Κβαντική πληροφορία

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Imports

```
from scipy.integrate import odeint
from scipy.interpolate import interp1d
from scipy.integrate import quad
from scipy.integrate import simpson
from scipy.optimize import fsolve
import numpy as np
import matplotlib.pyplot as plt
import warnings
warnings.filterwarnings('ignore')
```

Lane-Emden

Υδροστατική ισορροπία: $\frac{d}{dr}\left[\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right] = -4\pi G r^2 \rho(r)$

Ορίζοντας $P(r) = K\rho(r)^\gamma$ και $\rho(r) = \rho_o\theta^n$, $\xi = r/a$
με $n = \frac{1}{\gamma-1}$ και $a = \sqrt{\frac{K\gamma}{4\pi G(\gamma-1)}\rho_o^{\gamma-2}}$ προκύπτει:

$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} + \theta(\xi)^n = 0$
με αρχικές συνθήκες $\theta(0) = 1$ και $\theta'(0) = 0$

Αν $\theta'(\xi) = z$, τότε $z'(\xi) = \theta''(\xi) = -\frac{2}{\xi}\frac{d\theta(\xi)}{d\xi} - \theta(\xi)^n$

```
w_initial=[1,0]

def model(w,x,n):
    y=w[0]
    z=w[1]

    dw=[[],[]]

    dw[0]=z
    dw[1]=-2/x*z-y**n
    return dw

x=np.linspace(10**-8,10,2000)
```

Equations

$$h(\alpha k) = \left(\frac{4\pi\rho_o\alpha^3}{ak} \int_0^{\xi_R} \theta^n(\xi) \sin(\alpha k\xi)\xi d\xi \right)^2$$

$$\alpha k_{min} = \alpha \frac{\pi}{R} = \alpha \frac{\pi}{\alpha\xi_R} = \frac{\pi}{\xi_R}$$

$$h(\alpha k_{min}) = \left(\frac{4\pi\rho_o\alpha^3}{\alpha k_{min}} \int_0^{\xi_R} \theta^n(\xi) \sin(\alpha k_{min}\xi)\xi d\xi \right)^2 = \left(\frac{4\pi\rho_o\alpha^3}{\alpha k_{min}} \int_0^{\xi_R} \theta^n(\xi) \sin(\frac{\pi}{\xi_R}\xi)\xi d\xi \right)^2$$

$$\bar{f}(\alpha k) = \frac{h(\alpha k)}{h(\alpha k_{min})} = \left(\frac{\pi/\xi_R}{k}\right)^2 \frac{\left(\int_0^{\xi_R} \theta^n(\xi) \sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R} \theta^n(\xi) \sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2}$$

$$S = -4\pi a^{-3} \int_{\alpha k_{min}}^{\infty} \bar{f}(\alpha k) log(\bar{f}(\alpha k))(\alpha k)^2 d(\alpha k) = -4\pi \int_{\alpha k_{min}}^{\infty} \bar{f}(\alpha k) log(\bar{f}(\alpha k)) k^2 dk =$$

$$= -4\pi \int_{\alpha k_{min}}^{\infty} \left(\frac{\pi/\xi_R}{k}\right)^2 \frac{\left(\int_0^{\xi_R} \theta^n(\xi) \sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R} \theta^n(\xi) \sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2} log \left(\left(\frac{\pi/\xi_R}{k}\right)^2 \frac{\left(\int_0^{\xi_R} \theta^n(\xi) \sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R} \theta^n(\xi) \sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2} \right) k^2 dk$$

Equations and functions

$$h(\alpha k) = \left(\frac{4\pi\rho_o\alpha^3}{ak} \int_0^{\xi_R} \theta^n(\xi)\sin(\alpha k\xi)\xi d\xi \right)^2$$
$$h(\alpha k_{min}) = \left(\frac{4\pi\rho_o\alpha^3}{\alpha k_{min}} \int_0^{\xi_R} \theta^n(\xi)\sin(\frac{\pi}{\xi_R}\xi)\xi d\xi \right)^2$$

```
def hak_min(x,theta,n,wmin):
    if callable(theta):
        result=(theta(x)**n)*np.sin(wmin*x)*x
    else:
        result=(theta**n)*np.sin(wmin*x)*x
    return result

def hak(x,omega,theta,n):
    if callable(theta):
        result=(theta(x)**n)*np.sin(omega*x)*x
    else:
        result=(theta**n)*np.sin(omega*x)*x
    return result
```

Equations and functions

$$\bar{f}(\alpha k) = \frac{h(\alpha k)}{h(\alpha k_{min})} = \left(\frac{\pi/\xi_R}{k}\right)^2 \frac{\left(\int_0^{\xi_R} \theta^n(\xi)\sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R} \theta^n(\xi)\sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2}$$

```
def hak_min(x,theta,n,wmin):
    if callable(theta):
        result=(theta(x)**n)*np.sin(wmin*x)*x
    else:
        result=(theta**n)*np.sin(wmin*x)*x
    return result

def hak(x,omega,theta,n):
    if callable(theta):
        result=(theta(x)**n)*np.sin(omega*x)*x
    else:
        result=(theta**n)*np.sin(omega*x)*x
    return result
```

$CoefGamma = \sqrt{\frac{\gamma}{\gamma-1}}$ ε.ω. $\alpha k = \sqrt{\frac{K\gamma}{4\pi G(\gamma-1)\rho_o^{\gamma-2}}}k = \sqrt{\frac{K}{4\pi G\rho_o^{\gamma-2}}}\sqrt{\frac{\gamma}{\gamma-1}}k = const.\sqrt{\frac{\gamma}{\gamma-1}}k$ κι άρα η ανεξάρτητη μεταβλητή να γίνει από αk σε $\sqrt{\frac{K}{4\pi G\rho_o^{\gamma-2}}}k$ όπως δείχνει το Figure 1, διαιρόντας με $\sqrt{\frac{\gamma}{\gamma-1}}$.

Equations and functions

$$S=-4\ \pi\int_{\alpha k_{min}}^{\infty}\left(\frac{\pi/\xi_R}{k}\right)^2\frac{\left(\int_0^{\xi_R}\theta^n(\xi)\sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R}\theta^n(\xi)\sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2}log\left(\left(\frac{\pi/\xi_R}{k}\right)^2\frac{\left(\int_0^{\xi_R}\theta^n(\xi)\sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R}\theta^n(\xi)\sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2}\right)k^2dk$$

```
def S(omega,theta,n,wmin):
    if callable(theta):
        I1=quad(hak_min,xo,xr,args=(theta,n,wmin))
        paranomastis=I1[0]**2

        I2=quad(hak,xo,xr, args=(omega,theta,n))
        arithmitis=I2[0]**2

        paragontas=(wmin/omega)**2

        item=paragontas*arithmitis/paranomastis

        result=item*np.log(item)*omega**2
    else:
        I1=simpson(hak_min(xx,theta,n,wmin),xx)
        paranomastis=I1**2

        I2=simpson(hak(xx,omega,theta,n),xx)
        arithmitis=I2**2

        paragontas=(wmin/omega)**2

        item=paragontas*arithmitis/paranomastis

        result=item*np.log(item)*omega**2
    return result
```

Equations and functions

$$M = 4\pi \int_0^R \rho(r) r^2 dr = 4\pi \rho_o \alpha^3 \int_0^{\xi_R} \theta^n(\xi)\xi^2 d\xi$$

```
def Mass(x,theta,n):
    if callable(theta):
        result=(theta(x)**n)*(x**2)
    else:
        result=(theta**n)*(x**2)
    return result
```

Lane Emden + Figure 1

```
#Lane Emden + Figure 1
gamma=[1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9]

fig1,ax1=plt.subplots()
fig2,ax2=plt.subplots()
LinesColor=["red","black","blue"]
LinesStyle=["dashed","solid","dashdot"]
MarkerColor=["red","green","blue"]
counter=0

for g in gamma:
    n=1/(g-1)
    CoefGamma=np.sqrt(g/(g-1))

    y=odeint(model,w_initial,x,args=(n,))

    xx=x
    yy=y[:,0]

    ind=np.where(~np.isnan(yy))
    xx=xx[ind]
    yy=yy[ind]

    ax1.plot(xx,yy,label="γ=%.1f"%g)

    xr=xx[-1]
    wmin=np.pi/xr

    I1=simpson(hak_min(xx,yy,n,wmin),xx)
    paranomastis=I1**2

    omegas=np.arange(wmin,100,0.01)

    fakLIST=[]
    ws=[]
    if g==1.2 or g==1.4 or g==1.7:
        for w in omegas:
            paragontas=(wmin/(w))**2
            I1=simpson(hak(xx,w,yy,n),xx)
            arithmitis=I1**2

            fakLIST.append(paragontas*arithmitis/paranomastis)
            ws.append(w/CoefGamma)

        ax2.plot(ws,fakLIST,color=LinesColor[counter],linestyle=LinesStyle[counter],label="γ=%.1f"%g)
        ax2.scatter(ws[0],fakLIST[0],color=MarkerColor[counter],marker="v",zorder=2)
        ax2.legend(loc="best")
        counter=counter+1
```

Lane Emden + Figure 1

```
#Lane Emden
ax1.legend(loc="best")
ax1.set_title(r"Solutions for Lane-Emden")
ax1.set_ylabel(r"$\theta(\xi)$")
ax1.set_xlabel(r"$\xi$")
ax1.grid(True)
ax1.plot(np.linspace(0,10,10),10*[0],"k",linewidth=0.8)
ax1.set_xlim([0,10])
ax1.set_ylim([-0.5,1.2])

#Figure 1
ax2.set_xlabel(r"$k/\sqrt{\frac{4\pi G \rho_o^{\gamma-2}}{K}}$")
ax2.set_ylabel(r"$\bar{f}(\omega)$")
ax2.set_title(r"Plot "r"$\bar{f}(\omega)$" " vs " r"$ \omega=k/\sqrt{\frac{4\pi G \rho_o^{\gamma-2}}{K}}$")
ax2.set_xlim([0,1.5])
ax2.set_ylim([0,1.1])
ax2.set_yticks(np.arange(0,1.2,0.1))
ax2.legend(loc="best")

ax_NEW=ax2.twinx().twiny()
ax_NEW.set_xlim([0,1.5])
ax_NEW.set_ylim([0,1.1])
ax_NEW.set_yticks(np.arange(0,1.2,0.1))
ax_NEW.set_xticklabels([])
ax_NEW.set_yticklabels([])
```

Lane Emden

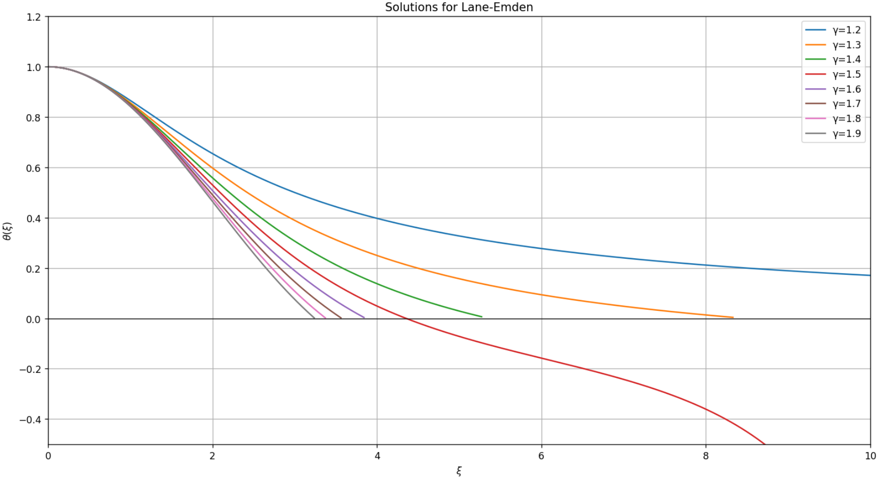
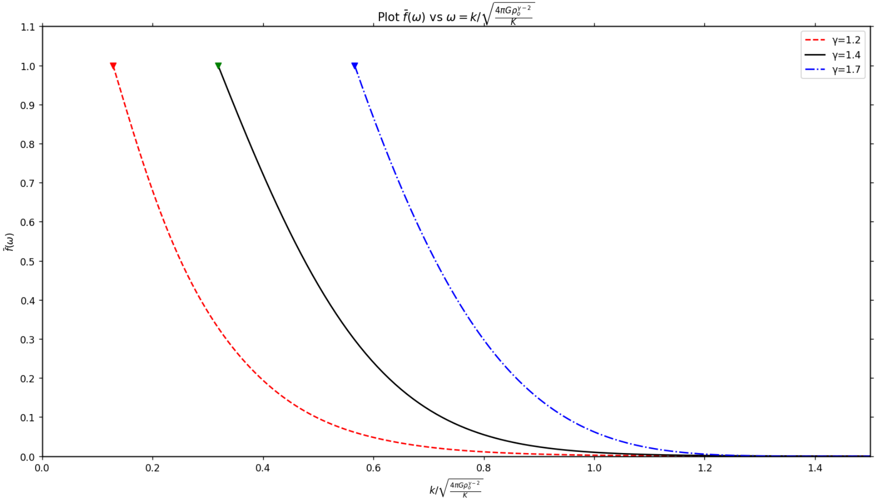

Solutions for Lane-Emden
γ=1.2
γ=1.3
γ=1.4
γ=1.5
γ=1.6
γ=1.7
γ=1.8
γ=1.9
θ(ξ)
ξ
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0
2
4
6
8
10

Figure 1



Plot $\bar{f}(\omega)$ vs $\omega = k/\sqrt{\frac{4\pi G\rho_o^{\gamma-2}}{K}}$
γ=1.2
γ=1.4
γ=1.7
$\bar{f}(\omega)$
$k/\sqrt{\frac{4\pi G\rho_o^{\gamma-2}}{K}}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.4

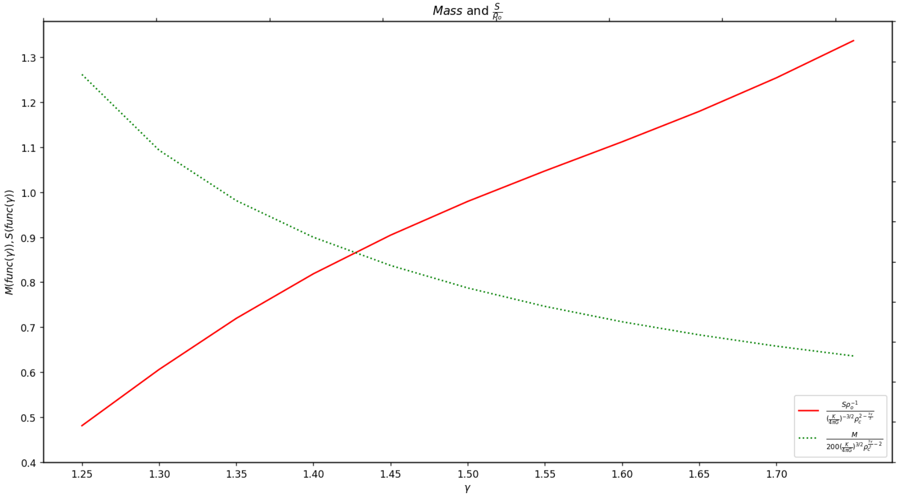
Figure 2

```
#Figure 2
listGamma=[]
Sro=[]
maza=[]
gamma=np.arange(1.25,1.8,0.05)
for g in gamma:
    n=1/(g-1)
    y=odeint(model,w_initial,x,args=(n,))

    xx=x
    yy=y[:,0]

    ind=np.where(~np.isnan(yy))
    xx=xx[ind]
    yy=yy[ind]

    theta=interp1d(xx,yy,kind="cubic",fill_value="extrapolate")

    xo=0
    xr=fsolve(theta,6)
    xr=xr[0]
    wmin=np.pi/xr

    I=quad(S,wmin,20,args=(theta,n,wmin))
    I=I[0]
    I=-I*np.pi*4*((g-1)/g)**(3/2)

    Sro.append(I)
    listGamma.append(g)

    Imass=simpson(Mass(xx,yy,n),xx)
    maza.append(4*np.pi/200*(g/(g-1))**(3/2)*Imass)
```

Figure 2

```
#Figure 2
fig3,ax3=plt.subplots()
ax3.plot(listGamma,Sro,color="red",\
         label=r"$\frac{S\rho_o^{-1}}{(\frac{K}{4\pi G})^{-3/2}\rho_c^{2-\frac{3\gamma}{2}}}$")
ax3.plot(listGamma,maza,color="green",linestyle="dotted",\
         label=r"$\frac{M}{200(\frac{K}{4\pi G})^{3/2}\rho_c^{\frac{3\gamma}{2}-2}}$")
ax3.legend()
ax3.set_xlabel(r"$\gamma$")
ax3.set_ylabel(r"$M(func(\gamma)),S(func(\gamma))$")
ax3.set_title(r"$Mass$"" and "r"$\frac{S}{\rho_o}$")
ax3.set_xticks(np.arange(1.25,1.75,0.05))
ax3.set_yticks(np.arange(0.4,1.4,0.1))

ax_NEW=ax3.twinx().twiny()
ax_NEW.set_xlim([0,1.5])
ax_NEW.set_ylim([0,1.1])
ax_NEW.set_yticks(np.arange(0,1.2,0.1))
ax_NEW.set_xticklabels([])
ax_NEW.set_yticklabels([])
```

Figure 2



$Mass$ and $\frac{S}{\rho_o}$
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
$M(func(\gamma)), S(func(\gamma))$
1.25
1.30
1.35
1.40
1.45
1.50
1.55
1.60
1.65
1.70
γ
$\frac{S\rho_o^{-1}}{(\frac{K}{4\pi G})^{-3/2}\rho_c^{2-\frac{3\gamma}{2}}}$
$\frac{M}{200(\frac{K}{4\pi G})^{3/2}\rho_c^{\frac{3\gamma}{2}-2}}$

Figure 3 και Figure 4

$$\rho(r) = \rho_o \theta(\xi)^n \xrightarrow{\rho_o=\rho_c}$$

$$\theta(\xi) = \left(\frac{\rho(r)}{\rho_c}\right)^{1/n}$$

$$\theta(0) = \left(\frac{\rho(0)}{\rho_c}\right)^{1/n} = \left(\frac{\rho_o}{\rho_c}\right)^{1/n} \Rightarrow \text{αλλαγή αρχικών συνθηκών.}$$

$$\begin{aligned}
\mathbf{M} &= 4\pi\rho_c\alpha^3 \int_0^{\xi_R} \theta(\xi)^n \xi^2 d\xi \\
&= 4\pi\rho_c \sqrt{\frac{K\gamma}{4\pi G(\gamma-1)}\rho_c^{\gamma-2}}^3 \int_0^{\xi_R} \theta(\xi)^n \xi^2 d\xi \\
&= 4\pi\rho_c \sqrt{\frac{K}{4\pi G}\rho_c^{\gamma-2}}^3 \sqrt{\frac{\gamma}{\gamma-1}}^3 \int_0^{\xi_R} \theta(\xi)^n \xi^2 d\xi \\
&= 4\pi(\frac{K}{4\pi G})^{3/2}\rho_c^{3\gamma/2-2}(\frac{\gamma}{\gamma-1})^{3/2} \int_0^{\xi_R} \theta(\xi)^n \xi^2 d\xi \\
&= 4\pi * const. * (\frac{\gamma}{\gamma-1})^{3/2} \int_0^{\xi_R} \theta(\xi)^n \xi^2 d\xi
\end{aligned}$$

$$\frac{S}{\alpha^3\rho_o} = \frac{-4\pi \displaystyle\int_{\alpha k_{min}}^{\infty} \left(\frac{\pi/\xi_R}{k}\right)^2 \frac{\left(\int_0^{\xi_R} \theta^n(\xi)\sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R} \theta^n(\xi)\sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2} log\left(\left(\frac{\pi/\xi_R}{k}\right)^2 \frac{\left(\int_0^{\xi_R} \theta^n(\xi)\sin(\alpha k\xi)\xi d\xi\right)^2}{\left(\int_0^{\xi_R} \theta^n(\xi)\sin(\frac{\pi}{\xi_R}\xi)\xi d\xi\right)^2}\right) k^2 dk}{\alpha^3\rho_o}$$

$$\frac{S}{\alpha^3\rho_o} = \frac{-4\pi * \text{Integral}_S}{(\frac{K}{4\pi G})^{3/2}\rho_c^{3\gamma/2-3}(\frac{\gamma}{\gamma-1})^{3/2}\frac{\rho_o}{\rho_c}\rho_c} = \frac{-4\pi * \text{Integral}_S * (\frac{K}{4\pi G})^{-3/2}\rho_c^{2-\frac{3\gamma}{2}}}{(\frac{\gamma}{\gamma-1})^{3/2}\frac{\rho_o}{\rho_c}} = \frac{-4\pi * \text{Integral}_S * const.}{(\frac{\gamma}{\gamma-1})^{3/2}\frac{\rho_o}{\rho_c}}$$

Figure 3 και Figure 4

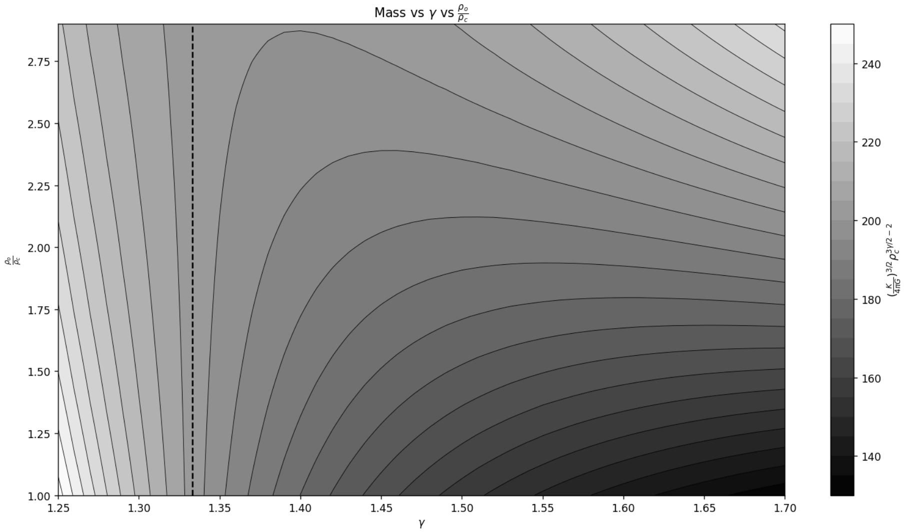
```
#Figure 3 + Figure 4
RoDiaRc=np.arange(1,2.95,0.01)
gamma=np.arange(1.2,1.75,0.01)
Entropy=np.zeros((len(RoDiaRc),len(gamma)),float)
mazaCnt=np.zeros((len(RoDiaRc),len(gamma)),float)

for i in range(len(RoDiaRc)):
    for j in range(len(gamma)):
        g=gamma[j]
        n=1/(g-1)
        w0=[RoDiaRc[i]**(1/n),0]

        y=odeint(model,w0,x,args=(n,))

        xx=x
        yy=y[:,0]

        index=np.where(~np.isnan(yy))
        xx=xx[index]
        yy=yy[index]

        theta=interp1d(xx,yy,kind="cubic",fill_value="extrapolate")

        xo=0
        xr=fsolve(theta,6)
        xr=xr[0]
        wmin=np.pi/xr
```

```
        I=simpson(Mass(xx,yy,n),xx)
        I=4*np.pi*(np.sqrt(g/(g-1)))**3*I

        mazaCnt[i][j]=I


        rCoef=RoDiaRc[i]
        gCoef=g/(g-1)
        listY=[]
        listX=[]
        h=(40-wmin)/(1000-1)
        wtest=np.arange(wmin,40+h,h)
        for wt in wtest:
                listY.append(S(wt,yy,n,wmin))
        listX=wtest

        I=simpson(listY,listX)
        I=-I*np.pi*4./(rCoef*gCoef**(3/2))
        Entropy[i][j]=I
```

Figure 3 και Figure 4

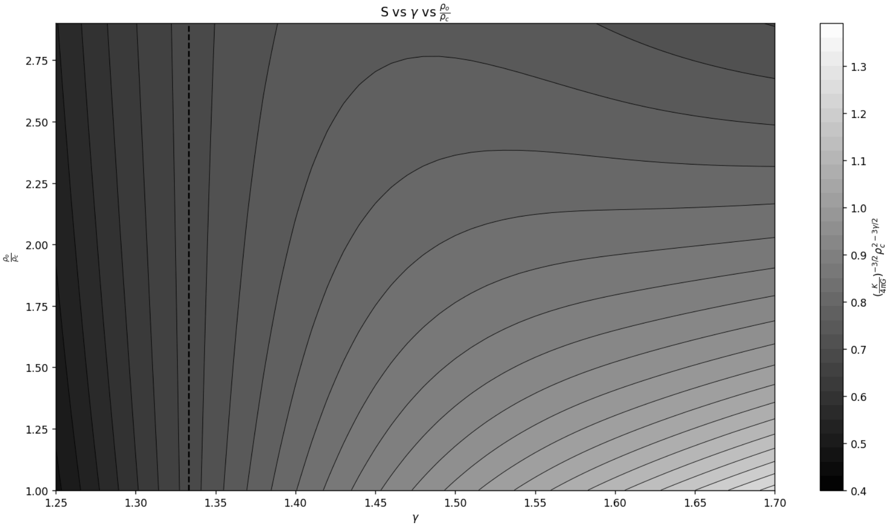
```
#Figure 3
plt.figure(4)
plt.contour(gamma,RoDiaRc,mazaCnt,levels=np.arange(130,255,5),linewidths=0.5, colors='k')
plt.contourf(gamma,RoDiaRc,mazaCnt,levels=np.arange(130,255,5),cmap="gist_gray")
plt.colorbar(ticks=np.arange(140,270,20),label=r"$(\frac{K}{4\pi G})^{3/2}\rho_c^{3\gamma/2-2}$")
plt.ylim([1,2.9])
plt.xlim([1.25,1.7])
ymin,ymax=plt.gca().get_ylim()
plt.plot([4/3,4/3],[ymin,ymax],"k--")
plt.xlabel(r"$\gamma$")
plt.ylabel(r"$\frac{\rho_o}{\rho_c}$")
plt.title("Mass vs "r"$\gamma$"" vs "r"$\frac{\rho_o}{\rho_c}$")

#Figure 4
plt.figure(5)
plt.contour(gamma,RoDiaRc,Entropy,levels=np.arange(0.4,1.4,0.03), linewidths=0.5, colors='k')
plt.contourf(gamma,RoDiaRc,Entropy,levels=np.arange(0.4,1.4,0.03),cmap="gist_gray")
plt.colorbar(ticks=np.arange(0.4,1.4,0.1),label=r"$(\frac{K}{4\pi G})^{-3/2}\rho_c^{2-3\gamma/2}$")
plt.ylim([1,2.9])
plt.xlim([1.25,1.7])
ymin,ymax=plt.gca().get_ylim()
plt.plot([4/3,4/3],[ymin,ymax],"k--")
plt.xlabel(r"$\gamma$")
plt.ylabel(r"$\frac{\rho_o}{\rho_c}$")
plt.title("S vs "r"$\gamma$"" vs "r"$\frac{\rho_o}{\rho_c}$")
```

Figure 3



Mass vs γ vs $\frac{\rho_o}{\rho_c}$
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
$\frac{\rho_o}{\rho_c}$
1.25
1.30
1.35
1.40
1.45
1.50
1.55
1.60
1.65
1.70
γ
240
220
200
180
160
140
$(\frac{K}{4\pi G})^{3/2}\rho_c^{3\gamma/2-2}$

Figure 4



S vs γ vs $\frac{\rho_o}{\rho_c}$
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
$\frac{\rho_o}{\rho_c}$
1.25
1.30
1.35
1.40
1.45
1.50
1.55
1.60
1.65
1.70
γ
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
$(\frac{K}{4\pi G})^{-3/2}\rho_c^{2-3\gamma/2}$

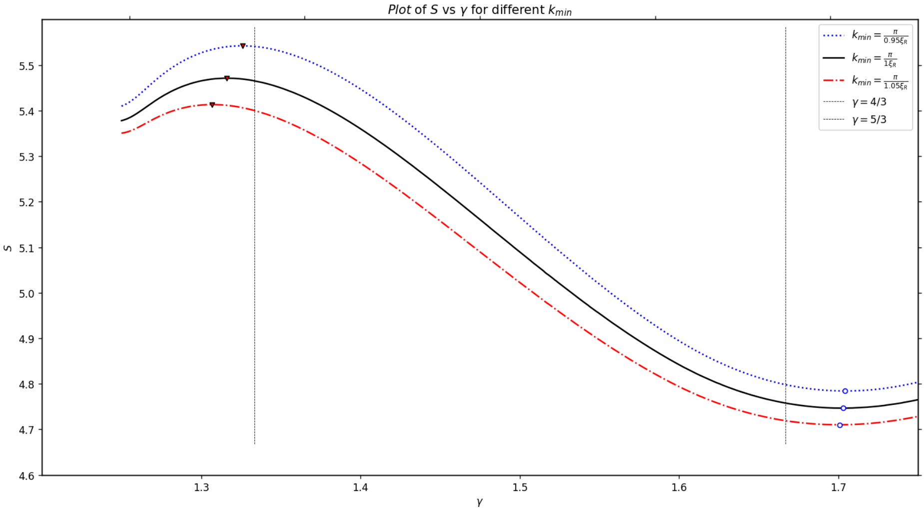
Figure 5

```
#Figure 5
GammaFig5=np.arange(1.25,1.8,0.001)
DataX=[]
DataY=[[],[],[]]
for g in GammaFig5:
    n=1/(g-1)
    y=odeint(model,w_initial,x,args=(n,))

    xx=x
    yy=y[:,0]

    ind=np.where(~np.isnan(yy))
    xx=xx[ind]
    yy=yy[ind]

    theta=interp1d(xx,yy,kind="cubic",fill_value="extrapolate")

    xo=0
    xr=fsolve(theta,6)
    xr=xr[0]

    wFIG5=[np.pi/xr/0.95,np.pi/xr,np.pi/xr/1.05]
    for i in range(len(wFIG5)):
        listY=[]

        wmin=wFIG5[i]
        h=(40-wmin)/(1000-1)
        wtest=np.arange(wmin,40+h,h)
        for wt in wtest:
                listY.append(S(wt,yy,n,wmin))

        I=simpson(listY,wtest)
        I=-I*np.pi*4
        DataY[i].append(I)

    DataX.append(g)
```

Figure 5

```
#Figure 5
fig4,ax4=plt.subplots()
ax4.plot(DataX,DataY[0],label=r"$k_{min}=\frac{\pi }{0.95\xi_R}$",linestyle="dotted",color="blue")
a1=DataY[0].index(max(DataY[0]))
a2=DataY[0].index(min(DataY[0]))
ax4.scatter(DataX[a1],DataY[0][a1],color="red",edgecolor="black",marker="v",s=20*2**0,zorder=2)
ax4.scatter(DataX[a2],DataY[0][a2],color="white",edgecolor="blue",s=20*2**0,zorder=2)

ax4.plot(DataX,DataY[1],label=r"$k_{min}=\frac{\pi }{1\xi_R}$",linestyle="solid",color="black")
a1=DataY[1].index(max(DataY[1]))
a2=DataY[1].index(min(DataY[1]))
ax4.scatter(DataX[a1],DataY[1][a1],color="red",edgecolor="black",marker="v",s=20*2**0,zorder=2)
ax4.scatter(DataX[a2],DataY[1][a2],color="white",edgecolor="blue",s=20*2**0,zorder=2)

ax4.plot(DataX,DataY[2],label=r"$k_{min}=\frac{\pi }{1.05\xi_R}$",linestyle="dashdot",color="red")
a1=DataY[2].index(max(DataY[2]))
a2=DataY[2].index(min(DataY[2]))
ax4.scatter(DataX[a1],DataY[2][a1],color="red",edgecolor="black",marker="v",s=20*2**0,zorder=2)
ax4.scatter(DataX[a2],DataY[2][a2],color="white",edgecolor="blue",s=20*2**0,zorder=2)

axes = plt.gca()
y_min, y_max = axes.get_ylim()
ax4.plot([4/3,4/3],[y_min,y_max],"k--",label="$\gamma=4/3$",linewidth=0.5)
ax4.plot([5/3,5/3],[y_min,y_max],"k--",label="$\gamma=5/3$",linewidth=0.5)
ax4.legend()
ax4.set_xlabel(r"$\gamma$")
ax4.set_ylabel(r"$S$")
ax4.set_title(r"$Plot$"" of "r"$S$"" vs "r"$\gamma$"" for different "r"$k_{min}$")
ax4.set_xlim([1.2,1.75])
ax4.set_ylim([4.6,5.6])
ax4.set_yticks(np.arange(4.6,5.6,0.1))
ax4.set_xticks(np.arange(1.3,1.8,0.1))

ax_NEW=ax4.twinx().twiny()
ax_NEW.set_xlim([1.25,1.75])
ax_NEW.set_ylim([4.6,5.6])
ax_NEW.set_yticks(np.arange(4.6,5.6,0.1))
ax_NEW.set_xticklabels([])
ax_NEW.set_yticklabels([])
```

Figure 5



Plot of S vs γ for different k_{min}
$k_{min} = \frac{\pi}{0.95\xi_R}$
$k_{min} = \frac{\pi}{1\xi_R}$
$k_{min} = \frac{\pi}{1.05\xi_R}$
$\gamma = 4/3$
$\gamma = 5/3$
S
γ
5.5
5.4
5.3
5.2
5.1
5.0
4.9
4.8
4.7
4.6
1.3
1.4
1.5
1.6
1.7

ΤΕΛΟΣ ΠΑΡΟΥΣΙΑΣΗΣ