level calculation. Despite this, the total number of drones was always larger than the demand. So the drawbacks can be compensated by rearranging the drones in real time.

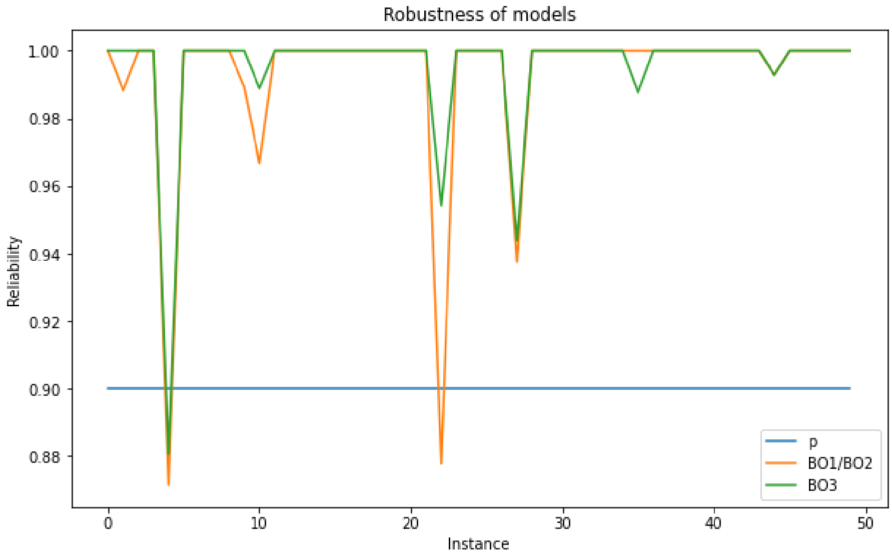

Figure 6. Robustness of the models for $p = 0.9$

Conclusion

In this seminar paper, we revisit the Emergent Specimen Delivery Problem of Petrenko (2024), but this time with the continuous demand and dependency structure between the doctor's offices' emergencies. The uncertainty is expressed through the joint chance constraints imposed on the entire system. We review the relaxations of such problems presented in the basic paper of Lejeune and Prékopa (2018). A "relaxation" means that we only need to compute univariate and bivariate probabilities instead of large, computationally intensive multivariate probabilities. We conduct computational experiments with three relaxations with only marginal probabilities. Even if the costs are larger, the models are proven to be robust to different demand realizations. The problem BO3 is more conservative than BO1 and BO2.

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