Lie Theory for Robotics

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Contents

0.1. Next todos

- Separate each chapter into “theory” and “derivations for specific groups”
- Move system linearization and Magnus expansion to section on dynamical systems

0.2. Research questions

1. How to define semi-simple group products? Can they be implemented using only a definition of group composition? Seems to be the case at least for SE2 and SE3 since the exponential shows up with the derivative of the exponential.
2. Equivariance and control-invariance for reduced-dimensionality asif.
3. Use Magnus expansion with closed-form dexpin expressions to work with implicit invariance on lie groups

0.3. Literature

- BOOKS:
  - Naive Lie Theory: [**stillwell_naive_2008**]
    - Matrix groups only
    - Relation between $SO(3)$ and S^3
  - Barfoot: [2]
    - Estimation in robotics
    - Some explicit formulas
  - Chirkijan: [5, 4]
    - Guassians and information theory
  - Agrachev: [1]
  - Geometrical Numerical Integration: [**blanes_concise_2016**]
- ARTICLES
  - Very basic Lie theory [9], rigorous mathematics.
  - A Micro Lie Theory [16], application-focused.
  - Quadrotor control [11]
  - IMU estimation: [10, 12]
  - Banana distribution: [**wheeler_relative_2018**]

Part I.

Theory

1. Introduction

Summary

- Treat parameterizations as regular groups
- Get rid of $\check{\mathbb{M}}$, define exp and log $\check{\mathfrak{m}} \leftrightarrow \mathbb{M}$
- Get rid of hat and vee for parameterizations
- Overview of notes.
- Advantages of on-manifold tools.
- Applications of Lie theory in robotics.

1.1. Numerical integration

1.2. Nonlinear control and estimation

1.3. Localization

1.4. Notation

	Set notation	Element notation
Group (matrix form)	$\mathbb{M}, \mathbb{N}$	$\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$
Group (param. form)	$\check{\mathbb{M}}, \check{\mathbb{N}}$	$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$
Algebra (matrix form)	$\mathfrak{m}, \mathfrak{n}$	$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$
Algebra (param. form)	$\check{\mathfrak{m}}, \check{\mathfrak{n}}$	$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$
Rotation matrices		$\boldsymbol{R}$
Rotation parameters		$\boldsymbol{q} = [q_w, q_x, q_y, q_z]$
Velocity parameters		$\boldsymbol{v} = [v_x, v_y, v_z]$
Translation parameters		$\boldsymbol{p} = [p_x, p_y, p_z]$
Angular velocity		$\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]$
Vectors $\mathbb{R}^n$		$\boldsymbol{u}$

2. Lie Groups

Summary

- Fundamental definitions and properties.
- Matrix Lie groups that appear in robotics.

2.1. Fundamentals

A Lie group is an object that is both a group and a smooth manifold. As will be illustrated in these notes, inheritence of these two sets of properties places Lie groups at a unique point where theory meets practice.

We recall the definitions of groups and smooth manifolds, respectively.

Definition 2.1 ([6]). *A **group** $(\mathbb{M}, \circ)$ is a set $\mathbb{M}$ closed under a binary operation $(\circ)$ such that*

- ***associativity** holds: $\boldsymbol{X} \circ (\boldsymbol{Y} \circ \boldsymbol{Z}) = (\boldsymbol{X} \circ \boldsymbol{Y}) \circ \boldsymbol{Z}$ for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{M}$,*
- *there is an **identity element** $e \in \mathbb{M}$ s.t. $e \circ \boldsymbol{X} = \boldsymbol{X} \circ e$ for all $\boldsymbol{X} \in \mathbb{M}$,*
- *for each element $\boldsymbol{X} \in \mathbb{M}$ there is an **inverse** $\boldsymbol{X}^{-1} \in \mathbb{M}$ s.t. $\boldsymbol{X}^{-1} \circ \boldsymbol{X} = \boldsymbol{X} \circ \boldsymbol{X}^{-1} = e$.*

Definition 2.2 ([3]). *A **smooth manifold** $(\mathbb{M}, \{c_i\})$ of dimension n is a set $\mathbb{M}$ and a family of injective mappings $c_i : U_i \subset \mathbb{R}^n \to \mathbb{M}$ of open sets U_i (called **charts**) such that*

1. *The charts cover the set: $\bigcup_i U_i = \mathbb{M}$,*
2. *For any pair i, j with $c_i(U_i) \cap c_j(U_j) =: W \neq \emptyset$, the sets $c_i^{-1}(W)$ and $c_j^{-1}(W)$ are open in $\mathbb{R}^n$, and the mappings $c_i^{-1} \circ c_j$ are differentiable.*

Figure of chart mappings

The definition of a Lie group is now straightforward.

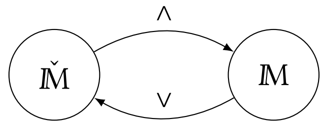


Figure 2.1.: The $\vee$ (hat) and $\wedge$ (vee) maps map between the matrix and parameter forms of a matrix Lie group.

Definition 2.3. *A **Lie group** of dimension n is a set $\mathbb{M}$ together with a binary operation $(\circ)$ and a family of injective mappings $c_i : U_i \subset \mathbb{R}^n \to X$ such that*

1. $(\mathbb{M}, \circ)$ *is a group,*
2. $(\mathbb{M}, \{c_i\})$ *is an n-dimensional smooth manifold.*

In the following we use $\mathbb{M}$ to refer both to the Lie group and to its underlying set. A mathematic object that satisfies the properties in Definition 2.3 is the **general linear group** $GL(n, \mathbb{C})$—the set of $n \times n$ invertible complex matrices with matrix multiplication $(\cdot)$ as the group operation. It turns out that many Lie groups of practical interest can be represented as sub-groups of $GL(n, \mathbb{C})$. In these notes we restrict attention to Matrix Lie groups.

Definition 2.4. *A **matrix Lie group** is a Lie group that is also a sub-group of $GL(n, \mathbb{C})$—the group of invertible matrices with complex coefficients.*

Lie theory is more straightforward to develop for matrix lie groups compared to a more general setting. Matrix Lie groups are however inefficient from a practical point of view since their representation is often redundant

We will distinguish between the matrix representation of a Lie group, which is useful for analytical purposes, and compact parameterized representations that are computationally more efficient. For a matrix Lie group $\mathbb{M}$ we denote the corresponding lower-dimensional representation $\check{\mathbb{M}}$. The mappings $\vee$ and $\wedge$ are used to convert between the two representations and are smooth group homomorphisms, i.e.

$$\boldsymbol{x} \circ \boldsymbol{y} = (\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}})^{\vee}, \quad \boldsymbol{x}, \boldsymbol{y} \in \check{\mathbb{M}}. \tag{2.1}$$

2.2. Examples of Lie Groups

In the following we introduce the Matrix Lie group forms of various Lie groups that are of practical interest in robotics. For every group we first introduce the matrix form $\mathbb{M}$ and present a lower-dimensional parameterization $\check{\mathbb{M}}$ that is homomorphic to $\mathbb{M}$. We then perform matrix inverseion and multiplication in $\mathbb{M}$ to obtain the forms of inverse and composition also in $\check{\mathbb{M}}$.

2.2.1. $\mathbb{E}(n)$: n-dimensional translations

We start with a group that is isometric to $\mathbb{R}^n$ under addition. For $\boldsymbol{p} \in \mathbb{R}^n$ consider the matrix

$$\boldsymbol{p}^\wedge := \begin{bmatrix} I_n & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}. \tag{2.2}$$

When multiplying two such matrices the result is

$$\begin{bmatrix} I_n & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \begin{bmatrix} I_n & \boldsymbol{p}' \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} = \begin{bmatrix} I_n & \boldsymbol{p} + \boldsymbol{p}' \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}, \tag{2.3}$$

i.e. the result is still a matrix of the form in (2.2).

Lower-dimensional representation: The Matrix lie group $\mathbb{E}(n)$ consists of matrices of the form (2.2), but those matrices are parameterized by n parameters, so $\check{\mathbb{E}}(n) = \mathbb{R}^n$—the regular Euclidean vector space in n dimensions.

Identity: $e_{\check{\mathbb{E}}(n)} = \mathbf{0}_n$ since $\mathbf{0}_n^\wedge$ is the identity matrix.

Inverse: Since $\begin{bmatrix} I_n & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} I_n & -\boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}$ the group inverse is negation: $\boldsymbol{p}^{-1} = -\boldsymbol{p}$.

Composition: The embedding of $\boldsymbol{p} \in \mathbb{R}^n$ in a matrix of the form (2.2) is such that addition in $\mathbb{R}^n$ corresponds to matrix multiplication in the embedding, i.e.

$$\boldsymbol{p}^\wedge \cdot (\boldsymbol{p}')^\wedge = (\boldsymbol{p} + \boldsymbol{p}')^\wedge. \tag{2.4}$$

Therefore group composition is addition: $\boldsymbol{p} \circ \boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{p}'$.

Action on $\mathbb{R}^n$: We can define an action of $\mathbb{E}(n)$ on $\mathbb{R}^n$ as translation. The action can be defined in terms of a matrix multiplication by associating $\boldsymbol{u} \in \mathbb{R}^n$ with a homogenous vector $\boldsymbol{u}^H = \begin{bmatrix} \boldsymbol{u} \\ 1 \end{bmatrix}$. Then

$$\langle \boldsymbol{p}^\wedge, \boldsymbol{u}^H \rangle_{\mathbb{E}(n)} = \begin{bmatrix} I_n & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u} + \boldsymbol{p} \\ 1 \end{bmatrix} = (\boldsymbol{u} + \boldsymbol{p})^H. \tag{2.5}$$

This first example of a matrix Lie group is not very interesting in itself. However, by expressing Euclidean space $\mathbb{R}^n$ as a Lie group the fundamental strengths of Lie theory becomes apparent: the ability to treat any Lie group in the same way as the linear space $\mathbb{R}^n$.

2.2.2. $\$\mathbb{O}(2)$: Two-dimensional rotations

This Lie group consists of 2x2 real matrices with determinant equal to one, or, equivalently, orthogonal matrices with positive determinant.

$$\$\mathbb{O}(2) = \left\{ \boldsymbol{R} = \begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix} \mid q_w^2 + q_z^2 = 1 \right\}. \tag{2.6}$$

Lower-dimensional representation: We use two parameters and one equality constraint to parameterize $\$\mathbb{O}(2)$.

$$\check{\$\mathbb{O}}(2) = \left\{ (q_w, q_z) \in \mathbb{R}^2 \mid q_w^2 + q_z^2 = 1 \right\}. \tag{2.7}$$

Identity: The identity element $e_{\check{\$\mathbb{O}}(2)} = (1, 0)$ correponds to the identity matrix $I_2 \in \$\mathbb{O}(2)$.

Inverse: The inverse is $(q_w, q_z)^{-1} = (q_w, -q_z)$ which corresponds to matrix transposition.

Composition:

$$\begin{aligned} (q_w, q_z) \circ (q_w', q_z') &= ((q_w, q_z)^\wedge \cdot (q_w', q_z')^\wedge)^\vee \\ &= \left(\begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix} \cdot \begin{bmatrix} q_w' & -q_z' \\ q_z' & q_w' \end{bmatrix} \right)^\vee = (q_w q_w' - q_z q_z', q_z q_w' + q_w q_z'). \end{aligned} \tag{2.8}$$

Action on $\mathbb{R}^2$: The group defines a rotation action on $\mathbb{R}^2$. For $\boldsymbol{R} \in \$\mathbb{O}(2)$ and $\boldsymbol{u} \in \mathbb{R}^2$ the action is matrix multiplication

$$\langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\$\mathbb{O}(2)} = \boldsymbol{R} \cdot u. \tag{2.9}$$

2.2.3. $\$\mathbb{E}(2)$: Planar poses

We now combine $\mathbb{E}(2)$ and $\$\mathbb{O}(2)$ into a group that simultaneously represents translation and rotation in two dimensions. The result is $\$\mathbb{E}(2)$—the special euclidean group in two dimensions.

In matrix form $\$\mathbb{E}(2)$ consists of matrices on the form

$$\$\mathbb{O}(2) = \left\{ \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \mid R \in \$\mathbb{O}(2) \right\}, \tag{2.10}$$

i.e. the identity matrix block in (2.2) has been replaced with a member of $\$\mathbb{O}(2)$. It follows that both $\$\mathbb{O}(2)$ and $\mathbb{E}(2)$ are sub-groups of $\$\mathbb{E}(2)$. In addition, $\mathbb{E}(2)$ is a normal subgroup[1] which implies that $\$\mathbb{E}(2)$ is a *semi-direct product* denoted $\$\mathbb{E}(2) \cong \$\mathbb{O}(2) \ltimes \mathbb{E}(2)$. Group products (direct and semi-direct) are discussed further below.

[1] If $X \in \mathbb{E}^2$ and $Y \in \$\mathbb{O}(2)$, then $YXY^{-1} \in \mathbb{E}^2$.

Lower-dimensional representation: Four parameters are required, two for each sub-group

$$\mathbb{S}\check{\mathbb{E}}(2) = \mathbb{S}\check{\mathbb{O}}(2) \times \check{\mathbb{E}}(2) = \Big\{((q_w, q_z), (p_x, p_y)) \,:\, (q_w, q_z) \in \mathbb{S}\check{\mathbb{O}}(2), (p_x, p_y) \in \check{\mathbb{E}}(2)\Big\}. \tag{2.11}$$

Identity: The identity element is inherited from the sub-groups: $e_{\mathbb{S}\check{\mathbb{E}}(2)} = \left(e_{\mathbb{S}\check{\mathbb{O}}(2)}, e_{\check{\mathbb{E}}(2)}\right) = ((1,0),(0,0))$.

Inverse: From matrix inverse it follows that $(\boldsymbol{R}, \boldsymbol{p})^{-1} = (\boldsymbol{R}^T, -\boldsymbol{R}^T \boldsymbol{p})$.

Composition: Matrix multiplication shows that composition in the lower-dimensional representation is

$$(\boldsymbol{R}, \boldsymbol{p}) \circ (\boldsymbol{R}', \boldsymbol{p}') = (\boldsymbol{R}\boldsymbol{R}', \boldsymbol{R}\boldsymbol{p}' + \boldsymbol{p}). \tag{2.12}$$

Action on $\boldsymbol{R}^2$: This group has a natural action on two-dimensional vectors that consists of rotation and translation. For $X := (\boldsymbol{R}, \boldsymbol{p}) \in \mathbb{S}\check{\mathbb{E}}(2)$ the action is

$$\langle \boldsymbol{X}, \boldsymbol{u} \rangle_{\mathbb{S}\mathbb{E}(2)} = \langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\mathbb{S}\mathbb{O}(2)} + \boldsymbol{p} = \boldsymbol{R}\boldsymbol{u} + \boldsymbol{p}. \tag{2.13}$$

That is, the vector $\boldsymbol{u}$ is first rotated through the action of the $\mathbb{S}\mathbb{O}(2)$ part of the state, and then subjected to a translation. Like in (2.5) this action can be written as a matrix multiplication if we associate $\boldsymbol{u}$ with its homogeneous counterpart $\boldsymbol{u}^H$:

$$\left\langle \boldsymbol{X}, \boldsymbol{u}^H \right\rangle = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}\boldsymbol{u} + \boldsymbol{p} \\ 1 \end{bmatrix}. \tag{2.14}$$

The action has a natural interpretation as a change of coordinates: if $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \in \mathbb{S}\mathbb{E}(2)$, then $\langle \boldsymbol{X}, \boldsymbol{u} \rangle$ represents the tranformation from a coordinate frame attached at $\boldsymbol{p}$ with unit vectors the columns of $\boldsymbol{R}$, to the global coordinate frame.

2.2.4. Groups representing three-dimensional rotations

As opposed to the 2D case where $\mathbb{S}\mathbb{O}(2)$ as defined above is the canonical way to represent rotations, the situation is more complicated in three dimensions. While $\mathbb{S}\mathbb{O}(2)$ generalizes to $\mathbb{S}\mathbb{O}(3)$ that consists of orthogonal 3×3 matrices with determinant 1, it is no longer as easy to construct a lower-dimensional representation. The usual choice is the unit quaternions, which are isomorphic to the matrix Lie group $\mathbb{S}\mathbb{U}(2)$. We begin by defining the matrix group $\mathbb{S}\mathbb{O}(3)$.

$\mathbb{SO}(3)$: three-dimensional rotations

$\mathbb{SO}(3)$ is a matrix Lie group that consists of 3×3 orthogonal matrices with determinant equal to one:

$$\mathbb{SO}(3) = \left\{ \boldsymbol{R} \in \mathbb{GL}(3) \mid \boldsymbol{R}^T \boldsymbol{R} = I, \det(\boldsymbol{R}) = 1 \right\}. \tag{2.15}$$

These matrices are usually referred to as **rotation matrices**.

There is no trivial low-dimensional parameterizations of this set, however, it is isometric to another group $\mathbb{SU}(2)$ that is in turn isometric to the unit quaternions $\mathbb{S}^3$ which can be used as a lower-dimensional representation of $\mathbb{SO}(3)$.

$$\mathbb{S}^3 = \left\{ (q_w, q_x, q_y, q_z) \, : \, q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1 \right\}. \tag{2.16}$$

However, the mapping is not 1-to-1, since both $\boldsymbol{q} := (q_w, q_x, q_y, q_z)$ and $-\boldsymbol{q}$ correspond to the same rotation matrix.

Action on $\mathbb{R}^3$ The action of $\boldsymbol{R} \in \mathbb{SO}(3)$ on $\boldsymbol{u} \in \mathbb{R}^3$ is rotation:

$$\langle \boldsymbol{R}, \boldsymbol{u} \rangle = \boldsymbol{R} \cdot \boldsymbol{u}. \tag{2.17}$$

$\mathbb{SU}(2)$ and its relation to the quaternion group $\mathbb{S}^3$

We can associate a quaternion $\boldsymbol{q} = q_w + q_x \boldsymbol{i} + q_y \boldsymbol{j} + q_z \boldsymbol{k}$ with the unitary matrix

$$\mathbb{SU}(2) = \left\{ \begin{bmatrix} q_w + iq_z & -q_x - iq_y \\ q_x - iq_y & q_w - iq_z \end{bmatrix} \mid q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1 \right\} \tag{2.18}$$

for which it holds that $A_{\mathrm{q}_1 * \mathrm{q}_2} = A_{\mathrm{q}_1} A_{\mathrm{q}_2}$. Thus the unit quaternions $\mathbb{S}^3$ are isomorphic to $\mathbb{SU}(2)$ and can therefore be viewed as a matrix Lie group.

By muliplying two elements in $\mathbb{SU}$ we retrieve quaternion multiplication:

$$\begin{bmatrix} q_w + iq_z & -q_x - iq_y \\ q_x - iq_y & q_w - iq_z \end{bmatrix} \begin{bmatrix} q_w' + iq_z' & -q_x' - iq_y' \\ q_x' - iq_y' & q_w' - iq_z' \end{bmatrix} = \begin{bmatrix} q_w'' + iq_z'' & -q_x'' - iq_y'' \\ q_x'' - iq_y'' & q_w'' - iq_z'' \end{bmatrix} \tag{2.19}$$

where

$$\begin{aligned} q_w'' &= q_w q_w' - q_x q_x' - q_y q_y' - q_z q_z', \\ q_x'' &= q_x q_w' + q_w q_x' + q_y q_z' - q_z q_y', \\ q_y'' &= q_y q_w' + q_w q_y' + q_z q_x' - q_x q_z', \\ q_z'' &= q_z q_w' + q_w q_z' + q_x q_y' - q_y q_x', \end{aligned} \tag{2.20}$$

which is exactly what is obtained by carrying out the usual quaternion multiplication

$$(q_w + q_x \boldsymbol{i} + q_y \boldsymbol{j} + q_z \boldsymbol{k}) * (q_w' + q_x' \boldsymbol{i} + q_y' \boldsymbol{j} + q_z' \boldsymbol{k}) \tag{2.21}$$

with the quaternion rules $\boldsymbol{i}\boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j}\boldsymbol{k} = \boldsymbol{i}, \boldsymbol{k}\boldsymbol{i} = \boldsymbol{j}$ and $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$.

Action on $\mathbb{R}^3$ A quaternion $\boldsymbol{q} = q_w + q_x\boldsymbol{i} + q_y\boldsymbol{j} + q_z\boldsymbol{k}$ acts on $\boldsymbol{u} := \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \in \mathbb{R}^3$ as quaternion rotation $\boldsymbol{q} * \boldsymbol{u} * \bar{\boldsymbol{q}}$ where $\boldsymbol{u}$ is associated with the quaternion $u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}$.

In terms of matrix multiplication operation can be written

$$\begin{aligned} &\begin{bmatrix} q_w + iq_z & -q_x - iq_y \\ q_x - iq_y & q_w - iq_z \end{bmatrix} \begin{bmatrix} iu_z & -u_x - iu_y \\ u_x - iu_y & iu_z \end{bmatrix} \begin{bmatrix} q_w - iq_z & q_x + iq_y \\ -q_x + iq_y & q_w + iq_z \end{bmatrix} \\ &\qquad = \begin{bmatrix} iu'_z & -u'_x - iu'_y \\ u'_x - iu'_y & iu'_z \end{bmatrix} \end{aligned} \tag{2.22}$$

for

$$\begin{aligned} u'_x &= (1 - 2(q_y^2 + q_z^2))u_x + 2(q_xq_y - q_wq_z)u_y + 2(q_xq_z + q_wq_y)u_z \\ u'_y &= 2(q_xq_y + q_wq_z)u_x + (1 - 2(q_x^2 + q_z^2))u_y + 2(q_yq_z - q_wq_x)u_z \\ u'_z &= 2(q_xq_z - q_wq_y)u_x + 2(q_wq_x + q_yq_z)u_y + (1 - 2(q_x^2 + q_y^2))u_z \end{aligned} \tag{2.23}$$

Since this is a linear mapping in u_x, u_y, u_z we can identify $\boldsymbol{q}$ with a matrix $\boldsymbol{R}(\boldsymbol{q})$ with coefficients

$$\boldsymbol{R}(\boldsymbol{q}) = \begin{bmatrix} (1 - 2(q_y^2 + q_z^2)) & 2(q_xq_y - q_wq_z) & 2(q_xq_z + q_wq_y) \\ 2(q_xq_y + q_wq_z) & (1 - 2(q_x^2 + q_z^2)) & 2(q_yq_z - q_wq_x) \\ 2(q_xq_z - q_wq_y) & 2(q_wq_x + q_yq_z) & (1 - 2(q_x^2 + q_y^2)) \end{bmatrix}. \tag{2.24}$$

Thus we can utilize the quaternion group $\mathbb{S}^3$ as the lower-dimensional representation of $\mathbb{SO}(3)$.

Useful quaternion identities

Axis-angle to quaternion The quaternion $\boldsymbol{q}$ representing the rotation about a unit axis $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$ for an angle α is

$$\boldsymbol{q} = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\left(\beta_x \boldsymbol{i} + \beta_y \boldsymbol{j} + \beta_z \boldsymbol{k}\right). \tag{2.25}$$

Two vectors to quaternion A quaternion $\boldsymbol{q}$ such that $\boldsymbol{qu} = \boldsymbol{v}$ for unit vectors $\boldsymbol{u}, \boldsymbol{v}$.

$$\boldsymbol{q} = \sqrt{\frac{1+s}{2}} + \sqrt{\frac{1-s}{2}}\left(\beta_x \boldsymbol{i} + \beta_y \boldsymbol{j} + \beta_z \boldsymbol{k}\right), \quad s = \boldsymbol{u}\cdot\boldsymbol{v},\ \boldsymbol{\beta} = \boldsymbol{u}\times\boldsymbol{v}. \tag{2.26}$$

Hopf fibration The quaternions can be parameterized as the product of a rotation $\boldsymbol{q}_\theta$ around the z axis and a quaternion that rotates $\boldsymbol{e}_z$ to $\boldsymbol{\beta} := \left[\beta_x, \beta_y, \beta_z\right] \in \mathbb{S}^2$ as

$$\boldsymbol{q} = \boldsymbol{q}_{\boldsymbol{\beta}} * \boldsymbol{q}_\theta, \quad \boldsymbol{q}_{\boldsymbol{\beta}} = \frac{1}{\sqrt{2(1+\beta_z)}}\left(1 + \beta_z - \boldsymbol{i}\beta_x + \boldsymbol{j}\beta_y\right), \ \boldsymbol{q}_\theta = \cos\left(\frac{\theta}{2}\right) + \boldsymbol{k}\sin\left(\frac{\theta}{2}\right). \tag{2.27}$$

The special case when $\beta_z = -1$ is a singularity and must be handled separately, for example by setting $\boldsymbol{q}_{\left[0,0,-1\right]} = \boldsymbol{i}$. The Hopf parameterization is a manifestation of the fact that $\mathbb{S}^3$ locally is a product of the spaces $\mathbb{S}^2$ and $\mathbb{S}^1$.

Proof of (2.26). From properties of the dot and cross products the sought-after rotation is about the axis $\boldsymbol{\beta} = \boldsymbol{u}\times\boldsymbol{v}$ for the angle α such that $s := \boldsymbol{u}\cdot\boldsymbol{v} = \cos(\alpha)$. The half-angle formulas then give that $\cos(\alpha/2) = \sqrt{(1+s)/2}$, and similarly for the sine part in (2.25). □

Summary of the three-dimensional rotation groups

In summary, $\mathbb{SO}(3)$ and $\mathbb{SU}(2)$ are both matrix Lie groups that represent rotations in three dimensions. $\mathbb{SU}(2)$ is isomorphic to the unit quaternions $\mathbb{S}^3$ which is a good choice for the lower-dimensional representation. We also use the unit quaternions as the lower-dimensional representation of $\mathbb{SO}(3)$ since a quaternion corresponds to a unique rotation matrix, and any rotation matrix corresponds to a unique (up to sign) quaternion.

2.2.5. $\mathbb{SE}(3)$: Three-dimensional poses

Just as $\mathbb{SE}(2)$ was constructed as a semi-simple product of $\mathbb{SO}(2)$ and $\mathbb{E}(2)$, an analogous construction can be done in three dimensions. We define $\mathbb{SE}(3)$ as as a semi-direct product $\mathbb{SE}(3) \cong \mathbb{SO}(3) \ltimes \mathbb{E}(3)$

$$\mathbb{SE}(3) = \left\{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix} \mid \boldsymbol{R} \in \mathbb{SO}(3), \boldsymbol{p} \in E(3)\right\}. \tag{2.28}$$

The construction is analogous to the construction of $\mathbb{SE}(3)$ above.

2.3. Product Groups

Discuss direct vs semi-direct products

compare e.g. $\mathbb{SE}(2) \cong \mathbb{SO}(2) \ltimes \mathbb{E}(2)$ and the direct product $\mathbb{SO}(2) \otimes \mathbb{E}(2)$.

2.4. Summary

The different Lie groups introduced above are the following:

	Matrix representation $\mathbb{M}$	Parameter representation $\check{\mathbb{M}}$
$E(n)$	$n+1 \times n+1$ as in (2.2)	$\mathbb{R}^n$
$\mathbb{SO}(2)$	2×2 orthogonal, determinant 1	$(q_w, q_z) \in \check{\mathbb{SO}}(2)$
$SU(2)$	2×2 unitary	$(q_w, q_x, q_y, q_z) \in \mathbb{S}^3$
$SO(3)$	3×3 orthogonal, determinant 1	$(q_w, q_x, q_y, q_z) \in \mathbb{S}^3$
$SE(2)$	$SO(2) \ltimes E(2)$ as in (2.10)	$\check{\mathbb{SO}}(2) \times \mathbb{R}^2$
$SE(3)$	$SE(3) \ltimes E(3)$ as in (2.28)	$\mathbb{S}^3 \times \mathbb{R}^3$

3. Lie Algebras

Summary

- Fundamental definitions and properties of Lie Algebras.
- The Lie Bracket.
- Hat and vee operators.
- Maybe: connection to Lie Derivative.

3.1. Lie Algebra definition

Definition 3.1. *A **Lie Algebra** is a vector space* $\mathfrak{m}$ *with a binary relation* $[\cdot,\cdot] : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ *called the **Lie bracket** that satisfies*

1. *Bilinearity:* $[\boldsymbol{A}, \beta\boldsymbol{B} + \gamma\boldsymbol{C}] = \beta[\boldsymbol{A}, \boldsymbol{B}] + \gamma[\boldsymbol{A}, \boldsymbol{C}]$, *and* $[\alpha\boldsymbol{A} + \beta\boldsymbol{B}, \boldsymbol{C}] = \alpha[\boldsymbol{A}, \boldsymbol{C}] + \beta[\boldsymbol{B}, \boldsymbol{C}]$,
2. $[\boldsymbol{A}, \boldsymbol{A}] = 0$,
3. *Jacobi's identity:* $[\boldsymbol{A}, [\boldsymbol{B}, \boldsymbol{C}]] + [\boldsymbol{C}, [\boldsymbol{A}, \boldsymbol{B}]] + [\boldsymbol{B}, [\boldsymbol{C}, \boldsymbol{A}]] = 0$.

3.2. The Lie bracket

Jacobis identity

Clean this up

We have the two flows $\phi^f(t,x)$ and $\phi^g(t,x)$ that are such that

$$\begin{aligned} \phi^f(0,x) &= x, \quad \frac{\partial}{\partial t}\phi^f(t,x) = f(\phi^f(t,x)), \\ \phi^g(0,x) &= x, \quad \frac{\partial}{\partial t}\phi^g(t,x) = g(\phi^g(t,x)). \end{aligned} \tag{3.1}$$

Consequently, we get the second derivative

$$\begin{aligned}\left[\frac{\partial^2}{\partial t^2}\phi^f(t,x)\right]_i = \frac{\partial\left[f(\phi^f(t,x)\right]_i}{\partial t} &= \left.\frac{\mathrm{d}f_i}{\mathrm{d}x_j}\right|_{x_j=[\phi^f(t,x)]_j} \frac{\partial}{\partial t}\left[\phi^f(t,x)\right]_j \\ &= \left.\frac{\mathrm{d}f_i}{\mathrm{d}x_j}\right|_{x_j=[\phi^f(t,x)]_j} \left[f(\phi^f(t,x))\right]_j .\end{aligned} \tag{3.2}$$

Thus we get

$$\frac{\partial^2}{\partial t^2}\phi^f(t,x) = \left.(f\cdot\nabla)\,f\right|_{\phi^f(t,x)} . \tag{3.3}$$

We are interested in the quantity

$$\phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\circ\phi^g(t,\cdot)\circ\phi^f(t,\cdot)\,[x] \tag{3.4}$$

for small t.

Note that by Taylor expansion,

$$\begin{aligned}\phi^f(t,x) &= \phi^f(0,x) + \left.\frac{\partial}{\partial t}\phi^f(t,x)\right|_{t=0} t + \left.\frac{\partial^2}{\partial t^2}\phi^f(t,x)\right|_{t=0}\frac{t^2}{2} + \mathcal{O}(t^3) \\ &= \phi^f(0,x) + t\,f(\phi^f(0,x)) + \frac{t^2}{2}\left(f(\phi^f(0,x))\cdot\nabla\right)f(\phi^f(0,x)) + \mathcal{O}(t^3) \\ &= x + t\,f(x) + \frac{t^2}{2}\left(f(x)\cdot\nabla\right)f(x) + \mathcal{O}(t^3).\end{aligned} \tag{3.5}$$

We also have that

$$g(x+t\alpha) = g(x) + t(\alpha\cdot\nabla)g(x) + \mathcal{O}(t^2). \tag{3.6}$$

Then we get, after omitting the (x) in $f(x)$ and $g(x)$,

$$
\begin{aligned}
&\phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\circ\phi^g(t,\cdot)\circ\phi^f(t,\cdot)\,[x] = \\
&\quad= \phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\circ\phi^g(t,\cdot)\left[x + f\,t + \frac{t^2}{2}(f\;\cdot\nabla)\,f\; + \mathcal{O}(t^3)\right] \\
&\quad= \phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\left[x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3)\right. \\
&\qquad\quad + \underbrace{t\,g\left(x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3)\right)}_{g\;+t(f\cdot\nabla)g+\mathcal{O}(t^2)} \\
&\qquad\quad + \frac{t^2}{2}\nabla g\left(x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3)\right) \\
&\qquad\qquad \left.\cdot g\left(\left(x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3)\right)\right) + \mathcal{O}(t^3)\right] \\
&\quad= \phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\left[x + t\{f + g(x)\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right] \\
&\quad= \phi^g(-t,\cdot)\left[x + t\{f+g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right. \\
&\qquad\quad - tf\left(x + t\{f+g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right) \\
&\qquad\quad + \frac{t^2}{2}\nabla f\left(x + t\{f+g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right) \\
&\qquad\qquad \cdot f\left(x + t\{f+g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right) \\
&\quad= \phi^g(-t,\cdot)\left[x + t\{g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - \nabla f\cdot(f+g) + \frac{1}{2}(f\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right] \\
&\quad= \phi^g(-t,\cdot)\left[x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right] \\
&\quad= x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3) \\
&\qquad\quad - tg\left(x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right) \\
&\qquad\quad + \frac{t^2}{2}\nabla g\left(x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right) \\
&\qquad\qquad \cdot g\left(x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right) \\
&\quad= x + t\{g-g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f - (g\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3) \\
&\quad= x + t^2\{(f\cdot\nabla)\,g - (g\cdot\nabla)\,f\} + \mathcal{O}(t^3) \\
&\quad= x + t^2[f,g](x) + \mathcal{O}(t^3).
\end{aligned}
$$

3.3. Application: Derive the Laguerre polynomials

This is an exercise from [9].

Consider the equation

$$xy'' + (1-x)y' + ny = 0,$$

we will show via Lie-algebraic concepts that a solution is given by

$$y = e^x \left(\frac{d}{dx}\right)^n e^{-x}x^n.$$

Letting $P = d/dx$ denote derivative and $Q = x$ multiplication by x the equation can be written

$$Ly = (P-I)QPy = -ny.$$

We consider the Lie algebra spanned by P, Q, I with commutator relationships

$$\begin{aligned} [P,Q]y = PQy - QPy = y + xy' - xy' = Iy, \quad \implies [P,Q] = I \\ [P,I] = [P,Q] = 0. \end{aligned}$$

We have from the bracket relation that $(P-I)Q = I + Q(P-I)$, consequently

$$\begin{aligned} [Q,(P-I)^n] &= Q(P-I)^n - (P-I)^n Q \\ &= \left(Q(P-I)^{n-1} - (P-I)^{n-1}Q\right)(P-I) - (P-I)^{n-1} \\ &= [Q,(P-I)^{n-1}](P-I) - (P-I)^{n-1}. \end{aligned}$$

From $[Q, P-I] = -I$ it follows by recursion that

$$[Q,(P-I)^n] = -n(P-I)^{n-1}.$$

Let $A_n = (P-I)^n Q^n$, then with the above

$$\begin{aligned} A_{n+1} &= (P-I)^{n+1}Q^{n+1} = (P-I)\left([(P-I)^n, Q] + Q(P-I)^n\right)Q^n \\ &= (P-I)\left\{n(P-I)^{n-1} + Q(P-I)^n\right\}Q^n = (n+A_1)A_n. \end{aligned}$$

Note that we have $L = A_1 P$ and that $PA_1 = P(P-I)Q = (P-I)QP + (P-I) = A_1 P + (P-I)$. It follows that

$$[A_1 P, A_1] = A_1 P A_1 - A_1^2 P = A_1(P-I) = L - A_1.$$

Using the bracket relation it follows that

$$L(A_1+n) = (A_1+n)L + [L, A_1+n] = (A_1+n)L + (L+n) - (A_n+n).$$

Proposition: If v_n is an eigenvector of L with eigenvalue $-n$, then $(A_1+n)v_n$ is an eigenvector with eigenvalue $-(n+1)$. **Proof**: We use the relation above to get

$$\begin{aligned} L(A_1+n)v_n &= (A_1+n)Lv_n + (L+n)v_n - (A_n+n)v_n \\ &= -n(A_1+n)v_n - (A_n+n)v_n. \end{aligned}$$

It follows via the relation $A_{n+1} = (A_1+n)A_n$ shown above that if v_0 is an eigenvector with eigenvalue 0, then $A_n v_0$ is an eigenvector with eigenvalue $-n$.

We have solved $Ly = -ny$, a solution is for instance

$$A_n v_0 = (P-I)^n Q^n 1 = \left(e^x \frac{d}{dx} e^{-x}\right)^n x^n = e^x \left(\frac{d}{dx}\right)^n e^{-x}x^n.$$

3.3.1. Hermite polynomials

Consider the equation

$$y'' + xy' - ny = 0.$$

We show that

$$y = e^{-x^2/2}\left(\frac{d}{dx}\right)^n e^{x^2/2}$$

is a solution.

Also the operators $P = d/dx$ and $Q = x + d/dx$ satisfy the same operations, in partiuclar $[P, Q] = I$. We have that

$$Ly = QPy = y'' + xy',$$

so we would like to solve

$$QPv = nv.$$

This is easy: suppose that $QPQ^{n-1}v_0 = (n-1)Q^{n-1}v_0$ which is true for $v_0 = 1$ at $n = 1$. Then,

$$\begin{aligned} QPQ^n v_0 &= QPQQ^{n-1}v_0 = Q([P,Q] + QP)Q^{n-1}v_0 \\ &= Q^n v_0 + Q^2 PQ^{n-1}v_0 = Q^n + (n-1)Q^n v_0 = nQ^n v_0. \end{aligned}$$

Thus it follows that the solution is $Q^n 1$, and using that

$$Q = e^{-x^2/2}\frac{d}{dx}e^{x^2/2}$$

the answer is obtained.

4. The Exponential Map

Summary

- The Exponential map and how it connects a Lie group to its Lie algebra.
- The Lie group logarithm, plus and minus operators.
- The structure of the Lie algebras corresponding to common Lie groups.

Need a nice derivation showing how lie algebra properties arise

4.1. One-Parameter Groups

Best way to prove that Lie Groups have Lie Algebras?

- In [9] it is shown that for **matrix** Lie groups the set $\{A \in \mathrm{End} V \ : \ \exp tA \in G \forall t\} = \cap_t t \exp^{-1}(G)$ is a Lie Algebra (i.e. closed under the bracket operation).

Dual viewpoint: solutions $\Phi(x,t)$ of ODEs correspond to one-parameter groups [9].
Connection to linear systems.

4.2. The Exponential Map

Definition 4.1. *The **Exponential map** of a matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $t \in \mathbb{R}$ is*

$$\mathrm{Exp}(\boldsymbol{A}) = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n}{n!} \in \mathbb{C}^{n\times n}. \tag{4.1}$$

Properties of the exponential map

For the exponential map in Definition 4.1 we have

$$\mathrm{Exp}(t\boldsymbol{A})\,\mathrm{Exp}(s\boldsymbol{A}) = \mathrm{Exp}((t+s)\boldsymbol{A}), \tag{4.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Exp}(t\boldsymbol{A}) = \boldsymbol{A}\,\mathrm{Exp}(t\boldsymbol{A}) = \mathrm{Exp}(t\boldsymbol{A})\boldsymbol{A}, \tag{4.2b}$$

$$\det(\mathrm{Exp}(\boldsymbol{A})) = e^{\mathrm{Tr}(\boldsymbol{A})}. \tag{4.2c}$$

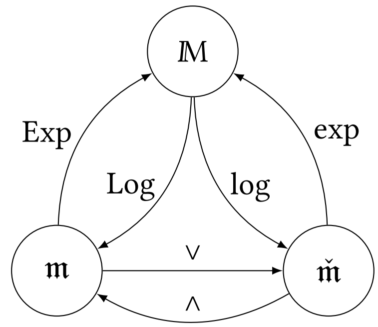


Figure 4.1.: Illustration of how the exponential maps connect a Lie Group $\mathbb{M}$, its Lie Algebra $\mathfrak{m}$, and the Lie Algebra parameterization which is the linear space $\check{\mathfrak{m}} \cong \mathbb{R}^n$.

The first two follow directly from the definition and are analogous to the scalar exponential map. Furthermore, (4.2a) implies that $\{\operatorname{Exp}(t\boldsymbol{A}) : t \in \mathbb{R}\}$ is a one-parameter subgroup of $\mathbb{M}$.

Not however that in general $\exp(\boldsymbol{A} + \boldsymbol{B}) \neq \exp(\boldsymbol{A}) \circ \exp(\boldsymbol{B})$ which is different from the scalar version. Equation (4.2c), known as Jacobi's identity, motivates a short proof:

Proof of (4.2c). It is easy to see that the eigenvalues of $\operatorname{Exp}(\boldsymbol{A})$ are the exponentials of the eigenvalues of $\boldsymbol{A}$. Since the determinant equals the product of the eigenvalues it follows that

$$\det(\operatorname{Exp}\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i(\operatorname{Exp}\boldsymbol{A}) = \prod_{i=1}^{n} e^{\lambda_i(\boldsymbol{A})} = e^{\sum_{i=1}^{n}\lambda_i(\boldsymbol{A})} = e^{\operatorname{Tr}(\boldsymbol{A})}. \tag{4.3}$$

□

4.2.1. Modern Definition

We contrast the algebraic Definition 4.1 with a more modern definition usually found in texts on differential geometry.

Definition 4.2. *The **exponential** of $\boldsymbol{A} \in \mathfrak{m}$ is*

$$\operatorname{Exp}\boldsymbol{A} := \gamma(1), \tag{4.4}$$

where γ is a one-parameter subgroup of $\mathbb{M}$ such that $\gamma'(0) = \boldsymbol{A}$.

This definition of course works even in the case that $\mathbb{M}$ is not a matrix Lie group, in fact it the same definition is used in more general differential geometry. We show that it coincides with Definition 4.1 for matrix Lie groups.

Proof. We know that $\gamma(0) = I$, $\gamma'(0) = \boldsymbol{A}$, and $\gamma(s)\gamma(t) = \gamma(s+t)$ by virtue of γ being a one-parameter sub-group.

$$\left(\gamma\left(\frac{h}{2}\right) - \gamma\left(-\frac{h}{2}\right)\right)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\gamma\left(\frac{h}{2}\right)^{n-k}\gamma\left(-\frac{h}{2}\right)^k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\gamma\left(\left(\frac{n}{2}-k\right)h\right). \tag{4.5}$$

For $h \to 0$ the left-hand side goes to $(h\gamma'(0))^n$, whereas the right-hand side is a finite-difference approximation of $h^n \gamma^{(n)}(0)$. It follows that $\gamma^n(0) = \boldsymbol{A}^n$, and hence a Taylor expansion around 0 gives

$$\gamma(1) = \sum_{k \geq 0} \frac{\gamma^{(n)}(0)(1-0)^n}{n!} = \sum_{k \geq 0} \frac{\boldsymbol{A}^n}{n!}. \tag{4.6}$$

□

A consequence of the proof above is that one-parameter subgroups of Lie groups are uniquely defined by their derivative at zero, and are therefore analogous to geodesics in Riemannian geometry.

4.3. The Lie Algebra of a Lie group

Definition 4.3. *For a matrix Lie group $\mathbb{M}$ the corresponding **matrix Lie algebra** $\mathfrak{m}$ is*

$$\mathfrak{m} = \{\boldsymbol{A} : \operatorname{Exp}(t\boldsymbol{A}) \in \mathbb{M} \ \forall t \in \mathbb{R}\}. \tag{4.7}$$

Just as for Lie groups, the matrix Lie algebras are typically parameterized by fewer than n^2 coefficients. In order to work with efficient parameterizations we therefore introduce a lower-dimensional parameterization denoted $\check{\mathfrak{m}}$. For this lower-dimensional representation we also define a lowercase exponential that maps from the parameterized lie algebra representation $\check{\mathfrak{m}}$ to the parameterized group representation $\check{\mathbb{M}}$:

$$\exp(\boldsymbol{a}) = \operatorname{Exp}(\boldsymbol{a}^\wedge). \tag{4.8}$$

The relationshop between the exponential maps and the hat and vee maps is shown in Figure 4.1.

Show that Lie algebra defined like this is indeed a Lie algebra(closed under bracket, jacobi, etc). Use property from previous chapter to show that as $t \to 0$ we obtain a tangent that is equal to the bracket. Group property is then enough to conclude.

4.4. The Logarithm

The matrix logarithm $\operatorname{Log} : \mathbb{M} \to \mathfrak{m}$ is defined as the inverse of the matrix exponential, and we also define lowercase $\log : \mathbb{M} \to \check{\mathfrak{m}}$ for mappings between the parameterized representations:

$$\begin{aligned} \operatorname{Log} \boldsymbol{X} &= \sum_{k \geq 1} (-1)^{k+1} \frac{(\boldsymbol{X} - \boldsymbol{I})^k}{k}, \\ \log \boldsymbol{X} &= (\operatorname{Log} \boldsymbol{X})^\vee. \end{aligned} \tag{4.9}$$

Show that it's the inverse of the exponential

4.5. Exponential and Logarithm derivations

To reveal the structure of a Lie algebra it is advantageous to study one-parameter subgroups

$$\boldsymbol{X}(t) = \operatorname{Exp}(t\boldsymbol{A}) \in \boldsymbol{X} \implies \boldsymbol{X}(0) = I,\ \boldsymbol{X}'(0) = \boldsymbol{A}. \tag{4.10}$$

The trajectory $\boldsymbol{X}(t)$ must satisfy a certain group constraint (e.g., orthogonal, unitary), which translates into a condition on the derivative. In essence, the Lie algebra consists of *tangent* elements to the Lie group. Once the structure of the Lie algebra is obtained, the form of the exponential can be found through Definition 4.1. We use this method to find the structure, exponential, and logartihm of the Lie algebras of several matrix Lie groups. For a non-matrix Lie group $\check{\mathbb{M}}$ that is homomorphic to a matrix Lie group $\mathbb{M}$ (such as $\check{\mathbb{SO}}(2)$ and $\mathbb{SO}(2)$, etc), we note that the exponentials and logarithms can be easily obtained as

$$\exp_{\check{\mathbb{M}}}(\boldsymbol{v}) = \left(\exp_{\mathbb{M}}(\boldsymbol{v})\right)^{\vee}, \tag{4.11a}$$

$$\log_{\check{\mathbb{M}}}(\boldsymbol{x}) = \log_{\mathbb{M}}(\hat{\boldsymbol{x}}). \tag{4.11b}$$

4.5.1. $\mathbb{E}(n)$

The matrices of $\mathbb{E}(n)$ defined in (2.2) are s.t. only the top n coefficients in the right-most column can vary. Consider a trajectory

$$\boldsymbol{X}(t) = \operatorname{Exp}(t\boldsymbol{A}) \in \mathbb{E}(n), \tag{4.12}$$

differentiating with respect to t then shows that

$$\begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \overset{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{A}. \tag{4.13}$$

From here it follows that the Lie algebra $\mathfrak{e}(n)$ of $\mathbb{E}(n)$ consists of matrices where the top n coefficients in the right-most column are non-zero:

$$\mathfrak{e}(n) = \left\{ \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}, \boldsymbol{v} \in \mathbb{R}^n \right\}. \tag{4.14}$$

It is straightforward to calculate the exponential of a matrix element $\boldsymbol{A} \in \mathfrak{e}(n)$

$$\operatorname{Exp} \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} := \sum_{k\geq 0} \frac{1}{k!} \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}^k = \begin{bmatrix} I_n & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}. \tag{4.15}$$

It follows that the exponential on $\mathbb{E}(n)$, and consequently also the logarithm, is the identity mapping.

The exponential and logarithm on $\mathbb{E}(n)$ are

$$\exp_{\mathbb{E}(n)} \boldsymbol{v} = \operatorname{Exp}_{\mathbb{E}(n)} \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} = \begin{bmatrix} I_n & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \in \mathbb{E}(n), \tag{4.16a}$$

$$\operatorname{Log}_{\mathbb{E}(n)} \begin{bmatrix} I_n & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \in \mathfrak{e}(n), \tag{4.16b}$$

$$\log_{\mathbb{E}(n)} \begin{bmatrix} I_n & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{p} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}^{\vee} = \boldsymbol{p} \in \check{\mathfrak{e}}(n). \tag{4.16c}$$

4.5.2. $\mathbb{SO}(2)$ and $\mathbb{SO}(3)$

For the special orthogonal groups $\mathbb{SO}(n)$ (for any dimension) the group constraint is orthogonality of the matrix: $\boldsymbol{X}^T\boldsymbol{X} = I_n$. Take a one-parameter subgroup $\boldsymbol{X}(t) \coloneqq \operatorname{Exp}(t\boldsymbol{A})$; it must then hold that

$$0 \overset{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)^T\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{X}'(0)^T\boldsymbol{X}(0) + \boldsymbol{X}(0)^T\boldsymbol{X}'(0) = \boldsymbol{A}^T + \boldsymbol{A}. \tag{4.17}$$

It follows that the Lie algebra $\mathfrak{so}(n)$ corresponding to $\mathbb{SO}(n)$ consists of **skew-symmetric matrices**.

$$\mathfrak{so}(n) = \left\{\boldsymbol{A} \in \mathbb{R}^{n\times n} \;:\; \boldsymbol{A}^T + \boldsymbol{A} = 0\right\}. \tag{4.18}$$

The 2×2 skew-symmetric matrices have only one degree of freedom, let this single parameter of $\check{\mathfrak{so}}(2)$ be denoted ω_z so that

$$\mathfrak{so}(2) = \left\{\begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \middle| \omega_z \in \mathbb{R}\right\}. \tag{4.19}$$

Take an element $\omega_z^\wedge \coloneqq \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \in \mathfrak{so}(2)$. The exponential of is easily calculated by noting that $(\omega_z^\wedge)^2 = -\omega_z^2 I_2$:

$$\begin{aligned} \operatorname{Exp}\omega_z^\wedge = \sum_{k\geq 0} \frac{\omega_z^{\wedge k}}{k!} = \left(1 - \frac{\omega_z^2}{2!} + \frac{\omega_z^4}{4!} - \dots\right) I_2 + \left(1 - \frac{\omega^2}{3!} + \frac{\omega^4}{5!} - \dots\right)\omega_z^\wedge &= \cos\omega_z I_2 + \omega_z \sin\omega_z\omega_z^\wedge \\ &= \begin{bmatrix} \cos\omega_z & -\sin\omega_z \\ \sin\omega_z & \cos\omega_z \end{bmatrix} \end{aligned} \tag{4.20}$$

Then $\operatorname{Exp} : \mathfrak{so}(2) \to \mathbb{SO}(2)$ and $\exp : \check{\mathfrak{so}}(2) \to \check{\mathbb{SO}}(2)$ are as follows:

The exponential and logarithm on $\mathbb{SO}(2)$ are

$$\exp_{\mathbb{SO}(2)}(\omega_w) = \operatorname{Exp}_{\mathbb{SO}(2)} \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} = \begin{bmatrix} \cos\omega_z & -\sin\omega_z \\ \sin\omega_z & \cos\omega_z \end{bmatrix}, \tag{4.21a}$$

$$\operatorname{Log}_{\mathbb{SO}(2)} \begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} \in \mathfrak{so}(2), \quad \alpha = \operatorname{arctan2}(q_z, q_w), \tag{4.21b}$$

$$\log_{\mathbb{SO}(2)} \begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}^{\vee} = \alpha \in \check{\mathfrak{so}}(2), \tag{4.21c}$$

where $\operatorname{arctan2}(y, x)$ is the angle between the ray passing through $(1, 0)$ and the ray (x, y).

We proceed with $\mathbb{SO}(3)$: skew-symmetric matrices of size 3×3 are parameterized by three parameters $\boldsymbol{\omega} \coloneqq (\omega_x, \omega_y, \omega_z)$ so that $\mathfrak{so}(3)$ consists of elements on the form

$$\hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{4.22}$$

Such a matrix has several interesting properties. First of all, for $\boldsymbol{u} \in \mathbb{R}^3$ left matrix multiplication of $\hat{\boldsymbol{\omega}}$ is equivalent to taking the vector cross product: $\hat{\boldsymbol{\omega}}\boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{u}$. As a result many properties of the cross product are inherited by the embedding $\mathbb{R}^3 \overset{\wedge}{\mapsto} \mathbb{R}^{3\times 3}$.

Properties of $\wedge$ on $\mathfrak{so}(3)$

For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$:

$$\hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\hat{\boldsymbol{a}} = -(\boldsymbol{a} \cdot \boldsymbol{b})\hat{\boldsymbol{a}}, \tag{4.23a}$$

$$\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}, \tag{4.23b}$$

$$\boldsymbol{a} \cdot (\hat{\boldsymbol{b}}\boldsymbol{c}) = \boldsymbol{b} \cdot (\hat{\boldsymbol{c}}\boldsymbol{a}), \tag{4.23c}$$

$$A\hat{\boldsymbol{b}} = \operatorname{Tr}(A)\hat{\boldsymbol{b}} - (A\boldsymbol{b})^{\wedge} - \hat{\boldsymbol{b}}A, \quad A \text{ symmetric } 3 \times 3 \text{ matrix}, \tag{4.23d}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{2}\left\langle \hat{\boldsymbol{a}}, \hat{\boldsymbol{b}} \right\rangle_F = -\frac{1}{2}\operatorname{Tr}\left(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\right). \tag{4.23e}$$

Proof of (4.23a). Consider $\hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}\boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c}))$. Expanding with the vector triple product gives

$$\hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}\boldsymbol{c} = \boldsymbol{a} \times ((\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}) = -(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{a} \times \boldsymbol{c} = -(\boldsymbol{a} \cdot \boldsymbol{b})\hat{\boldsymbol{a}}\boldsymbol{c}. \tag{4.24}$$

$\square$

We can use (4.23a) to obtain the exponential map on $\mathfrak{so}(3)$:

$$\begin{aligned}
\operatorname{Exp}\ \hat{\boldsymbol{\omega}} &= \sum_{k\geq 0}\frac{(\hat{\boldsymbol{\omega}})^k}{k!} = I + \hat{\boldsymbol{\omega}} + \frac{\hat{\boldsymbol{\omega}}^2}{2!} - \|\boldsymbol{\omega}\|^2\left(\frac{\hat{\boldsymbol{\omega}}}{3!} + \frac{\hat{\boldsymbol{\omega}}^2}{4!}\right) + \|\boldsymbol{\omega}\|^4\left(\frac{\hat{\boldsymbol{\omega}}}{5!} + \frac{\hat{\boldsymbol{\omega}}^2}{6!}\right) + \ldots \\
&= I + \left(1 - \frac{\|\boldsymbol{\omega}\|^2}{3!} + \frac{\|\boldsymbol{\omega}\|^4}{5!} - \ldots\right)\hat{\boldsymbol{\omega}} + \left(\frac{1}{2!} - \frac{\|\boldsymbol{\omega}\|^2}{4!} + \frac{\|\boldsymbol{\omega}\|^4}{6!} - \ldots\right)\hat{\boldsymbol{\omega}}^2 \\
&= I + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2.
\end{aligned} \tag{4.25}$$

To obtain the logarithm the expression

$$\boldsymbol{R} = I_3 + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2 \tag{4.26}$$

should be inverted. First note that due to $\hat{\boldsymbol{\omega}}$ being skew-symmetric:

$$\boldsymbol{R} - \boldsymbol{R}^T = \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2 - \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}}^T - \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}(\hat{\boldsymbol{\omega}}^T)^2 = 2\frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}}. \tag{4.27}$$

Secondly,

$$\operatorname{Tr}(\boldsymbol{R}) = 3 + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\operatorname{Tr}(\hat{\boldsymbol{\omega}}^2) = 3 - 2\frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\|\boldsymbol{\omega}\|^2 = 1 + 2\cos\|\boldsymbol{\omega}\|, \tag{4.28}$$

which makes it possible to write down an expression for the logarithm.

The exponential and logarithm on $\mathbb{SO}(3)$ are

$$\exp_{\mathbb{SO}(3)}\boldsymbol{\omega} = \operatorname{Exp}_{\mathbb{SO}(3)}\hat{\boldsymbol{\omega}} = I + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2, \tag{4.29a}$$

$$\operatorname{Log}_{\mathbb{SO}(3)}\boldsymbol{R} = \frac{\alpha}{\sin\alpha}\frac{\boldsymbol{R}-\boldsymbol{R}^T}{2}, \quad \alpha = \arccos\left(\frac{\operatorname{Tr}(\boldsymbol{R})-1}{2}\right). \tag{4.29b}$$

The lower-dimensional representation of $\mathbb{SO}(3)$ is $\mathbb{S}^3$, but as shown previously the $\wedge$ and $\vee$ mappings are not straightforward. In the next section we obtain the exponential and logarithm on $\mathbb{S}^3$ through its relation to $\mathbb{SU}(2)$.

4.5.3. $\mathbb{SU}(n)$

We proceed with $\mathbb{SU}(n)$. This family of Lie groups consists of complex unitary matrices that satisfy the group constraint $\boldsymbol{X}^*\boldsymbol{X} = I_n$, where $\boldsymbol{X}^*$ is the Hermitian transpose[1] of $\boldsymbol{X}$. For a one-parameter subgroup $\boldsymbol{X}(t) = \operatorname{Exp}(t\boldsymbol{A})$ we first note that due to (4.2c) the trace of $\boldsymbol{A}$ must be zero so that $\det\operatorname{Exp}(t\boldsymbol{A}) = 1$. We furthermore get that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)^*\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{X}'(0)^*\boldsymbol{X}(0) + \boldsymbol{X}(0)^*\boldsymbol{X}'(0) = \boldsymbol{A}^* + \boldsymbol{A}, \tag{4.30}$$

[1]The Hermitian transpose (also known as *conjugate transpose*) of $\boldsymbol{A}_{ij}$ is $\boldsymbol{A}^*_{ij} = \bar{\boldsymbol{A}}_{ji}$.

which implies that the Lie Algebra $\mathfrak{su}(n)$ consists of **skew-Hermitian matrices with vanishing trace**.

$$\mathfrak{su}(n) = \{A \in \mathbb{C}^{n\times n} \mid A^* + A = 0, \operatorname{Tr} A = 0\}. \tag{4.31}$$

The Lie algebra $\mathfrak{su}(2)$ is parameterized by three elements $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ that correspond to the skew-Hermitian matrix $\hat{\boldsymbol{\omega}} :=:= \frac{1}{2}\begin{bmatrix} i\omega_z & -\omega_x - i\omega_y \\ \omega_x - i\omega_y & -i\omega_z \end{bmatrix}$, where the factor $1/2$ is added for reasons that will become clear below. A simple calculation reveals that $\hat{\boldsymbol{\omega}}^2 = -\frac{\|\boldsymbol{\omega}\|}{4} I_2$ which can be used to evaluate the exponential.

$$\begin{aligned}
\operatorname{Exp}\hat{\boldsymbol{\omega}} &= \sum_{k\geq 0}\frac{\hat{\boldsymbol{\omega}}^k}{k!} = \sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\|\boldsymbol{\omega}\|}{2}\right)^{2*\lfloor\frac{k}{2}\rfloor}\hat{\boldsymbol{\omega}}^{(k \mod 2)} \\
&= \left(1 - \frac{(\|\boldsymbol{\omega}\|/2)^2}{2!} + \frac{(\|\boldsymbol{\omega}\|/2)^4}{4!} - \ldots\right) I + \left(1 - \frac{(\|\boldsymbol{\omega}\|/2)^2}{3!} + \frac{(\|\boldsymbol{\omega}\|/2)^4}{5!} + \ldots\right)\hat{\boldsymbol{\omega}} \\
&= \cos(\|\boldsymbol{\omega}\|/2) I + 2\frac{\sin\|\boldsymbol{\omega}\|/2}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} = \begin{bmatrix} \cos(\|\boldsymbol{\omega}\|/2) + i\frac{\omega_z\sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} & (-\omega_x - i\omega_y)\frac{\sin\|\boldsymbol{\omega}\|/2}{\|\boldsymbol{\omega}\|} \\ (\omega_x - i\omega_y)\frac{\sin\|\boldsymbol{\omega}\|/2}{\|\boldsymbol{\omega}\|} & \cos(\|\boldsymbol{\omega}\|/2) - i\frac{\omega_z\sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} \end{bmatrix}.
\end{aligned}$$

The upper-case exponential on $\mathbb{SU}(2)$ is:

$$\operatorname{Exp}\frac{1}{2}\begin{bmatrix} i\omega_z & -\omega_x - i\omega_y \\ \omega_x - i\omega_y & -i\omega_z \end{bmatrix} = \begin{bmatrix} \cos(\|\boldsymbol{\omega}\|/2) + i\frac{\omega_z\sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} & (-\omega_x - i\omega_y)\frac{\sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} \\ (\omega_x - i\omega_y)\frac{\sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} & \cos(\|\boldsymbol{\omega}\|/2) - i\frac{\omega_z\sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} \end{bmatrix}. \tag{4.32}$$

Since the mappings $\wedge : \mathbb{S}^3 \to \mathbb{SU}(2)$ and $\vee : \mathbb{SU}(2) \to \mathbb{S}^3$ are straightforward, we can also write down the exponentail and logarithm on $\mathbb{S}^3$. Since $\mathbb{S}^3$ is not itself a matrix Lie group, the uppercase exponential and logarithm do not have a meaning.

The exponential and logarithm on $\mathbb{S}^3$ are

$$\exp(\omega_x, \omega_y, \omega_z) = \left(\cos(\|\boldsymbol{\omega}\|/2), \frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2), \frac{\omega_y}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2), \frac{\omega_z}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right), \tag{4.33a}$$

$$\log(q_w, q_x, q_y, q_z) = \left(2\frac{\operatorname{arctan2}\left(\sqrt{q_x^2 + q_y^2 + q_z^2}, q_w\right)}{\sqrt{q_x^2 + q_y^2 + q_z^2}}\right) \times (q_x, q_y, q_z). \tag{4.33b}$$

From (4.33) the reason to divide the expression for $\hat{\boldsymbol{\omega}}$ by a factor 2 becomes apparent—$\|\boldsymbol{\omega}\|$ represents the rotation angle in radians. We provide a quick proof for the logarithm expression.

Proof of (12.5). Let $(q_w, q_x, q_y, q_z) = \exp(\omega_x, \omega_y, \omega_z)$. From (12.4) we have that

$$\sqrt{q_x^2 + q_y^2 + q_z^2} = \sqrt{\left(\frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right)^2 + \left(\frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right)^2 + \left(\frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right)^2} = \sin(\|\boldsymbol{\omega}\|/2). \tag{4.34}$$

It also follows from the same equation that

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{\|\boldsymbol{\omega}\|}{\sin(\|\boldsymbol{\omega}\|/2)}(q_x, q_y, q_z) = \frac{\operatorname{arctan2}(\sin\|\boldsymbol{\omega}\|, \cos\|\boldsymbol{\omega}\|)}{\sqrt{q_x^2+q_y^2+q_z^2}} \\
&= \frac{2\operatorname{arctan2}(\sin(\|\boldsymbol{\omega}\|/2), \cos(\|\boldsymbol{\omega}\|/2))}{\sqrt{q_x^2+q_y^2+q_z^2}} = \frac{2\operatorname{arctan2}\left(\sqrt{q_x^2+q_y^2+q_z^2}, q_w\right)}{\sqrt{q_x^2+q_y^2+q_z^2}}.
\end{aligned}
\tag{4.35}
$$

□

4.6. $\mathbb{SE}(2)$ and $\mathbb{SE}(3)$

For the semi-simple groups $\mathbb{SE}(2)$ and $\mathbb{SE}(3)$ we derive an identity that will be useful to construct the exponential maps.

Lemma 4.1. *Consider two matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ such that $\boldsymbol{B}^2 = \boldsymbol{B}\boldsymbol{A} = 0$. Then we have that*

$$
\operatorname{Exp}(\boldsymbol{A}+\boldsymbol{B}) = \operatorname{Exp}(\boldsymbol{A}) + \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{(k+1)!}\boldsymbol{B}. \tag{4.36}
$$

Proof. When we expand $(\boldsymbol{A}+\boldsymbol{B})^k$ all terms that contain a $\boldsymbol{B}$ before an $\boldsymbol{A}$, or multiple $\boldsymbol{B}$ in a row, vanish. As a result,

$$
\operatorname{Exp}(\boldsymbol{A}+\boldsymbol{B}) = I_n + \sum_{k=1}^{\infty}\frac{(\boldsymbol{A}+\boldsymbol{B})^k}{k!} = I_n + \sum_{k=1}^{\infty}\frac{\boldsymbol{A}^k + \boldsymbol{A}^{k-1}\boldsymbol{B}}{k!} = \operatorname{Exp}(\boldsymbol{A}) + \sum_{k=1}^{\infty}\frac{\boldsymbol{A}^{k-1}}{k!}\boldsymbol{B}. \tag{4.37}
$$

□

We can now easily derive the exponential maps for $\mathfrak{se}(2)$ and $\mathfrak{se}(3)$. For both groups we have the following structure:

$$
\boldsymbol{A} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{p} \\ \boldsymbol{0} & 0 \end{bmatrix}. \tag{4.38}
$$

Thus, it suffices to compute $S(\omega) := \sum_{k=0}^{\infty} \frac{\hat{\omega}}{(k+1)!}$ to obtain the exponential map for the semi-simple groups.

For $\mathfrak{se}(2)$, disregarding the trivial case $\boldsymbol{\omega} = 0$, we obtain

$$
\begin{aligned}
S(\omega_z) &= \sum_{k=0}^{\infty}\frac{\hat{\omega}_z^k}{(k+1)!} = (\hat{\omega}_z)^{-1}\left(\operatorname{Exp}\hat{\omega}_z - I\right) \\
&= \frac{1}{\omega_z^2}\begin{bmatrix} 0 & \omega_z \\ -\omega_z & 0 \end{bmatrix}\begin{bmatrix} \cos\omega_z - 1 & -\sin\omega_z \\ \sin\omega_z & \cos\omega_z - 1 \end{bmatrix} = \frac{1}{\omega_z}\begin{bmatrix} \sin\omega_z & \cos\omega_z - 1 \\ 1-\cos\omega_z & \sin\omega_z \end{bmatrix}.
\end{aligned}
\tag{4.39}
$$

This gives everything required to write down the exponential and logarithmic maps for $\mathbb{SE}(2)$.

The uppercase exponential and logarithm maps are Exp : $\mathfrak{se}(2) \to \mathrm{SE}(2)$ and Log : $\mathrm{SE}(2) \to \mathfrak{se}(2)$:

$$\exp_{\mathrm{SE}(2)}(\omega_z, \boldsymbol{v}) = \mathrm{Exp}_{\mathrm{SE}(2)} \begin{bmatrix} \hat{\omega}_z & \boldsymbol{v} \\ \mathbf{0}_{1\times 2} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Exp}_{\mathrm{SO}(2)}(\hat{\omega}_z) & S(\omega_z)\boldsymbol{v} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix}, \tag{4.40a}$$

$$\mathrm{Log}_{\mathrm{SE}(2)} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix} = \begin{bmatrix} \hat{\alpha} & (S(\omega_z))^{-1}\boldsymbol{p} \\ \mathbf{0}_{1\times 2} & 0 \end{bmatrix}, \tag{4.40b}$$

$$\log_{\mathrm{SE}(2)} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix} = \left(\alpha, (S(\omega_z))^{-1}\boldsymbol{p}\right) \in \check{\mathfrak{se}}(2), \tag{4.40c}$$

where $\alpha = \log_{\mathrm{SO}(2)} \boldsymbol{R}$.

Continuing with $\mathrm{SE}(3)$ we utilize (4.23a) to calculate

$$\sum_{k=0}^{\infty} \frac{\hat{\boldsymbol{\omega}}^k}{(k+1)!} = I_3 - \frac{1}{\|\boldsymbol{\omega}\|^2}\hat{\ } \sum_{k=2}^{\infty} \frac{\hat{\boldsymbol{\omega}}^k}{k!} = I_3 - \frac{1}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}\left(\mathrm{Exp}_{\mathrm{SO}(3)}(\hat{\boldsymbol{\omega}}) - I_3 - \hat{\boldsymbol{\omega}}\right). \tag{4.41}$$

From (4.25) we then have that

$$\begin{aligned} \mathrm{d}^l\left(\exp_{\mathrm{SO}(3)}\right)_{\boldsymbol{\omega}} &:= \sum_{k=0}^{\infty} \frac{\hat{\boldsymbol{\omega}}^k}{(k+1)!} = I_3 - \frac{1}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}\left(\left(I_3 + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{(1-\cos\|\boldsymbol{\omega}\|)}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2\right) - I_3 - \hat{\boldsymbol{\omega}}\right) \\ &= I_3 - \frac{(\sin\|\boldsymbol{\omega}\| - \|\boldsymbol{\omega}\|)}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2 + \frac{\|\boldsymbol{\omega}\|^2(1-\cos\|\boldsymbol{\omega}\|)}{\|\boldsymbol{\omega}\|^4}\hat{\boldsymbol{\omega}} \\ &= I_3 + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2 + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}. \end{aligned} \tag{4.42}$$

Applying Lemma 4.1 then gives the exponential.

The matrix exponential on $\mathfrak{se}(3)$ is

$$\exp_{\mathrm{SE}(3)}(\boldsymbol{\omega}, \boldsymbol{v}) = \mathrm{Exp}_{\mathrm{SE}(3)} \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Exp}_{\mathrm{SO}(3)}(\hat{\boldsymbol{\omega}}) & \mathrm{d}^l\left(\exp_{\mathrm{SO}(3)}\right)_{\boldsymbol{\omega}}\boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}, \tag{4.43a}$$

$$\mathrm{Log}_{\mathrm{SE}(3)} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\alpha}} & \left(\mathrm{d}^l\left(\exp_{\mathrm{SO}(3)}\right)_{\boldsymbol{\alpha}}\right)^{-1}\boldsymbol{p} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}, \tag{4.43b}$$

$$\log_{\mathrm{SE}(3)} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \left(\boldsymbol{\alpha}, \left(\mathrm{d}^l\left(\exp_{\mathrm{SO}(3)}\right)_{\boldsymbol{\alpha}}\right)^{-1}\boldsymbol{p}\right), \tag{4.43c}$$

where $\boldsymbol{\alpha} = \log_{\mathrm{SO}(3)} R$.

4.7. Baker–Campbell–Hausdorff formula

Prove BCH formula and provide context

$$\log\left(\exp \boldsymbol{a} \circ \exp \boldsymbol{b}\right) = \boldsymbol{a} + \boldsymbol{b} + \frac{1}{2}\left[\boldsymbol{a}, \boldsymbol{b}\right] + \frac{1}{12}\left[\boldsymbol{a}, \left[\boldsymbol{a}, \boldsymbol{b}\right]\right] - \frac{1}{12}\left[\boldsymbol{b}, \left[\boldsymbol{a}, \boldsymbol{b}\right]\right] + \dots \tag{4.44}$$

4.8. Plus and Minus Operators

Algorithms for optimization and numerical integration require taking small additative steps, but Lie groups are not closed under normal addition and subtraction. We can however define generalized addition and subtraction operators $\oplus, \ominus$ for Lie groups that behave similarly to how $+$ and $-$ operate on regular vector spaces.

The plus operations add an increment $\boldsymbol{a} \in \check{\mathfrak{m}}$ in the parameterized tangent space to an element $\boldsymbol{X} \in \mathbb{M}$ of the group, whereas the minus operators give the difference between two group elements as a vector in the parameterized tangent space.

$$\boldsymbol{X} \oplus_r \boldsymbol{a} = \boldsymbol{X} \circ \exp\left(\boldsymbol{a}\right) \in \check{\mathbb{M}}, \tag{right-plus}$$
$$Y \ominus_r \boldsymbol{X} = \log\left(\boldsymbol{X}^{-1} \circ Y\right) \in T_{\boldsymbol{X}}\check{\mathbb{M}} \cong \check{\mathfrak{m}}, \tag{right-minus}$$
$$\boldsymbol{a} \oplus_l \boldsymbol{X} = \exp\left(\boldsymbol{a}\right) \circ \boldsymbol{X} \in \check{\mathbb{M}}, \tag{left-plus}$$
$$Y \ominus_l \boldsymbol{X} = \log\left(Y \circ \boldsymbol{X}^{-1}\right) \in T_e\check{\mathbb{M}} \cong \check{\mathfrak{m}}. \tag{left-minus}$$

The plus operators are differentiated by the order: the right-plus has the tangent element at $\boldsymbol{X}$ while left-plus has the reverse order, meaning that the tangent element belongs to the tangent space at e.

Note that the derivates are defined in a way so that

$$\boldsymbol{X} \oplus_r (Y \ominus_r \boldsymbol{X}) = \boldsymbol{X} \circ \exp\log(\boldsymbol{X}^{-1}Y) = Y, \tag{4.45a}$$
$$(\boldsymbol{X} \oplus_r \boldsymbol{a}) \ominus_r \boldsymbol{X} = \log(\boldsymbol{X}^{-1} \circ \boldsymbol{X} \circ \exp\boldsymbol{a}) = \boldsymbol{a}, \tag{4.45b}$$
$$(Y \ominus_l \boldsymbol{X}) \oplus_l \boldsymbol{X} = \exp\log(Y \circ \boldsymbol{X}^{-1}) \circ \boldsymbol{X} = Y, \tag{4.45c}$$
$$(\boldsymbol{a} \oplus_l \boldsymbol{X}) \ominus_l \boldsymbol{X} = \log(\exp\boldsymbol{a} \circ \boldsymbol{X} \circ \boldsymbol{X}^{-1}) = \boldsymbol{a}. \tag{4.45d}$$

4.9. Homomorphy of Lie Groups implies Homomorphy of Lie Algebras

This is important since it implies that $SO(3)$ and S^3 can be treated analogously. A proof is in [9, Corr. 20].

4.10. The Adjoint

We define the **adjoint** $\mathrm{Ad}_X : \mathfrak{m} \to \mathfrak{m}$ of a matrix $\boldsymbol{A} \in \mathfrak{m}$ as

$$\mathrm{Ad}_X \boldsymbol{A} := \boldsymbol{X}\boldsymbol{A}\boldsymbol{X}^{-1}. \tag{4.46}$$

From the definition of the exponential map in (4.1) it can be seen that $\mathrm{Exp}(\mathrm{Ad}_X \boldsymbol{A}) \in \mathbb{M}$ if and only if $\mathrm{Exp}(\boldsymbol{A}) \in \mathbb{M}$, which implies that the lie algebra $\mathfrak{m}$ is closed under action of the adjoint.

The adjoint of a tangent matrix element $\boldsymbol{a} \in \check{\mathfrak{m}}$ is similary defined as a linear mapping $\mathbf{Ad}_X : \check{\mathfrak{m}} \to \check{\mathfrak{m}}$:

$$\mathbf{Ad}_X \boldsymbol{a} := (\mathrm{Ad}_X \hat{\boldsymbol{a}})^\vee = (\boldsymbol{X}\hat{\boldsymbol{a}}\boldsymbol{X}^{-1})^\vee. \tag{4.47}$$

For a given $\boldsymbol{X}$ this is a linear map, so $\mathbf{Ad}_X$ is an $n \times n$ matrix. The adjoint represents a coordinate change from the tangent space $T_X\check{\mathbb{M}}$ to the tangent space $T_e\check{\mathbb{M}} = \check{\mathfrak{m}}$ at the origin.

Remark 4.1. *Since the definition of* Ad *involves matrix multiplication it does not make sense for groups like* $\check{\mathbb{SO}}(2)$ *and* $\mathbb{S}^3$ *that are not matrix Lie groups. We can however still define the bold-face adjoint* $\boldsymbol{Ad}$ *on* $\check{\mathbb{M}}$ *as*

$$\boldsymbol{Ad}_x := \boldsymbol{Ad}_{\hat{x}}, \tag{4.48}$$

where $\wedge : \check{\mathbb{M}} \to \mathbb{M}$ *is a Lie group homomorphism that maps* $\check{\mathbb{M}}$ *into a matrix Lie group.*

Properties of the adjoint

The adjoints satisfy the following properties:

$$\mathbf{Ad}_X^{-1} = \mathbf{Ad}_{X^{-1}}, \tag{4.49a}$$
$$\mathbf{Ad}_X \mathbf{Ad}_Y = \mathbf{Ad}_{X \circ Y}, \tag{4.49b}$$
$$\exp\ \mathbf{Ad}_X\ \boldsymbol{a} = \boldsymbol{X} \circ \exp \boldsymbol{a} \circ \boldsymbol{X}^{-1}, \tag{4.49c}$$
$$\boldsymbol{X} \oplus_r \boldsymbol{a} = (\mathbf{Ad}_X \boldsymbol{a}) \oplus_l \boldsymbol{X}. \tag{4.49d}$$

The first two properties follow directly from the definition. Equation (4.49c) follows from

$$\exp \mathbf{Ad}_X \boldsymbol{a} = \mathrm{Exp}\ \ \mathrm{Ad}_X\ \ \hat{\boldsymbol{a}} = \sum_{k\geq 0} \frac{\left(\boldsymbol{X}\hat{\boldsymbol{a}}\boldsymbol{X}^{-1}\right)^k}{k} = \boldsymbol{X}\left(\sum_{k\geq 0}\frac{\hat{\boldsymbol{a}}^k}{k}\right)\boldsymbol{X}^{-1} = \boldsymbol{X}\exp(\boldsymbol{a})\boldsymbol{X}^{-1}. \tag{4.50}$$

We can then also show (4.49d)

$$\boldsymbol{X} \oplus_r \boldsymbol{a} \overset{\text{(right-plus)}}{=} \boldsymbol{X} \circ \exp(\boldsymbol{a}) = (\boldsymbol{X} \circ \exp(\boldsymbol{a}) \circ \boldsymbol{X}^{-1}) \circ \boldsymbol{X} \overset{(4.49c)}{=} \exp(\mathbf{Ad}_X \boldsymbol{a}) \circ \boldsymbol{X} \overset{\text{(left-plus)}}{=} (\mathbf{Ad}_X \boldsymbol{a}) \oplus_l \boldsymbol{X}. \tag{4.51}$$

Finally a result regarding the derivative of the adjoint.

Lemma 4.2.

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{Ad}_{\exp(\lambda(t)\boldsymbol{a})} = \lambda'(t)\, \mathrm{ad}_{\boldsymbol{a}}\, \boldsymbol{Ad}_{\exp(\lambda(t)\boldsymbol{a})}\,. \tag{4.52}$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{a})} &\overset{(5.22)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{\infty}\exp(\mathrm{ad}_{\lambda(t)\boldsymbol{a}}) \overset{(??)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{\infty}\exp(\lambda(t)\,\mathrm{ad}_{\boldsymbol{a}}) \overset{(5.21)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{\infty}\frac{\lambda(t)^k\,\mathrm{ad}_{\boldsymbol{a}}^k}{k!} \\ &= \lambda'(t)\sum_{k=1}^{\infty}\frac{\lambda(t)^{k-1}\,\mathrm{ad}_{\boldsymbol{a}}^k}{(k-1)!} = \lambda'(t)\,\mathrm{ad}_{\boldsymbol{a}}\sum_{k=1}^{\infty}\frac{\lambda(t)^{k-1}\,\mathrm{ad}_{\boldsymbol{a}}^{k-1}}{(k-1)!} = \lambda'(t)\,\mathrm{ad}_{\boldsymbol{a}}\,\mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{a})}\,.\end{aligned} \tag{4.53}$$

□

Examples calculating the adjoint for a couple of groups

5. Derivatives

Summary

- Definition of derivatives on manifolds.
- Differentiation rules.

Define derivatives w.r.t. matrix elements only, motivate that we can disregard parameterized expressions.

Definition 5.1. *The* ***right derivative*** *of* $f : \mathbb{M} \to \mathbb{N}$ *at* $\boldsymbol{X} \in \check{\mathbb{M}}$ *is a linear mapping* $\mathrm{d}^r f_X : T\mathbb{M}_X \to T\mathbb{N}_{f(X)}$ *such that:*

$$\mathrm{d}^r f_X := \lim_{\boldsymbol{a}\to 0} \frac{f(\boldsymbol{X} \oplus_r \boldsymbol{a}) \ominus_r f(\boldsymbol{X})}{\boldsymbol{a}} = \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(\boldsymbol{X})^{-1} \circ f(\boldsymbol{X} \circ \exp(\boldsymbol{a}))\right)}{\boldsymbol{a}}, \tag{5.1}$$

where $\boldsymbol{a} \in T_X\check{\mathbb{M}}$ *is a member of the parameterized Lie algebra and the division is component-wise.*
Similarly, the ***left derivative*** *is a linear mapping* $\mathrm{d}^r f_X : T\mathbb{M}_e \to T\mathbb{N}_e$ *such that*

$$\mathrm{d}^l f_X := \lim_{\boldsymbol{a}\to 0} \frac{f(\boldsymbol{X} \oplus_l \boldsymbol{a}) \ominus_l f(\boldsymbol{X})}{\boldsymbol{a}} = \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(\exp(\boldsymbol{a}) \circ \boldsymbol{X}) \circ f(\boldsymbol{X})^{-1}\right)}{\boldsymbol{a}}, \tag{5.2}$$

From the definition it can be seen that for small $\boldsymbol{a}$ it approximately holds that

$$f(\boldsymbol{X} \oplus_r \boldsymbol{a}) = f(\boldsymbol{X}) \oplus_r \left(\mathrm{d}^r f_X \boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^2)\right), \tag{5.3}$$

and for left-plus:

$$f(\boldsymbol{a} \oplus_l \boldsymbol{X}) = \left(\mathrm{d}^l f_X \boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^2)\right) \oplus_l f(\boldsymbol{X}). \tag{5.4}$$

From (5.3) and (5.4) we have that for small $\boldsymbol{a}$,

$$f(\boldsymbol{X}) \oplus_r (\mathrm{d}^r f_X \boldsymbol{a}) \overset{(5.3)}{=} f(\boldsymbol{X} \oplus_r \boldsymbol{a}) \overset{(4.49d)}{=} f\left(\mathbf{Ad}_X\, \boldsymbol{a} \oplus_l \boldsymbol{X}\right) \overset{(5.4)}{=} (\mathrm{d}^l f_X\, \mathbf{Ad}_X\, \boldsymbol{a}) \oplus_l f(\boldsymbol{X}). \tag{5.5}$$

Consequently,

$$\exp(\mathrm{d}^l f_X\, \mathbf{Ad}_X\, \boldsymbol{a}) = f(\boldsymbol{X}) \circ \exp(\mathrm{d}^r f_X \boldsymbol{a}) \circ f(\boldsymbol{X})^{-1} = \mathbf{Ad}_{f(X)} \exp(\mathrm{d}^r f_X \boldsymbol{a}), \tag{5.6}$$

and due to (4.49c) it follows that left and right derivatives are related through the adjoints via

$$\mathrm{d}^l f_X = \mathbf{Ad}_{f(X)}\ \mathrm{d}^r f_X\ \mathbf{Ad}_X^{-1}. \tag{5.7}$$

With the interpretation of the adjoints as coordinate changes this formula can be seen as follows: the derivative of f with respect to a tangent vector ${}^e\boldsymbol{a}$ at e can be obtained by

1. Convert ${}^{e}\boldsymbol{a}$ to a tangent vector at $\boldsymbol{X}$: ${}^{X}\boldsymbol{a} = \mathbf{Ad}_{X}^{-1}\,{}^{e}\boldsymbol{a} \in T_{X}\check{\mathbb{M}}$,
2. Map the tangent vector through the derivative: ${}^{X}\boldsymbol{b} = \mathrm{d}^{r} f_{X}\,{}^{X}\boldsymbol{a} \in T_{f(X)}\check{\mathbb{M}}$,
3. Convert the result back to a tangent vector at e: ${}^{e}\boldsymbol{b} = \mathbf{Ad}_{f(X)}\,{}^{X}\boldsymbol{b} \in T_{e}\check{\mathbb{M}}$.

Jacobians on Lie Groups satisfy the chain rule. Indeed, if $f(\boldsymbol{X}) = g \circ h(\boldsymbol{X})$ for some $g : \mathbb{M}' \to \mathbb{M}''$ and $h : \mathbb{M} \to \mathbb{M}'$ we have with $\boldsymbol{Z} := h(\boldsymbol{X})$

$$\begin{aligned}
\mathrm{d}^{r}(g \circ h)_{X} &= \lim_{\boldsymbol{a}\to 0} \frac{g\left(h(\boldsymbol{X} \oplus_{r} \boldsymbol{a})\right) \ominus_{r} g(h(\boldsymbol{X}))}{\boldsymbol{a}} \overset{(5.3)}{=} \lim_{\boldsymbol{a}\to 0} \frac{g\left(h(\boldsymbol{X}) \oplus_{r} \left(\mathrm{d}^{r} h_{X}\boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^{2})\right)\right) \ominus_{r} g(h(\boldsymbol{X}))}{\boldsymbol{a}} \\
&\overset{(5.3)}{=} \lim_{\boldsymbol{a}\to 0} \frac{\left(g(\boldsymbol{Z}) \oplus_{r} \left(\mathrm{d}^{r} g_{Z}\ \mathrm{d}^{r} h_{X}\boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^{2})\right)\right) \ominus_{r} g(h(\boldsymbol{X}))}{\boldsymbol{a}} \overset{(4.45b)}{=} \mathrm{d}^{r} g_{Z}\ \mathrm{d}^{r} h_{X}.
\end{aligned} \tag{5.8}$$

An analogous left chain rule can be developed in the same manner via (5.4) in lieu of (5.3).

Important formulas for Lie group derivatives

- Right derivative: $\mathrm{d}^{r} f_{X} := \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(X)^{-1} \circ f(X \circ \exp(\boldsymbol{a})\right)}{\boldsymbol{a}} \in T_{X}\mathbb{M}$,
- Left derivative: $\mathrm{d}^{l} f_{X} := \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(\exp(\boldsymbol{a}) \circ X) \circ f(X)^{-1}\right)}{\boldsymbol{a}} \in T_{e}\mathbb{M}$,
- Conversion between left and right jacobians: $\mathrm{d}^{l} f_{X} = \mathbf{Ad}_{f(X)}\ \mathrm{d}^{r} f_{X}\ \mathbf{Ad}_{X}^{-1}$,
- Right chain rule: $\mathrm{d}^{r}(g(h(\boldsymbol{X})))_{X} = \mathrm{d}^{r} g_{h(X)}\ \mathrm{d}^{r} h_{X}$,
- Left chain rule: $\mathrm{d}^{l}(g(h(\boldsymbol{X})))_{X} = \mathrm{d}^{l} g_{h(X)}\ \mathrm{d}^{l} h_{X}$.

5.1. Global Derivative

For a mapping $f : \mathbb{M} \to \mathbb{N}$ between two manifolds the classical way to define a derivative $\mathrm{D} f_{X}$ is as a mapping $T_{X}\mathbb{M} \to T_{f(X)}\mathbb{N}$ defined as

$$\mathrm{D} f_{X} \boldsymbol{B} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)), \qquad \begin{cases} \gamma(0) = \boldsymbol{X}, \\ \gamma'(0) = \boldsymbol{B}. \end{cases} \tag{5.9}$$

for $\boldsymbol{B} \in T_{X}\mathbb{M}$. Note that this definition wouldn't make sense for an arbitrary matrix $\boldsymbol{B}$; for γ to take values in $\mathbb{M}$ the derivative at zero must be on the form $\boldsymbol{B} = \boldsymbol{X}\hat{\boldsymbol{a}}$. Being in global matrix coordinates, $\mathrm{D} f_{X} \boldsymbol{B}$ typically does not exhibit the structure of the tangent space at $T_{f(X)}\mathbb{N}$. However, it can be mapped to the tangent space via group action, which yields an alternative way of defining the right and left derivatives.

$$
\begin{aligned}
\mathrm{d}^r f_X \boldsymbol{a} &:= \left(f(\boldsymbol{X})^{-1}\left(\mathrm{D} f_X\, \boldsymbol{X}\hat{\boldsymbol{a}}\right)\right)^\vee = \left(f(\boldsymbol{X})^{-1}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t))\right)\right)^\vee, \qquad \begin{cases}\gamma(0)=\boldsymbol{X},\\ \gamma'(0)=\boldsymbol{X}\hat{\boldsymbol{a}},\end{cases}\\
\mathrm{d}^l f_X \boldsymbol{a} &:= \left(\left(\mathrm{D} f_X\, \hat{\boldsymbol{a}}\boldsymbol{X}\right) f(\boldsymbol{X})^{-1}\right)^\vee = \left(\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t))\right) f(\boldsymbol{X})^{-1}\right)^\vee, \qquad \begin{cases}\gamma(0)=\boldsymbol{X},\\ \gamma'(0)=\hat{\boldsymbol{a}}\boldsymbol{X}.\end{cases}
\end{aligned} \tag{5.10}
$$

Show that these definitions agree with those above

5.2. Product rule

Consider a function $f(\boldsymbol{X}) = g(\boldsymbol{X}) \circ h(\boldsymbol{X})$, we utilize (5.3) to obtain

$$
\begin{aligned}
f(\boldsymbol{X}\oplus\boldsymbol{a}) &= \left(g(\boldsymbol{X}) \oplus \left(\mathrm{d}^r g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right) \circ \left(h(\boldsymbol{X}) \oplus \left(\mathrm{d}^r h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right)\\
&= g(\boldsymbol{X}) \circ \exp(\mathrm{d}^r g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)) \circ h(\boldsymbol{X}) \circ \exp(\mathrm{d}^r h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2))\\
&= g(\boldsymbol{X}) \circ h(\boldsymbol{X}) \circ \left(\mathrm{Ad}_{h(X)^{-1}} \exp\left(\mathrm{d}^r g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right) \circ \exp(\mathrm{d}^r h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2))\\
&\overset{(4.49c)}{=} g(\boldsymbol{X}) \circ h(\boldsymbol{X}) \circ \left(\exp \mathbf{Ad}_{h(X)^{-1}} \left(\mathrm{d}^r g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right) \circ \exp(\mathrm{d}^r h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2))\\
&\overset{(4.44)}{=} g(\boldsymbol{X}) \circ h(\boldsymbol{X}) \circ \exp\left(\mathbf{Ad}_{h(X)^{-1}}\, \mathrm{d}^r g_X \boldsymbol{a} + \mathrm{d}^r h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2))\right).
\end{aligned} \tag{5.11}
$$

From here we can conclude that

$$
\mathrm{d}^r(g\circ h)_X = \mathbf{Ad}_{h(X)^{-1}}\, \mathrm{d}^r g_X + \mathrm{d}^r h_X \tag{5.12}
$$

which is the product rule for Lie group derivatives.

Remark 5.1. *There is no Lie group equivalent of the rule of total derivative. Consider*

$$
\begin{aligned}
f(g(\boldsymbol{X}\oplus\boldsymbol{a}), h(\boldsymbol{X}\oplus\boldsymbol{a})) &\approx f\left(g(\boldsymbol{X}) \oplus \mathrm{d}^r g_X\, \boldsymbol{a}, h(\boldsymbol{X}) \oplus \mathrm{d}^r h_X\, \boldsymbol{a}\right)\\
&\approx f\left(g(\boldsymbol{X}), h(\boldsymbol{X}) \oplus \mathrm{d} h_X \boldsymbol{a}\right) \oplus \mathrm{d}^r f_g\, \mathrm{d}^r g_X \boldsymbol{a}\\
&\approx \left[f\left(g(\boldsymbol{X}), h(\boldsymbol{X})\right) \oplus \mathrm{d}^r f_h\, \mathrm{d}^r h_X \boldsymbol{a}\right] \oplus \mathrm{d}^r f_g\, \mathrm{d}^r g_X \boldsymbol{a}\\
&= f\left(g(\boldsymbol{X}), h(\boldsymbol{X})\right) \circ \left[\exp\left(\mathrm{d}^r f_h\, \mathrm{d}^r h_X \boldsymbol{a}\right) \circ \exp\left(\mathrm{d}^r f_g\, \mathrm{d}^r g_X \boldsymbol{a}\right)\right].
\end{aligned} \tag{5.13}
$$

That is, if $f(\boldsymbol{X}) = f(g(\boldsymbol{X}), h(\boldsymbol{X}))$ we typically have that

$$
\mathrm{d}^r(f(g(\boldsymbol{X}), h(\boldsymbol{X})))_X \neq \mathrm{d}^r f_{g(X)}\, \mathrm{d}^r g_X + \mathrm{d}^r f_{h(X)}\, \mathrm{d}^r h_X. \tag{5.14}
$$

However, from (4.44) it can be seen that if

$$
\left[\mathrm{d}^r f_{h(X)}\, \mathrm{d}^r h_X \boldsymbol{a}, \mathrm{d}^r f_{g(X)}\, \mathrm{d}^r g_X \boldsymbol{a}\right] = 0, \qquad \forall \boldsymbol{a}, \tag{5.15}
$$

then the rule of total derivatives applies. One important case when this holds is when f takes values in $\mathbb{E}(n)$ since matrix multiplication on $\mathbb{E}(n)$ corresponds to vector addition on $\mathbb{R}^n$ and hence all brackets are zero.

5.3. Lie Bracket

The Lie bracket between two tangent elements can be defined as the global derivative of the adjoint operator at identity. Consider the mapping $f(\boldsymbol{X}) := \mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b} = \boldsymbol{X}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1}$ and take a curve $\gamma(t) \in \mathbb{M}$ such that $\gamma(0) = \boldsymbol{X}$ and $\gamma'(0) = \hat{\boldsymbol{a}}$. From $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\gamma(t)^{-1} = 0$ it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)^{-1} = -\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1}$, hence

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)) = \gamma'(0)\hat{\boldsymbol{b}}\gamma(0)^{-1} - \gamma(0)\hat{\boldsymbol{b}}\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1} = \hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1} - \boldsymbol{X}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1}\hat{\boldsymbol{a}}\boldsymbol{X}^{-1}. \tag{5.16}$$

The derivatives of $\mathbf{Ad}_{\boldsymbol{X}}$ with respect to $\boldsymbol{X}$ are

$$\mathrm{D}(\mathbf{Ad}_X\,\boldsymbol{b})_X\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1} - \boldsymbol{X}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1}\hat{\boldsymbol{a}}X^{-1}, \tag{5.17a}$$

$$\mathrm{d}^r(\mathbf{Ad}_X\,\boldsymbol{b})_X\boldsymbol{a} = \left(\mathrm{D}\left(\mathbf{Ad}_X\,\boldsymbol{b}\right)_X\,\boldsymbol{X}\hat{\boldsymbol{a}}\right)^\vee = \mathbf{Ad}_X\,[\boldsymbol{a},\boldsymbol{b}] = [\mathbf{Ad}_X\,\boldsymbol{a},\mathbf{Ad}_X\,\boldsymbol{b}]\,, \tag{5.17b}$$

$$\mathrm{d}^l(\mathbf{Ad}_X\,\boldsymbol{b})_X\boldsymbol{a} = \left(\mathrm{D}\left(\mathbf{Ad}_X\,\boldsymbol{b}\right)_X\,\hat{\boldsymbol{a}}\boldsymbol{X}\right)^\vee = [\boldsymbol{a},\mathbf{Ad}_X\,\boldsymbol{b}]\,, \tag{5.17c}$$

whereas at $\boldsymbol{X} = e$ they simplify to

$$(\mathrm{D}(\mathbf{Ad}_X\,\boldsymbol{b})_{X=e}\hat{\boldsymbol{a}})^\vee = \mathrm{d}^r(\mathbf{Ad}_X\,\boldsymbol{b})_{X=e}\boldsymbol{a} = \mathrm{d}^l\,(\mathbf{Ad}_X\,\boldsymbol{b})_{X=e}\,\boldsymbol{a} = [\boldsymbol{a},\boldsymbol{b}]\,. \tag{5.18}$$

The lower-case adjoint is defined as

$$\begin{aligned}
\mathrm{ad}^0_{\boldsymbol{a}}\,\boldsymbol{b} &:= \boldsymbol{b}\\
\mathrm{ad}^1_{\boldsymbol{a}}\,\boldsymbol{b} &:= \mathrm{ad}_{\boldsymbol{a}}\,\boldsymbol{b} = [\boldsymbol{a},\boldsymbol{b}]\\
\mathrm{ad}^2_{\boldsymbol{a}}\,\boldsymbol{b} &:= [\boldsymbol{a},\mathrm{ad}_{\boldsymbol{a}}\,\boldsymbol{b}] = \underbrace{[\boldsymbol{a},[\boldsymbol{a},\boldsymbol{b}]]}_{2-\text{times}}\\
&\;\vdots\\
\mathrm{ad}^k_{\boldsymbol{a}}\,\boldsymbol{b} &:= [\boldsymbol{a},\mathrm{ad}^{k-1}_{\boldsymbol{a}}\,\boldsymbol{b}] = \underbrace{[\boldsymbol{a},[\boldsymbol{a},\ldots,[\boldsymbol{a},\boldsymbol{b}]]]}_{k\text{ times}}, \qquad k \geq 1.
\end{aligned} \tag{5.19}$$

From this definition it can be seen that for a scalar s,

$$\mathrm{ad}^k_{s\boldsymbol{a}} = s^k\,\mathrm{ad}_{\boldsymbol{a}}, \quad s \in \mathbb{R} \tag{5.20}$$

If we formally define the exponential of the adjoint as

$$\exp\mathrm{ad}_{\boldsymbol{a}} := \sum_{k=0}^{\infty}\frac{\mathrm{ad}^k_{\boldsymbol{a}}}{k!} \tag{5.21}$$

we can also show that the adjoint of the exponential equals the exponential of the adjoint.

$$\mathbf{Ad}_{\exp\boldsymbol{a}} = \exp\mathrm{ad}_{\boldsymbol{a}}\,. \tag{5.22}$$

Proof of (5.22). By expanding the left-hand side in (5.22) and letting $\boldsymbol{A} = \hat{\boldsymbol{a}}, \boldsymbol{B} = \hat{\boldsymbol{b}}$ we obtain

$$(\mathbf{Ad}_{\exp \boldsymbol{a}} \boldsymbol{b})^\wedge = \operatorname{Exp}(\boldsymbol{A})\boldsymbol{B}\operatorname{Exp}(-\boldsymbol{A}) = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\boldsymbol{A}^i\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{i!(k-i)!}. \tag{5.23}$$

We next show by induction that the summands in (5.23) and (5.21) are equal for each value of k. Equality evidently holds for the base case $k = 0$. Assume that it holds for $k-1$, i.e. that

$$\left(\frac{\mathrm{ad}_{\boldsymbol{a}}^{k-1} \boldsymbol{b}}{(k-1)!}\right)^\wedge = \sum_{i=0}^{k-1} \frac{\boldsymbol{A}^i\boldsymbol{B}(-\boldsymbol{A})^{k-1-i}}{i!(k-1-i)!}. \tag{5.24}$$

Then we have that

$$\begin{aligned}\left(\frac{\mathrm{ad}_{\boldsymbol{a}}^{k} \boldsymbol{b}}{k!}\right)^\wedge &= \frac{1}{k}\left[\boldsymbol{A}\frac{(\mathrm{ad}_{\boldsymbol{A}}^{k-1}\boldsymbol{B})}{(k-1)!} - \frac{(\mathrm{ad}_{\boldsymbol{A}}^{k-1}\boldsymbol{B})}{(k-1)!}\boldsymbol{A}\right] = \frac{1}{k}\left[\sum_{i=0}^{k-1}\frac{\boldsymbol{A}^{i+1}\boldsymbol{B}(-\boldsymbol{A})^{k-1-i}}{i!(k-1-i)!} + \sum_{i=0}^{k-1}\frac{\boldsymbol{A}^{i}\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{i!(k-1-i)!}\right] \\ &= \frac{1}{k}\left[\sum_{i=0}^{k-1}\frac{\boldsymbol{A}^{i}\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{i!(k-1-i)!} + \sum_{i=1}^{k}\frac{\boldsymbol{A}^{i}\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{(i-1)!(k-i)!}\right] = \frac{\boldsymbol{B}(-\boldsymbol{A})^k}{k!} + \sum_{i=1}^{k-1} c_i \boldsymbol{A}^i\boldsymbol{B}(-\boldsymbol{A})^{k-i} + \frac{\boldsymbol{A}^k\boldsymbol{B}}{k!},\end{aligned}$$

where $c_i = \frac{1}{k}\left(\frac{1}{i!(k-1-i)!} + \frac{1}{(i-1)!(k-i)!}\right)$ and it can be verified that $c_i = \frac{1}{i!(k-i)!}$ as required. □

Another useful identity is the following.

$$\mathbf{Ad}_X\,[\boldsymbol{a}, \boldsymbol{b}] = [\mathbf{Ad}_X\,\boldsymbol{a}, \mathbf{Ad}_X\,\boldsymbol{b}]\,. \tag{5.25}$$

5.4. Derivatives of the Exponential map

The derivatives of the exponential map is a fundamental expression that often shows up when manipulating derivatives on Lie groups. From (5.10) we have that

$$\mathrm{d}^r \exp_{\boldsymbol{a}} \boldsymbol{b} = \left(\exp(\boldsymbol{\gamma}(0))^{-1} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(\boldsymbol{\gamma}(t))\right)^\vee, \quad \boldsymbol{\gamma}(0) = \boldsymbol{a},\ \boldsymbol{\gamma}'(0) = \boldsymbol{b}. \tag{5.26}$$

To calculate this derivative consider a curve $\boldsymbol{\gamma}(t) \in \check{\mathfrak{m}}$ and the expression

$$\Gamma(\sigma, t) = \exp(\sigma\boldsymbol{\gamma}(t))^{-1}\frac{\partial}{\partial t}\exp(\sigma\boldsymbol{\gamma}(t)) = \operatorname{Exp}\left(-\sigma\hat{\boldsymbol{\gamma}}(t)\right)\frac{\partial}{\partial t}\operatorname{Exp}(\sigma\hat{\boldsymbol{\gamma}}(t)). \tag{5.27}$$

Take the derivative with respect to σ:

$$\begin{aligned}\frac{\partial}{\partial\sigma}\Gamma(\sigma,t) &= -\operatorname{Exp}\left(-\sigma\hat{\boldsymbol{\gamma}}(t)\right)\hat{\boldsymbol{\gamma}}(t)\frac{\partial}{\partial t}\operatorname{Exp}\left(\sigma\hat{\boldsymbol{\gamma}}(t)\right) + \operatorname{Exp}(-\sigma\hat{\boldsymbol{\gamma}}(t))\frac{\partial}{\partial t}\left[\hat{\boldsymbol{\gamma}}(t)\operatorname{Exp}(\sigma\hat{\boldsymbol{\gamma}}(t))\right] \\ &= \operatorname{Exp}(-\sigma\hat{\boldsymbol{\gamma}}(t))\hat{\boldsymbol{\gamma}}'(t)\operatorname{Exp}(\sigma\hat{\boldsymbol{\gamma}}(t)) = \mathrm{Ad}_{\operatorname{Exp}(-\sigma\hat{\boldsymbol{\gamma}}(t))}\hat{\boldsymbol{\gamma}}'(t) = \left(\mathbf{Ad}_{\exp(-\sigma\boldsymbol{\gamma}(t))}\boldsymbol{\gamma}'(t)\right)^\wedge \\ &\overset{(5.22)}{=} \left(\exp \mathrm{ad}_{-\sigma\boldsymbol{\gamma}(t)}\boldsymbol{\gamma}'(t)\right)^\wedge = \left(\sum_{k=0}^{\infty}\frac{\mathrm{ad}_{-\sigma\boldsymbol{\gamma}(t)}^k}{k!}\boldsymbol{\gamma}'(t)\right)^\wedge = \left(\sum_{k=0}^{\infty}\sigma^k\frac{\mathrm{ad}_{-\boldsymbol{\gamma}(t)}^k}{k!}\boldsymbol{\gamma}'(t)\right)^\wedge.\end{aligned} \tag{5.28}$$

Integrating from 0 to 1 with respect to σ and setting $t = 0$ then yields

$$\Gamma(1,0)^\vee = \int_0^1 \frac{\partial}{\partial\sigma}\Gamma(\sigma,0)^\vee \mathrm{d}\sigma = \sum_{k=0}^{\infty} \frac{\mathrm{ad}_{-\boldsymbol{\gamma}(0)}^k}{(k+1)!}\boldsymbol{\gamma}'(0). \tag{5.29}$$

From (5.27) we can see that $\Gamma(1,0)$ is equal to the right derivative of exp at $\gamma(0)$ in the direction $\gamma'(0)$.

The right- and left derivatives of the exponential map are

$$\begin{aligned} \mathrm{d}^r \exp_{\boldsymbol{a}} &= \frac{I - \exp(-\mathrm{ad}_{\boldsymbol{a}})}{\mathrm{ad}_{\boldsymbol{a}}} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}\,\mathrm{ad}_{\boldsymbol{a}}^k, \\ \mathrm{d}^l \exp_{\boldsymbol{a}} &= \frac{\exp \mathrm{ad}_a - I}{\mathrm{ad}_a} := \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\,\mathrm{ad}_{\boldsymbol{a}}^k. \end{aligned} \tag{5.30}$$

Through the Bernoulli numbers $B_0 = 0, B_1 = -1/2, B_2 = 1/6, \ldots$ that are defined as

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n. \tag{5.31}$$

we can also write the formal inverses of the derivatives of the exponential

$$\begin{aligned} \left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1} &= \frac{\mathrm{ad}_{\boldsymbol{a}}}{I - \exp(-\mathrm{ad}_{\boldsymbol{a}})} := \sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!}\,\mathrm{ad}_{\boldsymbol{a}}^n, \\ \left(\mathrm{d}^l \exp_{\boldsymbol{a}}\right)^{-1} &= \frac{\mathrm{ad}_{\boldsymbol{a}}}{\exp \mathrm{ad}_{\boldsymbol{a}} - I} := \sum_{n=0}^{\infty} B_n \frac{1}{n!}\,\mathrm{ad}_{\boldsymbol{a}}^n. \end{aligned} \tag{5.32}$$

Notably, at $\boldsymbol{a} = \boldsymbol{0}$ these are all equal to the identity matrix. Since $B_1 = -1/2$ and $B_n = 0$ for odd $n > 1$ it follows that

$$\left(\mathrm{d}^l \exp_{\boldsymbol{a}}\right)^{-1} = -\,\mathrm{ad}_{\boldsymbol{a}} + \left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1}. \tag{5.33}$$

While the derivatives (5.30) - (5.32) could be evaluated to arbitrary precision by adding enough terms, this is not a practical solution. Fortunately closed-form expressions can be obtained for all groups of interest, although doing so is tediuous. We devote the remainder of this section to that task.

We can now calculate $\left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1}$ for various groups.

5.4.1. $\mathbb{SO}(2)$

Consider tangent elements $\omega_z, \bar{\omega}_z$. The bracket on $\mathbb{SO}(2)$ is zero since

$$[\omega_z, \bar{\omega}_z] = \left(\begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \begin{bmatrix} 0 & -\bar{\omega}_z \\ \bar{\omega}_z & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\bar{\omega}_z \\ \bar{\omega}_z & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \right)^\vee = 0. \tag{5.34}$$

It follows that all terms vanish except for $n = 0$.

Lowercase adjoint and exponential derivatives on $\mathbb{SO}(2)$

$$\mathrm{ad}_{\omega_z} = 0, \tag{5.35}$$

$$\mathrm{d}^r \exp_{\omega_z} = \mathrm{d}^l \exp_{\omega_z} = \left(\mathrm{d}^r \exp_{\omega_z}\right)^{-1} = \left(\mathrm{d}^l \exp_{\omega_z}\right)^{-1} = 1. \tag{5.36}$$

5.4.2. $\mathbb{SO}(3)$

We know that $\mathrm{ad}_{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}$ and from (4.23a) that $\hat{\boldsymbol{\omega}}^3 = -\|\boldsymbol{\omega}\|^2\hat{\boldsymbol{\omega}}$. Thus $\mathrm{ad}^3_{\boldsymbol{\omega}} = -\|\boldsymbol{\omega}\|^2\,\mathrm{ad}_{\boldsymbol{\omega}}$ and we get,

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{B_n(-1)^n}{n!}\,\mathrm{ad}^n_{\boldsymbol{\omega}} &= \sum_{n=0}^{\infty}\frac{B_n(-1)^n}{n!}\hat{\boldsymbol{\omega}}^n = I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} + \sum_{n\geq 2}\frac{B_n(-1)^n}{n!}\,\mathrm{ad}^n_{\boldsymbol{\omega}} \\
&= I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} + \left(\frac{B_2}{2!}\,\mathrm{ad}^2_{\boldsymbol{\omega}} - \frac{B_4\|\boldsymbol{\omega}\|^2}{4!}\,\mathrm{ad}^2_{\boldsymbol{\omega}} + \frac{B_6\|\boldsymbol{\omega}\|^4}{6!}\,\mathrm{ad}^2_{\boldsymbol{\omega}} - \ldots\right) = I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} - \frac{1}{\|\boldsymbol{\omega}\|^2}\sum_{n\geq 1}\frac{B_{2n}(-1)^n\|\boldsymbol{\omega}\|^{2n}}{(2n)!}\,\mathrm{ad}^2_{\boldsymbol{\omega}} \\
&\overset{(9.20)}{=} I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} - \frac{1}{\|\boldsymbol{\omega}\|^2}\left(\frac{\|\boldsymbol{\omega}\|}{2}\cot\left(\frac{\|\boldsymbol{\omega}\|}{2}\right) - 1\right)\mathrm{ad}^2_{\boldsymbol{\omega}} = I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1+\cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\|\sin\|\boldsymbol{\omega}\|}\right)\mathrm{ad}^2_{\boldsymbol{\omega}},
\end{aligned}$$

where the half-angle formula $\cot x = (1+\cos x)/\sin x$ has been used. The left jacobian $\mathrm{d}^l \exp_{\boldsymbol{\omega}}$ was already calculated in (4.42) and since $\left(\mathrm{d}^r \exp_{\boldsymbol{\omega}}\right)^{-1} = \left[\left(\mathrm{d}^l \exp_{\boldsymbol{\omega}}\right)^{-1}\right]^T$. Due to the anti-symmetry of $\mathrm{ad}_{\boldsymbol{\omega}}$ it follows that also $\mathrm{d}^r \exp_{\boldsymbol{\omega}} = \left[\mathrm{d}^l \exp_{\boldsymbol{\omega}}\right]^T$ must hold.

Lowercase adjoint and exponential derivatives on $\mathbb{SO}(3)$

$$\mathrm{ad}_{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}, \tag{5.37}$$

$$\mathrm{d}^r \exp_{\boldsymbol{\omega}} = I_3 - \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}} + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2, \tag{5.38}$$

$$\mathrm{d}^l \exp_{\boldsymbol{\omega}} = I_3 + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}} + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2, \tag{5.39}$$

$$\left(\mathrm{d}^r \exp_{\boldsymbol{\omega}}\right)^{-1} = I_3 + \frac{\hat{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1+\cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\|\sin\|\boldsymbol{\omega}\|}\right)\hat{\boldsymbol{\omega}}^2, \tag{5.40}$$

$$\left(\mathrm{d}^l \exp_{\boldsymbol{\omega}}\right)^{-1} = I_3 - \frac{\hat{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1+\cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\|\sin\|\boldsymbol{\omega}\|}\right)\hat{\boldsymbol{\omega}}^2. \tag{5.41}$$

5.4.3. $\mathbb{SE}(2)$

We first calculate an expression for the bracket.

$$\left[\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix}, \begin{bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{\omega}_z \end{bmatrix}\right] = \left(\begin{bmatrix} 0 & -\omega_z & v_x \\ \omega_z & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\bar{\omega}_z & \bar{v}_x \\ \bar{\omega}_z & 0 & \bar{v}_y \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\bar{\omega}_z & \bar{v}_x \\ \bar{\omega}_z & 0 & \bar{v}_y \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\omega_z & v_x \\ \omega_z & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}\right)^\vee$$
$$= \begin{bmatrix} 0 & 0 & -\omega_z\bar{v}_y + \bar{\omega}_z v_y \\ 0 & 0 & \omega_z\bar{v}_x - \bar{\omega}_z v_x \\ 0 & 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} -\omega_z\bar{v}_y + \bar{\omega}_z v_y \\ \omega_z\bar{v}_x - \bar{\omega}_z v_x \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_z & v_y \\ \omega_z & 0 & -v_x \\ 0 & 0 & 0 \end{bmatrix}}_{\mathrm{ad}_{\boldsymbol{a}}}\begin{bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{\omega}_z \end{bmatrix}. \tag{5.42}$$

A quick calculation reveals that $\mathrm{ad}_{\boldsymbol{a}}^3 = -\omega_z^2\,\mathrm{ad}_{\boldsymbol{a}}$, which is exactly the relation we used for $\mathbb{SO}(3)$ above. Consequently the inverse derivatives must have the same form as on $\mathbb{SO}(3)$.

Lowercase adjoint and exponential derivatives on $\mathbb{SE}(2)$

Let $\boldsymbol{a} = \begin{bmatrix} v_x & v_y & \omega_z \end{bmatrix}^T$. Then,

$$\mathrm{ad}_{\boldsymbol{a}} = \begin{bmatrix} 0 & -\omega_z & v_y \\ \omega_z & 0 & -v_x \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.43}$$

$$\mathrm{d}^r\exp_{\boldsymbol{a}} = I_3 - \frac{1-\cos\omega_z}{\omega_z^2}\mathrm{ad}_{\boldsymbol{a}} + \frac{\omega_z - \sin\omega_z}{\omega_z^3}\mathrm{ad}_{\boldsymbol{a}}{}^2, \tag{5.44}$$

$$\mathrm{d}^l\exp_{\boldsymbol{a}} = I_3 + \frac{1-\cos\omega_z}{\omega_z^2}\mathrm{ad}_{\boldsymbol{a}} + \frac{\omega_z - \sin\omega_z}{\omega_z^3}\mathrm{ad}_{\boldsymbol{a}}{}^2, \tag{5.45}$$

$$\left(\mathrm{d}^r\exp_{\boldsymbol{a}}\right)^{-1} = I_3 + \frac{\mathrm{ad}_{\boldsymbol{a}}}{2} + \left(\frac{1}{\omega_z^2} - \frac{1+\cos\omega_z}{2\omega_z\sin\omega_z}\right)\mathrm{ad}_{\boldsymbol{a}}^2, \tag{5.46}$$

$$\left(\mathrm{d}^l\exp_{\boldsymbol{a}}\right)^{-1} = I_3 - \frac{\mathrm{ad}_{\boldsymbol{a}}}{2} + \left(\frac{1}{\omega_z^2} - \frac{1+\cos\omega_z}{2\omega_z\sin\omega_z}\right)\mathrm{ad}_{\boldsymbol{a}}^2. \tag{5.47}$$

5.4.4. $\mathbb{SE}(3)$

First we derive an expression for $\mathrm{ad}_{\boldsymbol{a}}$ utilizing that for the hat operator on $\mathbb{SO}(3)$, $\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}$

$$\left[\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}, \begin{bmatrix} \bar{\boldsymbol{v}} \\ \bar{\boldsymbol{\omega}} \end{bmatrix}\right]_{\mathbb{SE}(3)} = \left(\begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\bar{\boldsymbol{\omega}}} & \bar{\boldsymbol{v}} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \hat{\bar{\boldsymbol{\omega}}} & \bar{\boldsymbol{v}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix}\right)^\vee = \begin{bmatrix} [\boldsymbol{\omega}, \bar{\boldsymbol{\omega}}]^\wedge_{\mathbb{SO}(3)} & \hat{\boldsymbol{\omega}}\bar{\boldsymbol{v}} - \hat{\bar{\boldsymbol{\omega}}}\boldsymbol{v} \\ 0 & 0 \end{bmatrix}^\vee$$
$$= \begin{bmatrix} \hat{\boldsymbol{\omega}}\bar{\boldsymbol{v}} - \hat{\bar{\boldsymbol{\omega}}}\boldsymbol{v} \\ [\boldsymbol{\omega}, \bar{\boldsymbol{\omega}}]_{\mathbb{SO}(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\boldsymbol{v}} \\ 0 & \hat{\boldsymbol{\omega}} \end{bmatrix}}_{\mathrm{ad}_{\boldsymbol{a}}}\begin{bmatrix} \bar{\boldsymbol{v}} \\ \bar{\boldsymbol{\omega}} \end{bmatrix}. \tag{5.48}$$

We are interested in the powers $\mathrm{ad}_{\boldsymbol{a}}^k$ in order to evaluate the exponential derivatives. For $k \geq 1$

$$\mathrm{ad}_{\boldsymbol{a}}^k = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\boldsymbol{v}} \\ 0 & \hat{\boldsymbol{\omega}} \end{bmatrix}^k = \begin{bmatrix} \hat{\boldsymbol{\omega}}^k & \sum_{i=0}^{k-1} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-1-i} \\ 0 & \hat{\boldsymbol{\omega}}^k \end{bmatrix}. \tag{5.49}$$

Thus the left derivative of the exponential can be written

$$\mathrm{d}^l \exp_{\boldsymbol{a}} = \sum_{k=0}^{\infty} \frac{\mathrm{ad}_{\boldsymbol{a}}^k}{(k+1)!} = I + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \begin{bmatrix} \hat{\boldsymbol{\omega}}^k & \sum_{i=0}^{k-1} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-1-i} \\ 0 & \hat{\boldsymbol{\omega}}^k \end{bmatrix} = \begin{bmatrix} \mathrm{d}^l \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}} & Q^l(\boldsymbol{v}, \boldsymbol{\omega}) \\ 0 & \mathrm{d}^l \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}} \end{bmatrix},$$

where a closed-form expression for $Q^l(\boldsymbol{v}, \boldsymbol{\omega})$ can be painstakingly obtained through a series of sum manipulations. We first convert the formula to a form that is symmetric in i and k.

$$\begin{aligned} Q^l(\boldsymbol{v}, \boldsymbol{\omega}) &:= \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \sum_{i=0}^{k-1} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-1-i} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{1}{(k+2)!} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-i} \\ &= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \frac{1}{(k+2)!} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-i} = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(k+i+2)!} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k}. \end{aligned}$$

With the same steps the right derivative can be shown to be

$$Q^r(\boldsymbol{v}, \boldsymbol{\omega}) := \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \sum_{i=0}^{k-1} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-1-i} = \ldots = -\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+i}}{(k+i+2)!} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k} \tag{5.50}$$

and we can see that

$$Q^r(\boldsymbol{v}, \boldsymbol{\omega}) = Q^l(-\boldsymbol{v}, -\boldsymbol{\omega}) \tag{5.51}$$

which is convenient to know since calculating one of them is tedious enough.

In the following calculation the sum $\sum_{k,i \geq 0}$ is first split into parts $(k = i = 0)$, $(k = 0, i \geq 1)$, $(k \geq 1, i = 0)$ and $(k, i \geq 1)$, and then the resulting single sums are split into two sums $i = 0, 2, \ldots$ and $i = 1, 3, \ldots$. Also using that

$$\hat{\boldsymbol{\omega}}^{2k+1} = (-1)^k \|\boldsymbol{\omega}\|^{2k} \hat{\boldsymbol{\omega}}, \qquad \hat{\boldsymbol{\omega}}^{2k+2} = (-1)^k \|\boldsymbol{\omega}\|^{2k} \hat{\boldsymbol{\omega}}^2, \tag{5.52}$$

which follows from (4.23a), we get

$$\begin{aligned} Q^l(\boldsymbol{v}, \boldsymbol{\omega}) &= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=1}^{\infty} \frac{\hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}}}{(i+2)!} + \sum_{k=1}^{\infty} \frac{\hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^k}{(k+2)!} + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(i+k+2)!} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^k \\ &= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=0}^{\infty} \frac{\hat{\boldsymbol{\omega}}^{i+1} \hat{\boldsymbol{v}} + \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{i+1}}{(i+3)!} + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(i+k+4)!} \hat{\boldsymbol{\omega}}^{i+1} \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k+1} \\ &= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=0}^{\infty} \frac{\hat{\boldsymbol{\omega}}^{2i+1} \hat{\boldsymbol{v}} + \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{2i+1}}{(2i+3)!} + \sum_{i=0}^{\infty} \frac{\hat{\boldsymbol{\omega}}^{2i+2} \hat{\boldsymbol{v}} + \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{2i+2}}{(2i+4)!} + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(i+k+4)!} \hat{\boldsymbol{\omega}}^{i+1} \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k+1} \\ &= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=0}^{\infty} \frac{(-1)^i \|\boldsymbol{\omega}\|^{2i}}{(2i+3)!} \left(\hat{\boldsymbol{\omega}} \hat{\boldsymbol{v}} + \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}\right) + \sum_{i=0}^{\infty} \frac{(-1)^i \|\boldsymbol{\omega}\|^{2i}}{(2i+4)!} \left(\hat{\boldsymbol{\omega}}^2 \hat{\boldsymbol{v}} + \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^2\right) + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(i+k+4)!} \hat{\boldsymbol{\omega}}^{i+1} \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k+1}. \end{aligned}$$

The first two sums can now be evaluated in a fairly straightforward manner:

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+3)!}\|\boldsymbol{\omega}\|^{2i} = -\frac{1}{\|\boldsymbol{\omega}\|^3}\sum_{i=0}^{\infty}\frac{(-1)^{i+1}}{(2(i+1)+1)!}\|\boldsymbol{\omega}\|^{2(i+1)+1} = \frac{\|\boldsymbol{\omega}\|-\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}, \tag{5.53a}$$

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+4)!}\|\boldsymbol{\omega}\|^{2i} = \frac{1}{\|\boldsymbol{\omega}\|^4}\sum_{i=0}^{\infty}\frac{(-1)^{i+2}}{(2(i+2))!}\|\boldsymbol{\omega}\|^{2(i+2)} = \frac{\cos\|\boldsymbol{\omega}\|-1+\frac{\|\boldsymbol{\omega}\|^2}{2}}{\|\boldsymbol{\omega}\|^4}. \tag{5.53b}$$

The double sum requries additional work. Using $\hat{\boldsymbol{\omega}}\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}} = (-\boldsymbol{\omega}\cdot\boldsymbol{v})\hat{\boldsymbol{\omega}}$ from (4.23a) yields

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(k+i+4)!}\hat{\boldsymbol{\omega}}^{i+1}\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k+1} &= (-\boldsymbol{\omega}\cdot\boldsymbol{v})\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(k+i+4)!}\hat{\boldsymbol{\omega}}^{k+i+1} \overset{j=k+i}{=} (-\boldsymbol{\omega}\cdot\boldsymbol{v})\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{(j+4)!}\hat{\boldsymbol{\omega}}^{j+1} \\
&= -(\boldsymbol{\omega}\cdot\boldsymbol{v})\sum_{j=0}^{\infty}\frac{j+1}{(j+4)!}\hat{\boldsymbol{\omega}}^{j+1} = -(\boldsymbol{\omega}\cdot\boldsymbol{v})\sum_{j=0}^{\infty}\left(\frac{1}{(j+3)!}-\frac{3}{(j+4)!}\right)\hat{\boldsymbol{\omega}}^{j+1} \\
&= -(\boldsymbol{\omega}\cdot\boldsymbol{v})\left(\sum_{j=0}^{\infty}\left(\frac{1}{(2j+3)!}-\frac{3}{(2j+4)!}\right)\hat{\boldsymbol{\omega}}^{2j+1} + \sum_{j=0}^{\infty}\left(\frac{1}{(2j+4)!}-\frac{3}{(2j+5)!}\right)\hat{\boldsymbol{\omega}}^{2j+2}\right) \\
&\overset{(5.52)}{=} (\boldsymbol{\omega}\cdot\boldsymbol{v})\left(-\sum_{j=0}^{\infty}\left(\frac{(-1)^j}{(2j+3)!}\|\boldsymbol{\omega}\|^{2j}+3\frac{(-1)^j}{(2j+4)!}\|\boldsymbol{\omega}\|^{2j}\right)\hat{\boldsymbol{\omega}} + \sum_{j=0}^{\infty}\left(-\frac{(-1)^j}{(2j+4)!}\|\boldsymbol{\omega}\|^{2j}+3\frac{(-1)^j}{(2j+5)!}\|\boldsymbol{\omega}\|^{2j}\right)\hat{\boldsymbol{\omega}}^2\right).
\end{aligned}$$

The sums (5.53a) and (5.53b) appear again and can be re-used. The remaining sum with denominator $(2j+5)!$ evaluates to a higher-order sine expression as follows

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+5)!}\|\boldsymbol{\omega}\|^{2j} = \frac{1}{\|\boldsymbol{\omega}\|^5}\sum_{j=0}^{\infty}\frac{(-1)^{j+2}}{((2j+2)+1)}\|\boldsymbol{\omega}\|^{2(j+2)+1} = \frac{\sin\|\boldsymbol{\omega}\|-\|\boldsymbol{\omega}\|+\frac{\|\boldsymbol{\omega}\|}{6}}{\|\boldsymbol{\omega}\|^5}. \tag{5.54}$$

After collecting the various expressions the closed-form expression for Q^l can be written down

$$\begin{aligned}
Q^l(\boldsymbol{v},\boldsymbol{\omega}) = \frac{1}{2}\hat{\boldsymbol{v}} &+ \frac{\|\boldsymbol{\omega}\|-\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\left(\hat{\boldsymbol{\omega}}\hat{\boldsymbol{v}}+\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}-(\boldsymbol{\omega}\cdot\boldsymbol{v})\hat{\boldsymbol{\omega}}\right) \\
&+ \frac{\cos\|\boldsymbol{\omega}\|-1+\frac{\|\boldsymbol{\omega}\|^2}{2}}{\|\boldsymbol{\omega}\|^4}\left(\hat{\boldsymbol{\omega}}^2\hat{\boldsymbol{v}}+\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^2+(\boldsymbol{\omega}\cdot\boldsymbol{v})(3\hat{\boldsymbol{\omega}}-\hat{\boldsymbol{\omega}}^2)\right) \\
&- 3(\boldsymbol{\omega}\cdot\boldsymbol{v})\left(\frac{\|\boldsymbol{\omega}\|-\sin\|\boldsymbol{\omega}\|-\frac{\|\boldsymbol{\omega}\|^3}{6}}{\|\boldsymbol{\omega}\|^5}\right)\hat{\boldsymbol{\omega}}^2.
\end{aligned} \tag{5.55}$$

The Q matrix allows us to write down a closed-form expression for $\mathrm{d}^l\exp_{\boldsymbol{a}}$ on $\mathbb{SE}(3)$, and $\left(\mathrm{d}^l\exp_{\boldsymbol{a}}\right)^{-1}$ follows from noting that $\begin{bmatrix}A & B\\ 0 & A\end{bmatrix}^{-1} = \begin{bmatrix}A^{-1} & -A^{-1}BA^{-1}\\ 0 & A^{-1}\end{bmatrix}$ for A invertible.

Lowercase adjoint and exponential derivatives on $\mathbb{SE}(3)$

Let $\boldsymbol{a} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}$ and Q^l as in (5.55). Then

$$\mathrm{ad}_{\boldsymbol{a}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\boldsymbol{v}} \\ 0 & \hat{\boldsymbol{\omega}} \end{bmatrix}, \tag{5.56}$$

$$\mathrm{d}^r \exp_{\boldsymbol{a}} = \begin{bmatrix} J^r_{\mathbb{SO}(3)} & Q^l(-\boldsymbol{v}, -\boldsymbol{\omega}) \\ 0 & J^r_{\mathbb{SO}(3)} \end{bmatrix}, \tag{5.57}$$

$$\mathrm{d}^l \exp_{\boldsymbol{a}} = \begin{bmatrix} J^l_{\mathbb{SO}(3)} & Q^l(\boldsymbol{v}, \boldsymbol{\omega}) \\ 0 & J^l_{\mathbb{SO}(3)} \end{bmatrix} \tag{5.58}$$

$$\left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1} = \begin{bmatrix} \left(J^r_{\mathbb{SO}(3)}\right)^{-1} & -\left(J^r_{\mathbb{SO}(3)}\right)^{-1} Q^l(-\boldsymbol{v}, -\boldsymbol{\omega}) \left(J^r_{\mathbb{SO}(3)}\right)^{-1} \\ 0 & \left(J^r_{\mathbb{SO}(3)}\right)^{-1} \end{bmatrix}, \tag{5.59}$$

$$\left(\mathrm{d}^l \exp_{\boldsymbol{a}}\right)^{-1} = \begin{bmatrix} \left(J^l_{\mathbb{SO}(3)}\right)^{-1} & -\left(J^l_{\mathbb{SO}(3)}\right)^{-1} Q^l(\boldsymbol{v}, \boldsymbol{\omega}) \left(J^l_{\mathbb{SO}(3)}\right)^{-1} \\ 0 & \left(J^l_{\mathbb{SO}(3)}\right)^{-1} \end{bmatrix}. \tag{5.60}$$

where $J^{l/r}_{\mathbb{SO}(3)} = \left(\mathrm{d}^{l/r} \exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}}$ and $\left(J^{l/r}_{\mathbb{SO}(3)}\right)^{-1} = \left(\left(\mathrm{d}^{l/r} \exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}}\right)^{-1}$. Note that in these formulas $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{v}}$ denote the hat operator on $\mathbb{SO}(3)$.

5.5. Derivatives of common operations

Group composition We calculate the right derivatives using (5.1) and the left derivatives via (5.7).

$$\begin{aligned} \mathrm{d}^r(X \circ Y)_X &\overset{(5.1)}{=} \lim_{\boldsymbol{a} \to 0} \frac{\log\left((X \circ Y)^{-1} \circ X \circ \exp(\boldsymbol{a}) \circ Y\right)}{\boldsymbol{a}} = \lim_{\boldsymbol{a} \to 0} \frac{\log\left(Y^{-1} \circ \exp(\boldsymbol{a}) \circ Y\right)}{\boldsymbol{a}} \\ &\overset{(4.49c)}{=} \lim_{\boldsymbol{a} \to 0} \frac{\log \exp \mathbf{Ad}_{Y^{-1}} \boldsymbol{a}}{\boldsymbol{a}} = \mathbf{Ad}_{Y^{-1}}, \end{aligned} \tag{5.61}$$

$$\mathrm{d}^l(X \circ Y)_X \overset{(5.7)}{=} \mathbf{Ad}_{X \circ Y} \mathbf{Ad}_{Y^{-1}} \mathbf{Ad}_X^{-1} \overset{(4.49)}{=} I_n, \tag{5.62}$$

$$\mathrm{d}^r(X \circ Y)_Y \overset{(5.1)}{=} \lim_{\boldsymbol{a} \to 0} \frac{\log\left((X \circ Y)^{-1} \circ X \circ Y \circ \exp(\boldsymbol{a})\right)}{\boldsymbol{a}} = I_n, \tag{5.63}$$

$$\mathrm{d}^l(X \circ Y)_Y \overset{(5.7)}{=} \mathbf{Ad}_{X \circ Y} I_n \mathbf{Ad}_Y^{-1} \overset{(4.49)}{=} \mathbf{Ad}_X. \tag{5.64}$$

Group inverse

$$
\begin{aligned}
\mathrm{d}^r(X^{-1})_X &\overset{(5.1)}{=} \lim_{a\to 0} \frac{\log\left(X \circ (X \circ \exp(a))^{-1}\right)}{a} = \lim_{a\to 0} \frac{\log\left(X \circ \exp(-a) \circ X^{-1}\right)}{a} \\
&\overset{(4.49c)}{=} \frac{\log \exp \mathbf{Ad}_X - a}{a} = -\,\mathbf{Ad}_X .
\end{aligned} \tag{5.65}
$$

$$
\mathrm{d}^l(X^{-1})_X \overset{(5.7)}{=} -\,\mathbf{Ad}_{X^{-1}}\, \mathbf{Ad}_X\, \mathbf{Ad}_{X^{-1}} = -\,\mathbf{Ad}_{X^{-1}} . \tag{5.66}
$$

Logarithm From differentiating $a = \log \exp a$ using the chain rule we get $I = \mathrm{d}^r \log_{\exp a}\ \mathrm{d}^r \exp_a$, which implies that

$$
\mathrm{d}^r \log_X = \left[\mathrm{d}^r \exp_{\log X}\right]^{-1}, \tag{5.67}
$$

$$
\mathrm{d}^l \log_X = \left[\mathrm{d}^l \exp_{\log X}\right]^{-1}. \tag{5.68}
$$

$$
\tag{5.69}
$$

Plus and minus From the chain rule and the above we can also deduce the derivatives of the plus and minus maps

$$
\mathrm{d}^r(X \oplus_r a)_X = \mathrm{d}^r(X \circ \exp(a))_X \overset{(5.61)}{=} \mathbf{Ad}^{-1}_{\exp(a)}, \tag{5.70}
$$

$$
\mathrm{d}^r(X \oplus_r a)_a = \mathrm{d}^r(X \circ \exp(a))_a \overset{(5.8)}{=} \mathrm{d}^r(X \circ \exp(a))_{\exp a}\ \mathrm{d}^r \exp_a \overset{(5.63)}{=} \mathrm{d}^r \exp_a, \tag{5.71}
$$

$$
\mathrm{d}^r(Y \ominus_r X)_Y = \mathrm{d}^r\left(\log X^{-1} \circ Y\right)_Y \overset{(5.8)}{=} \mathrm{d}^r \log_{X^{-1}\circ Y}\ \mathrm{d}^r(X^{-1} \circ Y)_Y \overset{(5.63)}{=} \left[\mathrm{d}^r \exp_{Y\ominus_r X}\right]^{-1}, \tag{5.72}
$$

$$
\begin{aligned}
\mathrm{d}^r(Y \ominus_r X)_X &= \mathrm{d}^r\left(\log X^{-1} \circ Y\right)_X \overset{(5.8)}{=} \mathrm{d}^r \log_{X^{-1}\circ Y}\ \mathrm{d}^r(X^{-1} \circ Y)_{X^{-1}}\ \mathrm{d}^r(X^{-1})_X \\
&\overset{(5.67),(5.61),(5.65)}{=} \left[\mathrm{d}^r \exp_{Y\ominus_r X}\right]^{-1} \mathbf{Ad}_{Y^{-1}}(-\,\mathbf{Ad}_X) \overset{(4.49b)}{=} -\left[\mathrm{d}^r \exp_{Y\ominus_r X}\right]^{-1} \mathbf{Ad}_{Y^{-1}\circ X} \\
&= -\left[\mathrm{d}^r \exp_{Y\ominus_r X}\right]^{-1} \mathbf{Ad}^{-1}_{\exp Y\ominus_r X} \overset{(5.7)}{=} -\left[\mathrm{d}^l \exp_{Y\ominus_r X}\right]^{-1}.
\end{aligned} \tag{5.73}
$$

5.6. Trace on $\mathbb{SO}(3)$

This is a work in progress

Example: Left and right derivatives on SO(3)

Right derivatives

$$\frac{^{RQ}\partial RQ}{^{Q}\partial Q} = I, \quad \frac{^{RQ}\partial RQ}{^{R}\partial R} = Q^T, \quad \frac{^{R^T}\partial R^T}{^{R}\partial R} = -I \tag{5.74}$$

Left derivatives

$$\frac{^{e}\partial RQ}{^{e}\partial Q} = R, \quad \frac{^{e}\partial RQ}{^{e}\partial R} = I, \quad \frac{^{e}\partial R^T}{^{e}\partial R} = -I \tag{5.75}$$

Example: trace on SO(3)

Consider the mapping $\mathrm{Tr} : SO(3) \to \mathbb{R}$, we are interested its right derivatives. Using that $\mathrm{Exp}(\boldsymbol{\omega}) \approx I + [\boldsymbol{\omega}]_\times$ for small $\boldsymbol{\omega}$

$$\begin{aligned}\frac{^{\mathrm{Tr}(R)}\partial \mathrm{Tr}(R)}{^{R}\partial R} &= \lim_{\boldsymbol{\omega}\to 0} \frac{\mathrm{Tr}(R \oplus \boldsymbol{\omega}) - \mathrm{Tr}(R)}{\boldsymbol{\omega}} = \lim_{\boldsymbol{\omega}\to 0} \frac{\mathrm{Tr}(R(I + [\boldsymbol{\omega}]_\times) - R)}{\boldsymbol{\omega}} \\ &= \lim_{\boldsymbol{\omega}\to 0} \frac{\mathrm{Tr}(R[\boldsymbol{\omega}]_\times))}{\boldsymbol{\omega}} = \lim_{\boldsymbol{\omega}\to 0} \frac{\mathrm{Tr}((R - R^T)[\boldsymbol{\omega}]_\times))}{2\boldsymbol{\omega}} = \lim_{\boldsymbol{\omega}\to 0} \frac{(R^T - R)^\vee \cdot \boldsymbol{\omega}}{\boldsymbol{\omega}} \\ &= (R^T - R)^\vee \in \mathbb{R}^{1x3}.\end{aligned} \tag{5.76}$$

Another approach yielding the same result is to study the expression for $\boldsymbol{\omega} = uE_i$ for $i = 1, 2, 3$ and let $u \to 0$. It can be shown that the left derivative has the same expression, which also follows from (5.7):

$$\begin{aligned}\frac{^{e}\partial \mathrm{Tr}(R)}{^{e}\partial R} = \frac{^{\mathrm{Tr}(R)}\partial \mathrm{Tr}(R)}{^{R}R} \mathrm{Ad}_R^{-1} &= \left(R^T - R\right)^\vee R^T = \left[R\left(R^T - R\right)^\vee\right]^T \\ &= (R(R^T - R)R^T)^\vee = (R^T - R)^\vee.\end{aligned} \tag{5.77}$$

Furthermore, to get the derivative in another frame Q we get again from (5.7)

$$\frac{^{e}\partial \mathrm{Tr}(R)}{^{Q}\partial R} = \frac{^{e}\partial \mathrm{Tr}(R)}{^{e}\partial R} \mathrm{Ad}_Q = (R^T - R)^\vee Q = \left[Q^T(R^T - R)^\vee\right]^T = (Q^T(R^T - R)Q)^\vee. \tag{5.78}$$

5.7. On Automatic Differentiation

Consider a function $f : \mathbb{M} \to \mathbb{N}$ whose derivative we are interested in. Since autodiff tools are not aware of manifolds we can not directly obtain e.g. $\mathrm{d}^r f_{\boldsymbol{X}}$; here we discuss how to obtain on-manifold derivatives by only differentiating Euclidean functions. Since $\mathrm{d}^r(f(\boldsymbol{X} \oplus_r \boldsymbol{a}))_{\boldsymbol{a}=0} = \mathrm{d}^r f_{\boldsymbol{X}} \mathrm{d}\exp_0 = \mathrm{d}^r f_{\boldsymbol{X}}$ we can write

$$\mathrm{d}^r f_{\boldsymbol{X}} \boldsymbol{b} = \mathrm{d}^r(f(\boldsymbol{X} \oplus_r \boldsymbol{a}))_{\boldsymbol{a}=0} \boldsymbol{b} = \left(f(\boldsymbol{X} \oplus_r \boldsymbol{a})^{-1} \mathrm{D}(f(\boldsymbol{X} \oplus_r \boldsymbol{a}))_{\boldsymbol{a}} \hat{\boldsymbol{b}}\right)^\vee\Big|_{\boldsymbol{a}=0} = \left(f(\boldsymbol{X})^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\boldsymbol{X} \oplus (t\boldsymbol{b}))\right)^\vee. \tag{5.79}$$

Here the function $t \mapsto f(\boldsymbol{X} \oplus (t\boldsymbol{b}))$ maps a scalar to a matrix and can therefore be differentiated using regular tools, after which the expression can be evaluated to obtain $\mathrm{d}^r f_X \boldsymbol{b}$. Naturally, if the complete derivative $\mathrm{d}^r f_X$ is desired it can be obtained by repeating this procedure n times for each basis unit vector.

If f maps to a Euclidean space (i.e. $\mathbb{N} = \mathbb{R}^k$) this further simplifies to

$$\mathrm{d}^r f_X b = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\boldsymbol{X} \oplus (t\boldsymbol{b})), \quad f : \mathbb{M} \to \mathbb{R}^n, \tag{5.80}$$

which with some abuse of notation can be written as

$$\mathrm{d}^r f_X = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} f(\boldsymbol{X} \oplus \boldsymbol{b}). \tag{5.81}$$

5.7.1. Ceres Solver Local Parameterizations

A special case of when numerical derivatives are used is in the nonlinear optimizer Ceres. Being unaware of Lie groups, Ceres considers cost functions that are functions of some parameters, $f(\boldsymbol{x})$, and uses automatic differentiation of $f : \mathbb{R}^p \to \mathbb{R}^k$ with respect to $\boldsymbol{x} \in \mathbb{R}^p$ to figure out in what direction to move in order to minimize f. However, if $\boldsymbol{x}$ represents the coordinates of a manifold chart, i.e. $\hat{\boldsymbol{x}} = \boldsymbol{X}$ for $\boldsymbol{X} \in \mathbb{M}$, it is not desirable to directly apply an update in the direction of the gradient since this may lead the resulting point no longer being on the manifold.

Being unaware of the manifold structure, automatic differentiation can only evaluate $\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} f(\hat{\boldsymbol{x}})$, where we are using the hat and vee maps to denote conversions between elements of a Lie group $\mathbb{M}$ and its parameterization $\check{\mathbb{M}}$ as was done in Chapter 2. Ceres provides an interface for specifying cusom *local parameterizations* that enable on-manifold optimization. In the following we specify how a local parameterization for Lie Group optimization can be constructed.

According to (5.81) we can write a tangent space derivative for a Euclidean-valued function

$$\mathrm{d}^r f_X = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} f(\boldsymbol{X} \oplus \boldsymbol{b}) = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} f\left(\left((\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^\vee\right)^\wedge\right) = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{y}}\right|_{\boldsymbol{y}=\boldsymbol{x}} f(\hat{\boldsymbol{y}}) \times \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^\vee. \tag{5.82}$$

Thus, if $\left.\frac{\mathrm{d}}{\mathrm{d}y}\right|_{\boldsymbol{y}=\boldsymbol{x}} f(\hat{\boldsymbol{y}})$ is obtained through automatic differentiation it needs to be right-multiplied by a state-dependent matrix in order to obtain the tangent-space derivative. In Ceres parlance these matrices are called

- Local derivative: $\mathrm{d}^r f_X$, a $k \times n$ matrix,
- Global derivative: $\left.\frac{\mathrm{d}}{\mathrm{d}y}\right|_{\boldsymbol{y}=\boldsymbol{x}} f(\hat{\boldsymbol{y}})$, a $k \times p$ matrix,
- Jacobian: $\left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^\vee$, a $p \times n$ matrix

and it holds that (local derivative) = (global derivative) × (jacobian). The local parameterization for a Lie group can be specified as follows:

- Plus operation: $\boldsymbol{x} \boxplus \boldsymbol{b} \coloneqq (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^\vee$,

- Local dimension: $n = \|T\mathbb{M}\|$ tangent space dimension,
- Global dimension: $p = \|\check{\mathbb{M}}\|$ group parameterization dimension,
- Jacobian: $\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\Big|_{\boldsymbol{b}=0} (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^{\vee}$.

6. Dynamical Systems on Lie Groups

Having defined derivatives on Lie Groups we can introduce dynamical systems that evolve on Lie Groups. ODEs can be defined in multiple ways depending on how derivatives are handled. Consider a system $\boldsymbol{X}(t)$ whose state can be interpreted as a mapping $\mathbb{E}(1) \to \check{\mathbb{M}}$.

If ${}^{X}\boldsymbol{a}$ is the body derivative the system can be written with an equation involving the right derivative:

$$\mathrm{d}^{r}\boldsymbol{X}_t = {}^{X}\boldsymbol{a}(t), \quad {}^{X}\boldsymbol{a}(t) \in T_X\check{\mathbb{M}}. \tag{6.1}$$

The same system can be written in terms of the derivative ${}^{e}\boldsymbol{a}(t)$ in the global tangent frame $T_e\check{\mathbb{M}}$ using the left derivative:

$$\mathrm{d}^{l}\boldsymbol{X}_t = {}^{e}\boldsymbol{a}(t), \quad {}^{e}\boldsymbol{a}(t) \in T_e\check{\mathbb{M}}. \tag{6.2}$$

We can also write the global derivative as

$$\mathrm{D}\boldsymbol{X}_t = \boldsymbol{X} \circ {}^{X}\hat{\boldsymbol{a}}(t) = {}^{e}\hat{\boldsymbol{a}}(t) \circ \boldsymbol{X}, \tag{6.3}$$

where the derivative is with respect to the coefficients of the matrix $\boldsymbol{X}$. Naturally, as long as

$${}^{e}\boldsymbol{a}(t) = \mathbf{Ad}_{X(t)}\, {}^{X}\boldsymbol{a}(t) \tag{6.4}$$

all these formulations describe the same dynamical system, which can be readily seen from (5.7).

6.1. Dynamical systems and the exponential map

Consider the dynamical system $\boldsymbol{X}(t) = x_0 \circ \exp(t\boldsymbol{a})$, evaluating the right derivative with respect to t gives

$$\mathrm{d}^{r}\boldsymbol{X}_t = \lim_{\tau\to 0} \frac{\exp(t\boldsymbol{a})^{-1} \circ \exp((t+\tau)\boldsymbol{a})}{\tau} = \lim_{\tau\to 0} \frac{\exp(\tau\boldsymbol{a})}{\tau} = \boldsymbol{a}. \tag{6.5}$$

It follows that $\boldsymbol{X}_0 \circ \exp(t\boldsymbol{a})$ is the solution to the ordinary differential equation

$$\begin{cases} \mathrm{d}^{r}\boldsymbol{X}_t = \boldsymbol{a}, \\ \boldsymbol{X}(0) = \boldsymbol{X}_0. \end{cases} \tag{6.6}$$

Show connection with formulas that map velocities in a rotating frame

6.1.1. Geometric Numerical Integration

Geometric RK scheme

6.2. A Stabilizing Lie Group Controller

Consider the system

$$\begin{aligned} \mathrm{d}^r \boldsymbol{X}_t &= v \\ \mathrm{d}^r v_t &= u \end{aligned} \tag{6.7}$$

where u is a control input, and the objective of tracking a twice differentiable trajectory $\boldsymbol{X}_d(t)$ with first and second right-derivatives v_d and a_d. Consider the error

$$e_X := \boldsymbol{X}_d \ominus_r \boldsymbol{X}, \tag{6.8}$$

with derivative

$$\mathrm{d}^r(e_X)_t \overset{(5.72),(5.73)}{=} \left(\mathrm{d}^r \exp_{e_X}\right)^{-1} \mathrm{d}^r \boldsymbol{X}_d - \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} \mathrm{d}^r \boldsymbol{X} = \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} (\mathbf{Ad}_{\exp(e_X)}\, v_d - v). \tag{6.9}$$

Note that $\mathbf{Ad}_{\exp(e_X)} = \exp \mathrm{ad}_e = \sum_{k\geq 0} \frac{\mathrm{ad}_e}{k!}$ can typically be found on closed form via the usual expansion tricks. Let $e_v := \mathbf{Ad}_{\exp(e_X)}\, v_d - v$ be the velocity error in the body frame; we then have the double intergrator-like error system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e_X &= \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v, \\ \frac{\mathrm{d}}{\mathrm{d}t} e_v &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{Ad}_{\exp(e_X)}\, v_d\right) - u, \end{aligned} \tag{6.10}$$

Where we can further simplify

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{Ad}_{\exp(e_X)}\, v_d\right) \overset{(5.17c)}{=} \left[\mathrm{d}^l \exp_{e_X} \dot{e}_X, \mathbf{Ad}_{\exp(e_X)}\, v_d\right] + \mathbf{Ad}_{\exp(e_X)}\, \dot{v}_d = \left[e_v, \mathbf{Ad}_{\exp(e_X)}\, v_d\right] + \mathbf{Ad}_{\exp(e_X)}\, \dot{v}_d. \tag{6.11}$$

If we further consider an input on the form $u = \left[e_v, \mathbf{Ad}_{\exp(e_X)}\, v_d\right] + \mathbf{Ad}_{\exp(e_X)}\, \dot{v}_d + k_p \left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X + k_d e_v$ that cancels out the contribution from v_d and adds PD feedback terms the closed-loop dynamics become

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e_X &= \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v, \\ \frac{\mathrm{d}}{\mathrm{d}t} e_v &= -k_p \left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X - k_d e_v. \end{aligned} \tag{6.12}$$

Now consider a Lyapunov candidate function on the form

$$V = \frac{k_p}{2} \|e_X\|^2 + \frac{1}{2} \|e_v\|^2 + c \langle e_v, e_X \rangle \geq \frac{1}{2} \begin{bmatrix} \|e_X\| \\ \|e_v\| \end{bmatrix}^T \begin{bmatrix} k_p & -c \\ -c & 1 \end{bmatrix} \begin{bmatrix} \|e_X\| \\ \|e_v\| \end{bmatrix}, \tag{6.13}$$

where c is s.t. $k_p - c^2 \geq 0$ so that the matrix is positive definite. Its derivative evaluates to

$$\begin{aligned} \dot{V} &= k_p \left\langle e_X, \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v \right\rangle - k_p \left\langle e_v, \left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X \right\rangle - k_d \|e_v\|^2 + c \langle \dot{e}_v, e_X \rangle + c \langle e_v, \dot{e}_X \rangle \\ &= -k_d \|e_v\|^2 - c \left\langle k_p \left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X + k_d e_v, e_X \right\rangle + c \left\langle e_v, \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v \right\rangle \\ &= -k_d \|e_v\|^2 - c k_p \|e_X\|^2 - c k_d \langle e_v, e_X \rangle + c \left\langle e_v, \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v \right\rangle - c k_p \left\langle ((\mathrm{d}^l \exp_{e_X})^{-T} - I) e_X, e_X \right\rangle \\ &\leq -k_d \|e_v\|^2 - c k_p \|e_X\|^2 + c k_d \|e_v\| \|e_X\| + c \lambda_{\max} \left(\left(\mathrm{d}^l \exp_{e_X}\right)^{-1}\right) \|e_v\|^2 + c k_p \lambda_{\max} \left(\left(\mathrm{d}^l \exp_{e_X}\right)^{-1} - I\right) \|e_X\|^2. \end{aligned}$$

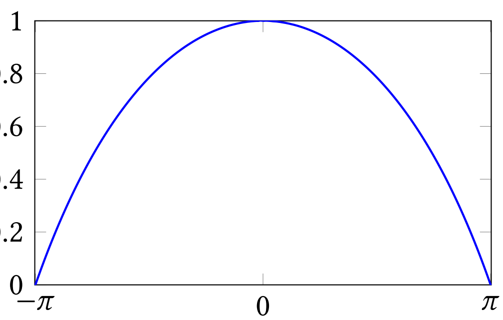


Figure 6.1.: Function $x \mapsto \frac{x}{2}\cot\frac{x}{2}$.

- Eigenvalues of $(\mathrm{d}^l \exp_{e_X})^{-1}$ can be shown to be on the form $\frac{\lambda}{e^\lambda - 1} = \sum_{k=0}^{\infty} \frac{B_n}{n!}\lambda^n$, where λ is an eigenvalue of ad_e.
- Zero is always an eigenvalue of ad_e since $\mathrm{ad}_e\, e = 0$ due to it being a commutator (the corresponding eigenvalue of $(\mathrm{d}^l \exp_e)^{-1}$ is 1
- Often, the eigenvalues of ad_e are purely imaginary. The corresponding eigenvalues of $(\mathrm{d}^l \exp_{e_X})^{-1}$ are

$$\frac{i\lambda}{e^{i\lambda} - 1} = \frac{i\lambda e^{-i\lambda/2}}{e^{i\lambda/2} - e^{-i\lambda/2}} = \frac{i\lambda e^{-i\lambda/2}}{2i \sin\lambda/2} = \lambda\frac{\cos\lambda/2 - i\sin\lambda/2}{2\sin\lambda/2} = \frac{\lambda}{2}\cot\frac{\lambda}{2} - i\frac{\lambda}{2}. \tag{6.14}$$

  That is, the real part is equal to $\frac{\lambda}{2}\cot\frac{\lambda}{2}$.
- For angular groups we should throttle the angular part of $\|e_X\|$ at $\pm\pi/2$ in order to avoid the region where the eigenvalues approach zero which otherwise would lead to sluggish convergence

The maximal real part for $\lambda \in [-\pi, \pi]$ is attained at $\lambda = 0$ and is equal to 1, as shown in Figure 6.1. Thus, for lie groups s.t. $\mathrm{ad}_{\boldsymbol{a}}$ has purely imaginary eigenvalues in the range $[-\pi, \pi]$ for all $\boldsymbol{a}$, it holds that $\left(\mathrm{d}^l \exp_{e_X}\right)^{-1}$ has no eigenvalue with real absolute magnitude larger than 1.

Let $\epsilon = \lambda_{\max}\left(\left(\mathrm{d}^l \exp_{e_X}\right)^{-1} - I\right)$; then we have

$$\dot{V} \leq -\begin{bmatrix}\|e_X\| \\ \|e_v\|\end{bmatrix}^T \begin{bmatrix} ck_p(1-\epsilon) & -\frac{ck_d}{2} \\ -\frac{ck_d}{2} & k_d - c\end{bmatrix} \begin{bmatrix}\|e_X\| \\ \|e_v\|\end{bmatrix} \tag{6.15}$$

Therefore, if

$$\begin{aligned} ck_p(1-\epsilon) + k_d - c &\geq 0 \\ ck_p(1-\epsilon) - \frac{c^2 k_d^2}{4} &\geq 0 \end{aligned} \tag{6.16}$$

6.3. Sensitivity Analysis

Consider again an ODE

$$\begin{aligned} \mathrm{d}^r \boldsymbol{x}_t &= f(t, \boldsymbol{x}), \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0. \end{aligned} \tag{6.17}$$

For a given initial condition $\boldsymbol{x}_0$ the solution at time $t \geq 0$ can be denoted $\phi(t; \boldsymbol{x}_0)$ where the *flow* operator $\phi : \mathbb{R} \times \mathbb{M} \to \mathbb{M}$ is s.t.

$$\begin{aligned} \phi(0; \boldsymbol{x}_0) &= \boldsymbol{x}_0, \\ \mathrm{d}^r \phi(t; \boldsymbol{x}_0)_t &= f\left(t, \phi(t; \boldsymbol{x}_0)\right). \end{aligned} \tag{6.18}$$

Remark 6.1. *Parameters and initial conditions are equivalent. A parameter-dependent system*

$$\begin{aligned} \mathrm{d}^r \boldsymbol{x}_t &= f(t, \boldsymbol{x}; p_0), \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned} \tag{6.19}$$

is equivalent to the parameter-free system on $\mathbb{M} \times \mathbb{R}^n$

$$\begin{aligned} \mathrm{d}^r (\boldsymbol{x}, p)_t &= \left(g(t, \boldsymbol{x}, p), 0\right), \qquad g(t, \boldsymbol{x}, p) := f(t, \boldsymbol{x}; p), \\ (\boldsymbol{x}, p)(t_0) &= (\boldsymbol{x}_0, p_0), \end{aligned} \tag{6.20}$$

where $g(t, \boldsymbol{x}, p) = f(t, \boldsymbol{x}; p)$*. Conversely, the system*

$$\begin{aligned} \mathrm{d}^r \boldsymbol{x}_t &= f(t, \boldsymbol{x}), \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned} \tag{6.21}$$

with a non-trivial initial condition is (locally) equivalent to the parameter-dependent system

$$\begin{aligned} \mathrm{d}^r \boldsymbol{a}_t &= g(t, \boldsymbol{a}; t_0, \boldsymbol{x}_0), \qquad g(t, \boldsymbol{a}; t_0, \boldsymbol{x}_0) := \left[\mathrm{d}^r \exp_{\boldsymbol{a}}\right]^{-1} f(t_0 + t, \boldsymbol{x}_0 \oplus_r \boldsymbol{a}), \\ \boldsymbol{a}(0) &= 0, \end{aligned} \tag{6.22}$$

with trivial initial conditions, in the sense that $\Phi^{\boldsymbol{x}}(t_0 + t; t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0 \oplus_r \Phi^{\boldsymbol{a}}(t; t_0, \boldsymbol{x}_0)$*.*

Due to the equivalence between parameters and initial conditions, it is sufficient to develop sensitivity for one type; we choose initial conditions as in (6.17).

6.3.1. Direct Method

Global derivative on matrix form $\Phi = \hat{\phi}$.

$$\dot{\Phi}(t; \boldsymbol{x}_0) = \Phi(t; \boldsymbol{x}_0) \left(\mathrm{d}^r \Phi(t; \boldsymbol{x}_0)_t\right)^{\wedge} = \Phi(t; \boldsymbol{x}_0) \hat{f}\left(t, \Phi(t; \boldsymbol{x}_0)\right). \tag{6.23}$$

Derivative of inverse

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t} \Phi(t; \boldsymbol{x}_0) \circ \Phi(t; \boldsymbol{x}_0)^{-1} &= \dot{\Phi}(t; \boldsymbol{x}_0) \Phi(t; \boldsymbol{x}_0)^{-1} + \Phi(t; \boldsymbol{x}_0) \frac{\mathrm{d}}{\mathrm{d}t} \Phi(t; \boldsymbol{x}_0)^{-1} \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}t} \phi(t; \boldsymbol{x}_0)^{-1} &= \Phi(t; \boldsymbol{x}_0)^{-1} \dot{\Phi}(t; \boldsymbol{x}_0) \Phi(t; \boldsymbol{x}_0)^{-1}. \end{aligned} \tag{6.24}$$

We can then evaluate how $\mathrm{d}^r\Phi(t;\boldsymbol{x}_0)_t$ depends on t by moving to global derivatives and changing the order of integration.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{d}^r\Phi(t;\boldsymbol{x}_0)_{\boldsymbol{x}_0}\boldsymbol{a}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi(t;\boldsymbol{x}_0)^{-1}\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\Phi(t;\boldsymbol{x}_0\oplus\tau\boldsymbol{a})\right)^{\vee} \\
&= \left(\left(-\Phi(t;\boldsymbol{x}_0)^{-1}\dot{\Phi}(t;\boldsymbol{x}_0)\Phi(t;\boldsymbol{x}_0)\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi(t;\boldsymbol{x}_0\oplus\tau\boldsymbol{a})+\Phi(t;\boldsymbol{x}_0)^{-1}\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\dot{\Phi}(t;\boldsymbol{x}_0\oplus\tau\boldsymbol{a})\right)^{\vee} \\
&= \left(-\hat{f}\,(t;\Phi(t;\boldsymbol{x}_0))\,\Phi(t;\boldsymbol{x}_0)^{-1}\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\Phi(t;\boldsymbol{x}_0\oplus\tau\boldsymbol{a})+\Phi(t;\boldsymbol{x}_0)^{-1}\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\Phi(t;\boldsymbol{x}_0\oplus\tau\boldsymbol{a})\hat{f}(t;\Phi(t;\boldsymbol{x}_0\oplus\tau\boldsymbol{a}))\right)^{\vee} \\
&= -\left(\hat{f}(t;\Phi(t;\boldsymbol{x}_0))\left(\mathrm{d}^r\Phi(t;\boldsymbol{x}_0)_{\boldsymbol{x}_0}\boldsymbol{a}\right)^{\wedge}+\left(\mathrm{d}^r\Phi(t;\boldsymbol{x}_0)_{\boldsymbol{x}_0}\boldsymbol{a}\right)^{\wedge}\hat{f}(t;\Phi(t;\boldsymbol{x}_0))+\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\hat{f}(t;\Phi(t;\boldsymbol{x}_0\oplus\tau\boldsymbol{a}))\right)^{\vee} \\
&= -\left[f(t;\Phi(t;\boldsymbol{x}_0)),\mathrm{d}^r\Phi(t;\boldsymbol{x}_0)_{\boldsymbol{x}_0}\boldsymbol{a}\right]+\mathrm{d}^r f(t,\Phi(t;\boldsymbol{x}_0))_{\boldsymbol{x}_0}\boldsymbol{a} \\
&= -\operatorname{ad}_{f(t;\Phi(t;\boldsymbol{x}_0))}\mathrm{d}^r\Phi(t;\boldsymbol{x}_0)_{\boldsymbol{x}_0}\boldsymbol{a}+\mathrm{d}^r f(t,\boldsymbol{x})_{\boldsymbol{x}=\Phi(t;\boldsymbol{x}_0)}\mathrm{d}^r\Phi(t;\boldsymbol{x}_0))_{\boldsymbol{x}_0}\boldsymbol{a}.
\end{aligned}
$$

The sensitivity $S(t) \coloneqq \mathrm{d}^r\Phi(t;\boldsymbol{x}_0)_{\boldsymbol{x}_0}$ satisfies the matrix-valued ODE

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}S(t) &= \left(-\operatorname{ad}_{f(t,\Phi(t;\boldsymbol{x}_0))}+\mathrm{d}^r f_{\boldsymbol{x}}|_{\boldsymbol{x}=\Phi(t;\boldsymbol{x}_0)}\right)S(t), \\
S(0) &= I.
\end{aligned}
\tag{6.25}
$$

6.3.2. Magnus Expansion Method

Pose that for a system $\mathrm{d}^r x_t = f(t,\boldsymbol{x}(t))$ we have that $\boldsymbol{x}(t) = \boldsymbol{x}_0 \circ \exp\Omega(t)$. Then

$$
\mathrm{d}^r\,(x(t))_{x_0} = \mathbf{Ad}_{\exp(-\Omega(t))} \tag{6.26}
$$

and we have that $\Omega(t)$ satisfies the equation

$$
f(t,\boldsymbol{x}(t)) = \mathrm{d}^r\boldsymbol{x}_t = \mathrm{d}^r\exp_{\Omega(t)}\Omega'(t), \tag{6.27}
$$

i.e.

$$
\Omega'(t) = \left[\mathrm{d}^r\exp_{\Omega(t)}\right]^{-1} f\,(t,\boldsymbol{x}(t)) \tag{6.28}
$$

This is a problem since the vector field is not invariant: this solution form can not work.

The sensitivity with respect to the initial condition is

$$\mathrm{d}^r \Phi(t; \boldsymbol{x}_0)_{\boldsymbol{x}_0} = \mathbf{Ad}_{\exp(-\Omega(t))} \tag{6.29}$$

where Ω satisfies the ODE

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Omega(t) &= \left[\mathrm{d}^r \exp_{\Omega(t)}\right]^{-1} f\left(t, \boldsymbol{x}(t)\right), \\ \Omega(0) &= 0. \end{aligned} \tag{6.30}$$

This is kind of weird because we don't differentiate f

6.3.3. Example

Compare the two sensitivity formulations

Example 6.1

If $\mathrm{d}^r \boldsymbol{x}_t = f(\boldsymbol{x}) \equiv \boldsymbol{a}$, then $\boldsymbol{x}(t) = \boldsymbol{x}_0 \exp(t\boldsymbol{a})$ and we get

$$\mathrm{d}^r(\boldsymbol{x}(t))_{\boldsymbol{x}_0} \overset{(5.61)}{=} \mathbf{Ad}_{\exp(-t\boldsymbol{a})}\,. \tag{6.31}$$

Direct Formulation We furthermore know from Lemma 4.2 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Ad}_{\exp(-t\boldsymbol{a})} = -\operatorname{ad}_{\boldsymbol{a}} \mathbf{Ad}_{\exp(-t\boldsymbol{a})}, \tag{6.32}$$

i.e. the sensitivity equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) = -\operatorname{ad}_{\boldsymbol{a}} S(t), \tag{6.33}$$

which was expected from (6.25) since f is constant.

Magnus Formulation Ω satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega(t) = \left[\mathrm{d}^r \exp_{\Omega(t)}\right]^{-1} \boldsymbol{a}, \quad \Omega(0) = 0. \tag{6.34}$$

Due to the semi-group property $\exp((t+\delta)\boldsymbol{a}) \approx \exp(t\boldsymbol{a}) \circ \exp(\delta \times \mathrm{d}^r \exp_{t\boldsymbol{a}} \boldsymbol{a})$, therefore $\mathrm{d}^r \exp_{t\boldsymbol{a}} \boldsymbol{a} = \boldsymbol{a}$, so it follows that $\Omega(t) = t\boldsymbol{a}$ is the unique solution to (6.34) which agrees with (6.29).

7. Probability Theory

7.1. Gaussian Distributions

Map tangent-space uncertainty through the group

7.2. The Banana Distribution

8. Equivariance

Write about equivariant systems

- Left-invariant, right-invariant, equivariant dynamical systems
- Grizzle: [8]
- Filtering: [7]

Part II.

Group-Specific Calculations

9. Common Matrix Lie Groups

Having gone through the foundational theory in the first part, we now turn our attention to specific groups and derive closed-form formulas for many of the concepts.

Discuss lower-dimensional parameterizations and isometries

9.1. Group and Algebra Structure of Matrix Lie Groups

We now introduce some common matrix Lie groups. Any subset of invertible square matrices that is closed under matrix multiplication is a Lie group. To find the structure of the corresponding Lie algebra it is useful to consider a trajectory

$$\boldsymbol{X}(t) = \operatorname{Exp}(t\boldsymbol{A}) \in \mathbb{M} \tag{9.1}$$

that satisfies $\boldsymbol{X}(0) = I$ and $\boldsymbol{X}'(0) = \boldsymbol{A}$.

The trajectory $\boldsymbol{X}(t)$ must satisfy a certain group constraint, which translates into a condition on $\boldsymbol{A}$. Since the Lie algebra of a group consists of all matrices $\boldsymbol{A}$ such that $\exp \boldsymbol{A} \in \mathbb{M}$ this yields the structure of the Lie algebra.

General Linear Group $\mathbb{GL}(n, F)$ The general linear group over a field F (here F is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$) is the most general Matrix lie group and contains all other groups as subgroups.

$$\mathbb{GL}(n, F) := \{\boldsymbol{A} \in F^{n\times n} \mid \det \boldsymbol{A} \neq 0\}. \tag{9.2}$$

The exponential map always produces invertible matrices, so the corresponding lie algebra is the space of all $n \times n$ matrices.

$$\mathfrak{gl}(n, F) = F^{n\times n}. \tag{9.3}$$

Any subset of $\mathbb{GL}(n, F)$ that is closed under matrix multiplication is also a matrix Lie group.

Euclidean Group $\mathbb{E}(n)$ The usual Euclidean vector space $\mathbb{R}^n$ can be embedded in matrices on the form $\begin{bmatrix} I_n & \boldsymbol{p} \\ 0_{1\times n} & 1 \end{bmatrix}$ for $\boldsymbol{p} \in \mathbb{R}^n$, so that matrix multiplication corresponds to addition in $\mathbb{R}^n$. Being a closed subset of $\mathbb{GL}(n, \mathbb{R})$ those matrices form a matrix Lie group.

To find the corresponding Lie algebra consider a trajectory

$$\boldsymbol{X}(t) = \begin{bmatrix} I_n & \boldsymbol{p}(t) \\ 0_{1\times n} & 1 \end{bmatrix} = \operatorname{Exp}(t\boldsymbol{A}) \in \mathbb{E}(n), \tag{9.4}$$

differentiating with respect to t then shows that

$$\begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{p}'(t) \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{A}. \tag{9.5}$$

From here it follows that the Lie algebra $\mathfrak{e}(n)$ of $\mathbb{E}(n)$ consists of matrices where only the top n coefficients in the right-most column are non-zero.

The Euclicean groups $\mathbb{E}(n)$ and corresponding Lie algebras $\mathfrak{e}(n)$:

$$\mathbb{E}(n) = \left\{\begin{bmatrix} I_n & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \in \mathbb{GL}(n+1, \mathbb{R}) \mid \boldsymbol{p} \in \mathbb{R}^n\right\}, \tag{9.6a}$$

$$\mathfrak{e}(n) = \left\{\begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}, \boldsymbol{v} \in \mathbb{R}^n\right\}. \tag{9.6b}$$

Orthogonal Groups $\mathbb{O}(n)$ and $\mathbb{SO}(n)$ The orthogonal matrices $\mathbb{O}(n)$ are real matrices $\boldsymbol{X}$ s.t. the inverse is equal to the transpose, i.e. $\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^T = I_n$. The special orthogonal matrices in addition have a determinant equal to 1. In robotics $\mathbb{SO}(n)$ is particularly useful in the $n = 2, 3$ cases since those correspond to rotation matrices in two and three dimensions.

Take a one-parameter subgroup $\boldsymbol{X}(t) \coloneqq \mathrm{Exp}(t\boldsymbol{A})$ and differentiate the group constraint $I_n = \boldsymbol{X}(t)^T\boldsymbol{X}(t)$:

$$0 \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)^T\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{X}'(0)^T\boldsymbol{X}(0) + \boldsymbol{X}(0)^T\boldsymbol{X}'(0) = \boldsymbol{A}^T + \boldsymbol{A}. \tag{9.7}$$

It follows that the Lie algebra $\mathfrak{so}(n)$ corresponding to $\mathbb{SO}(n)$ consists of **skew-symmetric matrices**.

Orthogonal groups and corresponding Lie algebras:

$$\mathbb{O}(n) = \left\{\boldsymbol{X} \in \mathbb{GL}(n, \mathbb{R}) \mid \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^T = I_n\right\}, \tag{9.8a}$$

$$\mathbb{SO}(n) = \left\{\boldsymbol{X} \in \mathbb{GL}(n, \mathbb{R}) \mid \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^T = I_n, \det\boldsymbol{X} = 1\right\}, \tag{9.8b}$$

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \left\{\boldsymbol{A} \in \mathbb{R}^{n\times n} \,:\, \boldsymbol{A}^T + \boldsymbol{A} = 0\right\}. \tag{9.8c}$$

Unitary Groups $\mathbb{U}(n)$ and $\mathbb{SU}(n)$ Unitary matrices $\boldsymbol{X}$ are characterized by the inverse being equal to the Hermitian transpose, i.e. $\boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^* = I$[1]. In the case of $\mathbb{SU}(n)$ the determinant is also required to be equal to 1.

For a one-parameter subgroup $\boldsymbol{X}(t) = \mathrm{Exp}(t\boldsymbol{A})$ constraint differentiation yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)^*\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{X}'(0)^*\boldsymbol{X}(0) + \boldsymbol{X}(0)^*\boldsymbol{X}'(0) = \boldsymbol{A}^* + \boldsymbol{A}. \tag{9.9}$$

This shows that $\mathfrak{u}(n)$ consists of **skew-Hermitian matrices**. In addition, due to the Jacobi identity (4.2c) $\det\mathrm{Exp}(t\boldsymbol{A}) = 1$ implies that $\mathrm{Tr}\boldsymbol{A} = 0$ i.e. $\mathfrak{su}(n)$ consists of **skew-Hermitian matrices with vanishing trace**.

[1]The Hermitian transpose (also known as *conjugate transpose*) of $\boldsymbol{X}_{ij}$ is $\boldsymbol{X}^*_{ij} = \bar{\boldsymbol{X}}_{ji}$.

Unitary groups and corresponding Lie algebras:

$$\mathbb{U}(n) = \{\boldsymbol{X} \in \mathbb{GL}(n, \mathbb{C}) \mid \boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^* = I_n\} \tag{9.10a}$$
$$\mathbb{SU}(n) = \{\boldsymbol{X} \in \mathbb{GL}(n, \mathbb{C}) \mid \boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^* = I_n, \det \boldsymbol{X} = 1\}, \tag{9.10b}$$
$$\mathfrak{u}(n) = \{\boldsymbol{A} \in \mathbb{C}^{n\times n} \mid \boldsymbol{A}^* + \boldsymbol{A} = 0\}, \tag{9.10c}$$
$$\mathfrak{su}(n) = \{\boldsymbol{A} \in \mathbb{C}^{n\times n} \mid \boldsymbol{A}^* + \boldsymbol{A} = 0, \operatorname{Tr}\boldsymbol{A} = 0\}. \tag{9.10d}$$

Special Euclidean Group $\mathbb{SE}(n)$ A "semi-simple product" between $\mathbb{SO}(n)$ and $\mathbb{E}(n)$ obtained by replacing the identity matrix in $\mathbb{E}(n)$ by a member of $\mathbb{SO}(n)$, i.e. $\mathbb{SE}(n)$ consists of matrices on the form $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0_{1\times n} & 1 \end{bmatrix}$ for $\boldsymbol{R} \in \mathbb{SO}(n)$. For a one-parameter subgroup $\boldsymbol{X}(t) = \begin{bmatrix} \boldsymbol{R}(t) & \boldsymbol{p}(t) \\ 0 & 1 \end{bmatrix} = \operatorname{Exp}(t\boldsymbol{A})$ differentiating shows that

$$\begin{bmatrix} \boldsymbol{R}'(t) & \boldsymbol{p}'(t) \\ 0 & 0 \end{bmatrix} = \boldsymbol{A}. \tag{9.11}$$

Since we already know that structure of $\boldsymbol{R}'(t)$ from (9.8c) it follows that $\boldsymbol{A}$ must be on the form

Special Euclidean groups and corresponding Lie algebras:

$$\mathbb{SE}(n) = \left\{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \in \mathbb{GL}(n+1, \mathbb{R}) \mid \boldsymbol{R} \in \mathbb{SO}(n), \boldsymbol{p} \in \mathbb{R}^n\right\}, \tag{9.12a}$$
$$\mathfrak{se}(n) = \left\{\begin{bmatrix} \boldsymbol{B} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \mid \boldsymbol{B} \in \mathfrak{so}(n), \boldsymbol{v} \in \mathbb{R}^n\right\}. \tag{9.12b}$$

9.2. Some Summation Formulas

Starting with the familiar Taylor expansions of cosine and sine

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x, \tag{9.13}$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x, \tag{9.14}$$

some higher-order formulas can be derived for $x \neq 0$ by dividing by a factor of x and subtracting the first summation terms.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{\sin x}{x}, \tag{9.15}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} x^{2n} = -\frac{1}{x^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \frac{1-\cos x}{x^2}, \tag{9.16}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)!} x^{2n} = -\frac{1}{x^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{x-\sin x}{x^3}, \tag{9.17}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+4)!} x^{2n} = \frac{1}{x^4} \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}, \tag{9.18}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+5)!} x^{2n} = \frac{1}{x^5} \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{\sin x - x + \frac{x^3}{6}}{x^5}. \tag{9.19}$$

Finally a sum involving the Bernoulli numbers will be useful for some groups of interest.

Proposition 9.1.

$$\sum_{n\geq 1} \frac{B_{2n}(-1)^n x^{2n}}{(2n)!} = \frac{x}{2}\cot\left(\frac{x}{2}\right). \tag{9.20}$$

Proof. By setting $x = iy$ and observing that $B_n = 0$ for odd $n > 1$ we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{B_{2n}(-1)^n x^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^n y^{2n}(-1)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{B_n y^n}{n!} - B_1 y \overset{(5.31)}{=} \frac{y}{e^y-1} + \frac{y}{2} = \frac{y}{2}\frac{e^y+1}{e^y-1} \\ &= \frac{ix}{2}\frac{1+e^{-iy}}{1-e^{-iy}} - 1 = \frac{ix}{2}\frac{e^{iy/2}+e^{-ix/2}}{e^{ix/2}-e^{-ix/2}} = \frac{ix}{2}\frac{\cos(x/2)}{i\sin(x/2)} = \frac{x}{2}\cot\left(\frac{x}{2}\right).\end{aligned} \tag{9.21}$$

□

10. $\mathbb{E}(n)$

Due to the isometry with $\mathbb{R}^n$ none of the properties of $\mathbb{E}(n)$ are particularly exciting, but we still list them for completenes..

$\mathbb{E}(n)$ parameterized by $\mathbb{R}^n$

Group Parameterization

$$\{\boldsymbol{p} \in \mathbb{R}^n\}, \quad \boldsymbol{p}^\wedge = \begin{bmatrix} 0_{n\times n} & \boldsymbol{p} \\ 0_{1\times n} & 1 \end{bmatrix}. \tag{10.1}$$

Algebra Parameterization

$$\{\boldsymbol{v} \in \mathbb{R}^n\}, \quad \boldsymbol{v}^\wedge = \begin{bmatrix} 0_{n\times n} & \boldsymbol{v} \\ 0_{1\times n} & 0 \end{bmatrix} \tag{10.2}$$

Group Operations

- Identity element: 0_n
- Inverse: $(\boldsymbol{p})^{-1} = -\boldsymbol{p}$
- Composition: $\boldsymbol{p} \circ \boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{p}'$

Adjoint

$$\mathbf{Ad}_{\boldsymbol{p}} = \begin{bmatrix} 0 \end{bmatrix}. \tag{10.3}$$

Exponential

$$\exp(\boldsymbol{v}) = \boldsymbol{v} \tag{10.4}$$

Logarithm

$$\log(\boldsymbol{p}) = \boldsymbol{p} \tag{10.5}$$

Lowercase adjoint

$$\mathrm{ad}_{\omega_z} = 0 \tag{10.6}$$

Derivatives of the Exponential

$$\mathrm{d}^r \exp_{\omega_z} = \mathrm{d}^l \exp_{\omega_z} = \left(\mathrm{d}^r \exp_{\omega_z}\right)^{-1} = \left(\mathrm{d}^l \exp_{\omega_z}\right)^{-1} = \begin{bmatrix} 1 \end{bmatrix}. \tag{10.7}$$

11. $\mathbb{SO}(2)$

Parameterization We use the isometry $\mathbb{SO}(2) \cong \mathbb{U}(1)$, where $\mathbb{U}(1)$ is the unitary group consisting of complex elements $c = q_w + iq_z$ with unit length, to parameterize elements of $\mathbb{SO}(2)$.

$$\check{\mathbb{SO}}(2) = \left\{(q_w, q_z) \in \mathbb{R}^2 \mid q_w^2 + q_z^2 = 1\right\}. \tag{11.1}$$

The hat and vee maps between the parameterization and matrix forms are thus as follows:

$$\begin{tikzcd} \check{\mathbb{SO}}(2) \ni (q_w, q_z) \arrow[r, bend left, "\wedge"] & \boldsymbol{R} = \begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix} \in \mathbb{SO}(2) \arrow[l, bend left, "\vee"] \end{tikzcd}$$

From the isometry to complex numbers it follows that the identity element is $(1, 0)$ and the inverse is $(q_w, q_z)^{-1} = (q_w, -q_z)$. A formula for group composition can either be obtained by expanding the complex product $(q_w + iq_z)(q'_w + iq_z)$ and identifying the coefficients, or by going via the matrix form:

$$(q_w, q_z) \circ (q'_w, q'_z) = ((q_w, q_z)^\wedge \cdot (q'_w, q'_z)^\wedge)^\vee = \left(\begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix} \cdot \begin{bmatrix} q'_w & -q'_z \\ q'_z & q'_w \end{bmatrix}\right)^\vee = (q_w q'_w - q_z q'_z, q_z q'_w + q_w q'_z).$$

Action on $\mathbb{R}^2$: In robotics applications it is convenient to define a rotational action on vectors in $\mathbb{R}^2$. For $\boldsymbol{R} \in \mathbb{SO}(2)$ and $\boldsymbol{u} \in \mathbb{R}^2$ the action is matrix multiplication

$$\langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\mathbb{SO}(2)} = \boldsymbol{R} \cdot u. \tag{11.2}$$

Lie Algebra For the special orthogonal groups $\mathbb{SO}(n)$ the group constraint is orthogonality of the matrix: $\boldsymbol{X}^T \boldsymbol{X} = I_n$. Take a one-parameter subgroup $\boldsymbol{X}(t) := \mathrm{Exp}(t\boldsymbol{A})$; it must then hold that

$$0 \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{X}(t)^T \boldsymbol{X}(t) \bigg|_{t=0} = \boldsymbol{X}'(0)^T \boldsymbol{X}(0) + \boldsymbol{X}(0)^T \boldsymbol{X}'(0) = \boldsymbol{A}^T + \boldsymbol{A}. \tag{11.3}$$

It follows that the Lie algebra $\mathfrak{so}(n)$ corresponding to $\mathbb{SO}(n)$ consists of **skew-symmetric matrices**.

$$\mathfrak{so}(n) = \left\{\boldsymbol{A} \in \mathbb{R}^{n \times n} \;:\; \boldsymbol{A}^T + \boldsymbol{A} = 0\right\}. \tag{11.4}$$

The 2×2 skew-symmetric matrices have only one degree of freedom, let this single parameter of $\check{\mathfrak{so}}(2)$ be denoted ω_z so that

$$\mathfrak{so}(2) = \left\{\begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \mid \omega_z \in \mathbb{R}\right\}, \tag{11.5}$$

and the algebra hat and vee maps become

$$\begin{tikzcd} \check{\mathfrak{so}}(2) \ni \omega_z \arrow[rr, bend left, "\wedge"] & & \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \in \mathfrak{so}(2) \arrow[ll, bend left, "\vee"] \end{tikzcd}$$

11.1. Derivations

Adjoint From the definition,

$$\mathbf{Ad}_{(q_w,q_z)}\,\omega_z = \left(\left((q_w,q_z)^\wedge \hat{\omega}_z \left(q_w,q_z\right)^{-1}\right)^\wedge\right)^\vee = \left(\begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix}\begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix}\begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix}\right)^\vee = \omega_z, \tag{11.6}$$

so it follows that $\mathbf{Ad}_{(q_w,q_z)} = \begin{bmatrix}1\end{bmatrix}$.

Exponential and Logarithm Take an element $\hat{\omega}_z := \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \in \mathfrak{so}(2)$. The exponential is calculated by noting that $(\omega_z^\wedge)^{2k} = (-1)^k \omega_z^{2k} I_2$:

$$\begin{aligned} \operatorname{Exp}\omega_z^\wedge = \sum_{k=0}^{\infty} \frac{(\omega_z^\wedge)^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(\omega_z^\wedge\right)^{2k}}{(2k)!} + \frac{\left(\omega_z^\wedge\right)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \omega_z^{2k}}{(2k)!} I_2 + \frac{(-1)^k \omega_z^{2k}}{(2k+1)!}\omega_z^\wedge \\ \overset{(9.13),(9.15)}{=} \cos\omega_z I_2 + \frac{\sin\omega_z}{\omega_z}\omega_z^\wedge = \begin{bmatrix} \cos\omega_z & -\sin\omega_z \\ \sin\omega_z & \cos\omega_z \end{bmatrix}. \end{aligned} \tag{11.7}$$

Thus $\exp(\omega_z) = (\cos\omega_z, \sin\omega_z)$ and consequently $\log(q_w, q_z) = \operatorname{arctan2}(q_z, q_w)$.

Derivatives of the Exponential Consider algebra elements $\omega_z, \bar{\omega}_z \in \check{\mathfrak{so}}(2)$. The bracket on $\check{\mathfrak{so}}(2)$ is zero since

$$[\omega_z, \bar{\omega}_z] = \left(\begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix}\begin{bmatrix} 0 & -\bar{\omega}_z \\ \bar{\omega}_z & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\bar{\omega}_z \\ \bar{\omega}_z & 0 \end{bmatrix}\begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix}\right)^\vee = 0. \tag{11.8}$$

It follows that all terms in (5.30) and (5.32) vanish except for $n = 0$, so the derivatives of the exponential are equal to $I_1 = \begin{bmatrix}1\end{bmatrix}$.

11.2. Summary

$\mathbb{SO}(2)$ parameterized by $\mathbb{U}(1)$

Group Parameterization

$$\left\{(q_w, q_z) \,:\, q_w^2 + q_z^2 = 1\right\}, \quad (q_w, q_z)^\wedge = \begin{bmatrix} q_w & -q_z \\ q_z & q_w \end{bmatrix}. \tag{11.9}$$

Algebra Parameterization

$$\{\omega_w \mid \omega_w \in [-\pi, \pi]\}, \quad \hat{\omega}_w = \begin{bmatrix} 0 & -\omega_w \\ \omega_w & 0 \end{bmatrix} \tag{11.10}$$

Group Operations

- Identity element: $(1, 0)$
- Inverse: $(q_w, q_z)^{-1} = (q_w, -q_z)$
- Composition: $(q_w, q_z) \circ (q_w', q_z') = (q_w q_w' - q_z q_z', q_z q_w' + q_w q_z')$

Adjoint

$$\mathbf{Ad}_{(q_w, q_z)} = \left[1\right]. \tag{11.11}$$

Exponential

$$\exp(\omega_w) = (\cos\omega_w, \sin\omega_w) \tag{11.12}$$

Logarithm

$$\log(q_w, q_z) = \operatorname{arctan2}(q_z, q_w) \tag{11.13}$$

Lowercase adjoint

$$\mathrm{ad}_{\omega_z} = 0 \tag{11.14}$$

Derivatives of the Exponential

$$\mathrm{d}^r \exp_{\omega_z} = \mathrm{d}^l \exp_{\omega_z} = \left(\mathrm{d}^r \exp_{\omega_z}\right)^{-1} = \left(\mathrm{d}^l \exp_{\omega_z}\right)^{-1} = \left[1\right]. \tag{11.15}$$

12. $\mathbb{SO}(3)$

Parameterization

12.1. Derivations

12.2. Summary

$\mathbb{SO}(3)$ parameterized by $\mathbb{S}^3$

Group Parameterization

$$
\begin{aligned}
&\left\{\boldsymbol{q} = (q_w, q_x, q_y, q_z) \,:\, q_w^2 + q_x^2 + q_z^2 + q_z^2 = 1\right\}, \\
&\boldsymbol{q}^\wedge = \boldsymbol{R}(\boldsymbol{q}) \coloneqq \begin{bmatrix} (1 - 2(q_y^2 + q_z^2)) & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & (1 - 2(q_x^2 + q_z^2)) & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_w q_x + q_y q_z) & (1 - 2(q_x^2 + q_y^2)) \end{bmatrix}.
\end{aligned} \tag{12.1}
$$

Algebra Parameterization

$$
\left\{\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \mid \omega_x, \omega_y, \omega_z \in [-\pi, \pi]\right\}, \quad \hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{12.2}
$$

Group Operations

- Identity element: $(1, 0, 0, 0)$
- Inverse: $(q_w, q_x, q_y, q_z)^{-1} = (q_w, -q_x, -q_y, -q_z)$
- Composition:

Adjoint

$$
\mathrm{Ad}_{\boldsymbol{q}} = \boldsymbol{R}(\boldsymbol{q}) \tag{12.3}
$$

Exponential

$$
\exp(\boldsymbol{\omega}) = \left(\cos(\|\boldsymbol{\omega}\|/2), \frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2), \frac{\omega_y}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2), \frac{\omega_z}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right), \tag{12.4}
$$

Logarithm

$$\log(\boldsymbol{q}) = \left(2\frac{\operatorname{arctan2}\left(\sqrt{q_x^2+q_y^2+q_z^2}, q_w\right)}{\sqrt{q_x^2+q_y^2+q_z^2}}\right) \times \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}. \tag{12.5}$$

Lowercase adjoint

$$\mathrm{ad}_{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}} \tag{12.6}$$

Derivatives of the Exponential

$$\mathrm{d}^r \exp_{\boldsymbol{\omega}} = I_3 - \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}} + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2, \tag{12.7a}$$

$$\mathrm{d}^l \exp_{\boldsymbol{\omega}} = I_3 + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}} + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2, \tag{12.7b}$$

$$\left(\mathrm{d}^r \exp_{\boldsymbol{\omega}}\right)^{-1} = I_3 + \frac{\hat{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1+\cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\|\sin\|\boldsymbol{\omega}\|}\right)\hat{\boldsymbol{\omega}}^2, \tag{12.7c}$$

$$\left(\mathrm{d}^l \exp_{\boldsymbol{\omega}}\right)^{-1} = I_3 - \frac{\hat{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1+\cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\|\sin\|\boldsymbol{\omega}\|}\right)\hat{\boldsymbol{\omega}}^2. \tag{12.7d}$$

13. $\mathbb{SE}(2)$

Parameterization

Action on $\mathbb{R}^2$:

Lie Algebra

13.1. Derivations

Adjoint

Exponential and Logarithm

Derivatives of the Exponential

13.2. Summary

$\mathbb{SE}(2)$ parameterized by $\mathbb{U}(1) \times \mathbb{R}^2$

Group Parameterization

$$\tag{13.1}$$

Algebra Parameterization

$$\tag{13.2}$$

Group Operations

- Identity element:
- Inverse:
- Composition:

Adjoint

$$\tag{13.3}$$

Exponential

$$\tag{13.4}$$

Logarithm

(13.5)

Lowercase adjoint

(13.6)

Derivatives of the Exponential

(13.7)

14. $\mathbb{SE}(3)$

Parameterization

Action on $\mathbb{R}^3$:

Lie Algebra

14.1. Derivations

Adjoint

Exponential and Logarithm

Derivatives of the Exponential

14.2. Summary

$\mathbb{SE}(3)$ parameterized by $\mathbb{S}^3 \times \mathbb{R}^3$

Group Parameterization

(14.1)

Algebra Parameterization

(14.2)

Group Operations

- Identity element:
- Inverse:
- Composition:

Adjoint

(14.3)

Exponential

(14.4)

Logarithm

$$ \tag{14.5} $$

Lowercase adjoint

$$ \tag{14.6} $$

Derivatives of the Exponential

$$ \tag{14.7} $$

Part III.

Applications

15. Application: Geometric Control

Summary

- Extend PD theory to Lie Groups.
- Quadrotor control on $SE(3)$

In the following we let $M = SO(3)$ be the Lie group consisting of rotation matrices with matrix multiplication being the group action. We also write $e = I_3$ for the identity element of the group.

Consider the rigid body dynamics

$$\dot{R} = R\,{}^R\hat{\boldsymbol{\omega}}, \tag{15.1a}$$

$$J\,{}^R\dot{\boldsymbol{\omega}} = -{}^R\hat{\boldsymbol{\omega}} J\,{}^R\boldsymbol{\omega} + u, \tag{15.1b}$$

where J is the moment of inertia, ${}^R\hat{\boldsymbol{\omega}} \in TM_R$ is the angular velocity in the body frame, and $R \in SO(3)$ is the attitude. By (6.4) we can see that the angular velocity ${}^e\hat{\boldsymbol{\omega}} \in TM_e$ in the inertial frame can be obtained as

$${}^e\boldsymbol{\omega} = \mathrm{Ad}_R\left({}^R\boldsymbol{\omega}\right) = R\,{}^R\boldsymbol{\omega}. \tag{15.2}$$

Using (??) we also see that ${}^e\hat{\boldsymbol{\omega}} = \widehat{\mathrm{Ad}_R\,{}^R\boldsymbol{\omega}} = R\,{}^R\boldsymbol{\omega}\,R^T$, so it follows that (15.1a) can be written as

$$\dot{R} = {}^e\hat{\boldsymbol{\omega}} R. \tag{15.3}$$

We assume that a smooth trajectory in the inertial frame is given by R_d and ${}^e\boldsymbol{\omega}_d$ satisfying the dynamics

$$\dot{R}_d = {}^e\hat{\boldsymbol{\omega}}_d R_d, \tag{15.4}$$

and the goal is to control u in (15.1) so that R and ${}^e\boldsymbol{\omega}$ are close to R_d and ${}^e\boldsymbol{\omega}_d$.

15.1. Error Functions

In general we would like to pick for $\tilde{e}_r = R_d \ominus R$ the error function $\frac{1}{2}\|\tilde{e}_r\|^2$ with derivative $\langle \tilde{e}_r, \tilde{e}_\omega \rangle$ for

$$\tilde{e}_\omega = \dot{\tilde{e}}_r = J_{R_d}^{R_d \ominus R}\,{}^{R_d}\boldsymbol{\omega}_d + J_R^{R_d \ominus R}\,{}^R\boldsymbol{\omega}. \tag{15.5}$$

This is general for any Lie group, and we can pick u to stabilize a double integrator system in the tangent space. However, the derivative of $\tilde{e}_\omega$ is cumbersome to evaluate and it is possible to arrive at a simpler formulation in $SO(3)$. Consider the error functions

$$\Psi(R, R_d) = 1 - \cos(\theta) = \frac{1 - \mathrm{Tr}(RR_d^T)}{2} = -\frac{1 - \langle R_d, R\rangle_F}{2}, \tag{15.6a}$$

$$e_r = \frac{1}{2}(R_d^T R - R^T R_d)^\vee, \tag{15.6b}$$

$$e_\omega = \boldsymbol{\omega} - R^T \boldsymbol{\omega}^d \in TSO(3)_R. \tag{15.6c}$$

It can be seen by (??) that e_r is a rescaling of $\tilde{e}_r$. The derivative of Ψ is $\langle e_r, e_\omega\rangle$ as above, indeed

$$\begin{aligned}
\dot{\Psi} &= -\frac{1}{2}\left(\left\langle R_d, \dot{R}\right\rangle_F + \left\langle \dot{R}_d, R\right\rangle_F\right) = -\frac{1}{2}\left(\langle R_d, R\hat{\boldsymbol{\omega}}\rangle_F + \langle \hat{\boldsymbol{\omega}}_d R_d, R\rangle_F\right) = \\
&= -\frac{1}{2}\left(\left\langle R^T R_d, \hat{\boldsymbol{\omega}}\right\rangle_F - \left\langle \hat{\boldsymbol{\omega}}_d^T R_d, R\right\rangle_F\right) = -\frac{1}{2}\left(\left\langle R^T R_d, \hat{\boldsymbol{\omega}}\right\rangle_F - \langle R_d, \hat{\boldsymbol{\omega}}_d R\rangle_F\right) \\
&\overset{(??)}{=} -\frac{1}{2}\left(\left\langle R^T R_d, \hat{\boldsymbol{\omega}}\right\rangle_F - \left\langle R_d, R\widehat{R^T \boldsymbol{\omega}_d}\right\rangle_F\right) = -\frac{1}{2}\left\langle R^T R_d, \hat{e}_\omega\right\rangle_F \\
&= \frac{1}{4}\left\langle R_d^T R - R^T R_d, \hat{e}_\omega\right\rangle_F \overset{(??)}{=} e_r \cdot e_\omega,
\end{aligned} \tag{15.7}$$

where we have used the property that the Frobenius product $\langle A, B\rangle_F = -\left\langle A^T, B\right\rangle_F$ for B skew-symmetric.

15.2. WIP: Derivative via Jacobians

It should also be possible to calculate the time derivative via (5.77), but it seems difficult to arrive at the same expression:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi &= -\frac{1}{2}\frac{{}^e\partial \mathrm{Tr}(RR_d^T)}{{}^e\partial RR_d^T}\left[\frac{{}^e\partial RR_d^T}{{}^e\partial R}\,{}^e\boldsymbol{\omega} + \frac{{}^e\partial RR_d^T}{{}^e\partial R_d}{}^e\boldsymbol{\omega}_d\right] \\
&= -\frac{1}{2}\left(R_d R^T - RR_d^T\right)^\vee \cdot [{}^e\boldsymbol{\omega} - R^e\boldsymbol{\omega}_d]
\end{aligned} \tag{15.8}$$

Using right derivatives instead:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi &= -\frac{1}{2}\frac{{}^{\mathrm{Tr}(RR_d^T)}\partial \mathrm{Tr}(RR_d^T)}{{}^{RR_d^T}\partial RR_d^T}\left[\frac{{}^{RR_d^T}\partial RR_d^T}{{}^{R}\partial R}{}^{R}\boldsymbol{\omega} + \frac{{}^{RR_d^T}\partial RR_d^T}{{}^{R_d}\partial R_d}{}^{R_d}\boldsymbol{\omega}_d\right] \\
&= -\frac{1}{2}\left(R_d R^T - RR_d^T\right)^\vee\left[R_d\,{}^{R}\boldsymbol{\omega} - {}^{R_d}\boldsymbol{\omega}_d\right]
\end{aligned} \tag{15.9}$$

15.3. Lyapunov Stability

We let the input be

$$u = -k_r e_r - k_\omega e_\omega + \widehat{R^T \boldsymbol{\omega}_d} J R^T \boldsymbol{\omega}_d + J R^T \dot{\boldsymbol{\omega}}_d. \tag{15.10}$$

and consider a Lyapunov candidate on the form

$$V = \frac{1}{2} e_\omega \cdot J e_\omega + k_r \Psi + c e_r \cdot J e_\omega \tag{15.11}$$

The derivative of the Lyapunov candidate then

Proposition 15.1. *It holds that*

$$J\dot{e}_\omega = -k_r e_r - k_\omega e_\omega + \big(Je_\omega + \big(2JR^T\omega_d - \operatorname{trace}(J)I\big)\,R^T\omega_d\big) \times e_\omega. \tag{15.12}$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Je_\omega &\overset{(15.20b)}{=} J\dot{\omega} - J\dot{R}^T\omega_d - JR^T\dot{\omega}_d \overset{(15.1)}{=} u - \hat{\omega}J\omega - J\,(R\hat{\omega})^T\,\omega_d - JR^T\dot{\omega}_d \\
&\overset{(15.21)}{=} -k_r e_r - k_\omega e_\omega + \widehat{R^T\,\omega_d}JR^T\,\omega_d - \hat{\omega}J\omega - J\hat{\omega}^T R^T\omega_d \\
&\overset{(15.20b),(??)}{=} -k_r e_r - k_\omega e_\omega + \widehat{R^T\,\omega_d}JR^T\,\omega_d - (\hat{e}_\omega + \widehat{R^T\omega_d})J(e_\omega + R^T\omega_d) \\
&\quad + J\left(\hat{e}_\omega + \cancelto{0}{\widehat{R^T\omega_d}}\right)R^T\omega_d \\
&\overset{(??)}{=} -k_r e_r - k_\omega e_\omega + \left(\widehat{Je_\omega} + \widehat{JR^T\omega_d} - \widehat{R^T\omega_d}J - J\widehat{R^T\omega_d}\right)e_\omega \\
&\overset{(??)}{=} -k_r e_r - k_\omega e_\omega + \big(Je_\omega + \big(2JR^T\omega_d - \operatorname{trace}(J)I\big)\,R^T\omega_d\big)^\wedge\, e_\omega.
\end{aligned}$$

□

We then get

$$\dot{V} = -k_\omega\|e_\omega\|^2 + c\dot{e}_r \cdot Je_\omega + ce_r \cdot J\dot{e}_\omega \tag{15.13}$$

It remains to bound the terms involving c. From (??) we have that $\|\omega\|_2^2 = \frac{1}{2}\|\hat{\omega}\|_F^2$. We also have

$$\frac{\mathrm{d}}{\mathrm{d}t}R_d^T R = R_d^T R\hat{\omega} + R_d^T\hat{\omega}_d^T R \overset{(??)}{=} R_d^T R\hat{\omega} - R_d^T\hat{\omega}_d R \overset{(??)}{=} R_d^T R\hat{\omega} - R_d^T R\,\widehat{R^T\omega_d} = R_d^T R\hat{e}_\omega. \tag{15.14}$$

and therefore we get that $\|\dot{\hat{e}}_r\|_F = \left\|\frac{1}{2}\left(R_d^T R\hat{e}_\omega + \hat{e}_\omega R^T R_d\right)\right\|_F \leq \|\hat{e}_\omega\|_F$, so it follows that

$$\|\dot{e}_r\|_2 \leq \|e_\omega\|_2 \qquad \Longrightarrow \quad \dot{e}_r \cdot Je_\omega \leq \lambda_M(J)\|e_\omega\|_2^2. \tag{15.15}$$

Finally, using that $\|e_r\| \leq 1$,

$$\begin{aligned}
J\dot{e}_\omega \cdot e_r &\overset{(15.12)}{=} \big(-k_r e_r - k_\omega e_\omega + \big(Je_\omega + \big(2JR^T\omega_d - \operatorname{trace}(J)I\big)\,R^T\omega_d\big) \times e_\omega\big) \cdot e_r \\
&\qquad\qquad \leq -k_r\|e_r\|^2 + k_\omega\|e_r\|\|e_\omega\| + \lambda_M(J)\|e_\omega\|^2 + B\|e_\omega\|\|e_r\|.
\end{aligned} \tag{15.16}$$

We can now bound the derivative as follows:

$$\dot{V} \leq -\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix}^T \begin{bmatrix} ck_r & -c(k_\omega + B)/2 \\ -c(k_\omega + B)/2 & k_\omega - 2c\lambda_M(J)\end{bmatrix}\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix}, \tag{15.17}$$

and it follows that if we choose c small enough then the matrix is positive definite and thus V decreases along trajectories of the closed-loop system.

15.4. Direction-driven Attitude Control on SO(3)

We pick two orthogonal unit-length directions b_1 and b_2 and define the following error function:

$$\Psi_i(R) = \frac{1}{2}\|Rb_i - R_d b_i\|^2 = 1 - (Rb_i)\cdot(R_d b_i). \tag{15.18}$$

The derivative of $\Psi_i(R)$ becomes

$$\begin{aligned}\dot{\Psi}_i(R) &= -\dot{R}b_i \cdot R_d b_i - Rb_i \cdot \dot{R}_d b_i \overset{(15.1a),(15.4)}{=} -R^R\hat{\boldsymbol{\omega}} b_i \cdot R_d b_i - Rb_i \cdot {}^e\hat{\boldsymbol{\omega}}_d R_d b_i \\ &= -{}^R\hat{\boldsymbol{\omega}} b_i \cdot R^T R_d b_i - b_i \cdot R^T\, {}^e\hat{\boldsymbol{\omega}}_d R_d b_i \overset{(??)}{=} -{}^R\hat{\boldsymbol{\omega}} b_i \cdot R^T R_d b_i - b_i \cdot \widehat{R^T\, {}^e\boldsymbol{\omega}_d} R^T R_d b_i \\ &\overset{(??)}{=} -{}^R\boldsymbol{\omega}\cdot\left(\hat{b}_i\, R^T R_d b_i\right) - R^T\, {}^e\boldsymbol{\omega}_d \cdot \widehat{R^T R_d b_i} b_i \overset{(??)}{=} \underbrace{\left({}^R\boldsymbol{\omega} - R^T\, {}^e\boldsymbol{\omega}_d\right)}_{e_\omega} \cdot \underbrace{\widehat{R^T R_d b_i} b_i}_{e_{r_i}},\end{aligned} \tag{15.19}$$

where we have defined two error functions

$$e_{r_i} = \widehat{R^T R_d b_i} b_i, \tag{15.20a}$$
$$e_\omega = {}^R\boldsymbol{\omega} - R^T\, {}^e\boldsymbol{\omega}_d, \tag{15.20b}$$

that are small when $R \approx R_d$ and when $\mathrm{Ad}_R\, {}^R\hat{\boldsymbol{\omega}} = R^R\hat{\boldsymbol{\omega}} \approx {}^e\boldsymbol{\omega}_d$, respectively.

15.5. Feedback Control

Given these error functions we consider the feedback control

$$u = -e_r - k_\omega e_\omega + \widehat{R^T\, {}^e\boldsymbol{\omega}_d} J R^T\, {}^e\boldsymbol{\omega}_d + J R^T\, {}^e\dot{\boldsymbol{\omega}}_d, \tag{15.21}$$

where

$$e_r = k_1 e_{r_1} + k_2 e_{r_2}, \tag{15.22}$$

and k_1, k_2, k_ω are positive gains. Take the candidate Lyapunov function

$$V = \frac{1}{2} e_\omega \cdot J e_\omega + k_1\Psi_1(R) + k_2\Psi_2(R) + cJe_\omega \cdot e_r. \tag{15.23}$$

In the following we drop the upper left superscripts and write $\boldsymbol{\omega} = {}^R\boldsymbol{\omega}$ and $\boldsymbol{\omega}_d = {}^e\boldsymbol{\omega}_d$.

15.6. Lyapunov lower bound

We would like to show that $V = 0$ implies that $\|e_r\|$ and $\|e_\omega\|$ are zero. The main challenge lies in bounding the terms containing Ψ_i. Note that

$$\|e_{r_i}\| = \|\widehat{R^T R_d b_i} b_i\| = \|R^T R_d b_i \times b_i\| = \sin\theta_i, \tag{15.24}$$

where θ_i is the angle between $R^T R_d b_i$ and b_i. Note that θ_i is always in the range $[0, \pi]$. Similarly,

$$\Psi_i(R) = 1 - R^T R_d b_i \cdot b_i = 1 - \cos\theta_i. \tag{15.25}$$

Utilizing this and $(a+b)^2 \leq 2(a^2+b^2)$ we get:

$$\begin{aligned}
\|e_r\|^2 &\overset{(15.22)}{=} \|k_1 e_{r_1} + k_2 e_{r_2}\|^2 \leq (k_1\|e_{r_1}\| + k_2\|e_{r_2}\|)^2 \overset{(15.24)}{=} (k_1 \sin\theta_1 + k_2 \sin\theta_2)^2 \\
&= \left(k_1\sqrt{1-\cos^2\theta_1} + k_2\sqrt{1-\cos^2\theta_2}\right)^2 \leq \left(k_1\sqrt{2(1-\cos\theta_1)} + k_2\sqrt{2(1-\cos\theta_2)}\right)^2 \\
&\overset{(15.25)}{=} 2\left(k_1\sqrt{\Psi_1(R)} + k_2\sqrt{\Psi_2(R)}\right)^2 \leq 4\min(k_1,k_2)\left(k_1\Psi_1(R) + k_2\Psi_2(R)\right).
\end{aligned}$$

We therefore get

$$V \geq \frac{1}{2}\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix}^T \begin{bmatrix}\frac{1}{2\min(k_1,k_2)} & -c\lambda_M(J) \\ -c\lambda_M(J) & \lambda_m(J)\end{bmatrix}\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix} \tag{15.26}$$

where the matrix is positive definite for small enough c.

15.7. Lyapunov derivative

We start with an intermediate result

Proposition 15.2. *It holds that*

$$J\dot{e}_\omega = -e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T\omega_d - \operatorname{trace}(J)I\right)R^T\omega_d\right) \times e_\omega. \tag{15.27}$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Je_\omega &\overset{(15.20b)}{=} J\dot{\omega} - J\dot{R}^T\omega_d - JR^T\dot{\omega}_d \overset{(15.1)}{=} u - \hat{\omega}J\omega - J\left(R\hat{\omega}\right)^T\omega_d - JR^T\dot{\omega}_d \\
&\overset{(15.21)}{=} -e_r - k_\omega e_\omega + \widehat{R^T\ \omega_d}JR^T\ \omega_d - \hat{\omega}J\omega - J\hat{\omega}^T R^T\omega_d \\
&\overset{(15.20b),(??)}{=} -e_r - k_\omega e_\omega + \widehat{R^T\ \omega_d}JR^T\ \omega_d - (\hat{e}_\omega + \widehat{R^T\omega_d})J(e_\omega + R^T\omega_d) \\
&\quad + J\cancelto{0}{\left(\hat{e}_\omega + \widehat{R^T\omega_d}\right)}R^T\omega_d \\
&\overset{(??)}{=} -e_r - k_\omega e_\omega + \left(\widehat{Je_\omega} + \widehat{JR^T\omega_d} - \widehat{R^T\omega_d}J - J\widehat{R^T\omega_d}\right)e_\omega \\
&\overset{(??)}{=} -e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T\omega_d - \operatorname{trace}(J)I\right)R^T\omega_d\right)^\wedge e_\omega.
\end{aligned}$$

□

Thus the derivative of V is

$$\dot{V} \overset{(15.19)}{=} e_\omega \cdot J\dot{e}_\omega + e_r \cdot e_\omega + cJ\dot{e}_\omega \cdot e_r + cJe_\omega \cdot \dot{e}_r \overset{(15.27)}{=} -k_\omega \|e_\omega\|^2 + cJ\dot{e}_\omega \cdot e_r + cJe_\omega \cdot \dot{e}_r, \tag{15.28}$$

so we would like to bound $J\dot{e}_\omega \cdot e_r$ and $Je_\omega \cdot \dot{e}_r$ in terms of $\|e_\omega\|$ and $\|e_r\|$. First we have

$$\frac{\mathrm{d}}{\mathrm{d}t} R_d^T R = R_d^T R\hat{\omega} + R_d^T \hat{\omega}_d^T R \overset{(??)}{=} R_d^T R\hat{\omega} - R_d^T \hat{\omega}_d R \overset{(??)}{=} R_d^T R\hat{\omega} - R_d^T R\,\widehat{R^T \omega_d} = R_d^T R\hat{e}_\omega. \tag{15.29}$$

Now, $e_{r_i} = \widehat{R^T R_d b_i} b_i$, so by linearity of the hat mapping and that $\|\hat{b}_i\| = \|b_i\| = 1$ it follows that

$$\dot{e}_{r_i} = \widehat{R_d^T R\hat{e}_\omega b_i}\; b_i = -\widehat{R_d^T R\hat{b}_i e_\omega}\; b_i, \qquad \Longrightarrow \|\dot{e}_{r_i}\| \le \|R_d^T R\|\|\hat{b}_i\|\|e_\omega\|\|b_i\| = \|e_\omega\|. \tag{15.30}$$

Thus, for $\lambda_M(J)$ the maximal eigenvalue of J,

$$\|Je_\omega \cdot \dot{e}_r\| \le \lambda_M(J)(k_1 + k_2)\|e_\omega\|^2. \tag{15.31}$$

Finally, we bound the last term, utilizing that $\|e_r\| \le k_1 + k_2$:

$$\begin{aligned} J\dot{e}_\omega \cdot e_r &\overset{(15.27)}{=} \left(-e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T \boldsymbol{\omega}_d - \mathrm{trace}(J)I\right) R^T \boldsymbol{\omega}_d\right) \times e_\omega\right) \cdot e_r \\ &\le -\|e_r\|^2 + k_\omega \|e_r\|\|e_\omega\| + \lambda_M(J)(k_1 + k_2)\|e_\omega\|^2 + B\|e_\omega\|\|e_r\|, \end{aligned} \tag{15.32}$$

where B is some number that upper bounds $\|\left(2JR^T \boldsymbol{\omega}_d - \mathrm{trace}(J)I\right) R^T \boldsymbol{\omega}_d\|$.

We can now bound the derivative as follows:

$$\dot{V} \le -\begin{bmatrix} \|e_r\| \\ \|e_\omega\| \end{bmatrix}^T \begin{bmatrix} c & -c(k_\omega + B)/2 \\ -c(k_\omega + B)/2 & k_\omega - 2c\lambda_M(J)(k_1 + k_2) \end{bmatrix} \begin{bmatrix} \|e_r\| \\ \|e_\omega\| \end{bmatrix}, \tag{15.33}$$

and it follows that if we choose c small enough then the matrix is positive definite and thus V decreases along trajectories of the closed-loop system.

Remaining steps:

- Show that undesired equilibria are unstable

16. Application: Model-Predictive Control

Consider a system $\boldsymbol{X}(t)$ evolving on a Matrix Lie group

$$\mathrm{d}^r \boldsymbol{X}_t = f(\boldsymbol{X}, u), \quad \boldsymbol{X} \in \mathbb{M}, \qquad f : \mathbb{M} \times U \to T\mathbb{M}. \tag{16.1}$$

We are interested in finding an approximate solution to the optimal control problem

$$\begin{cases} \min & \int_0^T \left\| \sqrt{Q(\tau)}(\boldsymbol{X}(\tau) \ominus_r \boldsymbol{X}_d(\tau)) \right\|_2^2 + \left\| \sqrt{R(\tau)}(u(\tau) - u_d(\tau)) \right\| \mathrm{d}\tau + \left\| \sqrt{Q(T)}(\boldsymbol{X}(T) \ominus_r x_d(T)) \right\|_2^2 \\ \text{s.t.} & (16.1) \\ & \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases}, \tag{16.2}$$

for positive semi-definite matrices Q and R.

We start by considering the dynamics around a nominal trajectory $(\boldsymbol{X}_l(t), u_l(t))$. Consider the error $\boldsymbol{a}_e = \boldsymbol{X}(t) \ominus_r \boldsymbol{X}_l(t)$. Since the error takes values in $T_{\boldsymbol{X}_l(t)}\mathbb{M} \cong \mathbb{R}^n$ the rule of total derivatives in Remark 5.1 applies and the error dynamics become

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{a}_e}{\mathrm{d}t} &= \mathrm{d}^r(\boldsymbol{a}_e)_t = \mathrm{d}^r(\boldsymbol{X} \ominus_r \boldsymbol{X}_l)_{\boldsymbol{X}} \mathrm{d}^r \boldsymbol{X}_t + \mathrm{d}^r(\boldsymbol{X} \ominus_r \boldsymbol{X}_l)_{\boldsymbol{X}_l} \, \mathrm{d}^r(\boldsymbol{X}_l)_t \\ &\overset{(5.72),(5.73)}{=} \left[\mathrm{d}^r \exp_{\boldsymbol{a}_e}\right]^{-1} f\left(\boldsymbol{X}_l \oplus_r \boldsymbol{a}_e, u_l + u_e\right) - \left[\mathrm{d}^l \exp_{\boldsymbol{a}_e}\right]^{-1} \, \mathrm{d}^r(\boldsymbol{X}_l)_t, \end{aligned} \tag{16.3}$$

Thus we can change coordinates and rewrite (16.2) as

$$\begin{cases} \min & \int_0^T \left\| \sqrt{Q(\tau)} \left((\boldsymbol{X}_l(\tau) \oplus_r \boldsymbol{a}_e(\tau)) \ominus_r \boldsymbol{X}_d(\tau) \right) \right\|_2^2 + \left\| \sqrt{R(\tau)}(u_l(\tau) + u_e(\tau) - u_d(\tau)) \right\| \mathrm{d}\tau, \\ \text{s.t.} & (16.3), \\ & \boldsymbol{a}_e(0) = \boldsymbol{X}_0 \ominus \boldsymbol{X}_l(0). \end{cases}, \tag{16.4}$$

This is now a regular optimal control problem and we can proceed by linearizing around $(\boldsymbol{a}_e, u_e) = (0, 0)$ to obtain the linear time-varying system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}_e = A(t)\boldsymbol{a}_e + B(t)u_e + E(t), \tag{16.5}$$

where, since $\mathrm{d}^r \exp_0 = \mathrm{d}^l \exp_0 = I$,

$$A(t) := \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}_e}\right|_{\boldsymbol{a}_e = 0} \left[\mathrm{d}^r \exp_{\boldsymbol{a}_e}\right]^{-1} f\left(\boldsymbol{X}_l(t) \oplus_r \boldsymbol{a}_e, u_l(t)\right), \tag{16.6}$$

$$B(t) := \left.\frac{\mathrm{d}}{\mathrm{d}u_e}\right|_{u_e = 0} f\left(\boldsymbol{X}_l(t), u_l(t) + u_e\right), \tag{16.7}$$

$$E(t) := f\left(\boldsymbol{X}_l(t), u_l(t)\right) - \mathrm{d}^r(\boldsymbol{X}_l)_t. \tag{16.8}$$

To facilitate evaluating the cost function we note that

$$(\boldsymbol{X}_l \oplus_r \boldsymbol{a}_e) \ominus_r \boldsymbol{X}_d = \log\left(\boldsymbol{X}_d^{-1} \circ \boldsymbol{X}_l \circ \exp(\boldsymbol{a}_e)\right) = \log(\exp(\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d) \circ \exp(\boldsymbol{a}_e)) \approx \boldsymbol{X}_l \ominus_r \boldsymbol{X}_d + \boldsymbol{a}_e(t), \tag{16.9}$$

where the last approximate step follows from the Baker-Campbell-Hausdorff formula (4.44).

We can thus write it on the form

$$\left\|\sqrt{Q}\left((\boldsymbol{X}_l \oplus_r \boldsymbol{a}_e) \ominus_r \boldsymbol{X}_d\right)\right\|_2^2 \approx \left(\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d + \boldsymbol{a}_e\right)^T Q \left(\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d + \boldsymbol{a}_e\right) = \boldsymbol{a}_e^T Q \boldsymbol{a}_e + 2\left(\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d\right)^T Q \boldsymbol{a}_e. \tag{16.10}$$

17. Application: State Estimation

17.1. IMU Model

An imu typically consists of a gyro measuring angular velocity, an accelerometer, and a magnetometer that estimates the orientation with respect to the earth magnetic field. Consider a body moving in the world described by the IMU frame to world frame transform $P_{WI} = (\boldsymbol{q}_{WI}, \boldsymbol{p}_{WI}) \in \$\mathbb{E}(3)$.

The gyro and accelerometer measurements of the IMU are then well modeled by the following:

$$\tilde{\boldsymbol{\omega}} = \mathrm{d}^r(\boldsymbol{q}_{WI})_t + \boldsymbol{b}_{\boldsymbol{\omega}} + \eta_{\boldsymbol{\omega}}, \tag{17.1a}$$

$$\tilde{\boldsymbol{a}} = \left[(\mathrm{d}^r)^2 P_{WI}\right]_{3:6} + \boldsymbol{q}_{WI}^{-1}\boldsymbol{g}_W + \boldsymbol{b}_{\boldsymbol{a}} + \eta_{\boldsymbol{a}}. \tag{17.1b}$$

In (17.1a), the first term is the actual body angular velocity, $\boldsymbol{b}_{\boldsymbol{\omega}}$ is a gyro bias, and $\eta_{\boldsymbol{\omega}}$ is white noise.

For the accelerometer model (17.1b), $\left[\mathrm{d}^2(P_{WI})_t\right]_{3:6}$ denotes the linear acceleration in the body frame (the last three components of the second derivative), $\boldsymbol{g}_W$ is the gravity in the world frame, and $\boldsymbol{b}_{\boldsymbol{a}}$ and $\eta_{\boldsymbol{a}}$ are bias and noise as above.

17.2. Complementary Filter for Attitude Estimation

Filter in [12] is of form

$$\dot{\hat{R}} = (\hat{R}\Omega + k_P \hat{R}\omega)\hat{R}, \quad \omega = vex\left(\frac{1}{2}\left(\tilde{R} - \tilde{R}^T\right)\right), \quad \tilde{R} = \hat{R}R_y.$$

That is, the natural dynamics are amended with a term $k_P\hat{R}\omega$ that induces stability of the observer. The quantitiy ω is a rotation quantity in body coordinates that corresponds to the anti-symmetric part of the empirical rotation error. It can be shown that for $R_y = R$ we have

$$\frac{d}{dt}\frac{1}{2}\mathrm{Tr}\left(I - \hat{R}^T R\right) = -\frac{k_P}{2}|\omega|^2.$$

Remark: Filter above is the *passive* form, if $R_y\Omega$ is used instead of $\hat{R}\Omega$ it is called *direct*.

This can be amended with an integrator that estimates a bias term in the gyro estimates.

17.3. TODO

1. Convert the attitude estimation filter from $SO(3)$ to S^3.

18. Nonlinear Least Squares

Like how Lie groups thread the line between linear and nonlinear manifolds, the same can be said for the role of nonlinear least squares in optimization, which is a type of optimization problem is rich enough for to model a wide variety of situations, yet structured enough to be amenable to practical algorithms.

A non-linear least squares problem has the general form

$$\min_{\boldsymbol{x}\in\mathbb{M}} \frac{1}{2}\sum_{i=1}^{N} \left\| r_i(\boldsymbol{x}) \right\|^2, \quad r_i : \mathscr{M} \to \mathbb{R}^{n_{r_i}}. \tag{18.1}$$

The manifold $\mathbb{M}$ can be a Lie group or a Lie group product $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) \in \mathbb{M}_1 \times \ldots \times \mathbb{M}_k$. For the latter case, typically not every residual depends on each member of the bundle, i.e. $r_i(\boldsymbol{x}) = r_i\left(\{\boldsymbol{x}_j\}_{j\in I_i}\right)$ where $I_i \subset \{1, \ldots, k\}$ is a subset of variables.

Remark 18.1. *An equivalent problem with a single residual is* $\min_{\boldsymbol{x}\in\mathbb{M}} \frac{1}{2}\|r(\boldsymbol{x})\|^2$ *for*

$$r(\boldsymbol{x}) = \begin{bmatrix} r_1(\boldsymbol{x}) \\ \vdots \\ r_k(\boldsymbol{x}) \end{bmatrix}. \tag{18.2}$$

Although the single residual formulation simplifies notation somewhat, in practice it is for large problems important to leverage the sparsity structure which is better exposed in (18.1).

18.1. Solution Sensitivity

In many applications the residuals $r_i(\boldsymbol{x})$ are obtained from data and are therefore associated with uncertainty. In this situation it is natural to ask how sensitive the optimal solution of the nonlinear least squares problem is to noise in the data. Assume that the noise associated with each residual is Gaussian and independent of other residuals, i.e. that

$$r_i(\boldsymbol{x}) \sim \mathcal{N}\left(\bar{r}_i(\boldsymbol{x}), I\right), \tag{18.3}$$

and consider a point $\bar{\boldsymbol{x}}$. We expand the objective using a Taylor approximation as

$$\min_{\bar{\boldsymbol{x}}\in\mathbb{M}} \frac{1}{2}\sum_{i=1}^{N} \left\| r_i(\bar{\boldsymbol{x}}) \right\|^2 \approx \min_{\boldsymbol{a}\in T_{\bar{\boldsymbol{x}}}\mathbb{M}} \frac{1}{2}\sum_{i=1}^{N} \left\| r_i(\bar{\boldsymbol{x}}) + \mathrm{d}^r(r_i)_{\bar{\boldsymbol{x}}}\boldsymbol{a} \right\|^2. \tag{18.4}$$

The optimal solution $\boldsymbol{x}^*$ of the left problem can be approximately retrieved from $\boldsymbol{a}^*$ as $\boldsymbol{x}^* = \bar{\boldsymbol{x}} \oplus_r \boldsymbol{a}^*$ assuming that $\boldsymbol{a}^*$ is small.

Letting $r_i := r_i(\bar{x})$ and $J_i := \mathrm{d}^r(r_i)_{\bar{x}}$ expanding the square and ignoring the constant term yields

$$\min_{\boldsymbol{a}\in T_{\bar{x}}\mathbb{M}} \sum_{i=1}^{N} \frac{1}{2}\boldsymbol{a}^T J_i^T J_i \boldsymbol{a} + \boldsymbol{a}^T J_i^T r_i = \min_{\boldsymbol{a}\in T_{\bar{x}}\mathbb{M}} \frac{1}{2}\boldsymbol{a}^T \left(\sum_{i=1}^{N} J_i^T J_i\right)\boldsymbol{a} + \boldsymbol{a}^T \sum_{i=1}^{N} J_i^T r_i. \tag{18.5}$$

The optimal solution of this problem can be obtained by setting the gradient w.r.t. $\boldsymbol{a}$ to zero and is

$$\boldsymbol{a}^* = -\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger} \left(\sum_{i=1}^{N} J_i^T r_i\right). \tag{18.6}$$

From this we can infer the sensitivity of $\boldsymbol{a}^*$ to noise in r_i: recalling that $\mathrm{Var}(Ax + By) = A\mathrm{Var}(x)A^T + B\mathrm{Var}(y)B^T$ we get that

$$\boldsymbol{a}^* \sim \mathcal{N}\left(-\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger} \left(\sum_{i=1}^{N} J_i^T \bar{r}_i\right), \left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger} \left(\sum_{i=1}^{N} J_i^T \Sigma_i J_i\right)\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\right). \tag{18.7}$$

For the special case when all r_i's have unit covariance, i.e. $\Sigma_i = I$, the expression simplifies to

$$\boldsymbol{a}^* \sim \mathcal{N}\left(-\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger} \left(\sum_{i=1}^{N} J_i^T \bar{r}_i\right), \left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\right). \tag{18.8}$$

Given a residual $r(x) \sim \mathcal{N}(\bar{r}(x), \Sigma)$ a residual with unit covariance can be obtained by left-multiplying with the square root information matrix $\sqrt{I} := \Sigma^{-1/2}$:

$$\sqrt{I}r(x) \sim \mathcal{N}\left(\sqrt{I}\bar{r}(x), I\right). \tag{18.9}$$

Scaling with $\sqrt{I}$ makes sense in many applications since it in effect scales the residual by the inverse noise magnitude.

For the unit covariance case $\Sigma_i = I$ the tangent space covariance of the optimal solution x^* is

$$\left(\sum_{i=1}^{N} \left(\mathrm{d}^r(r_i)_{x^*}\right)^T \mathrm{d}^r(r_i)_{x^*}\right)^{\dagger}. \tag{18.10}$$

18.2. Levenberg-Marquardt

Resources

- Original MINPACK manual: https://www.netlib.org/minpack/

LM Implementations

- Original MINPACK in fortran
- cminpack (ported from fortran) https://devernay.github.io/cminpack/
- Eigen unsupported (ported from cminpack)
- scipy (calls cminpack)
- Ceres
- GTSAM

Consider a Lie group optimization problem

$$\min_{\boldsymbol{x}} \frac{1}{2} \|f(\boldsymbol{x})\|^2, \quad f : \mathbb{M} \to \mathbb{R}^m. \tag{18.11}$$

We are interested in devising an iterative algorithm for minimizing this function.

Given a poimt $\boldsymbol{x}$ we can solve a local optimization problem to find step $\boldsymbol{a} \in T\mathbb{M}_{\boldsymbol{x}}$ that leads to an improved estimate $\boldsymbol{x} \oplus_r \boldsymbol{a}$. The optimization problem can be re-formulated in terms of $\boldsymbol{a}$ as

$$\arg\min_{\boldsymbol{a}} \|f(\boldsymbol{x} \oplus_r \boldsymbol{a})\|^2. \tag{18.12}$$

Since the problem is nonlinear we resort to linearization. To avoid stepping outside the region where the linearization is accurate we also limit the stepsize and obtain the new problem

$$\arg\min_{\boldsymbol{a} : \|D\boldsymbol{a}\| \leq \Delta} \|f(\boldsymbol{x}) + \mathrm{d}^r f_{\boldsymbol{x}} \boldsymbol{a}\|^2, \tag{18.13}$$

where D is a diagonal scaling matrix and Δ is a maximal step size.

To simplify notation let $J := \mathrm{d}^r f_{\boldsymbol{x}}$ and $r = f(\boldsymbol{x})$ which simplifies the objective function to $\|J\boldsymbol{a} + r\|^2$. We next show that this constrained problem can be transformed into an unconstrained problem.

Theorem 18.1. *A vector $\boldsymbol{a}^*$ is a global minimizer of*

$$\arg\min_{\|D\boldsymbol{a}\| \leq \Delta} \frac{1}{2} \|J\boldsymbol{a} + r\|^2. \tag{18.14}$$

if and only if there exists $\lambda \geq 0$ such that

$$J^T J + \lambda D^T D \succeq 0, \tag{18.15a}$$

$$(J^T J + \lambda D^T D)\boldsymbol{a} = -J^T r, \tag{18.15b}$$

$$\lambda (\|D\boldsymbol{a}\| - \Delta) = 0. \tag{18.15c}$$

We provide an argument based on duality to support this fact, see e.g. [14, Theorem 4.1] for a more rigorous proof.

Proof. Let the lagrangian of the problem be

$$L(\boldsymbol{a}, \lambda) = \frac{1}{2}\|J\boldsymbol{a} + r\|^2 + \frac{\lambda}{2}\left(\|D\boldsymbol{a}\|^2 - \Delta^2\right), \tag{18.16}$$

so that the optimization problem (18.13) equivalently can be written $\inf_{\boldsymbol{a}} \sup_{\lambda\geq 0} L(\boldsymbol{a}, \lambda)$, since the value of the inner problem is $+\infty$ when the constraint $\|D\boldsymbol{a}\| \leq \Delta$ is not satisfied.

Assuming that strong duality holds, the dual problem $\sup_{\lambda\geq 0} \inf_{\boldsymbol{a}} L(\boldsymbol{a}, \lambda)$ has the same optimal value. The inner infimum of the dual problem can be re-written as

$$\inf_{\boldsymbol{a}} L(\boldsymbol{a}, \lambda) = \frac{1}{2}\boldsymbol{a}^T(J^TJ + \lambda D^TD)\boldsymbol{a} + r^TJ\boldsymbol{a} - \lambda\frac{\Delta^2}{2}. \tag{18.17}$$

This inner problem has value $-\infty$ unless $J^TJ + \lambda D^TD \succeq 0$, so the outer supremum restricts λ to values that imply positive semi-definiteness. In this case the finite optimal value is attained for $\boldsymbol{a}$ such that $\left(J^TJ + \lambda D^TD\right)\boldsymbol{a} = -J^Tr$ which reduces the dual problem to

$$\sup_{\lambda\geq 0} -\frac{1}{2}\boldsymbol{a}^T\left(J^TJ + \lambda D^TD\right)\boldsymbol{a} - \lambda\frac{\Delta^2}{2}. \tag{18.18}$$

Also this problem has a closed-form solution: either the optimal solution is attained at the boundary, i.e. $\lambda = 0$, or it is attained at zero derivative w.r.t. λ which necessitates $\boldsymbol{a}^TD^TD\boldsymbol{a} + \Delta^2 = 0$. These two latter conditions imply that the complementarity condition $\lambda(\|D\boldsymbol{a}\| - \Delta) = 0$ holds. □

Equation (18.15b) represents the normal equations for the least-squares problem

$$\arg\min_{\boldsymbol{a}} \frac{1}{2}\boldsymbol{a}^T\left(J^TJ + \lambda D^TD\right)\boldsymbol{a} + r^TJ\boldsymbol{a}. \tag{18.19}$$

Equivalently, it can be written on the standard form

$$\arg\min_{\boldsymbol{a}} \left\|\begin{bmatrix} J \\ \sqrt{\lambda}D \end{bmatrix}\boldsymbol{a} + \begin{bmatrix} r \\ 0 \end{bmatrix}\right\|^2. \tag{18.20}$$

For numerical stability it is preferable to solve a least-squares problem instead of directly solving the normal equations.

The diagonal scaling matrix $D = \mathrm{Diag}(d_1, \dots, d_n)$ is typically chosen so that a component d_i is inversely proportional to the magnutide of the gradient in that direction, which has the effect of allowing larger steps in directions with low gradient. Common choices include $D = \sqrt{\mathrm{Diag}\left(\mathrm{diag}(J^TJ)\right)}$ and $d_i = \|[\mathrm{d}^r f_{\boldsymbol{x}}]_{:,i}\|$—the norm of the i:th column of the jacobian.

A complete Levenberg-Marquardt procedure is shown in Algorithm 1. The parameter λ that approximately satisfies the conditions in Theorem 18.1 can be obtained algorithmically [14, 13], with implementations available in e.g. MINPACK's `lmpar`[1].

The crucial parts of the algorithm are line 3 and 4; the best way to implement these operations depends on the size and sparsity structure of J.

[1] `https://www.netlib.org/minpack`

Algorithm 1: Single LM step

Data: Iteration variables: point $\boldsymbol{x}^k$, trust region Δ^k, scaling parameters d_i^k as diagonal matrix D^k

Result: Updated iteration variables $\boldsymbol{x}^{k+1}$, Δ^{k+1}, d_i^{k+1}

$r = f(\boldsymbol{x})$

$J = \mathrm{d}^r f_{\boldsymbol{x}}$

$\lambda = \texttt{lmpar}(J, r, D^k, \Delta^k)$ `// calculate LM parameter`

$\boldsymbol{a}_{\mathrm{LM}} = \arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} J \\ \sqrt{\lambda} D^k \end{bmatrix} \boldsymbol{a} + \begin{bmatrix} r \\ 0 \end{bmatrix} \right\|^2$ `// solve for increment step`

$\rho = \frac{\|r\|^2 - \|f(\boldsymbol{x} \oplus_r \boldsymbol{a}_{\mathrm{LM}})\|^2}{\|r\|^2 - \|r + J\boldsymbol{a}_{\mathrm{LM}}\|^2}$ `// actual to predicted reduction ratio`

if $\rho \leq 0.25$ **then**

  $\Delta^{k+1} = \Delta^k / 2$ `// decrease trust region`

else if $\rho \geq 0.75$ **then**

  $\Delta^{k+1} = 2\Delta^k$ `// increase trust region`

end

$d_i^{k+1} = \max\left(d_i^k, \left\| \left[\mathrm{d}^r f_{\boldsymbol{x}^{k+1}}\right]_{\cdot,i} \right\|\right)$ `// update scaling parameters`

if $\rho \leq 0.0001$ **then** $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k$ `// reject step`

else $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k \oplus_r \boldsymbol{a}_{\mathrm{LM}}$ `// accept step`

Calculation of the actual to predicted reduction ratio can be rewritten as

$$\rho = \frac{1 - \left(\frac{\|f(\boldsymbol{x} \oplus \boldsymbol{a}_{\mathrm{LM}})\|}{\|r\|}\right)^2}{\left(\frac{\|J\boldsymbol{a}\|}{\|r\|}\right)^2 + 2\left(\frac{\sqrt{\lambda}\|D\boldsymbol{a}\|}{\|r\|}\right)^2} \tag{18.21}$$

where we have used that $\|r\|^2 - \|r + J\boldsymbol{a}\|^2 = -2\boldsymbol{a}^T J r - \boldsymbol{a}^T J^T J \boldsymbol{a} = \|J\boldsymbol{a}\|^2 + 2\lambda\|D\boldsymbol{a}\|^2$ which is a consequence of (18.15b). This formulation has the benefit of avoiding subtraction of numbers of large magnitude which may cause floating point roundoff errors.

18.2.1. Solving the Least-Squares Problem

The least-squares problem

$$\arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} J \\ \sqrt{\lambda} D^k \end{bmatrix} \boldsymbol{a} + \begin{bmatrix} r \\ 0 \end{bmatrix} \right\|^2 \tag{18.22}$$

has structure which can be exploited to find a solution. Consider a QR decomposition with column pivoting of J s.t. $JP = QR$, where $P \in \mathbb{R}^{n \times n}$ is a permutation matrix, $Q \in \mathbb{R}^{n \times n}$ is orthogonal, and $R \in \mathbb{R}^{n \times n}$ is upper-diagonal. If $\boldsymbol{a}$ is a minimizer of (18.23) it is also a minimizer of

$$\arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} Q^T J P \\ \sqrt{\lambda} P^T D^k P \end{bmatrix} P^T \boldsymbol{a} + \begin{bmatrix} Q^T r \\ 0 \end{bmatrix} \right\|^2 = \arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} R \\ \sqrt{\lambda} P^T D^k P \end{bmatrix} P^T \boldsymbol{a} + \begin{bmatrix} Q^T r \\ 0 \end{bmatrix} \right\|^2. \tag{18.23}$$

Consider a second QR decomposition s.t.

$$\begin{bmatrix} R \\ \sqrt{\lambda} P^T D^k P \end{bmatrix} = \tilde{Q} \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} \tag{18.24}$$

where $\tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$ is orthogonal and $\tilde{R} \in \mathbb{R}^{n \times n}$ is upper-diagonal and has rank n. In these variables the least-squares problem takes the form

$$\arg\min_{\boldsymbol{a}} \left\| \tilde{Q} \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} P^T \boldsymbol{a} + \begin{bmatrix} Q^T r \\ 0 \end{bmatrix} \right\|^2 = \arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} P^T \boldsymbol{a} + \tilde{Q}^T \begin{bmatrix} Q^T r \\ 0 \end{bmatrix} \right\|^2 = \arg\min_{\boldsymbol{a}} \left\| \tilde{R} P^T \boldsymbol{a} + \tilde{Q}_{11}^T Q^T r \right\|^2,$$

and it is now apparent that the optimal solution is

$$\boldsymbol{a}_{\mathrm{LM}} = -P \tilde{R}^{-1} \tilde{Q}_{11}^T Q^T r. \tag{18.25}$$

When solving (18.23) repeatedly for different values of λ only the second QR decomposition needs to be re-computed.

18.2.2. Finding the LM Parameter

Follow [13]

Remark 18.2. *Ceres just uses $\lambda = 1/\Delta$.*
Dense strategy: mimic MINPACK with column QR decomposition
Sparse strategy: as ceres use $\lambda = 1/\Delta$ and use a sparse eigen solver for the step.

19. Pose Graph Optimization

19.1. Maximum Likelihood Estimation as Nonlinear Least Squares

Notation :

1. $X = \{x_i\}$ set of variables
2. $\hat{y}_j \in \mathbb{R}^{d_j}$
3. h_j measurement function s.t. $y_j \sim \mathcal{N}(h_j(X_j), \Sigma_j)$ for a (often small) subset of variables $X_j \subset X$.

Given a collection of measurements $\hat{y}_j$ we are interested in finding the **maximum-likelihood estimate** of the variables X. In the Gaussian setting this becomes

$$\begin{aligned} X^* = \arg\max_x \prod_j p_j(\hat{y}_j \mid X_j) &= \arg\max_x \prod_j \frac{1}{\sqrt{(2\pi)^{d_j}|\Sigma_j|}} \exp\left(-\frac{(\hat{y}_j - h_j(X_j))^T \Sigma_j^{-1} (\hat{y}_j - h_j(X_j))}{2}\right) \\ &= \arg\max_x \prod_j \exp\left(-\frac{(\hat{y}_j - h_j(X_j))^T \Sigma_j^{-1} (\hat{y}_j - h_j(X_j))}{2}\right) \end{aligned}$$

Maximizing $f(x)$ is equivalent to minimizing $2\log(-f(x))$. Taking the negative log and multiplying by two yields

$$X^* = \arg\min_x \sum_j (\hat{y}_j - h_j(X_j))^T \Sigma_j^{-1} (\hat{y}_j - h_j(X_j)) = \arg\min_x \sum_j \left\| \sqrt{I}_j \left(\hat{y}_j - h_j(X_j)\right) \right\|^2,$$

where $\sqrt{I}_j := \Sigma_j^{-1/2}$ is the **square root information matrix**. We have thus converted the maximum-likelihood estimation problem into a **least squares problem**. When the functions h_j involve 3D geometry the least squares problem is typically **nonlinear**.

The least-squares problem can be viewed as a **bipartite factor graph** where variables and measurements (factors) are nodes. By exploiting the graph structure updates can be made locally in the graph, but this requires sophisticated data structures and solvers [**dellaert_factor_2017**].

19.2. Measurement functions

19.2.1. Absolute pose measurement

Let the measurement be $\hat{y}_j = \log(\hat{P})$ where $\log : SE(3) \to \mathfrak{se}(3)$ is the logarithm on $SE(3)$ and $\hat{P}$ a pose measurement, then

$$h(P) = \log(P) \in \mathbb{R}^6.$$

Since $SE(3) = SO(3) \times \mathbb{R}^3$ we can use the same formula for individual measurements of orientation ($SO(3)$) or position ($\mathbb{R}^3$).

19.2.2. Relative pose measurement

Let the measurement be $\hat{y}_j = \log(\hat{P}_{12})$ where $\log : SE(3) \to \mathfrak{se}(3)$ is the logarithm on $SE(3)$ and $\hat{P}_{12}$ an estimate of the relative pose. Then

$$h(P_1, P_2) = \log(P_1^{-1} P_2) \in \mathbb{R}^6.$$

19.2.3. Rectified stereo landmark measurement

* $P \in SE(3)$ is the pose of the left camera (variable) * $l \in \mathbb{R}^3$ world location of a landmark (variable) * $P_{rl} \in SE(3)$ the pose of the right camera w.r.t. the left camera (known) * CM_l, CM_r camera projection matrices

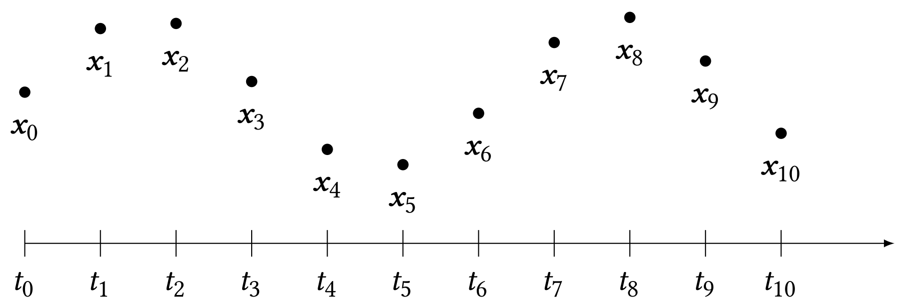
The landmark is projected to the left and right image pixel planes as

$$\lambda_l \tilde{\mathrm{x}}_l = CM_l P^{-1} l, \quad \lambda_r \tilde{\mathrm{x}}_r = CM_r P_{rl} P^{-1} l.$$

In a rectified system we have $y_l = y_r$, so we can let the 3-dimensional measurement be the pixel locations $\hat{x}_l, \hat{x}_r, \hat{y}_l$. The measurement function $h(P, l)$ is described by the equations above.

20. Splines on Lie Groups

This is largely based on [15, 17].



A B-spline interpolation of order k is a function $\boldsymbol{x}(t) = \sum_{i=0}^{N} B_{i,k}(t)\boldsymbol{x}_{\nu(i)}$ where $\boldsymbol{x}_{\nu(i)} \in \mathbb{R}^n$ are **control points** for **knots** t_i, and $B_{i,k}(t)$ are **basis functions** recursively defined as

$$\begin{aligned} B_{i,0}(t) &= \begin{cases} 1 & t_i \leq t < t_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \\ B_{i,k}(t) &= \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t). \end{aligned} \tag{20.1}$$

The following are some well-known properties of B-splines:

- $B_{i,k}(t)$ has finite support and is zero outside the interval $[t_i, t_{i+k+1})$,
- Inside this interval it is a piecewise polynomial of degree k,
- It is centered in the middle of that interval, it therefore makes sense to select k odd and $\nu(i) = i + (k+1)/2$ so that $x_{\nu(i)}$ coincides with the maximum of $B_{i,k}(t)$ (c.f. Figure 20.1),
- $\sum_i B_{i,k}(t) = 1$ for all t,

We are interested in an expression for the coefficients of the polynomial $B_{i,k}(t)$. We pose that for **a fixed interval** $t \in [t_{i^*}, t_{i^*+1})$ we have scalar cofficients $\alpha_{i,k}^{i}$ such that

$$B_{i,k}(t) = \sum_{l=0}^{k} \alpha_{i,k}^{l}\, u^l(t), \quad u(t) = \frac{t - t_{i^*}}{t_{i^*+1} - t_{i^*}} \tag{20.2}$$

where $i \in \{i^* - k, i^* - k + 1, \dots, i^*\}$ are the indices for which $B_{i,k}(t)$ is non-zero on $[t_{i^*}, t_{i^*+1})$ (c.f. Figure 20.1). If we also introduce

$$\begin{aligned} N_{i,k} &:= \begin{bmatrix} \alpha_{i,k}^0 & \alpha_{i,k}^1 & \cdots & \alpha_{i,k}^k \end{bmatrix}^T \in \mathbb{R}^{k+1} \\ M_{i^*,k} &:= \begin{bmatrix} N_{i^*-k,k} & N_{i^*-k+1,k} & \cdots & N_{i^*,k} \end{bmatrix} \in \mathbb{R}^{k+1,k+1} \end{aligned} \tag{20.3}$$

we can write the value of a spline $x(t)$ for $t \in [t_{i^*}, t_{i^*+1}]$ as

$$\boldsymbol{x}(t) = \sum_{j=0}^{n} B_{j,k}(t)\boldsymbol{x}_{\nu(j)} = \sum_{j=i^*-k}^{i^*} B_{j,k}(t)\boldsymbol{x}_{\nu(j)} = \sum_{j=i^*-k}^{i^*} \sum_{l=0}^{k} \alpha_{j,k}^l u^l \boldsymbol{x}_{\nu(j)} = \begin{bmatrix} 1 & u & \cdots & u^k \end{bmatrix} M_{i^*,k} \begin{bmatrix} \boldsymbol{x}_{\nu(i^*-k)} \\ \vdots \\ \boldsymbol{x}_{\nu(i^*)} \end{bmatrix}. \tag{20.4}$$

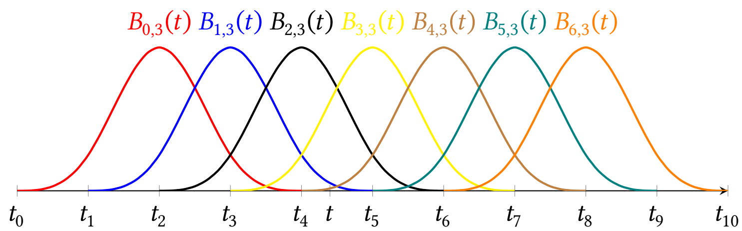


Figure 20.1.: For $k = 3$ and $t \in [t_4, t_5)$ the non-zero basis functions are B_1, B_2, B_3 and B_4. For interpolation purposes those basis functions should be matched with the control points $\boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5, \boldsymbol{x}_6$.

20.1. Splines on Lie Groups

We can re-arrange a B-spline into cumulative form

$$\sum_{i=0}^{N} B_{i,k}(t)\boldsymbol{x}_i = \tilde{B}_{0,k}(t)\boldsymbol{x}_0 + \sum_{i=1}^{N} \tilde{B}_{i,k}(t)(\boldsymbol{x}_i - \boldsymbol{x}_{i-1}) \tag{20.5}$$

where $\tilde{B}_{i,k}(t) = \sum_{j=i}^{N} B_{i,k}(t)$ and $\tilde{B}_{0,k}(t) \equiv 1$. The cumulative formulation lends itself well to a Lie group translation when $\boldsymbol{x} \in \boldsymbol{X}$ since we can replace plus and minus by the corresponding Lie group operations.

$$\boldsymbol{x}(t) = \exp\left[\tilde{B}_{0,k}(t)\log(\boldsymbol{x}_0)\right] \circ \prod_{i=1}^{N} \exp\left[\tilde{B}_{i,k}(t)(\boldsymbol{x}_i \ominus_r \boldsymbol{x}_{i-1})\right]. \tag{20.6}$$

20.1.1. Spline evaluation

Since for $t \in [t_{i^*}, t_{i^*+1})$ it holds that $\tilde{B}_{i,k}(t) = 1$ for $i \le i^* - k$ and $\tilde{B}_{i,k}(t) = 0$ for $i \ge i^* + 1$ we can simplify (20.6) into

$$\boldsymbol{x}(t) = \boldsymbol{x}_{i^*-k} \circ \prod_{j=i^*-k+1}^{i^*} \exp\left[\tilde{B}_{j,k}(t)\boldsymbol{v}_j\right], \qquad \boldsymbol{v}_j := \boldsymbol{x}_j \ominus_r \boldsymbol{x}_{j-1} = \log\left(\boldsymbol{x}_{j-1}^{-1} \circ \boldsymbol{x}_j\right). \tag{20.7}$$

Given the $N_{j,k}$:s we can evaluate $\tilde{B}_{j,k}(t)$ as

$$\begin{bmatrix}\tilde{B}_{i^*-k,k}(t) & \tilde{B}_{i^*-k+1,k}(t) & \dots & \tilde{B}_{i^*,k}(t)\end{bmatrix} = \begin{bmatrix}1 & u & \dots & u^k\end{bmatrix} \tilde{M}_{i^*,k}, \tag{20.8}$$

where $\tilde{M}_{i^*,k} \in \mathbb{R}^{k+1,k+1}$ is the column-wise reverse cumulative sum of $M_{i^*,k}$:

$$\tilde{M}_{i^*,k} = \begin{bmatrix}\sum_{j=i^*-k}^{i^*} N_{j,k} & \sum_{j=i^*-k+1}^{i^*} N_{j,k} & \dots & N_{i^*,k}\end{bmatrix} \tag{20.9}$$

20.1.2. First order derivative

To evaluate the derivative of a spline consider the formula

$$\boldsymbol{x}(t) = \boldsymbol{y}(t) \circ \boldsymbol{z}(t), \quad \boldsymbol{z}(t) := \exp\left(\lambda(t)\boldsymbol{v}\right). \tag{20.10}$$

Since $t \in \mathbb{R}$ we can, as discussed in Remark 5.1, evaluate the derivative of x w.r.t. t as

$$\mathrm{d}^r\boldsymbol{x}_t = \mathrm{d}^r(\boldsymbol{y} \circ \boldsymbol{z})_{\boldsymbol{y}}\, \mathrm{d}^r\boldsymbol{y}_t + \mathrm{d}^r(\boldsymbol{y} \circ \boldsymbol{z})_{\boldsymbol{z}}\, \mathrm{d}^r\boldsymbol{z}_t. \tag{20.11}$$

From the right-jacobian derivative rules (5.61), (5.63) we know that $\mathrm{d}^r(\boldsymbol{y}\circ\boldsymbol{z})_{\boldsymbol{y}} = \mathbf{Ad}_{\boldsymbol{z}^{-1}}$ and $\mathrm{d}^r(\boldsymbol{y}\circ\boldsymbol{z})_{\boldsymbol{z}} = I$. It therefore follows that

$$\mathrm{d}^r\boldsymbol{x}_t = \mathbf{Ad}_{\exp(-\lambda(t)\boldsymbol{v})}\, \mathrm{d}^r\boldsymbol{y}_t + \lambda'(t)\boldsymbol{v}. \tag{20.12}$$

This gives us a recursive procedure to calculate the derivative of a form (20.7) where we instead of matrix elements consider tangent elements $\boldsymbol{w}_i, \boldsymbol{v}_j \in \mathbb{R}^n$:

$$\begin{aligned}\boldsymbol{w}_{i^*-k} &= \mathbf{0},\\ \boldsymbol{w}_j &= \mathbf{Ad}_{\exp(-\tilde{B}_{j,k}(t)\boldsymbol{v}_j)}\, \boldsymbol{w}_{j-1} + \tilde{B}'_{j,k}(t)\boldsymbol{v}_j, \quad j = i^*+1, \dots, i^*+k,\\ \mathrm{d}^r\boldsymbol{x}_t &= \boldsymbol{w}_{i^*}.\end{aligned} \tag{20.13}$$

Due to using right jacobians this will result in a body velocity along the spline. If the world velocity is instead desired it can be obtained using

$$\mathrm{d}^l\boldsymbol{x}_t = \mathbf{Ad}_{x(t)}\, \mathrm{d}^r\boldsymbol{x}_t. \tag{20.14}$$

20.1.3. Second order derivative

The recursion in (20.13) can be differentiated a second time with respect to t to obtain the second order derivative. We use some properties of the adjoint to show

$$\begin{aligned} \mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{u}_1)}\boldsymbol{u}_2 &\overset{(5.22)}{=} \exp\left(\mathrm{ad}_{\lambda(t)\boldsymbol{u}_1}\right)\boldsymbol{u}_2 \overset{(5.20)}{=} \exp\left(\lambda(t)\,\mathrm{ad}_{\boldsymbol{u}_1}\right)\boldsymbol{u}_2 \overset{(5.21)}{=} \sum_{k=0}^{\infty}\frac{\lambda(t)^k\,\mathrm{ad}_{\boldsymbol{u}_1}^k}{k!}\boldsymbol{u}_2 \\ \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{u}_1)}\boldsymbol{u}_2 &= \lambda'(t)\sum_{k=1}^{\infty}\frac{\lambda(t)^{k-1}\,\mathrm{ad}_{\boldsymbol{u}_1}^k}{(k-1)!}\boldsymbol{u}_2 = \lambda'(t)\,\mathrm{ad}_{\boldsymbol{u}_1}\sum_{k=1}^{\infty}\frac{\lambda(t)^{k-1}\,\mathrm{ad}_{\boldsymbol{u}_1}^{k-1}}{(k-1)!}\boldsymbol{u}_2 \\ &= \lambda'(t)\left[\boldsymbol{u}_1, \sum_{k=0}^{\infty}\frac{\lambda(t)^k\,\mathrm{ad}_{\boldsymbol{u}_1}^k}{k!}\boldsymbol{u}_2\right] = \lambda'(t)\left[\boldsymbol{u}_1, \mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{u}_1)}\boldsymbol{u}_2\right]. \end{aligned} \tag{20.15}$$

With $\lambda(t) \to -\tilde{B}_{j,k}(t), \boldsymbol{u}_1 \to \boldsymbol{v}_j, \boldsymbol{u}_2 \to \boldsymbol{w}_{j-1}$ we get $\mathbf{Ad}_{\exp(-\tilde{B}_{j,k}(t)\boldsymbol{v}_j)}\boldsymbol{w}_{j-1} \overset{(20.13)}{=} \boldsymbol{w}_j - \tilde{B}'_{j,k}(t)\boldsymbol{v}_j$, and therefore by introducing $\boldsymbol{q}_j \coloneqq \frac{\mathrm{d}\boldsymbol{w}_j}{\mathrm{d}t}$:

$$\begin{aligned} \boldsymbol{q}_{i^*-k} &= \boldsymbol{0}, \\ \boldsymbol{q}_j &= \tilde{B}'_{j,k}(t)\left[\boldsymbol{w}_j, \boldsymbol{v}_j\right] + \mathbf{Ad}_{\exp(-\tilde{B}_{j,k}(t)\boldsymbol{v}_j)}\boldsymbol{q}_{j-1} + \tilde{B}^{(2)}_{j,k}(t)\boldsymbol{v}_j, \qquad j = i^*+1, \dots, i^*+k, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}^r\boldsymbol{x}_t &= \boldsymbol{q}_{i^*+k}. \end{aligned} \tag{20.16}$$

20.1.4. Derivatives w.r.t. Control Points

Finally it can be useful to express the derivative of $\boldsymbol{x}(t)$ with respect to the control point values $\boldsymbol{x}_j$. Recall that

$$\boldsymbol{x}(t) = \boldsymbol{x}_{i^*-k} \circ \prod_{j=i^*-k+1}^{i^*} \exp\left[\tilde{B}_{j,k}(t)\boldsymbol{v}_j\right], \tag{20.17}$$

so it is again just a matter of differentiating.

Let $\boldsymbol{s}_j = \tilde{B}_{j,k}(t)\boldsymbol{v}_j$, we then have that $\boldsymbol{x}(t) = x_{i^*-k} \circ \prod_{j=i^*-k+1}^{i^*} \exp(\boldsymbol{s}_j)$. Derivatives with respect to the terms are

$$\mathrm{d}^r(\boldsymbol{s}_j)_{\boldsymbol{x}_j} \overset{(5.72)}{=} \tilde{B}_{j,k}(t)\left[\mathrm{d}^r\exp_{\boldsymbol{v}_j}\right]^{-1} \quad\Longrightarrow\quad \mathrm{d}^r\left(\exp(\boldsymbol{s}_j)\right)_{\boldsymbol{x}_j} = \tilde{B}_{j,k}(t)\left[\mathrm{d}^r\exp_{\boldsymbol{s}_j}\right]\left[\mathrm{d}^r\exp_{\boldsymbol{v}_j}\right]^{-1}, \tag{20.18a}$$

$$\mathrm{d}^r(\boldsymbol{s}_j)_{x_{j-1}} \overset{(5.73)}{=} -\tilde{B}_{j,k}(t)\left[\mathrm{d}^l\exp_{\boldsymbol{v}_j}\right]^{-1} \quad\Longrightarrow\quad \mathrm{d}^r\left(\exp(\boldsymbol{s}_j)\right)_{\boldsymbol{x}_{j-1}} = -\tilde{B}_{j,k}(t)\left[\mathrm{d}^r\exp_{\boldsymbol{s}_j}\right]\left[\mathrm{d}^l\exp_{\boldsymbol{v}_j}\right]^{-1}. \tag{20.18b}$$

Thus the derivatives $\boldsymbol{r}_j \coloneqq \mathrm{d}^r\boldsymbol{x}(t)_{\boldsymbol{x}_j}$ of $\boldsymbol{x}$ become (where the $\bar{z}$'s are constant w.r.t. the differentiation variable)

$$\begin{aligned}
\boldsymbol{r}_{i^*} &= \mathrm{d}^r \left(\bar{z} \circ \exp\left[\boldsymbol{s}_{i^*}\right]\right)_{\boldsymbol{x}_{i^*}} \overset{(5.12)}{=} \mathrm{d}^r \exp\left(\boldsymbol{s}_{i^*}\right)_{\boldsymbol{x}_{i^*}} \overset{(20.18a)}{=} \tilde{B}_{i^*,k} \mathrm{d}^r \exp_{\boldsymbol{s}_{i^*}} \left[\mathrm{d}^r \exp_{\boldsymbol{v}_{i^*}}\right]^{-1}, \\
\boldsymbol{r}_j &= \mathrm{d}^r \left(\bar{z}_1 \circ \exp\left[\boldsymbol{s}_j\right] \circ \exp\left[\boldsymbol{s}_{j+1}\right] \circ \bar{z}_2\right)_{\boldsymbol{x}_j} \overset{(5.12)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}} \mathrm{d}^r \left(\bar{z}_1 \circ \exp\left[\boldsymbol{s}_j\right] \circ \exp\left[\boldsymbol{s}_{j+1}\right]\right)_{\boldsymbol{x}_j} \\
&\overset{(5.12)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}} \left(\mathbf{Ad}_{\exp(-\boldsymbol{s}_{j+1})} \mathrm{d}^r(\bar{z}_1 \circ \exp(\boldsymbol{s}_j))_{\boldsymbol{x}_j} + \mathrm{d}^r(\exp(\boldsymbol{s}_{j+1}))_{\boldsymbol{x}^j}\right) \\
&\overset{(5.12)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}} \left(\mathbf{Ad}_{\exp(-\boldsymbol{s}_{j+1})} \mathrm{d}^r(\exp(\boldsymbol{s}_j))_{\boldsymbol{x}_j} + \mathrm{d}^r(\exp(\boldsymbol{s}_{j+1}))_{\boldsymbol{x}^j}\right) \\
&\overset{(20.18)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}} \left(\tilde{B}_{j,k}(t)\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_{j+1})} \left[\mathrm{d}^r \exp_{\boldsymbol{s}_j}\right] \left[\mathrm{d}^r \exp_{\boldsymbol{v}_j}\right]^{-1} - \tilde{B}_{j+1,k}(t) \left[\mathrm{d}^r \exp_{\boldsymbol{s}_{j+1}}\right] \left[\mathrm{d}^l \exp_{\boldsymbol{v}_{j+1}}\right]^{-1}\right).
\end{aligned}$$

Can this be simplified?

Write down recursive scheme

20.2. General Coefficient Recursion

We seek an expression for $M_{i^*,k}$ which via (20.9) immediately gives $\tilde{M}_{i^*,k}$ that allows easy evaluation of the basis functions $\tilde{B}_{i,j}$. Inserting the basis expansion (20.2) into the recursive definition (20.1) yields

$$\begin{aligned}
\sum_{j=0}^{k} \alpha_{i,k}^j u^j &=: B_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t) \\
&= \left[\frac{t_{i^*} - t_i}{t_{i+k} - t_i} + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} u\right] B_{i,k-1}(t) + \left[\frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} u\right] B_{i+1,k-1}(t) \\
&= \frac{t_{i^*} - t_i}{t_{i+k} - t_i} \sum_{j=0}^{k-1} \alpha_{i,k-1}^j u^j + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} \sum_{j=0}^{k-1} \alpha_{i,k-1}^j u^{j+1} + \frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=0}^{k-1} \alpha_{i+1,k-1}^j u^j - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=0}^{k-1} \alpha_{i+1,k-1}^j u^{j+1} \\
&= \frac{t_{i^*} - t_i}{t_{i+k} - t_i} \sum_{j=0}^{k-1} \alpha_{i,k-1}^j u^j + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} \sum_{j=1}^{k} \alpha_{i,k-1}^{j-1} u^j + \frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=0}^{k-1} \alpha_{i+1,k-1}^j u^j - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=1}^{k} \alpha_{i+1,k-1}^{j-1} u^j \\
&= \sum_{j=0}^{k} \left[\frac{t_{i^*} - t_i}{t_{i+k} - t_i} \alpha_{i,k-1}^j + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} \alpha_{i,k-1}^{j-1} + \frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \alpha_{i+1,k-1}^j - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \alpha_{i+1,k-1}^{j-1}\right] u^j,
\end{aligned}$$

with the convention that $\alpha_{i,k}^j = 0$ for $j < 0$ and for $j > k$. By matching coefficients we therefore have that

$$\alpha_{i,k}^j = \underbrace{\frac{t_{i^*} - t_i}{t_{i+k} - t_i}}_{=:\tilde{\beta}_{i,i^*,k}} \alpha_{i,k-1}^j + \underbrace{\frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i}}_{=:\tilde{\gamma}_{i,i^*,k}} \alpha_{i,k-1}^{j-1} + \underbrace{\frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}}}_{1-\tilde{\beta}_{i+1,i^*,k}} \alpha_{i+1,k-1}^j - \underbrace{\frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}}}_{\tilde{\gamma}_{i+1,iz*,k}} \alpha_{i+1,k-1}^{j-1}, \tag{20.19}$$

or equivalently that for $N_{i,k}$ as in (20.3),

$$\begin{aligned} N_{i,k} &= \tilde{\beta}_i \begin{bmatrix} N_{i,k-1} \\ 0 \end{bmatrix} + \tilde{\gamma}_i \begin{bmatrix} 0 \\ N_{i,k-1} \end{bmatrix} + (1 - \tilde{\beta}_{i+1,i^*,k}) \begin{bmatrix} N_{i+1,k-1} \\ 0 \end{bmatrix} - \tilde{\gamma}_{i+1,i^*,k} \begin{bmatrix} 0 \\ N_{i+1,k-1} \end{bmatrix} \\ &= \begin{bmatrix} N_{ik-1} & N_{i+1,k-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\beta}_i \\ 1 - \tilde{\beta}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ N_{i,k-1} & N_{i+1,k-1} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{i,i^*,k} \\ -\tilde{\gamma}_{i+1,i^*,k} \end{bmatrix}, \end{aligned} \tag{20.20}$$

For convenience we re-define β and γ as

$$\beta_{j,i^*,k} := \tilde{\beta}_{i^*-j,i^*,k} = \frac{t_{i^*} - t_{i^*-j}}{t_{i^*-j+k} - t_{i^*-j}}, \quad \gamma_{j,i^*,k} := \tilde{\gamma}_{i^*-j,i^*,k} = \frac{t_{i^*+1} - t_{i^*}}{t_{i^*-j+k} - t_{i^*-j}} \tag{20.21}$$

Now we can write down a recursive formula for $M_{i^*,k}$ as given in (20.3):

$$\begin{aligned} M_{i^*,0} &= \begin{bmatrix} 1 \end{bmatrix}, \\ M_{i^*,k} &= \begin{bmatrix} M_{i^*,k-1} \\ 0 \end{bmatrix} \begin{bmatrix} 1 - \beta_{k-1,i^*,k} & \beta_{k-1,i^*,k} & 0 & \cdots & 0 \\ 0 & 1 - \beta_{k-2,i^*,k} & \beta_{k-2,i^*,k} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 - \beta_{0,i^*,k} & \beta_{0,i^*,k} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ M_{i^*,k-1} \end{bmatrix} \begin{bmatrix} -\gamma_{k-1,i^*,k} & \gamma_{k-1,i^*,k} & 0 & \cdots & 0 \\ 0 & -\gamma_{k-2,i^*,k} & \gamma_{k-2,i^*,k} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\gamma_{0,i^*,k} & \gamma_{0,i^*,k} \end{bmatrix}. \end{aligned} \tag{20.22}$$

However, close to the endpoints $\beta_{j,i^*,k}$ and $\gamma_{j,i^*,k}$ can no longer be evaluated. We can introduce artificial boundary knot points—$k-1$ to the left and $k-2$ on the right—to ensure that all splines have full support. Then β and γ can be computed using the expressions

$$\beta_{j,i^*,k} = \frac{t_{i^*} - t_{\max(i^*-j,0)}}{t_{\min(i^*-j+k,n)} - t_{\max(i^*-j,0)}}, \quad \gamma_{j,i^*,k} = \frac{t_{i^*+1} - t_{i^*}}{t_{\min(i^*-j+k,n)} - t_{\max(i^*-j,0)}} \tag{20.23}$$

that are valid for all indices $0 \leq i^* < n$.

20.3. Cardinal Cofficient Recursion

When all control points with indices $i^* - k + 1, \ldots, i^*$ are equally spaced such that $t_{i+1} - t_i = \Delta t$ for all i the expression can be simplified and $M_{i^*,k}$ no longer depends on i^* for interior points. In this case we have that

$$\beta_{j,i^*,k} = \frac{j\Delta t}{k\Delta t} = \frac{j}{k}, \quad \gamma_{j,i^*,k} = \frac{1}{k}. \tag{20.24}$$

We can use this to retrieve the first couple of matrices:

$$M_{i^*,0} = \begin{bmatrix}1\end{bmatrix}, \tag{20.25a}$$

$$M_{i^*,1} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1-\beta_{i^*,1} & \beta_{i^*,1}\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}-\gamma_1 & \gamma_1\end{bmatrix} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}, \tag{20.25b}$$

$$M_{i^*,2} = \begin{bmatrix}1 & 0\\-1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1-\beta_{i^*-1,2} & \beta_{i^*-1,2} & 0\\0 & 1-\beta_{i^*,2} & \beta_{i^*,2}\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}-\gamma_2 & \gamma_2 & 0\\0 & -\gamma_2 & \gamma_2\end{bmatrix} = \frac{1}{2!}\begin{bmatrix}1 & 1 & 0\\-2 & 2 & 0\\1 & -2 & 1\end{bmatrix} \tag{20.25c}$$

$$M_{i^*,3} = \ldots = \frac{1}{3!}\begin{bmatrix}1 & 4 & 1 & 0\\-3 & 0 & 3 & 0\\3 & -6 & 3 & 0\\-1 & 3 & -3 & 1\end{bmatrix}. \tag{20.25d}$$

Close to the boundary the formulas (20.23) should instead be applied.

21. Advanced control: controllability, lie brackets, and frobenius

- Difference is that vector fields are non-constant

22. Advanced: lidar odometry

22.1. Correspondence search

Assume that for the two clouds Q_{k-1} and Q_k edge and plane features $\mathscr{E}_{k-1}$, $\mathscr{H}_K$, $\mathscr{E}_k$, and $\mathscr{H}_k$ have been extracted.

Edge points in $\mathscr{E}_k$ should be matched with an edge in $\mathscr{E}_{k-1}$.

22.1.1. Edge matching

For a point $x_e \in \mathscr{E}_k$ select the n closest points S from $\mathscr{E}_{k-1}$, calculate the x-y-z covariance matrix and ensure that one eigenvalue is much larger than the other two. The fitting residual errors are

$$r_e(x_e) = (x_e - x_0) - ((x_e - x_0) \cdot \hat{d})\hat{d} = (I - \hat{d}\hat{d}^T)(x_e - x_0) \in \mathbb{R}^3$$

where x_0 is the centroid of S (assumed to be on the edge) and $\hat{d}$ is the normalized direction of the edge (eigenvector corresponding to largest eigenvalue of covariance matrix).

22.1.2. Plane matching

For a point $x_h \in \mathscr{H}_k$ select the n closest points from $\mathscr{H}_{k-1}$, calculate the x-y-z covariance matrix and ensure that one eigenvalue is much smaller than the other two. The fitting residual error is

$$r_h(x_h) = ((x_h - x_0) \cdot \hat{n})\,\hat{n} = (\hat{n}\hat{n}^T)\,(x_h - x_0) \in \mathbb{R}^3$$

where x_0 is the centroid of S (assumed to be in the plane) and $\hat{n}$ is the normalized normal of the plane (eigenvector corresponding to smallest eigenvalue of covariance matrix).

22.2. Optimization

For each detected correspondence we consider the transformed residuals

$$r_e(\exp(\omega\tau)x_e),$$

and

$$r_h(\exp(\omega\tau)x_h),$$

respectively, where $\omega \in \mathfrak{se}(3)$ represents the velocity during the sweep that is to be estimated, and τ is the time that has elapsed since the start of the sweep. That is, $\exp(\omega\tau)$ represents the relative pose between time t_k (start of the sweep), and time $t_k + \tau$ (time of point cloud collection).

23. The Magnus Expansion

Figure out what happens in the body-velocity ODE $\dot{\boldsymbol{x}} = \boldsymbol{x}A$

Remove things that are duplicated in Derivatives chapter

23.1. Preliminaries

The Bernoulli numbers are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n. \tag{23.1}$$

23.2. The Lie Group Adjoint

We can consider "higher order" Lie brackets and introduce an adjoint operator on the Lie Algebra to simplify notation. In particular, define:

$$\begin{aligned}
\mathrm{ad}_A^0 \, B &:= A \\
\mathrm{ad}_A^1 \, B &:= \mathrm{ad}_A \, B = [A, B] \\
\mathrm{ad}_A^2 \, B &:= [A, \mathrm{ad}_A \, B] = \underbrace{[A, [A, B]]}_{2-\text{times}} \\
&\vdots \\
\mathrm{ad}_A^k \, B &:= [A, \mathrm{ad}_A^{k-1} \, B] = \underbrace{[A, [A, \dots, [A, B]]]}_{k-\text{times}}, \qquad k \geq 1.
\end{aligned} \tag{23.2}$$

Note that this adjoint operator is different from the Lie group utilized above.

Define also the exponential of the adjoint as the formal expansion

$$\mathrm{Exp}(\mathrm{ad}_A) := \sum_{k \geq 0} \frac{\mathrm{ad}_A^k}{k!}. \tag{23.3}$$

Lemma 23.1.

$$\mathrm{Exp}(\mathrm{ad}_A) \; B = \mathrm{Exp}(A) \; B \; \mathrm{Exp}(-A). \tag{23.4}$$

Proof. By expanding the right-hand side in (23.4) we obtain

$$\sum_{k\geq 0}\sum_{i=0}^{k}\frac{A^iB(-A)^{k-i}}{i!(k-i)!}. \tag{23.5}$$

We next show by induction that the summands in (23.3) and (23.5) are equal for each value of k. Equality evidently holds for the base case $k = 0$. Assume that it holds for $k-1$, i.e. that

$$\frac{\mathrm{ad}_A^{k-1} B}{(k-1)!} = \sum_{i=0}^{k-1}\frac{A^iB(-A)^{k-1-i}}{i!(k-1-i)!}. \tag{23.6}$$

Then we have that

$$\begin{aligned}\frac{\mathrm{ad}_A^{k} B}{k!} &= \frac{1}{k}\left[A\frac{(\mathrm{ad}_A^{k-1} B)}{(k-1)!} - \frac{(\mathrm{ad}_A^{k-1} B)}{(k-1)!}A\right] = \frac{1}{k}\left[\sum_{i=0}^{k-1}\frac{A^{i+1}B(-A)^{k-1-i}}{i!(k-1-i)!} + \sum_{i=0}^{k-1}\frac{A^iB(-A)^{k-i}}{i!(k-1-i)!}\right]\\ &= \frac{1}{k}\left[\sum_{i=0}^{k-1}\frac{A^iB(-A)^{k-i}}{i!(k-1-i)!} + \sum_{i=1}^{k}\frac{A^iB(-A)^{k-i}}{(i-1)!(k-i)!}\right] = \frac{B(-A)^k}{k!} + \sum_{i=1}^{k-1}c_iA^iB(-A)^{k-i} + \frac{A^kB}{k!},\end{aligned}$$

where $c_i = \frac{1}{k}\left(\frac{1}{i!(k-1-i)!} + \frac{1}{(i-1)!(k-i)!}\right)$ and it can be verified that $c_i = \frac{1}{i!(k-i)!}$ as required. □

We are interested in solutions of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t), \quad \boldsymbol{x}(t) \in \boldsymbol{X}, A(t) \in \mathfrak{m}. \tag{23.7}$$

If A is constant in time the solution is given by the exponential, but in the general case the solution is more involved. The integrating factor technique used in scalar ODEs does not work since $A(t)$ and $\dot{A}(t)$ do not necessarily commute.

The Magnus Expansion approach is to posit that the solution of (23.7) is of the form

$$\boldsymbol{x}(t) = \mathrm{Exp}(\Omega(t)), \quad \Omega(t) \in \mathfrak{m}. \tag{23.8}$$

Consider $\boldsymbol{x}(t,\sigma) = \frac{\partial}{\partial t}[\mathrm{Exp}(\sigma\Omega(t))]\,\mathrm{Exp}(-\sigma\Omega(t))$. Differentiating with respect to σ yields

$$\begin{aligned}\frac{\partial}{\partial\sigma}\boldsymbol{x}(t,\sigma) = \frac{\partial}{\partial t}[\mathrm{Exp}(\sigma\Omega)\Omega]\,\mathrm{Exp}(-\sigma\Omega) - \frac{\partial}{\partial t}[\mathrm{Exp}(\sigma\Omega)]\,\Omega\,\mathrm{Exp}(-\sigma\Omega) = \mathrm{Exp}(\sigma\Omega)\frac{\mathrm{d}\Omega}{\mathrm{d}t}\,\mathrm{Exp}(-\sigma\Omega)\\ = \mathrm{Exp}(\mathrm{ad}_{\sigma\Omega})\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \sum_{k\geq 0}\frac{\sigma^k}{k!}\mathrm{ad}_\Omega^k\frac{\mathrm{d}\Omega}{\mathrm{d}t}.\end{aligned} \tag{23.9}$$

We can therefore write

$$\boldsymbol{x}(t,1) = \int_0^1\frac{\partial}{\partial\sigma}\boldsymbol{x}(t,\sigma)\mathrm{d}\sigma = \sum_{k\geq 0}\frac{\mathrm{ad}_\Omega^k}{(k+1)!}\frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{23.10}$$

and we have the following (where the second formulation follows by multiplying from the left with $\operatorname{Exp}(\Omega(t))$ and utilizing Lemma 23.1):

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Exp}(\Omega(t)) = \left(\mathrm{d}\operatorname{Exp}_{\Omega(t)}\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)\operatorname{Exp}(\Omega(t)) = \operatorname{Exp}(\Omega(t))\left(\mathrm{d}\operatorname{Exp}_{-\Omega(t)}\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right), \tag{23.11}$$

where, formally, $\mathrm{d}\operatorname{Exp}_\Omega : T\mathfrak{m}_\Omega \to \mathfrak{m}$ is the derivative of the exponential map $\operatorname{Exp} : \mathfrak{m} \to X$ around zero?

$$\mathrm{d}\operatorname{Exp}_\Omega = \sum_{k\geq 0}\frac{\mathrm{ad}_\Omega^k}{(k+1)!} = \frac{1}{\mathrm{ad}_\Omega}\sum_{k\geq 0}\frac{\mathrm{ad}_\Omega^{k+1}}{(k+1)!} = \frac{\operatorname{Exp}(\mathrm{ad}_\Omega) - I}{\mathrm{ad}_\Omega}. \tag{23.12}$$

Under certain conditions the linear operator $\mathrm{d}\operatorname{Exp}_\Omega$ can be inverted, and from the definition of the Bernoulli numbers in (23.1) we obtain

$$\mathrm{d}\operatorname{Exp}_\Omega^{-1} B = \frac{\mathrm{ad}_\Omega}{\operatorname{Exp}(\mathrm{ad}_\Omega) - I}B = \sum_{k\geq 0}\frac{B_k}{k!}\mathrm{ad}_\Omega^k B. \tag{23.13}$$

Theorem 23.1. *The solution of the time-varying ODE*

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{23.14}$$

is given by

$$\boldsymbol{x}(t) = \operatorname{Exp}(\Omega(t))\boldsymbol{x}_0, \tag{23.15}$$

where $\Omega(t)$ satisfies the initial-value problem

$$\Omega(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Omega(t) = \mathrm{d}\operatorname{Exp}_{\Omega(t)}^{-1} A(t) := \sum_{k\geq 0}\frac{B_k}{k!}\mathrm{ad}_{\Omega(t)}^k A(t). \tag{23.16}$$

Proof. Consider $\boldsymbol{y}(t) = \operatorname{Exp}(\Omega(t))\boldsymbol{x}_0$, by (23.11) it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y}(t) = \mathrm{d}\operatorname{Exp}_{\Omega(t)}(\Omega'(t))\operatorname{Exp}(\Omega(t))\boldsymbol{x}_0 = \mathrm{d}\operatorname{Exp}_{\Omega(t)}(\Omega'(t))\boldsymbol{y}(t), \tag{23.17}$$

and we see that $A(t) = \mathrm{d}\operatorname{Exp}_{\Omega(t)}(\Omega'(t))$. Applying the inverse operator results in the theorem statement. □

Remark 23.1. *For a body-velocity problem*

$$\dot{\boldsymbol{x}} = \boldsymbol{x}A, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{23.18}$$

the second expression in (23.11) *can be used to show that the solution is*

$$\boldsymbol{x}(t) = \boldsymbol{x}_0\operatorname{Exp}(\Omega(t)), \tag{23.19}$$

where Ω satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega(t) = \mathrm{d}\operatorname{Exp}_{-\Omega(t)}^{-1} A(t) = \sum_{k\geq 0}\frac{B_k}{k!}\mathrm{ad}_{-\Omega(t)}^k A(t), \quad \Omega(0) = 0. \tag{23.20}$$

Should verify this with a nilpotent Lie Algebra: The Lie group of invertible upper triangular matrices is the algebra of upper triangular invertible matrices!

The initial value problem for Ω is still challenging to solve. The **Magnus expansion** is obtained by setting $A = \epsilon\tilde{A}$ and expressing Ω as a series

$$\Omega(t) = \sum_{k\geq 1} \epsilon^k \Omega_k(t). \tag{23.21}$$

Inserting this in (23.16) and comparing powers of ϵ yields

$$\begin{aligned}
\Omega_1(t) &= \int_0^t A(s_1)\mathrm{d}s_1, \\
\Omega_2(t) &= -\frac{1}{2}\int_0^t [\Omega_1(s_1), A(s_1)]\,\mathrm{d}s_1 = \frac{1}{2}\int_0^t\int_0^{s_2} [A(s_1), A(s_2)]\,\mathrm{d}s_2\mathrm{d}s_1, \\
\Omega_3(t) &= \frac{1}{6}\int_0^t\int_0^{s_1}\int_0^{s_2} [A(s_1), [A(s_2), A(s_3)]] + [[A(s_1), A(s_2)], A(s_3)]\,\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1.
\end{aligned} \tag{23.22}$$

23.3. Example

Consider the initial value problem on $\mathbb{SE}(2)$:

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad \boldsymbol{x} \in \mathbb{SE}(2), \quad A \in \mathfrak{se}(2). \tag{23.23}$$

We assume that $A(t) = \hat{\tau}(t)$ is a known curve, i.e.

$$A(t) = \begin{bmatrix} 0 & -\theta(t) & u(t) \\ \theta(t) & 0 & v(t) \\ 0 & 0 & 0 \end{bmatrix} \tag{23.24}$$

According to Theorem 23.1 the solution is then

$$\boldsymbol{x}(t) = \mathrm{Exp}(\Omega(t))\boldsymbol{x}_0 = \mathrm{Exp}\left(\sum_{k\geq 0}\Omega_k(t)\right)\boldsymbol{x}_0. \tag{23.25}$$

The Lie algebra $\mathfrak{se}(2)$ is not nilpotent, so the exact solution requires the full Magnus expansion. Below we develop an approximate solution corresponding to the first two terms

$$\boldsymbol{x}(t) \approx \mathrm{Exp}\left(\int_0^t A(t)\mathrm{d}t + \frac{1}{2}\int_0^t\left(\int_0^{t_1}[A(t_1), A(t_2)]\,\mathrm{d}t_2\right)\mathrm{d}t_1\right)\boldsymbol{x}_0. \tag{23.26}$$

To find Ω_2 we consider the commutator of matrices in $\mathfrak{se}(2)$ on the form (23.24):

$$
\begin{aligned}
[A(t_1), A(t_2)] &= \begin{bmatrix} 0 & -\theta(t_1) & u(t_1) \\ \theta(t_1) & 0 & v(t_1) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\theta(t_2) & u(t_2) \\ \theta(t_2) & 0 & v(t_2) \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\theta(t_2) & u(t_2) \\ \theta(t_2) & 0 & v(t_2) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\theta(t_1) & u(t_1) \\ \theta(t_1) & 0 & v(t_1) \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\theta(t_1)\theta(t_2) & 0 & -\theta(t_1)v(t_2) \\ 0 & -\theta(t_1)\theta(t_2) & \theta(t_1)u(t_2) \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -\theta(t_1)\theta(t_2) & 0 & -\theta(t_2)v(t_1) \\ 0 & -\theta(t_1)\theta(t_2) & \theta(t_2)u(t_1) \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\theta(t_1)v(t_2) + \theta(t_2)v(t_1) \\ 0 & 0 & \theta(t_1)u(t_2) - \theta(t_2)u(t_1) \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

In the affine case where $\theta(t) = \theta_0 + a_\theta t$, and similarly for u and v, we get after evaluating the integrals:

$$
\begin{bmatrix} a(t) \\ b(t) \\ x(t) \\ y(t) \end{bmatrix} \approx \exp\left(\begin{bmatrix} \theta_0 t + a_\theta t\frac{t^2}{2} \\ u_0 t + a_u\frac{t^2}{2} + \theta_0 a_v \frac{t^3}{12} - a_\theta v_0 \frac{t^3}{12} \\ v_0 t + a_v \frac{t^2}{2} - \theta_0 a_u \frac{t^3}{12} + a_\theta u_0 \frac{t^3}{12} \end{bmatrix}\right).
$$

24. Advanced: Marginalization of nonlinear least squares

Objective is to remove a variable from the problem in a way so that

* The optimal solution is not effected * The first derivative at the optimal solution remains the same

For a nonlinear problem this is not possible, so we do it around a linearization point.

24.1. Lifted information matrix

Consider a set of variables $X = \{x_1, \dots, x_k\}$ where $x_i \in M_i$ and a square form

$$S = \frac{1}{2} \sum_j \left(h_j(X_j) - y_j\right)^T I_j \left(h_j(X_j) - y_j\right),$$

where $X_j = \{x_{j_1}, \dots x_{j_{n_j}}\} \subset X$ is a set of variables for the j:th measurement, and $h_j : X_j \mapsto h_j(X_j) \in \mathbb{R}^{p_j}$ are nonlinear measurement functions. Also let $I_j = \{j_1, \dots, j_{n_j}\}$ be the variable indices for measurement number j.

We are interested in marginalizing the expression S around a point $\{x_k = \mu_k\}$. Via Taylor expansion we obtain with $\mu_j = \begin{bmatrix} \mu_{j_1} & \cdots & \mu_{j_{n_j}} \end{bmatrix}$ being the measurement mean:

$$2S \approx \sum_j \left(h_j(\mu_j) + \sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i - y_j \right)^T I_j \left(h_j(\mu_j) + \sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i - y_j \right).$$

Here $[\mathrm{d}_i h_j]_{\mu_j} : TM_i \to \mathbb{R}^{p_j}$ is the differential of the measurement $h_j : \prod_{i \in I_j} M_i \to \mathbb{R}^{p_j}$ with respect to x_i evaluated at μ_j. The error differentials e_i are such that

$$x_i = \mu_i \oplus e_i = \mu_i \mathrm{Exp}_i(e_i) \quad \Longleftrightarrow \quad e_i = x_i \ominus \mu_i = \mathrm{Log}_i(\mu_i^{-1} x_i),$$

where $\mathrm{Exp} : \mathbb{R}^{n_i} \to M_i$ maps from coordinates in the tangent space to the manifold, and Log is the inverse mapping.

We now expand the sum

$$2S \approx \sum_j \left(h_j(\mu_j) - y_j\right)^T I_j \left(h_j(\mu_j) - y_j\right) + \sum_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right)^T I_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right)$$
$$+ 2 \sum_j \left(h_j(\mu_j) - y_j\right)^T I_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right)$$

and consider the term quadratic in e. We can augment the middle matrix with the differentials and

$$\begin{aligned}
&\sum_j \left(\sum_{i\in I_j}[\mathrm{d}_i h_j]_{\mu_j} e_i\right)^T I_j \left(\sum_{i\in I_j}[\mathrm{d}_i h_j]_{\mu_j} e_i\right) \\
&= \sum_j \begin{bmatrix} e_{j_1} & \cdots & e_{j_{n_j}} \end{bmatrix} \begin{bmatrix} [\mathrm{d}_{j_1} h_j]_{\mu_j}^T \\ \vdots \\ [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j}^T \end{bmatrix} I_j \begin{bmatrix} [\mathrm{d}_{j_1} h_j]_{\mu_j} & \cdots & [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j} \end{bmatrix} \begin{bmatrix} e_{j_1} \\ \vdots \\ e_{j_{n_j}} \end{bmatrix} \\
&= \sum_j \begin{bmatrix} e_{j_1} & \cdots & e_{j_{n_j}} \end{bmatrix} \begin{bmatrix} [\mathrm{d}_{j_1} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_1} h_j]_{\mu_j} & \cdots & [\mathrm{d}_{j_1} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j} \\ \vdots & \ddots & \vdots \\ [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_1} h_j]_{\mu_j} & \cdots & [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j} \end{bmatrix} \begin{bmatrix} e_{j_1} \\ \vdots \\ e_{j_{n_j}} \end{bmatrix} \\
&= \sum_{i_1,i_2} e_{i_1} \left[\sum_{j\,:\,i_1,i_2\in I_j} [\mathrm{d}_{i_1} h_j]_{\mu_j}^T I_j [\mathrm{d}_{i_2} h_j]_{\mu_j}\right] e_{i_2} = \begin{bmatrix} e_1 & \cdots & e_k \end{bmatrix} \Lambda \begin{bmatrix} e_1 \\ \vdots \\ e_k \end{bmatrix},
\end{aligned}$$

where Λ is the sum of **block-lifted information matrices** obtained by placing the blocks from cost functions at the appropriate places.

It follows that we can write S on the information form

$$S \sim \eta^T \mathrm{e} + \frac{1}{2}\mathrm{e}^T \Lambda \mathrm{e}$$

with Λ as above and η a similarly block-lifted column vector such that

$$\eta^T \mathrm{e} = \sum_{i=1}^k \left[\sum_{j\,:\,i\in I_j} (h_j(\mu_j) - y_j)^T I_j [\mathrm{d}_i h_j]_{\mu_j}\right] e_i.$$

24.2. Marginalization

We group the variables into e_α and e_β, where e_β is the variable to be removed, and write S on the general information form

$$S \sim \begin{bmatrix} \eta_\alpha^T & \eta_\beta^T \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta \end{bmatrix} + \frac{1}{2} \begin{bmatrix} e_\alpha^T & e_\beta^T \end{bmatrix} \begin{bmatrix} \Lambda_{\alpha\alpha} & \Lambda_{\alpha\beta} \\ \Lambda_{\beta\alpha} & \Lambda_{\beta\beta} \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta \end{bmatrix}$$

We can then expand the information matrix in the same way as in Joplin normal distribution to obtain

$$S \sim \begin{bmatrix} \eta_\alpha^T - \eta_\beta^T \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha} & \eta_\beta^T \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta + \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha} e_\alpha \end{bmatrix} + \frac{1}{2} \begin{bmatrix} e_\alpha \\ e_\beta + \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha} e_\alpha \end{bmatrix}^T \begin{bmatrix} \Lambda/\Lambda_{\beta\beta} & 0 \\ 0 & \Lambda_{\beta\beta} \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta + \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha} e_\alpha \end{bmatrix}.$$

That is, we have separated the expression into two quadratic expressions. A coordinate change $\kappa = e_\beta + \Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha} e_\alpha$ reveals that they can be solved independently. The sub-problem for e_α reads

$$\left(\eta_\alpha^T - \eta_\beta^T \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha}\right) e_a + \frac{1}{2} e_\alpha^T (\Lambda/\Lambda_{\beta\beta}) e_\alpha.$$

By completing the square we can write this as

$$\sim \frac{1}{2}\left(e_\alpha + (\Lambda/\Lambda_{\beta\beta})^{-1}\left(\eta_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\eta_\beta\right)\right)^T (\Lambda/\Lambda_{\beta\beta})\left(e_\alpha + (\Lambda/\Lambda_{\beta\beta})^{-1}\left(\eta_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\eta_\beta\right)\right)$$

which is the marginalized form of the problem.

24.3. Algorithm

Suppose we want to remove a variable x_k. Consider the set of factors F such that $k \in I_f$ for all $f \in F$, and the resulting blanket set of variables $X_F = \bigcup_{f\in F}\bigcup_{j\in f_j} X_j$.

Ceres provides evaluations $E_j = \sqrt{I_j}(h_j(\mu_j) - y_j)$ and gradient blocks $G_{ji} = \sqrt{I_j}[\mathrm{d}_i h_j]_{\mu_j}$. **It's fine if residuals are defined as the negative since the signs will cancel in multiplication**.

1. Find the information matrix Λ by summing over all factors in F. Sum blocks in Λ are of the form $[\mathrm{d}_{i_1} h_j]_{\mu_j}^T I_j [\mathrm{d}_{i_2} h_j]_{\mu_j} = G_{ji_1}^T G_{ji_2}$. 2. Find the mean vector η by summing over all factors in F. Sum segments in η are of the form $(h_j(\mu_j) - y_j)^T I_j [\mathrm{d}_i h_j]_{\mu_j} = E_j^T G_{ji}$. 3. Partition Λ and η as

$$\Lambda = \begin{bmatrix} \Lambda_{\neg k\neg k} & \Lambda_{\neg kk} \\ \Lambda_{k\neg k} & \Lambda_{kk} \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_{\neg k} \\ \eta_l \end{bmatrix}.$$

3. Calculate $\tilde{\Lambda} = \Lambda/\Lambda_k$ and $\gamma = -(\Lambda/\Lambda_k)^{-1}\left(\eta_{\neg k} - \Lambda_{\neg kk}\Lambda_{kk}^{-1}\eta_k\right)$ 4. Remove factors F and instead insert a new factor with cost function

$$(e_{\neg k} - \gamma)^T \tilde{\Lambda}(e_{\neg k} - \gamma) = \begin{bmatrix} e_1 - \gamma_1 \\ \vdots \\ e_n - \gamma_n \end{bmatrix}^T \tilde{\Lambda} \begin{bmatrix} e_1 - \gamma_1 \\ \vdots \\ e_n - \gamma_n \end{bmatrix} = \begin{bmatrix} \mathrm{Log}(\mu_1^{-1}x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_n^{-1}x_n) - \gamma_n \end{bmatrix}^T \tilde{\Lambda} \begin{bmatrix} \mathrm{Log}(\mu_1^{-1}x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_n^{-1}x_n) - \gamma_n \end{bmatrix}$$

24.4. Correction for singular information matrix

In the event that $\tilde{\Lambda}$ is singular we can not calculate γ. Instead consider the decomposition $\tilde{\Lambda} = UDU^T$, where D is a square diagonal matrix with only non-zero diagonal entries. We can then let

$$\gamma = -UD^{-1}U^T\left(\eta_{\neg k} - \eta_k\Lambda_{kk}^{-1}\Lambda_{k\neg k}\right)$$

and consider the cost

$$\left\| \sqrt{D}U^T \begin{bmatrix} \mathrm{Log}(\mu_1^{-1}x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_n^{-1}x_n) - \gamma_n \end{bmatrix} \right\|^2$$

which has a non-zero information matrix.

24.5. Marginalization factor in local frame

The above linearizes around a world point $\{\mu\}$. This makes sense if marginalizing a node that has an absolute factor, but perhaps not when it is only connected via relative factors and there may be a lot of drift. We can transform the measurements into a local frame by instead introducing the cost

$$\left\| \sqrt{D} U^T \begin{bmatrix} \mathrm{Log}(\mu_{01}^{-1} x_0^{-1} x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_{0n}^{-1} x_0^{-1} x_n) - \gamma_n \end{bmatrix} \right\|^2$$

where $\mu_{0i} = \mu_0^{-1} \mu_i$ are linearization points transformed into the local frame of x_0, which should be selected as a pose in the vicinity of the removed node.

* Only works if the measurements h_j are invariant to rigid transformations. This property holds for relative measurements such as relative poses and landmark triangulations.

* For a node with absolute factors the measurement h would have to be adjusted before building the *gamma* vector.

24.6. Example

Consider the least-squares problem

$$S = (x_1 - x_3 + 1)^2 + (x_2 - x_3 + 1)^2$$

where $h_1(\mathrm{x}) = x_1 - x_3$, $y_1 = -1$, $h_2(\mathrm{x}) = x_2 - x_3$, and $y_2 = -1$.

We expand in a Taylor form as above

$$\begin{aligned} S &\approx \left(h_1(\mathrm{x}_0) + \mathrm{d}h_1 \cdot (\mathrm{x} - \mathrm{x}_0) - y_1\right)^2 + \left(h_2(\mathrm{x}_0) + \mathrm{d}h_2 \cdot (\mathrm{x} - \mathrm{x}_0) - y_2\right)^2 \\ &= (\mathrm{d}h_1 \cdot (\mathrm{x} - \mathrm{x}_0))^2 + (\mathrm{d}h_2 \cdot (\mathrm{x} - \mathrm{x}_0))^2 \\ &\quad + 2\left[(h_1(\mathrm{x}_0) - y_1)\mathrm{d}h_1 \cdot (\mathrm{x} - \mathrm{x}_0) + (h_2(\mathrm{x}_0) - y_2)\mathrm{d}h_2 \cdot (\mathrm{x} - \mathrm{x}_0)\right] + C \end{aligned}$$

where C is a constant. Since both h are linear this expression is exact for any x_0 (can be verified).

We let $e_i = x_i - x_0^i$ and get

$$S = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \left(\mathrm{d}h_1^T \mathrm{d}h_1 + \mathrm{d}h_2^T \mathrm{d}h_2\right) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + 2\left[(h_1(\mathrm{x}_0) - y_1)\mathrm{d}h_1 + (h_2(\mathrm{x}_0) - y_2)\mathrm{d}h_2\right] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$+ 2\left[(x_1^0 - x_3^0 + 1)\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + (x_2^0 - x_3^0 + 1)\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\right] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$+ 2\begin{bmatrix} x_1^0 - x_3^0 + 1 & x_2^0 - x_3^0 + 1 & -x_1^0 - x_2^0 + 2x_3^0 - 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

We now marginalize out x_3 and identify the marginalized covariance

$$\tilde{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

and the marginalized mean

$$\tilde{\eta}^T = \eta_\alpha^T - \eta_\beta^T \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha} = \begin{bmatrix} x_1^0 - x_3^0 + 1 & x_2^0 - x_3^0 + 1 \end{bmatrix} - \begin{bmatrix} -x_1^0 - x_2^0 + 2x_3^0 - 2 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1^0 - x_2^0 & -x_1^0 + x_2^0 \end{bmatrix}.$$

The marginalized problem is now

$$\tilde{S} = \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix}^T \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix} + 2\begin{bmatrix} \frac{x_1^0 - x_2^0}{2} & \frac{-x_1^0 + x_2^0}{2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix}.$$

After expanding and removing constant terms this is equal to

$$\tilde{S} = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}(x_1 - x_2)^2.$$

That is, the problem is independent of the linearization point, as expected.

24.7. As part of least-squares problem

Now, for the least-squares problem that has this as a factor:

$$x_1^2 + (x_1 - x_2 + 1)^2 + (x_1 - x_3 + 1)^2 + (x_2 - x_3 + 1)^2$$

that the above as terms, marginalizing x_3 replaces the last two terms with the above, so the marginalized problem becomes

$$x_1^2 + (x_1 - x_2 + 1)^2 + \frac{1}{2}(x_1 - x_2)^2.$$

This has the same optimial solution $x_1 = 0, x_2 = 2/3$ and optimzal value $1/3$ as the full problem above.

https://www.wolframalpha.com/input/?i=Minimize%5Bx%5E2+%2B+%28x-y%2B1%29%5E2+%2B+%28x-z%2B1%29%5E2+%2B+%28y-z%2B1%29%5E2%5D

https://www.wolframalpha.com/input/?i=Minimize%5Bx%5E2+%2B+%28x-y%2B1%29%5E2+%2B+%28x-y%29%5E2%2F2%5D

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