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Reliability Assessment and Probabilistic Optimization in Structural Design

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To accomplish great things, we must not only act, but also dream, not only plan, but also believe.

- *Anatole France*

Abstract

Research in the field of reliability based design is mainly focused on two sub-areas: 1) The computation of the probability of failure and 2) its integration in the reliability based design optimization (RBDO) loop. Typically, new methods for the computation of the probability of failure improve both accuracy and efficiency, and the methods in optimization investigate approximations that aim at decoupling or eliminating the double loop arising in probabilistic optimization. Four papers are presented in this work, representing a contribution to both sub-areas.

In the first paper, a new Second Order Reliability Method (SORM) is presented. As opposed to the most commonly used SORMs, the presented approach is not limited to hyper-parabolic approximation of the performance function at the Most Probable Point (MPP). Instead, a full quadratic fit is used leading to a better approximation of the real performance function and therefore more accurate values of the probability of failure. Furthermore, the presented formulation is free of singularities and is generally more accurate than the existing methods, even for the hyper-parabolic case. Also, the method is simpler since it does not depend on the principal curvatures at the MPP and therefore does not involve a transformation of the Hessian in a rotated space with a Gram-Schmidt orthogonalization, as does the most widely used SORMs. The method is based on an asymptotic expansion of the sum of non-central chi-square variables taken from the literature, which applied to the current case yields an easy-to-use closed form expression for the probability of failure.

The second paper focuses on the integration of the expression for the probability of failure for general quadratic function, presented in the first paper, in RBDO. One important feature of the proposed approach is that it does not involve locating the MPP. Quadratic response surface models are fitted around the deterministic solution and probabilistic constraints are constructed using the proposed expression. The optimization problem is thereafter solved in a single loop, i.e. without any MPP search. Furthermore, the method can easily handle the case of varying variance of design variables using the same efficient formulation.

In the third paper, the expressions for the probability of failure based on general quadratic limit-state functions presented in the first paper are applied for the special case of a hyper-parabola. The expression is reformulated and simplified so that the probability of failure is only a function of three statistical measures: the Cornell reliability index, the skewness and the kurtosis of the hyper-parabola. These statistical measures are functions of the First-Order Reliability Index and the curvatures at the Most Probable Point. Furthermore, analytical sensitivities with respect to mean values of random variables and deterministic variables are presented. The sensitivities can be seen as the product of the sensitivities computed using the First-Order Reliability Method and a correction factor. The proposed expressions are studied and their applicability to Reliability-based Design Optimization demonstrated.

In the last paper, an approximate and efficient reliability method is proposed. Focus is on computational efficiency as well as intuitiveness for practicing engineers, especially regarding fatigue problems where volume methods are used to estimate the conditional probability of fatigue failure given a realization of design variables. The proposed method is generalized to classical load-strength structural reliability problems as well as problems involving epistemic uncertainties. The number of function evaluations to compute the probability of failure of the design under different types of uncertainties is a priori known to be $3n + 2$ in the proposed method, where n is the number of stochastic design variables. The necessary number of function evaluations for the reliability assessment, which may correspond to expensive Finite-Element computations or physical experiments, is therefore substantially lower in the proposed approach than in Monte Carlo methods, numerical multidimensional quadrature techniques as well as Most Probable Point methods such as the First- and Second-Order Reliability methods (FORM and SORM).

Keywords: SORM, Response Surface Single Loop (RSSL), MPP-free RBDO

Sammanfattning

Forskning inom området tillförlitlighetsbaserad optimering är främst inriktad på två delområden: 1) Beräkningen av sannolikheten för brott samt 2) användningen av dessa beräkningar i den tillförlitlighetsbaserade optimering (RBDO). Typiskt resulterar nya metoder för beräkning av sannolikheten för brott i både förbättrad noggrannhet och effektivitet. Metoder inom optimering undersöker approximationer som syftar till att effektivisera probabilistisk optimering. Fyra artiklar presenteras i detta arbete, motsvarande ett bidrag till båda delområdena.

I den första artikeln, presenteras en ny andra ordningens metod (SORM) för beräkning av sannolikheten för brott. I motsats till de vanligaste SORM metoderna, är den presenterade strategin inte begränsad till en hyper-parabolisk approximation av responsfunktionen vid den s.k. Mest Sannolika Punkten för brott (MPP). Istället används en fullständig kvadratisk anpassning vilket leder till en bättre approximation av den verkliga responsfunktionen och därmed mer exakta värden på sannolikheten för brott. De presenterade uttrycken har inga singulariteter, till skillnad från andra SORMs, och är generellt sätt noggrannare även för det hyper-paraboliska fallet. Dessutom är metoden enklare, eftersom den inte är beroende av huvudkrökningarna vid MPPn och därför krävs ingen transformation av Hessianen. Metoden är baserad på en asymptotisk expansion av summan av icke-centrala chitvå fördelningar tagen från litteraturen.

Den andra artikeln fokuserar på att integrera uttrycket för sannolikheten för brott för allmänna kvadratiska funktioner som presenterades i första artikeln inom RBDO. Ett viktigt inslag i den föreslagna strategin är att lokalisering av MPPn inte är nödvändig. Kvadratiska surrogatmodeller anpassas runt den deterministiska lösningen och probabilistiska bivillkor skapas med de föreslagna uttrycken. Optimeringsproblemet löses därefter direkt, dvs. utan MPP sökning. Dessutom kan metoden enkelt hantera fallet med varierande varianser med samma effektiva formulering.

I den tredje artikeln, tillämpas uttrycken för sannolikheten för brott för generella kvadratiska funktioner som presenterades i den första artikeln på specialfallet en hyper-parabel. Uttrycket omformuleras och förenklas så att sannolikheten för brott endast är en funktion av tre statistiska mått: Cornell tillförlitlighetsindex, skevhet och kurtosis av hyper-parabeln. Dessa statistiska mått är funktioner av första ordningens tillförlitlighetsindex och krökningarna vid MPPn. Dessutom presenteras analytiska känslighetsfunktioner med avseende på medelvärden av stokastiska variabler och deterministiska variabler. Känslighetsfunktionen kan ses som en produkt av känslighetsfunktionen beräknad med hjälp av första ordningens tillförlitlighetsindex och en korrektionsfaktorn. De föreslagna uttrycken studeras och deras tillämplighet på tillförlitlighetsbaserad optimering visas.

I den sista artikeln, föreslås en approximativ och effektiv tillförlitlighetsmetod. Fokus ligger på beräkningseffektivitet samt intuition för praktiserande ingenjörer, särskilt gällande utmattningsproblem där volymsmetoder används för att uppskatta den betingade sannolikheten för utmattningsbrott givet en realisering av de stokastiska design variablerna. Den föreslagna metoden generaliseras till det klassiska last-styrka tillförlitlighetsproblemet samt problem med epistemiska osäkerheter. Antalet funktionsutvärderingar för att beräkna sannolikheten för brott i en konstruktion under olika typer av osäkerheter är a priori känd att vara $3n + 2$ i den föreslagna metoden, där n är antalet stokastiska design variabler. Det nödvändiga antalet funktionsutvärderingar för att bedöma tillförlitligheten, vilka kan motsvaras av dyra finita element simuleringar eller fysikaliska experiment, är därför väsentligt lägre i den föreslagna metoden än i Monte Carlo-metoder, numeriska multidimensionella kvadratur tekniker samt

Mest Sannolika Punkten för brott (MPP) metoder som första- och andra- ordningens tillförlitlighetsmetoder (FORM och SORM).

Preface

The work presented in this thesis has been performed between April 2011 and April 2016 at the department of Solid Mechanics at KTH Royal Institute of Technology (KTH Hållfasthetslära). The work corresponds to three and a half years of full time research. The research was financially supported by Scania CV AB. The support is gratefully acknowledged.

I would like to express my sincerest appreciation to my supervisor Professor Mårten Olsson. I deeply appreciate his constant encouragement and excellent guidance through all these years. I also owe my deepest gratitude to my colleagues and to the exemplary work environment provided at the department of Solid Mechanics.

Finally, the most special thanks go to my parents, Lobna and Sayed, and my sister Rania, to whom I owe any success.

Stockholm, April 2016

Rami Mansour

List of appended papers

Paper A
A Closed-form Second-Order Reliability Method using Noncentral Chi-Squared Distributions
Rami Mansour and Mårten Olsson
ASME J Mech Des (2014) 136(10):101402-1-101402-10

Paper B
Response surface single loop reliability-based design optimization with higher-order reliability assessment
Rami Mansour and Mårten Olsson
J Struct Multidiscip Optim (2016).

Paper C
A novel closed-form Second-Order Reliability Method with probabilistic sensitivity analysis and application in Reliability-based Design Optimization
Rami Mansour and Mårten Olsson
Report 592, Department of Solid Mechanics, Royal Institute of Technology (KTH), Stockholm, Sweden. To be submitted for international publication.

Paper D
An efficient reliability method applied to classical load-strength uncertainties, aleatory fatigue problems and epistemic uncertainties
Rami Mansour and Mårten Olsson
Report 593, Department of Solid Mechanics, Royal Institute of Technology (KTH), Stockholm, Sweden. To be submitted for international publication.

In addition to the appended papers, the work has resulted in the following publications and presentations [1]:

Probabilistic fatigue design of gas turbine compressor blades under aleatory and epistemic uncertainty
Daniel Sandberg, Rami Mansour and Mårten Olsson
Report 587, Department of Solid Mechanics, Royal Institute of Technology (KTH), Stockholm, Sweden. To be submitted for international publication. (JP)

Reliability based optimisation of structures and systems
Rami Mansour and Mårten Olsson
Presented at *The Centre for Nuclear Energy Technology (CEKERT)/Westinhouse seminar*, Westinghouse Electric Sweden AB, Västerås, Sverige, 2015. (OP)

A direct method for optimal design under uncertainty
Rami Mansour and Mårten Olsson
Presented at *The Swedish Fatigue Network*(UTMIS), Trollhättan, Sweden, 2014. (OP)

A Single loop Method for Reliability Based Design Optimization
Rami Mansour and Mårten Olsson
Presented at *The 4th International Conference on Engineering Optimization* (EngOpt), Lisbon, Portugal, 2014. (EA,OP)

A novel closed-form Second-Order Reliability Method with probabilistic sensitivities and its application to Reliability-Based Design Optimization
Rami Mansour and Mårten Olsson
Presented at *International Conference on Engineering and Applied Sciences Optimization* (Opt-i), Kos Island, Greece, 2014. (EA,OP)

A new Second Order Reliability Method and its integration in Reliability Based Design Optimization
Rami Mansour and Mårten Olsson
Presented at 10th *World Congress on Structural and Multidisciplinary Optimization* (WCSMO), Orlando, Florida, 2013. (EA,OP,PP)

Reliability-based Design Optimization for structural problems
Rami Mansour and Mårten Olsson
Presented at 22nd *Swedish Mechanics conference, Svenska Mekanik Dagar* (SMD), Gothenburg, Sweden, 2011. (EA,OP)

[1] EA = Extended abstract, OP = Oral Presentation, PP = Proceeding paper, JP = Journal Paper.

The contribution of the authors to the appended paper is as follows:

Paper A
The conceptual idea was developed by Rami Mansour together with supervisor Mårten Olsson. Rami Mansour performed literature survey, mathematical derivation, numerical computations and major part of manuscript writing. Interpretation and checking of results as well as formulating the method for applications together with supervisor.

Paper B
During the work on the previous paper, both Rami Mansour and supervisor Mårten Olsson contributed to the idea of this paper. Rami Mansour performed numerical analysis and the major part of manuscript writing. Interpretation of results, advantages, limitations and industrial applicability of proposed method together with Mårten Olsson.

Paper C
The conceptual idea mainly developed by Rami Mansour under supervision of Mårten Olsson. Rami Mansour performed numerical analysis and major part of manuscript writing. Finalization of paper with Mårten Olsson.

Paper D
The conceptual idea mainly formulated by Mårten Olsson and further developed by Rami Mansour. The numerical algorithms and the major part of the manuscript were written by Rami Mansour. The finalization of the paper in cooperation with Mårten Olsson.

Contents

Introduction

Research in the field of reliability based design is mainly focused on two sub-areas: the computation of the probability of failure and its integration in the reliability-based design optimization (RBDO) loop. Typically, new methods for the computation of the probability of failure improve both accuracy and efficiency, and the methods in optimization investigate approximations that aim at decoupling or eliminating the double loop arising in probabilistic optimization. This work represents a contribution to each sub-area.

Probability of failure

In typical industrial applications, a performance function can be defined by the difference between a computed value, for instance the maximum stress in a structure, and an allowed value. When the performance function is equal to zero, it defines the limit-state, and a negative value corresponds to failure. When the performance function depends on stochastic variables, the problem of accurately and effectively computing the probability of failure arises. In applications involving reliability computations such as Reliability-based Design Optimization (RBDO), one differs between deterministic design variables $\mathbf{d}$, random design variables $\mathbf{x}$ and random design parameters $\mathbf{p}$. Denoting the vector of random design variables by $\mathbf{v} = \begin{bmatrix} \mathbf{x}^T & \mathbf{p}^T \end{bmatrix}^T$ and the vector of distribution parameters by $\boldsymbol{\theta}_{\mathbf{v}} = \begin{bmatrix} \boldsymbol{\theta}_{\mathbf{x}}{}^T & \boldsymbol{\theta}_{\mathbf{p}}{}^T \end{bmatrix}^T$, the limit-state function can be written as

$$g(\mathbf{d}, \mathbf{v}) = 0. \tag{1}$$

The probability of failure is then given by

$$P_{\mathrm{f}} = \Pr\left[g\left(\mathbf{d},\mathbf{v}\right) < 0\right] = \int\limits_{g(\mathbf{d},\mathbf{v})<0} f_{\mathbf{V}}\left(\mathbf{v},\boldsymbol{\theta}_{\mathbf{v}}\right) d\mathbf{v}, \tag{2}$$

where $f_{\mathbf{V}}\left(\mathbf{v},\boldsymbol{\theta}_{\mathbf{v}}\right)$ is the combined probability density function. Typically, the distribution parameters $\boldsymbol{\theta}_{\mathbf{x}}$ and $\boldsymbol{\theta}_{\mathbf{p}}$ are the mean values and standard deviations of $\mathbf{x}$ and $\mathbf{p}$, respectively. However, P_{f} cannot be computed exactly for the general case, since it necessitates the evaluation of a complicated multidimensional integral. Instead, the First-Order Reliability Method (FORM) and the Second-Order Reliability Method (SORM) aim at developing approximate expressions for P_{f}, or equivalently to approximate the generalized Reliability Index defined as

$$\beta = -\Phi^{-1}\left(P_{\mathrm{f}}\right). \tag{3}$$

For demonstration purpose, assume a problem in structural engineering where the probability of the stress $\sigma\left(\mathbf{x}\right)$ exceeding a maximum allowed value σ_{all} is to be computed. A choice of limit-state function is $g\left(\mathbf{x}\right) = \sigma_{\mathrm{all}} - \sigma\left(\mathbf{x}\right)$, where $g < 0$ represents failure. Assume further that the problem has two normally distributed random design variables x_1 and x_2 with mean values μ_{x_1} and μ_{x_1}, and standard deviations σ_{x_1} and σ_{x_2}, respectively. The integration region $g < 0$ in Eqn. (2) is visualized in Fig. 1, as the blue-colored region separated from the safe set by the non-linear limit-state function in red. The level curves of the integrand, i.e. the combined probability density function, are also shown in the figure. Observe the non-circular shape of the level curves indicating that $\sigma_{x_2} > \sigma_{x_1}$.

First-Order Reliability Method

In the First-Order Reliability Method (FORM), the probability of failure is approximated by the one computed using a linearised limit-state function in the normal standardized space $\mathbf{u}$ [12] at the so called Most Probable Point (MPP), i.e,

$$0 = g_{\mathbf{u}}\left(\mathbf{u}\right) \approx g_{\mathbf{u}}^{\mathrm{lin}}(\mathbf{u}) = \boldsymbol{\nabla} g_{\mathbf{u}}(\mathbf{u}_{\mathrm{MPP}})^{T}\left(\mathbf{u} - \mathbf{u}_{\mathrm{MPP}}\right), \tag{4}$$

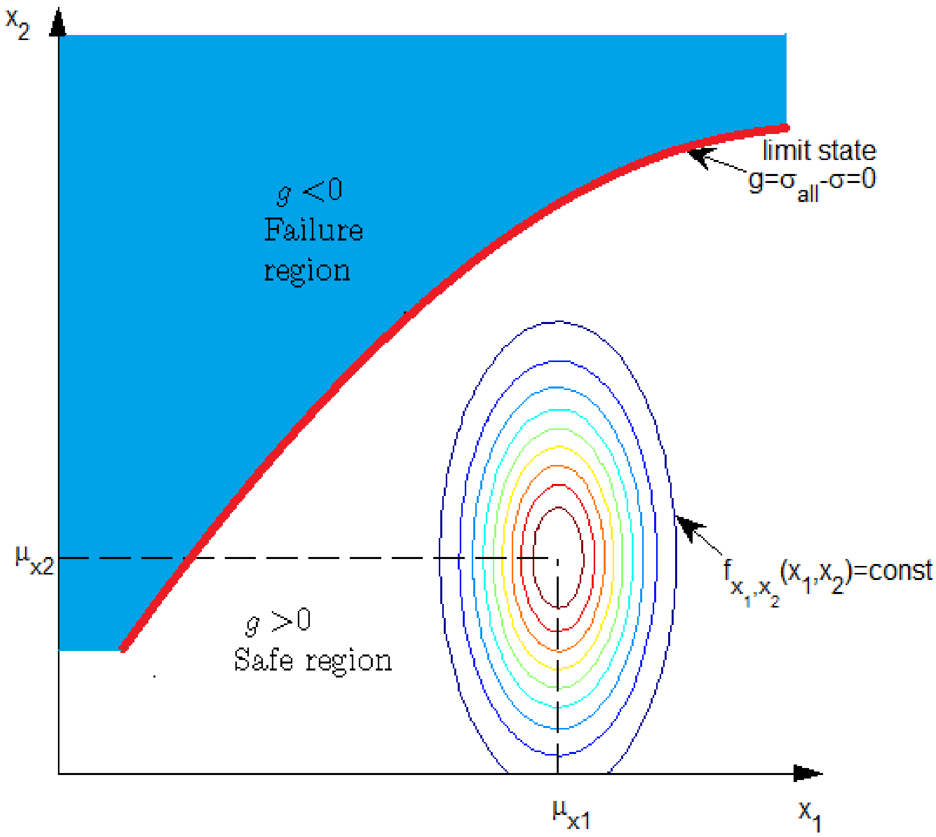

Figure 1: Visualization of the integration region (blue) for the computation of the probability of failure, and the combined probability density function in **x**-plane.

where $\nabla g_{\mathbf{u}}(\mathbf{u}_{\mathrm{MPP}})$ is the gradient vector at the MPP. As can be seen from Fig. 2, the MPP, also called the design point, is the point on the limit-state corresponding to the maximum probability density in the normal standardized space $\mathbf{u}$ [7]. This point is also the minimum distance from the origin to the limit-state and is therefore found by a minimization [5] problem according to

$$\begin{cases} \beta_{\mathrm{F}} = \min\limits_{\mathbf{u}} \left(\mathbf{u}^T\mathbf{u}\right)^{1/2} \\ \text{s.t. } g_{\mathbf{u}}(\mathbf{u}) = 0 \end{cases}. \tag{5}$$

In a rotationally transformed space $\mathbf{y}$ where the y_n-axis coincides with the vector from the origin to the MPP, the linearised limit-state function can simply be written as

$$g_{\mathbf{y}}^{\mathrm{lin}} = -y_n + \beta_{\mathrm{F}}. \tag{6}$$

The probability $\text{Prob}[g_{\mathbf{y}}^{\text{lin}} < 0]$ can be computed exactly according to

$$P_{f,\text{FORM}} = \Phi\left(-\beta_{\text{F}}\right). \tag{7}$$

The first-order approximation according to Eqn. (6) may be inaccurate for cases of high non-linearities in $g_{\mathbf{u}}$. For these cases Eqn. (7) is therefore inaccurate, and second-order approximations have been developed.

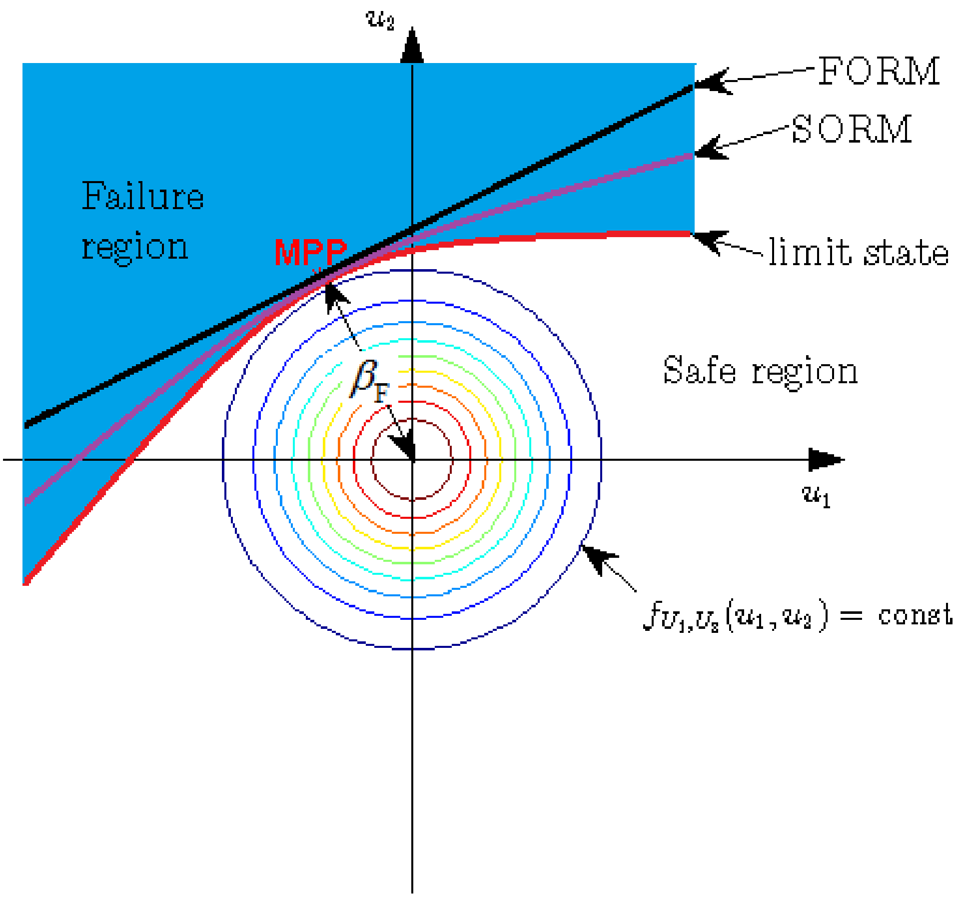

Figure 2: Visualization of the probability of failure and the combined probability density function in the $\mathbf{u}$-plane, as well as the First-Order Reliability Index.

Second-Order Reliability Methods

In the Second Order Reliability Method (SORM), the failure function is approximated by a second order function at the MPP according to

$$0 = g_{\mathbf{u}}(\mathbf{u}) \approx g_{\mathbf{u}}^{\text{quad}}(\mathbf{u}) = \nabla g_{\mathbf{u}}(\mathbf{u}_{\text{MPP}})^T(\mathbf{u}-\mathbf{u}_{\text{MPP}}) + \frac{1}{2}(\mathbf{u}-\mathbf{u}_{\text{MPP}})^T \nabla^2 g_{\mathbf{u}}(\mathbf{u}_{\text{MPP}})(\mathbf{u}-\mathbf{u}_{\text{MPP}}) \tag{8}$$

where $\nabla g_{\mathbf{u}}(\mathbf{u}_{\text{MPP}})$ is the gradient vector at the MPP and the symmetric matrix $\nabla^2 g_{\mathbf{u}}(\mathbf{u}_{\text{MPP}})$ is the matrix of second-order partial derivatives. An orthogonal transformation $\mathbf{y} = \mathbf{Hu}$ aiming at rotating the coordinate system to make the y_n axis coincide with the vector from the origin to the MPP is thereafter performed. Since the gradient vector and the vector from the origin to the MPP are proportional, i.e $\mathbf{u}_{\text{MPP}} = \beta_{\text{F}}\boldsymbol{\alpha}_{\text{MPP}}$, where $\boldsymbol{\alpha}_{\text{MPP}}$ is the unit directional vector to $\mathbf{u}_{\text{MPP}}$, the n-th row in $\mathbf{H}$ is equal to $\boldsymbol{\alpha}_{\text{MPP}}$. The failure function in the rotationally transformed space can be written as

$$0 = g_{\mathbf{y}}(\mathbf{y}) \approx -y_n + \beta_{\text{F}} + \frac{1}{2}(\mathbf{y}-\mathbf{y}_{\text{MPP}})^T \mathbf{M}(\mathbf{y}-\mathbf{y}_{\text{MPP}}), \tag{9}$$

where $(\mathbf{y}-\mathbf{y}_{\text{MPP}}) = (y_1, y_2, ..., y_n - \beta_{\text{F}})$ and the matrix $\mathbf{M}$ of second-order partial derivatives in the rotationally transformed space have been introduced. Partitioning according to [7]

$$\mathbf{M} = \begin{bmatrix} \tilde{\mathbf{M}} & \tilde{\mathbf{M}}_{1n} \\ \tilde{\mathbf{M}}_{1n}^T & \lambda \end{bmatrix} \tag{10}$$

where λ is a scalar and $\tilde{\mathbf{M}} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\tilde{\mathbf{M}}_{1n} \in \mathbb{R}^{(n-1)\times 1}$ yields with $\mathbf{y}^T = [\tilde{\mathbf{y}}^T, y_n]$

$$0 = g_{\mathbf{y}}(\mathbf{y}) \approx -y_n + \beta_{\text{F}} + \frac{1}{2}\tilde{\mathbf{y}}^T\tilde{\mathbf{M}}\tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T\tilde{\mathbf{M}}_{1n}(y_n - \beta_{\text{F}}) + \frac{1}{2}\lambda(y_n-\beta_{\text{F}})^2. \tag{11}$$

The probability content of Eqn. (9) and Eqn. (11) is complicated and is not applied in the most widely used SORM approaches. Therefore, traditionally Eqn. (11) has been replaced by

$$0 = g_{\mathbf{y}}(\mathbf{y}) \approx -y_n + \beta_{\text{F}} + \frac{1}{2}\tilde{\mathbf{y}}^T\tilde{\mathbf{M}}\tilde{\mathbf{y}}. \tag{12}$$

for which approximate closed form expressions for the probability of failure have been developed. According to [17] the inaccuracy resulting from replacing Eqn. (11) by Eqn. (12) has not been adequately investigated which may be considered as one of the main reasons why SORM has not been used extensively up to now. Equation (12) is thereafter rewritten as

$$y_n = \beta_{\mathrm{F}} + \frac{1}{2}\tilde{\mathbf{y}}^T\tilde{\mathbf{M}}\tilde{\mathbf{y}} \tag{13}$$

which after a further orthogonal transformation of the first $n-1$ variables $\tilde{\mathbf{y}}$ to $\mathbf{y}'$ brings y_n into diagonalized form

$$y_n = \beta_{\mathrm{F}} + \frac{1}{2}\sum_{i=1}^{n-1} k_i {y'_i}^2. \tag{14}$$

The principal curvatures $(k_1, k_2, ..., k_{n-1})$ at the MPP are therefore given by the eigenvalues of $\tilde{\mathbf{M}}$. It is important to note that, if the last two terms in Eqn. (11) are retained, diagonalizing $\tilde{\mathbf{M}}$ would yield

$$0 = g_{\mathbf{y}}(\mathbf{y}) \approx -y_n + \beta_{\mathrm{F}} + \frac{1}{2}\sum_{i=1}^{n-1} k_i {y'_i}^2 + (y_n - \beta_{\mathrm{F}})\sum_{i=1}^{n-1}\gamma_i {y'_i} + \frac{1}{2}\lambda(y_n - \beta_{\mathrm{F}})^2, \tag{15}$$

where γ_i is the i:th component of the product of $\tilde{\mathbf{M}}_{1n}^T$ and the orthogonal transformation matrix $(\tilde{\mathbf{y}} \rightarrow \mathbf{y}')$. This implies that in most SORMs, the general quadratic fit (15) is approximated by the parabolic expression (14). Using expression (14), Breitung [2] and Tvedt [14] proposed asymptotic approximation of the probability of failure. Both expressions are not valid for $\beta_{\mathrm{F}} k_i \leq -1$. Other expressions have been proposed by [3] and [6]. In general, these methods do not work well for negative or small curvature radii and neither for a large number of random variables [7, 17]. The approach proposed by [17] has been shown to be accurate for negative and small curvature radii as well as a large number of random variables. However, the method is based on a mean value of all curvature radii and therefore fails to satisfactory compute the probability of failure for curvature of different signs [10, 17].

Reliability-based Design Optimization

Description of the problem

Reliability-based design optimization (RBDO) is an optimization problem aiming at locating the optimal design with the variations of the design variables and parameters being considered [15]. The RBDO problem can typically be formulated [13] according to

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}}} C(\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}},\boldsymbol{\mu}_{\mathbf{p}}) \\ &\text{s.t Prob}\left[g_i\left(\mathbf{d},\mathbf{x},\mathbf{p}\right)<0\right] \leq P_{\text{f}i,\text{all}}, \quad i=1,...,n \\ &\mathbf{d}^{\text{Lower}} \leq \mathbf{d} \leq \mathbf{d}^{\text{Upper}},\ \boldsymbol{\mu}_{\mathbf{x}}^{\text{Lower}} \leq \boldsymbol{\mu}_{\mathbf{x}} \leq \boldsymbol{\mu}_{\mathbf{x}}^{\text{Upper}} \end{aligned} \tag{16}$$

where $\mathbf{d}$, $\mathbf{x}$, $\mathbf{p}$, $\boldsymbol{\mu}_{\mathbf{x}}$ and $\boldsymbol{\mu}_{\mathbf{p}}$ are the vectors of deterministic design variables, random design variables, random parameters, mean value of $\mathbf{x}$ and mean value of $\mathbf{p}$, respectively. The objective function C is to be minimized while the probability of satisfying the ith deterministic constraint g_i should be larger than the desired design reliability $r_{\text{d}i}$. The probability of failure is then $P_{\text{f}i} = 1 - r_{\text{d}i}$. The optimal design should be within the region given by the lower and upper bound on design variables, denoted with a superscript "Lower" and "Upper". The relations for the limits of the design region should be interpreted for each component of the vectors. The standard deviation of design variables $\boldsymbol{\sigma}_{\mathbf{x}}$ can either be constant or be allowed to vary. The RBDO problem according to Eqn. (16) can alternatively be formulated as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}}} C(\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}},\boldsymbol{\mu}_{\mathbf{p}}) \\ &\text{s.t } \beta_i\left(\mathbf{d},\mathbf{x},\mathbf{p}\right) - \beta_{\text{d}i} > 0, \quad i=1,...,n, \\ &\mathbf{d}^{\text{Lower}} \leq \mathbf{d} \leq \mathbf{d}^{\text{Upper}},\ \boldsymbol{\mu}_{\mathbf{x}}^{\text{Lower}} \leq \boldsymbol{\mu}_{\mathbf{x}} \leq \boldsymbol{\mu}_{\mathbf{x}}^{\text{Upper}} \end{aligned}, \tag{17}$$

where β_i and $\beta_{\text{d}i}$ are the generalized reliability index, and the desired generalized reliability index, respectively, given by

$$\begin{cases} \beta_i\left(\mathbf{d},\mathbf{x},\mathbf{p}\right) = -\Phi^{-1}\left(\text{Prob}\left[g_i\left(\mathbf{d},\mathbf{x},\mathbf{p}\right)<0\right]\right) \\ \beta_{\text{d}i} = \Phi^{-1}\left(r_{\text{d}i}\right) \end{cases}. \tag{18}$$

Please observe that in the formulation of Eqn. (17) a simple map of the probabilistic constraints has been performed according to Eqn. (18) and that this step does not involve any approximation.

Double loop method

The double loop method in RBDO is the most direct approach but also numerically expensive, since the structural reliability is estimated for each set of design variables evaluated by the optimization algorithm, see Fig. 3. In FORM and SORM, the reliability assessment is in itself an optimization problem. Therefore, the RBDO problem consists of two nested loops; an outer loop for the minimization of the objective function and an inner loop for evaluating the constraints, i.e. for the reliability assessment. Typically, the number of function evaluations using a double loop optimization is large.

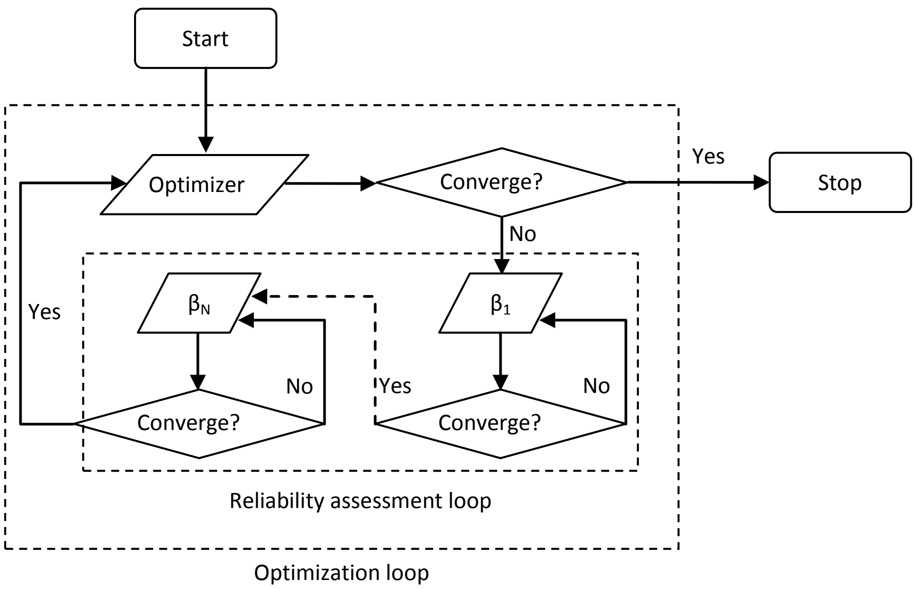

Figure 3: Flowchart of a double-loop RBDO method

Decoupling method

In order to avoid the expensive double loop RBDO, a decoupled approach aiming at separating the reliability assessment from the outer optimization loop is used. One method is to perform a linear sequential optimization loop where a linear approximation of the reliability index is constructed using information on sensitivities, i.e.

$$\beta_{\mathrm{F}}\left(\boldsymbol{\mu}_{\mathbf{x}}\right)=\beta_{\mathrm{F}}\left(\boldsymbol{\mu}_{\mathbf{x}}{}^{(k)}\right)+\left.\nabla_{\boldsymbol{\mu}_{\mathbf{x}}}\beta_{\mathrm{F}}\right|_{\boldsymbol{\mu}_{\mathbf{x}}=\boldsymbol{\mu}_{\mathbf{x}}{}^{(k)}}\left(\boldsymbol{\mu}_{\mathbf{x}}-\boldsymbol{\mu}_{\mathbf{x}}{}^{(k)}\right), \tag{19}$$

where $\boldsymbol{\mu}_{\mathbf{x}}^{(k)}$ is the k-th candidate optimal solution. A new candidate solution $\boldsymbol{\mu}_{\mathbf{x}}^{(k+1)}$is found based on the linear approximation of the reliability index Eqn. (19) which uses the sensitivities from the previous solution $\boldsymbol{\mu}_{\mathbf{x}}^{(k)}$. The process is repeated until convergence, see Fig. 4. The major difference between the double loop and the decoupled loop is that the reliabilities do not need to be computed for each set of design variables evaluated by the optimization algorithm; instead they are only updated after the convergence of the linear optimization problem.

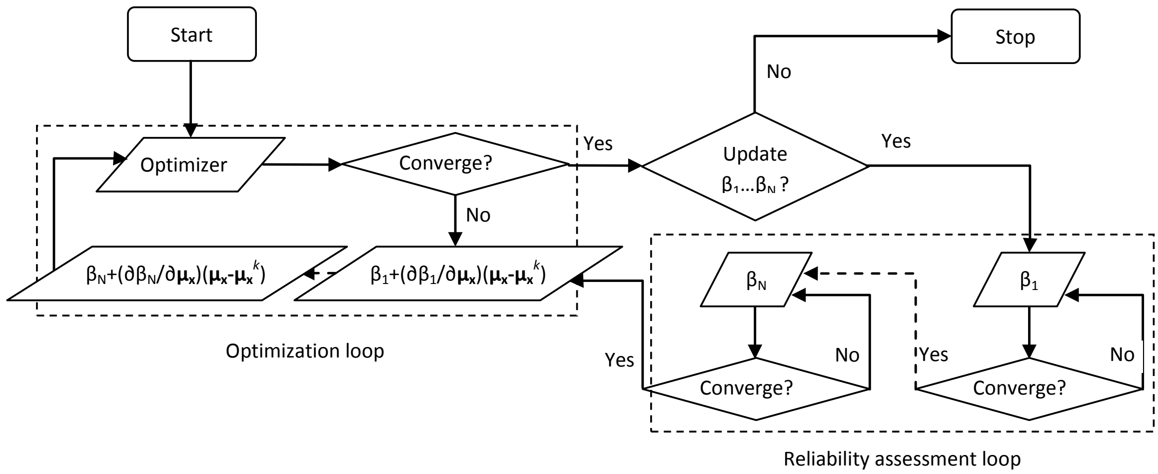

Figure 4: Flowchart of a decoupled-loop RBDO method

Another decoupled loop approach is the Sequential Optimization and Reliability Assessment (SORA) method [4], which approximates the probabilistic constraints by shifting the deterministic constraints towards the feasible region. An approximate optimal design is then found and the process is repeated until convergence. The SORA method can be formulated as

$$\begin{cases}\min\limits_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}}} f\left(\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}},\boldsymbol{\mu}_{\mathbf{p}}\right)\\ g_i\left(\mathbf{d},\mathbf{u}_{\mathbf{x}}-\mathbf{s}_i{}^{(k+1)},\mathbf{p}_{i\mathrm{iMPP}}{}^{(k)}\right)\geq 0, \quad i=1,...,n\\ \mathbf{d}^{\mathrm{Lower}}\leq\mathbf{d}\leq\mathbf{d}^{\mathrm{Upper}},\ \boldsymbol{\mu}_{\mathbf{x}}{}^{\mathrm{Lower}}\leq\boldsymbol{\mu}_{\mathbf{x}}\leq\boldsymbol{\mu}_{\mathbf{x}}{}^{\mathrm{Upper}}\end{cases} \tag{20}$$

where $\mathbf{s}_i{}^{(k+1)} = \boldsymbol{\mu}_\mathbf{X}{}^{(k)} - \boldsymbol{x}_{i\mathrm{iMPP}}{}^{(k)}$ and subscript iMPP designates the inverse MPP found by solving the optimization problem

$$\begin{cases} \mathbf{u}_{\mathrm{iMPP}} = \arg\min_{\mathbf{u}} g_\mathbf{u}(\mathbf{d},\mathbf{u}) \\ \mathrm{s.t} \ \ ||\mathbf{u}|| = \beta_\mathrm{d} \end{cases}, \tag{21}$$

where β_d is the desired reliability index. As can be seen, in cycle (k), the deterministic constraints are shifted towards the feasible region using the shifting vector $\mathbf{s}_i{}^{(k+1)}$.

Single loop method

In a single loop RBDO, the inner reliability assessment in the double loop method is avoided. Ideally, each probabilistic constraint is approximated by a function g^* of the design variables thus reducing RBDO to a deterministic optimization problem, see Fig. 5. The concept of

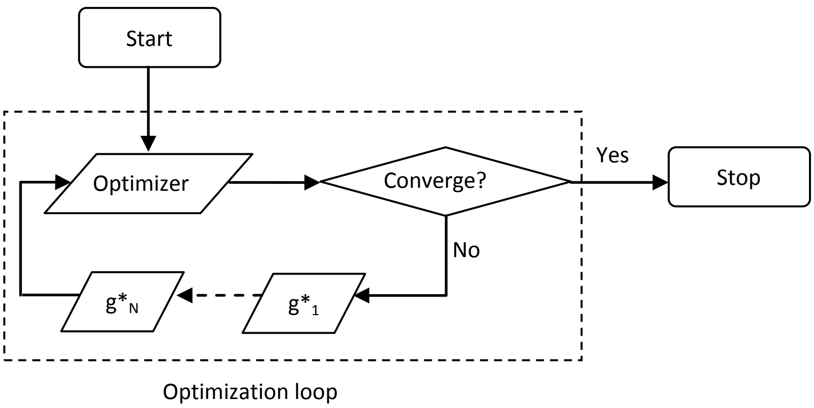

Figure 5: Flowchart of a single-loop RBDO method

reliable design space (RDS) presented by [13], is a complete single loop method as described in the flowchart. Another single-loop method which has been shown to perform well [1] regarding efficiency and convergence is the Single-Loop Approach (SLA) [8] formulated as

$$\begin{cases} \min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}) \\ \mathrm{s.t} \ \mathrm{g}_i(\mathbf{d},\mathbf{X}_i,\mathbf{P}_i) \geq 0, \quad i = 1,...,n \end{cases}, \tag{22}$$

where

$$\begin{cases} \mathbf{X}_i = \boldsymbol{\mu}_\mathbf{x} - \boldsymbol{\sigma}_\mathbf{x} \circ \beta_{\mathrm{d}i} \boldsymbol{\alpha}_{\mathbf{x}i}; \quad \mathbf{P}_i = \boldsymbol{\mu}_\mathbf{p} - \boldsymbol{\sigma}_\mathbf{p} \circ \beta_{\mathrm{d}i} \boldsymbol{\alpha}_{\mathbf{p}i}; \\ \boldsymbol{\alpha}_{\mathbf{x}i} = \boldsymbol{\sigma}_\mathbf{x} \circ \dfrac{\nabla_\mathbf{x} g_i\left(\mathbf{d}, \mathbf{x}_i, \mathbf{p}_i\right)}{\left\|\nabla_\mathbf{x} g_i\left(\mathbf{d}, \mathbf{x}_i, \mathbf{p}_i\right)\right\|}; \quad \boldsymbol{\alpha}_{\mathbf{p}i} = \boldsymbol{\sigma}_\mathbf{p} \circ \dfrac{\nabla_\mathbf{p} g_i\left(\mathbf{d}, \mathbf{x}_i, \mathbf{p}_i\right)}{\left\|\nabla_\mathbf{p} g_i\left(\mathbf{d}, \mathbf{x}_i, \mathbf{p}_i\right)\right\|}; \end{cases} \tag{23}$$

and $\circ$ denotes component-wise multiplication. The method uses the Karush-Kuhn Tucker (KKT) optimality condition of the inverse FORM approximation in the reliability loop in order to relate $\boldsymbol{\mu}_\mathbf{x}$ and $\boldsymbol{\mu}_\mathbf{p}$ to $\mathbf{X}$ and $\mathbf{P}$, respectively, according to Eqn. 23. As can be seen, the inner reliability assessment in the double loop method is avoided and the method has efficiency comparable to a deterministic optimization problem.

Proposed new methods

Methods for reliability assessment

Proposed new Second-Order Reliability Methods

As has been shown in the previous section, new methods which not only yield accurate results for the case of a hyper-paraboloid failure surface but also for the case of a general quadratic function are needed. The Second-Order Reliability Methods presented in this work attempt to fulfil both requirements.

In the proposed approach, the expression for the quadratic limit-state function in $\mathbf{u}$-space according to Eqn. (8) is rewritten as

$$g_\mathbf{u}^{\mathrm{quad}}\left(\mathbf{u}\right) = \mathbf{u}^T \mathbf{A} \mathbf{u} + \mathbf{b}^T \mathbf{u} + c \tag{24}$$

where

$$\begin{cases} \mathbf{A} = \dfrac{1}{2} \nabla^2 g(\mathbf{u}_{\mathrm{MPP}}) \\ \mathbf{b} = \nabla g(\mathbf{u}_{\mathrm{MPP}}) - \nabla^2 g(\mathbf{u}_{\mathrm{MPP}}) \mathbf{u}_{\mathrm{MPP}} \\ c = \dfrac{1}{2} {\mathbf{u}_{\mathrm{MPP}}}^T \nabla^2 g(\mathbf{u}_{\mathrm{MPP}}) \mathbf{u}_{\mathrm{MPP}} - \nabla g(\mathbf{u}_{\mathrm{MPP}})^T \mathbf{u}_{\mathrm{MPP}} \end{cases} . \tag{25}$$

The proposed expressions for the probability of failure [10] depend on the sign of the eigenvalues of the symmetric matrix $\mathbf{A}$. For the case where the eigenvalues $\overline{A}_{jj}$ have different signs, as well as for the case where an eigenvalue is equal to zero, the probability of failure is approximated by the closed-form expression

$$P_{\mathrm{f}} = \Phi(\kappa) - \varphi(\kappa) \left[\begin{array}{l} \dfrac{\sqrt{2}H_2(\kappa)}{3} \dfrac{m_3}{m_2\sqrt{m_2}} \\ + \dfrac{H_5(\kappa)}{9} \dfrac{{m_3}^2}{{m_2}^3} + \dfrac{H_3(\kappa)}{2} \dfrac{m_4}{{m_2}^2} \end{array} \right], \tag{26}$$

where H_i is the i:th probabilists' Hermite polynomial, φ and Φ are the standard normal probability and cumulative probability density function, respectively, and

$$\begin{cases} \kappa = -\dfrac{c + \sum\limits_j \overline{A}_{jj}}{\sqrt{\sum\limits_j \left(2\overline{A}_{jj}^2 + \overline{b}_j^2 \right)}} \\ m_r = \sum\limits_j \left(\overline{A}_{jj}^r + \dfrac{r}{4} \overline{A}_{jj}^{r-2} \overline{b}_j^2 \right), \quad r = 2, 3, 4 \ . \\ \overline{\mathbf{A}} = \mathbf{P}^T \mathbf{A} \mathbf{P}, \\ \overline{\mathbf{b}}^T = \mathbf{b}^T \mathbf{P} \end{cases} \tag{27}$$

Here $\mathbf{P}$ is an orthogonal matrix whose columns are the normalized eigenvectors of $\mathbf{A}$. For the case where all eigenvalues $\overline{A}_{jj}$ have the same sign, the probability of failure is approximated by

$$P_{\mathrm{f}} = \begin{cases} P, \ \mathrm{sign}\left(\overline{A}_{jj}\right) \cdot h > 0 \\ 1 - P, \ \mathrm{sign}\left(\overline{A}_{jj}\right) \cdot h < 0 \end{cases}, \tag{28}$$

where

$$P = \Phi\left(\kappa'\right) - \varphi\left(\kappa'\right) \left[\begin{array}{l} H_3(\kappa') \left(\begin{array}{l} \dfrac{{m_4}^2}{2{m_2}^2} - \dfrac{20{m_3}^2}{27{m_2}^3} \\ + \dfrac{2m_3}{9m_1m_2} \end{array} \right) \\ + H_1(\kappa') \left(-\dfrac{2{m_3}^2}{3{m_2}^3} + \dfrac{2m_3}{3m_1m_2} \right) \end{array} \right], \tag{29}$$

and

$$\begin{cases} h = 1 - \dfrac{2m_1 m_3}{3{m_2}^2} \\ \kappa' = \dfrac{|m_1|}{\sqrt{2h^2 m_2}} \left[\left(\left| \dfrac{q_0}{m_1} \right| \right)^h - 1 - \dfrac{h\,(h-1)\,m_2}{{m_1}^2} \right] \\ q_0 = \sum\limits_j \dfrac{\bar{b}_j^2}{4\bar{A}_{jj}} - c \end{cases}. \tag{30}$$

If the approximation according to Eqn. (14) is performed, the probability of failure can be written as a function of the Cornell [9] reliability index β_{C} defined as the ratio of the mean value μ_g and standard deviation σ_g of g, as well as skewness λ and kurtosis γ according to

$$P_{\mathrm{f}} = \Phi\left(-\beta_{\mathrm{C}}\right) - \varphi\left(-\beta_{\mathrm{C}}\right) \begin{bmatrix} \dfrac{H_2\left(-\beta_{\mathrm{C}}\right)}{6}\lambda + \dfrac{H_5\left(-\beta_{\mathrm{C}}\right)}{72}\lambda^2 \\ + \dfrac{H_3\left(-\beta_{\mathrm{C}}\right)}{24}\gamma \end{bmatrix}, \tag{31}$$

where

$$\begin{cases} \beta_{\mathrm{C}} = \dfrac{\mu_g}{\sigma_g} = \dfrac{\frac{1}{2}\sum\limits_{j=1}^{n-1} k_j + \beta_{\mathrm{F}}}{\sqrt{\frac{1}{2}\sum\limits_{j=1}^{n-1} {k_j}^2 + 1}} \\ \lambda = \dfrac{\sum\limits_{j=1}^{n-1} {k_j}^3}{\left(\frac{1}{2}\sum\limits_{j=1}^{n-1} {k_j}^2 + 1\right)^{3/2}} \\ \gamma = \dfrac{3\sum\limits_{j=1}^{n-1} {k_j}^4}{\left(\frac{1}{2}\sum\limits_{j=1}^{n-1} {k_j}^2 + 1\right)^{2}} \end{cases}. \tag{32}$$

The probability of failure according to Eqn. (31) is therefore a function of the Cornell Reliability Index, the skewness and the kurtosis of Eqn. (14); and these three statistical measures are functions of the First-Order Reliability Index and the curvatures at the MPP.

Proposed efficient reliability method

An efficient reliability assessment approach which is not based on the concept of MPP is also presented in this work. The proposed method necessitates only $3n + 2$ function evaluations of the performance function, where n is the number of stochastic variables, and is therefore suitable for problems where each function evaluation is computationally expensive. The probability of failure can be written as

$$P_{\mathrm{f}} = \int_{-\infty}^{\infty} P_{\mathrm{f,base}}\left(\mathbf{v}\right) f_{\mathbf{V}}\left(\mathbf{v}\right) d\mathbf{v}, \tag{33}$$

where a base probability of failure $P_{\mathrm{f,base}}$ is defined, which takes on different forms depending on the problem and the considered type of uncertainty. For the classical load-strength reliability problem [9] where the load $L\left(\mathbf{v}\right)$ is a function of random variables $\mathbf{v}$ and the strength S has a cumulative distribution function $F_S(S)$, the base probability of failure in Eqn. (33) is given by

$$P_{\mathrm{f,base}}\left(\mathbf{v}\right) = F_S\left(L\left(\mathbf{v}\right)\right). \tag{34}$$

For problems involving a fatigue probability of failure computed using a volume method such as the weakest link method [16], the base probability of failure is given by

$$P_{\mathrm{f,base}}\left(\mathbf{v}\right) = P_{\mathrm{f,WL}}\left(s_{\mathrm{eq}}\left(\mathbf{v}\right)\right), \tag{35}$$

where $P_{\mathrm{f,WL}}\left(s_{\mathrm{eq}}\left(\mathbf{v}\right)\right)$ is the Weakest Link (WL) fatigue probability of failure for a given realization of the variables $\mathbf{v}$ and $s_{\mathrm{eq}}\left(\mathbf{v}\right)$ is the equivalent stress. In the present approach, a surrogate model of the load $L\left(\mathbf{v}\right)$ in Eqn. (34) and the equivalent stress $s_{\mathrm{eq}}\left(\mathbf{v}\right)$ in Eqn. (35) which ensure proper evaluation of the integral in Eqn. (33), is proposed. The advantage of the method lies in its efficiency compared to MPP methods, as well as in its simplicity.

Reliability-based Design Optimization

The formulas for the probability of failure proposed in the previous subsection can be applied in Reliability-based Design Optimization (RBDO). The first approach is an efficient Response Surface Single-Loop (RSSL) [11] method presented in Paper B, which uses Eqn. (26) and Eqn. (28) to formulate the probabilistic contraints g^{*}. The RSSL method is capable of handling both constant and varying design variance. A flowchart of the method is presented in Fig. 6. As can be seen, the deterministic constraints $g_i(\mathbf{d}, \mathbf{x}, \mathbf{p})$, recalling the RBDO problem according to Eqn. (16), are approximated by general quadratic response surface models in $\mathbf{d}$, $\mathbf{x}$ and $\mathbf{p}$ around the deterministic solution. The probability of violating the deterministic constraint is thereafter approximated using the quadratic response surface, and therefore Eqn. (26) and Eqn. (28) can be applied directly without any MPP search. The second approach presented in Paper C is to use Eqn. (31) and to compute the sensitivities with respect to random and deterministic design variables. A decoupled-approach according to Eqn. (19) is thereafter used.

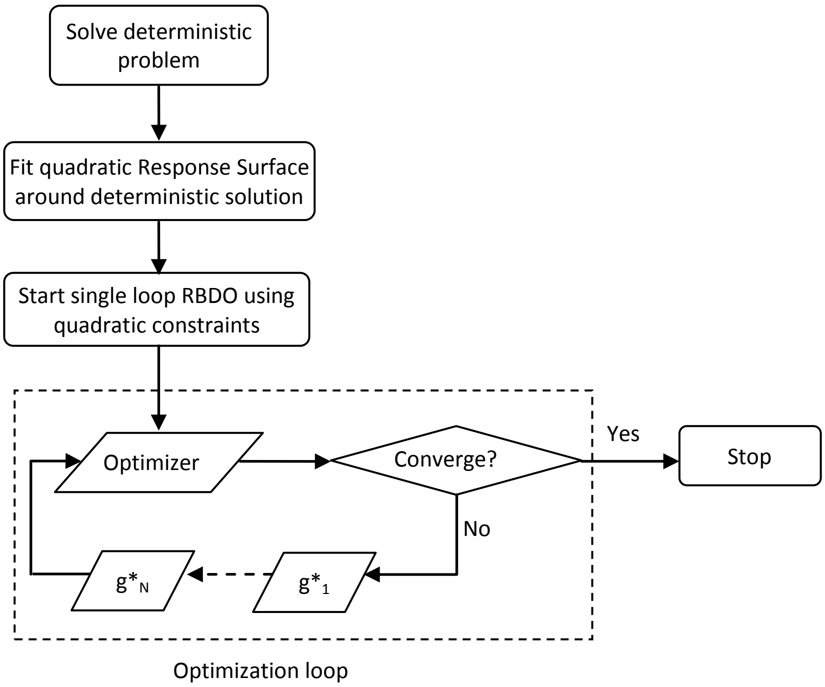

Figure 6: Flowchart of the proposed Response Surface Single-Loop (RSSL) method

Conclusion

The purpose of this work was partly to develop a new approximation for the probability of failure based on hyperparaboloid fit at the MPP in a rotationally transformed space which is more accurate than existing methods, free of singularities and easy-to-use; and partly to propose a closed form expression for the case of general quadratic performances. The expression of the probability of failure for the case of general quadratic performances together with the use of quadratic response surface models in RBDO applications, constitute the essence of the proposed Response Surface Single-Loop (RSSL) method, which has been shown to be an efficient and accurate MPP-free approach for solving RBDO problems. Furthermore, the proposed RSSL method can also handle problems with varying design variance with the same formulation, accuracy and efficiency. For problems where each evaluation of the performance function is computationally expensive, a new efficient method for the reliability assessment is also proposed.

Summary of appended papers

Paper A: *A Closed-form Second-Order Reliability Method using Noncentral Chi-Squared Distributions.*

In the Second-Order Reliability Method (SORM), the probability of failure is computed for an arbitrary performance function in arbitrarily distributed random variables. This probability is approximated by the probability of failure computed using a general quadratic fit made at the Most Probable Point (MPP). However, an easy-to-use, accurate and efficient closed form expression for the probability content of the general quadratic surface in normalized standard variables has not yet been presented. Instead, the most commonly used SORM approaches start with a relatively complicated rotational transformation. Thereafter, the last row and column of the rotationally transformed Hessian are neglected in the computation of the probability. This is equivalent to approximating the probability content of the general quadratic surface by the probability content of a hyper-parabola in a rotationally transformed space. The error made by this approximation may introduce unknown inaccuracies. Furthermore, the most commonly used closed-form expressions have one or more of the following drawbacks: They do not work well for small curvatures at the MPP and/or large number of random variables, neither do they work well for negative or strongly uneven curvatures at the MPP. The expressions may even present singularities. The purpose of this work is to present a simple, efficient and accurate closed form expression for the probability of failure, which does not neglect any component of the Hessian and does not necessitate the rotational transformation performed in the most common SORM approaches. Furthermore, when applied to industrial examples where quadratic response surfaces of the real performance functions are used, the proposed formulas can be applied directly to compute the probability of failure without locating the MPP, as opposed to the other FORM/SORM approaches. The method is based on an asymptotic expansion of the sum of non-central chi-squared variables taken from the literature. The two most widely used SORM approaches, an empirical SORM formula, as well as FORM, are compared to the proposed method with regards to accuracy and computational efficiency. All methods have also been compared when applied to a wide range of hyper-parabolic limit-state functions as well as to general quadratic limit-state functions in the rotationally transformed space, in order to quantify the error made by the approximation of the Hessian indicated above. In general, the presented method was the most accurate for almost all studied curvatures and number of random variables.

Paper B: *Response surface single loop reliability-based design optimization with higher-order reliability assessment.*
Reliability-based design optimization (RBDO) aims at determination of the optimal design in the presence of uncertainty. The available Single-Loop approaches for RBDO are based on the First-Order Reliability Method (FORM) for the computation of the probability of failure, along with different approximations in order to avoid the expensive inner loop aiming at finding the Most Probable Point (MPP). However, the use of FORM in RBDO may not lead to sufficient accuracy depending on the degree of nonlinearity of the limit-state function. This is demonstrated for an extensively studied reliability-based design for vehicle crashworthiness problem solved in this paper, where all RBDO methods based on FORM strongly violates the probabilistic constraints. The Response Surface Single Loop (RSSL) method for RBDO is proposed based on the higher order probability computation for quadratic models previously presented by the authors. The RSSL-method bypasses the concept of an MPP and has high accuracy and efficiency. The method can solve problems with both constant and varying standard deviation of design variables and is particularly well suited for typical industrial applications where general quadratic response surface models can be used. If the quadratic response surface models of the deterministic constraints are valid in the whole region of interest, the method becomes a true single loop method with accuracy higher than traditional SORM. In other cases, quadratic response surface models are fitted to the deterministic constraints around the deterministic solution and the RBDO problem is solved using the proposed single loop method.

Paper C: *A novel closed-form Second-Order Reliability Method with probabilistic sensitivity analysis and application in Reliability-based Design Optimization.*

In the Second-Order Reliability Method, the limit-state function is approximated by a hyper-parabola in standard normal and uncorrelated space. However, there is no exact closed form expression for the probability of failure based on a hyper-parabolic limit-state function and the existing approximate formulas in the literature have been shown to have major drawbacks. Furthermore, in applications such as Reliability-based Design Optimization, analytical expressions, not only for the probability of failure but also for probabilistic sensitivities, are highly desirable for efficiency reasons. In this work, the expressions for the probability of failure based on general quadratic limit-state functions previously presented by the authors are applied for the special case of a hyper-parabola. The expression is reformulated and simplified so that the probability of failure is only a function of three statistical measures: the Cornell Reliability Index, the skewness and the Kurtosis of the hyper-parabola. These statistical measures are functions of the First-Order Reliability Index and the curvatures at the Most Probable Point. Furthermore, analytical sensitivities with respect to mean values of random variables and deterministic variables are presented. The sensitivities can be seen as the product of the sensitivities computed using the First-Order Reliability Method and a correction factor. The proposed expressions are studied and their applicability to Reliability-based Design Optimization is demonstrated.

Paper D: *An efficient reliability method applied to classical load-strength uncertainties, aleatory fatigue problems and epistemic uncertainties.*

Reliability assessment is an important procedure in engineering design in which the probability of failure or equivalently the probability of success is computed based on some design criteria and model behaviour. In this paper, an approximate and efficient reliability method is proposed. Focus is on computational efficiency as well as intuitiveness for practising engineers, especially regarding fatigue problems where volume methods are used to estimate the conditional probability of fatigue failure given a realization of design variables. The proposed method is generalized to classical load-strength structural reliability problems as well as problems involving epistemic uncertainties. The number of function evaluations to compute the probability of failure of the design under different types of uncertainties is a priori known to be $3n + 2$ in the proposed method, where n is the number of stochastic design variables. The necessary number of function evaluations for the reliability assessment, which may correspond to expensive computations or physical experiments, is therefore substantially lower in the proposed approach than in Monte Carlo methods, numerical multidimensional quadratures techniques as well as Most Probable Point (MPP) methods such as the First- and Second-Order Reliability methods (FORM and SORM).

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