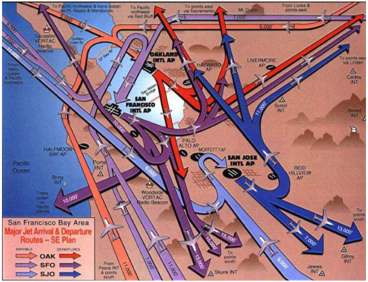
Distributed and Structured Control: The Impact of Youla-Kucera Parametrization

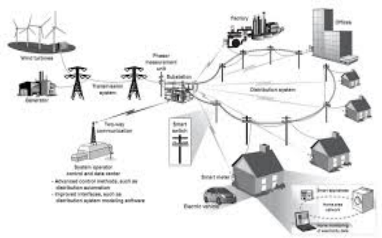
Petros G. Voulgaris

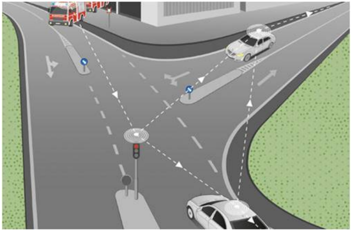
University of Illinois at Urbana-Champaign
Coordinated Science Laboratory

University of Science and Technology of China
October 18, 2019

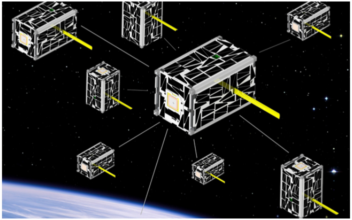
Modern Systems are Complex



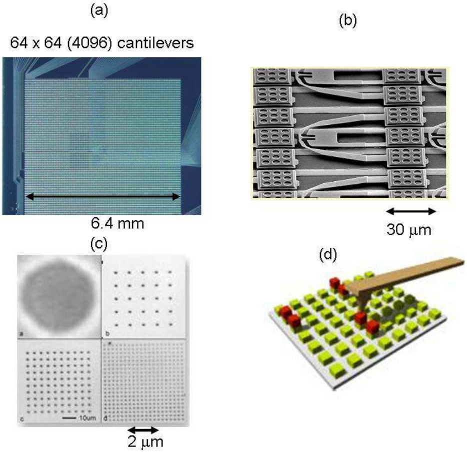




Modern Systems are Multi-Agent



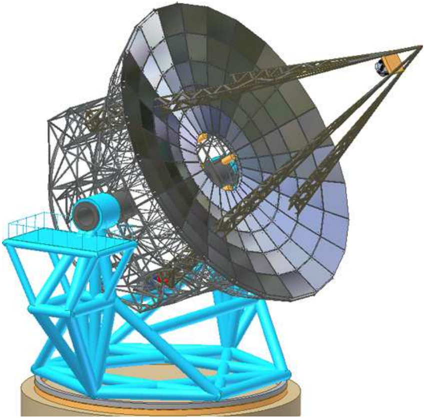
Coordination at Small Scale



(a),(b) 4096 array aimed for data storage system (IBM), (c) Bio arrays for biosensing (BioForce Nanosciences Inc.),

(d) A schematic of a single cantilever investigation array

Coordination at Large Scale

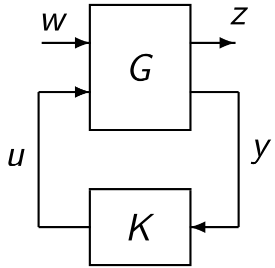


Atlas concept: $\sim 4,000$ mirror segments

Modern Systems Challenge: Efficient Coordination

- Multiagent coordination
- Centralized control virtually impossible
  - computational/processing complexity
  - cost and reliability
  - verification
  - $\vdots$
- Structured and distributed control action and information exchange is necessary
- How to design optimally such control mechanisms?

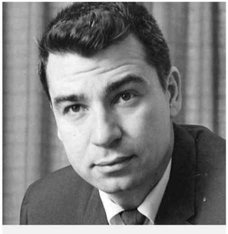
The Standard (Centralized) Framework



- Generalized plant $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ stabilizable
- Controller K stabilizes G
- Optimal control problem:

$$\inf_{K \text{ stabilizing}} \|\Phi\|, \quad \text{with} \quad \Phi : w \mapsto z$$

All Stabilizing Controllers



Dante C. Youla

Youla, D.C., Bongiorno, J.J., and Jabr, H.A. (1976). Modern Wiener-Hopf design of optimal controllers, Part I: The single-input case. *IEEE Trans. Auto. Control,* 21, 3-14.

Vladimir Kučera

Kučera, V. (1975). Stability of discrete linear control systems. In*: Proc. 6th IFAC World Congress*, paper 44.1. Boston, MA.

All Stabilizing Controllers cont.

- Given stabilizable $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$, $G_{22} : u \mapsto y$
- Youla-Kucera controller parametrization:

$$K = (Y_r - D_r Q)(X_r - N_r Q)^{-1} = (X_l - Q N_l)^{-1}(Y_l - Q D_l)$$

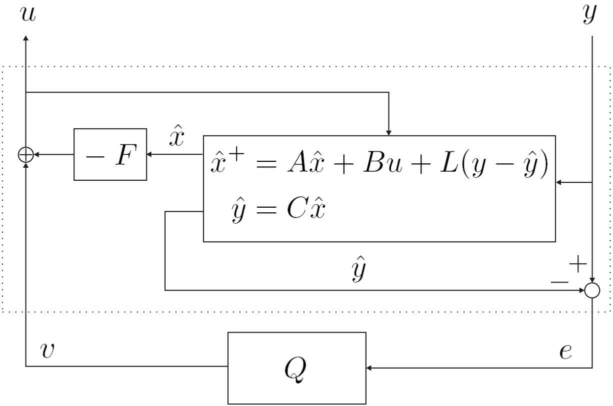
- Q is a free stable system: Y-K parameter
- Y_r, D_r, X_r, N_r right- and Y_l, D_l, X_l, N_l left- stable coprime factors of G_{22} :

$$G_{22} = N_r D_r^{-1} = D_l^{-1} N_l$$

$$\begin{bmatrix} X_l & -Y_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & Y_r \\ N_r & X_r \end{bmatrix} = I$$

Observer Based Construction

- $G_{22} \sim (A, B, C)$; (A, B) stabilizable and (A, C) detectable,
- Find F and L s.t. $A - BF$, $A - LC$ Hurwitz
- All stabilizing controllers:



All Closed Loop Maps

- Closed loop $z = \Phi w$

$$\begin{aligned} \Phi &= G_{11} + G_{12} K (I - G_{22} K)^{-1} G_{21} \\ &= H - U Q V \end{aligned}$$

with H, U, V stable

- Affine linear in Q
- For stable G: $H = G_{11},\ U = G_{12},\ V = G_{21}$

$$Q = -K(I - G_{22}K)^{-1}$$

All Closed Loop Maps cont.

$$\Phi = H - UQV$$

- $\inf_Q \|\Phi\|$ convex in Q
- Key in developing fundamantal insights and the theory of robust and optimal control in 80's and (early) 90's, e.g.,
  - $\mathcal{H}_\infty$ (Nehari's Theorem)
  - ℓ_1 (Duality Theorem)
  - $\mathcal{H}_2$ (Projection Theorem)
  - Multiobjective problems: $\mathcal{H}_2/\mathcal{H}_\infty$, $\mathcal{H}_2/\ell_1$, etc.
  - $\vdots$

Structural Constraints

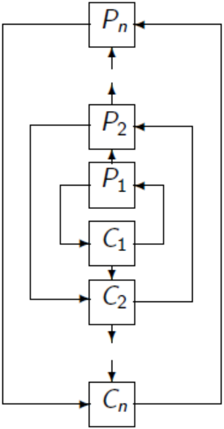
- What if K structured? e.g., decentralized:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} K_{11} & 0 & \dots \\ 0 & K_{22} & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

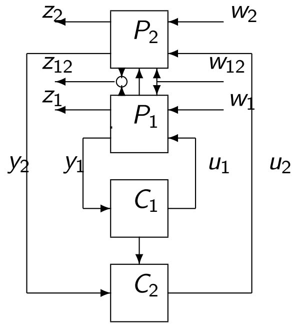
- Constraints on K do not reflect to convex constraints on Y-K parameter $Q \implies$ hard optimal control problems (e.g., Witsenhausen's counterexample)
- Are specific cases that can be solved satisfactorily?

Triangular (Nested) Structures Example

G_{22}, K lower triangular



A 2-Nested Structure Example



$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; z := \begin{bmatrix} z_1 \\ z_{12} \\ z_2 \end{bmatrix}, \; w := \begin{bmatrix} w_1 \\ w_{12} \\ w_2 \end{bmatrix}$$

$$u \mapsto y : G_{22} = \begin{bmatrix} g_{11} & 0 \\ g_{12} & g_{22} \end{bmatrix}, \qquad y \mapsto u : K = \begin{bmatrix} k_{11} & 0 \\ k_{12} & k_{22} \end{bmatrix}$$

Triangular Example cont.

- $K = -Q(I - G_{22}Q)^{-1}$ for stable G_{22}
- Given that

$$G_{22} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$$

then

$$K = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$$

iff

$$Q = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$$

- Convex constraint in Q !

Key Algebraic Property[1]

- If structure $\mathcal{S}$ of G_{22}, K preserved under

  (i) addition
  (ii) multiplication

  then structure $\mathcal{S}$ of K is also imposed on Y-K parameter Q:

$$K \in \mathcal{S} \iff Q \in \mathcal{S}$$

- Quadratic Invariance: a relaxation of (ii)

$$\mathcal{S} G_{22} \mathcal{S} \subset \mathcal{S}$$

  then

$$K \in \mathcal{S} \iff Q \in \mathcal{S}$$

[1] Voulgaris 01, Rotkowitz-Lall 06,...

Key Algebraic Property cont.

- Insight (stable G): $K = -Q(I - G_{22}Q)^{-1}$, or

$$Q = -K(I - G_{22}K)^{-1}$$

- Formal expansion of $(I - G_{22}K)^{-1}$

$$Q = -K(I + G_{22}K + (G_{22}K)(G_{22}K) + \dots)$$

$$= -K - KG_{22}K - (KG_{22}K)G_{22}K - \dots$$

- If $\mathcal{S}G_{22}\mathcal{S} \subset \mathcal{S}$, then

$$K \in \mathcal{S} \implies KG_{22}K \in \mathcal{S} \implies Q \in \mathcal{S}$$

- Similarly

$$Q \in \mathcal{S} \Longrightarrow K \in \mathcal{S}$$

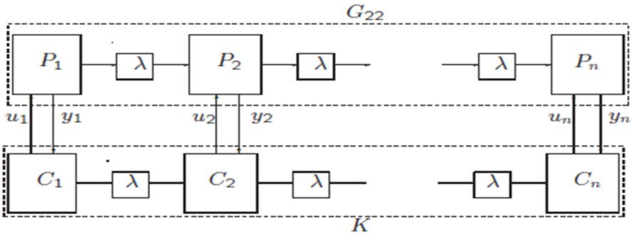
A Variety of Structures $\mathcal{S}$

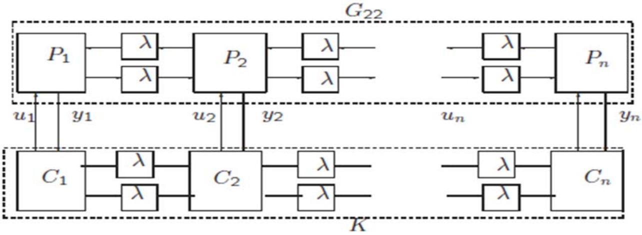
Several sparsity and delay patterns satisfy key algebraic property, or QI

- Triangular: $G_{22},\ K$ triangular matrices
  - IFPC; network congestion control; platoon formations; power systems
- Delayed interaction and observation sharing
  - MEMS; networked control; production lines
- Symmetric structures: $G_{22} = G_{22}^T$ and $K = K^T$
  - process control; large space structures; circuits
- $\vdots$

Delayed Interaction and Observation

λ: one-step delay in neighbor-to-neighbor interconnection





Delayed Interaction and Observation cont.

$$G_{22},\ K \ \sim \ \begin{bmatrix} * & & & & \\ \lambda * & * & & & \\ \lambda^2 * & \lambda * & * & & \\ \vdots & & \ddots & \ddots & \\ \lambda^{n-1} * & \cdots & \cdots & \lambda * & * \end{bmatrix}$$

$$G_{22},\ K \ \sim \ \begin{bmatrix} * & \lambda * & \lambda^2 * & \cdots & \lambda^{n-1} * \\ \lambda * & * & \lambda * & \cdots & \\ \lambda^2 * & \lambda * & * & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \lambda * \\ \lambda^{n-1} * & \cdots & \cdots & \lambda * & * \end{bmatrix}$$

Toeplitz Structures

$$G_{22} = \begin{bmatrix} g_1 & & & \\ \lambda g_2 & g_1 & & \\ \vdots & \ddots & \ddots & \\ \lambda^{n-1} g_n & \dots & \lambda g_2 & g_1 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & & & \\ \lambda k_2 & k_1 & & \\ \vdots & \ddots & \ddots & \\ \lambda^{n-1} k_n & \dots & \lambda k_2 & k_1 \end{bmatrix}$$

Constrained in Q Problems

- Under QI of structure $\mathcal{S}$

$$K \in \mathcal{S} \Longleftrightarrow Q \in \mathcal{S}$$

- If structure $\mathcal{S}$ is convex, optimal performance problem is convex in the Y-K paramter Q

$$\mu := \inf_{Q \in \mathcal{S}} \|H - UQV\|$$

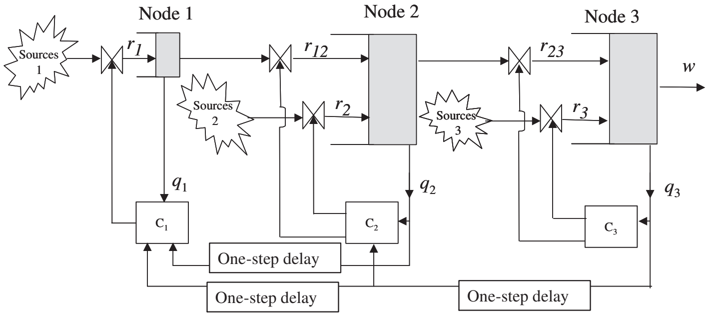
- How to solve these convex but infinite dimensional problems?

General Multiobjective Problems

- Cost
$$\inf_{Q\in\mathcal{S}} c_1 \left\|\Phi_1\right\|_1 + c_2 \left\|\Phi_2\right\|_2^2 + c_3 \left\|\Phi_3\right\|_{\mathcal{H}_\infty}$$
- Constraints:
$$\left\|\Phi_4\right\|_1 \leq c_4$$
$$\left\|\Phi_5\right\|_2 \leq c_5$$
$$\left\|\Phi_6\right\|_{\mathcal{H}_\infty} \leq c_6$$
$$+\text{Time Domain Constraints}$$
- Solution in terms of converging upper and lower bounds by LP/SDP[2]

[2] Qi-Salapaka-Voulgaris-Khammash 04

Optimal Control Design for a 3-Nodal Network



- Fluid model

  Node 1: $q_1(k+1) = q_1(k) + r_1(k) - r_{12}(k)$

  Node 2: $q_2(k+1) = q_2(k) + r_2(k) + r_{12}(k) - r_{23}(k)$

  Node 3: $q_3(k+1) = q_3(k) + r_3(k) + r_{23}(k) - w(k)$
- $z = [\ q_1,\ \ q_2,\ \ q_3,\ \ r_1 - w \cdot a_1,\ \ r_2 - w \cdot a_2,\ \ r_3 - w \cdot a_3\ ]^T$

3-Nodal Network cont.

- C_1: $\quad r_1 \; = f_1(q_1, \lambda q_2, \lambda^2 q_3)$
- C_2: $\begin{cases} r_{12} & = & f_{12}(q_2, \lambda q_3) \\ r_2 & = & f_2(q_2, \lambda q_3) \end{cases}$
- C_3: $\begin{cases} r_{23} & = & f_{23}(q_3) \\ r_3 & = & f_3(q_3) \end{cases}$

$$G_{22} := \begin{bmatrix} * & \lambda * & \lambda * & \lambda^2 * & \lambda^2 * \\ 0 & * & * & \lambda * & \lambda * \\ 0 & 0 & 0 & * & * \end{bmatrix}, \quad K := \begin{bmatrix} * & \lambda * & \lambda^2 * \\ 0 & * & \lambda * \\ 0 & * & \lambda * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}$$

Multiobjective Problem

$$\begin{array}{lll} \nu := & \inf & c_1\|\Phi(K)\|_1 + c_2\|\Phi(K)\|_2^2 \\ & \text{subject to} & \\ & & K \text{ is stabilizing} \\ & & K \text{ satisfies structural and delay constraints} \\ & & z_i (i = 4, 5, 6) \text{ satisfies prescribed TDCs.} \end{array}$$

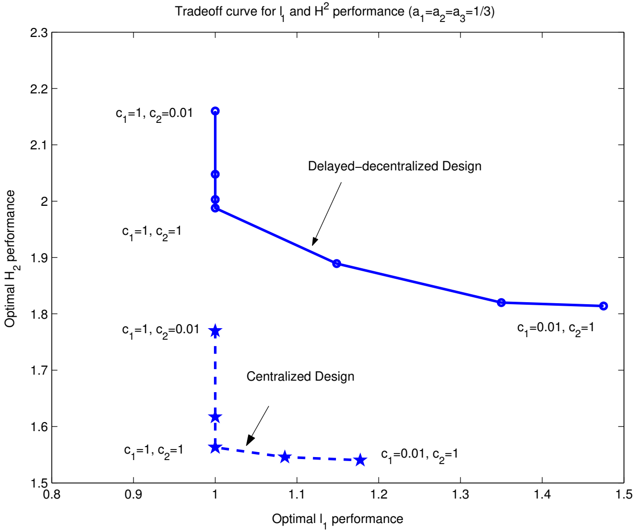
The state-space description of G_{22} is given by

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right] = \left[\begin{array}{cccccc} A_1 & 0 & 0 & B_1 & B_{12} & 0 \\ 0 & A_2 & 0 & 0 & B_2 & B_{23} \\ 0 & 0 & A_3 & 0 & 0 & B_3 \\ C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 & 0 \end{array}\right]$$

where

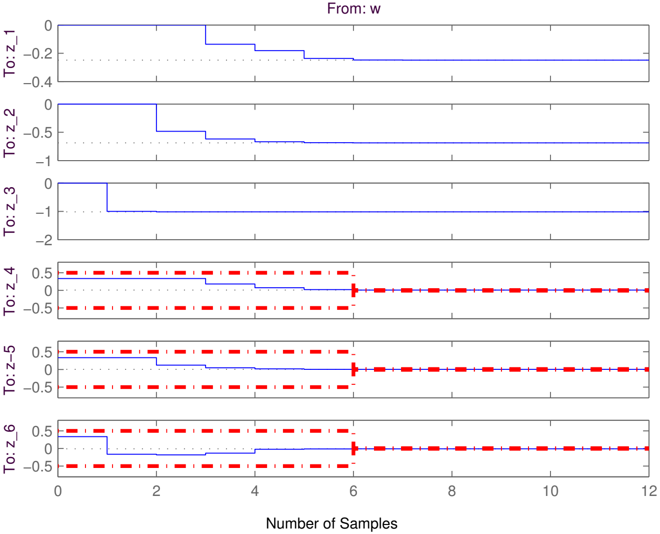
$$A_1 = A_2 = A_3 = 1, C_1 = C_2 = C_3 = 1$$
$$B_1 = 1,\ B_{12} = B_{23} = [-1 \ \ 0],\ B_2 = B_3 = [1 \ \ 1]$$

Performance Trade-Offs

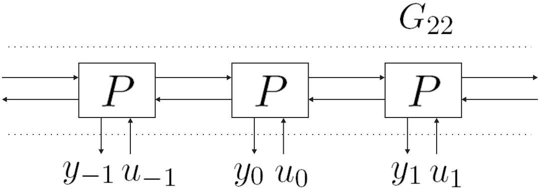


Decentralized controller order: 6

Step Responses

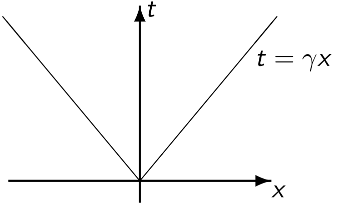


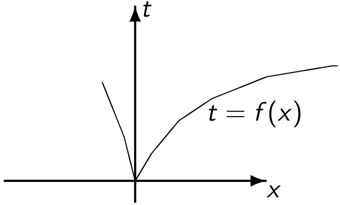
Extensions to Spatiotemporal Systems



- Complex systems often posses (or can be approximated) spatial invariance
- System G: $y(x,t) \;=\; \int\int g(x-\xi, t-\tau)\; u(\xi,\tau)\; \mathrm{d}\tau\; \mathrm{d}\xi$
- Greatly simplifies control design
- Optimal controller K: $u(x,t) \;=\; \int\int k(x-\xi, t-\tau)\; y(\xi,\tau)\; \mathrm{d}\tau\; \mathrm{d}\xi$
- Optimal control exhibits a "localization" property
- What if localization not enough?

Explicit Constraints

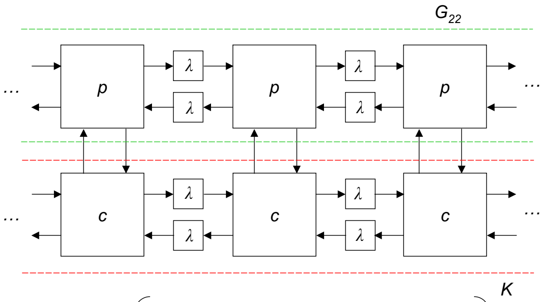




- Certain classes of constrained information problems can be converted to convex
- System G: $y(x,t) \; = \; \int\int g(x-\xi, t-\tau)\; u(\xi,\tau)\; \mathrm{d}\tau\; \mathrm{d}\xi$
- Funnel-Causality[3] $g(x,t) = 0,$ for $t < f(x)$, where $f(x)$ is a concave propagation function
- Cone Causality: $f(x) = \gamma x$

[3] Bamieh-Voulgaris 05

A Cone Causal SI System



$$G_{22} = \begin{pmatrix} \cdots & g_{-1}(\lambda)\,\lambda & g_0(\lambda) & g_1(\lambda)\,\lambda & \cdots \end{pmatrix}$$

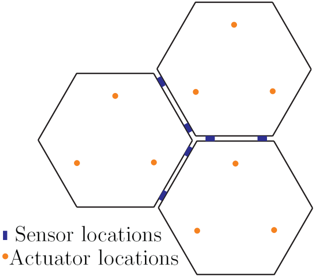
$$K = \begin{pmatrix} \cdots & k_{-1}(\lambda)\,\lambda & k_0(\lambda) & k_1(\lambda)\,\lambda & \cdots \end{pmatrix}$$

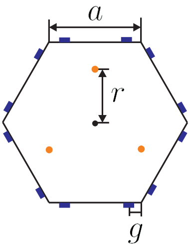
Explicit Constraints cont.

- Support of g: $S_g \; := \; \{(x,t);\; t \geq f(x)\}$: how fast a disturbance propagates
- If controller K: $u(x,t) \; = \; \int\int k(x-\xi, t-\tau)\; y(\xi,\tau)\, \mathrm{d}\tau\, \mathrm{d}\xi$ has support $S_k \supset S_g$ then problem is convex
- If actuation site ξ for $u(\xi,.)$ receives information $y(x,.)$ about site x at least as fast as $u(\xi,.)$ affects site x, then problem is convex

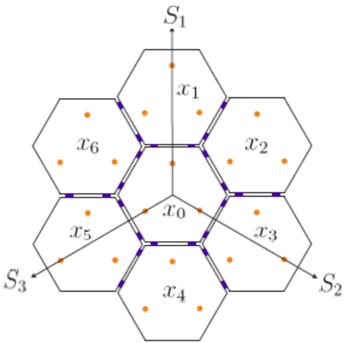
Atlas Giant Segmented Telescope Mirror

- The primary mirror: 25 meters in diameter, composed of 1000-4000 smaller segments
- Goal: less massive and cheaper, fast adjustment to counteract disturbances.





Spatially-invariant Formulation in 2-dimensional Space



x_i: the state of the ith segment

System dynamics:

$$\dot{x}_i = A_i x_i + B_{1i} w_i + B_{2i} u_i$$

Spatial shift operators: S_1, S_2, S_3

$$x_1 = S_1 x_0,\ x_3 = S_2 x_0,\ x_5 = S_3 x_0$$

Geometrical constraint:

$$S_i S_j S_k = 1\ \forall i \neq j \neq k.$$

$\mathcal{H}_2$ Performance[4]

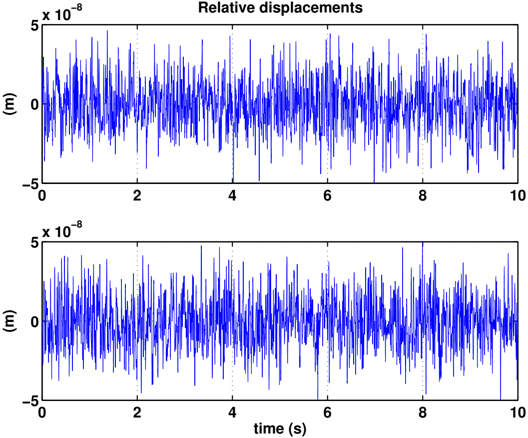
- RMS of relative displacements

	Exact Solution	First ring communication	First two ring communication
RMS (m)	$3.8447e-8$	$4.7550e-8$	$4.7487e-8$

[4] Jiang-Voulgaris-Holloway-Thompson 09

Simulation

- Simulation of a nineteen-segment system



Some Remarks

- Convexity is due to the specific algebraic structure of G_{22} and the imposed structure on K. What if special structure is not present?
- Covexification possible for a limited number of problems[5]; open problem for general structures
- Are there problems where no structure on K is imposed, yet optimal solution is structured due to the cost?

[5] Voulgaris 03

Y-K and Social Optimization Problems

- Social optimization type problems in many applications:
  stock market, production engineering, dynamic demand management in power systems, population dynamics, pricing, consensus, etc.
- Multiagent control problem coupled by a collective cost function
- Scalability to large number of agents?
- Mean Field game formulations
- Can Y-K offer effective ways for analysis and synthesis?

Decentralization and Selfishness

- Important to characterize when optimal solution in large network problems is decentralized and computed by solving local optimal control problems
- Social cost optimization measures typically involve the distance of each agent's output to the average output of the collective (e.g., Mean Field formulation [6])
- When decentralized and selfish behavior (i.e., disregard distance to the collective) is optimal?

[6] Huang-Caines-Malhame 07, Moon-Basar 14,...

Set Up

n decoupled LTI systems $\{G_i\}_{i=1}^{n}$

$$\begin{bmatrix} z_i \\ y_i \end{bmatrix} = G_i \begin{bmatrix} w_i \\ u_i \end{bmatrix}$$

Possibly centralized controller

$$u = Ky, \quad u = [u_i]_{1 \le i \le n}, \; y = [y_i]_{1 \le i \le n}$$

Closed loop

$$z = \Phi w, \quad z = [z_i]_{1 \le i \le n}, \; w = [w_i]_{1 \le i \le n}$$

Set Up cont.

Deviations from population average signals $e := [e_i]_{1 \le i \le n}$

$$e_i := z_i - \bar{z}$$

$$\bar{z} := \frac{1}{n}(z_1 + \cdots + z_n)$$

$$\Psi := w \mapsto e$$

Non-deviation from average signals $\xi := [\xi_i]_{1 \le i \le n}$

$$\Xi := w \mapsto \xi$$

Interested in $w \mapsto (e, \xi)$, i.e.,

$$\inf_{K} \left|\left| \begin{bmatrix} \Psi \\ \Xi \end{bmatrix} \right|\right|, \qquad \inf_{K : ||\Xi|| \le \gamma} ||\Psi|| \tag{1}$$

Example

$$
\begin{aligned}
x_i(k+1) &= A_i x_i(k) + B_{1i} w_{1i}(k) + B_{2i} u_i(k) \\
y_i(k) &= C_i x_i(k) + D_i w_{2i}(k) \\
z_i &= x_i, \ \bar{z} = \bar{x}
\end{aligned}
$$

- "Social" $\mathcal{H}_2$ cost :

$$\mathcal{E}[\sum_i (|x_i - \bar{x}|^2 + |u_i|^2)]$$

- Constrained "social" $\mathcal{H}_2$ cost:

$$\mathcal{E}[\sum_i (|x_i - \bar{x}|^2)], \quad \text{s.t. } \mathcal{E}[\sum_i |u_i|^2] \leq \gamma^2$$

- Optimizing social cost needs–in principle–a centralized solution $u(k) = (Ky)(k)$

Example cont.

- "Selfish" $\mathcal{H}_2$ cost (totally discard $\bar{x}$):

$$\mathcal{E}[|x_i|^2 + |u_i|^2]$$

- Constrained "selfish" $\mathcal{H}_2$ cost :

$$\mathcal{E}[|x_i|^2], \quad \text{s.t. } \mathcal{E}[|u_i|^2] \leq \gamma^2$$

- Optimizing selfish cost has a decentralized solution

$$u(k) = (Ky)(k), \; K = \mathrm{diag}\{K_i\}$$

- How optimizing for social cost relates to optimizing selfish cost?

Input-Output Approach

Youla-Kucera parametrization:

$$\Phi = w \mapsto z = H - UQV$$

$$H = \mathrm{diag}\{H_i\},\ U = \mathrm{diag}\{U_i\},\ V = \mathrm{diag}\{V_i\}$$

$Q = [Q_{ij}]_{1 \leq i,j \leq n}$ stable and full (not necessarily diagonal)

$$\Phi = \begin{bmatrix} H_1 - U_1 Q_{11} V_1 & -U_1 Q_{12} V_2 & \cdots \\ -U_2 Q_{21} V_1 & H_2 - U_2 Q_{22} V_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \qquad (2)$$

Input-Output Approach cont.

Define $\mathbf{1} := [1\ 1 \ldots 1]^T$, then

$$\Psi = (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)\Phi = (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)(H - UQV) : w \mapsto e$$

$$\Xi = H_\xi - U_\xi Q V_\xi : w \mapsto \xi$$

with

$$H = \mathrm{diag}\{H_i\},\ U = \mathrm{diag}\{U_i\},\ V = \mathrm{diag}\{V_i\}$$

$$H_\xi = \mathrm{diag}\{H_{\xi_i}\},\ U_\xi = \mathrm{diag}\{U_{\xi_i}\},\ V_\xi = \mathrm{diag}\{V_{\xi_i}\}$$

Problem to solve

$$\inf_Q \left|\left| \begin{bmatrix} \Psi \\ \Xi \end{bmatrix} \right|\right|, \quad \text{or} \quad \inf_{Q : ||\Xi|| \leq \gamma} ||\Psi|| \tag{3}$$

Identical Agents

For identical G_i:

$$H_i = \bar{H}, \quad U_i = \bar{U}, \quad V_i = \bar{V}$$

$$H_{\xi_i} = \bar{H}_\xi, \quad U_{\xi_i} = \bar{U}_\xi, \quad V_{\xi_i} = \bar{V}_\xi$$

Single-agent, "selfish" optimization (ignore $\bar{z}$):

$$\varphi^o := \inf_{q:||g(q)||\leq\gamma} ||f(q)||, \quad \text{or} \quad \varphi^o := \inf_q || \begin{bmatrix} f(q) \\ g(q) \end{bmatrix} || \qquad (4)$$

$$f(q) := \bar{H} - \bar{U}q\bar{V} = w_i \mapsto z_i$$

$$g(q) := \bar{H}_\xi - \bar{U}_\xi q\bar{V}_\xi = w_i \mapsto \xi_i$$

Resutls[7] ℓ_1-i, ℓ_∞-i, and ℓ_2-i ($\mathcal{H}_\infty$)

Social problem:

$$\psi^o := \inf_{Q:||\Xi||\leq\gamma} ||\Psi|| \tag{5}$$

Selfish problem:

$$\varphi^o := \inf_{q:||g(q)||\leq\gamma} ||f(q)|| \tag{6}$$

Theorem 1

Consider Problem (5).

1. *In the cases of ℓ_1-induced and ℓ_∞-induced norms, decentralized control K^o obtained by solving Problem (6) optimizes performance in Problem (5) for each n. Moreover, we have $\psi^o = 2\varphi^o(n-1)/n$.*
2. *In the case of $\mathcal{H}_\infty$, decentralized control K^o obtained by solving Problem (6) optimizes performance in Problem (5) for each n, with optimal cost $\psi^o = \varphi^o$.*

[7] Voulgaris-Elia 17

Remarks

- Optimal controller K^o is decentralized (diagonal) obtained by $Q^o = q^o I$, with q^o solution to selfish Problem (6)
- At optimal, social optimization is also selfish optimal
- This holds true for any finite n, as well as asymptotically as $n \to \infty$.

The $\mathcal{H}_2$ Norm Case

Social problem normalized per agent:

$$(\bar{\psi}^o)^2 := \inf_{Q:\frac{1}{n}||\Xi||^2 \leq \gamma^2} \frac{1}{n}||\Psi||^2 \tag{7}$$

Selfish, single-agent, problem:

$$(\bar{\varphi}^o)^2 := \inf_{q:||g(q)||^2 \leq \gamma^2} ||f(q)||^2 \tag{8}$$

- Decentralized control K^o obtained by solving Problem (8) optimizes a normalized by the number of agents performance in Problem (7) asymptotically as n increases in the sense $\lim_{n\to\infty} |(\bar{\psi}^o)^2 - (\bar{\varphi}^o)^2| = 0$.
- Optimal controller K^o is decentralized and selfish asymptotically as $n \to \infty$

Non Identical Agents: $\mathcal{H}_\infty$ control

Problem of interest:

$$\psi^o := \inf_{Q:||Q||\leq\gamma} ||\Psi|| \tag{9}$$

$$\Psi = (I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)\Phi$$

Assume sublinear growth: (true for identical agents)

$$||[U_1 \dots U_n]|| \leq \gamma_u n^\rho \tag{10}$$

$\gamma_u \geq 0$, $0 \leq \rho < 1$

Non Identical Agents: $\mathcal{H}_\infty$ control cont.

Consider M agents ignoring $\bar{z}$ that solve

$$\mu_M := \inf_{||Q|| \leq \gamma} ||\Pi_M H - \Pi_M U Q \Pi_M V|| \qquad (11)$$

$$\Pi_M H = \mathrm{diag}\{H_i\}_{i=1}^M, \ \Pi_M U = \mathrm{diag}\{U_i\}_{i=1}^M, \ \Pi_M V = \mathrm{diag}\{V_i\}_{i=1}^M$$

$$\mu^o := \limsup_{M \to \infty} \mu_M$$

Theorem 2

Under Assumption (10)

$$\lim_{n \to \infty} \psi^o = \mu^o$$

and an arbitrarily close to optimal decentralized controller can obtained by $Q^{o,M}$ solving (11) for sufficiently large M.

Non Identical Agents: $\mathcal{H}_2$ control

Problem of interest:

$$\psi^o := \inf_{Q:||Q||\leq\gamma} ||\Psi||_2 \tag{12}$$

$$\mu_M := \inf_{||Q||\leq\gamma} \frac{1}{\sqrt{M}} ||\Pi_M H - \Pi_M U Q \Pi_M V||_2 \tag{13}$$

$$\mu^o := \limsup_{M\to\infty} \mu_M$$

Theorem 3

Under assumption (10) we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \psi^o = \mu^o$$

and an arbitrarily close to optimal decentralized controller can obtained by $Q^{o,M}$ solving (13) for sufficiently large M.

Examples with $\mathcal{H}_\infty$ control

- Agents with identical dynamics (a single integrator/adder)

$$\begin{aligned} x_i(k+1) &= x_i(k) + w_{i1}(k) + u_i(k) \\ z_i(k) &= -x_i(k) + w_{i2}(k) \\ \xi_i(k) &= u_i(k) \\ y_i(k) &= -x_i(k) + w_{i2}(k) \end{aligned}$$

- Selfish controller to minimize the $\mathcal{H}_\infty$ norm

$$w_i = \begin{bmatrix} w_{i1} \\ w_{i2} \end{bmatrix} \mapsto \begin{bmatrix} z_i \\ \xi_i \end{bmatrix}$$

- The optimal $\mathcal{H}_\infty$ norm is 1.9021. The optimal (selfish) controller is a static gain $K_i = 0.61803$

Examples with $\mathcal{H}_\infty$ control, cont.

- Social Problem: Minimize $\mathcal{H}_\infty$ norm

$$w \mapsto \begin{bmatrix} z - \bar{z}\mathbf{1} \\ \xi \end{bmatrix}$$

- Optimal centralized controller for $i = 2$ agents that solves Social Problem, is given by

$$K(\lambda) = \begin{bmatrix} 0.34515\frac{\lambda+0.2056}{\lambda+0.1118} & -0.27288\frac{\lambda-0.006807}{\lambda+0.1118} \\ -0.27288\frac{\lambda-0.006807}{\lambda+0.1118} & 0.34515\frac{\lambda+0.2056}{\lambda+0.1118} \end{bmatrix}$$

which has the parallel structure, and the same optimal norm 1.9021.

Examples with $\mathcal{H}_\infty$ control, cont.

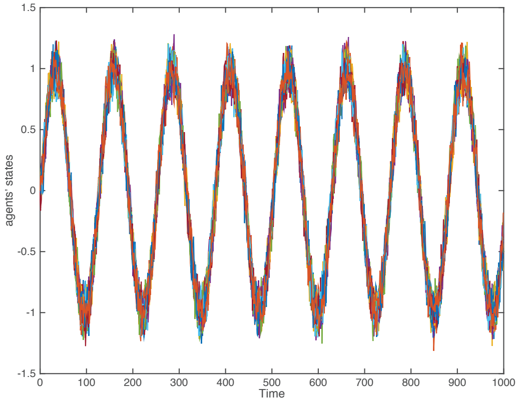


Figure 1: The state response of each agent with a decentralized controller $K_{dec} = I\,K_i$ for 30 agents when w_2 has a common sinusoidal component. The optimal $\mathcal{H}_\infty$ norm for $n = 30$ for both decentralized and centralized controllers is still 1.9021.

Agents with non identical dynamics

$$
\begin{aligned}
x_{i1}(k+1) &= x_{i1}(k) + x_{i2}(k) \\
x_{i2}(k+1) &= a_{i2} x_{i2}(k) + w_{i1}(k) + b_i u_i(k) \\
y_i(k) &= -x_{i1}(k) + w_{i2}(k)
\end{aligned}
$$

where $a_i \in [0, 2]$ and $b_i \in [0, 1]$, uniformly distributed

- Each agent computes its own selfish $\mathcal{H}_\infty$ controller for the generalized plant

$$
\begin{aligned}
x_{i1}(k+1) &= x_{i1}(k) + x_{i2}(k) \\
x_{i2}(k+1) &= a_{i2} x_{i2}(k) + w_{i1}(k) + b_i u_i(k) \\
z_{i1}(k) &= -x_{i1}(k) + w_{i2}(k) \\
\xi_i(k) &= u_i(k) \\
y_i(k) &= -x_{i1}(k) + w_{i2}(k)
\end{aligned}
$$

The optimal controller has order 2.

Agents with non identical dynamics cont.

- Considering 50 agents and apply decentralized selfish solution $K_{dec} = \mathrm{diag}\{K_i\}$ $i = 1, \ldots, 50$ leads to $\mathcal{H}_\infty$ social cost 13.1455
- Optimal centralized social solution leads to a social cost equal to 13.1422

Conclusions

- Y-K parametrization key enabler of powerful, input-output, approaches of analysis and synthesis in control systems
- Diverse performance and robustness measures: stochastic/deterministic
- Several classes of structured and distributed control problems shown to be tractable
- Algebraic characterization of tractability
- Y-K approach efficient to analyze agents' optimal behavior in social optimization problems
- Characterize when decentralized and selfish solutions are optimal and provide explicit single-agent corresponding problems (scalability)

Some New Directions

- Efficient realization of structured and distributed controllers[8]
- Unified state-space, operator framework and Y-K approach to optimal design[9]
- Extensions to nonlinear systems (e.g., switching dynamics)[10]

THANK YOU

[8] Naghnaeian-Voulgaris-Elia 18

[9] Naghnaeian-Voulgaris-Dullerud 17

[10] Naghnaeian-Hovakimyan-Voulgaris 12, Naghnaeian-Voulgaris 16