# Control of Large Collections of Switched Systems

**Petter Nilsson**, Necmiye Ozay

## Background

- General-purpose correct-by-construction synthesis methods rely on set reasoning
- Set manipulation intractable for large-scale systems. Must exploit symmetry when possible
- Best-case solution: propose analytical controller and prove that closed-loop system satisfies specification

Motivating example: **TCL scheduling**

- Thermostatically controlled loads (TCLs) are air condition units, refrigerators etc, that operate within a temperature range
- Local temperature constraints vs aggregate power consumption constraints

## Counting problem: Definition

Large number $i = 1, \ldots, N$ of subsystems $x_i \in \mathbb{R}^d$ with dynamics:

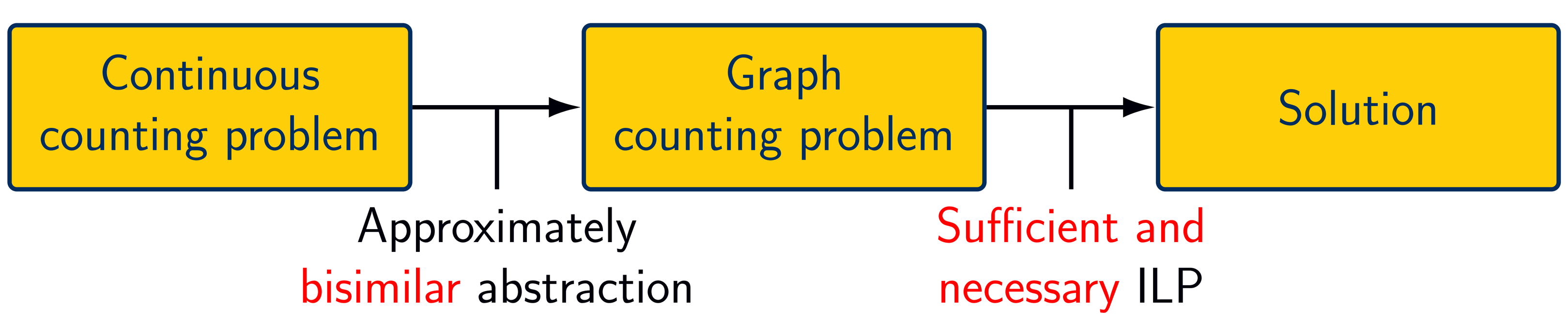
$$\dot{x}^i(t) = f_{\sigma^i(t)}(x^i(t), \delta^i(t)), \; \sigma^i : \mathbb{R} \to \{1, \ldots, M\}$$

Counting constraints $(X^l, R^l)$:

$$\sum_{i=1}^{N} \mathbb{1}_{X^l}\left(x^i(t), \sigma^i(t)\right) \leq R^l.$$

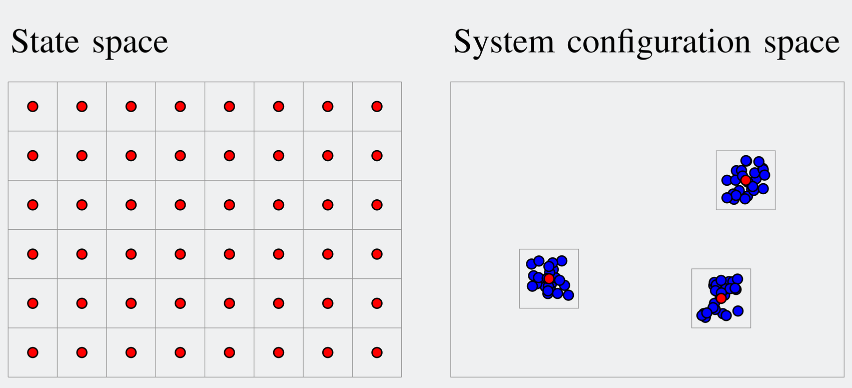
- Counting constraints for safety, aggregate consumption, balancing, etc
- Counting constraints are permutation invariant
- **Problem**: given initial state, construct a switching protocol $\{\sigma^i(t)\}_{n=1}^N$ such that the constraints are satisfied

## Synthesis for symmetric problem via abstraction and integer programming




### Re-use abstraction

- For mild heterogeneity, use same abstraction for similar subsystems. Can "abstract away" mild heterogeneity




- Standard abstraction: $\mathcal{O}\left(\frac{1}{\eta}\right)^{Nd}$
- Counting abstraction: $\mathcal{O}\left(\frac{1}{\eta}\right)^{d}$

- Necessary and sufficient conditions for feasibility of continuous problem via approximate simulation relations

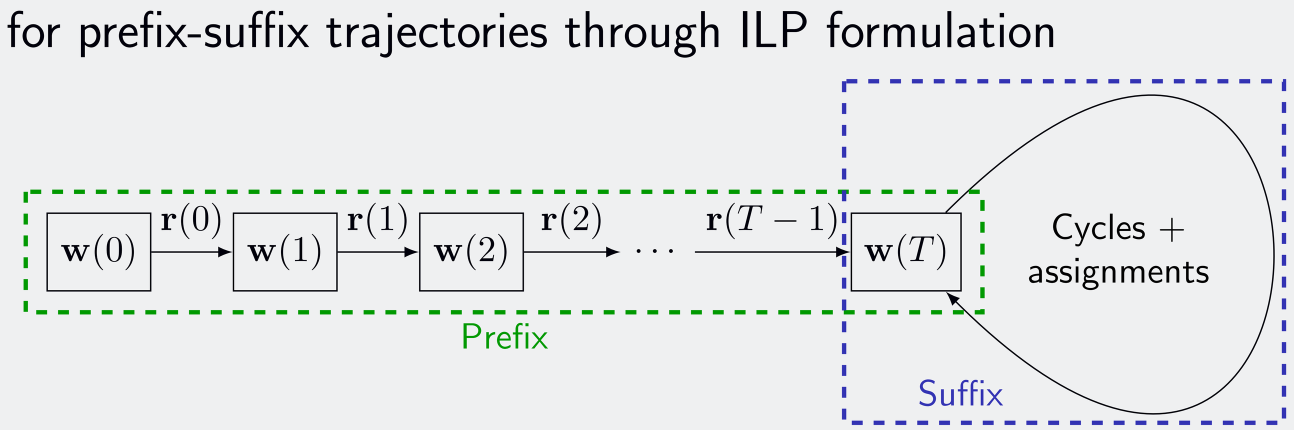
$$\beta(\epsilon, \tau) + \frac{\delta}{K}\left(e^{K\tau} - 1\right) + \frac{\eta}{2} \leq \epsilon,$$

### Aggregate counting dynamics and ILP solution

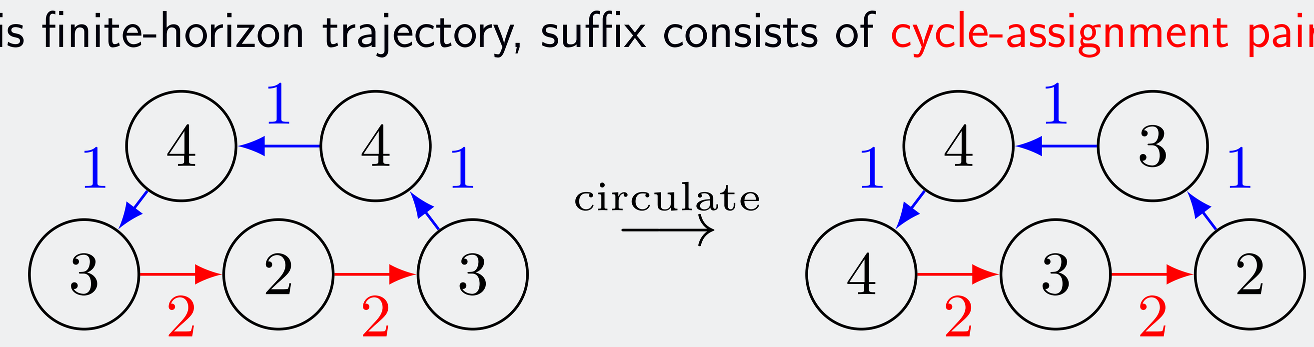
- Abstraction graph has nodes $\{\nu_k\}_{k=1}^K$, let $w_k(s) \in \mathbb{N}$: number of subsystems at node $\nu_k$. Aggregate dynamics:

$$w_k(s+1) = \sum_{m \in [M]} \sum_{k' \in \mathcal{N}_k^m} r_{k'}^m(s), \quad k \in [K],$$

- Search for prefix-suffix trajectories through ILP formulation




- Prefix is finite-horizon trajectory, suffix consists of cycle-assignment pairs




- Sufficient if prefix-horizon and cycle set is large enough
- If ILP too large: solve as relaxed LP and round solution

Petter Nilsson and Necmiye Ozay, **Control Synthesis for Large Collections of Systems with Mode-Counting Constraints**, *Proceedings of the International Conference on Hybrid Systems: Computation and Control*, pp. 205-214, 2016

## Analytic approach for heterogeneous problem with one-dimensional subsystems (e.g. TCLs)

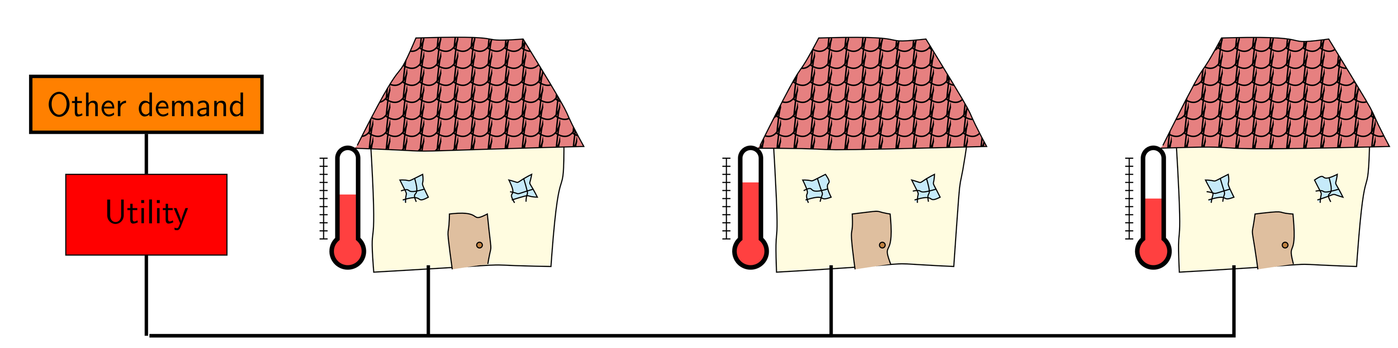
### Heterogeneous problem

- Strongly heterogeneous family of one-dimensional switched systems

$$\frac{\mathrm{d}}{\mathrm{d}t}x^i(t) = \begin{cases} f^i_{\texttt{off}}\left(x^i(t)\right) & \text{if } \sigma^i(t) = \texttt{off}, \\ f^i_{\texttt{on}}\left(x^i(t)\right) & \text{if } \sigma^i(t) = \texttt{on}, \end{cases}$$

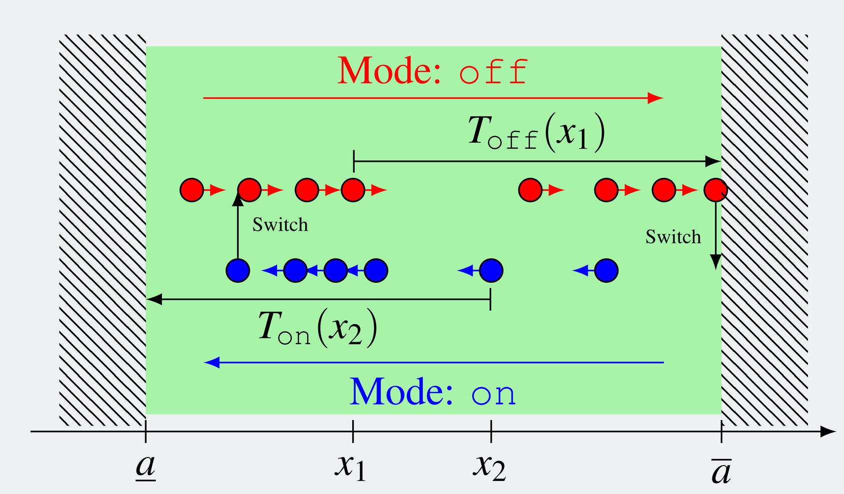
- Local constraints $\underline{a}^i \leq x^i(t) \leq \overline{a}^i$
- Mode-on-counting constraint: $\underline{K}_{\texttt{on}} \leq \sum_{i \,:\, \sigma^i(t) = \texttt{on}} 1 \leq \overline{K}_{\texttt{on}}$
- Feasibility condition

$$\sum_{i=1}^{N} \frac{\mathcal{L}^i_{\texttt{off}} T^i_{\texttt{on}}(\underline{a}^i)}{1 + \mathcal{L}^i_{\texttt{off}} T^i_{\texttt{on}}(\underline{a}^i)} > \underline{K}_{\texttt{on}}, \quad \sum_{i=1}^{N} \frac{\mathcal{L}^i_{\texttt{on}} T^i_{\texttt{off}}(\overline{a}^i)}{1 + \mathcal{L}^i_{\texttt{on}} T^i_{\texttt{off}}(\overline{a}^i)} > N - \overline{K}_{\texttt{on}} \qquad (1)$$




### "Lazy" feedback control strategy

- Only switch subsystems when a constraint is about to be violated
- Select subsystems based on times to exit $T^i_{\texttt{off}}$ and $T^i_{\texttt{on}}$

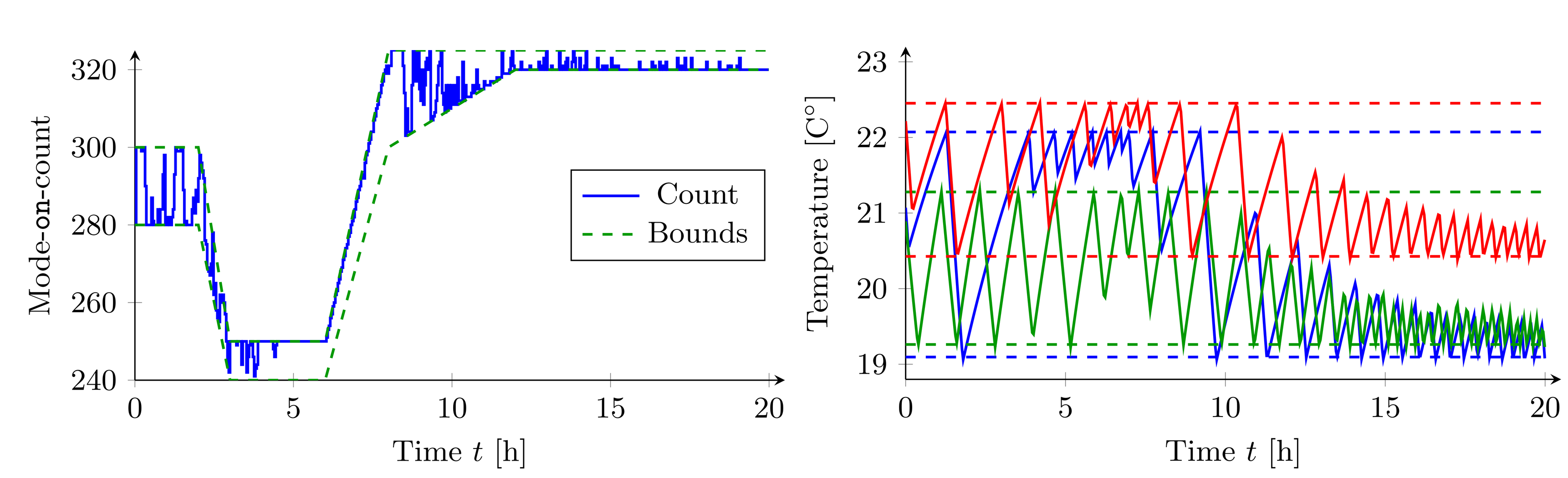



### Theorem

Sufficient: If (1) holds, then the proposed strategy solves the problem for any non-degenerate initial condition.

Necessary: If $f^i_{\texttt{on}}$ and $f^i_{\texttt{off}}$ are monotonically decreasing in $[\underline{a}^i, \overline{a}^i]$ and (1) is strictly violated, then the problem has no solution for any initial condition.

Simulation of 10,000 TCLs




Petter Nilsson and Necmiye Ozay, **On a class of maximal invariance inducing control strategies for large collections of switched systems**, *Proceedings of the International Conference on Hybrid Systems: Computation and Control*, pp. 187-196, 2017



pettni@umich.edu, necmiye@umich.edu