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Feasibility of DAGI Validation with Stabilizer Graph Codes

1. Reconstructing Subsystem Entropies from Provided Data/Code

Availability of Code and Data: The authors of *Engineering Holography with Stabilizer Graph Codes* have made their code openly available on GitHub [1]. This repository contains Python scripts (e.g. `Main_16.py`) that generate the small holographic code instance with **16 qubits** (12 boundary + 4 bulk) described in the paper [2]. The code can directly output the stabilizer **graph state** corresponding to the hyperbolic pentagon (HaPPY) code – for example, by selecting the provided set of local Clifford (Hadamard) gates to obtain the exact graph from the manuscript [3]. Running the code produces files like *GraphAM.pdf*, showing the final adjacency graph after contraction/measurement [4]. Using this output, we can reconstruct the **stabilizer generators or adjacency matrix** of the 16-qubit state. With the full stabilizer state in hand, it is feasible to compute **entropies of any boundary subregion**: for a pure stabilizer state, subsystem entropies can be derived from the stabilizer rank on that subset (or by partial trace if we convert to a statevector). In summary, **yes** – the provided code and data are sufficient to reconstruct the entanglement entropy of arbitrary boundary subsets and their mutual informations. The holographic code is known to obey the Ryu–Takayanagi entropy law, meaning the entropy $S(\partial E)$ of a boundary region is proportional to the length of the bulk geodesic (cut) γ that separates it from the rest [5]. This implies the necessary subregion entropy structure is present in the generated state (indeed the HaPPY code "exhibits … entanglement entropy constrained by the Ryu–Takayanagi formula" [6]). We can leverage that structure by explicitly computing $S(A)$ for various boundary sets A and verifying that it matches the expected cut-size (geodesic) from the code's tensor network model [7]. Such calculations are feasible either by linear algebra on the stabilizer check matrix or by constructing the 16-qubit state in a quantum simulator (e.g. Qiskit) and doing partial traces.

Mutual Information for Boundary Subregions: Once we can compute entropies for any chosen subset of boundary qubits, we can obtain **mutual information** values $I(\text{boundary subset} : \text{bulk qubit})$ for different subsets. For example, by taking one encoded bulk degree of freedom (say bulk qubit "A") and calculating the entropy of that bulk qubit, the entropy of a boundary region X, and the entropy of the combined system ${X \cup A}$, we get the mutual information $I(X:!A) = S(X) + S(A) - S(X \cup A)$. In practice – since the overall 16-qubit state is pure and the bulk qubits are entangled with the boundary – $I(X:!A)$ can be computed from just boundary entropy differences. The key point is that the code supplies the exact state needed to evaluate such measures **without additional experimental data**. We anticipate that for small boundary subsets (those not meeting the "entanglement wedge" size), the mutual information with a given bulk qubit will be zero (or negligible), whereas once the boundary region is large enough to reconstruct that bulk qubit, $I(X:!A)$ jumps to 1 ebit (for a perfectly encoded logical qubit). This behavior can indeed be checked by systematically varying $|X|$. The authors' toy model explicitly demonstrates that a bulk qubit can be recovered only from its **nearby boundary region** of sufficient size – e.g. **two bulk qubits (A,B) are recoverable from their adjacent 5-qubit boundary region** [8]. Therefore, by using the code's output state, we can calculate mutual informations $I(\text{any subset of ≤4 boundary qubits} : A)$ (expected to be ~0) and see it become nonzero at $|X|=5$ for the special region that touches bulk A's entanglement wedge.

2. Using Stabilizer Graph Outputs for Möbius-f_k Decomposition

The **stabilizer graph state** produced by the code can serve as the starting point to perform a **Möbius decomposition of multi-party information** (the *inclusion–exclusion* breakdown of mutual information among fragments). In the DAGI framework, one defines contributions $f(S)$ for each subset S of observers ("fragments") such that the total information gained by considering S together can be decomposed into unique contributions from each combination [9] [10]. In our context, S will refer to sets of boundary fragments collaborating to learn about a bulk bit. Using the entropies and mutual informations computed from the stabilizer state, we can **invert** the inclusion–exclusion relations to solve for the *intrinsic information* terms f_k (where f_k denotes the net information that is accessible **only** by joint observation of k boundary fragments and not by any smaller combination). Concretely, if we partition the relevant boundary region into, say, n fragments, we can compute $I(\text{bulk} : \text{any union of fragments})$ for all subsets and apply Möbius inversion to find each $f(X)$. For example, in a scenario with individual boundary qubits as fragments, $f_{{i}}$ would correspond to the information bulk qubit A has with fragment i alone (these are single-fragment mutual infos), $f_{{i,j}}$ corresponds to **2-fragment** synergy or redundancy, etc., and a positive $f_{{i,j,...,k}}$ indicates a *synergistic* information that only appears when those k fragments are combined [10]. By doing this for all fragment group sizes, we obtain the set ${f_1, f_2, ..., f_n}$ (summing over all specific subsets of a given size if needed). The key feasibility result here is that **the stabilizer formalism makes this computationally tractable** – stabilizer states allow efficient entropy calculation even for 12-qubit boundary partitions, so we can find all requisite mutual information values exactly (the state is small enough to enumerate subsets, or exploit symmetry to reduce the workload). In summary, the authors' stabilizer graph output **can be used directly to compute DAGI's Möbius-f_k terms** for the boundary–bulk information sharing. All necessary ingredients (entropy of every combination of boundary pieces with the bulk) are obtainable from the known quantum state. There is no indication of any roadblock in using their data for such an information-theoretic decomposition – it's essentially a post-processing step on the state's entropy table.

3. Unitary Circuits and Stabilizer States for Exact Entropies (Code Content)

Code includes circuit definitions: The published code provides an explicit constructive recipe for the holographic code state. In particular, the authors derive the **encoding circuit** that prepares the 16-qubit graph state corresponding to the hyperbolic pentagon code. The repository documentation confirms that gate sequences (controlled-Z gates and local Clifford operations) to build the state $|H\rangle$ are either given or can be generated by the script [11] [12]. Indeed, the paper even spells out the formula for the logical code state (Eq. (12) in the text, which involves preparing $|+\rangle^{\otimes 12}$ and applying CZ gates between specific qubit pairs) [13]. The code's *Option 2* for the 16-qubit instance explicitly applies the particular set of Hadamards corresponding to the published graph, yielding the **exact stabilizer state from the manuscript** [14]. Thus, we have at our disposal either (a) a list of **stabilizer generators** or (b) a concrete **quantum circuit** that produces the state. Both representations enable exact entropy computations. For instance, given the stabilizer check matrix for $|H\rangle$, one can determine the rank of the stabilizer subgroup acting on any subset of qubits, which directly gives the subsystem entropy (since for a pure stabilizer state on n qubits, the entropy $S(A)$ of subset A equals $|A| - \frac{1}{2}\text{rank}(S_A)$

where S_A are the stabilizer generators acting trivially on A). Alternatively, we could feed the circuit into a simulator to get the full state vector and then compute reduced density matrices for subsets (feasible for $2^{16}=65,536$ dimensions). In short, the codebase includes **all necessary circuit elements and state descriptions** to allow **exact calculation of entropies and mutual informations** – there is no need for random sampling or approximate simulation. This is a crucial enabler for the DAGI validation, because it means we can deterministically compute things like $I(X:!A)$ for *any* boundary set X and bulk qubit A. The code also outputs visualization of the graph structure (pre- and post-measurement) [4], which helps in identifying which boundary qubits are connected to which bulk nodes in the tensor network (useful for choosing relevant subregions to test). In summary, the **unitary encoding circuits and stabilizer graph** provided can be directly utilized to get exact entropy values needed for the analysis.

4. Subregion–Entanglement Plot and Fragment Sizes

Structure of Entropy vs Subregion Size: The small 12-boundary holographic code is expected to show a characteristic entropy curve as a function of boundary region size. In holographic error-correcting codes, **entanglement entropy initially grows with subregion size** (each additional boundary qubit adds entropy until a certain point) and then saturates or drops once the region becomes large enough to include a bulk degree of freedom's entanglement wedge [7] [15]. Specifically, for the 5-qubit region mentioned in the paper (the region that can recover 2 bulk qubits), the entropy $S(5)$ should reflect the RT formula: it equals the geodesic cut length (3 Bell pairs crossing) since the bulk content is recoverable [5] [16]. For any **smaller region (size <5)**, the region is entirely insufficient to see the bulk information, so we expect its entropy to be maximal (proportional to its size, as if those qubits were maximally mixed with the rest). This means the entropy plot vs. region size will look like a rising line up to $|X|=4$, and then a bend or plateau at $|X|=5$ where the entropy is lower than the trivial volume-law expectation. The authors' results imply a **broad range of subregion sizes** was considered conceptually – from very small up to half the boundary. In our 16-qubit state, we can calculate $S(L)$ for $L=1,\dots,6$ boundary qubits (beyond 6, it's symmetric by purification with bulk). This is indeed enough range to perform a **multiscale Partial Information Decomposition (PID)** analysis: we have data from the regime where no bulk info is accessible (1–4 qubits), the critical point (5 qubits), and beyond. The **subregion entanglement data are sufficiently rich** to extract higher-order information terms. Moreover, because the code is small, we can partition a given boundary region into multiple fragments in different ways (e.g. consider 5 single-qubit fragments versus a single 5-qubit block) to see how information combines. The paper's focus on *partial recovery from nearby boundary* guarantees that there is a non-trivial distribution of information among multiple boundary pieces – exactly the scenario where synergy can be analyzed. To summarize, the entropies as a function of subregion size follow the expected **holographic entropy cone** shape, and the available range of sizes (and ways to fragment the boundary) is sufficient to carry out a meaningful multiscale DAGI/PID analysis.

5. Experiment Design: DAGI Validation Coding Plan

Using the authors' code and data, we propose the following concrete steps for a coding agent to validate the DAGI predictions:

1. **Generate the Holographic Code State:** Run the provided `Main_16.py` (or an equivalent routine) in the configuration that produces the 16-qubit hyperbolic pentagon code state from the paper [14]. This will yield either the stabilizer adjacency matrix or a set of stabilizer generators for the state $|H\rangle$ (and optionally, a visual graph in *GraphAM.pdf* [4]). Confirm from the output which qubit

indices correspond to boundary nodes (should be labeled 1–12) and which to bulk (labeled A, B, C, D in the paper, likely mapped to indices or separate identifiers) [2] .

2. **Represent State in Python:** Using Python quantum libraries, construct an object for the stabilizer state. For example, one can use **Qiskit** to create a 16-qubit quantum circuit that applies CZ gates on all edges specified by the graph (the edge list is given in the paper's Eq. (12) and surrounding text [11]). Initialize all qubits (12 boundary + 4 bulk) to $|0\rangle$ or $|+\rangle$ as appropriate (the logical bulk qubits might start in $|0\rangle_{\text{bulk}}$ while boundary starts $|+\rangle$), then apply the sequence of CZ gates between qubits (u,v) as listed. Another approach is to use the **stabilizer table** directly: one can input the check matrix $(\mathbf{1} | \Gamma)$ (with Γ the adjacency matrix from GraphAM) into a stabilizer-state library. The goal is to have a **statevector or stabilizer object** that we can query for entropies.

3. **Bulk–Boundary Partitioning:** Identify one **bulk qubit of interest** (e.g. bulk qubit A) and its corresponding "nearby boundary" region. According to the paper, bulk A (and B) are recoverable from a specific 5-qubit boundary subset (call it region X) [8] . Determine which 5 boundary qubits make up this special region X (the figure and text indicate it's qubits 1–5 for two bulk qubits A,B in the example) [8] . We will focus on this boundary set $X` and its constituents. For clarity, further divide X into smaller **fragments**. The simplest choice is to treat each of the 5 individual boundary qubits in X as one fragment (so fragments = {1}, {2}, {3}, {4}, {5}). If desired, one could also consider grouping them in other ways, but single-qubit fragments give the finest-grained PID. Also note the other boundary qubits (6–12 in this scenario) which are not part of the recovery region – those should have essentially zero information about bulk A by themselves or in smaller groups, so they can serve as a control check.

4. **Compute Entropies and Mutual Information:** For each fragment (and combinations of fragments), compute the **von Neumann entropy** $S(\cdot)$ using the state representation. This can be done by tracing out all other qubits and finding the density matrix for the fragment(s) in question, then computing $-\mathrm{Tr}(\rho \log_2 \rho)$. However, since the full 16-qubit state is pure, it's easier to use relations: e.g., $S(\text{fragment}) = S(\text{complement})$, so $S(\text{fragment of boundary})$ can be obtained by calculating the entropy of the complementary subsystem (which includes the bulk and the rest of the boundary). Using the stabilizer formalism, an even more efficient method is available: determine the number of independent stabilizer generators that act only on the complement of the fragment – the remainder will equal the entropy (in qubits) of the fragment. After getting single-fragment entropies, compute **joint entropies** for multi-fragment groups (e.g. $S(i,j)$ for two boundary qubits i and j, etc., up to $S(1,2,3,4,5)$ for the entire region X). Similarly, compute entropies involving the bulk qubit A together with various fragments (for mutual infos). In practice, an easier route is to compute **mutual information** $I(A : Y)$ for any boundary set Y directly via $I(A:Y) = S(A) + S(Y) - S(AY)$. Here $S(A)$ (entropy of a single logical bulk qubit) is known (if bulk A is in a pure state initially, $S(A)$=0; if it's maximally mixed or one of two bulk qubits, treat accordingly), and $S(A Y)$ = entropy of the joint system of bulk A plus boundary subset Y (again obtainable from the stabilizer state by tracing out the complement). By iterating over all subsets $Y \subseteq X$ (as well as some outside X for comparison), we can tabulate $I(A:Y)$ for **all sizes** of interest (1 through 5 boundary qubits). This directly tells us how much information bulk A shares with any particular combination of boundary fragments. We expect $I(A:Y) \approx 0$ for all $|Y|<5$ when $Y \subseteq X$, and $I(A: X) = 1$ (one full bit) when

$Y=X$ (the full recovery set). For subsets Y that do not include all of X or are not contiguous, the mutual information should remain zero – reflecting the code's perfect secrecy for insufficient shares.

5. **Compute Möbius-f_k Terms:** Using the mutual information values from the previous step, perform a **Möbius inversion** to extract the contribution terms f_k. In essence, we treat the mutual information of bulk A with a union of fragments as the cumulative sum of information contributions from all combinations of those fragments. For example, in inclusion–exclusion form for five fragments:

$$I(A:\{1,2,3,4,5\}) = \sum_{i} f(\{i\}) + \sum_{i<j} f(\{i,j\}) + \sum_{i<j<k} f(\{i,j,k\}) + \cdots + f(\{1,2,3,4,5\}) \; ,$$

and similarly for any subset Y. By computing $I(A:Y)$ for every $Y \subseteq {1,2,3,4,5}$, we can solve for the unique f-values. In practice, start with singletons: $f({i}) = I(A:{i})$ for each fragment i (likely all zero here). Then for any pair ${i,j}$, use $I(A:{i,j}) = f({i}) + f({j}) + f({i,j})$ to find $f({i,j})$, and so on up to the 5-fragment set [9]. The result will be **$f_1, f_2, ..., f_5$** where f_k denotes the *total* information contribution from any k-way synergy. (We expect f_5 to be ~1 bit, representing the **5-party synergistic information** that only the full set ${1,2,3,4,5}$ can access, and all lower-order f_k (for $k<5$) to be ~0 in an ideal code.) These f-values directly tell us the distribution of information: a nonzero f_k (positive indicates synergy [10]) means an irreducible k-fragment cooperative effect is present. Negative f_k would indicate redundant information (not expected in this error-correcting code context), and zero means no new info gained by that combination beyond smaller groups [10].

1. **Plot the Running Synergy Ratio:** Finally, compute and plot the **running synergy ratio** $R_{\ge 3}(f)$. We define $R_{\ge 3}$ as the fraction of the bulk information that is contributed by **higher-order synergy** (tripartite and beyond). For example, $R_{\ge 3} = \frac{f_3 + f_4 + f_5}{I_{\text{total}}}$, where $I_{\text{total}} = I(A:{1,2,3,4,5})$ (which should equal 1 bit for a perfectly encoded bulk qubit). In our case, since we anticipate $f_5 \approx 1$ and all others 0, this ratio would jump from 0 to 100% once we include all five fragments. To illustrate the **multiscale emergence of synergy**, we can plot $R_{\ge 3}$ as a function of the number of fragments considered or the size of the boundary region. One way is to simulate a process of "adding fragments one by one" in an optimal order and at each step compute the portion of info that comes from $\ge$3-party effects. Initially, with fewer than 3 fragments, $R_{\ge3}=0$ by definition (not enough parties for higher-order synergy). As we get to 3, 4 fragments, we check if any triple synergy f_3 appears – likely it remains 0 until the full set of 5 is present. Thus the **plot will remain near zero** and then show a spike when the entanglement wedge is complete. This *running ratio* effectively visualizes the **threshold behavior**: before the threshold, any information about the bulk is either absent or encoded in at most pairwise correlations; beyond the threshold, essentially all the information is locked in a collective (high-order) form. The agent can produce a simple line or step plot: x-axis = "fraction of boundary included" or "number of fragments combined," y-axis = $R_{\ge3}$. We expect a clear qualitative signature (flat ~0, then rising sharply to 1).

Throughout these steps, the coding agent should rely **only on the authors' published code and data outputs** – no external experimental input is needed. Standard Python libraries (NumPy for linear algebra, Qiskit or Stim for stabilizer states, etc.) are sufficient to implement the entropy and mutual information calculations. The procedure above is concrete and algorithmic, so it can be directly turned into a script or Jupyter notebook, ensuring the experiment is **fully reproducible** with the given resources.

6. Testing the Gravitational Synergy Prediction

With the above plan, we can explicitly **test DAGI's gravitational information prediction** in this toy holographic code. The prediction is that *higher-order synergistic information* (requiring 3 or more parties) should remain essentially zero until the boundary region is large enough to include the bulk qubit's entanglement wedge (the error-correction threshold), at which point the previously "locked" information is suddenly accessible in combination. The results of our analysis will directly confirm or refute this:

- **For boundary fragments below the threshold size:** We expect to find $I(A:Y) = 0$ for any Y smaller than the required region (e.g. any 1, 2, 3, or 4-qubit subset had no mutual information with bulk A). Consequently, all **Möbius terms** f_k for $k<5$ will come out zero. In particular, **no non-zero tripartite or four-partite synergy** ($f_3=f_4=0$) will be detected. This aligns with the idea that no subset of fewer than 5 boundary qubits has any bit of the bulk information – so there's no "partial credit" information to distribute among smaller groups. The synergy ratio $R_{\ge3}(f)$ in this regime will be 0, indicating no multi-fragment effect present (the bulk info is completely inaccessible, rather than redundantly or fractionally stored).

- **At the entanglement wedge inclusion point (full region X):** We anticipate a **non-zero high-order term** – specifically $f_5 \approx 1$ bit – corresponding to the fact that only the *entire* collection of 5 boundary qubits can reconstruct bulk A. This is the hallmark of a **pure 5-partite synergy** (akin to a threshold secret-sharing: only the joint observation of all five pieces yields the secret). All the mutual information $I(A:X)$ comes as an irreducible 5-body correlation, which means the synergistic component f_{5} carries the full information and $f_{k<5}=0$. Thus, $R_{\ge3}$ jumps to 1 (or 100%) once the boundary region reaches the critical size. This outcome would quantitatively demonstrate the DAGI prediction in a clear way: **no information until the threshold, then suddenly all information is encoded in a collective fashion**. It mirrors the Ryu–Takayanagi behavior where the entropy of a region stops growing once it captures a bulk degree – here we see that reflected as "zero mutual information until you have the whole region, then one bit appears as a 5-way synergy."

- **Robustness check:** We can also verify that boundary regions larger than the minimal wedge (e.g. 6 qubits or more, which include region X plus extra qubits) do not introduce new independent synergy terms but rather continue to have the one bit of bulk info. In such cases, some of the f_k might redistribute (because adding an unrelated qubit that is maximally mixed can introduce trivial mutual information that gets subtracted out in inclusion-exclusion), but the essential f_{5} synergy for the core region should remain the only positive contribution. This confirms that once the bulk is reconstructible, adding more boundary (not in the minimal set) doesn't create further bulk info – it just adds uncorrelated entropy. The DAGI prediction to test is specifically that **higher-order (≥3) information remains zero up to the point of reconstruction, rather than gradually accumulating**; our experiment will clearly show a **sharp transition** rather than a smooth increase in multi-partite information. Any deviation (e.g. a partial information at 4 qubits or a gradual increase) would indicate the code is not perfectly error-correcting, but given it's a stabilizer code, we expect an ideal sharp threshold.

Feasibility Assessment: This entire validation experiment is **feasible and well within reach** using the authors' provided resources. We have a full description of the quantum state, enabling exact calculation of entropies and information measures. The system size (12 boundary qubits) is small enough for exhaustive subset analysis, and the stabilizer nature of the state greatly simplifies computations. The steps outlined can be automated, and the expected results (zero lower-order f_k, nonzero $f_{k\ge 5}$ at threshold) are aligned with both the authors' claims of partial bulk recoverability [15] and the theoretical DAGI framework's expectations. In conclusion, **the proposed DAGI validation is entirely feasible**: by leveraging the stabilizer graph code of Anglès Munné *et al.* (2024), a coding agent can definitively test and visualize how holographic quantum information is distributed across scales – confirming that **no multi-party synergy emerges until the boundary region is large enough to include the bulk, at which point a high-order synergy term activates**, exactly as the gravitational analogy predicts.

Sources: The analysis above is grounded in the details of the published code and paper: the code availability and usage instructions [1] [3], the structure of the 12+4 qubit holographic state [2], the authors' description of partial bulk recovery with a 5-qubit boundary [8], and the theoretical framework of information decomposition into f_k synergy measures [10]. These sources collectively support the feasibility of our approach and the interpretation of the expected outcomes.

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