RESIDUAL EQUIDISTRIBUTION OF MODULAR SYMBOLS AND COHOMOLOGY CLASSES FOR QUOTIENTS OF HYPERBOLIC n-SPACE

PETRU CONSTANTINESCU AND ASBJØRN CHRISTIAN NORDENTOFT

ABSTRACT. We provide a simple automorphic method using Eisenstein series to study the equidistribution of modular symbols modulo primes, which we apply to prove an average version of a conjecture of Mazur and Rubin. More precisely, we prove that modular symbols corresponding to a Hecke basis of weight 2 cusp forms are asymptotically jointly equidistributed mod p while we allow restrictions on the location of the cusps. Additionally, we prove the full conjecture in some particular cases using a connection to Eisenstein congruences. We also obtain residual equidistribution results for modular symbols where we order by the length of the corresponding geodesic. Finally, and most importantly, our methods generalise to equidistribution results for cohomology classes of finite volume quotients of n-dimensional hyperbolic space.

1. INTRODUCTION

Modular symbols are certain periods of weight 2 cusp forms introduced by Birch and Manin which are an indispensable tool for studying (twisted) L-functions of holomorphic cusp forms [26], [28] and for computing modular forms [9]. Modular symbols define elements of certain cohomology groups and the results of this paper thus fit into a bigger picture of the study of (co)homology of arithmetic groups, which has received a lot of attention recently [2], [5] due to their deep connections with number theory coming from [40].

Recently, Mazur and Rubin initiated the study of the arithmetic distribution of modular symbols and put forward a number of conjectures [30], which have received a lot of attention recently [35], [24], [50], [3], [32], [10], [8]. One of these conjectures (see [29]) predicts that (normalised) modular symbols should equidistribute among the residue classes modulo p. An average version of this conjecture was settled by Lee and Sun [24, Theorem I] recently using dynamical methods. In this paper we introduce a new automorphic method for studying the mod p distribution of modular symbols, which also applies to more general cohomology classes. As is the case in [24], we obtain an average version of the mod p conjecture of Mazur and Rubin (and its generalisations), but with further refinements. Using different arguments, we can actually prove the full conjecture in some special cases (specific p and specific cusp forms), see Section 3.

Our automorphic methods enable us to deal with a much more general setup compared to the work of Lee and Sun and thus we obtain a number of new results:

(1) First of all we obtain *joint* equidistribution for the mod p values of modular symbols (appropriately normalised) associated to a Hecke basis of weight 2 cusp forms restricted to cusps which lie in a *fixed* interval of $\mathbb{R}/\mathbb{Z}$.

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(2) We show that the values of (unnormalised) modular symbols restricted to cusps lying in a fixed interval of $\mathbb{R}/\mathbb{Z}$ equidistribute mod 1.
(3) We also obtain equidistribution results for modular symbols ordered by the length of the geodesic associated to the corresponding matrices (as opposed to the denominator of the cusp).
(4) Lastly (and most interestingly) we extend the equidistribution results to classes in the cohomology of general finite volume quotients of higher dimensional hyperbolic spaces.

We note that in the case of higher dimensional hyperbolic spaces there is interesting torsion in the cohomology. The breakthrough of Scholze [40] established that such torsion classes have associated Galois representations. This was actually our original motivation for studying the higher dimensional cases. Furthermore, Bergeron and Venkatesh [2] have conjectured that at least in the three dimensional case there is an abundance of torsion in the relevant cohomology group. In this paper we are able to shed light on the distribution properties of these cohomology classes. In Section 8 we will survey what is know about the dimensions of the cohomology groups, which our results apply to.

Remark 1.1. Our method is automorphic in nature and relies on the theory of Eisenstein series. It can be seen as a discrete version of the method introduced by Petridis and Risager in [33] for studying the distribution of modular symbols. They consider the perturbation of the family of characters χ^ε as $\varepsilon \to 0$, whereas we consider the discrete family χ^m for $m \in \mathbb{Z}$. In particular, we find it is interesting that residual equidistribution of modular symbols is an almost direct consequence of the meromorphic continuation of twisted Eisenstein series.

1.1. **Results for modular symbols.** Let us state the result in the simplest case for the 2 dimensional hyperbolic space in an arithmetic setup. We define the *modular symbol map* associated to a weight 2 and level N cusp form $f \in \mathcal{S}_2(\Gamma_0(N))$ as the map

$$\mathbb{Q} \ni r \mapsto \langle r, f \rangle := 2\pi i \int_r^{i\infty} f(z)dz, \tag{1.1}$$

where the contour integral is taken along a vertical line. One way to think about this map is as the Poincaré pairing on $\Gamma_0(N)\backslash\mathbb{H}^2$ between the 1-form $2\pi i f(z)dz$ and the homology class of paths containing the geodesic from r to $i\infty$.

Now assume that f is a Hecke newform with associated elliptic curve $E/\mathbb{Q}$ and $N \geq 3$. If we let Ω_+ and Ω_- be the real and imaginary Néron periods of E, then it is a fact that

$$\frac{1}{\Omega_\pm}\left(\langle r, f\rangle \pm \langle -r, f\rangle\right) \in \mathbb{Q} \tag{1.2}$$

for all $r \in \mathbb{Q}$ (see [30, Sec. 1]).

Given a prime p there is a minimal p-adic evaluation $v_p^\pm(f)$ of (1.2) among all $a/q \in \mathbb{Q}$ with $N|q$ (since the image under the modular symbols map is a lattice).

We define

$$\mathfrak{m}^\pm_{f,p}(r) := \frac{p^{-v_p^\pm(f)}}{\Omega_\pm}\left(\langle r, f\rangle \pm \langle -r, f\rangle\right),$$

which is p-integral for $r = a/q$ with $N|q$, but not all divisible by p. Then given a basis of newforms $f_1, \ldots, f_d$, we can consider the map $\mathbb{Q} \to (\mathbb{Z}/p\mathbb{Z})^{2d} \times (\mathbb{R}/\mathbb{Z})$ given by

$$\mathfrak{m}_{N,p}(r) := (\mathfrak{m}^+_{f_1,p}(r), \mathfrak{m}^-_{f_1,p}(r), \ldots, \mathfrak{m}^+_{f_d,p}(r), \mathfrak{m}^-_{f_d,p}(r), r)$$

as a random variable defined on the outcome space

$$\Omega_{Q,N} := \{a/q \mid 0 < a < q \leq Q, (a,q) = 1, N|q\}$$

endowed with the uniform probability measure. Then we have the following equidistribution result.

Theorem 1.2. *The random variables $\mathfrak{m}_{N,p}$ defined on the outcome spaces $\Omega_{Q,N}$ converge in distribution to the uniform distribution on $(\mathbb{Z}/p\mathbb{Z})^{2d} \times (\mathbb{R}/\mathbb{Z})$ as $Q \to \infty$. This means in concrete terms that for any fixed $\mathbf{a} \in (\mathbb{Z}/p\mathbb{Z})^{2d}$ and any interval $I \subset \mathbb{R}/\mathbb{Z}$, we have*

$$\frac{\#\left\{a/q \in \Omega_{Q,N} \cap I \mid (\mathfrak{m}^+_{f_1,p}(a/q), \ldots, \mathfrak{m}^-_{f_d,p}(a/q)) \equiv \mathbf{a} \bmod p\right\}}{\#\Omega_{Q,N}} = \frac{|I|}{p^{2d}} + o(1)$$

as $Q \to \infty$.

Secondly we can consider the map $\mathbb{Q} \to (\mathbb{R}/\mathbb{Z})^{2d+1}$ given by

$$\mathfrak{m}_{N,\mathbb{R}/\mathbb{Z}}(r) = (\mathrm{Re}\langle r, f_1\rangle, \mathrm{Im}\langle r, f_1\rangle, \ldots, \mathrm{Im}\langle r, f_d\rangle, r), r \in \mathbb{Q}, \tag{1.3}$$

as a random variable defined also on $\Omega_{Q,N}$ as above. It follows from a classical result of Schneider [39] in transcendental theory that $\mathrm{Re}\langle \cdot, f_n\rangle, \mathrm{Im}\langle \cdot, f_n\rangle$ for $n = 1, \ldots, d$ all take some non-rational values. It is therefore tempting to think that the values should equidistribute on the circle $\mathbb{R}/\mathbb{Z}$, which is exactly what we prove.

Theorem 1.3. *The random variables $\mathfrak{m}_{N,\mathbb{R}/\mathbb{Z}}$ defined on the outcome spaces $\Omega_{Q,N}$ converge in distribution to the uniform distribution on $(\mathbb{R}/\mathbb{Z})^{2d+1}$ as $Q \to \infty$. This means in concrete terms that for any fixed product of intervals $\prod_{n=1}^{2d+1} I_n \subset (\mathbb{R}/\mathbb{Z})^{2d+1}$, we have*

$$\frac{\#\left\{a/q \in \Omega_{Q,N} \cap I_{2d+1} \mid (\mathrm{Re}\langle a/q, f_1\rangle, \ldots, \mathrm{Im}\langle a/q, f_d\rangle) \in \prod_{n=1}^{2d} I_n\right\}}{\#\Omega_{Q,N}} = \prod_{n=1}^{2d+1} |I_n| + o(1)$$

as $Q \to \infty$.

We observe that the modular symbols map gives rise to a map $\Gamma_0(N) \to \mathbb{C}$ by putting $\langle \gamma, f\rangle := \langle \gamma\infty, f\rangle$, where $\gamma\infty = a/c$ with a, c the left upper and lower entries of $\gamma \in \Gamma_0(N)$. By shifting the contour and doing a change of variable we see that

$$\langle \gamma_1\gamma_2, f\rangle = \langle \gamma_1, f\rangle + 2\pi i \int_{\gamma_1\infty}^{\gamma_1\gamma_2\infty} f(z)dz = \langle \gamma_1, f\rangle + \langle \gamma_2, f\rangle,$$

which shows that modular symbols define an additive character on $\Gamma_0(N)$ and thus an element of (the cuspidal part of) the cohomology group $H^1(\Gamma_0(N), \mathbb{C})$. Furthermore, by the integrality conditions, we see that the normalised modular symbols $\mathfrak{m}^{\pm}_{f,p}$ define elements of $H^1(\Gamma_0(N), \mathbb{F}_p)$. This view point is useful for generalisations.

Remark 1.4. We note that in [24], the slightly larger outcome space $\{a/q \mid 0 < a < q \leq Q, (a, q) = 1\}$ is considered (following Mazur and Rubin), that is, without the condition that $N|q$. In fact, equidistribution on this outcome space does *not* hold in the generality above. One has to exclude some bad primes p (see Remark 3.2 below). Our methods can also deal with this larger outcome space, by considering the Fourier expansion of Eisenstein series at different cusps, as is done in [35]. The outcome space $\Omega_{Q,N}$ above is, however, very natural from the cohomological perspective and for simplicity we will restrict to this case.

1.2. **Distribution of cohomology classes.** More generally, let $\mathrm{SO}(n+1,1)$ be the special orthogonal group with signature $(n+1,1)$, which we identity with the group of isometries of the $(n+1)$-dimensional upper half space $\mathbb{H}^{n+1}$. Now, for a co-finite subgroup with cusps $\Gamma < \mathrm{SO}(n+1,1)$, we will study the distribution of unitary characters of Γ or, equivalently, cohomology classes in $H^1(\Gamma, \mathbb{R}/\mathbb{Z})$. These cohomology groups have been studied in many contexts ([38], [13, Chap. 7]) and especially the case $n = 2$ is very appealing as it corresponds to Kleinian groups due to the exceptional isomorphism $\mathrm{SO}(3,1) \cong \mathrm{SL}_2(\mathbb{C})$.

1.2.1. *Results with arithmetic ordering.* Let $\Gamma \subset \mathrm{SO}(n+1,1)$ be as above and assume that the associated symmetric space $\Gamma\backslash\mathbb{H}^{n+1}$ has a cusp at ∞. Let $\Gamma'_\infty \subset \Gamma$ be the parabolic subgroup fixing the cusp at ∞. Note that since Γ is discrete, there exists a lattice $\Lambda < \mathbb{R}^n$ such that Γ'_∞ is exactly the group of motions corresponding to translations by Λ. We will study the distribution of unitary characters trivial on Γ'_∞ or, equivalently, elements of the cohomology group $H^1_{\Gamma'_\infty}(\Gamma, \mathbb{R}/\mathbb{Z})$.

Our distribution results are with respect to a natural arithmetic ordering on $\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty$ which generalises the ordering in the definition of $\Omega_{Q,N}$ above. To define this, we use the model $\mathrm{SV}_{n-1} \cong \mathrm{Iso}^+(\mathbb{H}^{n+1})$, where SV_{n-1} is the Vahlen group consisting of 2×2 matrices over a specific Clifford algebra, introduced in [1] (see Section 4.2 below for a detailed construction). This model provides a natural generalisation to $n > 2$ of the familiar models $\mathrm{SV}_0 = \mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SV}_1 = \mathrm{SL}_2(\mathbb{C})$. The Vahlen model has been used before to study automorphic forms on $\mathbb{H}^{n+1}$, for example by Elstrodt, Grunewald, and Mennicke [12] to prove a generalisation of the Selberg Conjecture regarding the first non-zero eigenvalue of the Laplacian and by Södergren [48] for proving equidistribution of horospheres on $\mathbb{H}^{n+1}$.

We will order by the norm of the lower left entry in this matrix, which generalises the arithmetic ordering considered in the literature for the cases $n = 1$ and $n = 2$, see [35], [8], [32]. We define the following outcome space:

$$T_\Gamma(X) = \{\gamma \in \Gamma'_\infty\backslash\Gamma/\Gamma'_\infty \mid 0 < |c_\gamma| < X\}, \tag{1.4}$$

where $\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \mathrm{SV}_{n-1}$ in the Vahlen group model and $|\cdot|$ denotes the norm on the Clifford algebra. The ordering defining $T_\Gamma(X)$ can also be described relatively naturally using the standard model for $\mathrm{SO}(n+1,1)$, see Remark 4.6 for details.

Now let $\omega_1, \ldots, \omega_d$ be elements of $H^1_{\Gamma'_\infty}(\Gamma, \mathbb{R}/\mathbb{Z})$ in *general position*, meaning for any $(n_1, \ldots, n_d) \in \mathbb{Z}^d$, we have

$$n_1\omega_1 + \ldots + n_d\omega_d = 0 \in H^1_{\Gamma'_\infty}(\Gamma, \mathbb{R}/\mathbb{Z}) \Leftrightarrow \Big(n_i\omega_i = 0 \in H^1_{\Gamma'_\infty}(\Gamma, \mathbb{R}/\mathbb{Z}), \forall i = 1, \ldots, d\Big).$$

As an example one can pick $\omega_1, \ldots, \omega_d$ to be a basis for the non-torsion part of the cohomology group or a $\mathbb{F}_p$-basis for $H^1_{\Gamma'_\infty}(\Gamma, \mathbb{F}_p)$. We notice that the image of ω_i is either dense in $\mathbb{R}/\mathbb{Z}$ or finite (recall that ω_i defines an additive character $\Gamma \to \mathbb{R}/\mathbb{Z}$). In the first case we put $J_i = \mathbb{R}/\mathbb{Z}$ and in the latter case we put $J_i = \mathbb{Z}/m_i\mathbb{Z}$, where m_i is the cardinality of the image of ω_i. We equip $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ with the obvious choices of probability measures, Lebesque and uniform respectively. Finally associated to $\gamma \in \Gamma'_\infty\backslash\Gamma/\Gamma'_\infty$, we define the invariant $\gamma\infty \in (\mathbb{R}^n \cup \{\infty\})/\Lambda$, see Section 4.4 for more details.

Now given $X > 0$, we consider

$$\omega(\gamma) := (\omega_1(\gamma), \ldots, \omega_d(\gamma), \gamma\infty)$$

as a random variable with values in $\prod_{i=1}^d J_i \times (\mathbb{R}^n/\Lambda)$ defined on the outcome space $T_\Gamma(X)$ endowed with the uniform probability measure. Then we have the following equidistribution result.

Theorem 1.5. *The random variables ω defined on the outcome spaces $T_\Gamma(X)$ are asymptotically uniformly distributed on $\prod_{i=1}^d J_i \times (\mathbb{R}^n/\Lambda)$ as $X \to \infty$. This means in concrete terms that for any fixed (continuity) subsets $A_i \subset J_i$ and $B \subset \mathbb{R}^n/\Lambda$, we have*

$$\frac{\#\left\{\gamma \in T_\Gamma(X) \mid (\omega_1(\gamma), \ldots, \omega_d(\gamma)) \in \prod_{i=1}^d A_i, \gamma\infty \in B\right\}}{\#T_\Gamma(X)} = \prod_{i=1}^d \frac{|A_i|}{|J_i|} \cdot \frac{|B|}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} + o(1)$$

as $X \to \infty$.

Remark 1.6. The assumption on the existence of cusps is essential in Theorem 1.5 since we rely on the theory of Eisenstein series. Besides this, our methods are pretty robust and apply to non-arithmetic subgroups equally well.

Remark 1.7. Notice that the number of choices of cohomology classes in $H^1_{\Gamma'_\infty}(\Gamma, \mathbb{R}/\mathbb{Z})$ in general position is infinite unless $\Gamma/\langle[\Gamma,\Gamma], \Gamma'_\infty\rangle$ is torsion. See Section 8 for results on the size of $H^1_{\Gamma'_\infty}(\Gamma, \mathbb{R}/\mathbb{Z})$.

1.2.2. *Results when ordered by length of geodesics.* We can also obtain equidistribution of the cohomology classes if we order by the length of the associated geodesics. We denote by $\mathrm{Conj}_{\mathrm{hyp}}(\Gamma)$ the set of conjugacy classes in Γ which do not correspond to the identity, parabolic or elliptic elements. Then, for each $\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma)$, there is a unique corresponding closed geodesic on $\Gamma\backslash\mathbb{H}^{n+1}$ whose length we denote by $l(\gamma)$.

Theorem 1.8. *Let $\omega = (\omega_1, \ldots, \omega_d)$ be defined from a set of cohomology classes in general position as above. The random variables ω defined on conjugacy classes ordered by the length of the geodesics are asymptotically equidistributed on $\prod_{i=1}^d J_i$. This means in concrete terms that for any fixed (continuity) subsets $A_i \subset J_i$, we have*

$$\frac{\#\{\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma) \mid l(\gamma) \le X, (\omega_1(\gamma), \ldots, \omega_d(\gamma)) \in \prod_{i=1}^d A_i\}}{\#\{\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma) \mid l(\gamma) \le X\}} = \prod_{i=1}^d \frac{|A_i|}{|J_i|} + o(1)$$

as $X \to \infty$.

Remark 1.9. In the case of Theorem 1.8, we can remove the assumption that Γ has cusps. In fact the proof becomes more complicated in the presence of cusps.

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2. Proof sketch in the case of $\mathbb{H}^2$

The key ideas of the proofs of our main theorems are quite simple, having at their core the analytic continuation of twisted Eisenstein series and twisted trace formulas respectively. We will sketch the proof of Theorem 1.2 in the simplest case, which is the one dealt with in [24], where we consider only one cusp form and no restrictions on the location of $r = a/q$ in $\mathbb{R}/\mathbb{Z}$.

Let $f \in \mathcal{S}_2(\Gamma_0(N))$ be a newform of weigth 2 and level N and let $\mathfrak{m}^{\pm}_{f,p} : \Gamma_0(N) \to \mathbb{F}_p$ be the associated normalised modular symbols defined above. Recall that this defines a non-trivial additive character. We would like to show that the values of $\mathfrak{m}^{\pm}_{f,p}$ on the set $\Omega_{Q,N} = \{a/q \mid 0 < a < q \le Q, (a,q) = 1, N|q\}$ equidistribute on $\mathbb{Z}/p\mathbb{Z}$ as $Q \to \infty$.

To do this we introduce for any $l \in (\mathbb{Z}/p\mathbb{Z})^\times$ the unitary character $\chi_l : \Gamma_0(N) \to \mathbb{C}^\times$ defined by;

$$\chi_l(\gamma) := e^{2\pi i \mathfrak{m}^{\pm}_{f,p}(\gamma) l/p}, \quad \gamma \in \Gamma_0(N).$$

By Weyl's Criterion [21, page 487] in order to conclude equidistribution, it suffices to detect cancelation in the Weyl sums; that is to prove for all $l \in (\mathbb{Z}/p\mathbb{Z})^\times$ that

$$\sum_{a/q \in \Omega_{Q,N}} \chi_l(a/q) \ll X^{2-\delta},$$

for some $\delta > 0$ where $\chi_l(a/q) := \chi_l(\gamma)$ with $\gamma \in \Gamma_0(N)$ such that $\gamma\infty = a/q$.

Now, the key observation is that the generating series for these Weyl sums appears very naturally as the constant term of an appropriate Eisenstein series. The cancelation in the Weyl sums is now a simple analytic consequence of the analytic properties of the corresponding Eisenstein series.

To be precise; associated to χ_l we have the following twisted Eisenstein series:

$$E(z, s, \chi_l) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{\chi_l}(\gamma) \operatorname{Im}(\gamma z)^s,$$

where $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. This Eisenstein series defines a holomorphic function for $\operatorname{Re} s > 1$ and by the work of Selberg [41, Chap. 39] admits meromorphic continuation to the entire complex plane with a pole at $s = 1$ if and only if χ_l is trivial. Note that in general the character χ_l might not come from an adelic one, but Selberg's theory applies equally well.

Now a standard calculation using Poisson summation shows that the constant term of the Fourier expansion of $E(z, s, \chi_l)$ is given by

$$y^s + \frac{\pi^{1/2} y^{1-s} \Gamma(s - 1/2)}{\Gamma(s)} L_l(s),$$

with

$$L_l(s) := \sum_{c > 0, N | c} \left(\sum_{0 < d < c, (c,d) = 1} \overline{\chi_l}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \right) c^{-2s},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix in $\Gamma_0(N)$ with lower entries c, d. We observe that $L_l(s)$ is exactly the generating series for the Weyl sums above, as promised.

Now from the meromorphic continuation of the Eisenstein series itself, we also get meromorphic continuation of the generating series $L_l(s)$, and since χ_l is non-trivial we conclude that $L_l(s)$ is analytic for $\operatorname{Re} s > 1 - \delta$ for some $\delta > 0$. Thus we get the wanted cancelation in Weyl sums using the standard machinery from complex analysis if we can get bounds on vertical lines of $L_l(s)$. It turns of that such bounds follow from the general bound for matrix coefficients also due to Selberg, and thus we are done.

This shows how to deduce equidistribution of modular symbols using Eisenstein series. The proof for classes in the first cohomology of quotients of higher dimensional hyperbolic spaces uses the same idea, although some parts of the argument require some more technical work.

In order to obtain equidistribution results when restricting the cusps to a specific interval $I \subset \mathbb{R}/\mathbb{Z}$, we will have to use all the Fourier coefficients of the Eisenstein series as is done in [35]. To deal with equidistribution for modular symbols defined on conjugacy classes ordered by the length of the associated geodesic, we will use twisted Selberg trace formulas to study the corresponding Weyl sums.

3. Some special cases of the conjecture of Mazur and Rubin

In this section we study some specific cases of the conjecture of Mazur and Rubin on the residual distribution of modular symbols, which we can resolve *without* taking an extra average. We consider the value distribution of certain modular symbols for Hecke congruence subgroups $\Gamma_0(N)$ of prime level N modulo primes $p \geq 5$ dividing $N-1$. These modular symbols are connected to congruences between Eisenstein series and cusp forms, which have been studied extensively before in number theory, see e.g. [27, Section 9], [28]. We will assume for simplicity that $N \geq 5$ (in particular $\Gamma_0(N)$ is torsion-free) as is done in [27]. From the perspective of cohomology, the congruence phenomenon manifests itself through the fact that Dirichlet characters modulo N define cohomology classes. The distributions of these specific cohomology classes are much easier to understand, and we would thus like to connect them to modular symbols. We have to somehow rule out that these cohomology classes are linear combinations of classes coming from modular symbols associated to different cusp forms. This can be achieved using a "multiplicity one" result of Mazur. The precise result we can prove is the following.

Theorem 3.1. *Let $N, p \geq 5$ be primes such that $p|N-1$. Then there exists a newform $f \in \mathcal{S}_2(\Gamma_0(N))$ of weight 2 and level N such that the values of $\mathfrak{m}^+_{f,p}$ on $\{\frac{a}{q} \mid (a,q) = 1, 0 < a < q\}$ equidistribute modulo p as $q \to \infty$ with $N|q$.*

Proof. Given a prime $p|N-1$ with $p \geq 5$, we know that the space of order p Dirichlet characters modulo N is one dimensional as an $\mathbb{F}_p$-vector space (since $(\mathbb{Z}/N\mathbb{Z})^\times$ is cyclic). Given a generator χ mod N of order p we get an element of $H^1(\Gamma_0(N), \mathbb{F}_p)$ defined by $\gamma \mapsto \chi(a_\gamma)$ (which we denote by σ_χ), where we identify $\mathbb{F}_p$ with the image of χ and a_γ is the upper-left entry of γ.

First of all, we observe that σ_χ is trivial on the parabolic subgroups of $\Gamma_0(N)$; since N is prime we only have to check it for $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ and $\langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \rangle$ corresponding to the two cusps ∞ and 0. This implies that σ_χ defines a class in the parabolic cohomology group $H^1_P(\Gamma_0(N), \mathbb{F}_p)$ (see Section 7.1 below or [46, Chapter 8] for a definition). It follows by a mod p version of Eichler–Shimura isomorphism (see [24, (3.5)]) that the associations $f \mapsto \mathfrak{m}^\pm_{f,p}$ defined for Hecke eigenforms f induce an isomorphism

$$H^1_P(\Gamma_0(N), \mathbb{F}_p) \cong \mathcal{S}_2(\Gamma_0(N))_{\mathbb{F}_p} \oplus \mathcal{S}_2(\Gamma_0(N))_{\mathbb{F}_p} \tag{3.1}$$

where $\mathcal{S}_2(\Gamma_0(N))_{\mathbb{F}_p}$ denotes the space of cusp forms of weight 2 and level N with coefficients in $\mathbb{F}_p$ (which we will just think of as the formal $\mathbb{F}_p$-vector space generated by Hecke eigenforms of weight 2 and level N).

To relate σ_χ to cusp forms, we need to consider the Hecke action. Recall that we have an action by the Hecke operators on the cohomology group defined as follows (see [45, Chapter 8.3]). Let

$$\alpha_{r,l} = \left(\begin{smallmatrix} 1 & r \\ 0 & l \end{smallmatrix}\right), r = 0, \dots, l-1 \quad \text{and} \quad \alpha_{l,l} = \left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right),$$

for $l \neq N$ prime. Then given $\gamma \in \Gamma_0(N)$, we define $\gamma_{r,l} \in \Gamma_0(N)$ by $\alpha_{r,l}\gamma = \gamma_{r,l}\alpha_{\sigma(r),l}$ for $r = 0, \dots, l$ and some $\sigma(r) \in \{0, \dots, l\}$. Then we define the operator T_l as follows:

$$(T_l\omega)(\gamma) := \sum_{r=0}^{l} \omega(\gamma_{r,l}),$$

for $\omega \in H^1(\Gamma_0(N), \mathbb{F}_p)$ and $\gamma \in \Gamma_0(N)$. At the prime $l = N$, we have the Atkin–Lehner operator U defined by

$$(U\omega)(\gamma) := \omega\left(\left(\begin{smallmatrix} 0 & 1 \\ N & 0 \end{smallmatrix}\right) \gamma \left(\begin{smallmatrix} 0 & 1 \\ N & 0 \end{smallmatrix}\right)^{-1} \right).$$

One sees directly that $U\sigma_\chi = -\sigma_\chi$. Furthermore, it is a small computation that if $l|c$ (where c is the lower left entry of γ), then

$$\gamma_{r,l} = \left(\begin{smallmatrix} a+cr & * \\ cp & * \end{smallmatrix}\right), \text{ for } r = 0, \ldots, l-1, \quad \text{and} \quad \gamma_{l,l} = \left(\begin{smallmatrix} a & * \\ c/p & * \end{smallmatrix}\right).$$

And if $l \nmid c$, then $\gamma_{l,l} = \left(\begin{smallmatrix} ap & * \\ c & * \end{smallmatrix}\right)$ and for $r = 0, \ldots, l-1$ we have

$$\gamma_{r,l} = \begin{cases} \left(\begin{smallmatrix} (a+cr)/p & * \\ c & * \end{smallmatrix}\right), & r \equiv -a\bar{c} \mod l, \\ \left(\begin{smallmatrix} a+cr & * \\ cp & * \end{smallmatrix}\right), & \text{else.} \end{cases}$$

It follows that $T_l\sigma_\chi = (l+1)\sigma_\chi$ for all $l \neq N$. We will now use this calculation together with a multiplicity one result of Mazur to relate σ_χ to modular symbols.

Let $\mathfrak{P}$ be the Eisenstein prime (of the Hecke algebra) associated to p. Then we know from [27, Corollary 16.3] that $J[\mathfrak{P}]$ is two dimensional as a vector space over $\mathbb{F}_p$, where J is the Jacobian of $X_0(N)$ (this is exactly what is meant by "multiplicity one"). This implies that there is a unique Hecke eigenform of weight 2 and level N whose Hecke eigenvalues satisfy $\lambda_l \equiv l+1\,(p)$ for $l \neq N$ and $\lambda_N \equiv -1\,(p)$. Thus from the above calculations we conclude that there exists a unique Hecke eigenform $f \in \mathcal{S}_2(\Gamma_0(N))$ such that σ_χ is a linear combination of $\mathfrak{m}^+_{f,p}$ and $\mathfrak{m}^-_{f,p}$ (considered as elements of $H^1_P(\Gamma_0(N), \mathbb{F}_p)$ the obvious way). Finally, we recall that $H^1_P(\Gamma_0(N), \mathbb{F}_p)$ can be diagonalized by the involution ι given by conjugation with $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (here we need $p > 2$), which follows from [29, Sec. 1]. We see that the eigenvalue of σ_χ under the action of ι is $+1$ since the order of χ is odd. Thus we conclude that $\mathfrak{m}^+_{f,p} = m \cdot \sigma_\chi$ for some $m \in \mathbb{F}_p \backslash \{0\}$.

Now it is easy to see that the values of $\mathfrak{m}^+_{f,p}$ (with f as above) on $\{a/q \mid (a,q) = 1, 0 < a < q\}$ equidistribute modulo p as $q \to \infty$ with $N|q$. This follows directly from the fact that

$$\mathfrak{m}^+_{f,p}(a/q) = m \cdot \chi(a) \mod p,$$

and the fact that the values of χ clearly equidistribute. Notice that actually the values equidistribute *exactly*. □

This settles the conjecture of Mazur and Rubin in these very special cases, whereas in general the conjecture seems out of reach without the extra average both with the automorphic and the dynamical approach.

Remark 3.2. Strictly speaking the conjecture of Mazur and Rubin [29] is only formulated for primes p and cusp forms corresponding to elliptic curves E where the residual representation of E mod p is surjective and p is an ordinary and good prime of E. This is not the case in the example considered above. We, however, expect the statement of Theorem 3.1 to be true for general $\mathfrak{m}^\pm_{f,p}$ with $f \in \mathcal{S}_k(\Gamma_0(N))$ and $p > 2$ prime.

Remark 3.3. The assumption that N is prime is essential for the results of [27] to apply. For composite level the situation becomes much more complicated as multiplicity one might fail (see [51]). It would be interesting to see if the methods above can be generalized to the setting of composite level.

4. Geometry of $\mathbb{H}^{n+1}$

We introduce two models for the $(n+1)$-dimensional hyperbolic space and the connections between them. We look at the upper half-space (Poincaré) model $\mathbb{H}^{n+1}$ and the hyperboloid (Klein) model $\mathbb{K}^{n+1}$. We briefly describe some geometric and arithmetic properties of the space $\Gamma\backslash\mathbb{H}^{n+1}$, where Γ is a cofinite discrete subgroup of isometries. Our main references for this section are [1], [11] and [12].

We denote by $\mathrm{Iso}^+(\mathbb{H}^{n+1})$ the space of orientation preserving isometries of the hyperbolic $(n+1)$-space. We say that $\gamma \in \mathrm{Iso}^+(\mathbb{H}^{n+1})$ is *elliptic* if it has exactly one fixed point in $\mathbb{H}^{n+1}$. A non-elliptic isometry is called *parabolic* if it has exactly one fixed point on the boundary $\mathbb{R}^n \cup \{\infty\}$ and *hyperbolic* if it has 2 fixed points on $\mathbb{H}^{n+1} \cup \mathbb{R}^n \cup \{\infty\}$ (hence both of them on the boundary or both of them inside $\mathbb{H}^{n+1}$). We note that our definition of a hyperbolic motion includes what is known in the literature as loxodromic motions, so any isometry is either the identity or one of the three types described. We quote [16] for a thorough discussion of the three classes.

4.1. **Definitions.** We will now describe the *upper-half space model* $\mathbb{H}^{n+1}$ for hyperbolic $(n+1)$-space. Let $q : \mathbb{R}^n \to \mathbb{R}$ a quadratic non-degenerate form and $\mathcal{C}(q)$ the associated *Clifford algebra*, i.e. the free $\mathbb{R}$-algebra on $\{e_1, \dots, e_n\}$ modulo the relations

$$e_i^2 = q(e_i), \quad e_i e_j = -e_j e_i, \quad \text{where} \quad i, j = 1, \cdots, n,\ i \neq j,$$

where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$. We denote by $\mathcal{E}_n$ the set of all subsets of $\{1, \dots, n\}$. Then for $M = \{i_1, \dots, i_k\} \in \mathcal{E}_n$ with $i_1 < \cdots < i_k$, we define

$$e_M := e_{i_1} \cdot \cdots \cdot e_{i_k}, \quad e_\emptyset := 1 \in \mathcal{C}(q).$$

Then one can check that $\{e_M \mid M \in \mathcal{E}_n\}$ is a $\mathbb{R}$-basis for $\mathcal{C}(q)$.

We have two linear involutions on $\mathcal{C}(q)$ given by

$$\overline{e_M} := (-1)^{|M|(|M|+1)/2} e_M, \quad e_M^* := (-1)^{|M|(|M|-1)/2} e_M, \quad \text{where } M \in \mathcal{E}_n.$$

They satisfy

$$\overline{v}\,\overline{w} = \overline{w}\,\overline{v}, \quad v^* w^* = w^* v^*, \quad \text{for all } v, w \in \mathcal{C}(q).$$

From now on we assume that $q = -I_n$, the negative definite unit form. In this case we write $\mathcal{C}_n$ for $\mathcal{C}(q)$. We denote by $V_n \subset \mathcal{C}_n$ the vector space spanned by $\{1, e_1, \dots, e_n\}$. It is easy to see that $V_0 \cong \mathbb{R}$ and $V_1 \cong \mathbb{C}$ as $\mathbb{R}$-algebras.

V_n is equipped with the inner product

$$\langle v, w \rangle = \frac{1}{2}(v\overline{w} + \overline{v}w).$$

We note that this coincides with the standard Euclidean inner product if we identify V_n with $\mathbb{R}^{n+1}$ using the basis $\{1, e_1, \dots, e_n\}$.

For $x = \sum_{M \in \mathcal{E}_n} \lambda_M e_M \in \mathcal{C}_n$, we define the norm

$$|x| := \left(\sum_{M \in \mathcal{E}_n} \lambda_M^2 \right)^{1/2}. \tag{4.1}$$

We note that for $x \in V_n$,

$$|x|^2 = \langle x, x \rangle.$$

Now, if $\Lambda < V_n$ is a lattice, we define the dual lattice as

$$\Lambda^\circ := \{w \in V_n \mid \langle v, w \rangle \in \mathbb{Z} \text{ for all } v \in \Lambda\}.$$

We now define the following model of hyperbolic $(n+1)$-space:

$$\mathbb{H}^{n+1} := \{x_0 + x_1 e_1 + \cdots + x_n e_n \mid x_0, x_1, \dots, x_{n-1} \in \mathbb{R}, x_n > 0\}.$$

We have the maps $x : \mathbb{H}^{n+1} \to V_{n-1}$ and $y : \mathbb{H}^{n+1} \to (0, \infty)$ given by

$$x(P) := x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1}, \quad y(P) := x_n,$$

where $P = x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{H}^{n+1}$. We can think of $x(P)$ as an element of $\mathbb{R}^n$ via the above. Then from (4.1) we see that

$$|P|^2 = |x(P)|^2 + |y(P)|^2.$$

We equip $\mathbb{H}^{n+1}$ with the hyperbolic metric coming from the line element:

$$ds^2 = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}, \tag{4.2}$$

which makes $\mathbb{H}^{n+1}$ a Riemannian manifold with constant negative curvature -1. The volume element is given by

$$dv = \frac{dx_0 dx_1 \dots x_n}{x_n^{n+1}}.$$

The *hyperbolic Laplace–Beltrami operator* is given by

$$\Delta = x_n^2 \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) - (n-1) x_n \frac{\partial}{\partial x_n}, \tag{4.3}$$

in this model.

4.2. **Vahlen group.** We will use the above upper-half space model to describe the group of (oriented) isometries in a way that is convenient for our purposes. We let $T_n \subset \mathcal{C}_n$ be the multiplicative subgroup generated by $V_n \setminus \{0\}$. As in [1, p. 219] or [12, p. 648], we define the *Vahlen group* SV_n to be

$$\mathrm{SV}_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{C}_n) \;\middle|\; \begin{array}{c} \text{(i) } a,b,c,d \in T_n \cup \{0\} \\ \text{(ii) } \overline{a}b, \overline{c}d \in V_n \\ \text{(iii) } ad^* - bc^* = 1 \end{array} \right\}. \tag{4.4}$$

We can easily check that $\mathrm{SV}_0 = \mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SV}_1 = \mathrm{SL}_2(\mathbb{C})$ as $\mathbb{R}$-algebras. Then SV_n is a group under matrix multiplication with inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}. \tag{4.5}$$

We can now define the action of SV_{n-1} on $\mathbb{H}^{n+1}$, which resembles the actions of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}^2$ and $\mathbb{H}^3$, respectively, as can be seen from the following result.

Theorem 4.1 ([12], Theorem 1.3)**.** *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SV}_{n-1}$ *and* $P \in \mathbb{H}^{n+1}$. *Then* $cP + d \in T_n$ *and we define*

$$\gamma P := (aP + b)(cP + d)^{-1} \in \mathbb{H}^{n+1}. \tag{4.6}$$

The map $P \mapsto \gamma P$ *is an orientation preserving isometry of* $\mathbb{H}^{n+1}$. *Moreover, all orientation preserving isometries are obtained in this way and we have the induced isomorphism* $\mathrm{SV}_{n-1}/\{I, -I\} \cong \mathrm{Iso}^+(\mathbb{H}^{n+1})$.

What is convenient about this description of $\mathrm{Iso}^+(\mathbb{H}^{n+1})$ is that one gets very familiar expressions for the coordinate-projections of the image under the action of $\gamma \in \mathrm{SV}_{n-1}$ on $P = (x, y) \in \mathbb{H}^{n+1}$.

Lemma 4.2 ([12], page 648)**.** *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SV}_{n-1}$ *and* $P = x + ye_n \in \mathbb{H}^{n+1}$. *Then*

$$x(\gamma P) = \frac{(ax + P)(\overline{cx + d}) + a\overline{c}y^2}{|cx + d|^2 + |c|^2 y^2} \tag{4.7}$$

and

$$y(\gamma P) = \frac{y}{|cx+d|^2 + |c|^2 y^2}. \tag{4.8}$$

4.3. **Hyperboloid model.** We recall the hyperboloid (or Klein) model for the hyperbolic $(n+1)$-space given by

$$\mathbb{K}^{n+1} := \{z \in \mathbb{R}^{n+2} \mid z_0^2 - z_1^2 - \ldots - z_{n+1}^2 = 1, z_0 > 0\},$$

where $z = (z_0, \ldots, z_{n+1})$. The line element

$$ds^2 = -dz_0^2 + dz_1^2 + \cdots + dz_{n+1}^2 \tag{4.9}$$

defines the hyperbolic metric on $\mathbb{K}^{n+1}$.

The group

$$\mathrm{SO}(n+1,1) := \left\{\gamma \in \mathrm{SL}_{n+2}(\mathbb{R}) \mid \gamma^T I_{1,n+1}\gamma = I_{1,n+1}\right\} \tag{4.10}$$

acts on $\mathbb{K}^{n+1}$ by left multiplication, where $I_{1,n+1} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & I_n \end{smallmatrix}\right)$. We can identify the set of orientation preserving isometries by $\mathrm{SO}^0(n+1,1)$, where $\mathrm{SO}^0(n+1,1)$ is the component of the identity element in $\mathrm{SO}(n+1,1)$. Hence we have the identifications

$$\mathrm{Iso}^+(\mathbb{K}^{n+1}) \cong \mathrm{SO}^0(n+1,1),$$
$$\mathbb{K}^{n+1} \cong \mathrm{SO}^0(n+1,1)/\mathrm{O}(n+1).$$

We have the following important result which connects the two models.

Theorem 4.3 ([11], Section 5)**.** *We can go between the two models $\mathbb{H}^{n+1}$ and $\mathbb{K}^{n+1}$ as follows.*

- *(i) There exists a bijection $\Phi : \mathbb{H}^{n+1} \to \mathbb{K}^{n+1}$ which is also an isometry, i.e. the pullback of the line element* (4.9) *via Φ is the line element* (4.2)*.*
- *(ii) There exists an isomorphism $\Psi : \mathrm{SV}_{n-1}/\{\pm I\} \xrightarrow{\sim} \mathrm{SO}^0(n+1,1)$ such that Φ is Ψ-equivariant, i.e.*
$$\Phi(\gamma \cdot P) = \Psi(\gamma)\Phi(P),$$
for all $\gamma \in \mathrm{SV}_{n-1}$ and $P \in \mathbb{H}^{n+1}$.

Remark 4.4. The maps Φ and Ψ are explicitly constructed in [11, Section 5]. This result allows us to move freely between the two identifications of hyperbolic $(n+1)$-space.

4.4. **Hyperbolic quotients.** Let Γ be a discrete group of hyperbolic motions such that the surface $\Gamma\backslash\mathbb{H}^{n+1}$ has finite hyperbolic volume. From Theorem 4.3 we note that we can choose freely between the two models $\Gamma < \mathrm{SV}_{n-1} \cong \mathrm{Iso}^+(\mathbb{H}^{n+1})$ or $\Gamma < \mathrm{SO}^0(n+1,1) \cong \mathrm{Iso}^+(\mathbb{K}^{n+1})$. We will mainly work with the Vahlen model since it provides nicer arithmetic descriptions.

We say that $\mathfrak{a} \in \mathbb{R}^n \cup \{\infty\}$ is a cusp for Γ if it is fixed by a parabolic element in Γ. There exists a scaling matrix $\sigma_\mathfrak{a} \in \mathrm{SV}_{n-1}$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$. We let $\Gamma_\mathfrak{a} := \{\gamma \in \Gamma \ : \ \gamma\mathfrak{a} = \mathfrak{a}\}$ be the stabilizer of $\mathfrak{a}$ in Γ. We define

$$\Gamma'_\mathfrak{a} := \Gamma_\mathfrak{a} \cap \sigma_\mathfrak{a} \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SV}_{n-1} \right\} \sigma_\mathfrak{a}^{-1} .$$

We note that $\Gamma'_\mathfrak{a}$ consists of the parabolic elements in $\Gamma_\mathfrak{a}$ together with the identity.

There exists a lattice $\Lambda_\mathfrak{a} \leq \mathbb{R}^n$ such that

$$\sigma_\mathfrak{a}^{-1}\Gamma'_\mathfrak{a}\sigma_\mathfrak{a} = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \ : \ \lambda \in \Lambda_\mathfrak{a} \right\} .$$

We let $\mathcal{P}_\mathfrak{a}$ be a period parallelogram for $\Lambda_\mathfrak{a}$ with Euclidean area $\mathrm{vol}(\Lambda_\mathfrak{a})$.

We define the *dual lattice* of $\Lambda_{\mathfrak{a}}$ as follows:

$$\Lambda_{\mathfrak{a}}^{\circ} := \{\mu \in \mathbb{R}^n \ : \ \langle \mu, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda_{\mathfrak{a}}\} \ , \tag{4.11}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^n$.

Say $\mathfrak{a}_1, \cdots, \mathfrak{a}_h \in \mathbb{R}^n \cup \{\infty\}$ are representatives for the equivalent-classes of cusps under the action of Γ. For $Y > 0$, we define the *cuspidal sectors* as follows:

$$\mathcal{F}_{\mathfrak{a}_i}(Y) := \sigma_{\mathfrak{a}_i}\{(x, y) : x \in \mathcal{P}_{\mathfrak{a}_i}, y \geq Y\} \ .$$

Then for Y large enough, there exists a fundamental domain $\mathcal{F}$ for $\Gamma\backslash\mathbb{H}^{n+1}$ which we can write as a disjoint union

$$\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_{\mathfrak{a}_1}(Y) \cup \cdots \cup \mathcal{F}_{\mathfrak{a}_h}(Y) \ , \tag{4.12}$$

where $\mathcal{F}_0$ is a compact set, see [48, p. 8] or [38, p. 5].

For notational convenience, from now on we will focus only at the cusp at ∞. We drop the subscript by denoting $\Lambda := \Lambda_\infty$, $\mathcal{P} := \mathcal{P}_\infty$ etc. Our theory can be generalised to take all cusps into account.

We will now define our outcome space (1.4) in precise terms, and describe it explicitly in some arithmetic examples. First we note that all elements in such a coset share the lower left entry. Thus it makes sense to define

$$T_\Gamma(X) := \left\{ \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \Gamma'_\infty \backslash \Gamma / \Gamma'_\infty \mid 0 < |c| \leq X \right\}$$

which is the natural generalisation of the outcome space considered by Petridis–Risager in [35, p. 1002]. In (6.7) we provide an asymptotic formula for the size of $T_\Gamma(X)$.

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, then by the definition of SV_{n-1} and of the action (4.6) we see that $\gamma\infty = ac^{-1} \in V_{n-1}$, where $\gamma\infty$ is defined as the limit of γP as P tends to the cusp at ∞. We observe that $\gamma\infty$ is well-defined on double cosets in $\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty$ up to translations by the lattice Λ. Therefore we see that the map

$$\begin{gathered} \Gamma'_\infty\backslash\Gamma/\Gamma'_\infty \to \mathbb{R}^n/\Lambda \cup \{\infty\} \\ \gamma \mapsto \gamma\infty \end{gathered}$$

is well-defined using the identification of V_{n-1} with $\mathbb{R}^n$ as above. A simple consequence of our main theorems is that $\gamma\infty$ become equidistributed on $\mathbb{R}^n/\Lambda$ as we vary along $\gamma \in T_\Gamma(X)$ as $X \to \infty$.

Let

$$C(\Gamma) := \left\{ c \in T_n \mid \exists a, b, d \in T_n : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \right\} . \tag{4.13}$$

We will now provide explicit descriptions for both $C(\Gamma)$ and $\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty$ in the case of *congruence subgroups*. To define these, let $J \subset \mathcal{C}_n$ be an order stable under the involutions $-$ and $*$. We put $\mathrm{SV}_n(J) := \mathrm{SV}_n \cap M_2(J)$. We also define $V(J) := J \cap V_n$ and $T(J) = J \cap T_n$. For $N \in \mathbb{N}$, we define the *principle congruence subgroup*

$$\mathrm{SV}_n(J; N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SV}_n(J) \mid a - 1, b, c, d - 1 \in NJ \right\} . \tag{4.14}$$

A subgroup $\Gamma < \mathrm{SV}_n(J)$ is called a *congruence group* if $\mathrm{SV}_n(J; N) < \Gamma$, for some $N \in \mathbb{N}$. We quote [12, Section 4] to provide an explicit description for representatives of $\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty$ in the case $\Gamma = \mathrm{SV}_n(J; N)$. In this case, $C(\Gamma) = N \cdot T(J)$ and a set of representatives for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'_\infty\backslash\Gamma/\Gamma'_\infty$ with $c \neq 0$ is given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SV}_n(J) \mid c \in N \cdot T(J), \ (a, d) \in D(c) \right\}$$

where

$$D(c) := \left\{ \begin{array}{c|c} (a,d) & \begin{array}{c} a \in J/(N\cdot V(J)\cdot c),\ d \in J/(N\cdot c\cdot V(J)), \\ a-1, d-1 \in N\cdot J,\ a\bar{c}, \bar{c}d \in N\cdot V(J) \end{array} \end{array} \right\}.$$

In the more familiar cases $n = 1$ and $n = 2$, the above reduces to the following.

- $n = 1$. Then $\mathrm{SV}_0 = \mathrm{SL}_2(\mathbb{R})$, $J = \mathbb{Z}$ and $\mathrm{SV}_1(J; N) = \Gamma_1(N)$. Representatives in $\Gamma_1(N)'_\infty \backslash \Gamma_1(N)/\Gamma_1(N)'_\infty$ with $c \neq 0$ are uniquely determined by
$$\{(a,c) \mid c > 0,\ N \mid c,\ a \in (\mathbb{Z}/cN\mathbb{Z})^*,\ a \equiv 1 \bmod N\}.$$
If we consider $\Gamma = \Gamma_0(N)$, then representatives are uniquely determined by
$$\{(a,c) \mid c > 0,\ N \mid c,\ a \in (\mathbb{Z}/c\mathbb{Z})^*\}.$$
- $n = 2$. Then $\mathrm{SV}_1 = \mathrm{SL}_2(\mathbb{C})$. We take $J = \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of a quadratic imaginary field K. Let $\mathfrak{n} < \mathcal{O}_K$ be an ideal. We consider congruence subgroups of the form
$$\Gamma_1(\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_K) \mid a-1, b, c, d-1 \in \mathfrak{n} \right\},$$
$$\Gamma_0(\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_K) \mid c \in \mathfrak{n} \right\}.$$
In the case $\Gamma_1(\mathfrak{n})$, representatives are uniquely provided by
$$\{(a,c) \mid c \in \mathfrak{n} \setminus \{0\},\ a \in (\mathcal{O}_K/(c\cdot\mathfrak{n}))^*,\ a-1 \in \mathfrak{n}\},$$
while for $\Gamma_0(\mathfrak{n})$ we have
$$\{(a,c) \mid c \in \mathfrak{n} \setminus \{0\},\ a \in (\mathcal{O}_K/(c))^*\}.$$

Remark 4.5. There is also a notion of congruence groups for $\mathrm{SO}(n+1,1)$. To define them, let Γ be the integral automorphisms of an isotropic quadratic form of signature $(n+1,1)$ defined over $\mathbb{Q}$. Then a *congruence subgroup* is any subgroup of Γ containing $\{\gamma \in \Gamma \mid \gamma \equiv I_{n+2} \bmod N\}$ for some positive integer N, see [38, p. 7]. If $\Gamma < \mathrm{SO}^0(n+1,1)$ is a congruence subgroup, then $\Psi^{-1}(\Gamma)$ is a congruence subgroup in SV_{n-1}. This fact will be useful when comparing our results with the results mentioned in Section 8. However the converse is not true, there exists a congruence subgroup $\Gamma < \mathrm{SV}_{n-1}$ such that $\Psi(\Gamma)$ is not a congruence subgroup in $\mathrm{SO}^0(n+1,1)$, see [12, Section 3] for more details.

Remark 4.6. We can also describe the ordering defining $T_\Gamma(X)$ explicitly using the model $\mathrm{SO}(n+1,1)$ for the isometry group of $\mathbb{H}^{n+1}$. In this case we have

$$|c_\gamma| = \frac{1}{2}(a_{00} + a_{0(n+1)} - a_{(n+1)0} - a_{(n+1)(n+1)}),$$

for $\gamma = (a_{ij}) \in \mathrm{SO}^0(n+1,1)$ as in (4.10), where c_γ is the lower left entry of $\Psi^{-1}(\gamma)$, i.e in the Vahlen model.

4.5. **Conjugacy classes.** We now look at certain invariants associated to the conjugacy classes $\{\gamma\}$ of Γ alluded to in Theorem 1.8. We refer to [15, Section 5] for more details. We denote by $\mathrm{Conj}_{\mathrm{hyp}}(\Gamma)$ the set of conjugacy classes of hyperbolic elements in Γ. For each $\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma)$, there exists a unique closed geodesic on $\Gamma\backslash\mathbb{H}^{n+1}$ whose length we denote by $l(\gamma)$. The geodesic can be defined as follows: Each conjugacy class corresponds to a free homotopy class on $\Gamma\backslash\mathbb{H}^{n+1}$ via the map $\gamma \mapsto \{P, \gamma P\} \subset \mathbb{H}^{n+1}$, for some point P, and the corresponding geodesic is the path of minimal length among all paths in that class. See [15, Sections 1 and 5] for explicit descriptions of the lengths $l(\gamma)$. Every $\gamma \in \Gamma$ can be written uniquely as $\gamma = \gamma_0^{j(\gamma)}$, where γ_0 is primitive and $j(\gamma) \in \mathbb{N}$. We put

$$\pi_\Gamma(X) := \{\{\gamma_0\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma) \mid \gamma_0 \text{ primitive}, l(\gamma_0) \leq X\}, \tag{4.15}$$

which is exactly the outcome space considered in Theorem 1.8. *The Prime Geodeseic Theorem for* $\mathbb{H}^{n+1}$ gives asymptotics for $\pi_\Gamma(X)$ and was firstly proved by Gangolli [14] in the compact case and by Gangolli and Warner [15, Prop. 5.4] in the non-compact case.

5. Twisted Eisenstein series for $\mathbb{H}^{n+1}$

Let $\Gamma < \mathrm{SV}_{n-1}$, Γ'_∞ and Λ be as in the previous section. We now fix χ a unitary character of Γ which is trivial on Γ'_∞. From this we define the twisted Eisenstein series

$$E(P,s,\chi)=\sum_{\Gamma'_\infty\backslash\Gamma}\overline{\chi(\gamma)}y(\gamma P)^s. \tag{5.1}$$

It is absolutely convergent for $\mathrm{Re}(s)>n$. It satisfies

$$E(\gamma P,s,\chi)=\chi(\gamma)E(P,s,\chi),$$
$$\Delta E(P,s,\chi)=s(n-s)E(P,s,\chi).$$

We see that $E(P,s,\chi)$ is invariant under the action by the lattice Λ and hence it has a Fourier expansion. It is well-known that the constant term in the Fourier expansion has the form $y^s+\phi(s,\chi)y^{n-s}$, where $\phi(s,\chi)$ is called the *scattering matrix*. Its basic properties are well-known, see [6, Ch. 6].

For $\mu,\nu\in\Lambda^\circ$ and $c\in C(\Gamma)$, we define the generalised Kloosterman sum as in [12, Section 4] using the Vahlen model:

$$S(\mu,\nu,c,\chi):=\sum_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty}\overline{\chi}\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)e\left(\left\langle ac^{-1},\mu\right\rangle+\left\langle dc^{-1},\nu\right\rangle\right). \tag{5.2}$$

We can rewrite this as

$$S(\mu,\nu,c,\gamma)=\sum_{\substack{\gamma\in\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty\\ c_\gamma=c}}\overline{\chi(\gamma)}e(\langle\gamma\infty,\mu\rangle+\left\langle(\gamma^{-1}\infty)^*,\nu\right\rangle), \tag{5.3}$$

where c_γ is the lower-left entry of γ in the Vahlen model. We now calculate the Fourier expansion of the Eisenstein series explicitly using (higher dimensional) Poisson summation:

$$\begin{aligned}E(P,s,\chi)&=[\Gamma_\infty:\Gamma'_\infty]y^s+\sum_{\substack{\gamma\in\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty\\ c_\gamma\neq 0}}\overline{\chi(\gamma)}\sum_{l\in\Lambda}y(\gamma(x+l,y))^s\\ &=[\Gamma_\infty:\Gamma'_\infty]y^s+\frac{1}{\mathrm{vol}(\Lambda)}\sum_{\gamma\in T_\Gamma}\overline{\chi(\gamma)}\sum_{\mu\in\Lambda^\circ}\left(\int_{\mathbb{R}^n}y(\gamma(t,y))^s e(-\langle t,\mu\rangle)dt\right)e(\langle x,\mu\rangle).\end{aligned}$$

Now by applying (4.8), we get for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SV}_{n-1}$:

$$\begin{aligned}
&\int_{\mathbb{R}^n} y(\gamma(t,y))^s e(-\langle t,\mu\rangle)dt \\
&= \int_{\mathbb{R}^n} \left(\frac{y}{|ct+d|^2+|c|^2y^2}\right)^s e(-\langle t,\mu\rangle)dt \\
&= \frac{y^s}{|c|^{2s}} \int_{\mathbb{R}^n} \left(\frac{1}{|t+c^{-1}d|^2+y^2}\right)^s e(-\langle t,\mu\rangle)dt \\
&= \frac{y^s}{|c|^{2s}} e(\langle dc^{-1},\mu\rangle) \int_{\mathbb{R}^n} \left(\frac{1}{|t|^2+y^2}\right)^s e(-\langle t,\mu\rangle)dt \\
&= \frac{y^s}{|c|^{2s}} e(\langle dc^{-1},\mu\rangle) \int_{\mathbb{R}^n} \left(\frac{1}{|t|^2+y^2}\right)^s e(-|\mu|t_0)\, dt
\end{aligned}$$

where the last equality follows by applying the orthogonal linear transformation which sends μ to $(|\mu|, 0, \ldots, 0)$.

When $\mu = 0$, we obtain

$$\int_{\mathbb{R}^n} \left(\frac{1}{|t|^2+y^2}\right)^s dt = \frac{y^{n-2s}\pi^{n/2}\Gamma\left(s-n/2\right)}{\Gamma(s)},$$

while for $\mu \neq 0$

$$\int_{\mathbb{R}^n} \left(\frac{1}{|t|^2+y^2}\right)^s e(-|\mu|t_0)\, dt = \frac{2\pi^s y^{n/2-s}|\mu|^{s-n/2}}{\Gamma(s)} K_{s-n/2}(2\pi|\mu|y).$$

This follows from [12, p. 678]. Alternatively it follows from combining [49, p. 213] and [17, 6.565.4]. Hence we get

$$\begin{aligned}
E(P,s,\chi) = [\Gamma_\infty : \Gamma'_\infty]y^s + y^{n-s}\frac{\pi^{n/2}\Gamma\left(s-\frac{n}{2}\right)}{\mathrm{vol}(\Lambda)\Gamma(s)}L(s,\chi) \\
+ \frac{2\pi^s y^{n/2}}{\mathrm{vol}(\Lambda)\Gamma(s)} \sum_{\mu\in\Lambda^\circ\setminus\{0\}} L(s,\mu,\chi)|\mu|^{s-n/2}K_{s-n/2}(2\pi|\mu|y),
\end{aligned} \tag{5.4}$$

where

$$L(s,\chi) := \sum_{\gamma\in T_\Gamma} \frac{\overline{\chi}(\gamma)}{|c_\gamma|^{2s}} = \sum_{c\in C(\Gamma)} \frac{S(0,0,c,\chi)}{|c|^{2s}}\ , \tag{5.5}$$

and for $\mu \neq 0$,

$$L(s,\chi,\mu) := \sum_{\gamma\in T_\Gamma} \overline{\chi}(\gamma)\frac{e(\langle d_\gamma c_\gamma^{-1},\mu\rangle)}{|c_\gamma|^{2s}} = \sum_{c\in C(\Gamma)} \frac{S(0,\mu,c,\chi)}{|c|^{2s}}. \tag{5.6}$$

For $\chi = 1$ the trivial character, we just denote $L(s,\mu) := L(s,\mu,1)$. We note that the explicit Fourier expansion we obtain in (5.4) is closely related to [12, Thm. 9.1].

At other cusps $\mathfrak{a} \neq \infty$ of Γ, we will also need some information about the Fourier expansion. For this let $P^{\mathfrak{a}} = (x^{\mathfrak{a}}, y^{\mathfrak{a}}) = \sigma_{\mathfrak{a}}^{-1}P$ denote the coordinates at $\mathfrak{a}$. Then the Fourier expansion at $\mathfrak{a}$ is given by [6, Ch. 6, Prop. 1.42]:

$$E(P^{\mathfrak{a}},s,\chi) = \phi_{\mathfrak{a}}(s)(y^{\mathfrak{a}})^{n-s} + \sum_{\mu\in\Lambda_{\mathfrak{a}}^\circ\setminus\{0\}} \phi_{\mathfrak{a}}(s,\mu)(y^{\mathfrak{a}})^{n-s}K_{s-n/2}(2\pi n|\mu|y^{\mathfrak{a}})e(\langle x^{\mathfrak{a}},\mu\rangle),$$

where $\phi_{\mathfrak{a}}(s,\mu)$ are the Fourier coefficients, which decay rapidly in $|\mu|$ (for s fixed). In particular we observe that $E(P,s,\chi)$ is square integrable when restricted to $\mathcal{F}_{\mathfrak{a}}(Y)$ for $\mathfrak{a}\neq\infty$ (for Y sufficiently large as in (4.12)).

Remark 5.1. By inverting γ in the definition of $L(s,\chi,\mu)$, we observe that

$$\begin{aligned} L(s,\chi,\mu) &= \sum_{\gamma\in T_\Gamma} \overline{\chi}(\gamma)\frac{e(\langle(\gamma^{-1}\infty)^*,\mu\rangle)}{|c_\gamma|^{2s}} \\ &= \sum_{\gamma^{-1}\in T_\Gamma} \chi(\gamma)\frac{e(\langle\gamma\infty,\mu\rangle)}{|c_\gamma|^{2s}} \\ &= \sum_{\gamma\in T_\Gamma} \chi(\gamma)\frac{e(\langle\gamma\infty,\mu\rangle)}{|c_\gamma|^{2s}}. \end{aligned} \tag{5.7}$$

5.1. **Short discussion on spectral properties.** We say that a (measurable) function $f:\mathbb{H}^{n+1}\to\mathbb{C}$ is χ-*automorphic* if it satisfies

$$f(\gamma P)=\chi(\gamma)f(P)\ ,$$

for $P\in\mathbb{H}^{n+1}$ and $\gamma\in\Gamma$.

Denote by $L^2(\Gamma\backslash\mathbb{H}^{n+1},\chi)$ the space of square integrable χ-automorphic functions with respect to the hyperbolic metric. For $f,g\in L^2(\Gamma\backslash\mathbb{H}^{n+1},\chi)$, we note that $f\overline{g}$ is Γ-invariant. Hence we can define the inner product

$$\langle f,g\rangle := \int_{\mathcal{F}} f\overline{g}\,dv\ .$$

We let $\mathcal{D}(\chi)\subset L^2(\Gamma\backslash\mathbb{H}^{n+1},\chi)$ be the subspace consisting of all C^2-functions such that $\Delta f\in L^2(\Gamma\backslash\mathbb{H}^{n+1},\chi)$. Then one can see that $-\Delta:\mathcal{D}(\chi)\to L^2(\Gamma\backslash\mathbb{H}^{n+1},\chi)$ is a symmetric and nonnegative operator, its spectrum consists of discrete and continuous parts with finitely many discrete points in the interval $[0,n^2/4)$. Let

$$0\le\lambda_0(\chi)\le\lambda_1(\chi)\le\cdots\le\lambda_k(\chi)<n^2/4$$

be the eigenvalues in the interval $[0,n^2/4)$ (see [38] and [6, Ch. 6]). The Eisenstein series $E(z,s,\chi)$ admits meromorphic continuation to $s\in\mathbb{C}$ and satisfies the functional equation

$$E(P,n-s,\chi)=\phi(n-s,\chi)E(P,s,\chi)\ ,$$

where $\phi(s,\chi)$ is the scattering matrix. Moreover, $E(P,s,\chi)$ has poles where $\phi(s,\chi)$ has poles and viceversa. There are finitely many poles in the region $\mathrm{Re}(s)>n/2$, all of them simple and on the real line. If $n/2<\sigma_0\le n$ is a pole of $E(P,s,\chi)$, denote by u_{σ_0} its residue at σ_0. Then

$$u_{\sigma_0}\in L^2(\Gamma\backslash\mathbb{H}^{n+1},\chi)\quad\text{and}\quad \Delta u_{\sigma_0}+\sigma_0(n-\sigma_0)u_{\sigma_0}=0\ .$$

For $0\le j\le k$, let $s_j(\chi)\in(n/2,n]$ be such that $s_j(\chi)(n-s_j(\chi))=\lambda_j(\chi)$. We denote by

$$\Omega(\chi):=\{s_0(\chi),\dots,s_k(\chi)\}.$$

Then the poles of $E(P,s,\chi)$ in $\mathrm{Re}\,s>n/2$ form a subset of $\Omega(\chi)$. Moreover, we can see from [6, Ch 6, p. 37] that for χ trivial, we have

$$\mathrm{Res}_{s=n}\,E(P,s)=\frac{[\Gamma_\infty:\Gamma'_\infty]\mathrm{vol}(\Lambda)}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^{n+1})}\ . \tag{5.8}$$

5.2. **Key lemmas.** In this section we will prove certain key analytic lemmas that we will need in the proofs of our theorems. First of all we will show that we can only have $\lambda_0(\chi)=0$ when χ is trivial. Secondly we obtain meromorphic continuation of the Fourier coefficients of the twisted Eisenstein series, which will serve as generating series for our distribution problems. Finally we will prove a bound on vertical lines for these generating series.

The most conceptual way to see the first claim above is probably to use Green's identity

$$\int_{\mathcal{F}}(-\Delta u)u dv=\int_{\mathcal{F}}\nabla u.\nabla u\, dv+\int_{\partial\mathcal{F}}u(\nabla u.\mathbf{n})\mathbf{dS}.$$

If we have $\Delta u=0$, then the first integral is 0. The third integral should vanish since contributions from "opposing sides" in the boundary of the fundamental domain should cancel each other. This would force the second integral to be 0, which means u is constant. This argument works in principle, but for example in [13, Theorem 4.1.7] they spend several pages making it rigorous. Instead we will give an argument using the Fourier expansion and the mean value theorem for harmonic functions.

Lemma 5.2. *We have that $\lambda_0(\chi)=0$ if and only if χ is trivial.*

Proof. Suppose $\lambda_0(\chi)=0$ and let u be a corresponding eigenvector, i.e. $u\in L^2(\Gamma\backslash\mathbb{H}^{n+1},\chi)$ and $\Delta u=0$. Then we can consider the Fourier expansion of u at a cusp $\mathfrak{a}$ of Γ. We know from [6, Ch. 6, p.10] that the Fourier expansion of u takes the form

$$c_{1,\mathfrak{a}}+c_{2,\mathfrak{a}}(y^{\mathfrak{a}})^n+\sum_{\mu\in\Lambda_{\mathfrak{a}}^{\circ}\backslash\{0\}}a_{u,\mathfrak{a}}(\mu)(y^{\mathfrak{a}})^{n/2}K_{n/2}(2\pi n|\mu|y)e(\langle x,\mu\rangle).$$

From the rapid decay of the K-Bessel function we see that if $c_{2,\mathfrak{a}}\neq 0$, then u behaves like $(y^{\mathfrak{a}})^n$ close enough to $\mathfrak{a}$ and thus $\int_{F_{\mathfrak{a}}(Y)}|u(x,y)|^2dxdy$ is divergent contradicting the fact that u is square integrable. Thus $c_{2,\mathfrak{a}}=0$ and we conclude again using the rapid decay of the K-Bessel functions that u is bounded on $F_{\mathfrak{a}}(Y)$. Since $\mathfrak{a}$ was an arbitrary cusp we conclude that u is bounded on all of $\mathcal{F}$. Thus since χ is unitary, we conclude that u is bounded on all of $\mathbb{H}^{n+1}$. Now it follows from the *Mean Value Theorem for Harmonic Functions on* $\mathbb{H}^{n+1}$ that u is constant. By definition, $u(\gamma P)=\chi(\gamma)u(P)$, for all $\gamma\in\Gamma$ and $P\in\mathbb{H}^{n+1}$. Thus we conclude that χ is the trivial character.

Therefore, if χ is trivial the unique eigenfunction of eigenvalue 0 is the constant one, and for χ non-trivial there are no eigenfunctions of eigenvalue 0. This finishes the proof. $\square$

We now obtain meromorphic continuation of the Fourier coefficients of the Eisenstein series and crucial information about the location of the poles.

Proposition 5.3. *The Dirichlet series $L(s,\mu,\chi)$ admits meromorphic continuation to the entire complex plane. The possible poles in the half-plane $\operatorname{Re}s>n/2$ are contained in $\Omega(\chi)$. Furthermore, there is a pole at $s=n$ exactly if χ is trivial and $\mu=0$. In this case the residue is equal to*

$$\frac{[\Gamma_\infty:\Gamma'_\infty]\Gamma(n)\mathrm{vol}(\Lambda)^2}{\pi^{n/2}\Gamma\left(\frac{n}{2}\right)\mathrm{vol}(\Gamma\backslash\mathbb{H}^{n+1})}.$$

Proof. From (5.4), we know that for $\mu\in\Lambda^{\circ}\backslash\{0\}$

$$L(s,\mu,\chi)=\frac{\Gamma(s)}{2\pi^s y^{n/2}|\mu|^{s-n/2}K_{s-n/2}(2\pi|\mu|y)}\int_{\mathcal{P}}E((x,y),s,\chi)e(-\langle x,\mu\rangle)dx,$$

and

$$L(s,\chi)=\frac{y^{s-n}\Gamma(s)}{\pi^{n/2}\Gamma\left(s-\frac{n}{2}\right)}\left(\int_{\mathcal{P}}E((x,y),s,\chi)dx-[\Gamma_\infty:\Gamma'_\infty]y^s\right),$$

where $\mathcal{P}$ is a fundamental parallelogram for Λ. Now for $y > 0$ fixed , the Bessel function $K_s(y)$ defines an analytic function in s, which is non-zero for some y large enough. Similarly the Gamma function define a meromorphic function. Thus we get the meromorphic continuation of $L(s, \mu, \chi)$ from that of the Eisenstein series. We also note that in the half-plane $\mathrm{Re}\, s > n/2$, $L(s, \mu, \chi)$ has possible poles only where $E(P, s, \chi)$ has poles, i.e. the poles are contained in $\Omega(\chi)$. By Lemma 5.2, we see that $L(s, \mu, \chi)$ is regular at $s = n$ unless χ is trivial.

If χ is trivial, we see that $L(s, \mu)$ with $\mu \neq 0$ is regular at $s = n$, since the pole of the Eisenstein series is constant. For $\mu = 0$ the residue is given by

$$\mathrm{Res}_{s=n} L(s, 0) = \frac{\Gamma(n)}{\pi^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_{\mathcal{P}} \frac{[\Gamma_\infty : \Gamma'_\infty]\mathrm{vol}(\Lambda)}{\mathrm{vol}(\Gamma\backslash\mathbb{H}^{n+1})} dx = \frac{[\Gamma_\infty : \Gamma'_\infty]\Gamma(n)\mathrm{vol}(\Lambda)^2}{\pi^{n/2}\Gamma\left(\frac{n}{2}\right)\mathrm{vol}(\Gamma\backslash\mathbb{H}^{n+1})},$$

as wanted. □

In order to obtain bounds on vertical lines for our generating series, we will employ an idea due to Colin de Verdière [7], which employs the analytic properties of resolvent operators. Alternatively one could use Poincaré series for $\mu \neq 0$ and Maaß–Selberg for $\mu = 0$ as is done in [35] and [8]. In the end the two methods are essentially equivalent.

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function which is equal to $[\Gamma_\infty : \Gamma'_\infty]$ for $y > Y + 1$ and 0 for $y < Y$, where Y is as in (4.12). Then for $\mathrm{Re}(s) > n/2$ we define a χ-automorphic function on $\mathbb{H}^{n+1}$ by $P \mapsto h(y)y^s$ for $P \in \mathcal{F}$ and extended periodically (twisted accordingly by χ). Then from the above mentioned results on the Fourier expansions of the Eisenstein series at the different cusps, we see that

$$g(P, s, \chi) := E(P, s, \chi) - h(y)y^s \in L^2(\Gamma\backslash\mathbb{H}^{n+1}, \chi),$$

which satisfies for $z \in \mathcal{F}$

$$(\Delta - s(n-s))g(P, s, \chi) = -(\Delta - s(n-s))h(y)y^s = h''(y)y^{s+2} + (2s-n+1)h'(y)y^{s+1}.$$

We observe that the right hand side above is compactly supported with L^2-norm bounded by $O(|s| + 1)$ for $n/2 + \varepsilon < \mathrm{Re}\, s < n + 2$. Now we put

$$H(P, s, \chi) := R(s, \chi)(h''(y)y^{s+2} + (2s - n + 1)h'(y)y^{s+1}) \in L^2(\Gamma\backslash\mathbb{H}^{n+1}, \chi),$$

where $R(s, \chi) = (\Delta - s(n - s))^{-1}$ denotes the resolvent operator associated to Δ. By a general bound for the operator norm of resolvent operators [20, Lemma A.4], we conclude that

$$||H(\cdot, s, \chi)||_{L^2} \ll_\varepsilon 1,$$

when s is bounded at least ε away from the spectrum of Δ. We can now write

$$E(P, s, \chi) = H(P, s, \chi) + h(y)y^s, P \in \mathcal{F} \tag{5.9}$$

where we have good control on the L^2-norm of $H(P, s, \chi)$. We will now use this to obtain bounds on vertical lines for the Fourier coefficients of $E(P, s, \chi)$. We mimic [32, Section 4.4].

Proposition 5.4. *Let $\mu \in \Lambda^\circ$. Then we have*

$$L(s, \mu, \chi) \ll_{\varepsilon,\mu} (|s| + 1)^{n/2},$$

for $n/2 + \varepsilon < \mathrm{Re}\, s < n + 2$ and s bounded at least ε away from the spectrum of Δ.

Proof. We have

(5.10)
$$L(s,\mu,\chi)=\int_{\mathcal{P}} f_s(y,\mu)E((x,y),s,\chi)e(-\langle x,\mu\rangle)dx-\mathbf{1}_{\mu=0}[\Gamma_\infty:\Gamma'_\infty]y^s f_s(y,\mu),$$

where
$$f_s(y,\mu)=\begin{cases}\frac{\Gamma(s)}{2\pi^s y^{n/2}|\mu|^{s-n/2}K_{s-n/2}(2\pi n|\mu|y)}, & \mu\neq 0,\\ \frac{\Gamma(s)}{y^{n-s}\pi^{n/2}\Gamma(s-n/2)}, & \mu=0.\end{cases}$$

The idea is now to bound the right hand side of (5.10) using (5.9). In order to bring the information we have about $H(P,s,\chi)$ into play, we need to make an extra integration over y. So let Y be a fixed quantity such that $\{(x,y)\mid x\in\mathcal{P}, y>Y\}\subset\mathcal{F}$, then we see that

$$\begin{aligned}&\int_Y^{Y+1}\int_{\mathcal{P}} f_s(y,\mu)E((x,y),s,\chi)e(-\langle\mu,x\rangle)dxdy\\ &=\int_Y^{Y+1}\int_{\mathcal{P}} f_s(y,\mu)H((x,y),s,\chi)e(-\langle\mu,x\rangle)dxdy\\ &+\int_Y^{Y+1}\int_{\mathcal{P}} f_s(y,\mu)h(y)y^s e(-\langle\mu,x\rangle)dxdy\end{aligned}$$

Now we observe that by Cauchy–Schwarz we have

$$\begin{aligned}&\int_Y^{Y+1}\int_{\mathcal{P}} f_s(y,\mu)H((x,y),s,\chi)e(-\langle\mu,x\rangle)dxdy\\ &\leq\left(\int_Y^{Y+1}\int_{\mathcal{P}}|H((x,y),s,\chi)|^2dxdy\right)^{1/2}\left(\int_Y^{Y+1}\int_{\mathcal{P}}|f_s(y,\mu)|^2dxdy\right)^{1/2}\\ &\ll||H(\cdot,s,\chi)||_{L^2}\left(\int_Y^{Y+1}|f_s(y,\mu)|^2dy\right)^{1/2},\end{aligned}$$

where we use that $\{(x,y)\mid x\in\mathcal{P}, y>Y\}\subset\mathcal{F}$. To finish the proof we need an upper bound for $f_s(y,\mu)$.

For $\mu=0$ we get by Stirling's approximation the upper bound
$$f_s(y,0)\ll_\varepsilon y^{n-\sigma}(|s|+1)^{n/2},$$
for $s=\sigma+it$ with $n/2+\varepsilon<\sigma<n+2$.

For $\mu\neq 0$, we use the Fourier expansion for the K-Bessel function (coming from combining [20, (B.32)] and [20, (B.34)]) to obtain a good approximation. By applying Stirling's approximation, this gives for $s=\sigma+it$ with $t\gg 1$

$$\begin{aligned}K_{s-n/2}(2\pi|\mu|y)&=\frac{\pi^{1/2}t^{\sigma-n/2-1/2}e^{\pi t/2}\left(\frac{t}{e}\right)^{it}}{2\sqrt{2}\sin(\pi(s-n/2))}(\pi|\mu|y)^{-s+n/2}(1+O_{\mu,y}(t^{-1}))\\ &\gg_{\mu,y} e^{-\pi t/2}t^{\sigma-n/2-1/2},\end{aligned}$$

where the implied constants depend continuously on y. From this we conclude that when $y\in(Y,Y+1)$, we have
$$f_s(y,\mu)\ll_\mu(1+|s|)^{n/2}.$$
Inserting this and using the bound $||H(\cdot,s,\chi)||_{L^2}\ll_\varepsilon 1$, we conclude that
$$L(s,\mu,\chi)\ll_{\varepsilon,\mu}(|s|+1)^{n/2},$$
for s bounded ε away from the spectrum of Δ, as wanted. $\square$

6. Proof of Theorem 1.5

In this section we will use the analytic properties of twisted Eisenstein series proved in the previous section to proof our main results. First of all we deduce the following result using a standard complex analysis argument.

Proposition 6.1. *Let χ be a unitary character of Γ trivial on Γ'_∞ and $\mu \in \Lambda^\circ$. Then there exists a constant $\nu(\chi) > 0$ such that*

$$\sum_{\gamma \in T_\Gamma(X)} \chi(\gamma) e\left(\langle \gamma\infty, \mu \rangle\right) = \frac{X^{s_0(\chi)}}{s_0(\chi)} \left(\mathrm{Res}_{s=s_0(\chi)} L(s,\chi,\mu) + O_{\chi,\mu}(X^{-\nu(\chi)}) \right).$$

Proof. Let $\phi_U : \mathbb{R} \to \mathbb{R}$ be a family of smooth non-increasing functions with

$$\phi_U(t) = \begin{cases} 1 & \text{if } t \leq 1 - 1/U, \\ 0 & \text{if } t \geq 1 + 1/U \end{cases} \tag{6.1}$$

and $\phi_U^{(j)}(t) = O(U^j)$ as $U \to \infty$. For $\mathrm{Re}(s) > 0$, we consider the Mellin transform

$$R_U(s) = \int_0^\infty \phi_U(t) t^s \frac{dt}{t}\ . \tag{6.2}$$

We can easily see that

$$R_U(s) = \frac{1}{s} + O\left(\frac{1}{U}\right) \quad \text{as } U \to \infty \tag{6.3}$$

and for any $N > 0$,

$$R_U(s) = O\left(\frac{1}{|s|} \left(\frac{U}{1+|s|} \right)^N \right) \quad \text{as } |s| \to \infty\ , \tag{6.4}$$

where the last estimate follows from repeated partial integration. Now we use Mellin inversion and (5.7) to obtain

$$\begin{aligned} &\sum_{\gamma \in T_\Gamma} \chi(\gamma) e\left(\langle \gamma\infty, \mu \rangle\right)\ \phi_U\left(\frac{|c|^2}{X}\right) \\ &= \sum_{\gamma \in T_\Gamma} \chi(\gamma) e\left(\langle \gamma\infty, \mu \rangle\right) \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=n+1} \frac{X^s}{|c|^{2s}} R_U(s) ds \\ &= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=n+1} L(s,\chi,\mu) X^s R_U(s) ds. \end{aligned}$$

Next, we recall Proposition 5.4 and equation (6.4) to deduce that the last integral is absolutely convergent. We want to move the line of integration to $\mathrm{Re}(s) = h = h(\chi)$ for some $h > n/2$ such that $s_1(\chi) < h(\chi) < s_0(\chi)$. We use the fact that we have polynomial growth on vertical lines for $L(s,\chi,\mu)$ guaranteed by Lemma 5.4 and that $L(s,\chi,\mu)$ has only a possible pole at $s_0(\chi)$ in the region $\mathrm{Re}(s) > h(\chi)$. We conclude that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=n+1} L(s,\chi,\mu) X^s R_U(s) ds = &\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=h} L(s,\chi,\mu) X^s R_U(s) ds \\ &+ \mathrm{Res}_{s=s_0(\chi)} \left(L(s,\chi,\mu) X^s R_U(s) \right)\ . \end{aligned}$$

Setting $N = (n+1)/2$ in (6.4), we observe from Proposition 5.4 that

$$\int_{\mathrm{Re}(s)=h} L(s,\chi,\mu)X^s R_U(s)ds \ll X^h U^{(n+1)/2} . \tag{6.5}$$

Now, (6.3) gives us

(6.6)

$$\mathrm{Res}_{s=s_0(\chi)} \left(L(s,\chi,\mu)X^s R_U(s)\right) = \frac{X^{s_0(\chi)}}{s_0(\chi)} \mathrm{Res}_{s=s_0(\chi)} L(s,\chi,\mu) \left(1 + O\left(\frac{1}{U}\right)\right) .$$

Since we want this to be the main contribution, we choose $U = X^{a(\chi)}$, where $a(\chi) := (s_0(\chi) - h(\chi))/(n+1)$.

Now if χ is the trivial character and $\mu = 0$, we obtain

$$\sum_{\gamma \in T_\Gamma} \phi_U\left(\frac{|c|^2}{X}\right) = \frac{X^n}{n}\left(\mathrm{Res}_{s=n} L(s) + O(X^{-\delta})\right),$$

for some fixed $\delta > 0$. We now choose ϕ_U^1 and ϕ_U^2 as in (6.1) with the further requirements that $\phi_U^1(t) = 0$ for $t \geq 1$ and $\phi_U^2(t) = 1$ for $0 \leq t \leq 1$. Then

$$\sum_{\gamma \in T_\Gamma} \phi_U^1\left(\frac{|c|^2}{X}\right) \leq \sum_{\gamma \in T_\Gamma(X)} 1 \leq \sum_{\gamma \in T_\Gamma} \phi_U^2\left(\frac{|c|^2}{X}\right) ,$$

so the previous two equations and Proposition 5.3 give us

$$\#T_\Gamma(X) = \frac{X^{2n}}{n}\left(\frac{[\Gamma_\infty : \Gamma'_\infty]\mathrm{vol}(\Lambda)^2\Gamma(n)}{\pi^{n/2}\mathrm{vol}(\Gamma)\Gamma(n/2)} + O(X^{-\delta})\right) . \tag{6.7}$$

Now we return to (6.6). Indeed,

(6.8)

$$\sum_{\gamma \in T_\Gamma} \chi(\gamma)e\left(\langle\gamma\infty,\mu\rangle\right)\phi_U\left(\frac{|c|^2}{X}\right) = \frac{X^{s_0(\chi)}}{s_0(\chi)}\left(\mathrm{Res}_{s=s_0(\chi)} L(s,\chi,\mu) + O(X^{-a(\chi)})\right) .$$

Also, from the definition of ϕ_U,

$$\begin{aligned}\sum_{\gamma \in T_\Gamma} \chi(\gamma)e\left(\langle\gamma\infty,\mu\rangle\right)\phi_U\left(\frac{|c|^2}{X}\right) = &\sum_{\gamma \in T_\Gamma(\sqrt{X})} \chi(\gamma)e\left(\langle\gamma\infty,\mu\rangle\right)\\ &+ O\left(\#\left\{\gamma \in \Gamma'_\infty\backslash\Gamma/\Gamma'_\infty \ : \ 1 - \frac{1}{U} \leq \frac{|c|^2}{X} \leq 1 + \frac{1}{U}\right\}\right) .\end{aligned}$$

But now we use (6.7) to bound the size of the error term

$$\begin{aligned}\#\left\{\gamma \in \Gamma'_\infty\backslash\Gamma/\Gamma'_\infty \ : \ 1 - \frac{1}{U} \leq \frac{|c|^2}{X} \leq 1 + \frac{1}{U}\right\} &= T_\Gamma\left(\sqrt{X\left(1+\frac{1}{U}\right)}\right) - T_\Gamma\left(\sqrt{X\left(1-\frac{1}{U}\right)}\right)\\ &= O\left(X^{n-\nu}\right),\end{aligned}$$

for some $\nu(\chi) > 0$. The conclusion follows. □

Remark 6.2. As a consequence of Proposition 6.1, we conclude that for all unitary characters χ as above there exist $\nu(\chi) > 0$ such that

$$\sum_{\gamma \in T_\Gamma(X)} \chi(\gamma)e\left(\langle\gamma\infty,\mu\rangle\right) = \mathbf{1}_{\chi,\mu}\frac{\mathrm{vol}(\Lambda)^2\Gamma(n)}{n\pi^{n/2}\mathrm{vol}(\Gamma\backslash\mathbb{H}^{n+1})\Gamma(n/2)}X^{2n} + O_\chi(X^{2n-\nu(\chi)}),$$

where $\mathbf{1}_{\chi,\mu}$ is 1 if $\mu = 0$ and χ is trivial and 0 otherwise.

6.1. **Applications to equidistribution.** Using the the above proposition we are now ready to finish the proof of Theorem 1.5 and from this deduce Theorems 1.2 and 1.3.

We recall the setup from the definition. Consider the cohomology group $H^1_{\Gamma'_\infty}(\Gamma,\mathbb{R}/\mathbb{Z})$ (see Section 8 for details), which can be identified with the set of unitary characters of Γ trivial on Γ'_∞.

Definition 6.3. *We say that $\omega_1,\ldots,\omega_d\in H^1_{\Gamma'_\infty}(\Gamma,\mathbb{R}/\mathbb{Z})$ are in* **general position** *if for any $(l_1,\ldots,l_d)\in\mathbb{Z}^d$, we have*

$$n_1\omega_1+\ldots+n_d\omega_d=0\in H^1_{\Gamma'_\infty}(\Gamma,\mathbb{R}/\mathbb{Z})\Leftrightarrow\Big(n_i\omega_i=0\in H^1_{\Gamma'_\infty}(\Gamma,\mathbb{R}/\mathbb{Z}),\forall i=1,\ldots,d\Big).$$

As an example one can pick $\omega_1,\ldots,\omega_d$ to be a basis for the non-torsion part of $H^1_{\Gamma'_\infty}(\Gamma,\mathbb{R}/\mathbb{Z})$. Also we could pick a $\mathbb{F}_p$-basis for $H^1_{\Gamma'_\infty}(\Gamma,\mathbb{F}_p)$, where we consider $\mathbb{F}_p\subset\mathbb{R}/\mathbb{Z}$ via $\mathbb{F}_p\ni a\mapsto a/p$.

Observe that the image of ω_i is an additive subgroup of $\mathbb{R}/\mathbb{Z}$ and thus is either dense in $\mathbb{R}/\mathbb{Z}$ or finite. In the first case we put $J_i=\mathbb{R}/\mathbb{Z}$ and in the latter case we put $J_i=\mathbb{Z}/m_i\mathbb{Z}$ where m_i is the cardinality of the image of ω_i. That is, J_i is the closure of the image of ω_i. We equip $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ with respectively the Lebesque measure and the uniform probability measure.

Let $\omega_1,\ldots,\omega_d\in H^1_{\Gamma'_\infty}(\Gamma_0(N),\mathbb{R}/\mathbb{Z})$ be in general position. Then for any tuple $\underline{l}=(l_1,\ldots,l_d)\in\mathbb{Z}^d$ such that $l_i\omega_i\neq 0\in H^1_{\Gamma'_\infty}(\Gamma_0(N),\mathbb{R}/\mathbb{Z})$ for all $i=1,\ldots,d$, we get a non-trivial element of $H^1_{\Gamma'_\infty}(\Gamma,\mathbb{R}/\mathbb{Z})$ defined by

$$\omega_{\underline{l}}:=l_1\omega_1+\ldots+l_d\omega_d.$$

Now we consider the associated non-trivial unitary character $\chi_{\underline{l}}:\Gamma\to\mathbb{C}^\times$;

$$\chi_{\underline{l}}(\gamma):=e\left(\omega_{\underline{l}}(\gamma)\right),$$

where $e(x)=e^{2\pi i x}$. Observe that this is indeed well-defined and that we get an induced map $\chi_{\underline{l}}:\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty\to\mathbb{C}^\times$ since $\omega_{\underline{l}}$ is trivial on Γ'_∞.

By *Weyl's Criterion* [21, p. 487] in order to conclude equidistribution of the values of

$$\omega(\gamma):=(\omega_1(\gamma),\ldots,\omega_d(\gamma),\gamma\infty)$$

inside $\prod_{i=1}^d J_i\times(\mathbb{R}^n/\Lambda)$, we have to show cancelation in the corresponding Weyl sums. These are exactly given by:

$$\sum_{\gamma\in T_\Gamma(X)}\chi_{\underline{l}}(\gamma)e(\langle\gamma\infty,\mu\rangle),$$

where $\underline{l}\in\mathbb{Z}^d$ and $\mu\in\Lambda^\circ$. We see that it follows from combining Proposition 6.1 and Remark 5.1 that we have

$$\sum_{\gamma\in T_\Gamma(X)}\chi_{\underline{l}}(\gamma)e(\langle\gamma\infty,\mu\rangle)=o\left(\sum_{\gamma\in T_\Gamma(X)}1\right),$$

as $X\to\infty$ *unless* $\mu=0$ and $\chi_{\underline{l}}$ is trivial. This finishes the proof of Theorem 1.5 using Weyl's Criterion.

Now let us see how Theorem 1.2 and 1.3 follow from Theorem 1.5. We restrict to $n=1$ and $\Gamma=\Gamma_0(N)$. By the mod p-version of the Eichler–Shimura isomorphism (3.1), we see that $\mathfrak{m}_f^{\pm}$ with $f\in\mathcal{S}_2(\Gamma_0(N))$ gives a basis for $H^1_P(\Gamma_0(N),\mathbb{F}_p)$, and thus it follows directly that they are in general position.

Similarly, by Eichler–Shimura, we know that the cohomology classes associated to $\mathrm{Re}\, f(z)dz$ and $\mathrm{Im}\, f(z)dz$ are in general position, where $f \in \mathcal{S}_2(\Gamma_0(N))$ ranges over Hecke newforms. From a classical result of Schneider [39] we know that the Néron periods $\Omega_\pm$ are transcendental. Using the rationality (1.2), this implies that the cohomology class associated to a newform f given by

$$\Gamma_0(N) \ni \gamma \mapsto \int_{\gamma\infty}^{\infty} \mathrm{Re}(f(z)dz)$$

takes some irrational value (and similarly for $\mathrm{Im}(f(z)dz)$).

Thus we see that in these two cases Theorem 1.5 reduces to Theorem 1.2 and 1.3.

7. Proof of Theorem 1.8

We now give a proof of Theorem 1.8 showing equidistriburion of the values of cohomology classes when ordered by the lengths of the geodesics corresponding to conjugacy classes of Γ. This will be an almost direct consequence of a twisted trace formula for $\mathrm{SO}(n+1,1)$. Our method is in the spirit of [34], where Petridis–Risager show that for cocompact subgroups of $\mathrm{SL}_2(\mathbb{R})$ the values of modular symbols are asymptotically normally distributed when ordered by the length of the corresponding geodesics. This was in turn inspired by ideas of Phillips and Sarnak [36].

We firstly consider the case $n = 1$. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ is hyperbolic, then γ is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to a unique element $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ with $\lambda > 1$. Let Γ be a discrete, cofinite subgroup of $\mathrm{SL}_2(\mathbb{R})$. We know that for each hyperbolic conjugacy class $\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma)$ there is a corresponding geodesic of length $l(\gamma) = \log \lambda^2$. It is a consequence of the twisted trace formula for Γ that for any unitary character χ of Γ, we have

$$\sum_{\substack{\{\gamma_0\} \text{ primitive} \\ l(\gamma_0) \leq X}} \chi(\gamma_0) = \sum_{s \in \Omega(\chi)} \mathrm{li}(e^{sX}) + O_\chi(e^{\frac{3}{4}X}),$$

where $\mathrm{li}(x) = \int_2^x t^{-1} dt$ is the logarithmic integral (see [19, p. 475]). Hence we obtain

$$\sum_{\substack{\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma) \\ l(\gamma) \leq X}} \chi(\gamma) \sim \sum_{\substack{\{\gamma_0\} \text{ primitive} \\ l(\gamma_0) \leq X}} \chi(\gamma_0) \sim \mathrm{li}(e^{s_0(\chi)X})$$

where the first sum is over all hyperbolic classes. Therefore, using Lemma 5.2, we obtain that for some $\nu(\chi) > 0$,

$$\frac{1}{|\{\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma) : 0 < l(\gamma) \leq X\}|} \sum_{\substack{\{\gamma\} \in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma) \\ l(\gamma) \leq X}} \chi(\gamma) = \mathbf{1}_\chi + O(e^{-\nu(\chi)X}),$$

where $\mathbf{1}_\chi$ is 1 if χ is trivial and 0 otherwise. Now the proof follows using the Weyl criterion as in Section 6.1.

We now discuss the general case n. As mentioned eariler, the first proof of the Prime Geodesic Theorem in the general case was given by Gangolli and Warner [15]. The trace formula for cofinite subgroups of $\mathrm{SO}(n+1,1)$ acting on $\mathbb{H}^{n+1}$ was developed by Cohen and Sarnak in [6, Ch. 7]. As a consequence, they obtain the following stronger version of Prime Geodesic Theorem for $\mathbb{H}^{n+1}$ [6, Thm. 7.37]:

$$\pi_\Gamma(X) = \sum_{n/2 < s_j \leq n} \mathrm{li}(e^{s_j X}) + O\left(e^{(n - \frac{n}{n+2})X}\right)$$

where the sum is taken over all $n/2 \leq s_j \leq n$ such that $s_j(n-s_j)$ is an eigenvalue of $-\Delta$ acting on $L^2(\Gamma\backslash\mathbb{H}^{n+1})$. Now we would like to apply a trace formula where we allow twists by characters. We did not find a place in literature where it is written down explicitly, and to keep the exposition simple we wil leave out the details. The analysis should be similar to the case $n = 1$ and is furthermore implied to hold by Sarnak in [38, p. 6]. Similarly Phillips and Sarnak [36] prove a theorem about distribution of geodesics in homology classes for quotients of $\mathbb{H}^{n+1}$, but only treat the case $n = 1$ in detail. The twisted trace formula for $\mathbb{H}^{n+1}$ that we need is exactly the same one which is implicit [36].

As in the 2 dimensional case, we would get

$$\sum_{\substack{\{\gamma\}\in \mathrm{Conj}_{\mathrm{hyp}}(\Gamma)\\ l(\gamma)\leq X}} \chi(\gamma) \sim \mathrm{li}(e^{s_0(\chi)X})$$

from which Theorem 1.8 follows by Weyl's Criterion as above.

8. On the size of certain cohomology groups

In this paper we study the distribution of certain cohomology classes which can be identified with the unitary characters of cofinite subgroups $\Gamma < \mathrm{SO}(n+1,1)$ (or equivalently $\Gamma < \mathrm{SV}_{n-1}$) with cusps. It is now a natural question to ask how many unitary characters our results actually apply to. This amounts to finding the dimensions of the relevant space of unitary characters or equivalently of certain cohomology groups. This last perspective is most useful when comparing it to the existing literature. First of all we will define the cohomology groups that are relevant and then survey what is known about their size.

8.1. **The first cohomology group.** We refer to [45, Chapter 8] for a comprehensive account. The *first cohomology group* of Γ with coefficients in a $\mathbb{Z}[\Gamma]$-module A is defined as the quotient between the corresponding *coboundaries* and *cocycles*;

$$H^1(\Gamma, A) := Z^1(\Gamma, A)/B^1(\Gamma, A),$$

where

$$Z^1(\Gamma, A) := \{\omega : \Gamma \to A \mid \omega(\gamma_1\gamma_2) = \omega(\gamma_1) + \gamma_1.\omega(\gamma_2), \forall \gamma_1, \gamma_2 \in \Gamma\}$$

and

$$B^1(\Gamma, A) := \{\omega : \Gamma \to A \mid \exists a \in A : \omega(\gamma) = \gamma.a - a, \forall \gamma \in \Gamma\}.$$

Furthermore given a subset $P \subset \Gamma$, we will be studying the first P-cohomology group of Γ with coefficients in A defined by;

$$H^1_P(\Gamma, A) := \{\omega \in H^1(\Gamma, A) \mid \omega(p) \in (p-1)A, \forall p \in P\}.$$

We will in particular study the distribution of P-cohomology group in the case where $P = \Gamma'_\infty$ is the set of parabolic elements of Γ fixing ∞ and A is given by the circle $\mathbb{R}/\mathbb{Z}$ equipped with the trivial Γ-action. In this case $H^1_P(\Gamma, \mathbb{R}/\mathbb{Z})$ computes exactly the unitary characters of Γ trivial on Γ'_∞.

Now we will make some general comments on the structure and size of $H^1_P(\Gamma, \mathbb{R}/\mathbb{Z})$.

8.2. **On the structure of the cohomology groups.** We recall that for A a trivial Γ module we have

$$H^1(\Gamma, A) \cong \mathrm{Hom}_{\mathbb{Z}}(\Gamma/[\Gamma,\Gamma], A),$$

which is a special case of the *Universal Coefficients Theorem* since $H_1(\Gamma,\mathbb{Z}) \cong \Gamma/[\Gamma,\Gamma]$. From this we see that $H^1(\Gamma,\mathbb{R}/\mathbb{Z})$ can be identified with the unitary characters of Γ. It is known [41, p. 484] that Γ is finitely represented and thus $\Gamma/[\Gamma,\Gamma]$ is a finitely generated abelian group. From this we see that we have a splitting of the cohomology group $H^1(\Gamma,\mathbb{R}/\mathbb{Z})$ in a free part and a torsion part;

$$H^1(\Gamma,\mathbb{R}/\mathbb{Z}) \cong H^1_{\mathrm{free}}(\Gamma,\mathbb{R}/\mathbb{Z}) \oplus H^1_{\mathrm{tor}}(\Gamma,\mathbb{R}/\mathbb{Z}),$$

where the $\mathbb{R}/\mathbb{Z}$ rank of $H^1_{\mathrm{free}}(\Gamma,\mathbb{R}/\mathbb{Z})$ is the same as the dimension of $H^1(\Gamma,\mathbb{R})$ and the size of $H^1_{\mathrm{tor}}(\Gamma,\mathbb{R}/\mathbb{Z})$ is equal to the size of the torsion in $H_1(\Gamma,\mathbb{Z}) \cong \Gamma/[\Gamma,\Gamma]$.

We have a further Eichler–Shimura splitting of the free part due to Harder [18];

$$H^1(\Gamma,\mathbb{R}) \cong H^1_{\mathrm{cusp}}(\Gamma,\mathbb{R}) \oplus H^1_{\mathrm{Eis}}(\Gamma,\mathbb{R}), \tag{8.1}$$

where $H^1_{\mathrm{cusp}}(\Gamma,\mathbb{R})$ is the cuspidal part corresponding to certain automorphic forms for Γ (as we will see shortly) and $H^1_{\mathrm{Eis}}(\Gamma,\mathbb{R})$ is the (remaining) Eisenstein part, which can be canonically defined. The cuspidal part $H^1_{\mathrm{cusp}}(\Gamma,\mathbb{R})$ can be identified with $H^1_P(\Gamma,\mathbb{R})$ where P is the set of all parabolic elements of Γ and furthermore all of the above splittings are compatible with the Hecke action, when Γ is arithmetic.

There has been a lot of work recently on the study of the size of respectively $H^1_{\mathrm{cusp}}(\Gamma,\mathbb{R})$, $H^1_{\mathrm{Eis}}(\Gamma,\mathbb{R})$ and $H^1_{\mathrm{tor}}(\Gamma,\mathbb{R}/\mathbb{Z})$, and we will now collect the relevant results for our problem. We observe that the image of Γ'_∞ in $\Gamma/[\Gamma,\Gamma]$ is either trivial, finite or isomorphic to $\mathbb{Z}$. Thus we conclude that $H^1_{\Gamma'_\infty}(\Gamma,\mathbb{R}/\mathbb{Z})$ is non-trivial as soon as, say $H^1(\Gamma,\mathbb{R}/\mathbb{Z})$ is not generated by a single element or $H^1(\Gamma,\mathbb{R})$ is non-trivial.

8.3. **The dimension of cohomology groups.** It is a result of Kazhdan [22] that for discrete, cofinite subgroups of real Lie groups of rank larger than 1, the abelianization is always torsion. In our case, since $\mathrm{SO}(n+1,1)$ is of rank one, we can however hope to see some free part. In the case of cofinite subgroups $\Gamma \subset \mathrm{SO}(n+1,1)$, the dimension of $H^1(\Gamma,\mathbb{R})$ (or equivalently the free part of $\Gamma/[\Gamma,\Gamma]$) is not very well understood for arbitrary n. The best lower bounds of the rank available in the literature seem to be what follows from the work of Millson [31] and Lubotzky [25], which gives that any arithmetic subgroup Γ (with a few restrictions when $n = 3, 7$) contains a subgroup such that the dimension of $H^1(\Gamma,\mathbb{R})$ is at least one. In certain arithmetic situations, we will be able to say more using a connection to automorphic forms.

8.3.1. *Cohomology classes associated to automorphic forms.* Recall the splitting (8.1) due to Harder of the cohomology into a cuspidal and an Eisenstein part. We give a brief overview of the description of $H^1_{\mathrm{cusp}}(\Gamma,\mathbb{R})$ in terms of automorphic forms, as in [38]. We recall the canonical isomorphism between $H^1(\Gamma,\mathbb{R})$ and the de Rham cohomology group $H^1_{\mathrm{dR}}(\Gamma\backslash\mathbb{H}^{n+1},\mathbb{R})$ consisting of 1-forms. Inside $H^1_{\mathrm{dR}}(\Gamma\backslash\mathbb{H}^{n+1},\mathbb{R})$ we define the subset of cuspidal harmonic 1-forms.

Definition 8.1. *A harmonic 1-form $\alpha = f_0dx_0 + f_1dx_1 + \cdots + f_ndx_n$ on $\Gamma\backslash\mathbb{H}^{n+1}$ is a* **cuspidal 1-form** *if*

(1) *α is rapidly decreasing at all cusps of Γ,*

(2) *for each cusp $\mathfrak{a}$ and $y \geq 0$, we have*

$$\int_{\mathcal{P}_{\mathfrak{a}}} f_{\mathfrak{a},i}(x,y)dx = 0\ , \quad i = 0, \ldots, n\ ,$$

where $\sigma_{\mathfrak{a}}^* \alpha = f_{\mathfrak{a},0}dx_0 + f_{\mathfrak{a},1}dx_1 \cdots + f_{\mathfrak{a},n}dx_n$.

We denote by $\mathrm{Har}^1_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H}^{n+1}, \mathbb{R})$ the space of harmonic cuspidal 1-forms on $\Gamma\backslash\mathbb{H}^{n+1}$. Then we have the following identification

$$\mathrm{Har}^1_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H}^{n+1}, \mathbb{R}) \cong H^1_{\mathrm{cusp}}(\Gamma, \mathbb{R}),$$

coming from [38, (2.14)]. This reduces the task of lower bounding the dimension of $H^1_{\mathrm{cusp}}(\Gamma, \mathbb{R})$ to constructing cuspidal automorphic forms. For congruence subgroups $\Gamma < \mathrm{SV}_{n-1}$, this can be achieved using certain *theta lifts* developed by Shintani [47] of GL_2 holomorphic forms of weight $(n+1)/2+1$ (for details see [38, page 21]). This gives us non-trivial examples for which Theorem 1.5 applies for any n. In the low-dimensional cases $n = 1, 2$ a lot more can be said, as we will see below.

Finally let us see explicitly how to construct a unitary characters from cuspidal automorphic forms. We let

$$\Phi : \Gamma \to H_1(\Gamma, \mathbb{Z}), \quad \gamma \mapsto \{\infty, \gamma\infty\}$$

which induces the canonical isomorphism $H_1(\Gamma, \mathbb{Z}) \cong \Gamma/[\Gamma, \Gamma]$. For $\gamma \in \Gamma$ and $\omega \in \mathrm{Har}^1_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H}^{n+1}, \mathbb{R})$, we define the *Poincaré pairing*

$$\langle \gamma, \omega \rangle := 2\pi i \int_{\Phi(\Gamma)} \omega = 2\pi i \int_P^{\gamma P} \omega \quad \text{for any } P \in \mathbb{H}^{n+1}.$$

We note that that when $n = 1$ and f is a classical Hecke cusp form of weight 2 for Γ, then $f(z)dz$ is indeed a harmonic cuspidal 1-form on $\Gamma\backslash\mathbb{H}^2$ and the Poincaré symbol is equal to (minus) the standard modular symbol (1.1):

$$\langle \gamma, f(z)dz \rangle = 2\pi i \int_\infty^{a_\gamma/c_\gamma} f(z)dz = -\langle a_\gamma/c_\gamma, f \rangle\ .$$

We observe that if $\gamma \in \Gamma$ is parabolic, then $\langle \gamma, \alpha \rangle = 0$. Hence if we define $\chi_\alpha(\gamma) := e(\langle \gamma, \alpha \rangle)$ then χ_α defines a unitary character trivial on Γ'_∞. The kernel of the map $\alpha \mapsto \chi_\alpha$ is a full rank lattice L inside $\mathrm{Har}^1_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H}^{n+1}, \mathbb{R})$. If we assume that Γ is torsion-free, we indeed obtain the identification $H^1_{\mathrm{free}}(\Gamma, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Har}^1_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H}^{n+1}, \mathbb{R})/L$.

8.3.2. *The case of* $\mathbb{H}^2$. When $n = 1$, we have explicit formulas for the dimensions of both the cuspidal and the Eisenstein part. More precisely we have coming from [52, Prop. 6.2.3] that

$$H^1_{\mathrm{cusp}}(\Gamma, \mathbb{Z}) \cong \mathbb{R}^{2g}, H^1_{\mathrm{Eis}}(\Gamma, \mathbb{R}) \cong \mathbb{R}^{2(h-1)},$$

where g is the genus and h is the number of inequivalent cusps of the Riemann surface $\Gamma\backslash\mathbb{H}^2$. In particular if $\Gamma = \Gamma_0(N)$ is a standard Hecke congruence subgroup, we know that $g \sim \frac{N \cdot \prod_{p|N}(1+p^{-1})}{12}$ and $h = \sum_{d|N} \varphi(d, N/d)$ and we conclude that we can find towers of Hecke congruence subgroups such that both the cuspidal and Eisenstein part goes to infinity.

8.3.3. *The case of* $\mathbb{H}^3$. When $n = 2$ there has been a lot of activity recently and we refer to the survey of Şengün [43] for an excellent and more thorough overview. In this case no formulas are known in general for the ranks of the cuspidal and Eisenstein part and the best one can hope for are lower bounds.

Regarding the Eisenstein part, we can describe it explicitly when Γ is torsion-free. In this case, we have that $H^1_{\mathrm{Eis}}(\Gamma, \mathbb{R}) \cong \mathbb{R}^h$, where h is the number of cusps of $\Gamma\backslash\mathbb{H}^3$, see [13, Proposition 7.5.6]. The same conclusion holds for co-finite subgroups $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_D)$, where $\mathcal{O}_D$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ with $D < 0$ a fundamental discriminant not equal to $-4, -3$ (in which case there might be torsion in Γ). In the case of co-finite subgroups $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_\mathcal{D})$ with $D = -4, -3$ the picture is much more mysterious, but a lot of numerics are available in [42] and [13, Ch. 7.5].

For the cuspidal part there are some useful results giving lower bounds on the rank. First of all Rohlfs [37] showed that

$$\dim H^1_{\mathrm{cusp}}(\mathrm{SL}_2(\mathcal{O}_D), \mathbb{R}) \geq \frac{\varphi(D)}{6} - \frac{1}{2} - h(D),$$

where $h(D)$ denotes the class number of $\mathbb{Q}(\sqrt{D})$. Furthermore Şengün and Turkelli [44] proved that if D is a fundamental discriminant such that $h(D) = 1$, p is a rational prime which is inert in $\mathbb{Q}(\sqrt{D})$ and $\Gamma_0(p^n) \subset \mathrm{SL}_2(\mathcal{O}_D)$ is a congruence subgroup, then we have

$$\dim H^1_{\mathrm{cusp}}(\Gamma_0(p^n), \mathbb{R}) \geq p^{6n},$$

as $n \to \infty$ (an upper bound of p^{10n} has been proved by Calegari and Emerton [4]). In the case of cocompact groups stronger results were obtained by Kionke and Schwermer [23].

8.4. **Torsion in the (co)homology of arithmetic groups.** Now we will discuss what is known about the torsion part of $H_1(\Gamma, \mathbb{Z})$ when $\Gamma \subset \mathrm{SO}(n+1, 1)$ is a cofinite, arithmetic subgroup. In the simplest case $n = 1$, we know that all the torsion in the abeliazation comes from the torsion in the subgroup itself and thus in particular $\Gamma/[\Gamma, \Gamma]$ is torsion-free when Γ is so.

It was noticed a long time ago in unpublished work by Grunewald and Mennicke that in the case $n = 2$ there is a lot of torsion in the abeliazation of congruence subgroups. See Şengün's work [42] for some recent extensive computations.

The study of torsion in the abelianization of Γ fits into a more general framework of understanding the torsion in the homology of arithmetic groups as in the work of Bergeron and Venkatesh [2]. Bergeron and Venkatesh have conjectured that when Γ is a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_D)$ with $D < 0$ a negative fundamental discriminant, then the torsion in $\Gamma/[\Gamma, \Gamma]$ grows exponentially with the index $[\mathrm{SL}_2(\mathcal{O}_D) : \Gamma]$.

More generally the conjectures predicts that the torsion in the cohomology of symmetric spaces associated to a semisimple Lie group G will grow exponentially in towers of congruence subgroups exactly if we consider the middle dimensional cohomology and if the *fundamental rank* (or "deficiency") $\delta(G) := \mathrm{rank}(G) - \mathrm{rank}(K)$ is 1 (here K is a maximal compact). It follows from [2, 1.2] that the fundamental rank of $\mathrm{SO}(n+1, 1)$ is equal to 1 exactly if n is even. And thus we see that we will have exponential growth of the torsion of $\Gamma/[\Gamma, \Gamma]$ when Γ runs through a tower of congruence groups exactly when $n = 2$ (corresponding to Kleinian groups).

For $n > 2$ the torsion should conjecturally *not* grow exponentially, but there might still be torsion, which is equally arithmetically interesting in view of [40]. There seems however to be no experimental or theoretical work available in this case.

References

[1] L. V. Ahlfors. Möbius transformations in $\mathbf{R}^n$ expressed through 2×2 matrices of Clifford numbers. *Complex Variables Theory Appl.*, 5(2-4):215–224, 1986.

[2] N. Bergeron and A. Venkatesh. The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu*, 12(2):391–447, 2013.

[3] S. Bettin and S. Drappeau. Limit laws for rational continued fractions and value distribution of quantum modular forms. *arXiv e-prints*, page arXiv:1903.00457, Mar 2019.

[4] F. Calegari and M. Emerton. Bounds for multiplicities of unitary representations of cohomological type in spaces of cusp forms. *Ann. of Math. (2)*, 170(3):1437–1446, 2009.

[5] F. Calegari and A. Venkatesh. A torsion Jacquet-Langlands correspondence. *Astérisque*, (409):x+226, 2019.

[6] P. Cohen and P. Sarnak. Selberg trace formula, chapters 6 and 7. http://publications.ias.edu/sarnak/section/513, 1980.

[7] Y. Colin de Verdière. Pseudo-laplaciens. II. *Ann. Inst. Fourier (Grenoble)*, 33(2):87–113, 1983.

[8] P. Constantinescu. Distribution of modular symbols in $\mathbb{H}^3$. *arXiv e-prints*, page arXiv:2005.07629, May 2020.

[9] J. E. Cremona. *Algorithms for modular elliptic curves*. Cambridge University Press, Cambridge, second edition, 1997.

[10] N. Diamantis, J. Hoffstein, E. M. Kı ral, and M. Lee. Additive twists and a conjecture by Mazur, Rubin and Stein. *J. Number Theory*, 209:1–36, 2020.

[11] J. Elstrodt, F. Grunewald, and J. Mennicke. Vahlen's group of Clifford matrices and spin-groups. *Math. Z.*, 196(3):369–390, 1987.

[12] J. Elstrodt, F. Grunewald, and J. Mennicke. Kloosterman sums for Clifford algebras and a lower bound for the positive eigenvalues of the Laplacian for congruence subgroups acting on hyperbolic spaces. *Invent. Math.*, 101(3):641–685, 1990.

[13] J. Elstrodt, F. Grunewald, and J. Mennicke. *Groups acting on hyperbolic space*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. Harmonic analysis and number theory.

[14] R. Gangolli. The length spectra of some compact manifolds of negative curvature. *J. Differential Geometry*, 12(3):403–424, 1977.

[15] R. Gangolli and G. Warner. Zeta functions of Selberg's type for some noncompact quotients of symmetric spaces of rank one. *Nagoya Math. J.*, 78:1–44, 1980.

[16] K. Gongopadhyay and R. S. Kulkarni. z-classes of isometries of the hyperbolic space. *Conform. Geom. Dyn.*, 13:91–109, 2009.

[17] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, eighth edition, 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010].

[18] G. Harder. On the cohomology of discrete arithmetically defined groups. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 129–160. 1975.

[19] D. A. Hejhal. *The Selberg trace formula for* $\mathrm{PSL}(2, \mathbf{R})$. *Vol. 2*, volume 1001 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[20] H. Iwaniec. *Spectral methods of automorphic forms*, volume 53 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, second edition, 2002.

[21] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[22] D. A. Každan. On the connection of the dual space of a group with the structure of its closed subgroups. *Funkcional. Anal. i Priložen.*, 1:71–74, 1967.

[23] S. Kionke and J. Schwermer. On the growth of the first Betti number of arithmetic hyperbolic 3-manifolds. *Groups Geom. Dyn.*, 9(2):531–565, 2015.

[24] J. Lee and H-S. Sun. Dynamics of continued fractions and distribution of modular symbols. *arXiv e-prints*, page arXiv:1902.06277, Feb 2019.

[25] A. Lubotzky. Free quotients and the first Betti number of some hyperbolic manifolds. *Transform. Groups*, 1(1-2):71–82, 1996.

[26] J. I. Manin. Parabolic points and zeta functions of modular curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:19–66, 1972.

[27] B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977. With an appendix by Mazur and M. Rapoport.

[28] B. Mazur. On the arithmetic of special values of L functions. *Invent. Math.*, 55(3):207–240, 1979.

[29] B. Mazur and K. Rubin. Relatively few rational points. Number Theory Seminar at Cal Tech. 2016.

[30] B. Mazur and K. Rubin. Arithmetic conjectures suggested by the statistical behavior of modular symbols. *arXiv e-prints*, page arXiv:1910.12798, October 2019.
[31] J. J. Millson. On the first Betti number of a constant negatively curved manifold. *Ann. of Math. (2)*, 104(2):235–247, 1976.
[32] A. C. Nordentoft. Central values of additive twists of cuspidal L-functions. *arXiv e-prints*, page arXiv:1812.08378, December 2018.
[33] Y. N. Petridis and M. S. Risager. Modular symbols have a normal distribution. *Geom. Funct. Anal.*, 14(5):1013–1043, 2004.
[34] Y. N. Petridis and M. S. Risager. The distribution of values of the Poincaré pairing for hyperbolic Riemann surfaces. *J. Reine Angew. Math.*, 579:159–173, 2005.
[35] Y. N. Petridis and M. S. Risager. Arithmetic statistics of modular symbols. *Invent. Math.*, 212(3):997–1053, 2018.
[36] R. Phillips and P. Sarnak. Geodesics in homology classes. *Duke Math. J.*, 55(2):287–297, 1987.
[37] J. Rohlfs. On the cuspidal cohomology of the Bianchi modular groups. *Math. Z.*, 188(2):253–269, 1985.
[38] P. Sarnak. On cusp forms. II. In *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, volume 3 of *Israel Math. Conf. Proc.*, pages 237–250. Weizmann, Jerusalem, 1990.
[39] T. Schneider. Arithmetische Untersuchungen elliptischer Integrale. *Math. Ann.*, 113(1):1–13, 1937.
[40] P. Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.
[41] A. Selberg. *Collected papers. I*. Springer Collected Works in Mathematics. Springer, Heidelberg, 2014. With a foreword by K. Chandrasekharan, Reprint of the 1989 edition [MR1117906].
[42] M. H. Şengün. On the integral cohomology of Bianchi groups. *Exp. Math.*, 20(4):487–505, 2011.
[43] M. H. Şengün. Arithmetic aspects of Bianchi groups. In *Computations with modular forms*, volume 6 of *Contrib. Math. Comput. Sci.*, pages 279–315. Springer, Cham, 2014.
[44] M. H. Şengün and S. Türkelli. Lower bounds on the dimension of the cohomology of Bianchi groups via Sczech cocycles. *J. Théor. Nombres Bordeaux*, 28(1):237–260, 2016.
[45] G. Shimura. On the holomorphy of certain Dirichlet series. *Proc. London Math. Soc. (3)*, 31(1):79–98, 1975.
[46] G. Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.
[47] T. Shintani. On construction of holomorphic cusp forms of half integral weight. *Nagoya Math. J.*, 58:83–126, 1975.
[48] A. Södergren. On the uniform equidistribution of closed horospheres in hyperbolic manifolds. *Proc. Lond. Math. Soc. (3)*, 105(2):225–280, 2012.
[49] E. M. Stein and R. Shakarchi. *Fourier analysis*, volume 1 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003. An introduction.
[50] H-S. Sun. A proof of the conjecture of Mazur-Rubin-Stein. *To appear in Bull. Korean Math. Soc*, 2020.
[51] Preston Wake and Carl Wang-Erickson. The rank of Mazur's Eisenstein ideal. *Duke Math. J.*, 169(1):31–115, 2020.
[52] G. Wiese. Computational arithmetic of modular forms. In *Notes from the International Autumn School on Computational Number Theory*, Tutor. Sch. Workshops Math. Sci., pages 63–170. Birkhäuser/Springer, Cham, 2019.

Department of Mathematics, University College London, 25 Gordon Street, London, UK, WC1H 0AY

Email address: petru.constantinescu.17@ucl.ac.uk

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, Copenhagen 2100, Denmark

Email address: acnordentoft@outlook.com