# On a class of maximal invariance inducing control strategies for large collections of switched systems

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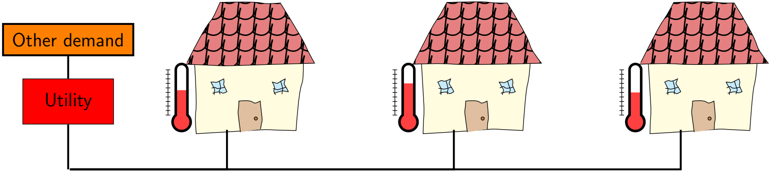
# Formal methods in control

- Fundamental problem in synthesis and verification: curse of dimensionality
- Very large systems: no hope for general-purpose methods. Need suitable abstractions
- We are interested in controlling large collections of similar switched systems
  - HSCC '16: symmetric state-space abstraction to enable synthesis
  - This year: propose controller, verify that it fulfills specification
    - Controller based on time domain abstraction to "abstract away" heterogeneity

# Thermostatically Controlled Load (TCL) Scheduling

- A TCL can be in mode `on` or `off`
- (Local) state constraints: Each TCL should maintain temperature within a desired temperature range
- (Global) counting constraint: Aggregate electricity consumption should be controlled over time
- The flexibility in individual specifications can be leveraged to control aggregate demand to for instance mitigate fluctuations




- How to schedule on/off cycles to meet both local and global constraints?

# TCL Mode-Counting Problem

- $N$ (heterogeneous) subsystems $x^i \in \mathbb{R}$ with switched dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} x^i(t) = \begin{cases} f^i_{\texttt{off}}\left(x^i(t)\right) & \text{if } \sigma^i(t) = \texttt{off}, \\ f^i_{\texttt{on}}\left(x^i(t)\right) & \text{if } \sigma^i(t) = \texttt{on}, \end{cases}$$

- Local (heterogeneous) state constraints:

$$\underline{a}^i \le x^i(t) \le \overline{a}^i \quad \forall t \ge 0 \tag{1}$$

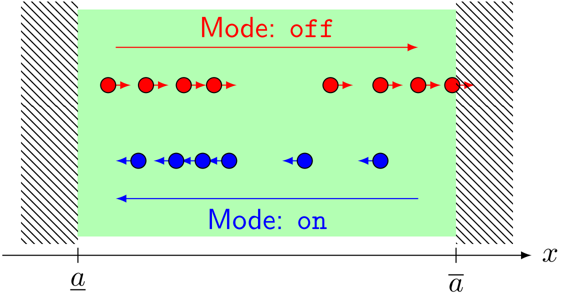
- Global counting constraint:

$$\underline{K}_{\texttt{on}} \le \sum_{i:\ \sigma^i(t) = \texttt{on}} 1 \le \overline{K}_{\texttt{on}} \quad \forall t \ge 0 \tag{2}$$

- Objective: find switching strategy $\{\sigma^i(\cdot)\}_{i=1}^N$ that enforces (1)-(2).

# Problem characteristics

- Assumption: $f^i_{\texttt{on}}$ strictly negative, $f^i_{\texttt{off}}$ strictly positive




- Aggregate system: $N$ states and $2^N$ modes, but very structured
- Global counting constraints vs. local safety constraints
- Either type of constraints is trivial to satisfy on its own

# Related work

- Control strategy but no guarantees
  - H Hao et al. (2015). "Aggregate Flexibility of Thermostatically Controlled Loads". In: *IEEE Trans Power Syst* 30.1, pp. 189–198
- Schedulability conditions in linear case
  - T X Nghiem et al. (2011). "Green scheduling of control systems for peak demand reduction". In: *Proc. IEEE CDC*, pp. 5131–5136
- More general state constraints (polyhedron), less general dynamics (constant rate); LP to find schedule
  - R Alur et al. (2013). "Safe schedulability of bounded-rate multi-mode systems". In: *Proc. HSCC*

## Our contribution

Closed-form necessary and (almost) sufficient schedulability conditions in monotone, nonlinear and heterogeneous case, plus associated control strategy.

# Outline

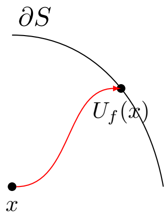
# Time to exit, general case

## Definition

Given a set $S$, for $x \in S$ and some $f$, the **time to exit** $T_f(x)$ is the time it takes for the flow of $f$ starting in $x$ to reach $\partial S$:

$$T_f(x) = \inf \{\tau : \phi_f(x, \tau) \in \partial S\}.$$

For $x \in S$ and $f$, the **exit point** $U_f(x)$ is $U_f(x) = \phi_f(x, T_f(x))$.



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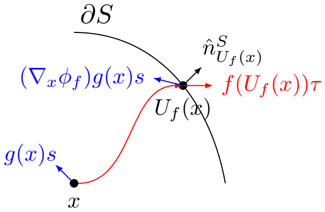


## Proposition

The Lie derivative of $T_f(x)$ with respect to $g$, $\mathcal{L}_g T_f(x)$, is

$$-\frac{\left(\hat{n}^S_{U_f(x)}\right)^T \left(\nabla_x \phi_f(x, T_f(x))\right) g(x)}{\left(\hat{n}^S_{U_f(x)}\right)^T f(U_f(x))}.$$

# Time to exit for 1D TCL system

Easy to show that $\mathcal{L}_{f^i_{\mathtt{off}}} T_{f^i_{\mathtt{off}}}(x) = -1$ and $\mathcal{L}_{f^i_{\mathtt{on}}} T_{f^i_{\mathtt{on}}}(x) = -1$.

## Proposition

$$\mathcal{L}_{f^i_{\mathtt{off}}} T_{f^i_{\mathtt{on}}}(x) = -\frac{f^i_{\mathtt{off}}(x)}{f^i_{\mathtt{on}}(x)}, \quad \mathcal{L}_{f^i_{\mathtt{on}}} T_{f^i_{\mathtt{off}}}(x) = -\frac{f^i_{\mathtt{on}}(x)}{f^i_{\mathtt{off}}(x)}.$$

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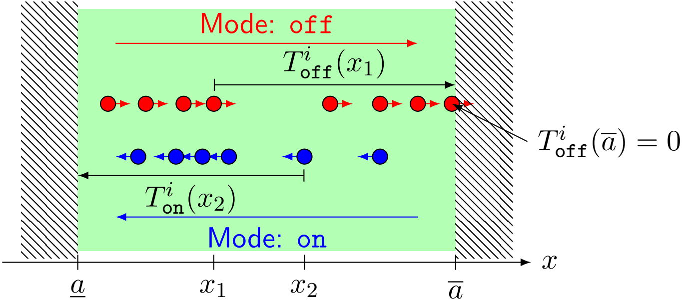

## Proposed strategy

Switch subsystem $i$ at a time instant $t$ if one of the following conditions occur:

1. If $T^i_{\mathtt{off}}(x^i(t)) = 0$, switch subsystem $i$ to on,
2. If $T^i_{\mathtt{on}}(x^i(t)) = 0$, switch subsystem $i$ to $\mathtt{off}$,
3. If $\sum_{i:\ \sigma^i(t^+)=\mathtt{on}} 1 > \overline{K}_{\mathtt{on}}$ for $t^+ > t$, select the subsystem $j$ in mode on with the largest time to $\mathtt{off}$-exit, i.e.
$$j = \underset{k\,:\,\sigma^k(t)=\mathtt{on}}{\arg\max}\ T^k_{\mathtt{off}}(x^k(t)),$$
and switch it to $\mathtt{off}$. If the bound is still violated at $t^+$, repeat step 3.
4. If $\sum_{i:\ \sigma^i(t^+)=\mathtt{on}} 1 < \underline{K}_{\mathtt{on}}$ for $t^+ > t$, do the dual of 3.

# Strategy illustration

1. If $T^i_{\texttt{off}}(x^i(t)) = 0$, switch subsystem $i$ to on,
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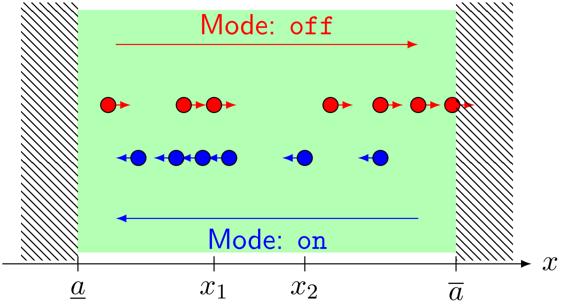

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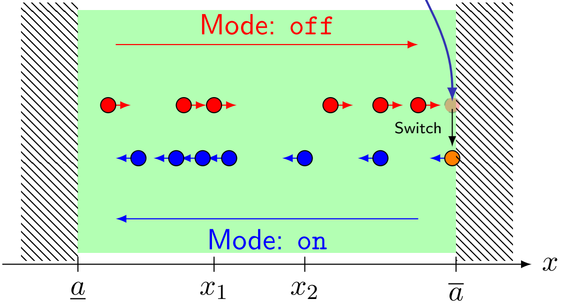

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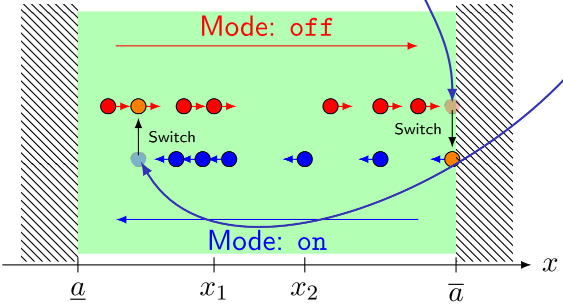

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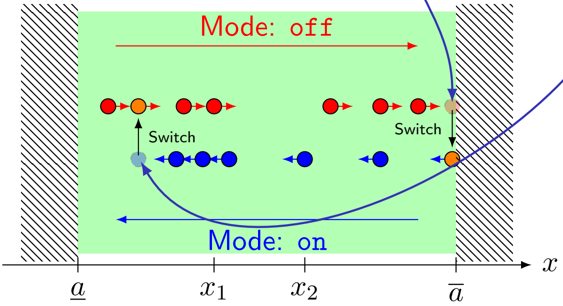



- Time to exit way to assess the imminence of constraint violation in heterogeneous collection

# Theoretical results

$$\sum_{i=1}^{N} \frac{\mathcal{L}^i_{\texttt{off}} T^i_{\texttt{on}}(\underline{a}^i)}{1+\mathcal{L}^i_{\texttt{off}} T^i_{\texttt{on}}(\underline{a}^i)} > \underline{K}_{\texttt{on}}, \quad \sum_{i=1}^{N} \frac{\mathcal{L}^i_{\texttt{on}} T^i_{\texttt{off}}(\overline{a}^i)}{1+\mathcal{L}^i_{\texttt{on}} T^i_{\texttt{off}}(\overline{a}^i)} > N - \overline{K}_{\texttt{on}} \qquad (3)$$

## Theorem

*If* (3) *holds, then the proposed strategy solves the problem for any non-degenerate initial condition.*

## Assumption 1

The functions $f^i_{\texttt{on}}$ and $f^i_{\texttt{off}}$ are monotonically decreasing in $[\underline{a}^i, \overline{a}^i]$.

## Theorem

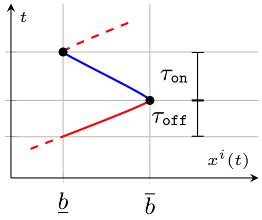
*Under Assumption 1; if* (3) *is strictly violated, then the problem has no solution for any initial condition.*

# Proof sketch

- Consider lower boundaries $\underline{a}^i$: "need to keep few systems in on but at least $\underline{K}_{\mathtt{on}}$"
- What happens during a "cycle" $\underline{b} \to \overline{b} \to \underline{b}$?

**Lemma**

$$\frac{\tau_{\mathtt{on}}}{\tau_{\mathtt{on}} + \tau_{\mathtt{off}}} \leq \frac{\mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}\left(\underline{b}\right)}{1 + \mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}\left(\underline{b}\right)}$$



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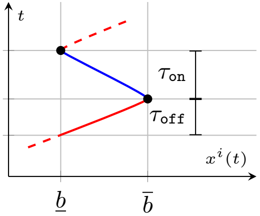
duty cycle monot. decreasing




- Necessity: if $\sum_{i=1}^N \frac{\mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}(\underline{a}^i)}{1+\mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}(\underline{a}^i)} < \underline{K}_{\mathtt{on}}$, aggregate duty cycles smaller than allowed by $\underline{K}_{\mathtt{on}}$
- Sufficiency: strategy selects systems such that Zeno behavior at $\underline{a}^i$ contradicts $\sum_{i=1}^N \frac{\mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}(\underline{a}^i)}{1+\mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}(\underline{a}^i)} > \underline{K}_{\mathtt{on}}$

# Sufficient conditions in more general settings

- With uncertainty

$$\sum_{i=1}^{N} \min_{d^i \in D^i} \frac{f^i_{\mathtt{off}}(\underline{a}^i, d^i)}{-f^i_{\mathtt{on}}(\underline{a}^i, d^i) + f^i_{\mathtt{off}}(\underline{a}^i, d^i)} > \underline{K}_{\mathtt{on}}, \text{ and,}$$

$$\sum_{i=1}^{N} \min_{d^i \in D^i} \frac{f^i_{\mathtt{on}}(\overline{a}^i, d^i)}{-f^i_{\mathtt{on}}(\overline{a}^i, d^i) + f^i_{\mathtt{off}}(\overline{a}^i, d^i)} > N - \overline{K}_{\mathtt{on}}.$$

- Subsystems are $> 1$D

$$\sum_{i=1}^{N} \min_{\underline{a}^i \in \partial S^i_{\mathtt{off}}} \frac{\mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}\left(\underline{a}^i\right)}{1 + \mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}\left(\underline{a}^i\right)} > \underline{K}_{\mathtt{on}}, \text{ and,}$$

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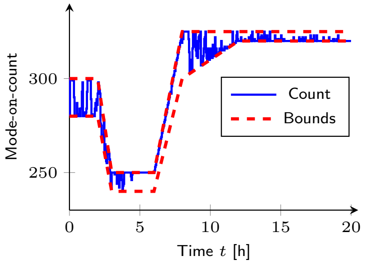
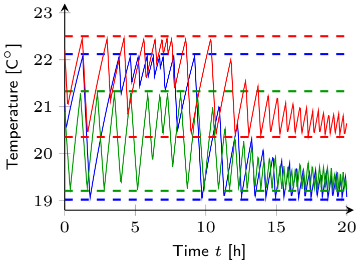
# Simulation of 1,000 TCL's

- TCL dynamics: $\frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = -a(\theta_i(t) - \theta_a) - bP_m \times \mathbb{1}_{\{\texttt{on}\}}(\sigma_i(t))$
- 1,000 subsystems with randomly sampled parameters
- Analytical bounds: $\underline{K}_{\texttt{on}} \leq 323$ and $\overline{K}_{\texttt{on}} \geq 250$
  - Can track any signal taking values in $[250, 323]$



# Outline

# Maximizing the time of invariance

- We characterized (almost) exactly the situations when the problem has an infinite-horizon solution
- Resulted in aggregate flexibility $(323 - 250)/1000 \approx 7\%$
- In practice aggregate flexibility upwards of 60% may be required but for short durations

## Extended TCL Mode-Counting Problem

- Local (heterogeneous) state constraints:

$$\underline{a}^i \leq x^i(t) \leq \overline{a}^i \quad \forall t \in [0, T_I] \tag{4}$$

- Global counting constraint:

$$\underline{K}_{\mathtt{on}} \leq \sum_{i:\ \sigma^i(t)=\mathtt{on}} 1 \leq \overline{K}_{\mathtt{on}} \quad \forall t \in [0, T_I] \tag{5}$$

- Objective: find switching strategy $\{\sigma^i(\cdot)\}_{i=1}^N$ that enforces (4)-(5) and s.t. $T_I$ is maximized

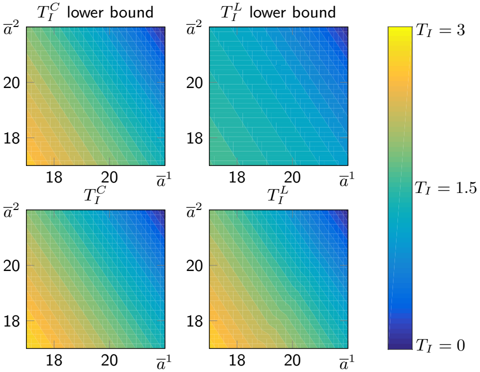
- **Time of invariance** $T_I$ will be a function of initial condition
- Results in this work identify when $T_I = +\infty$

# Problem approach

- Method to get guaranteed lower bounds on time of invariance for two strategies
  - Strategy $C$: "fast-switching" strategy useful in analysis
  - Strategy $L$: variant of "lazy-switching" strategy presented earlier
- Bounds obtained by re-formulating as an optimal control problem and using an analytical lower estimate of the value function

# Results

- Illustrations for 2-D system




- Large-scale numerical example with Strategy $L$:
  - Guaranteed time of invariance $0.86h$
  - Achieved time of invariance: $0.96h$
  - Time of invariance with temperature-driven switching (IEEE Power Syst. 30.1): $0.67h$

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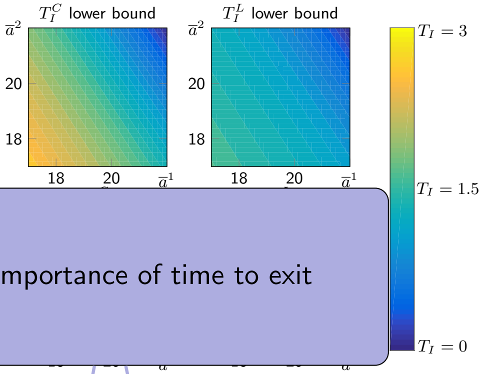


Shows importance of time to exit

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# Summary

- Solved the TCL mode-counting problem in the infinite-horizon case
  - Maximal controlled invariant set is either empty or equal (up to closure) to constraint set
  - Closed-form solution, no need for set computations: applicable to arbitrary number of subsystems
  - Time to exit as unifying measure for heterogeneous collection
- Current work: approximately maximal solutions with guarantees when infinite-horizon problem lacks solution
- Future work: use theoretical results to make informed decisions in high-level load distribution algorithm, tight conditions for problem generalizations

Thank you for your attention