Radial barrier for obstacle avoidance

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January 12, 2019

1 Radial transformation

The hero dynamics are

$$\dot{x}_h = \beta v_h, \quad \dot{v}_h = \beta(u - f v_h) \tag{1}$$

and the enemy follows

$$\dot{x}_e = \beta v_e, \tag{2}$$

where β is the refresh rate of the monitor.

We consider the relative dynamics $x_r = x_h - x_e$ so that $\dot{x}_r = \beta(v_h - v_e)$ and $\ddot{x}_r = \beta^2(u - f v_h)$ and the radial transformation $r = \|x_r\|$. The relative radius dynamics become

$$\begin{aligned} \dot{r} &= \frac{\langle x_r, \dot{x}_r \rangle}{r}, \\ \ddot{r} &= \frac{\|\dot{x}_r\|^2}{r} + \frac{\langle x_r, \ddot{x}_r \rangle}{r} - \frac{\langle x_r, \dot{x}_r \rangle^2}{r^3} = \frac{\|\dot{x}_r\|^2}{r} + \frac{\langle x_r, \ddot{x}_r \rangle}{r} - \frac{\dot{r}^2}{r}. \end{aligned} \tag{3}$$

2 Barrier function

We consider the system

$$\ddot{r} = u \tag{4}$$

which for $u = \bar{u}$ and $\dot{r}(0) = -v_0 \leq 0$ has solutions $r(t) = r_0 - v_0 t + \bar{u}\frac{t^2}{2}$. The minimum is attained for $t^* = v_0/\bar{u}$ with value $r(t^*) = r_0 - v_0^2/2\bar{u}$. It follows that the barrier for invariance of $\{r \geq \bar{r}\}$ is

$$h(r, \dot{r}) = r - \frac{\dot{r}^2}{2\bar{u}} - \bar{r}, \tag{5}$$

where $\bar{u}$ is a maximal input.

3 Barrier constraint

Instead plugging in $\ddot{r}$ from above, the constraint to satisfy whenever $\dot{r} < 0$ becomes

$$\dot{h} + \alpha h = \alpha h + \dot{r} + \frac{\dot{r}^3}{r\bar{u}} - \frac{\dot{r}\|\dot{x}_r\|^2}{r\bar{u}} - \frac{\dot{r}}{\bar{u}}\frac{\langle x_r, \ddot{x}_r \rangle}{r} \geq 0. \tag{6}$$

We can expand the term involving $\ddot{r}$ and rearrange to obtain

$$\alpha h + \dot{r} + \frac{\dot{r}^3}{r\bar{u}} - \frac{\dot{r}\|\dot{x}_r\|^2}{r\bar{u}} + \frac{f\dot{r}\beta^2}{\bar{u}}\frac{\langle x_r, v_h \rangle}{r} - \frac{\dot{r}\beta^2}{\bar{u}}\frac{\langle x_r, u \rangle}{r} \geq 0. \tag{7}$$

That is, the component $\frac{\langle x_r, u \rangle}{r}$ of the input along the relative direction must be larger than a lower bound. We can create a minimally invasive safety filter by selecting the closest input that satisfies this inequality (remark that $\dot{r}$ is negative whenever this is so this is a lower bound on u).