Stable Matchings: an Integer Linear Programming (ILP) Application

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Introduction to ILP: Subset Sum

Structure of an ILP:
- Linear constraints consisting of equalities / inequalities
- An optional linear objective function
- All variables must take on integer values

Ex. Subset Sum: Given a set $S = \{a_1, a_2, \ldots a_n\}$ and a target t, determine if there exists a subset of S such that all the values sum to t.

Useful Definitions:
- $x_i = 1$ represent the i^{th} element of S is included in the subset and $x_i = 0$ otherwise.

Linear Constraints:
- At least one element of the original set should be included in the subset:

$$\sum_{i=1}^{n} x_i \geq 1$$

- The Sum of all included elements should equal the target:

$$\sum_{i=1}^{n} x_i * S_i = t$$

Ex.

$S = \{1, 4, 2, 9, 3, 12\}$. $t = 8$

$$X = \begin{array}{cccccc} 1 & 4 & 2 & 9 & 3 & 12 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 \end{array}$$

$1*1 + 1*4 + 0*2 + 0*9 + 1*3 + 0*12 = 8$

So, a subset of S which sums to 8 exists, and it is $\{1, 4, 3\}$.

Stable Marriage

Problem Statement: Given a set M of men and a set W of women with ranked preference lists, give a matching where there is no **blocking pair**. A blocking pair is defined as a pair (m, w) such that m and w would rather be matched with each other than their current partner

Useful Definitions:
- $x(m, w) = 1$ if man m is married to woman w, $x(m, w) = 0$ otherwise
- $P_w(m, w)$ is the set of men that women w prefers less than m
- $P_m(m, w)$ is the set of women that m prefers more than w

Women:		Men:	
1	CBA	A	321
2	CBA	B	213
3	BCA	C	231

$$X = \begin{array}{c|ccc} & A & B & C \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}$$

Linear Constraints:
- Each man can marry one woman, each woman can marry one man:

For each man m: For each woman w:

$$\sum_{w \in W} x(m, w) = 1 \qquad \sum_{m \in M} x(m, w) = 1$$

- Ensure that there is no blocking pair

For each pair (m, w):

$$\sum_{m' \in P_w(m,w)} x(m', w) - \sum_{w' \in P_m(m,w)} x(m, w') \leq 0$$

Objective Function
- function to generate a woman-optimal matching

$$\sum_{m \in M} \sum_{w \in W} x(m, w) * (\text{index of } m \text{ in } w\text{'s preferences})$$

The CS Match

Problem Statement: Given a set S of students and a set C of courses, each with rankings of the opposite group featuring ties and incomplete lists, return a matching where there is no **blocking pair** (s, c) where both would rather be matched with each other than their current assignment.

Useful Definitions:
- $x_{sc} = 1$ if student s is matched to course c, $x_{sc} = 0$ otherwise
- $P_c(c, s)$ is the set of courses that s prefers to c
- $P_s(c, s)$ is the set of students that c prefers to s

Linear Constraints
- Each student can only be matched to one course:

For each student s:

$$\sum_{c \in C} x_{sc} \leq 1$$

- Each course has a capacity of students:

For each course c:

$$\sum_{s \in S} x_{sc} \leq capacity$$

- Ensure that there is no blocking pair

For each pair (s, c):

$$capacity\left(1 - \sum_{c' \in P_c(c,s)} x_{sc'}\right) \leq \sum_{s' \in P_s(c,s)} x_{s'c}$$

Ex.

Students:	Courses:
1: AB	A: 1324
2 : AB	cap = 2
3 : BA	B : 2134
4 : AB	cap = 2

$$X = \begin{array}{c|cc} & A & B \\ \hline 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 0 & 1 \end{array}$$

Explanation: The left side of the inequality will be 0 if the student is in a course they prefer to c and will be the capacity of c otherwise. The right side will be the number of students matched to c that c prefers to s. Therefore, this inequality will be false when s would prefer to be in course c and there is space, but they are not in the course.

Objective Function:
- Match as many students to courses as possible

$$\sum_{s \in S} \sum_{c \in C} x_{sc}$$

The Weighted CS Match

Problem Statement: As an extension to the CS match problem, we purpose the student-weighted CS match problem which allows students to have weighted preference lists. We then select the stable matching which maximizes student's rankings.

Modifications:
- Students input their course preferences as weights
- These weights are normalized for each student

Useful Definitions:
- w_{sc} the weight with which student s prefers course c

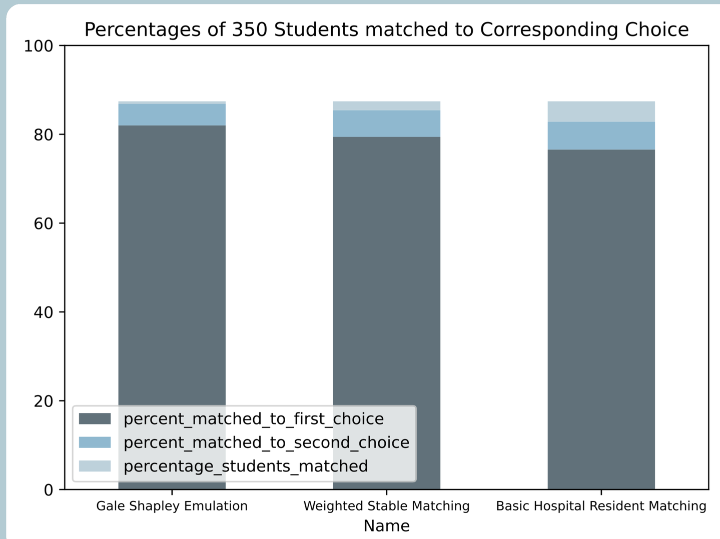
Linear Constraints: see the CS Match

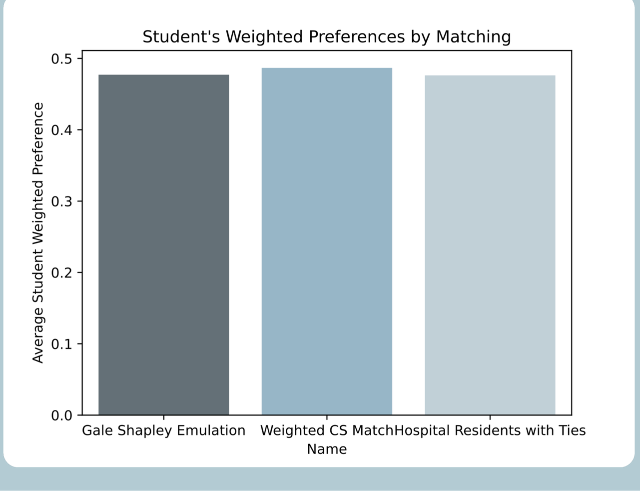
Objective Function:

$$large\_constant * \sum_{s \in S} \sum_{c \in C} x_{sc} + \sum_{s \in S} \sum_{c \in C} x_{sc} * w_{sc}$$

Analysis

We found that there are pros and cons to each method of stable matching students to courses. The Gale-Shapley Emulation ILP tends to match more students to their first or second choice class, but fewer to their first choice.





Furthermore, we see that the weighted stable matching ILP does find a solution which maximizes the students' weighted preferences for 350 students. This results in fewer students being matched to their first choice than in Gale-Shapley.

Future Directions

In [3], Pini *et al.* prove that incorporating weights in stable matching problems leads to increased manipulation opportunity, using versions of the weighted stable matching problem that modify the definition of a blocking pair to include weights. An extension of the student courses matching problem to maximize weights is a proof of whether this increases student's ability to obtain a better matching by falsely reporting their preferences. For example, consider a student who desires their first choice 70% and their second choice 30%. By assigning all their weight to one course, they may have a higher probability of matching to that course. However, if there is no stable matching with that student and their first choice, now they have decreased probability of matching to their second choice. So, an extension to this project is a rigorous investigation of this phenomenon.

Works Cited

[1] Gusfield, D. & Irving, R. W. (1989). The stable marriage problem : structure and algorithms. MIT Press.
[2] Delorme, M., García, S., Gondzio, J., Kalcsics, J., Manlove, D., & Pettersson, W. (2019). Mathematical models for stable matching problems with ties and incomplete lists, European Journal of Operational Research
[3] Pini, M. S., Rossi, F., Venable, K. B., & Walsh, T. (2011). Weights in stable marriage problems increase manipulation opportunities. TARK XIII. Association for Computing Machinery. 200–204.

Acknowledgments

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