Invariant Sets of Two-Dimensional Affine Switched Systems

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Outline

Concepts in Control Theory

- Dynamical system model $\dot{x} = f(x, t, u)$, where u is some *input* parameter.
- Basic question: What input should we choose to make the system behave as desired?
- Issues: stability, robustness, safety guarantees, optimal control.

Switched systems

- Special class of hybrid systems, i.e. systems with both continuous and discrete components.
- General switched system:

$$\begin{cases} \dot{x} = f_{u(t)}(x, t), \\ u(t) \in \mathcal{P}. \end{cases}$$

  - $u : \mathbb{R}^+ \to \mathcal{P}$ is the *switching sequence* or *switching control*.
  - $\{f_p \mid p \in \mathcal{P}\}$ are the *admissible vector fields*.
- Challenges with switched systems (Liberzon and Morse, 1999).
  - Find a stabilizing switching sequence.
  - Determine if a system is stable under constrained switching.
  - Determine if a system is stable under arbitrary switching.
    - Lyapunov function methods (Dayawansa and Martin, 1999)
    - Properties of the generated Lie algebra (Agrachev and Liberzon, 2001).
    - Geometrical methods.

Planar, linear case

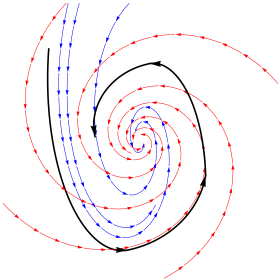
- *Focused* system:

$$\dot{x} = u(t)Ax + (1 - u(t))Bx,$$
$$u(t) \in [0, 1], \quad A, B \in \mathbb{R}^{2\times 2}.$$

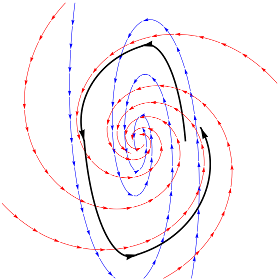
- When is this system stable?
- Evident necessary condition: A and B are stable matrices.

Graphical examples

Asymptotically stable



Unstable

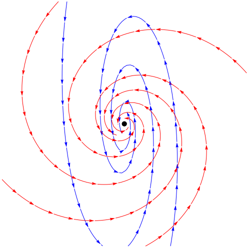


- Stability for a pair of matrices (A, B) can be checked using the results in (Balde et al., 2009).

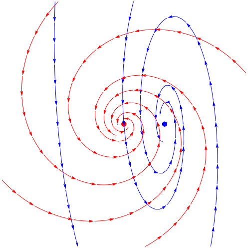
Generalization: separated attraction points.

- *Defocused* system:

$$\dot{x} = uA(x - x_c) + (1-u)Bx,$$
$$u \in [0,1], \quad A, B \in \mathbb{R}^{2\times 2}.$$



$\longrightarrow$



- Not possible for a single point to be stable under arbitrary switching.

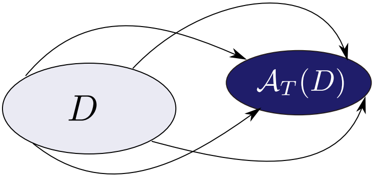
Definitions

Definition

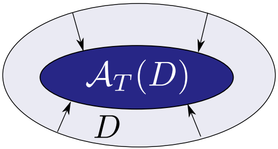
The *reachable set* $\mathcal{A}_T(D)$ from an initial set D consists of all the points reachable from D in time T using any switching sequence $u : [0, T] \to \mathcal{P}$.

Definition

A set D is *invariant* if $\mathcal{A}_T(D) \subset D$ for all $T > 0$.



Reachable set.



D is invariant.

Outline

Assumptions

$$\dot{x} = uAx + (1-u)Bx, \quad u \in [0,1]. \tag{1}$$

$$\dot{x} = uA(x - x_c) + (1-u)Bx, \quad u \in [0,1]. \tag{2}$$

Assumption

The origin is asymptotically stable for the focused system (1).

Assumption

The defocused system (2) *is Lie bracket generating, i.e.* $dim(L_x(A(x-x_c), Bx)) = 2$ *for all* $x \in \mathbb{R}^2$.

Main result

Define a 'smallest' invariant set Ω for the defocused system as

$$\Omega = \bigcap_{D \text{ invariant}} D. \tag{3}$$

Theorem

Consider the defocused system (2) and suppose the two assumptions hold. Then the invariant set Ω defined in (3) is bounded and non-empty and has the following properties.

- *When considering $u(\cdot)$ as a control signal, Ω is completely controllable.*
- *Ω is open and simply connected with a boundary $\partial\Omega$ that is piecewise C^∞.*
- *If furthermore both system matrices A and B have eigenvalues in $\mathbb{C} \setminus \mathbb{R}$, then Ω is attractive in the sense that $cl(\Omega) = cl(\lim_{T\to\infty} \mathcal{A}_{\geq T}(y))$ for all $y \in \mathbb{R}^2$.*

Boundedness

- Idea: Take a common Lyapunov function for the focused system, show that it is decreasing also along the trajectories of the defocused system for large $\|x\|$.
- A common Lyapunov function $V(x)$ exists (Dayawansa and Martin, 1999) s.t.

$$\nabla V \cdot Ax \leq -\frac{1}{4}\|x\|, \quad \nabla V \cdot Bx \leq -\frac{1}{4}\|x\|, \quad V(\alpha x) = \alpha^2 V(x).$$

- It can be shown that there exists R s.t.

$$\nabla V(x) \cdot A(x - x_c) < 0,$$

  when $\|x\| > R$.
- It follows that a bounded invariant set D_B exists and $\Omega \subset D_B$.

Non-emptyness

- Idea: The constant switching sequence $u \equiv 0$ will lead to convergence towards 0. Therefore 0 must be in the closure of any invariant set.
- Kreners theorem: Reachable sets for bracket-generating systems have non-empty interior.
- Define a non-empty set

$$D_0 = \text{int}(\mathcal{A}_{\leq \varepsilon}(0)).$$

- By the preceding D_0 is open, non-empty and must be contained in any invariant set.
- Then also $\Omega = \bigcap_{D \text{ invariant}} D$ is non-empty.

Attractiveness in the case of non-real eigenvalues

- Characterize Ω as a reachable set: $\Omega = \text{int}(\mathcal{A}(Q^-))$.
- $Q = Q^- \cup Q^+$ is the set of *colinearity*.
  - $Q = \{x \in \mathbb{R}^2 : \det(A(x - x_c),\ Bx) = 0\}$.
  - Q is generically either an ellipse or a hyperbola.
- Double matrix normal forms:

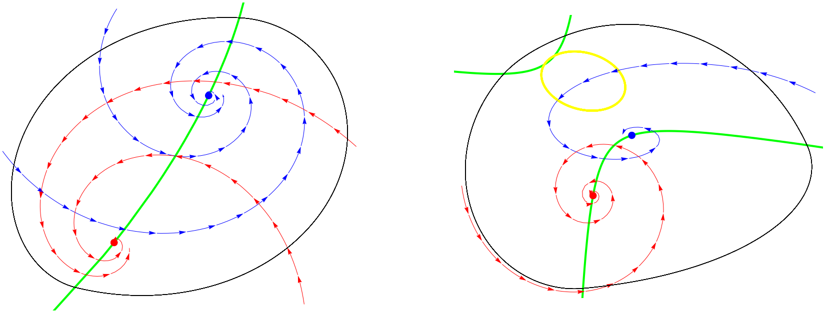
$$\dot{x} = u \begin{bmatrix} -\rho_A & -1/E \\ E & -\rho_A \end{bmatrix} \left(x - \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \right) + (1-u) \begin{bmatrix} -\rho_B & -1 \\ 1 & -\rho_B \end{bmatrix} x.$$

- Construct $\mathcal{A}(Q^-)$ for two different cases and verify attractiveness.
  - $E > 0$: fields turn in the same direction. Use Poincaré-Benedixson-like theorem for differential inclusions (Filippov and Arscott, 1988).
  - $E < 0$: fields turn in opposite directions. Conclude by geometrical reasoning.

Outline

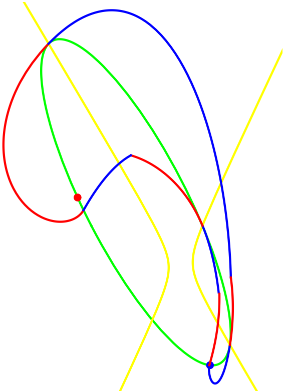
$E > 0$ examples

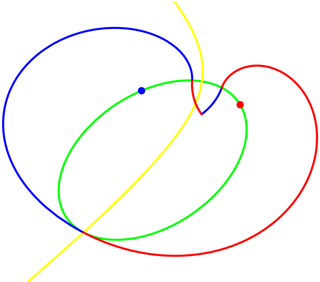
- The boundary of Ω is the black curve.
- $\partial\Omega$ is C^1 and piecewise C^∞.



$E < 0$ examples

- The boundary of Ω is in red and blue, it can consist of up to 8 pieces.

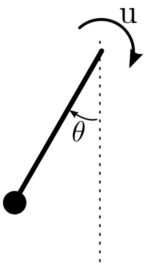




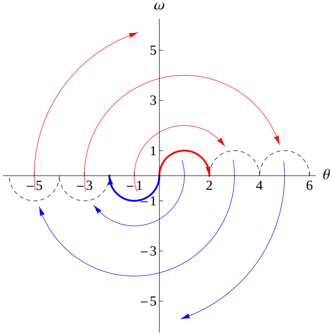
Outline

Optimal synthesis: Theory

- Problem: For an initial point y, find for every point $x \in \Omega$ the minimal time trajectory from y to x.
- For a 2D system the optimal synthesis can be represented as a graph in which a unique trajectory arrives in every point.
- Example: Linearized forced pendulum (Boscain and Piccoli, 2005, p. 63).



$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\omega} = -\theta + u, \\ u \in [-1, 1]. \end{cases}$$



- All time-optimal trajectories are *extremals*, characterized by Pontryagin's Maximum Principle (PMP).
- The function θ^γ:
  - Let v^γ be the solution at time 0 to the problem

$$\begin{cases} \frac{d}{ds} v^\gamma(s) = (\nabla F \mid_{\gamma(s)} + u(s) \nabla G \mid_{\gamma(s)}) v^\gamma(s), \\ v^\gamma(t) = G(\gamma(t)). \end{cases}$$

  - Define θ^γ as

$$\theta^\gamma(t) = \arg(v^\gamma(t)) - \arg(v^\gamma(0)).$$

- Using the PMP it can be shown that

Theorem

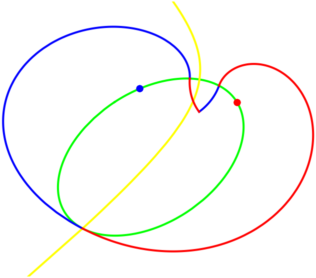
Assume that γ is an extremal trajectory and let $\bar{t}$ be the first time its switching function is zero, i.e. $\bar{t} = \min\{t \geq 0 \mid \phi(t) = 0\}$. If $\bar{\theta} = \theta^\gamma(\bar{t})$ the switching function is zero for time t iff $\theta^\gamma(t) = \bar{\theta} + n\pi$, $n \in \mathbb{N}$.

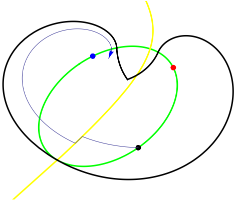
Optimal synthesis in Ω

- Consider the system:

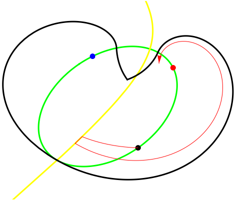
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = u \begin{bmatrix} -0.42 & 1.33 \\ -0.75 & -0.42 \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \cos(3) \\ \sin(3) \end{bmatrix} \right) + (1-u) \begin{bmatrix} -0.6 & -1.5 \\ 1.5 & -0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- $E < 0$.
- Ω looks like this:

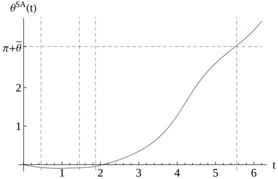




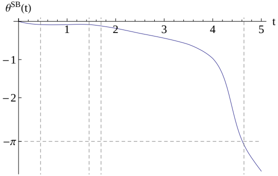
Trajectory γ^{SA}.



Trajectory γ^{SB}.

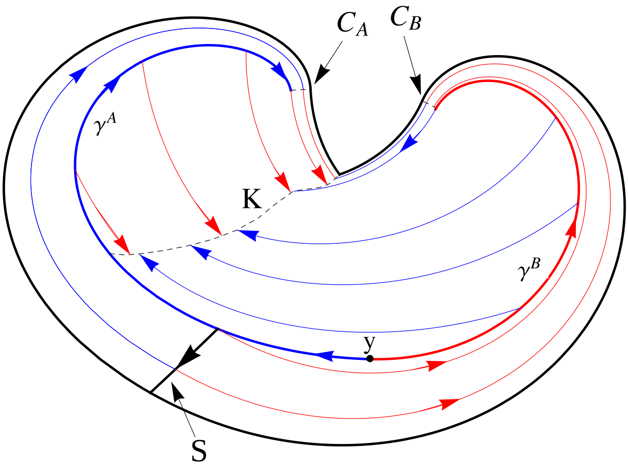


The function θ^{SA}.



The function θ^{SB}.

Result

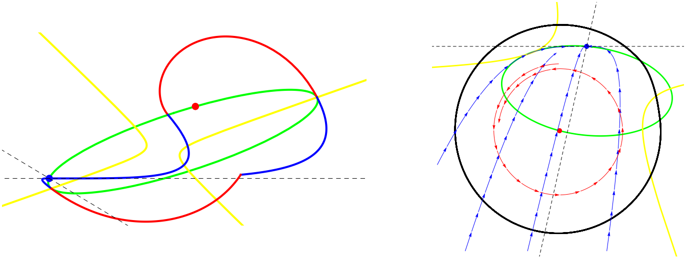


Minimal time trajectories starting in y.

Outline

Real eigenvalues

- Behaviour is harder to classify.
- Have not been able to prove ‘attractiveness’.



Summary

Conclusions:

- Have found a 'smallest' invariant set Ω.
- Complete characterization and attractiveness property for eigenvalues in $\mathbb{C} \setminus \mathbb{R}$.

Future work:

- Real eigenvalue cases.
- General results about optimal syntheses.
- Generalizations:
  - Non-linear systems.
  - Higher dimensions.
  - Switched systems with more than two subsystems.

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