Lie Theory for Robotics

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0.1. New material/Potential papers

1. Pose graph marginalization
2. Monotonicity and implicit invariance on Lie groups
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    - Lie group PD controller (use linearization)
    - Lie group MPC
    - Lie group EKF as generalization of IEKF
    - Optimization and group-linearity? Is spline fitting linear?

0.2. Literature

- BOOKS:
    - Barfoot: [3]
        * Estimation in robotics
        * Some explicit formulas
    - Chirkijan: [9, 8]
        * Guassians and information theory
    - Agrachev: [1]
    - Marsden: [21]
- ARTICLES
    - Very basic Lie theory [16], rigorous mathematics.
    - A Micro Lie Theory [24], application-focused.
    - Quadrotor control [18]
    - IMU estimation: [17, 19]

Part I.

Theory

1. Introduction

Summary

- Treat parameterizations as regular groups
- Get rid of $\check{\mathbb{M}}$, define exp and log $\check{\mathfrak{m}} \leftrightarrow \mathbb{M}$
- Get rid of hat and vee for parameterizations
- Overview of notes.
- Advantages of on-manifold tools.
- Applications of Lie theory in robotics.

1.1. Numerical integration

1.2. Nonlinear control and estimation

1.3. Localization

1.4. Notation

	Set notation	Element notation
Group (matrix form)	$\mathbb{M}, \mathbb{N}$	$\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$
Group (param. form)	$\check{\mathbb{M}}, \check{\mathbb{N}}$	$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$
Algebra (matrix form)	$\mathfrak{m}, \mathfrak{n}$	$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$
Algebra (param. form)	$\check{\mathfrak{m}}, \check{\mathfrak{n}}$	$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$
Rotation matrices		$\boldsymbol{R}$
Rotation parameters		$\boldsymbol{q} = [q_w, q_x, q_y, q_z]$
Velocity parameters		$\boldsymbol{v} = [v_x, v_y, v_z]$
Translation parameters		$\boldsymbol{p} = [p_x, p_y, p_z]$
Angular velocity		$\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]$
Vectors $\mathbb{R}^n$		$\boldsymbol{u}$

2. Lie Groups

Summary

- Fundamental definitions and properties.
- Matrix Lie groups that appear in robotics.

A Lie group is an object that is both a group and a smooth manifold. As will be illustrated in these notes, inheritence of these two sets of properties places Lie groups at a unique point where theory meets practice.

2.1. Fundamentals

We recall the definitions of groups and smooth manifolds, respectively.

Definition 2.1 ([11]). *A **group** $(\mathbb{M}, \circ)$ is a set $\mathbb{M}$ closed under a binary operation $(\circ)$ such that*

- ***associativity** holds: $\boldsymbol{X} \circ (\boldsymbol{Y} \circ \boldsymbol{Z}) = (\boldsymbol{X} \circ \boldsymbol{Y}) \circ \boldsymbol{Z}$ for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{M}$,*
- *there is an **identity element** $e \in \mathbb{M}$ s.t. $e \circ \boldsymbol{X} = \boldsymbol{X} \circ e$ for all $\boldsymbol{X} \in \mathbb{M}$,*
- *for each element $\boldsymbol{X} \in \mathbb{M}$ there is an **inverse** $\boldsymbol{X}^{-1} \in \mathbb{M}$ s.t. $\boldsymbol{X}^{-1} \circ \boldsymbol{X} = \boldsymbol{X} \circ \boldsymbol{X}^{-1} = e$.*

Definition 2.2 ([7]). *A **smooth manifold** $(\mathbb{M}, \{c_i\})$ of dimension n is a set $\mathbb{M}$ and a family of injective mappings $c_i : U_i \subset \mathbb{R}^n \to \mathbb{M}$ of open sets U_i (called **charts**) such that*

1. *The charts cover the set: $\bigcup_i U_i = \mathbb{M}$,*
2. *For any pair i, j with $c_i(U_i) \cap c_j(U_j) =: W \neq \emptyset$, the sets $c_i^{-1}(W)$ and $c_j^{-1}(W)$ are open in $\mathbb{R}^n$, and the mappings $c_i^{-1} \circ c_j$ are differentiable.*

Figure of chart mappings

The definition of a Lie group is now straightforward.

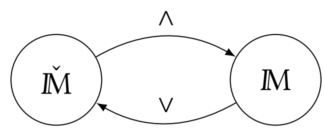


Figure 2.1.: The $\vee$ (hat) and $\wedge$ (vee) maps map between the matrix and parameter forms of a matrix Lie group.

Definition 2.3. *A* ***Lie group*** *of dimension n is a set $\mathbb{M}$ together with a binary operation $(\circ)$ and a family of injective mappings $c_i : U_i \subset \mathbb{R}^n \to X$ such that*

1. $(\mathbb{M}, \circ)$ *is a group,*
2. $(\mathbb{M}, \{c_i\})$ *is an n-dimensional smooth manifold.*

In the following we use $\mathbb{M}$ to refer both to the Lie group and to its underlying set. A mathematic object that satisfies the properties in Definition 2.3 is the **general linear group** $GL(n, \mathbb{C})$—the set of $n \times n$ invertible complex matrices with matrix multiplication $(\cdot)$ as the group operation. It turns out that many Lie groups of practical interest can be represented as sub-groups of $GL(n, \mathbb{C})$. In these notes we focus exclusively on Matrix Lie groups, and Lie groups that are isomorphic to Matrix Lie groups.

Definition 2.4. *A* ***matrix Lie group*** *is a Lie group that is also a sub-group of $GL(n, \mathbb{C})$—the group of invertible matrices with complex coefficients.*

Lie theory is more straightforward to develop for matrix Lie groups compared to a more general setting. Matrix Lie groups are however inefficient from a practical point of view since their representation is often redundant. For this first part of the book we focus exclusively on matrix Lie groups. In Part II we use isometries between matrix Lie groups and more concice non-matrix Lie groups to obtain closed-form formulas for the latter.

3. Lie Algebras

Summary

- Fundamental definitions and properties of Lie Algebras.
- The Lie Bracket.
- Hat and vee operators.
- Maybe: connection to Lie Derivative.

3.1. Lie Algebra definition

Definition 3.1. *A **Lie Algebra** is a vector space $\mathfrak{m}$ with a binary relation $[\cdot,\cdot] : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ called the **Lie bracket** that satisfies*

1. *Bilinearity:* $[\boldsymbol{A}, \beta\boldsymbol{B} + \gamma\boldsymbol{C}] = \beta[\boldsymbol{A}, \boldsymbol{B}] + \gamma[\boldsymbol{A}, \boldsymbol{C}]$, *and* $[\alpha\boldsymbol{A} + \beta\boldsymbol{B}, \boldsymbol{C}] = \alpha[\boldsymbol{A}, \boldsymbol{C}] + \beta[\boldsymbol{B}, \boldsymbol{C}]$,
2. $[\boldsymbol{A}, \boldsymbol{A}] = 0$,
3. *Jacobi's identity:* $[\boldsymbol{A}, [\boldsymbol{B}, \boldsymbol{C}]] + [\boldsymbol{C}, [\boldsymbol{A}, \boldsymbol{B}]] + [\boldsymbol{B}, [\boldsymbol{C}, \boldsymbol{A}]] = 0$.

3.2. The Lie bracket

Jacobis identity

Clean this up

We have the two flows $\phi^f(t, x)$ and $\phi^g(t, x)$ that are such that

$$\begin{aligned} \phi^f(0, x) &= x, \quad \frac{\partial}{\partial t}\phi^f(t, x) = f(\phi^f(t, x)), \\ \phi^g(0, x) &= x, \quad \frac{\partial}{\partial t}\phi^g(t, x) = g(\phi^g(t, x)). \end{aligned} \tag{3.1}$$

Consequently, we get the second derivative

$$\left[\frac{\partial^2}{\partial t^2}\phi^f(t,x)\right]_i = \frac{\partial\left[f(\phi^f(t,x)\right]_i}{\partial t} = \left.\frac{\mathrm{d}f_i}{\mathrm{d}x_j}\right|_{x_j=[\phi^f(t,x)]_j}\frac{\partial}{\partial t}\left[\phi^f(t,x)\right]_j$$
$$= \left.\frac{\mathrm{d}f_i}{\mathrm{d}x_j}\right|_{x_j=[\phi^f(t,x)]_j}\left[f(\phi^f(t,x))\right]_j. \tag{3.2}$$

Thus we get

$$\frac{\partial^2}{\partial t^2}\phi^f(t,x) = \left.(f\cdot\nabla)\,f\right|_{\phi^f(t,x)}. \tag{3.3}$$

We are interested in the quantity

$$\phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\circ\phi^g(t,\cdot)\circ\phi^f(t,\cdot)\,[x] \tag{3.4}$$

for small t.

Note that by Taylor expansion,

$$\begin{aligned}\phi^f(t,x) &= \phi^f(0,x) + \left.\frac{\partial}{\partial t}\phi^f(t,x)\right|_{t=0} t + \left.\frac{\partial^2}{\partial t^2}\phi^f(t,x)\right|_{t=0}\frac{t^2}{2} + \mathcal{O}(t^3)\\ &= \phi^f(0,x) + t\,f(\phi^f(0,x)) + \frac{t^2}{2}\left(f(\phi^f(0,x))\cdot\nabla\right)f(\phi^f(0,x)) + \mathcal{O}(t^3)\\ &= x + t\,f(x) + \frac{t^2}{2}\left(f(x)\cdot\nabla\right)f(x) + \mathcal{O}(t^3).\end{aligned} \tag{3.5}$$

We also have that

$$g(x+t\alpha) = g(x) + t(\alpha\cdot\nabla)g(x) + \mathcal{O}(t^2). \tag{3.6}$$

Then we get, after omitting the (x) in $f(x)$ and $g(x)$,

$$
\begin{aligned}
&\phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\circ\phi^g(t,\cdot)\circ\phi^f(t,\cdot)\,[x] = \\
&\quad= \phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\circ\phi^g(t,\cdot)\left[x + f\,t + \frac{t^2}{2}(f\ \cdot\nabla) f\ + \mathcal{O}(t^3)\right] \\
&\quad= \phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\Big[x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3) \\
&\qquad + \underbrace{t\,g\left(x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3)\right)}_{g\ +t(f\cdot\nabla)g+\mathcal{O}(t^2)} \\
&\qquad + \frac{t^2}{2}\nabla g\left(x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3)\right) \\
&\qquad\qquad \cdot g\left(\left(x + tf + \frac{t^2}{2}(f\cdot\nabla)\,f + \mathcal{O}(t^3)\right)\right) + \mathcal{O}(t^3)\Big] \\
&\quad= \phi^g(-t,\cdot)\circ\phi^f(-t,\cdot)\left[x + t\{f + g(x)\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right] \\
&\quad= \phi^g(-t,\cdot)\Big[x + t\{f + g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3) \\
&\qquad - tf\left(x + t\{f + g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right) \\
&\qquad + \frac{t^2}{2}\nabla f\left(x + t\{f + g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right) \\
&\qquad\qquad \cdot f\left(x + t\{f + g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3)\right) \\
&\quad= \phi^g(-t,\cdot)\left[x + t\{g\} + t^2\left\{\frac{1}{2}(f\cdot\nabla)\,f + (f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - \nabla f\cdot(f + g) + \frac{1}{2}(f\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right] \\
&\quad= \phi^g(-t,\cdot)\left[x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right] \\
&\quad= x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3) \\
&\qquad - tg\left(x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right) \\
&\qquad + \frac{t^2}{2}\nabla g\left(x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right) \\
&\qquad\qquad \cdot g\left(x + t\{g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f\right\} + \mathcal{O}(t^3)\right) \\
&\quad= x + t\{g - g\} + t^2\left\{(f\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g - (g\cdot\nabla)\,f - (g\cdot\nabla)\,g + \frac{1}{2}(g\cdot\nabla)\,g\right\} + \mathcal{O}(t^3) \\
&\quad= x + t^2\{(f\cdot\nabla)\,g - (g\cdot\nabla)\,f\} + \mathcal{O}(t^3) \\
&\quad= x + t^2[f,g](x) + \mathcal{O}(t^3).
\end{aligned}
$$

3.3. Application: Derive the Laguerre polynomials

This is an exercise from [16].

Consider the equation

$$xy'' + (1-x)y' + ny = 0,$$

we will show via Lie-algebraic concepts that a solution is given by

$$y = e^x \left(\frac{d}{dx}\right)^n e^{-x} x^n.$$

Letting $P = d/dx$ denote derivative and $Q = x$ multiplication by x the equation can be written

$$Ly = (P-I)QPy = -ny.$$

We consider the Lie algebra spanned by P, Q, I with commutator relationships

$$\begin{aligned}[P,Q]y = PQy - QPy = y + xy' - xy' = Iy, \quad \implies [P,Q] = I \\ [P,I] = [P,Q] = 0.\end{aligned}$$

We have from the bracket relation that $(P-I)Q = I + Q(P-I)$, consequently

$$\begin{aligned}[Q,(P-I)^n] &= Q(P-I)^n - (P-I)^n Q \\ &= \left(Q(P-I)^{n-1} - (P-I)^{n-1}Q\right)(P-I) - (P-I)^{n-1} \\ &= [Q,(P-I)^{n-1}](P-I) - (P-I)^{n-1}.\end{aligned}$$

From $[Q, P-I] = -I$ it follows by recursion that

$$[Q,(P-I)^n] = -n(P-I)^{n-1}.$$

Let $A_n = (P-I)^n Q^n$, then with the above

$$\begin{aligned}A_{n+1} &= (P-I)^{n+1}Q^{n+1} = (P-I)\left([(P-I)^n, Q] + Q(P-I)^n\right)Q^n \\ &= (P-I)\left\{n(P-I)^{n-1} + Q(P-I)^n\right\}Q^n = (n + A_1)A_n.\end{aligned}$$

Note that we have $L = A_1 P$ and that $PA_1 = P(P-I)Q = (P-I)QP + (P-I) = A_1P + (P-I)$. It follows that

$$[A_1P, A_1] = A_1PA_1 - A_1^2 P = A_1(P-I) = L - A_1.$$

Using the bracket relation it follows that

$$L(A_1+n) = (A_1+n)L + [L, A_1+n] = (A_1+n)L + (L+n) - (A_n+n).$$

Proposition: If v_n is an eigenvector of L with eigenvalue $-n$, then $(A_1+n)v_n$ is an eigenvector with eigenvalue $-(n+1)$. **Proof**: We use the relation above to get

$$\begin{aligned}L(A_1+n)v_n &= (A_1+n)Lv_n + (L+n)v_n - (A_n+n)v_n \\ &= -n(A_1+n)v_n - (A_n+n)v_n.\end{aligned}$$

It follows via the relation $A_{n+1} = (A_1+n)A_n$ shown above that if v_0 is an eigenvector with eigenvalue 0, then $A_n v_0$ is an eigenvector with eigenvalue $-n$.

We have solved $Ly = -ny$, a solution is for instance

$$A_n v_0 = (P-I)^n Q^n 1 = \left(e^x \frac{d}{dx} e^{-x}\right)^n x^n = e^x \left(\frac{d}{dx}\right)^n e^{-x} x^n.$$

3.3.1. Hermite polynomials

Consider the equation

$$y'' + xy' - ny = 0.$$

We show that

$$y = e^{-x^2/2}\left(\frac{d}{dx}\right)^n e^{x^2/2}$$

is a solution.

Also the operators $P = d/dx$ and $Q = x + d/dx$ satisfy the same operations, in partiuclar $[P, Q] = I$. We have that

$$Ly = QPy = y'' + xy',$$

so we would like to solve

$$QPv = nv.$$

This is easy: suppose that $QPQ^{n-1}v_0 = (n-1)Q^{n-1}v_0$ which is true for $v_0 = 1$ at $n = 1$. Then,

$$\begin{aligned} QPQ^n v_0 &= QPQQ^{n-1}v_0 = Q([P,Q] + QP)Q^{n-1}v_0 \\ &= Q^n v_0 + Q^2 PQ^{n-1}v_0 = Q^n + (n-1)Q^n v_0 = nQ^n v_0. \end{aligned}$$

Thus it follows that the solution is $Q^n 1$, and using that

$$Q = e^{-x^2/2}\frac{d}{dx}e^{x^2/2}$$

the answer is obtained.

4. The Exponential Map

Summary

- The Exponential map and how it connects a Lie group to its Lie algebra.
- The Lie group logarithm, plus and minus operators.
- The structure of the Lie algebras corresponding to common Lie groups.

Need a nice derivation showing how lie algebra properties arise

4.1. One-Parameter Groups

Best way to prove that Lie Groups have Lie Algebras?

- In [16] it is shown that for **matrix** Lie groups the set $\{A \in \mathrm{End}V \ : \ \exp tA \in G \forall t\} = \cap_t t \exp^{-1}(G)$ is a Lie Algebra (i.e. closed under the bracket operation).

Dual viewpoint: solutions $\Phi(x,t)$ of ODEs correspond to one-parameter groups [16].
Connection to linear systems.

4.2. The Exponential Map

Definition 4.1. *The **Exponential map** of a matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $t \in \mathbb{R}$ is*

$$\mathrm{Exp}(\boldsymbol{A}) = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n}{n!} \in \mathbb{C}^{n\times n}. \tag{4.1}$$

Properties of the exponential map

For the exponential map in Definition 4.1 we have

$$\mathrm{Exp}(t\boldsymbol{A})\,\mathrm{Exp}(s\boldsymbol{A}) = \mathrm{Exp}((t+s)\boldsymbol{A}), \tag{4.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Exp}(t\boldsymbol{A}) = \boldsymbol{A}\,\mathrm{Exp}(t\boldsymbol{A}) = \mathrm{Exp}(t\boldsymbol{A})\boldsymbol{A}, \tag{4.2b}$$

$$\det(\mathrm{Exp}(\boldsymbol{A})) = e^{\mathrm{Tr}(\boldsymbol{A})}. \tag{4.2c}$$

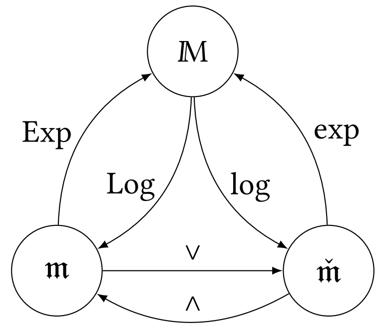


Figure 4.1.: Illustration of how the exponential maps connect a Lie Group $\mathbb{M}$, its Lie Algebra $\mathfrak{m}$, and the Lie Algebra parameterization which is the linear space $\check{\mathfrak{m}} \cong \mathbb{R}^n$.

The first two follow directly from the definition and are analogous to the scalar exponential map. Furthermore, (4.2a) implies that $\{\mathrm{Exp}(t\boldsymbol{A}) : t \in \mathbb{R}\}$ is a one-parameter subgroup of $\mathbb{M}$.

Not however that in general $\exp(\boldsymbol{A} + \boldsymbol{B}) \neq \exp(\boldsymbol{A}) \circ \exp(\boldsymbol{B})$ which is different from the scalar version. Equation (4.2c), known as Jacobi's identity, motivates a short proof:

Proof of (4.2c). It is easy to see that the eigenvalues of $\mathrm{Exp}(\boldsymbol{A})$ are the exponentials of the eigenvalues of $\boldsymbol{A}$. Since the determinant equals the product of the eigenvalues it follows that

$$\det(\mathrm{Exp}\,\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i(\mathrm{Exp}\,\boldsymbol{A}) = \prod_{i=1}^{n} e^{\lambda_i(\boldsymbol{A})} = e^{\sum_{i=1}^{n} \lambda_i(\boldsymbol{A})} = e^{\mathrm{Tr}(\boldsymbol{A})}. \tag{4.3}$$

□

4.2.1. Modern Definition

We contrast the algebraic Definition 4.1 with a more modern definition usually found in texts on differential geometry.

Definition 4.2. *The* ***exponential*** *of $\boldsymbol{A} \in \mathfrak{m}$ is*

$$\mathrm{Exp}\,\boldsymbol{A} \coloneqq \gamma(1), \tag{4.4}$$

where γ is a one-parameter subgroup of $\mathbb{M}$ such that $\gamma'(0) = \boldsymbol{A}$.

This definition of course works even in the case that $\mathbb{M}$ is not a matrix Lie group, in fact it the same definition is used in more general differential geometry. We show that it coincides with Definition 4.1 for matrix Lie groups.

Proof. We know that $\gamma(0) = I$, $\gamma'(0) = \boldsymbol{A}$, and $\gamma(s)\gamma(t) = \gamma(s+t)$ by virtue of γ being a one-parameter sub-group.

$$\left(\gamma\left(\frac{h}{2}\right) - \gamma\left(-\frac{h}{2}\right)\right)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\gamma\left(\frac{h}{2}\right)^{n-k}\gamma\left(-\frac{h}{2}\right)^k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\gamma\left(\left(\frac{n}{2}-k\right)h\right). \tag{4.5}$$

For $h \to 0$ the left-hand side goes to $(h\gamma'(0))^n$, whereas the right-hand side is a finite-difference approximation of $h^n \gamma^{(n)}(0)$. It follows that $\gamma^n(0) = \boldsymbol{A}^n$, and hence a Taylor expansion around 0 gives

$$\gamma(1) = \sum_{k\geq 0} \frac{\gamma^{(n)}(0)(1-0)^n}{n!} = \sum_{k\geq 0} \frac{\boldsymbol{A}^n}{n!}. \tag{4.6}$$

□

A consequence of the proof above is that one-parameter subgroups of Lie groups are uniquely defined by their derivative at zero, and are therefore analogous to geodesics in Riemannian geometry.

4.3. The Lie Algebra of a Lie group

Definition 4.3. *For a matrix Lie group $\mathbb{M}$ the corresponding **matrix Lie algebra** $\mathfrak{m}$ is*

$$\mathfrak{m} = \{\boldsymbol{A} : \operatorname{Exp}(t\boldsymbol{A}) \in \mathbb{M}\ \forall t \in \mathbb{R}\}. \tag{4.7}$$

Just as for Lie groups, the matrix Lie algebras are typically parameterized by fewer than n^2 coefficients. In order to work with efficient parameterizations we therefore introduce a lower-dimensional parameterization denoted $\check{\mathfrak{m}}$. For this lower-dimensional representation we also define a lowercase exponential that maps from the parameterized lie algebra representation $\check{\mathfrak{m}}$ to the parameterized group representation $\check{\mathbb{M}}$:

$$\exp(\boldsymbol{a}) = \operatorname{Exp}(\boldsymbol{a}^\wedge). \tag{4.8}$$

The relationshop between the exponential maps and the hat and vee maps is shown in Figure 4.1.

Show that Lie algebra defined like this is indeed a Lie algebra(closed under bracket, jacobi, etc). Use property from previous chapter to show that as $t \to 0$ we obtain a tangent that is equal to the bracket. Group property is then enough to conclude.

4.4. The Logarithm

The matrix logarithm $\operatorname{Log} : \mathbb{M} \to \mathfrak{m}$ is defined as the inverse of the matrix exponential, and we also define lowercase $\log : \mathbb{M} \to \check{\mathfrak{m}}$ for mappings between the parameterized representations:

$$\begin{aligned} \operatorname{Log} \boldsymbol{X} &= \sum_{k\geq 1} (-1)^{k+1} \frac{(\boldsymbol{X}-I)^k}{k}, \\ \log \boldsymbol{X} &= (\operatorname{Log} \boldsymbol{X})^\vee. \end{aligned} \tag{4.9}$$

Show that it's the inverse of the exponential

4.5. Plus and Minus Operators

Algorithms for optimization and numerical integration require taking small additative steps, but Lie groups are not closed under normal addition and subtraction. We can however define generalized addition and subtraction operators $\oplus, \ominus$ for Lie groups that behave similarly to how $+$ and $-$ operate on regular vector spaces.

The plus operations add an increment $\boldsymbol{a} \in \check{\mathfrak{m}}$ in the parameterized tangent space to an element $\boldsymbol{X} \in \mathbb{M}$ of the group, whereas the minus operators give the difference between two group elements as a vector in the parameterized tangent space.

$$\boldsymbol{X} \oplus_r \boldsymbol{a} = \boldsymbol{X} \circ \exp(\boldsymbol{a}) \in \check{\mathbb{M}}, \tag{right-plus}$$
$$\boldsymbol{Y} \ominus_r \boldsymbol{X} = \log\left(\boldsymbol{X}^{-1} \circ \boldsymbol{Y}\right) \in T_{\boldsymbol{X}}\check{\mathbb{M}} \cong \check{\mathfrak{m}}, \tag{right-minus}$$
$$\boldsymbol{a} \oplus_l \boldsymbol{X} = \exp(\boldsymbol{a}) \circ \boldsymbol{X} \in \check{\mathbb{M}}, \tag{left-plus}$$
$$\boldsymbol{Y} \ominus_l \boldsymbol{X} = \log\left(\boldsymbol{Y} \circ \boldsymbol{X}^{-1}\right) \in T_e\check{\mathbb{M}} \cong \check{\mathfrak{m}}. \tag{left-minus}$$

The plus operators are differentiated by the order: the right-plus has the tangent element at $\boldsymbol{X}$ while left-plus has the reverse order, meaning that the tangent element belongs to the tangent space at e.

Note that the derivates are defined in a way so that

$$\boldsymbol{X} \oplus_r (\boldsymbol{Y} \ominus_r \boldsymbol{X}) = \boldsymbol{X} \circ \exp\log(\boldsymbol{X}^{-1}\boldsymbol{Y}) = \boldsymbol{Y}, \tag{4.10a}$$
$$(\boldsymbol{X} \oplus_r \boldsymbol{a}) \ominus_r \boldsymbol{X} = \log(\boldsymbol{X}^{-1} \circ \boldsymbol{X} \circ \exp \boldsymbol{a}) = \boldsymbol{a}, \tag{4.10b}$$
$$(\boldsymbol{Y} \ominus_l \boldsymbol{X}) \oplus_l \boldsymbol{X} = \exp\log(\boldsymbol{Y} \circ \boldsymbol{X}^{-1}) \circ \boldsymbol{X} = \boldsymbol{Y}, \tag{4.10c}$$
$$(\boldsymbol{a} \oplus_l \boldsymbol{X}) \ominus_l \boldsymbol{X} = \log(\exp \boldsymbol{a} \circ \boldsymbol{X} \circ \boldsymbol{X}^{-1}) = \boldsymbol{a}. \tag{4.10d}$$

4.6. Homomorphy of Lie Groups implies Homomorphy of Lie Algebras

This is important since it implies that $SO(3)$ and S^3 can be treated analogously. A proof is in [16, Corr. 20].

4.7. The Adjoint

We define the **adjoint** $\mathrm{Ad}_{\boldsymbol{X}} : \mathfrak{m} \to \mathfrak{m}$ of a matrix $\boldsymbol{A} \in \mathfrak{m}$ as

$$\mathrm{Ad}_{\boldsymbol{X}} \boldsymbol{A} := \boldsymbol{X}\boldsymbol{A}\boldsymbol{X}^{-1}. \tag{4.11}$$

From the definition of the exponential map in (4.1) it can be seen that $\mathrm{Exp}(\mathrm{Ad}_{\boldsymbol{X}} \boldsymbol{A}) \in \mathbb{M}$ if and only if $\mathrm{Exp}(\boldsymbol{A}) \in \mathbb{M}$, which implies that the lie algebra $\mathfrak{m}$ is closed under action of the adjoint.

The adjoint of a tangent matrix element $\boldsymbol{a} \in \check{\mathfrak{m}}$ is similary defined as a linear mapping $\mathbf{Ad}_X : \check{\mathfrak{m}} \to \check{\mathfrak{m}}$:

$$\mathbf{Ad}_X\, \boldsymbol{a} := (\mathrm{Ad}_X\, \hat{\boldsymbol{a}})^\vee = (\boldsymbol{X}\hat{\boldsymbol{a}}\boldsymbol{X}^{-1})^\vee. \tag{4.12}$$

For a given $\boldsymbol{X}$ this is a linear map, so $\mathbf{Ad}_X$ is an $n \times n$ matrix. The adjoint represents a coordinate change from the tangent space $T_X\check{\mathbb{M}}$ to the tangent space $T_e\check{\mathbb{M}} = \check{\mathfrak{m}}$ at the origin.

Remark 4.1. *Since the definition of* Ad *involves matrix multiplication it does not make sense for groups like* $\check{\mathbb{SO}}(2)$ *and* $\mathbb{S}^3$ *that are not matrix Lie groups. We can however still define the bold-face adjoint* $\boldsymbol{Ad}$ *on* $\check{\mathbb{M}}$ *as*

$$\boldsymbol{Ad}_x := \boldsymbol{Ad}_{\hat{x}}, \tag{4.13}$$

where $\wedge : \check{\mathbb{M}} \to \mathbb{M}$ *is a Lie group homomorphism that maps* $\check{\mathbb{M}}$ *into a matrix Lie group.*

Properties of the adjoint

The adjoints satisfy the following properties:

$$\mathbf{Ad}_X^{-1} = \mathbf{Ad}_{X^{-1}}, \tag{4.14a}$$

$$\mathbf{Ad}_X\, \mathbf{Ad}_Y = \mathbf{Ad}_{X \circ Y}, \tag{4.14b}$$

$$\exp\ \mathbf{Ad}_X\ \boldsymbol{a} = \boldsymbol{X} \circ \exp \boldsymbol{a} \circ \boldsymbol{X}^{-1}, \tag{4.14c}$$

$$\boldsymbol{X} \oplus_r \boldsymbol{a} = (\mathbf{Ad}_X\, \boldsymbol{a}) \oplus_l \boldsymbol{X}. \tag{4.14d}$$

The first two properties follow directly from the definition. Equation (4.14c) follows from

$$\exp \mathbf{Ad}_X\, \boldsymbol{a} = \mathrm{Exp}\ \ \mathrm{Ad}_X\ \ \hat{\boldsymbol{a}} = \sum_{k \geq 0} \frac{\left(\boldsymbol{X}\hat{\boldsymbol{a}}\boldsymbol{X}^{-1}\right)^k}{k} = \boldsymbol{X}\left(\sum_{k \geq 0} \frac{\hat{\boldsymbol{a}}^k}{k}\right)\boldsymbol{X}^{-1} = \boldsymbol{X} \exp(\boldsymbol{a})\boldsymbol{X}^{-1}. \tag{4.15}$$

We can then also show (4.14d)

$$\boldsymbol{X} \oplus_r \boldsymbol{a} \overset{\text{(right-plus)}}{=} \boldsymbol{X} \circ \exp(\boldsymbol{a}) = (\boldsymbol{X} \circ \exp(\boldsymbol{a}) \circ \boldsymbol{X}^{-1}) \circ \boldsymbol{X} \overset{(4.14c)}{=} \exp(\mathbf{Ad}_X\, \boldsymbol{a}) \circ \boldsymbol{X} \overset{\text{(left-plus)}}{=} (\mathbf{Ad}_X\, \boldsymbol{a}) \oplus_l \boldsymbol{X}. \tag{4.16}$$

Finally a result regarding the derivative of the adjoint.

Lemma 4.1.

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{Ad}_{\exp(\lambda(t)\boldsymbol{a})} = \lambda'(t)\, \mathrm{ad}_{\boldsymbol{a}}\, \boldsymbol{Ad}_{\exp(\lambda(t)\boldsymbol{a})}\,. \tag{4.17}$$

Proof.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{a})} &\overset{(5.22)}{=} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} \exp(\mathrm{ad}_{\lambda(t)\boldsymbol{a}}) \overset{(5.20)}{=} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} \exp(\lambda(t)\, \mathrm{ad}_{\boldsymbol{a}}) \overset{(5.21)}{=} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} \frac{\lambda(t)^k\, \mathrm{ad}_{\boldsymbol{a}}^k}{k!} \\ &= \lambda'(t) \sum_{k=1}^{\infty} \frac{\lambda(t)^{k-1}\, \mathrm{ad}_{\boldsymbol{a}}^k}{(k-1)!} = \lambda'(t)\, \mathrm{ad}_{\boldsymbol{a}} \sum_{k=1}^{\infty} \frac{\lambda(t)^{k-1}\, \mathrm{ad}_{\boldsymbol{a}}^{k-1}}{(k-1)!} = \lambda'(t)\, \mathrm{ad}_{\boldsymbol{a}}\, \mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{a})}\,. \end{aligned} \tag{4.18}$$

□

4.8. Group-Linear Functions

Definition 4.4. *A function $f : \mathbb{M} \to \mathbb{N}$ is said to be* ***group-linear*** *if*

$$f(\boldsymbol{x} \circ \boldsymbol{y}) = f(\boldsymbol{x}) f(e)^{-1} f(\boldsymbol{y}). \tag{4.19}$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{M}$.

Differentiating (4.19) w.r.t. $\boldsymbol{y}$ at $\boldsymbol{y} = e$ shows that

$$\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{x}} = \mathrm{d}^{\mathrm{r}} f_e, \tag{4.20}$$

i.e. the right derivative is constant, and consequently all higher-order derivatives are zero.

If f is group-linear, then

$$f(\boldsymbol{x} \oplus \boldsymbol{a}) = f(\boldsymbol{x}) \oplus_r (F\boldsymbol{a}), \quad F = \mathrm{d}^{\mathrm{r}} f_e. \tag{4.21}$$

holds exactly for all $\boldsymbol{x} \in \mathbb{M}, \boldsymbol{a} \in \mathbb{R}^{\dim \mathfrak{m}}$.

Linear functions are characterized by being determined by how they act on differences.

$$\begin{aligned} f(\boldsymbol{y})^{-1} \circ f(\boldsymbol{x}) = f(\boldsymbol{y})^{-1} \circ f(\boldsymbol{y} \exp(\boldsymbol{x} \ominus_r \boldsymbol{y})) = f(\boldsymbol{y})^{-1} \circ f(\boldsymbol{y}) \circ f(e)^{-1} \circ f(\exp \boldsymbol{x} \ominus_r \boldsymbol{y}) \\ = f(e)^{-1} \circ f(\boldsymbol{y}^{-1} \circ \boldsymbol{x}), \end{aligned} \tag{4.22}$$

i.e. $f(\boldsymbol{y})^{-1} \circ f(\boldsymbol{x})$ is a function only of $\boldsymbol{y}^{-1}\boldsymbol{x}$.

Propagating Differences Let $\eta = \boldsymbol{x} \ominus_r \boldsymbol{y}$ denote the difference between $\boldsymbol{x}$ and $\boldsymbol{y}$. Consider mapping $\boldsymbol{x}$ and $\boldsymbol{y}$ through a linear function f,

$$\eta^+ = f(\boldsymbol{x}) \ominus_r f(\boldsymbol{y}) = \log\left(f(e)^{-1} \circ f(\boldsymbol{y}^{-1}\boldsymbol{x})\right) = \log\left(f(e)^{-1} \circ f(\exp \eta)\right). \tag{4.23}$$

Exponentiating gives

$$\exp \eta^+ = f(e)^{-1} \circ f(\exp \eta), \tag{4.24}$$

and expanding around $\eta = 0$ results in

$$\exp \eta^+ = f(e)^{-1} \circ f(e) \oplus \mathrm{d}^{\mathrm{r}} f_e \, \mathrm{d}^{\mathrm{r}} \exp_0 \eta = \exp(\mathrm{d}^{\mathrm{r}} f_e \eta), \tag{4.25}$$

so the difference η can be easily propagated through a linear f as

$$\eta^+ = F\eta, \quad F = \mathrm{d}^{\mathrm{r}} f_e. \tag{4.26}$$

5. Derivatives

Summary

- Definition of derivatives on manifolds.
- Differentiation rules.

Define derivatives w.r.t. matrix elements only, motivate that we can disregard parameterized expressions.

Definition 5.1. *The **right derivative** of $f : \mathbb{M} \to \mathbb{N}$ at $\boldsymbol{X} \in \check{\mathbb{M}}$ is a linear mapping $\mathrm{d}^{\mathrm{r}} f_X : T\mathbb{M}_X \to T\mathbb{N}_{f(X)}$ such that:*

$$\mathrm{d}^{\mathrm{r}} f_X \coloneqq \lim_{\boldsymbol{a}\to 0} \frac{f(\boldsymbol{X} \oplus_r \boldsymbol{a}) \ominus_r f(\boldsymbol{X})}{\boldsymbol{a}} = \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(\boldsymbol{X})^{-1} \circ f(\boldsymbol{X} \circ \exp(\boldsymbol{a}))\right)}{\boldsymbol{a}}, \tag{5.1}$$

where $\boldsymbol{a} \in T_X\check{\mathbb{M}}$ is a member of the parameterized Lie algebra and the division is component-wise.

*Similarly, the **left derivative** is a linear mapping $\mathrm{d}^{\mathrm{r}} f_X : T\mathbb{M}_e \to T\mathbb{N}_e$ such that*

$$\mathrm{d}^{\mathrm{l}} f_X \coloneqq \lim_{\boldsymbol{a}\to 0} \frac{f(\boldsymbol{X} \oplus_l \boldsymbol{a}) \ominus_l f(\boldsymbol{X})}{\boldsymbol{a}} = \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(\exp(\boldsymbol{a}) \circ \boldsymbol{X}) \circ f(\boldsymbol{X})^{-1}\right)}{\boldsymbol{a}}, \tag{5.2}$$

From the definition it can be seen that for small $\boldsymbol{a}$ it approximately holds that

$$f(\boldsymbol{X} \oplus_r \boldsymbol{a}) = f(\boldsymbol{X}) \oplus_r \left(\mathrm{d}^{\mathrm{r}} f_X \boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^2)\right), \tag{5.3}$$

and for left-plus:

$$f(\boldsymbol{a} \oplus_l \boldsymbol{X}) = \left(\mathrm{d}^{\mathrm{l}} f_X \boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^2)\right) \oplus_l f(\boldsymbol{X}). \tag{5.4}$$

From (5.3) and (5.4) we have that for small $\boldsymbol{a}$,

$$f(\boldsymbol{X}) \oplus_r (\mathrm{d}^{\mathrm{r}} f_X \boldsymbol{a}) \overset{(5.3)}{=} f(\boldsymbol{X} \oplus_r \boldsymbol{a}) \overset{(4.14d)}{=} f\left(\mathbf{Ad}_X\, \boldsymbol{a} \oplus_l \boldsymbol{X}\right) \overset{(5.4)}{=} (\mathrm{d}^{\mathrm{l}} f_X\, \mathbf{Ad}_X\, \boldsymbol{a}) \oplus_l f(\boldsymbol{X}). \tag{5.5}$$

Consequently,

$$\exp(\mathrm{d}^{\mathrm{l}} f_X\, \mathbf{Ad}_X\, \boldsymbol{a}) = f(\boldsymbol{X}) \circ \exp(\mathrm{d}^{\mathrm{r}} f_X \boldsymbol{a}) \circ f(\boldsymbol{X})^{-1} = \mathbf{Ad}_{f(X)} \exp(\mathrm{d}^{\mathrm{r}} f_X \boldsymbol{a}), \tag{5.6}$$

and due to (4.14c) it follows that left and right derivatives are related through the adjoints via

$$\mathrm{d}^{\mathrm{l}} f_X = \mathbf{Ad}_{f(X)}\ \mathrm{d}^{\mathrm{r}} f_X\ \mathbf{Ad}_X^{-1}\,. \tag{5.7}$$

With the interpretation of the adjoints as coordinate changes this formula can be seen as follows: the derivative of f with respect to a tangent vector ${}^{e}\boldsymbol{a}$ at e can be obtained by

1. Convert ${}^e\boldsymbol{a}$ to a tangent vector at $\boldsymbol{X}$: ${}^X\boldsymbol{a} = \mathbf{Ad}_X^{-1}\,{}^e\boldsymbol{a} \in T_X\check{\mathbb{M}}$,
2. Map the tangent vector through the derivative: ${}^X\boldsymbol{b} = \mathrm{d}^{\mathrm{r}}\, f_X\, {}^X\boldsymbol{a} \in T_{f(X)}\check{\mathbb{M}}$,
3. Convert the result back to a tangent vector at e: ${}^e\boldsymbol{b} = \mathbf{Ad}_{f(X)}\,{}^X\boldsymbol{b} \in T_e\check{\mathbb{M}}$.

Jacobians on Lie Groups satisfy the chain rule. Indeed, if $f(\boldsymbol{X}) = g \circ h(\boldsymbol{X})$ for some $g : \mathbb{M}' \to \mathbb{M}''$ and $h : \mathbb{M} \to \mathbb{M}'$ we have with $\boldsymbol{Z} := h(\boldsymbol{X})$

$$\begin{aligned}\mathrm{d}^{\mathrm{r}}(g \circ h)_X &= \lim_{\boldsymbol{a}\to 0} \frac{g\left(h(\boldsymbol{X} \oplus_r \boldsymbol{a})\right) \ominus_r g(h(\boldsymbol{X}))}{\boldsymbol{a}} \overset{(5.3)}{=} \lim_{\boldsymbol{a}\to 0} \frac{g\left(h(\boldsymbol{X}) \oplus_r \left(\mathrm{d}^{\mathrm{r}}\, h_X \boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^2)\right)\right) \ominus_r g(h(\boldsymbol{X}))}{\boldsymbol{a}} \\ &\overset{(5.3)}{=} \lim_{\boldsymbol{a}\to 0} \frac{\left(g(\boldsymbol{Z}) \oplus_r \left(\mathrm{d}^{\mathrm{r}}\, g_Z\ \mathrm{d}^{\mathrm{r}}\, h_X \boldsymbol{a} + \mathcal{O}(\|\boldsymbol{a}\|^2)\right)\right) \ominus_r g(h(\boldsymbol{X}))}{\boldsymbol{a}} \overset{(4.10b)}{=} \mathrm{d}^{\mathrm{r}}\, g_Z\ \mathrm{d}^{\mathrm{r}}\, h_X.\end{aligned} \tag{5.8}$$

An analogous left chain rule can be developed in the same manner via (5.4) in lieu of (5.3).

Important formulas for Lie group derivatives

- Right derivative: $\mathrm{d}^{\mathrm{r}}\, f_X := \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(\boldsymbol{X})^{-1} \circ f(\boldsymbol{X} \circ \exp(\boldsymbol{a})\right)}{\boldsymbol{a}} \in T_X\mathbb{M}$,
- Left derivative: $\mathrm{d}^{\mathrm{l}}\, f_X := \lim_{\boldsymbol{a}\to 0} \frac{\log\left(f(\exp(\boldsymbol{a}) \circ \boldsymbol{X}) \circ f(\boldsymbol{X})^{-1}\right)}{\boldsymbol{a}} \in T_e\mathbb{M}$,
- Conversion between left and right jacobians: $\mathrm{d}^{\mathrm{l}}\, f_X = \mathbf{Ad}_{f(X)}\ \mathrm{d}^{\mathrm{r}}\, f_X\ \mathbf{Ad}_X^{-1}$,
- Right chain rule: $\mathrm{d}^{\mathrm{r}}(g(h(\boldsymbol{X})))_X = \mathrm{d}^{\mathrm{r}}\, g_{h(X)}\ \mathrm{d}^{\mathrm{r}}\, h_X$,
- Left chain rule: $\mathrm{d}^{\mathrm{l}}(g(h(\boldsymbol{X})))_X = \mathrm{d}^{\mathrm{l}}\, g_{h(X)}\ \mathrm{d}^{\mathrm{l}}\, h_X$.

5.1. Global Derivative

For a mapping $f : \mathbb{M} \to \mathbb{N}$ between two manifolds the classical way to define a derivative $\mathrm{D}f_X$ is as a mapping $T_X\mathbb{M} \to T_{f(X)}\mathbb{N}$ defined as

$$\mathrm{D}f_X\boldsymbol{B} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)), \qquad \begin{cases}\gamma(0) = \boldsymbol{X}, \\ \gamma'(0) = \boldsymbol{B}.\end{cases} \tag{5.9}$$

for $\boldsymbol{B} \in T_X\mathbb{M}$. Note that this definition wouldn't make sense for an arbitrary matrix $\boldsymbol{B}$; for γ to take values in $\mathbb{M}$ the derivative at zero must be on the form $\boldsymbol{B} = \boldsymbol{X}\hat{\boldsymbol{a}}$. Being in global matrix coordinates, $\mathrm{D}f_X\boldsymbol{B}$ typically does not exhibit the structure of the tangent space at $T_{f(X)}\mathbb{N}$. However, it can be mapped to the tangent space via group action, which yields an alternative way of defining the right and left derivatives.

$$\begin{aligned}
\mathrm{d}^{\mathrm{r}} f_X \boldsymbol{a} &:= \left(f(\boldsymbol{X})^{-1}\left(\mathrm{D} f_X\, \boldsymbol{X}\hat{\boldsymbol{a}}\right)\right)^{\vee} = \left(f(\boldsymbol{X})^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\gamma(t))\right)\right)^{\vee}, &&\begin{cases}\gamma(0) = \boldsymbol{X},\\ \gamma'(0) = \boldsymbol{X}\hat{\boldsymbol{a}},\end{cases}\\
\mathrm{d}^{\mathrm{l}} f_X \boldsymbol{a} &:= \left(\left(\mathrm{D} f_X\, \hat{\boldsymbol{a}}\boldsymbol{X}\right) f(\boldsymbol{X})^{-1}\right)^{\vee} = \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\gamma(t))\right) f(\boldsymbol{X})^{-1}\right)^{\vee}, &&\begin{cases}\gamma(0) = \boldsymbol{X},\\ \gamma'(0) = \hat{\boldsymbol{a}}\boldsymbol{X}.\end{cases}
\end{aligned} \tag{5.10}$$

Show that these definitions agree with those above

5.2. Product rule

Consider a function $f(\boldsymbol{X}) = g(\boldsymbol{X}) \circ h(\boldsymbol{X})$, we utilize (5.3) to obtain

$$\begin{aligned}
f(\boldsymbol{X} \oplus \boldsymbol{a}) &= \left(g(\boldsymbol{X}) \oplus \left(\mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right) \circ \left(h(\boldsymbol{X}) \oplus \left(\mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right)\\
&= g(\boldsymbol{X}) \circ \exp(\mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)) \circ h(\boldsymbol{X}) \circ \exp(\mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2))\\
&= g(\boldsymbol{X}) \circ h(\boldsymbol{X}) \circ \left(\mathrm{Ad}_{h(X)^{-1}} \exp\left(\mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right) \circ \exp(\mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2))\\
&\overset{(4.14c)}{=} g(\boldsymbol{X}) \circ h(\boldsymbol{X}) \circ \left(\exp \mathbf{Ad}_{h(X)^{-1}}\left(\mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right)\right) \circ \exp(\mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2))\\
&\overset{(5.61)}{=} g(\boldsymbol{X}) \circ h(\boldsymbol{X}) \circ \exp\left(\mathbf{Ad}_{h(X)^{-1}} \mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a} + \mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a} + \mathcal{O}(\boldsymbol{a}^2)\right).
\end{aligned} \tag{5.11}$$

From here we can conclude that

$$\mathrm{d}^{\mathrm{r}}(g \circ h)_X = \mathbf{Ad}_{h(X)^{-1}} \mathrm{d}^{\mathrm{r}} g_X + \mathrm{d}^{\mathrm{r}} h_X \tag{5.12}$$

which is the product rule for Lie group derivatives.

Remark 5.1. *There is no Lie group equivalent of the rule of total derivative. Consider*

$$\begin{aligned}
f(g(\boldsymbol{X} \oplus \boldsymbol{a}), h(\boldsymbol{X} \oplus \boldsymbol{a})) &\approx f\left(g(\boldsymbol{X}) \oplus \mathrm{d}^{\mathrm{r}} g_X\, \boldsymbol{a}, h(\boldsymbol{X}) \oplus \mathrm{d}^{\mathrm{r}} h_X\, \boldsymbol{a}\right)\\
&\approx f\left(g(\boldsymbol{X}), h(\boldsymbol{X}) \oplus \mathrm{d}h_X \boldsymbol{a}\right) \oplus \mathrm{d}^{\mathrm{r}} f_g\ \mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a}\\
&\approx \left[f\left(g(\boldsymbol{X}), h(\boldsymbol{X})\right) \oplus \mathrm{d}^{\mathrm{r}} f_h\ \mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a}\right] \oplus \mathrm{d}^{\mathrm{r}} f_g\ \mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a}\\
&= f\left(g(\boldsymbol{X}), h(\boldsymbol{X})\right) \circ \left[\exp\left(\mathrm{d}^{\mathrm{r}} f_h\ \mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a}\right) \circ \exp\left(\mathrm{d}^{\mathrm{r}} f_g\ \mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a}\right)\right].
\end{aligned} \tag{5.13}$$

That is, if $f(\boldsymbol{X}) = f(g(\boldsymbol{X}), h(\boldsymbol{X}))$ we typically have that

$$\mathrm{d}^{\mathrm{r}}(f(g(\boldsymbol{X}), h(\boldsymbol{X})))_X \neq \mathrm{d}^{\mathrm{r}} f_{g(X)}\, \mathrm{d}^{r} g_X + \mathrm{d}^{\mathrm{r}} f_{h(X)}\, \mathrm{d}^{r} h_X. \tag{5.14}$$

However, from (5.61) *it can be seen that if*

$$\left[\mathrm{d}^{\mathrm{r}} f_{h(X)}\ \mathrm{d}^{\mathrm{r}} h_X \boldsymbol{a}, \mathrm{d}^{\mathrm{r}} f_{g(X)}\ \mathrm{d}^{\mathrm{r}} g_X \boldsymbol{a}\right] = 0, \qquad \forall \boldsymbol{a}, \tag{5.15}$$

then the rule of total derivatives applies. One important case when this holds is when f takes values in $\mathbb{R}^n$ since all brackets then are zero.

5.3. Lie Bracket as the Derivative of the Adjoint

The Lie bracket between two tangent elements can be defined as the global derivative of the adjoint operator at identity. Consider the mapping $f(\boldsymbol{X}) \coloneqq \mathbf{Ad}_{\boldsymbol{X}}\, \boldsymbol{b} = \boldsymbol{X}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1}$ and take a curve $\gamma(t) \in \mathbb{M}$ such that $\gamma(0) = \boldsymbol{X}$ and $\gamma'(0) = \hat{\boldsymbol{a}}$. From $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\gamma(t)^{-1} = 0$ it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)^{-1} = -\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1}$, hence

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)) = \gamma'(0)\hat{\boldsymbol{b}}\gamma(0)^{-1} - \gamma(0)\hat{\boldsymbol{b}}\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1} = \hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1} - \boldsymbol{X}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1}\hat{\boldsymbol{a}}\boldsymbol{X}^{-1}. \tag{5.16}$$

The derivatives of $\mathbf{Ad}_{\boldsymbol{X}}$ with respect to $\boldsymbol{X}$ are

$$\mathrm{D}(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b})_{\boldsymbol{X}}\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1} - \boldsymbol{X}\hat{\boldsymbol{b}}\boldsymbol{X}^{-1}\hat{\boldsymbol{a}}X^{-1}, \tag{5.17a}$$

$$\mathrm{d}^{\mathrm{r}}(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b})_{\boldsymbol{X}}\boldsymbol{a} = \left(\mathrm{D}\left(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b}\right)_{\boldsymbol{X}}\boldsymbol{X}\hat{\boldsymbol{a}}\right)^{\vee} = \mathbf{Ad}_{\boldsymbol{X}}\left[\boldsymbol{a},\boldsymbol{b}\right] = \left[\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{a}, \mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b}\right], \tag{5.17b}$$

$$\mathrm{d}^{\mathrm{l}}(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b})_{\boldsymbol{X}}\boldsymbol{a} = \left(\mathrm{D}\left(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b}\right)_{\boldsymbol{X}}\hat{\boldsymbol{a}}\boldsymbol{X}\right)^{\vee} = \left[\boldsymbol{a}, \mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b}\right], \tag{5.17c}$$

whereas at $\boldsymbol{X} = e$ they simplify to

$$\left(\mathrm{D}(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b})_{\boldsymbol{X}=e}\hat{\boldsymbol{a}}\right)^{\vee} = \mathrm{d}^{\mathrm{r}}(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b})_{\boldsymbol{X}=e}\boldsymbol{a} = \mathrm{d}^{\mathrm{l}}\left(\mathbf{Ad}_{\boldsymbol{X}}\,\boldsymbol{b}\right)_{\boldsymbol{X}=e}\boldsymbol{a} = \left[\boldsymbol{a},\boldsymbol{b}\right]. \tag{5.18}$$

The lower-case adjoint is defined as

$$\begin{aligned}
\mathrm{ad}^0_{\boldsymbol{a}}\,\boldsymbol{b} &:= \boldsymbol{b}\\
\mathrm{ad}^1_{\boldsymbol{a}}\,\boldsymbol{b} &:= \mathrm{ad}_{\boldsymbol{a}}\,\boldsymbol{b} = [\boldsymbol{a},\boldsymbol{b}]\\
\mathrm{ad}^2_{\boldsymbol{a}}\,\boldsymbol{b} &:= [\boldsymbol{a}, \mathrm{ad}_{\boldsymbol{a}}\,\boldsymbol{b}] = \underbrace{[\boldsymbol{a},[\boldsymbol{a},\boldsymbol{b}]]}_{2-\text{times}}\\
&\;\vdots\\
\mathrm{ad}^k_{\boldsymbol{a}}\,\boldsymbol{b} &:= [\boldsymbol{a}, \mathrm{ad}^{k-1}_{\boldsymbol{a}}\,\boldsymbol{b}] = \underbrace{[\boldsymbol{a},[\boldsymbol{a},\ldots,[\boldsymbol{a},\boldsymbol{b}]]]}_{k\ \text{times}}, \qquad k \geq 1.
\end{aligned} \tag{5.19}$$

From this definition it can be seen that for a scalar s,

$$\mathrm{ad}^k_{s\boldsymbol{a}} = s^k\,\mathrm{ad}_{\boldsymbol{a}}, \qquad s \in \mathbb{R} \tag{5.20}$$

If we formally define the exponential of the adjoint as

$$\exp \mathrm{ad}_{\boldsymbol{a}} \coloneqq \sum_{k=0}^{\infty} \frac{\mathrm{ad}^k_{\boldsymbol{a}}}{k!} \tag{5.21}$$

we can also show that the adjoint of the exponential equals the exponential of the adjoint.

$$\mathbf{Ad}_{\exp \boldsymbol{a}} = \exp \mathrm{ad}_{\boldsymbol{a}}\,. \tag{5.22}$$

Proof of (5.22). By expanding the left-hand side in (5.22) and letting $\boldsymbol{A} = \hat{\boldsymbol{a}}, \boldsymbol{B} = \hat{\boldsymbol{b}}$ we obtain

$$(\mathbf{Ad}_{\exp \boldsymbol{a}}\, \boldsymbol{b})^\wedge = \operatorname{Exp}(\boldsymbol{A})\boldsymbol{B}\operatorname{Exp}(-\boldsymbol{A}) = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\boldsymbol{A}^i\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{i!(k-i)!}. \tag{5.23}$$

We next show by induction that the summands in (5.23) and (5.21) are equal for each value of k. Equality evidently holds for the base case $k = 0$. Assume that it holds for $k-1$, i.e. that

$$\left(\frac{\mathrm{ad}_{\boldsymbol{a}}^{k-1}\boldsymbol{b}}{(k-1)!}\right)^\wedge = \sum_{i=0}^{k-1} \frac{\boldsymbol{A}^i\boldsymbol{B}(-\boldsymbol{A})^{k-1-i}}{i!(k-1-i)!}. \tag{5.24}$$

Then we have that

$$\begin{aligned}\left(\frac{\mathrm{ad}_{\boldsymbol{a}}^{k}\boldsymbol{b}}{k!}\right)^\wedge &= \frac{1}{k}\left[\boldsymbol{A}\frac{(\mathrm{ad}_{\boldsymbol{A}}^{k-1}\boldsymbol{B})}{(k-1)!} - \frac{(\mathrm{ad}_{\boldsymbol{A}}^{k-1}\boldsymbol{B})}{(k-1)!}\boldsymbol{A}\right] = \frac{1}{k}\left[\sum_{i=0}^{k-1}\frac{\boldsymbol{A}^{i+1}\boldsymbol{B}(-\boldsymbol{A})^{k-1-i}}{i!(k-1-i)!} + \sum_{i=0}^{k-1}\frac{\boldsymbol{A}^{i}\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{i!(k-1-i)!}\right]\\ &= \frac{1}{k}\left[\sum_{i=0}^{k-1}\frac{\boldsymbol{A}^{i}\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{i!(k-1-i)!} + \sum_{i=1}^{k}\frac{\boldsymbol{A}^{i}\boldsymbol{B}(-\boldsymbol{A})^{k-i}}{(i-1)!(k-i)!}\right] = \frac{\boldsymbol{B}(-\boldsymbol{A})^k}{k!} + \sum_{i=1}^{k-1} c_i\boldsymbol{A}^i\boldsymbol{B}(-\boldsymbol{A})^{k-i} + \frac{\boldsymbol{A}^k\boldsymbol{B}}{k!},\end{aligned}$$

where $c_i = \frac{1}{k}\left(\frac{1}{i!(k-1-i)!} + \frac{1}{(i-1)!(k-i)!}\right)$ and it can be verified that $c_i = \frac{1}{i!(k-i)!}$ as required. □

As a consequence,

$$\mathbf{Ad}_{\lambda \exp(\boldsymbol{a})}\, \boldsymbol{a} = \exp \mathrm{ad}_{\lambda \boldsymbol{a}}\, \boldsymbol{a} = \boldsymbol{a}. \tag{5.25}$$

Another useful identity is the following.

$$\mathbf{Ad}_X\,[\boldsymbol{a}, \boldsymbol{b}] = [\mathbf{Ad}_X\,\boldsymbol{a}, \mathbf{Ad}_X\,\boldsymbol{b}]\,. \tag{5.26}$$

5.4. Derivatives of the Exponential map

The derivatives of the exponential map is a fundamental expression that often shows up when manipulating derivatives on Lie groups. From (5.10) we have that

$$\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \boldsymbol{b} = \left(\exp(\boldsymbol{\gamma}(0))^{-1} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(\boldsymbol{\gamma}(t))\right)^\vee, \quad \boldsymbol{\gamma}(0) = \boldsymbol{a},\ \boldsymbol{\gamma}'(0) = \boldsymbol{b}. \tag{5.27}$$

To calculate this derivative consider a curve $\boldsymbol{\gamma}(t) \in \check{\mathfrak{m}}$ and the expression

$$\Gamma(\sigma, t) = \exp(\sigma\boldsymbol{\gamma}(t))^{-1}\frac{\partial}{\partial t}\exp(\sigma\boldsymbol{\gamma}(t)) = \operatorname{Exp}\left(-\sigma\hat{\boldsymbol{\gamma}}(t)\right)\frac{\partial}{\partial t}\operatorname{Exp}(\sigma\hat{\boldsymbol{\gamma}}(t)). \tag{5.28}$$

Take the derivative with respect to σ:

$$\begin{aligned}\frac{\partial}{\partial\sigma}\Gamma(\sigma,t) &= -\operatorname{Exp}\left(-\sigma\hat{\boldsymbol{\gamma}}(t)\right)\hat{\boldsymbol{\gamma}}(t)\frac{\partial}{\partial t}\operatorname{Exp}\left(\sigma\hat{\boldsymbol{\gamma}}(t)\right) + \operatorname{Exp}(-\sigma\hat{\boldsymbol{\gamma}}(t))\frac{\partial}{\partial t}\left[\hat{\boldsymbol{\gamma}}(t)\operatorname{Exp}(\sigma\hat{\boldsymbol{\gamma}}(t))\right]\\ &= \operatorname{Exp}(-\sigma\hat{\boldsymbol{\gamma}}(t))\hat{\boldsymbol{\gamma}}'(t)\operatorname{Exp}(\sigma\hat{\boldsymbol{\gamma}}(t)) = \mathrm{Ad}_{\operatorname{Exp}(-\sigma\hat{\boldsymbol{\gamma}}(t))}\,\hat{\boldsymbol{\gamma}}'(t) = \left(\mathbf{Ad}_{\exp(-\sigma\boldsymbol{\gamma}(t))}\,\boldsymbol{\gamma}'(t)\right)^\wedge\\ &\overset{(5.22)}{=} \left(\exp \mathrm{ad}_{-\sigma\boldsymbol{\gamma}(t)}\,\boldsymbol{\gamma}'(t)\right)^\wedge = \left(\sum_{k=0}^{\infty}\frac{\mathrm{ad}_{-\sigma\boldsymbol{\gamma}(t)}^k}{k!}\boldsymbol{\gamma}'(t)\right)^\wedge = \left(\sum_{k=0}^{\infty}\sigma^k\frac{\mathrm{ad}_{-\boldsymbol{\gamma}(t)}^k}{k!}\boldsymbol{\gamma}'(t)\right)^\wedge.\end{aligned} \tag{5.29}$$

Integrating from 0 to 1 with respect to σ and setting $t = 0$ then yields

$$\Gamma(1,0)^\vee = \int_0^1 \frac{\partial}{\partial\sigma}\Gamma(\sigma,0)^\vee \mathrm{d}\sigma = \sum_{k=0}^{\infty} \frac{\mathrm{ad}^k_{-\boldsymbol{\gamma}(0)}}{(k+1)!}\boldsymbol{\gamma}'(0). \tag{5.30}$$

From (5.28) we can see that $\Gamma(1,0)$ is equal to the right derivative of exp at $\gamma(0)$ in the direction $\gamma'(0)$.

The right- and left derivatives of the exponential map are

$$\begin{aligned} \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} &= \frac{I - \exp(-\,\mathrm{ad}_{\boldsymbol{a}})}{\mathrm{ad}_{\boldsymbol{a}}} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}\,\mathrm{ad}^k_{\boldsymbol{a}}, \\ \mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}} &= \frac{\exp \mathrm{ad}_a - I}{\mathrm{ad}_a} := \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\,\mathrm{ad}^k_{\boldsymbol{a}}. \end{aligned} \tag{5.31}$$

From the relation (5.20) it can be seen that

$$\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} = \mathrm{d}^{\mathrm{l}} \exp_{-\boldsymbol{a}}, \tag{5.32}$$

another useful identity that follows from the definition and (5.22) is

$$\mathbf{Ad}_{\exp(-\boldsymbol{a})} = I - \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \mathrm{ad}_{\boldsymbol{a}}. \tag{5.33}$$

Through the Bernoulli numbers $B_0 = 1, B_1 = -1/2, B_2 = 1/6, \ldots$ that are defined as

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n. \tag{5.34}$$

we can also write the formal inverses of the derivatives of the exponential

$$\begin{aligned} \left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1} &= \frac{\mathrm{ad}_{\boldsymbol{a}}}{I - \exp(-\,\mathrm{ad}_{\boldsymbol{a}})} := \sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!}\,\mathrm{ad}^n_{\boldsymbol{a}}, \\ \left(\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}}\right)^{-1} &= \frac{\mathrm{ad}_{\boldsymbol{a}}}{\exp \mathrm{ad}_{\boldsymbol{a}} - I} := \sum_{n=0}^{\infty} B_n \frac{1}{n!}\,\mathrm{ad}^n_{\boldsymbol{a}}. \end{aligned} \tag{5.35}$$

Notably, at $\boldsymbol{a} = \mathbf{0}$ these are all equal to the identity matrix. Since $B_1 = -1/2$ and $B_n = 0$ for odd $n > 1$ it follows that

$$\left(\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}}\right)^{-1} = -\,\mathrm{ad}_{\boldsymbol{a}} + \left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1}. \tag{5.36}$$

From these definitions it follows that

$$[\boldsymbol{a}, \boldsymbol{b}] = 0 \quad \Longrightarrow \quad \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{b}} \boldsymbol{a} = \mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{b}} = \left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{b}}\right)^{-1} = \left(\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{b}}\right)^{-1} = \boldsymbol{a}, \tag{5.37}$$

which in particular holds for the case $\boldsymbol{b} = \lambda\boldsymbol{a}$. As a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(t\boldsymbol{a}) = \mathrm{d}^r \exp_{t\boldsymbol{a}} \boldsymbol{a} = \boldsymbol{a}. \tag{5.38}$$

While the derivatives (5.31) - (5.35) could be evaluated to arbitrary precision by adding enough terms, this is not a practical solution. Fortunately closed-form expressions can be obtained for all groups of interest, but we postpone those calculations to the next part.

5.5. Derivatives of common operations

Group composition We calculate the right derivatives using (5.1) and the left derivatives via (5.7).

$$\begin{aligned}\mathrm{d}^{\mathrm{r}}(X\circ Y)_X &\overset{(5.1)}{=} \lim_{\boldsymbol{a}\to 0}\frac{\log\left((X\circ Y)^{-1}\circ X\circ\exp(\boldsymbol{a})\circ Y\right)}{\boldsymbol{a}} = \lim_{\boldsymbol{a}\to 0}\frac{\log\left(Y^{-1}\circ\exp(\boldsymbol{a})\circ Y\right)}{\boldsymbol{a}} \\ &\overset{(4.14c)}{=} \lim_{\boldsymbol{a}\to 0}\frac{\log\exp \mathbf{Ad}_{Y^{-1}}\boldsymbol{a}}{\boldsymbol{a}} = \mathbf{Ad}_{Y^{-1}},\end{aligned} \tag{5.39}$$

$$\mathrm{d}^{\mathrm{l}}(X\circ Y)_X \overset{(5.7)}{=} \mathbf{Ad}_{X\circ Y}\,\mathbf{Ad}_{Y^{-1}}\,\mathbf{Ad}_X^{-1} \overset{(4.14)}{=} I_n, \tag{5.40}$$

$$\mathrm{d}^{\mathrm{r}}(X\circ Y)_Y \overset{(5.1)}{=} \lim_{\boldsymbol{a}\to 0}\frac{\log\left((X\circ Y)^{-1}\circ X\circ Y\circ\exp(\boldsymbol{a})\right)}{\boldsymbol{a}} = I_n, \tag{5.41}$$

$$\mathrm{d}^{\mathrm{l}}(X\circ Y)_Y \overset{(5.7)}{=} \mathbf{Ad}_{X\circ Y}\,I_n\,\mathbf{Ad}_Y^{-1} \overset{(4.14)}{=} \mathbf{Ad}_X\,. \tag{5.42}$$

Group inverse

$$\begin{aligned}\mathrm{d}^{\mathrm{r}}(X^{-1})_X &\overset{(5.1)}{=} \lim_{\boldsymbol{a}\to 0}\frac{\log\left(X\circ(X\circ\exp(\boldsymbol{a}))^{-1}\right)}{\boldsymbol{a}} = \lim_{\boldsymbol{a}\to 0}\frac{\log\left(X\circ\exp(-\boldsymbol{a})\circ X^{-1}\right)}{\boldsymbol{a}} \\ &\overset{(4.14c)}{=} \frac{\log\exp \mathbf{Ad}_X -\boldsymbol{a}}{\boldsymbol{a}} = -\mathbf{Ad}_X\,.\end{aligned} \tag{5.43}$$

$$\mathrm{d}^{\mathrm{l}}(X^{-1})_X \overset{(5.7)}{=} -\mathbf{Ad}_{X^{-1}}\,\mathbf{Ad}_X\,\mathbf{Ad}_{X^{-1}} = -\mathbf{Ad}_{X^{-1}}\,. \tag{5.44}$$

Logarithm From differentiating $\boldsymbol{a} = \log\exp\boldsymbol{a}$ using the chain rule we get $I = \mathrm{d}^{\mathrm{r}}\log_{\exp\boldsymbol{a}}\ \mathrm{d}^{\mathrm{r}}\exp_{\boldsymbol{a}}$, which implies that

$$\mathrm{d}^{\mathrm{r}}\log_X = \left[\mathrm{d}^{\mathrm{r}}\exp_{\log X}\right]^{-1}, \tag{5.45}$$

$$\mathrm{d}^{\mathrm{l}}\log_X = \left[\mathrm{d}^{\mathrm{l}}\exp_{\log X}\right]^{-1}. \tag{5.46}$$

$$\tag{5.47}$$

Plus and minus From the chain rule and the above we can also deduce the derivatives of the plus and minus maps

$$\mathrm{d}^{\mathrm{r}}(\boldsymbol{X}\oplus_r \boldsymbol{a})_{\boldsymbol{X}} = \mathrm{d}^{\mathrm{r}}(\boldsymbol{X}\circ\exp(\boldsymbol{a}))_{\boldsymbol{X}} \overset{(5.39)}{=} \mathbf{Ad}^{-1}_{\exp(\boldsymbol{a})}, \tag{5.48}$$

$$\mathrm{d}^{\mathrm{r}}(\boldsymbol{X}\oplus_r \boldsymbol{a})_{\boldsymbol{a}} = \mathrm{d}^{\mathrm{r}}(\boldsymbol{X}\circ\exp(\boldsymbol{a}))_{\boldsymbol{a}} \overset{(5.8)}{=} \mathrm{d}^{\mathrm{r}}(\boldsymbol{X}\circ\exp(\boldsymbol{a}))_{\exp\boldsymbol{a}}\ \mathrm{d}^{\mathrm{r}}\exp_{\boldsymbol{a}} \overset{(5.41)}{=} \mathrm{d}^{\mathrm{r}}\exp_{\boldsymbol{a}}, \tag{5.49}$$

$$\mathrm{d}^{\mathrm{r}}(\boldsymbol{Y}\ominus_r \boldsymbol{X})_{\boldsymbol{Y}} = \mathrm{d}^{\mathrm{r}}\left(\log \boldsymbol{X}^{-1}\circ\boldsymbol{Y}\right)_{\boldsymbol{Y}} \overset{(5.8)}{=} \mathrm{d}^{\mathrm{r}}\log_{\boldsymbol{X}^{-1}\circ\boldsymbol{Y}}\ \mathrm{d}^{\mathrm{r}}(\boldsymbol{X}^{-1}\circ\boldsymbol{Y})_{\boldsymbol{Y}} \overset{(5.41)}{=} \left[\mathrm{d}^{\mathrm{r}}\exp_{\boldsymbol{Y}\ominus_r\boldsymbol{X}}\right]^{-1}, \tag{5.50}$$

$$\begin{aligned}\mathrm{d}^{\mathrm{r}}(\boldsymbol{Y}\ominus_r \boldsymbol{X})_{\boldsymbol{X}} &= \mathrm{d}^{\mathrm{r}}\left(\log \boldsymbol{X}^{-1}\circ\boldsymbol{Y}\right)_{\boldsymbol{X}} \overset{(5.8)}{=} \mathrm{d}^{\mathrm{r}}\log_{\boldsymbol{X}^{-1}\circ\boldsymbol{Y}}\ \mathrm{d}^{\mathrm{r}}(\boldsymbol{X}^{-1}\circ\boldsymbol{Y})_{\boldsymbol{X}^{-1}}\ \mathrm{d}^{\mathrm{r}}(\boldsymbol{X}^{-1})_{\boldsymbol{X}} \\ &\overset{(5.45),(5.39),(5.43)}{=} \left[\mathrm{d}^{\mathrm{r}}\exp_{\boldsymbol{Y}\ominus_r\boldsymbol{X}}\right]^{-1}\mathbf{Ad}_{\boldsymbol{Y}^{-1}}(-\mathbf{Ad}_{\boldsymbol{X}}) \overset{(4.14b)}{=} -\left[\mathrm{d}^{\mathrm{r}}\exp_{\boldsymbol{Y}\ominus_r\boldsymbol{X}}\right]^{-1}\mathbf{Ad}_{\boldsymbol{Y}^{-1}\circ\boldsymbol{X}} \\ &= -\left[\mathrm{d}^{\mathrm{r}}\exp_{\boldsymbol{Y}\ominus_r\boldsymbol{X}}\right]^{-1}\mathbf{Ad}^{-1}_{\exp\boldsymbol{Y}\ominus_r\boldsymbol{X}} \overset{(5.7)}{=} -\left[\mathrm{d}^{\mathrm{l}}\exp_{\boldsymbol{Y}\ominus_r\boldsymbol{X}}\right]^{-1}.\end{aligned} \tag{5.51}$$

5.6. On Automatic Differentiation

Consider a function $f : \mathbb{M} \to \mathbb{N}$ whose derivative we are interested in. Since autodiff tools are not aware of manifolds we can not directly obtain e.g. $\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}}$; here we discuss how to obtain on-manifold derivatives by only differentiating Euclidean functions. Since $\mathrm{d}^{\mathrm{r}}(f(\boldsymbol{X}\oplus_r \boldsymbol{a}))_{\boldsymbol{a}=0} = \mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}}\mathrm{d}\exp_0 = \mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}}$ we can write

$$\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}}\boldsymbol{b} = \mathrm{d}^{\mathrm{r}}(f(\boldsymbol{X}\oplus_r \boldsymbol{a}))_{\boldsymbol{a}=0}\boldsymbol{b} = \left(f(\boldsymbol{X}\oplus_r \boldsymbol{a})^{-1}\mathrm{D}(f(\boldsymbol{X}\oplus_r \boldsymbol{a}))_{\boldsymbol{a}}\hat{\boldsymbol{b}}\right)^{\vee}\Big|_{\boldsymbol{a}=0} = \left(f(\boldsymbol{X})^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\boldsymbol{X}\oplus(t\boldsymbol{b}))\right)^{\vee}. \tag{5.52}$$

Here the function $t \mapsto f(\boldsymbol{X}\oplus(t\boldsymbol{b}))$ maps a scalar to a matrix and can therefore be differentiated using regular tools, after which the expression can be evaluated to obtain $\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}}\boldsymbol{b}$. Naturally, if the complete derivative $\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}}$ is desired it can be obtained by repeating this procedure n times for each basis unit vector.

If f maps to a Euclidean space (i.e. $\mathbb{N} = \mathbb{R}^k$) this further simplifies to

$$\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}} b = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\boldsymbol{X}\oplus(t\boldsymbol{b})), \quad f : \mathbb{M}\to\mathbb{R}^n, \tag{5.53}$$

which with some abuse of notation can be written as

$$\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{X}} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\Big|_{\boldsymbol{b}=0} f(\boldsymbol{X}\oplus\boldsymbol{b}). \tag{5.54}$$

Pretty sure this will immediately yield the correct derivative

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\Big|_{\boldsymbol{b}=0} \log\left(f(\boldsymbol{X})^{-1}f(\boldsymbol{X}\oplus\boldsymbol{b})\right) \tag{5.55}$$

5.6.1. Ceres Solver Local Parameterizations

A special case of when numerical derivatives are used is in the nonlinear optimizer Ceres. Being unaware of Lie groups, Ceres considers cost functions that are functions of some parameters, $f(\boldsymbol{x})$, and uses automatic differentiation of $f : \mathbb{R}^p \to \mathbb{R}^k$ with respect to $\boldsymbol{x} \in \mathbb{R}^p$ to figure out in what direction to move in order to minimize f. However, if $\boldsymbol{x}$ represents the coordinates of a manifold chart, i.e. $\hat{\boldsymbol{x}} = \boldsymbol{X}$ for $\boldsymbol{X} \in \mathbb{M}$, it is not desirable to directly apply an update in the direction of the gradient since this may lead the resulting point no longer being on the manifold.

Being unaware of the manifold structure, automatic differentiation can only evaluate $\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} f(\hat{\boldsymbol{x}})$, where we are using the hat and vee maps to denote conversions between elements of a Lie group $\mathbb{M}$ and its parameterization $\check{\mathbb{M}}$ as was done in Chapter 2. Ceres provides an interface for specifying cusom *local parameterizations* that enable on-manifold optimization. In the following we specify how a local parameterization for Lie Group optimization can be constructed.

According to (5.54) we can write a tangent space derivative for a Euclidean-valued function

$$\mathrm{d}^{\mathrm{r}} f_X = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} f(\boldsymbol{X} \oplus \boldsymbol{b}) = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} f\left(\left((\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^{\vee}\right)^{\wedge}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{y}}\right|_{\boldsymbol{y}=\boldsymbol{x}} f(\hat{\boldsymbol{y}}) \times \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^{\vee}. \tag{5.56}$$

Thus, if $\left.\frac{\mathrm{d}}{\mathrm{d}y}\right|_{\boldsymbol{y}=\boldsymbol{x}} f(\hat{\boldsymbol{y}})$ is obtained through automatic differentiation it needs to be right-multiplied by a state-dependent matrix in order to obtain the tangent-space derivative. In Ceres parlance these matrices are called

- Local derivative: $\mathrm{d}^{\mathrm{r}} f_X$, a $k \times n$ matrix,
- Global derivative: $\left.\frac{\mathrm{d}}{\mathrm{d}y}\right|_{\boldsymbol{y}=\boldsymbol{x}} f(\hat{\boldsymbol{y}})$, a $k \times p$ matrix,
- Jacobian: $\left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^{\vee}$, a $p \times n$ matrix

and it holds that (local derivative) = (global derivative) × (jacobian). The local parameterization for a Lie group can be specified as follows:

- Plus operation: $\boldsymbol{x} \boxplus \boldsymbol{b} \coloneqq (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^{\vee}$,
- Local dimension: $n = \|T\mathbb{M}\|$ tangent space dimension,
- Global dimension: $p = \|\check{\mathbb{M}}\|$ group parameterization dimension,
- Jacobian: $\left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=0} (\hat{\boldsymbol{x}} \oplus \boldsymbol{b})^{\vee}$.

5.7. Baker–Campbell–Hausdorff formula

We next prove the Baker-Campbell-Hausdorff formula that is a powerful analytical tool. One form of it reads

$$\log\left(\exp(\boldsymbol{a}) \circ \exp(\boldsymbol{b})\right) = \boldsymbol{a} + \left(\int_0^1 \psi\left(\exp(\mathrm{ad}_{\boldsymbol{a}}) \exp(t\, \mathrm{ad}_{\boldsymbol{b}})\right) \mathrm{d}t\right) \boldsymbol{b}, \tag{5.57}$$

where

$$\psi(x) = \frac{x \log x}{x-1} = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} (x-1)^k. \tag{5.58}$$

which converges for $\|x\| < 1$.

Proof. Consider $z(t) \coloneqq \log(\exp(\boldsymbol{a})\exp(t\boldsymbol{b}))$ so that the sought-after expression is $z(1)$. The integral expression is obtained by calculating the derivative of z. Clearly $z(0) = \boldsymbol{a}$ so only the derivative of z w.r.t. t is required to obtain the integral expression. The differentiation rules from Section 5.4-5.5 give

$$\begin{aligned} z'(t) = \mathrm{d}^r \log_{\exp(\boldsymbol{a})\exp(t\boldsymbol{b})} \mathrm{d}^r \exp_{t\boldsymbol{b}} \boldsymbol{b} = \left(\mathrm{d}^r \exp_z\right)^{-1} \boldsymbol{b} = \frac{\mathrm{ad}_z}{1 - \exp(-\mathrm{ad}_z)}\boldsymbol{b} \\ = \frac{\exp(\mathrm{ad}_z)\,\mathrm{ad}_z}{\exp(\mathrm{ad}_z) - 1} b = \psi(\exp(\mathrm{ad}_z)). \end{aligned} \tag{5.59}$$

Finally, applying (5.22) to $\exp z = \exp(\boldsymbol{a})\exp(t\boldsymbol{b})$ gives

$$\exp \mathrm{ad}_z = \exp \mathrm{ad}_{\boldsymbol{a}} \exp \mathrm{ad}_{t\boldsymbol{b}} = \exp\left(\mathrm{ad}_{\boldsymbol{a}}\right) \exp\left(t\, \mathrm{ad}_{\boldsymbol{b}}\right) \tag{5.60}$$

which completes the proof. □

The first terms of the BCH formula read as follows:

$$\log\left(\exp \boldsymbol{a} \circ \exp \boldsymbol{b}\right) = \boldsymbol{a} + \boldsymbol{b} + \frac{1}{2}[\boldsymbol{a}, \boldsymbol{b}] + \frac{1}{12}[\boldsymbol{a}, [\boldsymbol{a}, \boldsymbol{b}]] - \frac{1}{12}[\boldsymbol{b}, [\boldsymbol{a}, \boldsymbol{b}]] + \dots . \tag{5.61}$$

Both the forumal above are cumbersome to apply in practice. However, a convenient first-order approximation in $\boldsymbol{b}$ can be obtained by using the approximation $z = \boldsymbol{a} + \mathcal{O}(\boldsymbol{b})$ to obtain $\left(\mathrm{d}^r \exp_z\right)^{-1} \boldsymbol{b} = \left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1} \boldsymbol{b} + \mathcal{O}(\boldsymbol{b}^2)$. Inserting this into (5.59) then results in

$$z'(t) = \left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1} \boldsymbol{b} + \mathcal{O}\left(\boldsymbol{b}^2\right) \tag{5.62}$$

which in turn gives the first-order approximation

$$\boxed{z(t) = \log\left(\exp \boldsymbol{a} \exp \boldsymbol{b}\right) = \boldsymbol{a} + \left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1} \boldsymbol{b} + \mathcal{O}\left(\boldsymbol{b}^2\right).} \tag{5.63}$$

6. Dynamical Systems on Lie Groups

Having defined Lie group derivatives a logical next step is to consider differential equations on Lie groups, i.e. solutions $\boldsymbol{x}(t)$ to

$$\begin{aligned} \mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t &= f(t, \boldsymbol{x}(t)), \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0. \end{aligned} \tag{6.1}$$

The same system can due to (5.7) be written with the left derivative as

$$\mathrm{d}^l \boldsymbol{x}_t = f^l(t, \boldsymbol{x}(t)), \quad f^l(t, \boldsymbol{x}(t)) = \mathbf{Ad}_{\boldsymbol{x}(t)}\, f^R(t, \boldsymbol{x}(t)), \tag{6.2}$$

or with the global derivative as

$$\mathrm{D}\boldsymbol{X}_t = \boldsymbol{X}(t)\hat{f}(t, \boldsymbol{X}(t)) = \hat{f}^l(t, \boldsymbol{X}(t))\boldsymbol{X}(t), \tag{6.3}$$

where $\boldsymbol{X}(t)$ is a matrix Lie group. For a constant f^l we recognize the typical form $\dot{x} = Ax$ of a linear system.

In this chapter we study various properties of (6.1): its linearization, sensitivity with respect to initial conditions, monotonicity, and finally a method to analyze it via a system on $\mathbb{R}^n$.

For a given initial condition $\boldsymbol{x}_0$ the solution of (6.1) at time $t \geq 0$ can be denoted $\phi(t; \boldsymbol{x}_0)$ where the *flow* operator $\phi : \mathbb{R} \times \mathbb{M} \to \mathbb{M}$ is s.t.

$$\begin{aligned} \phi(0; \boldsymbol{x}_0) &= \boldsymbol{x}_0, \\ \mathrm{d}^{\mathrm{r}}\, \phi(t; \boldsymbol{x}_0)_t &= f\left(t, \phi(t; \boldsymbol{x}_0)\right). \end{aligned} \tag{6.4}$$

Remark 6.1. *Parameters and initial conditions are equivalent in the following sense. A parameter-dependent system*

$$\begin{aligned} \mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t &= f(t, \boldsymbol{x}; p_0), \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned} \tag{6.5}$$

is equivalent to the parameter-free system on $\mathbb{M} \times \mathbb{R}^n$

$$\begin{aligned} \mathrm{d}^{\mathrm{r}}(\boldsymbol{x}, p)_t &= \left(g(t, \boldsymbol{x}, p), 0\right), \qquad g(t, \boldsymbol{x}, p) := f(t, \boldsymbol{x}; p), \\ (\boldsymbol{x}, p)(t_0) &= (\boldsymbol{x}_0, p_0), \end{aligned} \tag{6.6}$$

where $g(t, \boldsymbol{x}, p) = f(t, \boldsymbol{x}; p)$*. Conversely, the system*

$$\begin{aligned} \mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t &= f(t, \boldsymbol{x}), \\ \boldsymbol{x}(t_0) &= \boldsymbol{x}_0, \end{aligned} \tag{6.7}$$

with a non-trivial initial condition is (locally) equivalent to the parameter-dependent system

$$\begin{aligned} \mathrm{d}^{\mathrm{r}}\, \boldsymbol{a}_t &= g(t, \boldsymbol{a}; t_0, \boldsymbol{x}_0), \qquad g(t, \boldsymbol{a}; t_0, \boldsymbol{x}_0) := \left[\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right]^{-1} f(t_0 + t, \boldsymbol{x}_0 \oplus_r \boldsymbol{a}), \\ \boldsymbol{a}(0) &= 0, \end{aligned} \tag{6.8}$$

with trivial initial conditions, in the sense that $\Phi^{\boldsymbol{x}}(t_0 + t; t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0 \oplus_r \Phi^{\boldsymbol{a}}(t; t_0, \boldsymbol{x}_0)$*.*

6.1. Sensitivity Analysis

Next we study the sensitivity of $\phi(t; \boldsymbol{x}_0)$ with respect to the initial conditon. Due to the equivalence between parameters and initial conditions as discussed in Remark 6.1, sensitivity with respect to the initial conditons can also be used to figure the sensitivity with respect to paramters.

Global derivative on matrix form $\Phi = \hat{\phi}$.

$$\dot{\Phi}(t; \boldsymbol{x}_0) = \Phi(t; \boldsymbol{x}_0) \left(\mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_t \right)^{\wedge} = \Phi(t; \boldsymbol{x}_0) \hat{f} \left(t, \Phi(t; \boldsymbol{x}_0) \right). \tag{6.9}$$

Derivative of inverse

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t} \Phi(t; \boldsymbol{x}_0) \circ \Phi(t; \boldsymbol{x}_0)^{-1} &= \dot{\Phi}(t; \boldsymbol{x}_0) \Phi(t; \boldsymbol{x}_0)^{-1} + \Phi(t; \boldsymbol{x}_0) \frac{\mathrm{d}}{\mathrm{d}t} \Phi(t; \boldsymbol{x}_0)^{-1} \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}t} \phi(t; \boldsymbol{x}_0)^{-1} &= \Phi(t; \boldsymbol{x}_0)^{-1} \dot{\Phi}(t; \boldsymbol{x}_0) \Phi(t; \boldsymbol{x}_0)^{-1}. \end{aligned} \tag{6.10}$$

We can then evaluate how $\mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_t$ depends on t by moving to global derivatives and changing the order of integration.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_{\boldsymbol{x}_0} \boldsymbol{a} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\Phi(t; \boldsymbol{x}_0)^{-1} \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \right|_{\tau=0} \Phi(t; \boldsymbol{x}_0 \oplus_r \tau \boldsymbol{a}) \right)^{\vee} \\ &\quad = \left(\left(-\Phi(t; \boldsymbol{x}_0)^{-1} \dot{\Phi}(t; \boldsymbol{x}_0) \Phi(t; \boldsymbol{x}_0) \right)^{-1} \frac{\mathrm{d}}{\mathrm{d}\tau} \Phi(t; \boldsymbol{x}_0 \oplus_r \tau \boldsymbol{a}) + \Phi(t; \boldsymbol{x}_0)^{-1} \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \right|_{\tau=0} \dot{\Phi}(t; \boldsymbol{x}_0 \oplus_r \tau \boldsymbol{a}) \right)^{\vee} \\ &\quad = \left(-\hat{f} \left(t; \Phi(t; \boldsymbol{x}_0) \right) \Phi(t; \boldsymbol{x}_0)^{-1} \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \right|_{\tau=0} \Phi(t; \boldsymbol{x}_0 \oplus_r \tau \boldsymbol{a}) \right)^{\vee} \\ &\quad + \left(\Phi(t; \boldsymbol{x}_0)^{-1} \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \right|_{\tau=0} \Phi(t; \boldsymbol{x}_0 \oplus_r \tau \boldsymbol{a}) \hat{f}(t; \Phi(t; \boldsymbol{x}_0 \oplus_r \tau \boldsymbol{a})) \right)^{\vee} \\ &\quad = - \left(\hat{f}(t; \Phi(t; \boldsymbol{x}_0)) \left(\mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_{\boldsymbol{x}_0} \boldsymbol{a} \right)^{\wedge} + \left(\mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_{\boldsymbol{x}_0} \boldsymbol{a} \right)^{\wedge} \hat{f}(t; \Phi(t; \boldsymbol{x}_0)) + \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \right|_{\tau=0} \hat{f}(t; \Phi(t; \boldsymbol{x}_0 \oplus_r \tau \boldsymbol{a})) \right)^{\vee} \\ &\quad = - \left[f(t; \Phi(t; \boldsymbol{x}_0)), \mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_{\boldsymbol{x}_0} \boldsymbol{a} \right] + \mathrm{d}^{\mathrm{r}} \, f(t, \Phi(t; \boldsymbol{x}_0))_{\boldsymbol{x}_0} \boldsymbol{a} \\ &\quad = -\operatorname{ad}_{f(t; \Phi(t; \boldsymbol{x}_0))} \mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_{\boldsymbol{x}_0} \boldsymbol{a} + \mathrm{d}^{\mathrm{r}} \, f(t, \boldsymbol{x})_{\boldsymbol{x} = \Phi(t; \boldsymbol{x}_0)} \, \mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0))_{\boldsymbol{x}_0} \boldsymbol{a}. \end{aligned}$$

The sensitivity $S(t) \coloneqq \mathrm{d}^{\mathrm{r}} \, \Phi(t; \boldsymbol{x}_0)_{\boldsymbol{x}_0}$ satisfies the matrix-valued ODE

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} S(t) &= \left(-\operatorname{ad}_{f(t, \Phi(t; \boldsymbol{x}_0))} + \mathrm{d}^{\mathrm{r}} \, f_{\boldsymbol{x}} |_{\boldsymbol{x} = \Phi(t; \boldsymbol{x}_0)} \right) S(t), \\ S(0) &= I. \end{aligned} \tag{6.11}$$

6.1.1. Example

Example 6.1

If $\mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t = f(\boldsymbol{x}) \equiv \boldsymbol{a}$, then $\boldsymbol{x}(t) = \boldsymbol{x}_0 \exp(t\boldsymbol{a})$ and we get

$$\mathrm{d}^{\mathrm{r}}(\boldsymbol{x}(t))_{\boldsymbol{x}_0} \overset{(5.39)}{=} \mathbf{Ad}_{\exp(-t\boldsymbol{a})} \,. \tag{6.12}$$

We furthermore know from Lemma 4.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{Ad}_{\exp(-t\boldsymbol{a})} = -\operatorname{ad}_{\boldsymbol{a}} \mathbf{Ad}_{\exp(-t\boldsymbol{a})}, \tag{6.13}$$

i.e. the sensitivity equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t) = -\operatorname{ad}_{\boldsymbol{a}} S(t), \tag{6.14}$$

which was expected from (6.11) since f is constant.

6.2. Group-Linear Systems

Linear systems enjoy many nice properties compared to their nonlinear counterparts. In [2] a useful generalization of linear systems to Lie groups was introduced that we present here. A defining feature of linear systems is the superposition property—that solutions are linear w.r.t. the initial condition. Another way to state this property is that the dynamics of differences between trajectories does not depend on the trajectories themselves—only on the value of the difference. The following definition extends this idea to Lie group systems.

Definition 6.1. *The system*

$$\mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t = f(\boldsymbol{x}(t)) \tag{6.15}$$

*is **group-linear** if for any two system trajectories $\boldsymbol{x}_1, \boldsymbol{x}_2$ it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{x}_1(t) \ominus_r \boldsymbol{x}_2(t) = g\left(\boldsymbol{x}_1(t) \ominus_r \boldsymbol{x}_2(t)\right) \tag{6.16}$$

for some function $g(\cdot)$.

In the following we characterize properties of group-linear systems. Let the difference between two trajectories be $\boldsymbol{a} = \boldsymbol{y} \ominus_r \boldsymbol{x}$; the time derivative of $\boldsymbol{a}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a} = \left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1} f(\boldsymbol{y}) - \left(\mathrm{d}^{l} \exp_{\boldsymbol{a}}\right)^{-1} f(\boldsymbol{x}) = \left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1} f(\boldsymbol{x} \oplus_r \boldsymbol{a}) - \left(\mathrm{d}^{l} \exp_{\boldsymbol{a}}\right)^{-1} f(\boldsymbol{x}). \tag{6.17}$$

For this to not be a function of $\boldsymbol{x}$, substituting e for $\boldsymbol{x}$ should not change the expression:

$$\left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1} f(\boldsymbol{x} \oplus_r \boldsymbol{a}) - \left(\mathrm{d}^{l} \exp_{\boldsymbol{a}}\right)^{-1} f(\boldsymbol{x}) = \left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1} f(e \oplus_r \boldsymbol{a}) - \left(\mathrm{d}^{l} \exp_{\boldsymbol{a}}\right)^{-1} f(e). \tag{6.18}$$

This condition simplifies by noting that

$$\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \left(\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}}\right)^{-1} = \mathbf{Ad}^{-1}_{\exp(\boldsymbol{a})} \mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}} \mathbf{Ad}_{\boldsymbol{a}} (\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}})^{-1} = \mathbf{Ad}_{\exp(-\boldsymbol{a})}, \tag{6.19}$$

which gives the following condition on f for group-linearity:

$$f(\boldsymbol{x} \oplus_r \boldsymbol{a}) = \mathbf{Ad}_{\exp(-\boldsymbol{a})} \left(f(\boldsymbol{x}) - f(e)\right) + f(\exp \boldsymbol{a}). \tag{6.20}$$

A property of functions f that satisfy (6.20) is that they are completely determined by the value and differential at identity. We show this using two lemmas

Lemma 6.1. *For f satisfying (6.20) we have*

$$-\operatorname{ad}_{f(\boldsymbol{x})} + \mathrm{d}^{\mathrm{r}} f_{\boldsymbol{x}} = -\operatorname{ad}_{f(e)} + \mathrm{d}^{\mathrm{r}} f_e. \tag{6.21}$$

Proof. The directional derivative of (6.20) w.r.t. $\boldsymbol{a}$ at $\boldsymbol{a} = 0$ in the direction $\boldsymbol{b}$ is

$$\begin{aligned}
\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{x}} \boldsymbol{b} &= \mathrm{d}^{\mathrm{r}} \left(\mathbf{Ad}_X (f(\boldsymbol{x}) - f(e))\right)_{X=\exp(-0)} (-\boldsymbol{b}) + \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{b} \\
&= [\mathbf{Ad}_X(-\boldsymbol{b}), \mathbf{Ad}_X(f(\boldsymbol{x}) - f(e))]|_{X=e} + \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{b} = [f(\boldsymbol{x}) - f(e), \boldsymbol{b}] + \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{b}.
\end{aligned}$$

Since the direction $\boldsymbol{b}$ is arbitrary the result follows. □

Lemma 6.2. *For f satisfying (6.20)*

$$f(\exp \boldsymbol{a}) = f(e) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a}. \tag{6.22}$$

In other words, the value $f(\exp \boldsymbol{a})$ is completely determined by $f(e)$ and $\mathrm{d}^{\mathrm{r}} f_e$.

Proof. Consider $\gamma(t) = \exp(t \log(\boldsymbol{x}))$ which is s.t. $\gamma(0) = e$ and $\gamma(1) = \boldsymbol{x}$. The value of f along γ satisfies

$$\mathrm{d}^{\mathrm{r}} f(\gamma(t))_t = \mathrm{d}^{\mathrm{r}} f_{\gamma(t)} \mathrm{d}^{\mathrm{r}} \gamma_t = \left(\mathrm{d}^{\mathrm{r}} f_e + \operatorname{ad}_{f(\gamma(t)) - f(e)}\right) \mathrm{d}^{\mathrm{r}} \exp_{t \log \boldsymbol{x}} \log \boldsymbol{x}.$$

Noting that due to (5.38), $\mathrm{d}^{\mathrm{r}} \exp_{t \log \boldsymbol{x}} \log \boldsymbol{x} = \log \boldsymbol{x}$. Letting $z = f(\gamma(t)) - f(e)$ shows that $f(\boldsymbol{x}) = f(e) + z(1)$ where z solves

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} z &= \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a} - \operatorname{ad}_{\boldsymbol{a}} z, \\
z(0) &= 0.
\end{aligned} \tag{6.23}$$

This linear ODE has solution $z(t)$ s.t. $\operatorname{ad}_{\boldsymbol{a}} z(t) = (I - \exp(-t \operatorname{ad}_{\boldsymbol{a}})) \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a}$. Thus by (5.31), $z(1) = \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a}$. □

Combining the result of Lemma 6.2 with (6.20) allows us to summarize.

Theorem 6.1. *A system $\mathrm{d}^{\mathrm{r}} \boldsymbol{x}_t = f(\boldsymbol{x}(t))$ is **group-linear** if and only if f satisfies*

$$f(\boldsymbol{x} \oplus_r \boldsymbol{a}) = \boldsymbol{Ad}_{\exp(-\boldsymbol{a})} f(\boldsymbol{x}) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \left(-\operatorname{ad}_{f(e)} + \mathrm{d}^{\mathrm{r}} f_e\right) \boldsymbol{a}, \quad \forall \boldsymbol{x} \in \mathbb{M}, \boldsymbol{a} \in \mathbb{R}^{\dim \mathfrak{m}}, \tag{6.24}$$

in which case for any two trajectories $\boldsymbol{x}_1, \boldsymbol{x}_2$ of the system, the difference $\eta := \boldsymbol{x}_1 \ominus_r \boldsymbol{x}_2$ satisfies the linear ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} \eta = \left(-\operatorname{ad}_{f(e)} + \mathrm{d}^{\mathrm{r}} f_e\right) \eta. \tag{6.25}$$

Note that the difference ODE (6.25) resembles the more general sensitivity ODE (6.11). To highlight the importance of this result assume that $\bar{\boldsymbol{x}}$ is a fixed-point of a group-linear system. Then a trajectory $\boldsymbol{x}$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is determined by

$$\boldsymbol{x}(t) = \bar{\boldsymbol{x}} \oplus \eta(t), \tag{6.26}$$

for $\eta(0) = \boldsymbol{x}_0 \ominus_r \bar{\boldsymbol{x}}$ and $\dot{\eta}$ satisfying (6.25). Thus it is sufficient to integrate a linear ODE on $\mathbb{R}^n$ to find a solution to the Lie group ODE.

Proof. Inserting (6.22) into (6.20) gives

$$\begin{aligned} f(\boldsymbol{y} \oplus_r \boldsymbol{a}) &= \mathbf{Ad}_{\exp(-\boldsymbol{a})} \left(f(\boldsymbol{y}) - f(e)\right) + f(e) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a} \\ &= \mathbf{Ad}_{\exp(-\boldsymbol{a})} f(\boldsymbol{y}) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a} + (I - \mathbf{Ad}_{-\exp(-\boldsymbol{a})}) f(e), \end{aligned} \tag{6.27}$$

which via (5.33) gives (6.24). Furthermore, inserting (6.22) into (6.17) (after substituting $\boldsymbol{y} = e$) gives

$$\begin{aligned} \dot{\boldsymbol{a}} &= (\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}})^{-1} f(\exp \boldsymbol{a}) - (\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}})^{-1} f(e) \\ &= (\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}})^{-1} \left(f(e) + \mathrm{d}^{\mathrm{r}} \exp_a \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a}\right) - (\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}})^{-1} f(e) \\ &= \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a} + \left(\left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1} - \left(\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{a}}\right)^{-1}\right) f(e) \\ &\overset{(5.36)}{=} \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a} + \mathrm{ad}_{\boldsymbol{a}} f(e) = \mathrm{d}^{\mathrm{r}} f_e \boldsymbol{a} - \mathrm{ad}_{f(e)} \boldsymbol{a}. \end{aligned} \tag{6.28}$$

□

For systems on $\mathbb{R}^n$ the linearity condition simplifies to

$$f(\boldsymbol{x} + \boldsymbol{a}) = f(\boldsymbol{x}) + \nabla f_{\boldsymbol{x}=0} \boldsymbol{a}, \tag{6.29}$$

i.e. f being a linear mapping. Remark that if (6.29) holds for a single $\boldsymbol{x}$, then it holds for all $\boldsymbol{x}$. This property does not however translate to general group-linearity: group-linear systems can satisfy (6.24) for some $\boldsymbol{x}$ without being group-linear (which requires satisfaction for all $\boldsymbol{x}$). We show this with an example.

Example 6.2

Let $f(\boldsymbol{x}) = \boldsymbol{x} \ominus e$ for $\boldsymbol{x} \in \mathbb{SO}(3)$, then $f(e) = 0$ and $\mathrm{d}^{\mathrm{r}} f_{\boldsymbol{x}=e} = I$ and it holds that

$$f(e \oplus \boldsymbol{a}) = \boldsymbol{a} = \mathbf{Ad}_{\exp(-\boldsymbol{a})} f(e) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \left(-\mathrm{ad}_{f(e)} + \mathrm{d}^{\mathrm{r}} f_e\right) \boldsymbol{a}, \tag{6.30}$$

since $\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \boldsymbol{a} \overset{(5.38)}{=} \boldsymbol{a}$. Thus f satisfies the linearity condition (6.24) around e. For systems on $\mathbb{R}^n$ this would be enough to conclude that the system is linear.
However, f is not group-linear. Figure 6.2 shows the value of $\boldsymbol{a}_i = \boldsymbol{y}_i \ominus \boldsymbol{x}_i$ for two pairs of trajectories such that $\boldsymbol{a}_1(0) = \boldsymbol{a}_2(0)$ but with different values for $\boldsymbol{x}_i(0)$. As can be seen the dynamics are different which shows that global linearity does not hold.

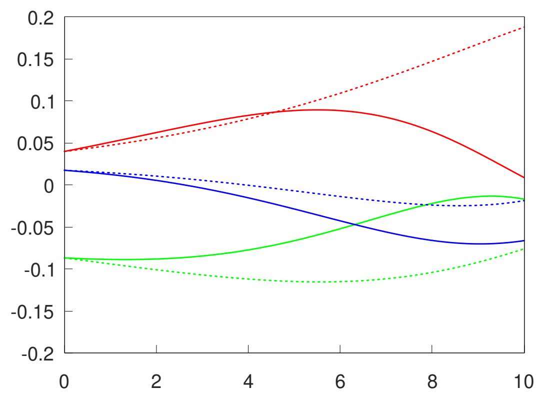


Figure 6.1.: Component values of $\boldsymbol{y} \ominus_r \boldsymbol{x}$ for two pairs of trajectories $(\boldsymbol{x}_i, \boldsymbol{y}_i)$ of the system $\mathrm{d^r}\,\boldsymbol{x}_t = \boldsymbol{x} \ominus_r e$. Although the values start out identical they diverge, indicating that the system is not group-linear.

6.2.1. System Linearization

Take $\mathrm{d^r}\,\boldsymbol{x}_t = f(\boldsymbol{x}(t))$ and a nominal curve $\boldsymbol{x}_l(t)$ that does not necessarily satisfy the dynamics. Consider the difference $\boldsymbol{a} = \boldsymbol{x} \ominus_r \boldsymbol{x}_l$; the dynamics of $\boldsymbol{a}$ become

$$\dot{\boldsymbol{a}} = (\mathrm{d^r}\exp_{\boldsymbol{a}})^{-1} f(\boldsymbol{x}_l \oplus_r \boldsymbol{a}) - (\mathrm{d^l}\exp_{\boldsymbol{a}})^{-1}\,\mathrm{d^r}(\boldsymbol{x}_l)_t. \tag{6.31}$$

This is a system on $\mathbb{R}^{\dim \mathfrak{m}}$, so it can be linearized as usual into $\dot{z} = A(t)z + K(t)$ where

$$\begin{aligned} A(t) &= \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}}\right|_{\boldsymbol{a}=0} \left(\left(\mathrm{d^r}\exp_{\boldsymbol{a}}\right)^{-1} f\left(\boldsymbol{x}_l \oplus_r \boldsymbol{a}\right) - \left(\mathrm{d^l}\exp_{\boldsymbol{a}}\right)^{-1} \mathrm{d^r}(\boldsymbol{x}_l)_t \right), \\ K(t) &= f(\boldsymbol{x}_l) - \mathrm{d^r}(\boldsymbol{x}_l)_t. \end{aligned} \tag{6.32}$$

We seek to simplify the expression for $A(t)$. First note that

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}} \operatorname{ad}_{\boldsymbol{a}} \boldsymbol{b} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}} - \operatorname{ad}_{\boldsymbol{b}} \boldsymbol{a} = -\operatorname{ad}_{\boldsymbol{b}}. \tag{6.33}$$

Furthermore, for higher-order brackets $k > 1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}}\right|_{\boldsymbol{a}} \operatorname{ad}^k_{\boldsymbol{a}} \boldsymbol{b} = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}}\right|_{\boldsymbol{a}} \left[\boldsymbol{a}, \operatorname{ad}^{k-1}_{\boldsymbol{a}} \boldsymbol{b}\right] = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}_1}\right|_{\boldsymbol{a}_1=\boldsymbol{a}} \operatorname{ad}_{\boldsymbol{a}_1}(\operatorname{ad}^{k-1}_{\boldsymbol{a}} \boldsymbol{b}) + \operatorname{ad}_{\boldsymbol{a}} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}} \operatorname{ad}^{k-1}_{\boldsymbol{a}} = -\operatorname{ad}_{\operatorname{ad}^{k-1}_{\boldsymbol{a}} \boldsymbol{b}} + \operatorname{ad}_{\boldsymbol{a}} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}} \operatorname{ad}^{k-1}_{\boldsymbol{a}}.$$

In particular, at $\boldsymbol{a} = 0$ only the $k = 1$ term is non-zero. Then, from the definition (5.35) of $\mathrm{d^r}\exp^{-1}_{\boldsymbol{a}}$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}}\right|_{\boldsymbol{a}=0} (\mathrm{d^r}\exp_{\boldsymbol{a}})^{-1}\boldsymbol{b} = \left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}}\right|_{\boldsymbol{a}=0} \sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!} \operatorname{ad}^n_{\boldsymbol{a}} \boldsymbol{b} = -\frac{1}{2} \operatorname{ad}_{\boldsymbol{b}}, \tag{6.34}$$

and from (5.36) it follows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{a}}\right|_{\boldsymbol{a}=0} (\mathrm{d^l}\exp_{\boldsymbol{a}})^{-1}\boldsymbol{b} = \frac{1}{2} \operatorname{ad}_{\boldsymbol{b}}. \tag{6.35}$$

We can now simplify (6.32) and retrieve an expression for the linearized system.

A tangent-space linearization of the system $\mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t = f(\boldsymbol{x}(t))$ around $\boldsymbol{x}_l(t)$ is given by

$$\dot{z} = A(t)z + K(t), \tag{6.36}$$

where

$$\begin{aligned} A(t) &= -\frac{1}{2}\operatorname{ad}_{f(\boldsymbol{x}_l(t))} + \mathrm{d}^{\mathrm{r}}\, f_{\boldsymbol{x}_l(t)} - \frac{1}{2}\operatorname{ad}_{\mathrm{d}^{\mathrm{r}}(\boldsymbol{x}_l(t))_t}, \\ K(t) &= f(\boldsymbol{x}_l(t)) - \mathrm{d}^{\mathrm{r}}(\boldsymbol{x}_l(t))_t. \end{aligned} \tag{6.37}$$

In the special case when $\boldsymbol{x}_l$ is dynamically feasible, i.e. $\mathrm{d}^{\mathrm{r}}(\boldsymbol{x}_l(t))_t = f(\boldsymbol{x}_l(t))$, we retrieve the familiar expressions

$$\begin{aligned} A(t) &= -\operatorname{ad}_{f(\boldsymbol{x}_l(t))} + \mathrm{d}^{\mathrm{r}}\, f_{\boldsymbol{x}_l(t)}, \\ K(t) &= 0. \end{aligned} \tag{6.38}$$

Letting $\bar{x}(t) = \boldsymbol{x}_l(t) \oplus_r \boldsymbol{a}(t)$, where $\boldsymbol{a}(t)$ is a linearized tangent space trajectory, be the linearized (group) trajectory. We retrieve its dynamics as

$$\begin{aligned} \mathrm{d}^{\mathrm{r}}(\bar{x})_t &= \mathrm{d}^{\mathrm{r}}(\boldsymbol{x}_l \oplus \boldsymbol{a})_t \\ &\overset{(5.48),(5.49)}{=} \mathbf{Ad}_{\exp(-\boldsymbol{a})}\, \mathrm{d}^{\mathrm{r}}(\boldsymbol{x}_l)_t + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \dot{\boldsymbol{a}} = \mathbf{Ad}_{\exp(-\boldsymbol{a})}\, f(\boldsymbol{x}_l) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}(-\operatorname{ad}_{f(\boldsymbol{x}_l)} + \mathrm{d}^{\mathrm{r}}\, f_{\boldsymbol{x}_l})\boldsymbol{a} \\ &\overset{(5.33)}{=} \left(I - \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \operatorname{ad}_{\boldsymbol{a}}\right) f(\boldsymbol{x}_l) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \left(-\operatorname{ad}_{f(\boldsymbol{x}_l)} + \mathrm{d}^{\mathrm{r}}\, f_{\boldsymbol{x}_l}\right) \boldsymbol{a} \\ &= f(\boldsymbol{x}_l) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \mathrm{d}^{\mathrm{r}}\, f_{\boldsymbol{x}_l} \boldsymbol{a}. \end{aligned} \tag{6.39}$$

That is, the (group) linearization of $\mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t = f(\boldsymbol{x}(t))$ around a dynamically feasible $\boldsymbol{x}_l(t)$ is given by the system $\mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t = f_l(t, \boldsymbol{x}(t))$, where

$$f_l(t, \boldsymbol{x}) = f(\boldsymbol{x}_l(t)) + \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{x} \ominus_r \boldsymbol{x}_l(t)} \mathrm{d}^{\mathrm{r}}\, f_{\boldsymbol{x}_l(t)} \left(\boldsymbol{x} \ominus_r \boldsymbol{x}_l(t)\right). \tag{6.40}$$

Note that the time dependence of f_l is only through $\boldsymbol{x}_l(t)$, so if $\boldsymbol{x}_l$ is a fixed point then the dynamics of the linearized system do not depend on time. This system satisfies the group linearity condition (6.24) at $\boldsymbol{x}_l$, but is *not* a group-linear system (c.f. Example 6.2).

6.3. Monotonicity

Very much a work in progress

Monotonicity is a useful property of dynamical systems that can be leveraged in order to bound the envelope of possible behaviors by a small number of extremal trajectories. For instance, a forward-traveling vehicle that is trying to stop is always better of the less it accelerates, which means that it is sufficient to analyze it's minimal acceleration in order to determine whether it can stop in time.

Monotonicity on $\mathbb{R}^n$ Monotonicity is usually defined with respect to a *cone*—a set with the property that $0 \in K$ and $\boldsymbol{x} \in K \implies \alpha K \in K$ for $\alpha \geq 0$. For a cone we can define an ordering $\preceq_K$ such that

$$x \preceq_K y \iff y - x \in K. \tag{6.41}$$

Monotonicity of a function f can then be defined as the following property:

$$x \preceq_K y \implies f(x) \preceq_K f(y). \tag{6.42}$$

Monotonicity on Lie groups The usual notion of monotonicity only applies for *ordered spaces*, which is a property that is not present in the usual Lie groups used in robotics. Indeed, for a circle ordering makes little sense. However, the tangent space of a Lie group is monotone which makes it possible to define a notion of *local monotonicity* in a way that is analogous to the Euclidean case.

Definition 6.2. *A function $f : \mathbb{M} \to \mathbb{N}$ is locally monotone around $\boldsymbol{Z} \in \mathbb{M}$ with respect to a cone $K \subset \mathfrak{m} \cong \mathbb{R}^n$ if for all $\boldsymbol{a}, \boldsymbol{b}$ that are sufficiently small it holds that*

$$\boldsymbol{a} \preceq_K \boldsymbol{b} \implies f(\boldsymbol{Z} \oplus_r \boldsymbol{a}) \ominus_r f(\boldsymbol{Z}) \preceq_K f(\boldsymbol{Z} \oplus_r \boldsymbol{b}) \ominus_r f(\boldsymbol{Z}). \tag{6.43}$$

When $\mathbb{M}$ and $\mathbb{N}$ are Euclidean spaces $\boldsymbol{Z}$ can be set to zero to retrieve the original definition.

- Define mixed monotonicity corresponding to Def (6.2).
- Derive a jacobian condition on f for mixed monotonicity that is analogous to sign-stability?
- Create a dynamical system that over-approximates reach sets of one of these forms:
  - MID-DOWN-UP: $\mathrm{A}(\boldsymbol{X}, l, u) = \{\boldsymbol{Y} : l \preceq_K \boldsymbol{Y} \ominus_r \boldsymbol{X} \preceq_K u\}$
  - MID-SINGLE: $\mathrm{A}(\boldsymbol{X}, k) = \{\boldsymbol{Y} : -k \preceq_K \boldsymbol{Y} \ominus_r \boldsymbol{X} \preceq_K k\}$
  - MID-RADIUS: $\mathrm{A}(\boldsymbol{X}, r) = \{\boldsymbol{Y} : \|Y \ominus_r \boldsymbol{X}\| < r\}$
  - The values l, u need to be twisted as part of the mapping

The derivative of the mapping $\boldsymbol{a} \mapsto f(\boldsymbol{Z} \oplus_r \boldsymbol{a}) \ominus_r f(\boldsymbol{Z})$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is

$$\mathrm{d}^{\mathrm{r}} \left(f(\boldsymbol{Z} \oplus_r \boldsymbol{a}) \ominus_r f(\boldsymbol{Z}) \right)_{\boldsymbol{a}} = \left[\mathrm{d}^{\mathrm{r}} \exp_{f(\boldsymbol{Z} \oplus_r \boldsymbol{a}) \ominus_r f(\boldsymbol{Z})} \right]^{-1} \mathrm{d}^{\mathrm{r}} f_{\boldsymbol{Z} \oplus_r \boldsymbol{a}} \; \mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}} \tag{6.44}$$

It follows that this is what we need to make sign-stable, so the decomposition should depend on it. Challenge is that the decomposition may have to depend on both $\boldsymbol{a}$ and on $\boldsymbol{Z}$.

Reach mapping: Set $\left\{ \boldsymbol{X} : \underline{\boldsymbol{a}} \preceq_K \boldsymbol{X} \ominus_r \boldsymbol{Z} \preceq_K \overline{\boldsymbol{a}} \right\}$
Decomposition function g s.t. $f(\boldsymbol{Z} \oplus_r \boldsymbol{a}) \ominus_r f(\boldsymbol{Z}) = g_{\boldsymbol{Z}}(\boldsymbol{a}, \boldsymbol{a})$
Mapped set:

$$\left\{ \boldsymbol{X} : g_{\boldsymbol{Z}}(\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}) \preceq_K \boldsymbol{X} \ominus_r f(\boldsymbol{Z}) \preceq_K g_{\boldsymbol{Z}}(\overline{\boldsymbol{a}}, \underline{\boldsymbol{a}}) \right\}. \tag{6.45}$$

- How to go from monotonicity of $f : \mathbb{M} \to \mathbb{R}^m$ to monotonicity of the flow $\phi : \mathbb{M} \to \mathbb{M}$?
- How is decomposition function done in practice? Like in Necmiyes paper?

6.4. The Magnus Expansion

For the case when the right-hand side in (6.1) only depends on t,

$$\mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t = \boldsymbol{a}(t), \tag{6.46}$$

which is the Lie group equivalent of a linear time-varying (LTV) system, we can posit that the solution be on the form

$$\boldsymbol{x}(t) = \exp(\Omega(t))\boldsymbol{x}_0, \quad \Omega(t) \in \mathbb{M}. \tag{6.47}$$

From the differentiation rules it follows that

$$\boldsymbol{a}(t) = \mathrm{d}^{\mathrm{r}}\, \boldsymbol{x}_t = \mathrm{d}^{\mathrm{r}} \exp_{\Omega(t)} \frac{\mathrm{d}}{\mathrm{d}t}\Omega(t), \tag{6.48}$$

which yields an ODE for $\Omega(t)$.

Theorem 6.2. *The solution of the time-varying ODE (6.46) is given by*

$$\boldsymbol{x}(t) = \exp(\Omega(t))\boldsymbol{x}_0, \tag{6.49}$$

where $\Omega(t)$ satisfies the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Omega(t) &= \left(\mathrm{d}^{\mathrm{r}} \exp_{\Omega(t)}\right)^{-1} \boldsymbol{a}(t), \\ \Omega(0) &= 0. \end{aligned} \tag{6.50}$$

The initial value problem for Ω may still be challenging to solve in case an expression for $\left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{a}}\right)^{-1}$ is not available. The **Magnus expansion** is obtained by setting $\boldsymbol{a} = \epsilon\tilde{\boldsymbol{a}}$ and expressing Ω as a series

$$\Omega(t) = \sum_{k\geq 1} \epsilon^k \Omega_k(t). \tag{6.51}$$

Inserting this in (6.50) and comparing powers of ϵ yields

$$\begin{aligned} \Omega_1(t) &= \int_0^t \boldsymbol{a}(s_1)\mathrm{d}s_1, \\ \Omega_2(t) &= -\frac{1}{2}\int_0^t [\Omega_1(s_1), \boldsymbol{a}(s_1)]\, \mathrm{d}s_1 = \frac{1}{2}\int_0^t \int_0^{s_2} [\boldsymbol{a}(s_1), \boldsymbol{a}(s_2)]\, \mathrm{d}s_2 \mathrm{d}s_1, \\ \Omega_3(t) &= \frac{1}{6}\int_0^t \int_0^{s_1} \int_0^{s_2} [\boldsymbol{a}(s_1), [\boldsymbol{a}(s_2), \boldsymbol{a}(s_3)]] + [[\boldsymbol{a}(s_1), \boldsymbol{a}(s_2)]\, , \boldsymbol{a}(s_3)]\, \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1, \end{aligned} \tag{6.52}$$

and so on for higher powers of k.

6.4.1. Example

Consider the initial value problem on $\mathbb{SE}(2)$:

$$\dot{X}(t) = A(t)X(t), \quad X(0) = X_0, \quad X \in \mathbb{SE}(2), \quad A \in \mathfrak{se}(2). \tag{6.53}$$

We assume that $A(t) = \hat{a}(t)$ is a known curve, i.e.

$$A(t) = \begin{bmatrix} 0 & -\theta(t) & u(t) \\ \theta(t) & 0 & v(t) \\ 0 & 0 & 0 \end{bmatrix} \tag{6.54}$$

According to Theorem 6.2 the solution is then

$$x(t) = \operatorname{Exp}(\Omega(t))x_0 = \operatorname{Exp}\left(\sum_{k\geq 0} \Omega_k(t)\right) x_0. \tag{6.55}$$

The Lie algebra $\mathfrak{se}(2)$ is not nilpotent, so the exact solution requires the full Magnus expansion. Below we develop an approximate solution corresponding to the first two terms

$$x(t) \approx \operatorname{Exp}\left(\int_0^t A(t)\mathrm{d}t + \frac{1}{2}\int_0^t \left(\int_0^{t_1} [A(t_1), A(t_2)]\,\mathrm{d}t_2\right)\mathrm{d}t_1\right) x_0. \tag{6.56}$$

To find Ω_2 we consider the commutator of matrices in $\mathfrak{se}(2)$ on the form (6.54):

$$\begin{aligned}
[A(t_1), A(t_2)] &= \begin{bmatrix} 0 & -\theta(t_1) & u(t_1) \\ \theta(t_1) & 0 & v(t_1) \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\theta(t_2) & u(t_2) \\ \theta(t_2) & 0 & v(t_2) \\ 0 & 0 & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & -\theta(t_2) & u(t_2) \\ \theta(t_2) & 0 & v(t_2) \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\theta(t_1) & u(t_1) \\ \theta(t_1) & 0 & v(t_1) \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\theta(t_1)\theta(t_2) & 0 & -\theta(t_1)v(t_2) \\ 0 & -\theta(t_1)\theta(t_2) & \theta(t_1)u(t_2) \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -\theta(t_1)\theta(t_2) & 0 & -\theta(t_2)v(t_1) \\ 0 & -\theta(t_1)\theta(t_2) & \theta(t_2)u(t_1) \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\theta(t_1)v(t_2) + \theta(t_2)v(t_1) \\ 0 & 0 & \theta(t_1)u(t_2) - \theta(t_2)u(t_1) \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

In the affine case where $\theta(t) = \theta_0 + a_\theta t$, and similarly for u and v, we get after evaluating the integrals:

$$\begin{bmatrix} a(t) \\ b(t) \\ x(t) \\ y(t) \end{bmatrix} \approx \exp\left(\begin{bmatrix} \theta_0 t + a_\theta t\frac{t^2}{2} \\ u_0 t + a_u\frac{t^2}{2} + \theta_0 a_v\frac{t^3}{12} - a_\theta v_0\frac{t^3}{12} \\ v_0 t + a_v\frac{t^2}{2} - \theta_0 a_u\frac{t^3}{12} + a_\theta u_0\frac{t^3}{12} \end{bmatrix}\right).$$

7. Probability Theory

This material is largely inspired by [4] which uses the left-normal.

Definition 7.1. *The (right) expected value of a random variable $\boldsymbol{X}$ taking values on a Lie group $\mathbb{M}$ is defined as*

$$\mathbb{E}_r \boldsymbol{X} = \arg\min_{\mu} \mathbb{E}\|\boldsymbol{X} \ominus_r \mu\|^2. \tag{7.1}$$

Alternatively, the expected value can be implicitly defined as

$$\mathbb{E}_r \boldsymbol{X} = \mu \quad \Longleftrightarrow \quad \mathbb{E}(\boldsymbol{X} \ominus_r \mu) = 0, \tag{7.2}$$

as shown in [15]. From the definition it follows that left-composition with a deterministic $\boldsymbol{y}$ satisfies

$$\mathbb{E}_r(\boldsymbol{y} \circ \boldsymbol{X}) = \boldsymbol{y} \circ \mathbb{E}_r(\boldsymbol{X}). \tag{7.3}$$

Definition 7.2. *The (right) covariance of a random variable $\boldsymbol{X}$ taking values on a Lie group $\mathbb{M}$ is defined as*

$$\mathbb{V}\mathrm{ar}_r \boldsymbol{X} = \mathbb{E}\left[(\boldsymbol{X} \ominus_r \mathbb{E}\boldsymbol{X})(\boldsymbol{X} \ominus_r \mathbb{E}\boldsymbol{X})^T\right]. \tag{7.4}$$

The right covariance is invariant under left group action due to (7.3):

$$\mathbb{V}\mathrm{ar}_r(\boldsymbol{y} \circ \boldsymbol{X}) = \mathbb{E}\left[((\boldsymbol{y} \circ \boldsymbol{X}) \ominus_r \mathbb{E}(\boldsymbol{y} \circ \boldsymbol{X}))((\boldsymbol{y} \circ \boldsymbol{X}) \ominus_r \mathbb{E}(\boldsymbol{y} \circ \boldsymbol{X}))^T\right] = \mathbb{V}\mathrm{ar}_r \boldsymbol{X}. \tag{7.5}$$

7.1. Normal Distributions

A variable $\boldsymbol{x} \in \mathbb{M}$ is said to follow a **right-normal** distribution $\mathcal{N}_r(\hat{\boldsymbol{x}}, \Sigma)$ if

$$\boldsymbol{x} \ominus_r \hat{\boldsymbol{x}} \in \mathcal{N}(0, \Sigma), \tag{7.6}$$

which is equivalent to $\boldsymbol{x}$ being on the form

$$\boldsymbol{x} = \hat{\boldsymbol{x}} \oplus_r \boldsymbol{\alpha} = \hat{\boldsymbol{x}} \circ \exp(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \mathcal{N}(0, \Sigma). \tag{7.7}$$

Conversely, $\boldsymbol{x} \in \mathbb{M}$ is said to follow a **left-normal** distribution $\mathcal{N}_l(\hat{\boldsymbol{x}}, \Sigma)$ if

$$\boldsymbol{x} \ominus_l \hat{\boldsymbol{x}} \in \mathcal{N}(0, \Sigma). \tag{7.8}$$

which is equivalent to

$$\boldsymbol{x} = \hat{\boldsymbol{x}} \oplus_l \boldsymbol{\alpha} = \exp(\boldsymbol{\alpha}) \circ \hat{\boldsymbol{x}}, \quad \boldsymbol{\alpha} \in \mathcal{N}(0, \Sigma). \tag{7.9}$$

Right-normal and left-normal distributions are quite different in the kinds of uncertainty they can represent. For the right-normal the uncertainty lives in the tangent space at $\hat{\boldsymbol{x}}$, i.e. the body frame, whereas the left-normal models uncertainty in the global frame. Depending on the application one may be more suited than the other. The left-normal distribution is for example appropriate for odometry applications where uncertainty builds up in the pose of the vehicle, whereas the right-normal may be more suitable when modeling the uncertainty of a landmark that is being tracked in the body frame.

The expected value of a function $f : \mathbb{M} \to \mathbb{R}^k$ of a variable $\boldsymbol{x} \sim \mathcal{N}_r(0, \Sigma)$ can be calculated as

$$\mathbb{E} f(\boldsymbol{x}) = \mathbb{E} f(\hat{\boldsymbol{x}} \oplus_r \boldsymbol{\alpha}) = \int f(\hat{\boldsymbol{x}} \oplus_r \boldsymbol{\alpha}) \frac{1}{\sqrt{(2\pi)^m \det \Sigma}} \exp\left(-\frac{1}{2} \boldsymbol{\alpha}^T \Sigma^{-1} \boldsymbol{\alpha}\right) \mathrm{d}\boldsymbol{\alpha}, \tag{7.10}$$

and similarly for the left-normal.

Plot left-normal and right-normals on SE2

Convert left-normal to right-normal and vice versa (approximations)

What about left-right normals (noise on both sides)?

7.2. Propagating Uncertainty

In applications a probability distribution is maintained to represent the uncertainty associated with the state. As time passes and new information is obtained the uncertainty needs to be updated. In this section we study how uncertainty propagates through group composition, which later is used to derive an ODE over probability distributions, and through Bayesian updates.

7.2.1. Distribution of Compositions

For a normal distribution on a Euclidean vector space it is straightforward to obtain the distribution of $\boldsymbol{a} + \boldsymbol{b}$ when $\boldsymbol{a}$ and $\boldsymbol{b}$ are normal. Unfortunately this becomes more involved on Lie groups due to non-commutativity and the exponential map property that $\exp \boldsymbol{a} \exp \boldsymbol{b} \neq \exp(\boldsymbol{a} + \boldsymbol{b})$.

We seek an approximation of the distribution of $\boldsymbol{y} = \hat{\boldsymbol{x}}_1 \circ \exp(\boldsymbol{\alpha}_1) \circ \hat{\boldsymbol{x}}_2 \circ \exp(\boldsymbol{\alpha}_2)$ for independent variables $\boldsymbol{\alpha}_1 \sim \mathcal{N}(0, P_1)$ and $\boldsymbol{\alpha}_2 \sim \mathcal{N}(0, P_2)$. First note that

$$\boldsymbol{y} = \hat{\boldsymbol{x}}_1 \hat{\boldsymbol{x}}_2 \hat{\boldsymbol{x}}_2^{-1} \exp(\boldsymbol{\alpha}_1) \boldsymbol{x}_2 \exp(\boldsymbol{\alpha}_2) \overset{(4.14c)}{=} \hat{\boldsymbol{x}}_1 \hat{\boldsymbol{x}}_2 \exp\left(\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1\right) \exp(\boldsymbol{\alpha}_2) \tag{7.11}$$

Since the composed variable is not necessarily right-normal the best we can do is to match moments, i.e. find $\hat{\boldsymbol{y}}$ and P_y s.t.

$$\begin{aligned} \hat{\boldsymbol{y}} &= \mathbb{E}_r \left[\hat{\boldsymbol{x}}_1 \hat{\boldsymbol{x}}_2 \exp\left(\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1\right) \exp(\boldsymbol{\alpha}_2)\right] = \hat{\boldsymbol{x}}_1 \hat{\boldsymbol{x}}_2 \mathbb{E}_r \left[\exp\left(\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1\right) \exp(\boldsymbol{\alpha}_2)\right], \\ P_y &= \mathbb{V}\mathrm{ar}_r \left[\hat{\boldsymbol{x}}_1 \hat{\boldsymbol{x}}_2 \exp\left(\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1\right) \exp(\boldsymbol{\alpha}_2)\right] = \mathbb{V}\mathrm{ar}_r \left[\exp\left(\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1\right) \exp(\boldsymbol{\alpha}_2)\right]. \end{aligned} \tag{7.12}$$

The exponential product can be expanded into a single exponential through the BCH formula (5.61), yielding

$$\exp\left(\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1\right) \exp(\boldsymbol{\alpha}_2) = \exp\left(\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \frac{1}{2}\left[\mathbf{Ad}_{\boldsymbol{x}_2^{-1}} \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\right] + \ldots\right), \tag{7.13}$$

but the expression inside the exponential is **not** zero-mean. This suggests that exactly matching moments requires iterative methods. Instead a reasonable approximation is to disregard the higher-order terms in the BCH expansion (the fourth order term is the first one with a non-zero mean).

Let $\boldsymbol{x}_1 \sim \mathcal{N}_r(\hat{\boldsymbol{x}}_1, P_1)$ and $\boldsymbol{x}_2 \sim \mathcal{N}_r(\hat{\boldsymbol{x}}_2, P_2)$. Then it approximately holds that

$$\boldsymbol{x}_1 \circ \boldsymbol{x}_2 \sim \mathcal{N}_r \left(\hat{\boldsymbol{x}}_1 \circ \hat{\boldsymbol{x}}_2, \mathbf{Ad}_{\hat{\boldsymbol{x}}_2^{-1}} P_1 \left(\mathbf{Ad}_{\hat{\boldsymbol{x}}_2^{-1}} \right)^T + P_2 \right). \tag{7.14}$$

If P_2 (and thus $\boldsymbol{\alpha}_2$) is "small" an alternative approximation is obtained via (5.63) that directly yields a zero-mean expression.

$$\boldsymbol{x}_1 \circ \boldsymbol{x}_2 \sim \mathcal{N}_r \left(\hat{\boldsymbol{x}}_1 \circ \hat{\boldsymbol{x}}_2, \mathbf{Ad}_{\hat{\boldsymbol{x}}_2^{-1}} P_1 \left(\mathbf{Ad}_{\hat{\boldsymbol{x}}_2^{-1}} \right)^T + \left(\mathrm{d}^r \exp_{\mathbf{Ad}_{\hat{\boldsymbol{x}}_2^{-1}}} \right)^{-1} P_2 \left(\mathrm{d}^r \exp_{\mathbf{Ad}_{\hat{\boldsymbol{x}}_2^{-1}}} \right)^{-T} \right). \tag{7.15}$$

An analogous result is obtained for the left-normal by noting that

$$\boldsymbol{y} = \exp(\boldsymbol{\alpha}_1)\boldsymbol{x}_1 \exp(\boldsymbol{\alpha}_2)\boldsymbol{x}_1^{-1}\boldsymbol{x}_1\boldsymbol{x}_2 \overset{(4.14c)}{=} \exp(\boldsymbol{\alpha}_1) \exp\left(\mathbf{Ad}_{\boldsymbol{x}_1} \boldsymbol{\alpha}_2\right) \boldsymbol{x}_1\boldsymbol{x}_2. \tag{7.16}$$

Let $\boldsymbol{x}_1 \sim \mathcal{N}_l(\hat{\boldsymbol{x}}_1, P_1)$ and $\boldsymbol{x}_2 \sim \mathcal{N}_l(\hat{\boldsymbol{x}}_2, P_2)$. Then it approximately holds that

$$\boldsymbol{x}_1 \circ \boldsymbol{x}_2 \sim \mathcal{N}_l \left(\hat{\boldsymbol{x}}_1 \circ \hat{\boldsymbol{x}}_2, P_1 + \mathbf{Ad}_{\hat{\boldsymbol{x}}_1} P_2 \left(\mathbf{Ad}_{\hat{\boldsymbol{x}}_1} \right)^T \right). \tag{7.17}$$

7.2.2. Bayesian Updates

Bayesian methods maintain a probability distribution that is updated when new measurements arrive. For a prior $\boldsymbol{x} \sim \mathcal{N}_r(\hat{\boldsymbol{x}}, P)$ and a conditional measurement $y \mid \boldsymbol{x} \sim \mathcal{N}(h(\boldsymbol{x}), R)$, the posteriori distribution of $\boldsymbol{x}|y$ represents the updated belief about x. We show how to calculate the posterior distribution.

By definition, $e \coloneqq \boldsymbol{x} \ominus_r \hat{\boldsymbol{x}} \sim \mathcal{N}(0, P)$. We expand the measurement model h around $\hat{\boldsymbol{x}}$

$$h(\boldsymbol{x}) \approx h + He, \quad h \coloneqq h(\hat{\boldsymbol{x}}),\ H \coloneqq \mathrm{d}^r h_{\boldsymbol{x}=\hat{\boldsymbol{x}}}. \tag{7.18}$$

If the measurement model is linear the expansion is exact; for nonlinear models this is an approximation that is necessary to maintain the Gaussian property of the distribution. The variable $z = y - He$ is distributed according to $\mathcal{N}(h, R)$ and is independent of e. Therefore the joint distribution of e and z is

$$\begin{bmatrix} e \\ z \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ h \end{bmatrix}, \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \right). \tag{7.19}$$

Furthermore $\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ H & I \end{bmatrix} \begin{bmatrix} e \\ z \end{bmatrix}$ which means that the joint distribution of e and y is

$$\begin{bmatrix} e \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ h \end{bmatrix}, \begin{bmatrix} I & 0 \\ H & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ H & I \end{bmatrix}^T \right) = \mathcal{N}\left(\begin{bmatrix} 0 \\ h \end{bmatrix}, \begin{bmatrix} P & PH^T \\ HP & HPH^T + R \end{bmatrix} \right). \tag{7.20}$$

Next the well-known formula for the marginalization of a normal distribution can be applied which gives that

$$(\boldsymbol{x} \ominus_r \hat{\boldsymbol{x}}) \mid y \sim \mathcal{N}\left(PH^T\left(HPH^T+R\right)^{-1}(y-h(\hat{\boldsymbol{x}})), P-PH^T\left(HPH^T+R\right)^{-1}HP\right). \tag{7.21}$$

Since this normal is typically not zero-mean it does not describe a right-normal distribution over $\boldsymbol{x}$. Matching the first and second moments gives with $K = PH^T(HPH^T+R)^{-1}$:

$$\begin{aligned} \hat{\boldsymbol{x}}^+ &= \mathbb{E}_r\left(\hat{\boldsymbol{x}}\exp\left(K\left(y-h(\hat{x})\right)+(I-KH)Pw\right)\right) = \hat{\boldsymbol{x}}\mathbb{E}_r\left(\exp\left(K\left(y-h(\hat{x})\right)+(I-KH)Pw\right)\right), \\ P^+ &= \mathbb{V}\mathrm{ar}_r\left(\hat{\boldsymbol{x}}\exp\left(K\left(y-h(\hat{x})\right)+(I-KH)Pw\right)\right) = \mathbb{V}\mathrm{ar}_r\left(\exp\left(K\left(y-h(\hat{x})\right)+(I-KH)Pw\right)\right), \end{aligned} \tag{7.22}$$

where $w \in \mathcal{N}(0, I)$. As before obtaining a closed-form expression for the posterior of $\boldsymbol{x}$ requires approximation.

Consider a prior $\boldsymbol{x} \sim \mathcal{N}_r(\hat{\boldsymbol{x}}, P)$ and a measurement $y \sim \mathcal{N}(h(\boldsymbol{x}), R)$. An approximate of the posterior distribution is the right-normal $\mathcal{N}_r(\hat{\boldsymbol{x}}^+, P^+)$ described by

$$\begin{aligned} \hat{\boldsymbol{x}}^+ &= \hat{\boldsymbol{x}} \oplus_r K\left(y-h(\hat{\boldsymbol{x}})\right), \\ P^+ &= (I-KH)\,P, \end{aligned} \tag{7.23}$$

where $H = \mathrm{d}^r h_{\boldsymbol{x}=\hat{\boldsymbol{x}}}$ and $K \coloneqq PH^T\left(HPH^T+R\right)^{-1}$ is known as the (Kalman) gain matrix.

Joseph stabilization The covariance update can be stabilized to ensure symmetry of P^+ under possible round-off errors by noting that $PH^TK^T = K(HPH^T+R)K^T = KHPH^TK^T + KRK^T$ which gives an algebraically equivalent update equation

$$P^+ = (I-KH)P(I-KH)^T + KRK^T \tag{7.24}$$

known as the *Joseph stabilized* version.

Square root form Yet another alternative is to maintain a description of the square root covariance S s.t. $P = SS^T$. Starting from the Joseph version

$$P^+ = (I-KH)SS^T(I-KH)^T + K\sqrt{R}\sqrt{R}^T K^T = \begin{bmatrix}(I-KH)S & K\sqrt{R}\end{bmatrix}\begin{bmatrix}(I-KH)S & K\sqrt{R}\end{bmatrix}^T. \tag{7.25}$$

Letting $\tilde{Q}\tilde{R} = \begin{bmatrix}(I-KH)S & K\sqrt{R}\end{bmatrix}^T$ be a QR decomposition then gives $P^+ = \tilde{R}^T\tilde{Q}^T\tilde{Q}\tilde{R} = \tilde{R}^T\tilde{R}$, which is equivalent to the square root update

$$S^+ = \tilde{R}^T. \tag{7.26}$$

8. Equivariance

- Left-invariant, right-invariant, equivariant dynamical systems

Literature:

- Grizzle on structure of control systems with symmetries [14]
- Modern paper on equivariant filtering [12]

Seems like there are two procedures for leveraging equivariance in dynamical systems: 1. A transitive action and lifting dynamics to G (van Goor). 2. A proper and free action and projecting dynamics to M/G (Grizzle).

One of the primary reasons to study Lie groups is to take advantage of the symmetry that comes with being a group. In this chapter we formalize the concept of symmetries and discuss how it can be leveraged.

Up until now we have taken $\mathbb{M}$ to be a Lie group, but in this chapter it denotes any manifold. On the other hand, $\mathscr{G}$ denotes a Lie group that *acts* on the manifold. A special case is when $\mathbb{M} = \mathscr{G}$ and the action is that of group composition.

Definition 8.1. *Consider a manifold $\mathbb{M}$ and a Lie group $\mathscr{G}$. A function $\phi : \mathscr{G} \times \mathbb{M} \to \mathbb{M}$ is a (left)* ***action*** *if*

- $\phi_e(\boldsymbol{x}) = \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \mathbb{M}$
- $\phi_h \circ \phi^g(\boldsymbol{x}) = \phi_{h \circ g}(\boldsymbol{x})$ *for all* $h, g \in \mathscr{G}$ *and all* $\boldsymbol{x} \in \mathbb{M}$

When the actions are understood from the context we write $g.\boldsymbol{x} = \phi^g \boldsymbol{x}$ to denote the action of g on $\boldsymbol{x}$. Some properties of actions are defined as follows.

Definition 8.2. *Consider an action ϕ of $\mathscr{G}$ on $\mathbb{M}$. It is said to be*

- ***Free*** *if $g.\boldsymbol{x} = \boldsymbol{x}$ if and only if $g = e$,*
- ***Proper*** *if $(g, \boldsymbol{x}) \mapsto (\boldsymbol{x}, g.\boldsymbol{x})$ is proper (the pre-image of compact sets is compact),*
- ***Transitive*** *if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{M}$ there exists $g \in \mathscr{G}$ such that $\boldsymbol{y} = \mathscr{G}\boldsymbol{x}$.*

For $\boldsymbol{x} \in \mathbb{M}$ let $\mathscr{G}\boldsymbol{x}$ denote the **orbit** at $\boldsymbol{x}$—the set of points that is "reachable" from $\boldsymbol{x}$ using the action by $\mathscr{G}$. It is known that if an action is free and proper, then the quotient space

$$\mathbb{M}/\mathscr{G} := \{\mathscr{G}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{M}\} \tag{8.1}$$

is a smooth manifold [21, Proposition 9.3.2]. When ϕ is transitive the quotient space is a singleton; in this case $\mathbb{M}$ is said to be a **homogeneous space**.

We next define equivariance of a function with respect to two actions by the same Lie group.

$$\begin{tikzcd} \mathbb{M} \arrow[r, "f"] \arrow[d, "\phi^M"'] & \mathbb{N} \arrow[d, "\phi^N"] \\ \mathbb{M} \arrow[r, "f"] & \mathbb{N} \end{tikzcd}$$

Figure 8.1.: Commutation diagram of an equivariant function $f : \mathbb{M} \to \mathbb{N}$.

Definition 8.3. *Let $\mathcal{G}$ be a Lie group that acts on two manifolds $\mathbb{M}$ and $\mathbb{N}$ via actions $\phi^{\mathbb{M}}$ and $\phi^{\mathbb{N}}$. Then a function $f : \mathbb{M} \to \mathbb{N}$ is **equivariant** with respect to $\left(\phi^{\mathbb{M}}, \phi^{\mathbb{N}}\right)$ if*

$$f \circ \phi_g^M = \phi_g^N \circ f, \quad \forall g \in \mathcal{G}. \tag{8.2}$$

Equivariance is illustrated by the commutation diagram in Figure 8.1. A special case of equivariance is **invariance**, which is when $\phi_g^N \boldsymbol{x} = \boldsymbol{x}$ is the trivial action.

Equivariance and invariance are useful since the behavior of an equivariant function $f : \mathbb{M} \to \mathbb{N}$ is completely characterized by ϕ^M/ϕ^N and the restricted function $f : \mathbb{M}/G \to \mathbb{N}$., where $\mathbb{M}/G$ is a lower-dimensional space.

8.1. Equivariant Dynamical Systems

For a function equivariance holds if function application and application by ϕ^g commute. A dynamical system is a type of function that propagetes the state into the future via integration. Thus it makes sense to define equivariance for a dynamical system as commutation of integration and application of ϕ^g. In other words, it should hold that $\boldsymbol{x}(t) = \phi^g \boldsymbol{y}(t)$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the solutions to the following two ODEs

$$\begin{cases} \mathrm{d}^r \boldsymbol{x}_t = f(t, \boldsymbol{x}(t)) \\ \boldsymbol{x}(0) = \phi^g \boldsymbol{x}_0. \end{cases}, \qquad \begin{cases} \mathrm{d}^r \boldsymbol{y}_t = f(t, \boldsymbol{y}(t)) \\ \boldsymbol{y}(0) = \boldsymbol{x}_0. \end{cases} \tag{8.3}$$

The first system starts at $\phi^g \boldsymbol{x}_0$, so $\boldsymbol{x}(t)$ represents application of ϕ^g followed by integration, whereas $\phi^g \boldsymbol{y}(t)$ represents integration followed by application of ϕ^g.

We now derive a condition on the derivative f that ensures equivariance, i.e. $\boldsymbol{x}(t) = \phi^g \boldsymbol{y}(t)$ for all t. This evidently holds for $t = 0$. Assume for induction that it holds at t and consider a small τ, then we approximately have

$$\boldsymbol{x}(t+\tau) = (\phi^g \boldsymbol{y}(t)) \exp\left(\tau f\left(t, \phi^g \boldsymbol{y}(t)\right)\right), \quad \boldsymbol{y}(t+\tau) = \boldsymbol{y}(t) \exp\left(\tau f\left(t, \boldsymbol{y}(t)\right)\right), \tag{8.4}$$

i.e. equality $\boldsymbol{x}(t+\tau) = \phi^g \boldsymbol{y}(t+\tau)$ requires

$$(\phi^g \boldsymbol{y}(t)) \exp\left(\tau f\left(t, \phi^g \boldsymbol{y}(t)\right)\right) = \phi^g \left(\boldsymbol{y}(t) \exp(\tau f(t, \boldsymbol{y}(t)))\right). \tag{8.5}$$

$$\begin{tikzcd} \mathbb{M} \arrow[r, "f"] \arrow[d, "\phi^g"'] & \mathfrak{m} \arrow[d, "\mathrm{d}\phi^g"] \\ \mathbb{M} \arrow[r, "f"] & \mathfrak{m} \end{tikzcd}$$

Figure 8.2.: Commutation diagram for an equivariant dynamical system.

This evidently holds for $\tau = 0$, and differentiation w.r.t. τ at $\tau = 0$ yields

$$f(t, \phi^g \boldsymbol{y}(t)) = \mathrm{d}^r (\phi^g)_{\boldsymbol{y}(t)} f(t, \boldsymbol{y}(t)), \tag{8.6}$$

which is a condition on f that implies equivariance for the corresponding dynamical system.

Definition 8.4. *A system*

$$\mathrm{d}^r \boldsymbol{x}_t = f(t, \boldsymbol{x}(t)) \tag{8.7}$$

is ***equivariant*** *with respect to a $\mathscr{G}$-action ϕ^g on $\mathbb{M}$ if*

$$f(t, \phi^g \boldsymbol{x}) = \mathrm{d}^r (\phi^g)_{\boldsymbol{x}} f(t, \boldsymbol{x}), \quad \forall g \in \mathscr{G}. \tag{8.8}$$

Example 8.1

Consider

$$\mathrm{d}^r \boldsymbol{x}_t = f(t), \tag{8.9}$$

i.e. a system where the right-hand side is not a function of $\boldsymbol{x}$. This is common for robotic systems—the dynamics usually do not depend on the location of the robot in the world. Typically $f(t)$ is the solution of some higher-order force balance dynamics.
Consider the action of a group $\mathscr{G}$ on $\mathbb{M}$, since f does not depend on $\boldsymbol{x}$ the equivariance condition simplifies to

$$\mathrm{d}^r (\phi^g)_{\boldsymbol{x}} = I, \tag{8.10}$$

i.e. the right derivative of the action w.r.t. the state must be the identity matrix. This is common for left actions like changing the reference coordinate system but does typically not hold for right actions.

8.1.1. Equivariance and Lyapunov Functions

We show how equivariance in a dynamical system can be applied in a Lyapunov context. Consider a function $h : M \to \mathbb{R}$ that is G-**invariant**, i.e. for all $g \in G$ it holds that

$$h \equiv h \circ \phi^g. \tag{8.11}$$

We show that if h is G-invariant, and f is G-equivariant, then $\mathscr{L}_f h(X) \coloneqq \langle \mathrm{d}h_X, f(X)\rangle$ is G-invariant as well. By differentiating (8.11) at X it follows that

$$\mathrm{d}h_X = \mathrm{d}h_{\phi^g X} \circ \mathrm{d}\phi^g_X \quad \Leftrightarrow \quad \mathrm{d}h_{\phi^g X} = \mathrm{d}h_X \circ \left(\mathrm{d}\phi^g_X\right)^{-1}. \tag{8.12}$$

Then indeed we have,

$$\mathscr{L}_f h(\phi^g X) = \left\langle \mathrm{d}h_{\phi^g X}, (f \circ \phi^g)(X)\right\rangle = \left\langle \mathrm{d}h_X \circ (\mathrm{d}\phi^g_X)^{-1}, (\mathrm{d}\phi^g_X \circ f)(X)\right\rangle = \langle \mathrm{d}h_X, f(X)\rangle, \tag{8.13}$$

or in other words, $\mathscr{L}_f h$ is G-invariant:

$$\mathscr{L}_f h \circ \phi^g \equiv \mathscr{L}_f h. \tag{8.14}$$

Thus, if the dynamics are G-equivariant and h is G-invariant, it is enough to assertain that a Lyapunov condition holds on a quotient space:

$$\mathscr{L}_f h(X) \leq 0 \qquad \forall X \in M/G, \tag{8.15}$$

since this implies that the property holds on all of M.

8.2. Equivariance of Lie Group Control Systems

If M itself is a Lie Group we can write a second-order system on control-affine form as

$$\begin{bmatrix} \mathrm{d}X_t \\ \mathrm{d}\omega_t \end{bmatrix} = f\,(X, \omega, u) \coloneqq \begin{bmatrix} \omega \\ e(X, \omega)u \end{bmatrix}, \tag{8.16}$$

where $f : M \times TM \times U \to TM \times TTM$. For a group G acting on M via $\phi^g : M \to M$ we can introduce a lifted action $\bar{\phi}^g : M \times TM \to M \times TM$ in a natural way by mapping the tangent element through the differential $\mathrm{d}\phi^g_X : T_X M \to T_{\phi^g X} M$ of ϕ^g:

$$\bar{\phi}^g \begin{bmatrix} X \\ \omega \end{bmatrix} = \begin{bmatrix} \phi^g X \\ \mathrm{d}\phi^g_X \omega \end{bmatrix}, \qquad \mathrm{d}\bar{\phi}^g_{X,\omega} \begin{bmatrix} \omega \\ a \end{bmatrix} = \begin{bmatrix} \mathrm{d}\phi^g_X \omega \\ \mathrm{d}\phi^g_X a \end{bmatrix}, \tag{8.17}$$

since $\mathrm{d}(\mathrm{d}\phi^g_X \omega)_\omega = \mathrm{d}\phi^g_X$. With this as the action the we can apply the equivariance ideas from above. We also introduce a separate action ψ^g that represents the action of $g \in G$ on an element of the input space U. We say that the control system is G-equivariant if

$$f\left(\phi^g X, \mathrm{d}\phi^g_X \omega, \psi^g u\right) = \mathrm{d}\bar{\phi}^g_X f(X, \omega, u), \tag{8.18}$$

which is illustrated as a commutation property in Figure 8.1.

Proposition 8.1. *Assume that the control system (8.16) satisfies the equi-variance property (8.18). Then, if $u(X, \omega)$ is G-equivariant with respect to $(\bar{\phi}^g, \psi^g)$, i.e.*

$$u(\phi^g X, \mathrm{d}\phi^g_X \omega) = \psi^g u(X, \omega), \tag{8.19}$$

then the closed-loop system is also G-equivariant.

Proof. From the (8.18) and (8.19) we have

$$f(\phi^g X, \mathrm{d}\phi^g_X \omega, u(\phi^g X, \mathrm{d}\phi^g_X \omega)) = f(\phi^g, \mathrm{d}\phi^g_X \omega, \psi^g u(X, \omega)) = \mathrm{d}\bar{\phi}^g_X f(X, \omega, u(X, \omega)), \tag{8.20}$$

which shows equivariance of the closed-loop system. $\square$

Part II.

Robotics Lie Groups

9. Classical Lie Groups

Having gone through the foundational theory in the first part, we now turn our attention to specific groups and derive closed-form expressions for the most important formulas that can be used in applications.

First we introduce what are commonly known as the *classical* Lie groups, which are all families of matrix groups parameterized by the matrix size n. A general recipe to figure out the structure of the Lie algebra associated with a particular Lie group is to consider a trajectory of the form

$$\boldsymbol{X}(t) = \operatorname{Exp}(t\boldsymbol{A}) \in \mathbb{M} \tag{9.1}$$

that evidently satisfies $\boldsymbol{X}(0) = I$ and $\boldsymbol{X}'(0) = \boldsymbol{A}$.

The trajectory $\boldsymbol{X}(t)$ must satisfy the constraints that define the group, which translates into conditions on $\boldsymbol{A}$. Since the Lie algebra of a group consists of all matrices $\boldsymbol{A}$ such that $\exp \boldsymbol{A} \in \mathbb{M}$ this reveals the structure of the Lie algebra.

General Linear Group $\mathbb{GL}(n, F)$ The general linear group over a field F (in the following F is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$) is the largest matrix Lie group and contains all other matrix Lie groups as subgroups.

$$\mathbb{GL}(n, F) := \{\boldsymbol{A} \in F^{n\times n} \mid \det \boldsymbol{A} \neq 0\}. \tag{9.2}$$

The exponential map always produces invertible matrices, so the corresponding Lie algebra is the space of all $n \times n$ matrices.

$$\mathfrak{gl}(n, F) = F^{n\times n}. \tag{9.3}$$

Any subset of $\mathbb{GL}(n, F)$ that is closed under matrix multiplication and matrix inversion is also a matrix Lie group.

Translation Group $\mathbb{T}(n)$ The usual Euclidean vector space $\mathbb{R}^n$ can be embedded in matrices on the form $\begin{bmatrix} I_n & \boldsymbol{p} \\ 0_{1\times n} & 1 \end{bmatrix}$ for $\boldsymbol{p} \in \mathbb{R}^n$, so that matrix multiplication in $\mathbb{T}(n)$ corresponds to addition in $\mathbb{R}^n$:

$$\begin{bmatrix} I_n & \boldsymbol{p} \\ 0_{1\times n} & 1 \end{bmatrix}\begin{bmatrix} I_n & \boldsymbol{p}' \\ 0_{1\times n} & 1 \end{bmatrix} = \begin{bmatrix} I_n & \boldsymbol{p}+\boldsymbol{p}' \\ 0_{1\times n} & 1 \end{bmatrix}. \tag{9.4}$$

Being a closed subset of $\mathbb{GL}(n, \mathbb{R})$, those matrices form a matrix Lie group.

To find the corresponding Lie algebra consider a trajectory

$$\boldsymbol{X}(t) = \begin{bmatrix} I_n & \boldsymbol{p}(t) \\ 0_{1\times n} & 1 \end{bmatrix} = \operatorname{Exp}(t\boldsymbol{A}) \in \mathbb{T}(n). \tag{9.5}$$

Differentiating with respect to t shows that

$$\begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{p}'(t) \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{A}, \tag{9.6}$$

which means that the Lie algebra $\mathfrak{t}(n)$ of $\mathbb{T}(n)$ consists of matrices where only the top n coefficients in the right-most column are non-zero.

The translation groups $\mathbb{T}(n)$ and corresponding Lie algebras $\mathfrak{t}(n)$ are:

$$\mathbb{T}(n) = \left\{ \begin{bmatrix} I_n & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \in \mathbb{GL}(n+1,\mathbb{R}) \mid \boldsymbol{p} \in \mathbb{R}^n \right\}, \tag{9.7a}$$

$$\mathfrak{t}(n) = \left\{ \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{v} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}, \boldsymbol{v} \in \mathbb{R}^n \right\}. \tag{9.7b}$$

Orthogonal Groups $\mathbb{O}(n)$ and $\mathbb{SO}(n)$ The orthogonal matrices $\mathbb{O}(n)$ are real matrices $\boldsymbol{X}$ s.t. the inverse is equal to the transpose: $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^T = I_n$. The special orthogonal matrices in addition have a determinant equal to 1. In robotics $\mathbb{SO}(n)$ is particularly useful in the $n = 2, 3$ cases since those correspond to rotation matrices in two and three dimensions and are therefore natural representations of angles.

Take a one-parameter subgroup $\boldsymbol{X}(t) := \mathrm{Exp}(t\boldsymbol{A}) \in \mathbb{SO}(n)$ and differentiate the group constraint $I_n = \boldsymbol{X}(t)^T\boldsymbol{X}(t)$:

$$0 \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)^T\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{X}'(0)^T\boldsymbol{X}(0) + \boldsymbol{X}(0)^T\boldsymbol{X}'(0) = \boldsymbol{A}^T + \boldsymbol{A}. \tag{9.8}$$

It follows that the Lie algebra $\mathfrak{so}(n)$ of $\mathbb{SO}(n)$ consists of **skew-symmetric matrices**.

Orthogonal groups and corresponding Lie algebras:

$$\mathbb{O}(n) = \left\{ \boldsymbol{X} \in \mathbb{GL}(n,\mathbb{R}) \mid \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^T = I_n \right\}, \tag{9.9a}$$

$$\mathbb{SO}(n) = \left\{ \boldsymbol{X} \in \mathbb{GL}(n,\mathbb{R}) \mid \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^T = I_n, \det \boldsymbol{X} = 1 \right\}, \tag{9.9b}$$

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \left\{ \boldsymbol{A} \in \mathbb{R}^{n\times n} \,:\, \boldsymbol{A}^T + \boldsymbol{A} = 0 \right\}. \tag{9.9c}$$

Unitary Groups $\mathbb{U}(n)$ and $\mathbb{SU}(n)$ Unitary matrices $\boldsymbol{X}$ are complex matrices characterized by the inverse being equal to the Hermitian transpose, i.e. $\boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^* = I$[1]. In the case of $\mathbb{SU}(n)$ the determinant is also required to be equal to 1.

For a one-parameter subgroup $\boldsymbol{X}(t) = \mathrm{Exp}(t\boldsymbol{A})$ constraint differentiation yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)^*\boldsymbol{X}(t)\Big|_{t=0} = \boldsymbol{X}'(0)^*\boldsymbol{X}(0) + \boldsymbol{X}(0)^*\boldsymbol{X}'(0) = \boldsymbol{A}^* + \boldsymbol{A}. \tag{9.10}$$

[1]The Hermitian transpose (also known as *conjugate transpose*) of $\boldsymbol{X}_{ij}$ is $\boldsymbol{X}^*_{ij} = \bar{\boldsymbol{X}}_{ji}$.

This shows that $\mathfrak{u}(n)$ consists of **skew-Hermitian matrices**. In addition, due to the Jacobi identity (4.2c) $\det \mathrm{Exp}(t\boldsymbol{A}) = 1$ implies that $\mathrm{Tr}\boldsymbol{A} = 0$, i.e. $\mathfrak{su}(n)$ consists of **skew-Hermitian matrices with vanishing trace**.

Unitary groups and corresponding Lie algebras:

$$\mathbb{U}(n) = \{\boldsymbol{X} \in \mathbb{GL}(n,\mathbb{C}) \mid \boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^* = I_n\} \tag{9.11a}$$

$$\mathbb{SU}(n) = \{\boldsymbol{X} \in \mathbb{GL}(n,\mathbb{C}) \mid \boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^* = I_n, \det \boldsymbol{X} = 1\}, \tag{9.11b}$$

$$\mathfrak{u}(n) = \{\boldsymbol{A} \in \mathbb{C}^{n\times n} \mid \boldsymbol{A}^* + \boldsymbol{A} = 0\}, \tag{9.11c}$$

$$\mathfrak{su}(n) = \{\boldsymbol{A} \in \mathbb{C}^{n\times n} \mid \boldsymbol{A}^* + \boldsymbol{A} = 0, \mathrm{Tr}\boldsymbol{A} = 0\}. \tag{9.11d}$$

Euclidean Groups $\mathbb{E}(n)$ and $\mathbb{SE}(n)$ These groups are formed via a "semi-direct" product between $\mathbb{O}(n)$ (resp. $\mathbb{SO}(n)$) and $\mathbb{T}(n)$ obtained by replacing the identity matrix in $\mathbb{T}(n)$ by a member of $\mathbb{O}(n)$ (resp. $\mathbb{SO}(n)$). For example, $\mathbb{SE}(n)$ consists of matrices on the form $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0_{1\times n} & 1 \end{bmatrix}$ for $\boldsymbol{R} \in \mathbb{SO}(n)$. For a one-parameter subgroup $\boldsymbol{X}(t) = \begin{bmatrix} \boldsymbol{R}(t) & \boldsymbol{p}(t) \\ 0 & 1 \end{bmatrix} = \mathrm{Exp}(t\boldsymbol{A})$ differentiation shows that

$$\begin{bmatrix} \boldsymbol{R}'(t) & \boldsymbol{p}'(t) \\ 0 & 0 \end{bmatrix} = \boldsymbol{A}. \tag{9.12}$$

Since we already know the structure of $\boldsymbol{R}'(t)$ from (9.9c) the form of $\boldsymbol{A}$ can be determined.

Euclidean groups and corresponding Lie algebras:

$$\mathbb{E}(n) = \left\{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \in \mathbb{GL}(n+1,\mathbb{R}) \mid \boldsymbol{R} \in \mathbb{O}(n), \boldsymbol{p} \in \mathbb{R}^n\right\}, \tag{9.13a}$$

$$\mathbb{SE}(n) = \left\{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \in \mathbb{GL}(n+1,\mathbb{R}) \mid \boldsymbol{R} \in \mathbb{SO}(n), \boldsymbol{p} \in \mathbb{R}^n\right\}, \tag{9.13b}$$

$$\mathfrak{e}(n) = \mathfrak{se}(n) = \left\{\begin{bmatrix} \boldsymbol{B} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \mid \boldsymbol{B}^T + \boldsymbol{B} = 0_{n\times n}, \boldsymbol{v} \in \mathbb{R}^n\right\}. \tag{9.13c}$$

Symplectic Groups $\mathbb{Sp}(n)$ Are these useful?

Write about symplectic groups maybe

9.1. Lower-Dimensional Representations

Although matrix Lie groups are convenient to analyze in comparison to their non-matrix counterparts, a matrix representation is often not the best practical choice. One reason is that it is often unnecessarily large: $\mathbb{SO}(3)$ for example consists of $3x3$ matrices with 9 elements, but the manifold has just three degrees of freedom. Computational gains can therefore be had by selecting a more parsimonious parameterization.

A principled way of parameterizing a matrix Lie group is by leveraging a Lie group isomorphism between the matrix Lie group of interest and another group with a more compact representation. An isomorphism is a bijective function $f : \mathbb{M} \to \mathbb{N}$ that is homomorphic with respect to the group operation, i.e. f is required to be one-to-one and onto, and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{M}$ it must hold that

$$f(\boldsymbol{x} \circ \boldsymbol{y}) = f(\boldsymbol{x}) \circ f(\boldsymbol{y}). \tag{9.14}$$

When $\mathbb{N}$ is a matrix Lie group $f(\boldsymbol{x}) \circ f(\boldsymbol{y})$ is simply matrix multiplication. If an isomorphism exists between $\mathbb{M}$ and $\mathbb{N}$ we say that the two groups are isomorphic and write $\mathbb{M} \cong \mathbb{N}$. It should be clear that the presence of an isomorphism makes it possible to move freely between elements of $\mathbb{M}$ and $\mathbb{N}$, and means that we can have the best of both worlds: the analysis framework of matrix Lie groups and the compactness of the non-matrix isomorphic groups.

Leveraging isomorphism is utilized as a recipe in the following. As a parameterization of $\mathbb{SO}(3)$ the established choice $\mathbb{S}^3$—the unit quaternions represented by four scalars, and we utilize that $\mathbb{SO}(2) \cong \mathbb{U}(1)$ to parameterize $\mathbb{SO}(2)$ with just two scalars. Like for Lie algebras we denote by the hat ($\wedge$) and vee ($\vee$) maps conversions between a matrix Lie groups and its parameterization. That is, if $f : \check{\mathbb{M}} \to \mathbb{M}$ is an isomorphism and $\mathbb{M}$ is a matrix Lie group, then

$$\begin{aligned}
\wedge &: \check{\mathbb{M}} \to \mathbb{M}, \\
\boldsymbol{x} &\overset{\wedge}{\mapsto} \boldsymbol{x}^\wedge := f(\boldsymbol{x}), \\
\vee &: \mathbb{M} \to \check{\mathbb{M}}, \\
\boldsymbol{X} &\mapsto \boldsymbol{X}^\vee := f^{-1}(\boldsymbol{X}).
\end{aligned} \tag{9.15}$$

If two Lie groups are isomorphic so are their Lie algebras. We can therefore select the same parameterization of the Lie algebras so that for instance

$$\left(\exp_{\check{M}} \boldsymbol{x}\right)^\wedge = \exp_M \left(\boldsymbol{x}^\wedge\right), \tag{9.16}$$

which in turn can be utilized to find an expression for exp on $\check{\mathbb{M}}$.

9.2. Mathematical Preliminaries

Our goal is to study the most important groups for robotics: $\mathbb{SO}(2), \mathbb{SO}(3), \mathbb{SE}(2)$, and $\mathbb{SE}(3)$. In the remainder of this chapter we provide certain formulas that are useful towards that end.

Starting with the familiar Taylor expansions of cosine and sine

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x, \tag{9.17}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x, \tag{9.18}$$

some higher-order formulas can be derived for $x \neq 0$ by dividing by a factor of $\boldsymbol{x}$ and subtracting the first summation terms.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{\sin x}{x}, \tag{9.19}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} x^{2n} = -\frac{1}{x^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \frac{1-\cos x}{x^2}, \tag{9.20}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)!} x^{2n} = -\frac{1}{x^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{x-\sin x}{x^3}, \tag{9.21}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+4)!} x^{2n} = \frac{1}{x^4} \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}, \tag{9.22}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+5)!} x^{2n} = \frac{1}{x^5} \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{\sin x - x + \frac{x^3}{6}}{x^5}. \tag{9.23}$$

From the sum representation it is easy to see that all these expressions have a finite limit at zero.

A sum involving the Bernoulli numbers will be useful for some groups of interest.

Proposition 9.1.

$$\sum_{n=1}^{\infty} \frac{B_{2n}(-1)^n x^{2n}}{(2n)!} = \frac{x}{2} \cot\left(\frac{x}{2}\right). \tag{9.24}$$

Proof. By setting $x = iy$ and observing that $B_n = 0$ for odd $n > 1$ we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^n x^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} \frac{B_{2n}(-1)^n y^{2n}(-1)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{B_n y^n}{n!} - B_1 y \overset{(5.34)}{=} \frac{y}{e^y - 1} + \frac{y}{2} = \frac{y}{2} \frac{e^y+1}{e^y-1} \\ &= \frac{ix}{2} \frac{1+e^{-iy}}{1-e^{-iy}} - 1 = \frac{ix}{2} \frac{e^{iy/2} + e^{-ix/2}}{e^{ix/2} - e^{-ix/2}} = \frac{ix}{2} \frac{\cos(x/2)}{i \sin(x/2)} = \frac{x}{2} \cot\left(\frac{x}{2}\right). \end{aligned} \tag{9.25}$$

□

Finally we derive an identity that will be useful to construct the exponential maps for semi-simple groups.

Lemma 9.1. *Consider two matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$ such that $\boldsymbol{B}^2 = \boldsymbol{B}\boldsymbol{A} = 0$. Then we have that*

$$\operatorname{Exp}(\boldsymbol{A} + \boldsymbol{B}) = \operatorname{Exp}(\boldsymbol{A}) + \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{(k+1)!} \boldsymbol{B}. \tag{9.26}$$

Proof. When we expand $(\boldsymbol{A}+\boldsymbol{B})^k$ all terms that contain a $\boldsymbol{B}$ before an $\boldsymbol{A}$, or multiple $\boldsymbol{B}$ in a row, vanish. As a result,

$$\operatorname{Exp}(\boldsymbol{A} + \boldsymbol{B}) = I_n + \sum_{k=1}^{\infty} \frac{(\boldsymbol{A}+\boldsymbol{B})^k}{k!} = I_n + \sum_{k=1}^{\infty} \frac{\boldsymbol{A}^k + \boldsymbol{A}^{k-1}\boldsymbol{B}}{k!} = \operatorname{Exp}(\boldsymbol{A}) + \sum_{k=1}^{\infty} \frac{\boldsymbol{A}^{k-1}}{k!} \boldsymbol{B}. \tag{9.27}$$

□

10. $\mathbb{SO}(2)$: The 2D Rotation Group

Let $\boldsymbol{R}$ denote an element of $\mathbb{SO}(2)$, commonly referred to as a *2D rotation matrix*.

In robotics applications it is convenient to define a rotational action on vectors in $\mathbb{R}^2$. For $\boldsymbol{R} \in \mathbb{SO}(2)$ and $\boldsymbol{u} \in \mathbb{R}^2$ the action is matrix multiplication:

$$\langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\mathbb{SO}(2)} = \boldsymbol{R} \cdot \boldsymbol{u}. \tag{10.1}$$

By (9.9c) the corresponding Lie algebra $\mathfrak{so}(2)$ consists of the 2×2 skew-symmetric matrices, which have just one degree of freedom. Let this single parameter of $\mathfrak{so}(2)$ be denoted ω_z so that

$$\mathfrak{so}(2) = \left\{ \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \mid \omega_z \in \mathbb{R} \right\}, \tag{10.2}$$

and the Lie algebra hat and vee maps become

$$\begin{tikzcd} \mathbb{R}^1 \ni \begin{bmatrix} \omega_z \end{bmatrix} \arrow[rr, bend left, "\wedge"] & & \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \in \mathfrak{so}(2) \arrow[ll, bend left, "\vee"] \end{tikzcd}$$

10.1. Formulas

Adjoint From the definition:

$$\mathbf{Ad}_{\boldsymbol{R}} \begin{bmatrix} \omega_z \end{bmatrix} = \left(\boldsymbol{R} \omega_z^\wedge \boldsymbol{R}^{-1} \right)^\vee = \left(\boldsymbol{R} \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \boldsymbol{R}^T \right)^\vee = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix}^\vee = \begin{bmatrix} \omega_z \end{bmatrix}, \tag{10.3}$$

so $\mathbf{Ad}_{\boldsymbol{R}} = \begin{bmatrix} 1 \end{bmatrix}$.

Exponential and Logarithm Take an element $\begin{bmatrix} \omega_z \end{bmatrix} \in \mathfrak{so}(2)$; the exponential is calculated by noting that $(\omega_z^\wedge)^{2k} = (-1)^k \omega_z^{2k} I_2$:

$$\begin{aligned} \operatorname{Exp} \omega_z^\wedge &= \sum_{k=0}^{\infty} \frac{(\omega_z^\wedge)^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(\omega_z^\wedge\right)^{2k}}{(2k)!} + \frac{\left(\omega_z^\wedge\right)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \omega_z^{2k}}{(2k)!} I_2 + \frac{(-1)^k \omega_z^{2k}}{(2k+1)!} \omega_z^\wedge \\ &\overset{(9.17),(9.19)}{=} \cos \omega_z I_2 + \frac{\sin \omega_z}{\omega_z} \omega_z^\wedge = \begin{bmatrix} \cos \omega_z & -\sin \omega_z \\ \sin \omega_z & \cos \omega_z \end{bmatrix}, \end{aligned} \tag{10.4}$$

which we recognize as the 2D rotation matrix that rotates counter-clockwise with angle ω_z. With $\boldsymbol{R} = \begin{bmatrix}\cos\omega_z & -\sin\omega_z \\ \sin\omega_z & \cos\omega_z\end{bmatrix}$ it follows that

$$\mathrm{Tr}(\boldsymbol{R} + \boldsymbol{R}^T) = 2\cos\omega_z, \tag{10.5}$$

from where it follows that

$$\log\boldsymbol{R} = \arccos\left(\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{R}^T + \boldsymbol{R}\right)\right). \tag{10.6}$$

Derivatives of the Exponential Consider algebra elements $\omega_z, \bar{\omega}_z \in \mathfrak{so}(2)$. The bracket on $\mathfrak{so}(2)$ is zero since

$$[\omega_z, \bar{\omega}_z] = \left(\begin{bmatrix}0 & -\omega_z \\ \omega_z & 0\end{bmatrix}\begin{bmatrix}0 & -\bar{\omega}_z \\ \bar{\omega}_z & 0\end{bmatrix} - \begin{bmatrix}0 & -\bar{\omega}_z \\ \bar{\omega}_z & 0\end{bmatrix}\begin{bmatrix}0 & -\omega_z \\ \omega_z & 0\end{bmatrix}\right)^\vee = 0. \tag{10.7}$$

It follows that all terms in (5.31) and (5.35) vanish except for $n = 0$, so the derivatives of the exponential are equal to $I_1 = \left[1\right]$.

$\mathbb{SO}(2)$ formula sheet

Consists of 2×2 rotation matrices $\boldsymbol{R}$ that act on $\mathbb{R}^2$ via $\boldsymbol{v} \mapsto \boldsymbol{R}\boldsymbol{v}$.

Algebra Parameterization

$$\{\omega_w \mid \omega_w \in [-\pi, \pi]\}, \qquad (\omega_w)^\wedge = \begin{bmatrix}0 & -\omega_w \\ \omega_w & 0\end{bmatrix} \in \mathfrak{so}(2). \tag{10.8}$$

Adjoint

$$\mathbf{Ad}_{\boldsymbol{R}} = \left[1\right]. \tag{10.9}$$

Exponential and Logarithm

$$\exp(\omega_w) = \begin{bmatrix}\cos\omega_w & -\sin\omega_w \\ \sin\omega_w & \cos\omega_w\end{bmatrix}, \tag{10.10a}$$

$$\log(\boldsymbol{R}) = \arccos\left(\frac{1}{2}\mathrm{Tr}(\boldsymbol{R}^T + \boldsymbol{R})\right). \tag{10.10b}$$

Bracket and Lowercase adjoint

$$[\omega_z, \omega_z'] = 0 \tag{10.11a}$$

$$\mathrm{ad}_{\omega_z} = 0 \tag{10.11b}$$

Derivatives of the Exponential

$$\mathrm{d}^r\exp_{\omega_z} = \mathrm{d}^l\exp_{\omega_z} = \left(\mathrm{d}^r\exp_{\omega_z}\right)^{-1} = \left(\mathrm{d}^l\exp_{\omega_z}\right)^{-1} = \left[1\right]. \tag{10.12}$$

10.2. Parameterization via Isomorhphism with $\mathbb{U}(1)$

We use the isomorphism $\mathbb{SO}(2) \cong \mathbb{U}(1)$, where $\mathbb{U}(1)$ is the unitary group consisting of complex elements $c = \omega_w + \omega_z \boldsymbol{i} \in \mathbb{C}$ with unit length, to parameterize elements of $\mathbb{SO}(2)$.

$$\mathbb{U}(1) = \{c \in \mathbb{C} \mid |c| = c\bar{c} = 1\}. \tag{10.13}$$

The hat and vee maps between the parameterization and matrix forms are

$$\begin{tikzcd} \mathbb{U}(1) \ni c \arrow[r, bend left, "\wedge"] & \boldsymbol{R} = \begin{bmatrix} \mathrm{Re}(c) & -\mathrm{Im}(c) \\ \mathrm{Im}(c) & \mathrm{Re}(c) \end{bmatrix} \in \mathbb{SO}(2) \arrow[l, bend left, "\vee"] \end{tikzcd}$$

and it can be verified that this is indeed a group isomorphism. Due to the simplicity of the isomorphism it follows that the lowercase exponential on $\mathbb{U}(1)$ is

$$\exp \omega_z = \cos \omega_z + \sin \omega_z \boldsymbol{i}, \tag{10.14}$$

and consequently the lowercase logarithm can be taken as

$$\log c = \operatorname{arctan2}(\mathrm{Im}(c), \mathrm{Re}(c)). \tag{10.15}$$

The following table summarizes the properties of $\mathbb{U}(1)$. Note that the Lie algebra is shared with $\mathbb{SO}(2)$ so operators such as $\mathbf{Ad}$ and $\mathrm{d}^{\mathrm{r}} \exp$ are identical to $\mathbb{SO}(2)$ and not repeated here.

$\mathbb{U}(1)$ formula sheet

Group Parameterization

$$\mathbb{U}(1) = \{c \in \mathbb{C} \mid |c| = 1\} \tag{10.16}$$

Group Operations

- Identity element: $1 + 0\boldsymbol{i} \in \mathbb{C}$,
- Inverse: $c^{-1} = \bar{c}$, where bar denotes complex conjugate,
- Composition: $c \circ c' = \mathrm{Re}(c)\mathrm{Re}(c') - \mathrm{Im}(c)\mathrm{Im}(c') + i\,(\mathrm{Re}(c)\mathrm{Im}(c') + \mathrm{Im}(c)\mathrm{Re}(c'))$.

$\mathbb{U}(1)$ is isomorphic to $\mathbb{SO}(2)$ via $\wedge : \mathbb{U}(1) \to \mathbb{SO}(2)$

$$c^{\wedge} = \begin{bmatrix} \mathrm{Re}(c) & -\mathrm{Im}(c) \\ \mathrm{Im}(c) & \mathrm{Re}(c) \end{bmatrix} \tag{10.17}$$

and therefore inherits Lie algebra properties from $\mathbb{SO}(2)$.

Rotation Action on $\boldsymbol{u} \in \mathbb{R}^2$

$$\langle c, \boldsymbol{u} \rangle_{\mathbb{U}(1)} = \begin{bmatrix} \mathrm{Re}(c) & -\mathrm{Im}(c) \\ \mathrm{Im}(c) & \mathrm{Re}(c) \end{bmatrix} \boldsymbol{u}. \tag{10.18}$$

Exponential and Logarithm

$$\exp(\omega_w) = \cos\omega_w + \sin\omega_w \boldsymbol{i}, \tag{10.19a}$$
$$\log(c) = \operatorname{arctan2}\left(\operatorname{Im}(c), \operatorname{Re}(c)\right). \tag{10.19b}$$

11. $\mathbb{SO}(3)$: The 3D Rotation Group

As with $\mathbb{SO}(2)$ we denote by $\boldsymbol{R}$ an element of $\mathbb{SO}(3)$; these matrices are commonly known as *3D rotation matrices*. Also like with $\mathbb{SO}(2)$ we define an action on $\mathbb{R}^3$ to be that of matrix multiplication; for $\boldsymbol{u} \in \mathbb{R}^3$

$$\langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\mathbb{SO}(3)} = \boldsymbol{R} \cdot \boldsymbol{u}, \tag{11.1}$$

which corresponds to a rotation of $\boldsymbol{R}$.

The Lie algebra $\mathfrak{so}(3)$ consists of skew-symmetric matrices

$$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \right\}. \tag{11.2}$$

It is defined by just three elements $\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$ which means that the hat and vee maps can be defined as

$$\begin{tikzcd} \mathbb{R}^3 \ni \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \arrow[r, bend left, "\wedge"] & \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_z & \omega_x & 0 \end{bmatrix} \in \mathfrak{so}(2) \arrow[l, bend left, "\vee"] \end{tikzcd}$$

Matrices on this form have several interesting properties. First of all, for $\boldsymbol{u} \in \mathbb{R}^3$ left matrix multiplication by $\hat{\boldsymbol{\omega}}$ is equivalent to taking the vector cross product: $\hat{\boldsymbol{\omega}}\boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{u}$. As a result many properties of the cross product are inherited by the embedding $\mathbb{R}^3 \overset{\wedge}{\mapsto} \mathbb{R}^{3\times 3}$.

Properties of $\wedge$ on $\mathfrak{so}(3)$

For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$:

$$\hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\hat{\boldsymbol{a}} = -(\boldsymbol{a} \cdot \boldsymbol{b})\hat{\boldsymbol{a}}, \tag{11.3a}$$

$$\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}, \tag{11.3b}$$

$$\boldsymbol{a} \cdot (\hat{\boldsymbol{b}}\boldsymbol{c}) = \boldsymbol{b} \cdot (\hat{\boldsymbol{c}}\boldsymbol{a}), \tag{11.3c}$$

$$A\hat{\boldsymbol{b}} = \mathrm{Tr}(A)\hat{\boldsymbol{b}} - (A\boldsymbol{b})^\wedge - \hat{\boldsymbol{b}}A, \quad A \text{ symmetric } 3 \times 3 \text{ matrix}, \tag{11.3d}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{2}\left\langle \hat{\boldsymbol{a}}, \hat{\boldsymbol{b}} \right\rangle_F = -\frac{1}{2}\mathrm{Tr}\left(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}\right). \tag{11.3e}$$

Proof of (11.3a). Consider $\hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}\boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c}))$. Expanding with the vector triple product gives

$$\hat{\boldsymbol{a}}\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}\boldsymbol{c} = \boldsymbol{a} \times ((\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}) = -(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{a} \times \boldsymbol{c} = -(\boldsymbol{a} \cdot \boldsymbol{b})\hat{\boldsymbol{a}}\boldsymbol{c}. \tag{11.4}$$

□

11.1. Formulas

Adjoint Take any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$, since the cross product satisfies $(\boldsymbol{R}\boldsymbol{u}) \times (\boldsymbol{R}\boldsymbol{v}) = \boldsymbol{R}(\boldsymbol{u} \times \boldsymbol{v})$ it follows that

$$(\boldsymbol{R}\boldsymbol{\omega}^\wedge)\boldsymbol{u} = (\boldsymbol{R}\boldsymbol{\omega}) \times \boldsymbol{u} = \boldsymbol{R}(\boldsymbol{\omega} \times \boldsymbol{R}^T\boldsymbol{u}) = \boldsymbol{R}\boldsymbol{\omega}^\wedge\boldsymbol{R}\boldsymbol{u}, \tag{11.5}$$

and hence $(\boldsymbol{R}\boldsymbol{\omega})^\wedge = \boldsymbol{R}\boldsymbol{\omega}^\wedge\boldsymbol{R}^T$. From (4.12) therefore

$$\mathbf{Ad}_{\boldsymbol{R}}\, \boldsymbol{\omega} := (\boldsymbol{R}\boldsymbol{\omega}^\wedge\boldsymbol{R}^T)^\vee = \boldsymbol{R}\boldsymbol{\omega}, \tag{11.6}$$

so $\mathbf{Ad}_{\boldsymbol{R}} = \boldsymbol{R}$.

Exponential and Logarithm We can use (11.3a) to obtain the exponential map on $\mathbb{SO}(3)$:

$$\begin{aligned} \exp \boldsymbol{\omega} := \operatorname{Exp}\ \hat{\boldsymbol{\omega}} = \sum_{k \geq 0} \frac{(\hat{\boldsymbol{\omega}})^k}{k!} &= I + \hat{\boldsymbol{\omega}} + \frac{\hat{\boldsymbol{\omega}}^2}{2!} - \|\boldsymbol{\omega}\|^2 \left(\frac{\hat{\boldsymbol{\omega}}}{3!} + \frac{\hat{\boldsymbol{\omega}}^2}{4!}\right) + \|\boldsymbol{\omega}\|^4 \left(\frac{\hat{\boldsymbol{\omega}}}{5!} + \frac{\hat{\boldsymbol{\omega}}^2}{6!}\right) + \ldots \\ &= I + \left(1 - \frac{\|\boldsymbol{\omega}\|^2}{3!} + \frac{\|\boldsymbol{\omega}\|^4}{5!} - \ldots\right)\hat{\boldsymbol{\omega}} + \left(\frac{1}{2!} - \frac{\|\boldsymbol{\omega}\|^2}{4!} + \frac{\|\boldsymbol{\omega}\|^4}{6!} - \ldots\right)\hat{\boldsymbol{\omega}}^2 \\ &= I + \frac{\sin \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{1 - \cos \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2. \end{aligned} \tag{11.7}$$

To obtain the logarithm the expression

$$\boldsymbol{R} = I_3 + \frac{\sin \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{1 - \cos \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2 \tag{11.8}$$

should be inverted. First note that due to $\hat{\boldsymbol{\omega}}$ being skew-symmetric:

$$\boldsymbol{R} - \boldsymbol{R}^T = \frac{\sin \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}} + \frac{1 - \cos \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}}^2 - \frac{\sin \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}}^T - \frac{1 - \cos \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}(\hat{\boldsymbol{\omega}}^T)^2 = 2\frac{\sin \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\hat{\boldsymbol{\omega}}. \tag{11.9}$$

Secondly,

$$\operatorname{Tr}(\boldsymbol{R}) = 3 + \frac{1 - \cos \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\operatorname{Tr}(\hat{\boldsymbol{\omega}}^2) = 3 - 2\frac{1 - \cos \|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\|\boldsymbol{\omega}\|^2 = 1 + 2\cos \|\boldsymbol{\omega}\|, \tag{11.10}$$

which makes it possible to write down an expression for the logarithm.

$$\log \boldsymbol{R} = \left(\frac{\alpha}{\sin \alpha}\frac{\boldsymbol{R} - \boldsymbol{R}^T}{2}\right)^\vee, \quad \alpha = \arccos\left(\frac{\operatorname{Tr}\boldsymbol{R}) - 1}{2}\right). \tag{11.11}$$

Derivatives of the Exponential We know from (11.3a) that $[\boldsymbol{\omega}, \boldsymbol{\omega}'] = \boldsymbol{\omega}^\wedge \boldsymbol{\omega}'$, hence that $\mathrm{ad}_{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}$, and from (11.3a) that $\hat{\boldsymbol{\omega}}^3 = -\|\boldsymbol{\omega}\|^2 \hat{\boldsymbol{\omega}}$. Thus $\mathrm{ad}_{\boldsymbol{\omega}}^3 = -\|\boldsymbol{\omega}\|^2 \,\mathrm{ad}_{\boldsymbol{\omega}}$ and we get,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{B_n(-1)^n}{n!} \mathrm{ad}_{\boldsymbol{\omega}}^n &= \sum_{n=0}^{\infty} \frac{B_n(-1)^n}{n!} \hat{\boldsymbol{\omega}}^n = I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} + \sum_{n\geq 2} \frac{B_n(-1)^n}{n!} \mathrm{ad}_{\boldsymbol{\omega}}^n \\
&= I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} + \left(\frac{B_2}{2!} \mathrm{ad}_{\boldsymbol{\omega}}^2 - \frac{B_4\|\boldsymbol{\omega}\|^2}{4!} \mathrm{ad}_{\boldsymbol{\omega}}^2 + \frac{B_6\|\boldsymbol{\omega}\|^4}{6!} \mathrm{ad}_{\boldsymbol{\omega}}^2 - \ldots \right) = I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} - \frac{1}{\|\boldsymbol{\omega}\|^2} \sum_{n\geq 1} \frac{B_{2n}(-1)^n \|\boldsymbol{\omega}\|^{2n}}{(2n)!} \mathrm{ad}_{\boldsymbol{\omega}}^2 \\
&\overset{(9.24)}{=} I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} - \frac{1}{\|\boldsymbol{\omega}\|^2} \left(\frac{\|\boldsymbol{\omega}\|}{2} \cot\left(\frac{\|\boldsymbol{\omega}\|}{2}\right) - 1 \right) \mathrm{ad}_{\boldsymbol{\omega}}^2 = I_3 + \frac{\mathrm{ad}_{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1 + \cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\| \sin\|\boldsymbol{\omega}\|} \right) \mathrm{ad}_{\boldsymbol{\omega}}^2,
\end{aligned}$$

where the half-angle formula $\cot(x/2) = (1 + \cos x)/\sin x$ has been used. The left jacobian $\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{\omega}}$ was already calculated in (13.5) and since $\left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{\omega}}\right)^{-1} = \left[\left(\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{\omega}}\right)^{-1}\right]^T$. Due to the anti-symmetry of $\mathrm{ad}_{\boldsymbol{\omega}}$ it follows that also $\mathrm{d}^r \exp_{\boldsymbol{\omega}} = \left[\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{\omega}}\right]^T$ must hold.

$\mathbb{SO}(3)$ formula sheet

Group definition

$$\mathbb{SO}(3) = \left\{ \boldsymbol{R} \in \mathbb{GL}(3, \mathbb{R}) \mid \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}^T\boldsymbol{R} = I_3, \det \boldsymbol{R} = 1 \right\}. \tag{11.12}$$

Action on $\mathbb{R}^3$: Left multiplication by rotation matrix

$$\langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\mathbb{SO}(3)} = \boldsymbol{R}\boldsymbol{u}. \tag{11.13}$$

Algebra parameterization

$$\left\{ \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \mid \omega_x, \omega_y, \omega_z \in [-\pi, \pi] \right\}, \quad \hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathfrak{so}(3). \tag{11.14}$$

Adjoint

$$\mathbf{Ad}_{\boldsymbol{R}} = \boldsymbol{R} \tag{11.15}$$

Exponential and Logarithm

$$\exp \boldsymbol{\omega} = I + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|} \hat{\boldsymbol{\omega}} + \frac{1 - \cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2} \hat{\boldsymbol{\omega}}^2, \tag{11.16a}$$

$$\log \boldsymbol{R} = \left(\frac{\alpha}{\sin\alpha} \frac{\boldsymbol{R} - \boldsymbol{R}^T}{2} \right)^\vee, \quad \alpha = \arccos\left(\frac{\mathrm{Tr}(\boldsymbol{R}) - 1}{2} \right). \tag{11.16b}$$

Bracket and Lowercase adjoint

$$[\boldsymbol{\omega}, \boldsymbol{\omega}'] = \hat{\boldsymbol{\omega}} \boldsymbol{\omega}' \tag{11.17a}$$

$$\mathrm{ad}_{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}} \tag{11.17b}$$

Derivatives of the Exponential

$$\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{\omega}} = I_3 - \frac{1 - \cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}} + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2, \tag{11.18a}$$

$$\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{\omega}} = I_3 + \frac{1 - \cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\hat{\boldsymbol{\omega}} + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\hat{\boldsymbol{\omega}}^2, \tag{11.18b}$$

$$\left(\mathrm{d}^{\mathrm{r}} \exp_{\boldsymbol{\omega}}\right)^{-1} = I_3 + \frac{\hat{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1 + \cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\|\sin\|\boldsymbol{\omega}\|}\right)\hat{\boldsymbol{\omega}}^2, \tag{11.18c}$$

$$\left(\mathrm{d}^{\mathrm{l}} \exp_{\boldsymbol{\omega}}\right)^{-1} = I_3 - \frac{\hat{\boldsymbol{\omega}}}{2} + \left(\frac{1}{\|\boldsymbol{\omega}\|^2} - \frac{1 + \cos\|\boldsymbol{\omega}\|}{2\|\boldsymbol{\omega}\|\sin\|\boldsymbol{\omega}\|}\right)\hat{\boldsymbol{\omega}}^2. \tag{11.18d}$$

11.2. Parameterization via Isomorphism with $\mathbb{S}^3$

As opposed to the 2D case where $\mathbb{U}(1)$ was a fairly straightforward isomorphic choice, the situation is more complicated in three dimensions. The usual choice is the group of unit quaternions

$$\mathbb{S}^3 = \left\{\boldsymbol{q} = q_w + q_x\boldsymbol{i} + q_y\boldsymbol{j} + q_z\boldsymbol{k} \,:\, q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1\right\}, \tag{11.19}$$

which forms a double cover of $\mathbb{SO}(3)$. While the double cover mapping between $\mathbb{S}^3$ and $\mathbb{SO}(3)$ is fairly complicated, $\mathbb{S}^3$ is isomorphic to $\mathbb{SU}(2)$ which is also a matrix Lie group. We therefore go via $\mathbb{SU}(2)$ to derive formulas for $\mathbb{S}^3$.

11.2.1. $\mathbb{SU}(2)$ is isomorphic to $\mathbb{S}^3$

We can associate a quaternion $\boldsymbol{q} = q_w + q_x\boldsymbol{i} + q_y\boldsymbol{j} + q_z\boldsymbol{k}$ with the unitary matrix

$$\begin{bmatrix} q_w + iq_z & -q_x - iq_y \\ q_x - iq_y & q_w - iq_z \end{bmatrix} \tag{11.20}$$

for which it holds that $A_{\boldsymbol{q}_1 * \boldsymbol{q}_2} = A_{\boldsymbol{q}_1} A_{\boldsymbol{q}_2}$. Thus $\mathbb{S}^3 \cong \mathbb{SU}(2)$ and the matrix structure of $\mathbb{SU}(2)$ be leveraged to derive quaternion formulas.

Muliplying two elements in $\mathbb{SU}(2)$ yields:

$$\begin{bmatrix} q_w + iq_z & -q_x - iq_y \\ q_x - iq_y & q_w - iq_z \end{bmatrix}\begin{bmatrix} q_w' + iq_z' & -q_x' - iq_y' \\ q_x' - iq_y' & q_w' - iq_z' \end{bmatrix} = \begin{bmatrix} q_w'' + iq_z'' & -q_x'' - iq_y'' \\ q_x'' - iq_y'' & q_w'' - iq_z'' \end{bmatrix} \tag{11.21}$$

where

$$\begin{aligned} q_w'' &= q_w q_w' - q_x q_x' - q_y q_y' - q_z q_z', \\ q_x'' &= q_x q_w' + q_w q_x' + q_y q_z' - q_z q_y', \\ q_y'' &= q_y q_w' + q_w q_y' + q_z q_x' - q_x q_z', \\ q_z'' &= q_z q_w' + q_w q_z' + q_x q_y' - q_y q_x', \end{aligned} \tag{11.22}$$

which is exactly what is obtained by carrying out the usual quaternion multiplication

$$(q_w + q_x\boldsymbol{i} + q_y\boldsymbol{j} + q_z\boldsymbol{k}) * (q'_w + q'_x\boldsymbol{i} + q'_y\boldsymbol{j} + q'_z\boldsymbol{k}) \tag{11.23}$$

with the quaternion rules $\boldsymbol{i}\boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j}\boldsymbol{k} = \boldsymbol{i}, \boldsymbol{k}\boldsymbol{i} = \boldsymbol{j}$ and $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$.

11.2.2. $\mathbb{S}^3$ forms a double cover of $\mathbb{SO}(3)$

A quaternion $\boldsymbol{q} = q_w + q_x\boldsymbol{i} + q_y\boldsymbol{j} + q_z\boldsymbol{k}$ acts on $\boldsymbol{u} := \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \in \mathbb{R}^3$ as quaternion rotation $\boldsymbol{q} * \boldsymbol{u} * \bar{\boldsymbol{q}}$ where $\boldsymbol{u}$ is associated with the quaternion $u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}$.

In terms of matrix multiplication operation can be written

$$\begin{aligned} \begin{bmatrix} q_w + iq_z & -q_x - iq_y \\ q_x - iq_y & q_w - iq_z \end{bmatrix} \begin{bmatrix} iu_z & -u_x - iu_y \\ u_x - iu_y & iu_z \end{bmatrix} \begin{bmatrix} q_w - iq_z & q_x + iq_y \\ -q_x + iq_y & q_w + iq_z \end{bmatrix} \\ = \begin{bmatrix} iu'_z & -u'_x - iu'_y \\ u'_x - iu'_y & iu'_z \end{bmatrix} \end{aligned} \tag{11.24}$$

for

$$\begin{aligned} u'_x &= (1 - 2(q_y^2 + q_z^2))u_x + 2(q_xq_y - q_wq_z)u_y + 2(q_xq_z + q_wq_y)u_z \\ u'_y &= 2(q_xq_y + q_wq_z)u_x + (1 - 2(q_x^2 + q_z^2))u_y + 2(q_yq_z - q_wq_x)u_z \\ u'_z &= 2(q_xq_z - q_wq_y)u_x + 2(q_wq_x + q_yq_z)u_y + (1 - 2(q_x^2 + q_y^2))u_z \end{aligned} \tag{11.25}$$

Since this is a linear mapping in u_x, u_y, u_z we can identify $\boldsymbol{q}$ with a matrix $\boldsymbol{R}(\boldsymbol{q})$ with coefficients

$$\boldsymbol{R}(\boldsymbol{q}) = \begin{bmatrix} 1 - 2(q_y^2 + q_z^2) & 2(q_xq_y - q_wq_z) & 2(q_xq_z + q_wq_y) \\ 2(q_xq_y + q_wq_z) & 1 - 2(q_x^2 + q_z^2) & 2(q_yq_z - q_wq_x) \\ 2(q_xq_z - q_wq_y) & 2(q_wq_x + q_yq_z) & 1 - 2(q_x^2 + q_y^2) \end{bmatrix}, \tag{11.26}$$

which is a mapping from $\mathbb{S}^3$ to $\mathbb{SO}(3)$. However, the mapping is not one-to-one since both $\boldsymbol{q}$ and $-\boldsymbol{q}$ map to the same rotation matrix: $\boldsymbol{R}(\boldsymbol{q}) = \boldsymbol{R}(-\boldsymbol{q})$. This is what is meant when saying that $\mathbb{S}^3$ forms a *double cover* of $\mathbb{SO}(3)$.

Inverse mapping...

11.3. Formulas

To obtain the exponential map on $\mathbb{S}^3$ we go via $\mathbb{SU}(2)$. The Lie algebra $\mathfrak{su}(2)$ is parameterized by three elements $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ that can be associated with the skew-Hermitian matrix

$$\boldsymbol{\omega}^\wedge := \frac{1}{2}\begin{bmatrix} i\omega_z & -\omega_x - i\omega_y \\ \omega_x - i\omega_y & -i\omega_z \end{bmatrix}, \tag{11.27}$$

where the factor $1/2$ is added for reasons that will become clear below. A simple calculation reveals that $\hat{\boldsymbol{\omega}}^2 = -\frac{\|\boldsymbol{\omega}\|}{4} I_2$ which can be used to evaluate the exponential.

$$
\begin{aligned}
\operatorname{Exp} \hat{\boldsymbol{\omega}} &= \sum_{k\geq 0} \frac{\hat{\boldsymbol{\omega}}^k}{k!} = \sum_{k\geq 0} \frac{1}{k!} \left(-\frac{\|\boldsymbol{\omega}\|}{2}\right)^{2*\lfloor \frac{k}{2} \rfloor} \hat{\boldsymbol{\omega}}^{(k \mod 2)} \\
&= \left(1 - \frac{(\|\boldsymbol{\omega}\|/2)^2}{2!} + \frac{(\|\boldsymbol{\omega}\|/2)^4}{4!} - \dots\right) I + \left(1 - \frac{(\|\boldsymbol{\omega}\|/2)^2}{3!} + \frac{(\|\boldsymbol{\omega}\|/2)^4}{5!} + \dots\right) \hat{\boldsymbol{\omega}} \\
&= \cos(\|\boldsymbol{\omega}\|/2) I + 2\frac{\sin\|\boldsymbol{\omega}\|/2}{\|\boldsymbol{\omega}\|} \hat{\boldsymbol{\omega}} = \begin{bmatrix} \cos(\|\boldsymbol{\omega}\|/2) + i\frac{\omega_z \sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} & (-\omega_x - i\omega_y)\frac{\sin\|\boldsymbol{\omega}\|/2}{\|\boldsymbol{\omega}\|} \\ (\omega_x - i\omega_y)\frac{\sin\|\boldsymbol{\omega}\|/2}{\|\boldsymbol{\omega}\|} & \cos(\|\boldsymbol{\omega}\|/2) - i\frac{\omega_z \sin(\|\boldsymbol{\omega}\|/2)}{\|\boldsymbol{\omega}\|} \end{bmatrix}.
\end{aligned}
$$

Since the mappings $\wedge : \mathbb{S}^3 \to \mathbb{SU}(2)$ and $\vee : \mathbb{SU}(2) \to \mathbb{S}^3$ are straightforward, we can identify the exponentail and logarithm on $\mathbb{S}^3$.

$$
\exp(\omega_x, \omega_y, \omega_z) = \left(\cos(\|\boldsymbol{\omega}\|/2), \frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2), \frac{\omega_y}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2), \frac{\omega_z}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right), \tag{11.28a}
$$

$$
\log(q_w, q_x, q_y, q_z) = \left(2\frac{\operatorname{arctan2}\left(\sqrt{q_x^2 + q_y^2 + q_z^2}, q_w\right)}{\sqrt{q_x^2 + q_y^2 + q_z^2}}\right) \times (q_x, q_y, q_z). \tag{11.28b}
$$

From (11.28) the reason to divide the expression for $\hat{\boldsymbol{\omega}}$ by a factor 2 becomes apparent—$\|\boldsymbol{\omega}\|$ represents the rotation angle in radians. We provide a quick proof for the logarithm expression.

Proof of (11.28b). Let $(q_w, q_x, q_y, q_z) = \exp(\omega_x, \omega_y, \omega_z)$. From (11.28a) we have that

$$
\sqrt{q_x^2 + q_y^2 + q_z^2} = \sqrt{\left(\frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right)^2 + \left(\frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right)^2 + \left(\frac{\omega_x}{\|\boldsymbol{\omega}\|}\sin(\|\boldsymbol{\omega}\|/2)\right)^2} = \sin(\|\boldsymbol{\omega}\|/2). \tag{11.29}
$$

It also follows from the same equation that

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{\|\boldsymbol{\omega}\|}{\sin(\|\boldsymbol{\omega}\|/2)}(q_x, q_y, q_z) = \frac{\operatorname{arctan2}(\sin\|\boldsymbol{\omega}\|, \cos\|\boldsymbol{\omega}\|)}{\sqrt{q_x^2 + q_y^2 + q_z^2}} \\
&= \frac{2\operatorname{arctan2}(\sin(\|\boldsymbol{\omega}\|/2), \cos(\|\boldsymbol{\omega}\|/2))}{\sqrt{q_x^2 + q_y^2 + q_z^2}} = \frac{2\operatorname{arctan2}\left(\sqrt{q_x^2 + q_y^2 + q_z^2}, q_w\right)}{\sqrt{q_x^2 + q_y^2 + q_z^2}}.
\end{aligned} \tag{11.30}
$$

□

$\mathbb{S}^3$ formula sheet

Group definition

$$
\mathbb{S}^3 = \left\{\boldsymbol{q} = q_w + q_x\boldsymbol{i} + q_y\boldsymbol{j} + q_z\boldsymbol{k} \ : \ q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1\right\} \tag{11.31}
$$

- Identity element: 1
- Inverse: $(q_w + q_x \boldsymbol{i} + q_z \boldsymbol{j} + q_w \boldsymbol{k})^{-1} = q_w - q_x \boldsymbol{i} - q_z \boldsymbol{j} - q_w \boldsymbol{k}$
- Composition via (11.22)

$\mathbb{S}^3$ forms a double cover of $\mathbb{SO}(3)$ via (11.26) and therefore inherits Lie algebra properties from $\mathbb{SO}(3)$.

Rotation action on $\mathbb{R}^3$ Defined by (11.25).

Exponential and Logarithm

$$\exp \boldsymbol{\omega} = \cos\left(\frac{\|\boldsymbol{\omega}\|}{2}\right) + \frac{1}{\|\boldsymbol{\omega}\|} \sin\left(\frac{\|\boldsymbol{\omega}\|}{2}\right) \left(\omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}\right), \tag{11.32a}$$

$$\log \boldsymbol{q} = \left(2 \frac{\operatorname{arctan2}\left(\sqrt{q_x^2 + q_y^2 + q_z^2}, q_w\right)}{\sqrt{q_x^2 + q_y^2 + q_z^2}}\right) \times \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}. \tag{11.32b}$$

We conclude with some additional quaternion formulas that may be of practical interest.

Useful quaternion identities

Axis-angle to quaternion The quaternion $\boldsymbol{q}$ representing the rotation about a unit axis $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$ for an angle α is

$$\boldsymbol{q} = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) \left(\beta_x \boldsymbol{i} + \beta_y \boldsymbol{j} + \beta_z \boldsymbol{k}\right), \tag{11.33}$$

which follows from (11.32a) and the interpretation of $\boldsymbol{\omega}$ as angles.

Two vectors to quaternion A quaternion $\boldsymbol{q}$ such that $\boldsymbol{q}\boldsymbol{u} = \boldsymbol{v}$ for unit vectors $\boldsymbol{u}, \boldsymbol{v}$.

$$\boldsymbol{q} = \sqrt{\frac{1+s}{2}} + \sqrt{\frac{1-s}{2}} \left(\beta_x \boldsymbol{i} + \beta_y \boldsymbol{j} + \beta_z \boldsymbol{k}\right), \quad s = \boldsymbol{u} \cdot \boldsymbol{v}, \;\; \boldsymbol{\beta} = \boldsymbol{u} \times \boldsymbol{v}. \tag{11.34}$$

Hopf fibration The quaternions can be parameterized as the product of a rotation $\boldsymbol{q}_\theta$ around the z axis and a quaternion that rotates $\boldsymbol{e}_z$ to $\boldsymbol{\beta} := \left[\beta_x, \beta_y, \beta_z\right] \in \mathbb{S}^2$ as

$$\boldsymbol{q} = \boldsymbol{q}_{\boldsymbol{\beta}} * \boldsymbol{q}_\theta, \quad \boldsymbol{q}_{\boldsymbol{\beta}} = \frac{1}{\sqrt{2(1+\beta_z)}} \left(1 + \beta_z - \boldsymbol{i}\beta_x + \boldsymbol{j}\beta_y\right), \;\; \boldsymbol{q}_\theta = \cos\left(\frac{\theta}{2}\right) + \boldsymbol{k} \sin\left(\frac{\theta}{2}\right). \tag{11.35}$$

The special case when $\beta_z = -1$ is a singularity and must be handled separately, for example by setting $\boldsymbol{q}_{\left[0,\,0,\,-1\right]} = \boldsymbol{i}$. The Hopf parameterization is a manifestation of the fact that $\mathbb{S}^3$ locally is a product of the spaces $\mathbb{S}^2$ and $\mathbb{S}^1$.

Proof of (11.34). From properties of the dot and cross products the sought-after rotation is about the axis $\boldsymbol{\beta} = \boldsymbol{u} \times \boldsymbol{v}$ for the angle α such that $s := \boldsymbol{u} \cdot \boldsymbol{v} = \cos(\alpha)$. The half-angle formulas then give that $\cos(\alpha/2) = \sqrt{(1+s)/2}$, and similarly for the sine part in (11.33). $\square$

12. $\mathbb{SE}(2)$: The 2D Rigid Motion Group

The special Euclidean group in two dimensions, $\mathbb{SE}(2)$, is formed as a semi-direct product between $\mathbb{SO}(2)$ and $\mathbb{T}(2)$, and consists of matrices on the form

$$\mathbb{SE}(2) = \mathbb{SO}(2) \ltimes \mathbb{E}(2) = \left\{ \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \mid \boldsymbol{R} \in \mathbb{SO}(2) \right\}, \tag{12.1}$$

This group has a natural action on two-dimensional vectors that consists of rotation and translation. For $\boldsymbol{X} \in \mathbb{SE}(2)$ the action is

$$\langle \boldsymbol{X}, \boldsymbol{u} \rangle_{\mathbb{SE}(2)} = \langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\mathbb{SO}(2)} + \boldsymbol{p} = \boldsymbol{R}\boldsymbol{u} + \boldsymbol{p}. \tag{12.2}$$

That is, the vector $\boldsymbol{u}$ is first rotated by the $\mathbb{SO}(2)$ component, and then subjected to a translation. This action can be written as a matrix multiplication if we associate $\boldsymbol{u}$ with its homogeneous counterpart $\boldsymbol{u}^H = \begin{bmatrix} \boldsymbol{u}^T & 1 \end{bmatrix}^T$:

$$\langle \boldsymbol{X}, \boldsymbol{u} \rangle = \boldsymbol{X}\boldsymbol{u}^H = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}\boldsymbol{u} + \boldsymbol{p} \\ 1 \end{bmatrix}. \tag{12.3}$$

The action has a natural interpretation as a change of coordinates: if $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \in \mathbb{SE}(2)$, then $\langle \boldsymbol{X}, \boldsymbol{u} \rangle$ represents the transformation from a coordinate frame attached at $\boldsymbol{p}$ with unit vectors the columns of $\boldsymbol{R}$, to the global coordinate frame.

The Lie algebra $\mathfrak{se}(2)$ has three degrees of freedom; we parameterize it with the following hat and vee maps:

$$\begin{tikzcd}
\mathbb{R}^3 \ni \begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} \arrow[r, bend left, "\wedge"] & \begin{bmatrix} 0 & -\omega_z & v_x \\ \omega_z & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{se}(2) \arrow[l, bend left, "\vee"]
\end{tikzcd}$$

12.1. Formulas

Adjoint Take $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix}$ and recall the hat formula $\hat{\omega}_z = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix}$ and adjoint $(\mathbf{Ad}_{\boldsymbol{R}}\, \omega_z)^\wedge = \boldsymbol{R}\hat{\omega}_z\boldsymbol{R}^T = \hat{\omega}_z$ from $\mathbb{SO}(2)$. Evaluating (4.12) gives

$$\begin{aligned} \mathbf{Ad}_X \begin{bmatrix} \boldsymbol{v} \\ \omega_z \end{bmatrix} &= \left(\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}_z & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix}^{-1} \right)^\vee = \left(\begin{bmatrix} \boldsymbol{R}\hat{\omega}_z & \boldsymbol{R}\boldsymbol{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{p} \\ 0 & 1 \end{bmatrix} \right)^\vee \\ &= \begin{bmatrix} \boldsymbol{R}\hat{\omega}_z\boldsymbol{R}^T & \boldsymbol{R}\boldsymbol{v} - \boldsymbol{R}\hat{\omega}_z\boldsymbol{R}^T\boldsymbol{p} \\ 0 & 1 \end{bmatrix}^\vee = \begin{bmatrix} \boldsymbol{R}\boldsymbol{v} - \hat{\omega}_z\boldsymbol{p} \\ \omega_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & -\hat{1}\,\boldsymbol{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \omega_z \end{bmatrix}, \end{aligned} \tag{12.4}$$

which exposes the adjoint as the matrix $\begin{bmatrix} \boldsymbol{R} & -\hat{1}\,\boldsymbol{p} \\ 0 & 1 \end{bmatrix}$.

Exponential and Logarithm We use Lemma 9.1 to derive the exponential map. The Lie algebra elements have structure

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}, \quad \boldsymbol{B} = \begin{bmatrix} \hat{\omega}_z & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \mathbf{0} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}. \tag{12.5}$$

Thus, it suffices to compute $S(\omega) \coloneqq \sum_{k=0}^{\infty} \frac{\hat{\boldsymbol{\omega}}}{(k+1)!}$ to obtain the exponential map for the semi-simple groups. Disregarding the trivial case $\omega_z = 0$, we obtain

$$\begin{aligned} S(\omega_z) &= \sum_{k=0}^{\infty} \frac{\hat{\omega}_z^k}{(k+1)!} = (\hat{\omega}_z)^{-1} (\operatorname{Exp} \hat{\omega}_z - I) \\ &= \frac{1}{\omega_z^2} \begin{bmatrix} 0 & \omega_z \\ -\omega_z & 0 \end{bmatrix} \begin{bmatrix} \cos\omega_z - 1 & -\sin\omega_z \\ \sin\omega_z & \cos\omega_z - 1 \end{bmatrix} = \frac{1}{\omega_z} \begin{bmatrix} \sin\omega_z & \cos\omega_z - 1 \\ 1 - \cos\omega_z & \sin\omega_z \end{bmatrix}. \end{aligned} \tag{12.6}$$

Lemma 9.1 now gives the exponential.

$$\exp(\boldsymbol{A}) = \exp(\boldsymbol{B} + \boldsymbol{C}) = \exp(\boldsymbol{B}) + \begin{bmatrix} S(\omega_z) & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{C} = \begin{bmatrix} \exp_{\mathbb{SO}(2)} \omega_z & S(\omega_z)\boldsymbol{v} \\ 0 & 1 \end{bmatrix} \tag{12.7}$$

Derivatives of the Exponential We first calculate an expression for the bracket.

$$\begin{aligned} \left[\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix}, \begin{bmatrix} \bar{v}_x \\ v_y \\ \bar{\omega}_z \end{bmatrix} \right] &= \left(\begin{bmatrix} 0 & -\omega_z & v_x \\ \omega_z & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\bar{\omega}_z & \bar{v}_x \\ \bar{\omega}_z & 0 & \bar{v}_y \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\bar{\omega}_z & \bar{v}_x \\ \bar{\omega}_z & 0 & \bar{v}_y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_z & v_x \\ \omega_z & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} \right)^\vee \\ &= \begin{bmatrix} 0 & 0 & -\omega_z\bar{v}_y + \bar{\omega}_z v_y \\ 0 & 0 & \omega_z\bar{v}_x - \bar{\omega}_z v_x \\ 0 & 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} -\omega_z\bar{v}_y + \bar{\omega}_z v_y \\ \omega_z\bar{v}_x - \bar{\omega}_z v_x \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_z & v_y \\ \omega_z & 0 & -v_x \\ 0 & 0 & 0 \end{bmatrix}}_{\mathrm{ad}_{\boldsymbol{a}}} \begin{bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{\omega}_z \end{bmatrix}. \end{aligned} \tag{12.8}$$

A quick calculation reveals that $\mathrm{ad}_{\boldsymbol{a}}^3 = -\omega_z^2\, \mathrm{ad}_{\boldsymbol{a}}$, which is exactly the relation we used for $\mathbb{SO}(3)$ above. Consequently the inverse derivatives must have the same form as on $\mathbb{SO}(3)$.

$\mathbb{SE}(2)$ formula sheet

Consists of 3×3 matrices $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix}$ that act on $\mathbb{R}^2$ via $\boldsymbol{u} \mapsto \boldsymbol{R}\boldsymbol{u} + \boldsymbol{p}$.

Algebra Parameterization

$$\left\{ \begin{bmatrix} \boldsymbol{v} \\ \omega_z \end{bmatrix} \mid \boldsymbol{v} \in \mathbb{R}^2, \omega_z \in [-\pi, \pi] \right\}, \quad \begin{bmatrix} \boldsymbol{v} \\ \omega_z \end{bmatrix}^\wedge = \begin{bmatrix} \hat{\omega}_z & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2). \tag{12.9}$$

Adjoint

$$\mathbf{Ad}_X = \begin{bmatrix} \boldsymbol{R} & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ 0 & 1 \end{bmatrix} \tag{12.10}$$

Exponential and Logarithm

Let $S(\omega_z)$ be as in (12.6),

$$\exp_{\mathbb{SE}(2)}\left(\begin{bmatrix} \boldsymbol{v} \\ \omega_z \end{bmatrix} \right) = \begin{bmatrix} \exp_{\mathbb{SO}(2)}(\omega_z) & S(\omega_z)\boldsymbol{v} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix}, \tag{12.11a}$$

$$\log_{\mathbb{SE}(2)} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} = \begin{bmatrix} S(\alpha)^{-1}\boldsymbol{p} \\ \alpha \end{bmatrix}, \quad \alpha = \log_{\mathbb{SO}(2)}(\boldsymbol{R}). \tag{12.11b}$$

Bracket and Lowercase Adjoint

$$\begin{aligned} \left[\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix}, \begin{bmatrix} v'_x \\ v'_y \\ \omega'_z \end{bmatrix} \right] &= \begin{bmatrix} -\omega_z v'_y + \omega'_z v_y \\ \omega_z v'_x - \omega'_z v_x \\ 0 \end{bmatrix}, \\ \mathrm{ad}_{\boldsymbol{a}} &= \begin{bmatrix} 0 & -\omega_z & v_y \\ \omega_z & 0 & -v_x \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{12.12}$$

Derivatives of the Exponential

Let $\boldsymbol{a} = \begin{bmatrix} v_x & v_y & \omega_z \end{bmatrix}^T$. Then,

$$\mathrm{d}^r \exp_{\boldsymbol{a}} = I_3 - \frac{1 - \cos\omega_z}{\omega_z^2} \mathrm{ad}_{\boldsymbol{a}} + \frac{\omega_z - \sin\omega_z}{\omega_z^3} \mathrm{ad}_{\boldsymbol{a}}{}^2, \tag{12.13}$$

$$\mathrm{d}^l \exp_{\boldsymbol{a}} = I_3 + \frac{1 - \cos\omega_z}{\omega_z^2} \mathrm{ad}_{\boldsymbol{a}} + \frac{\omega_z - \sin\omega_z}{\omega_z^3} \mathrm{ad}_{\boldsymbol{a}}{}^2, \tag{12.14}$$

$$\left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1} = I_3 + \frac{\mathrm{ad}_{\boldsymbol{a}}}{2} + \left(\frac{1}{\omega_z^2} - \frac{1 + \cos\omega_z}{2\omega_z \sin\omega_z} \right) \mathrm{ad}_{\boldsymbol{a}}^2, \tag{12.15}$$

$$\left(\mathrm{d}^l \exp_{\boldsymbol{a}}\right)^{-1} = I_3 - \frac{\mathrm{ad}_{\boldsymbol{a}}}{2} + \left(\frac{1}{\omega_z^2} - \frac{1 + \cos\omega_z}{2\omega_z \sin\omega_z} \right) \mathrm{ad}_{\boldsymbol{a}}^2. \tag{12.16}$$

12.2. Parameterization via Isomorphism with $\mathbb{U}(1) \ltimes \mathbb{R}^2$

Since $\mathbb{SE}(2) = \mathbb{SO}(2) \ltimes \mathbb{T}(2)$, and we know that $\mathbb{SO}(2) \cong \mathbb{U}(1)$ and $\mathbb{T}(2) \cong \mathbb{R}^2$, a natural parameterization of $\mathbb{SE}(2)$ is via the isomorphism

$$\mathbb{SE}(2) \cong \mathbb{U}(1) \ltimes \mathbb{R}^2, \tag{12.17}$$

where the group operation in $\mathbb{U}(1) \ltimes \mathbb{R}^2$ is defined analogously to $\mathbb{SE}(2)$. For $c, c' \in \mathbb{U}(1)$ and $\boldsymbol{p}, \boldsymbol{p}' \in \mathbb{R}^2$ this becomes

$$(c, \boldsymbol{p}) \circ (c', \boldsymbol{p}') = \left(c \circ c', \langle c, \boldsymbol{p}' \rangle_{\mathbb{U}(1)} + \boldsymbol{p}\right). \tag{12.18}$$

The hat and vee maps between $\mathbb{SE}(2)$ and $\mathbb{U}(1) \ltimes \mathbb{R}^2$ are

$$\begin{tikzcd}
\mathbb{U}(1) \ltimes \mathbb{R}^2 \ni (c, \boldsymbol{p}) \arrow[r, bend left, "\wedge"] & \begin{bmatrix} c^\wedge & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \in \mathbb{SE}(2) \arrow[l, bend left, "\vee"]
\end{tikzcd}$$

from where the exponential and log maps follow from (12.11).

$\mathbb{U}(1) \ltimes \mathbb{R}^2$ as a parameterization of $\mathbb{SE}(2)$

Group Definition The parameterization of $\mathbb{U}(1) \ltimes \mathbb{R}^2$ is

$$\left\{(c, \boldsymbol{p}) \in \mathbb{C} \times \mathbb{R}^2\right\} \tag{12.19}$$

and the group operation is defined in (12.18). $\mathbb{U}(1) \ltimes \mathbb{R}^2$ is isomorphic to $\mathbb{SE}(2)$ and inherits its Lie algebra properties.

Frame Transformation Action on $\boldsymbol{u} \in \mathbb{R}^2$

$$\langle (c, \boldsymbol{p}), \boldsymbol{u} \rangle_{\mathbb{U}(1) \ltimes \mathbb{R}^2} = \langle c, \boldsymbol{u} \rangle_{\mathbb{U}(1)} + \boldsymbol{p}. \tag{12.20}$$

Exponential and Logarithm

$$\exp \begin{bmatrix} \boldsymbol{v} \\ \omega_z \end{bmatrix} = \left(\exp_{\mathbb{U}(1)}(\omega_z), S(\omega_z)\boldsymbol{v}\right), \tag{12.21a}$$

$$\log(c, \boldsymbol{p}) = \begin{bmatrix} S(\alpha)^{-1}\boldsymbol{p} \\ \alpha \end{bmatrix}, \quad \alpha = \log_{\mathbb{U}(1)} c. \tag{12.21b}$$

13. $\mathbb{SE}(3)$: The 3D Rigid Motion Group

Moving to three dimensions changes little—the properties of $\mathbb{SE}(3)$ are quite similar to those of $\mathbb{SE}(2)$. It is formed as a semi-direct product between $\mathbb{SO}(3)$ and $\mathbb{T}(3)$

$$\mathbb{SE}(3) = \mathbb{SO}(3) \ltimes \mathbb{T}(3) = \left\{ \boldsymbol{X} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \mid \boldsymbol{R} \in \mathbb{SO}(3), \boldsymbol{p} \in \mathbb{R}^3 \right\}. \tag{13.1}$$

A rotation plus translation action on $\boldsymbol{u} \in \mathbb{R}^3$ is defined by

$$\langle \boldsymbol{X}, \boldsymbol{u} \rangle_{\mathbb{SE}(3)} = \langle \boldsymbol{R}, \boldsymbol{u} \rangle_{\mathbb{SO}(3)} + \boldsymbol{p} = \boldsymbol{R}\boldsymbol{u} + \boldsymbol{p}. \tag{13.2}$$

The Lie algebra has six degrees of freedom that we parameterize by

$$\mathbb{R}^6 \ni \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \underset{\vee}{\overset{\wedge}{\rightleftarrows}} \begin{bmatrix} \boldsymbol{\omega}^\wedge & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

where $\boldsymbol{v}, \boldsymbol{\omega} \in \mathbb{R}^3$ are the linear and angular components, respectively.

13.1. Formulas

Adjoint From the definition (4.12),

$$\begin{aligned}
\mathbf{Ad}_X \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} &= \left(\begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0 & 1 \end{bmatrix}^{-1} \right)^\vee = \begin{bmatrix} \boldsymbol{R}\boldsymbol{\omega}^\wedge \boldsymbol{R}^T & -\boldsymbol{R}\boldsymbol{\omega}^\wedge \boldsymbol{R}^T \boldsymbol{p} + \boldsymbol{R}\boldsymbol{v} \\ 0 & 1 \end{bmatrix}^\vee \\
&= \begin{bmatrix} \boldsymbol{R}\boldsymbol{v} - \boldsymbol{R}\boldsymbol{\omega}^\wedge \boldsymbol{R}^T \boldsymbol{p} \\ (\boldsymbol{R}\boldsymbol{\omega}^\wedge \boldsymbol{R})^\vee \end{bmatrix} \overset{(11.15)}{=} \begin{bmatrix} \boldsymbol{R}\boldsymbol{v} - (\boldsymbol{R}\boldsymbol{\omega})^\wedge \boldsymbol{p} \\ \boldsymbol{R}\boldsymbol{\omega} \end{bmatrix} \overset{(11.3b)}{=} \begin{bmatrix} \boldsymbol{R}\boldsymbol{v} + \boldsymbol{p}^\wedge \boldsymbol{R}\boldsymbol{\omega} \\ \boldsymbol{R}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p}^\wedge \boldsymbol{R} \\ 0 & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix},
\end{aligned}$$

thus the uppercase adjoint is

$$\mathbf{Ad}_X = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p}^\wedge \boldsymbol{R} \\ 0 & \boldsymbol{R} \end{bmatrix}. \tag{13.3}$$

Exponential and Logarithm Looking to again utilize Lemma 9.1 we calculate

$$S(\boldsymbol{\omega}) := \sum_{k=0}^{\infty} \frac{\hat{\boldsymbol{\omega}}^k}{(k+1)!} = I_3 - \frac{1}{\|\boldsymbol{\omega}\|^2} \sum_{k=2}^{\infty} \frac{\hat{\boldsymbol{\omega}}^k}{k!} = I_3 - \frac{1}{\|\boldsymbol{\omega}\|^2} \hat{\boldsymbol{\omega}} \left(\mathrm{Exp}_{\mathbb{SO}(3)}(\hat{\boldsymbol{\omega}}) - I_3 - \hat{\boldsymbol{\omega}} \right). \tag{13.4}$$

From the $\mathbb{SO}(3)$ exponential in (11.16a),

$$\begin{aligned} S(\boldsymbol{\omega}) &= I_3 - \frac{1}{\|\boldsymbol{\omega}\|^2} \hat{\boldsymbol{\omega}} \left(\left(I_3 + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|} \hat{\boldsymbol{\omega}} + \frac{(1-\cos\|\boldsymbol{\omega}\|)}{\|\boldsymbol{\omega}\|^2} \hat{\boldsymbol{\omega}}^2 \right) - I_3 - \hat{\boldsymbol{\omega}} \right) \\ &= I_3 - \frac{(\sin\|\boldsymbol{\omega}\| - \|\boldsymbol{\omega}\|)}{\|\boldsymbol{\omega}\|^3} \hat{\boldsymbol{\omega}}^2 + \frac{\|\boldsymbol{\omega}\|^2(1-\cos\|\boldsymbol{\omega}\|)}{\|\boldsymbol{\omega}\|^4} \hat{\boldsymbol{\omega}} \\ &= I_3 + \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3} \hat{\boldsymbol{\omega}}^2 + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2} \hat{\boldsymbol{\omega}}. \end{aligned} \tag{13.5}$$

Remark that for $\mathbb{SE}(3)$ it happens that $S(\boldsymbol{\omega}) = \mathrm{d}^{\mathrm{l}} \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}}$, which is a difference from $\mathbb{SE}(2)$ where there is no direct relation between $S(\omega_z)$ and the derivatives of the exponential on $\mathbb{SO}(2)$.

Applying Lemma 9.1 then gives the sought-after $\mathbb{SE}(3)$ exponential

$$\exp_{\mathbb{SE}(3)}(\boldsymbol{\omega}, \boldsymbol{v}) = \mathrm{Exp}_{\mathbb{SE}(3)} \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} = \begin{bmatrix} \exp_{\mathbb{SO}(3)}(\boldsymbol{\omega}) & S(\boldsymbol{\omega})\boldsymbol{v} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}, \tag{13.6}$$

and consequently the logarithm

$$\log_{\mathbb{SE}(3)} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \left(\boldsymbol{\alpha}, S(\boldsymbol{\alpha})^{-1} \boldsymbol{p} \right), \tag{13.7}$$

where $\boldsymbol{\alpha} = \log_{\mathbb{SO}(3)} R$.

Derivatives of the Exponential First we derive an expression for $\mathrm{ad}_{\boldsymbol{a}}$ utilizing that for the hat operator on $\mathbb{SO}(3)$, $\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}$

$$\begin{aligned} \left[\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}, \begin{bmatrix} \bar{\boldsymbol{v}} \\ \bar{\boldsymbol{\omega}} \end{bmatrix} \right]_{\mathbb{SE}(3)} &= \left(\begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\bar{\boldsymbol{\omega}}} & \bar{\boldsymbol{v}} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \hat{\bar{\boldsymbol{\omega}}} & \bar{\boldsymbol{v}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \right)^{\vee} = \begin{bmatrix} [\boldsymbol{\omega}, \bar{\boldsymbol{\omega}}]^{\wedge}_{\mathbb{SO}(3)} & \hat{\boldsymbol{\omega}}\bar{\boldsymbol{v}} - \hat{\bar{\boldsymbol{\omega}}}\boldsymbol{v} \\ 0 & 0 \end{bmatrix}^{\vee} \\ &= \begin{bmatrix} \hat{\boldsymbol{\omega}}\bar{\boldsymbol{v}} - \hat{\bar{\boldsymbol{\omega}}}\boldsymbol{v} \\ [\boldsymbol{\omega}, \bar{\boldsymbol{\omega}}]_{\mathbb{SO}(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\boldsymbol{v}} \\ 0 & \hat{\boldsymbol{\omega}} \end{bmatrix}}_{\mathrm{ad}_{\boldsymbol{a}}} \begin{bmatrix} \bar{v} \\ \bar{\boldsymbol{\omega}} \end{bmatrix}. \end{aligned} \tag{13.8}$$

We are interested in the powers $\mathrm{ad}_{\boldsymbol{a}}^k$ in order to evaluate the exponential derivatives. For $k \geq 1$

$$\mathrm{ad}_{\boldsymbol{a}}^k = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\boldsymbol{v}} \\ 0 & \hat{\boldsymbol{\omega}} \end{bmatrix}^k = \begin{bmatrix} \hat{\boldsymbol{\omega}}^k & \sum_{i=0}^{k-1} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-1-i} \\ 0 & \hat{\boldsymbol{\omega}}^k \end{bmatrix}. \tag{13.9}$$

Thus the left derivative of the exponential can be written

$$\mathrm{d}^l \exp_{\boldsymbol{a}} = \sum_{k=0}^{\infty} \frac{\mathrm{ad}_{\boldsymbol{a}}^k}{(k+1)!} = I + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \begin{bmatrix} \hat{\boldsymbol{\omega}}^k & \sum_{i=0}^{k-1} \hat{\boldsymbol{\omega}}^i \hat{\boldsymbol{v}} \hat{\boldsymbol{\omega}}^{k-1-i} \\ 0 & \hat{\boldsymbol{\omega}}^k \end{bmatrix} = \begin{bmatrix} \mathrm{d}^l \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}} & Q^l(\boldsymbol{v}, \boldsymbol{\omega}) \\ 0 & \mathrm{d}^l \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}} \end{bmatrix},$$

where a closed-form expression for $Q^l(\boldsymbol{v},\boldsymbol{\omega})$ can be painstakingly obtained through a series of sum manipulations. We first convert the formula to a form that is symmetric in i and k.

$$
\begin{aligned}
Q^l(\boldsymbol{v},\boldsymbol{\omega}) &:= \sum_{k=1}^{\infty}\frac{1}{(k+1)!}\sum_{i=0}^{k-1}\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k-1-i} = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{1}{(k+2)!}\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k-i}\\
&= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{1}{(k+2)!}\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k-i} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(k+i+2)!}\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k}.
\end{aligned}
$$

With the same steps the right derivative can be shown to be

$$
Q^r(\boldsymbol{v},\boldsymbol{\omega}) := \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\sum_{i=0}^{k-1}\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k-1-i} = \dots = -\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+i}}{(k+i+2)!}\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k} \tag{13.10}
$$

and we can see that Q^r is conveniently obtained from Q^l by

$$
Q^r(\boldsymbol{v},\boldsymbol{\omega}) = Q^l(-\boldsymbol{v},-\boldsymbol{\omega}) \tag{13.11}
$$

which is convenient to know since calculating one of them is tedious enough.

In the following calculation the sum $\sum_{k,i\geq 0}$ is first split into parts $(k=i=0)$, $(k=0, i\geq 1)$, $(k\geq 1, i=0)$ and $(k,i\geq 1)$, and then the resulting single sums are split into two sums $i=0,2,\dots$ and $i=1,3,\dots$. Also using that

$$
\hat{\boldsymbol{\omega}}^{2k+1} = (-1)^k\|\boldsymbol{\omega}\|^{2k}\hat{\boldsymbol{\omega}}, \qquad \hat{\boldsymbol{\omega}}^{2k+2} = (-1)^k\|\boldsymbol{\omega}\|^{2k}\hat{\boldsymbol{\omega}}^2, \tag{13.12}
$$

which follows from (11.3a), we get

$$
\begin{aligned}
Q^l(\boldsymbol{v},\boldsymbol{\omega}) &= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=1}^{\infty}\frac{\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}}{(i+2)!} + \sum_{k=1}^{\infty}\frac{\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^k}{(k+2)!} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(i+k+2)!}\hat{\boldsymbol{\omega}}^i\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^k\\
&= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=0}^{\infty}\frac{\hat{\boldsymbol{\omega}}^{i+1}\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{i+1}}{(i+3)!} + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(i+k+4)!}\hat{\boldsymbol{\omega}}^{i+1}\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k+1}\\
&= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=0}^{\infty}\frac{\hat{\boldsymbol{\omega}}^{2i+1}\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{2i+1}}{(2i+3)!} + \sum_{i=0}^{\infty}\frac{\hat{\boldsymbol{\omega}}^{2i+2}\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{2i+2}}{(2i+4)!} + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(i+k+4)!}\hat{\boldsymbol{\omega}}^{i+1}\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k+1}\\
&= \frac{1}{2}\hat{\boldsymbol{v}} + \sum_{i=0}^{\infty}\frac{(-1)^i\|\boldsymbol{\omega}\|^{2i}}{(2i+3)!}\left(\hat{\boldsymbol{\omega}}\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}\right) + \sum_{i=0}^{\infty}\frac{(-1)^i\|\boldsymbol{\omega}\|^{2i}}{(2i+4)!}\left(\hat{\boldsymbol{\omega}}^2\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^2\right) + \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(i+k+4)!}\hat{\boldsymbol{\omega}}^{i+1}\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k+1}.
\end{aligned}
$$

The first two sums are given in (9.21) and (9.22):

$$
\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+3)!}\|\boldsymbol{\omega}\|^{2i} = \frac{\|\boldsymbol{\omega}\| - \sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}, \qquad \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+4)!}\|\boldsymbol{\omega}\|^{2i} = \frac{\cos\|\boldsymbol{\omega}\| - 1 + \frac{\|\boldsymbol{\omega}\|^2}{2}}{\|\boldsymbol{\omega}\|^4}. \tag{13.13}
$$

The double sum requries additional work. Using $\hat{\boldsymbol{\omega}}\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}} = (-\boldsymbol{\omega}\cdot\boldsymbol{v})\hat{\boldsymbol{\omega}}$ from (11.3a) yields

$$
\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(k+i+4)!}\hat{\boldsymbol{\omega}}^{i+1}\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^{k+1} = (-\boldsymbol{\omega}\cdot\boldsymbol{v})\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(k+i+4)!}\hat{\boldsymbol{\omega}}^{k+i+1} \overset{j=k+i}{=} (-\boldsymbol{\omega}\cdot\boldsymbol{v})\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{(j+4)!}\hat{\boldsymbol{\omega}}^{j+1},
$$

and we can evaluate the sum as

$$\begin{aligned}
\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{(j+4)!}\hat{\boldsymbol{\omega}}^{j+1} &= -\sum_{j=0}^{\infty}\frac{j+1}{(j+4)!}\hat{\boldsymbol{\omega}}^{j+1} = -\sum_{j=0}^{\infty}\left(\frac{1}{(j+3)!}-\frac{3}{(j+4)!}\right)\hat{\boldsymbol{\omega}}^{j+1} \\
&= -\sum_{j=0}^{\infty}\left(\frac{1}{(2j+3)!}-\frac{3}{(2j+4)!}\right)\hat{\boldsymbol{\omega}}^{2j+1}+\sum_{j=0}^{\infty}\left(\frac{1}{(2j+4)!}-\frac{3}{(2j+5)!}\right)\hat{\boldsymbol{\omega}}^{2j+2} \\
&\overset{(13.12)}{=} -\sum_{j=0}^{\infty}\left(\frac{(-1)^j}{(2j+3)!}\|\boldsymbol{\omega}\|^{2j}+3\frac{(-1)^j}{(2j+4)!}\|\boldsymbol{\omega}\|^{2j}\right)\hat{\boldsymbol{\omega}}+\sum_{j=0}^{\infty}\left(-\frac{(-1)^j}{(2j+4)!}\|\boldsymbol{\omega}\|^{2j}+3\frac{(-1)^j}{(2j+5)!}\|\boldsymbol{\omega}\|^{2j}\right)\hat{\boldsymbol{\omega}}^2.
\end{aligned}$$

The sums in (13.13) appear again and can be re-used, and the remaining sum with denominator $(2j+5)!$ was given in (9.23):

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+5)!}\|\boldsymbol{\omega}\|^{2j}=\frac{\sin\|\boldsymbol{\omega}\|-\|\boldsymbol{\omega}\|+\frac{\|\boldsymbol{\omega}\|}{6}}{\|\boldsymbol{\omega}\|^5}. \tag{13.14}$$

After collecting the various expressions the closed-form expression for Q^l can be written down

$$\begin{aligned}
Q^l(\boldsymbol{v},\boldsymbol{\omega}) &= \frac{1}{2}\hat{\boldsymbol{v}}+\frac{\|\boldsymbol{\omega}\|-\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^3}\left(\hat{\boldsymbol{\omega}}\hat{\boldsymbol{v}}+\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}-(\boldsymbol{\omega}\cdot\boldsymbol{v})\hat{\boldsymbol{\omega}}\right) \\
&+\frac{\cos\|\boldsymbol{\omega}\|-1+\frac{\|\boldsymbol{\omega}\|^2}{2}}{\|\boldsymbol{\omega}\|^4}\left(\hat{\boldsymbol{\omega}}^2\hat{\boldsymbol{v}}+\hat{\boldsymbol{v}}\hat{\boldsymbol{\omega}}^2+(\boldsymbol{\omega}\cdot\boldsymbol{v})(3\hat{\boldsymbol{\omega}}-\hat{\boldsymbol{\omega}}^2)\right) \\
&-3(\boldsymbol{\omega}\cdot\boldsymbol{v})\left(\frac{\|\boldsymbol{\omega}\|-\sin\|\boldsymbol{\omega}\|-\frac{\|\boldsymbol{\omega}\|^3}{6}}{\|\boldsymbol{\omega}\|^5}\right)\hat{\boldsymbol{\omega}}^2.
\end{aligned} \tag{13.15}$$

The Q matrix allows us to write down a closed-form expression for $\mathrm{d}^l\exp_{\boldsymbol{a}}$ on $\mathbb{SE}(3)$, and $\left(\mathrm{d}^l\exp_{\boldsymbol{a}}\right)^{-1}$ follows from noting that $\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ 0 & \boldsymbol{A}\end{bmatrix}^{-1}=\begin{bmatrix}\boldsymbol{A}^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\\ 0 & \boldsymbol{A}^{-1}\end{bmatrix}$ for $\boldsymbol{A}$ invertible.

$\mathbb{SE}(3)$ formula sheet

Consists of 3×3 matrices $\boldsymbol{X}=\begin{bmatrix}\boldsymbol{R} & \boldsymbol{p}\\ 0 & 1\end{bmatrix}$ that act on $\mathbb{R}^3$ via $\boldsymbol{u}\mapsto\boldsymbol{R}\boldsymbol{u}+\boldsymbol{p}$.

Algebra Parameterization

$$\left\{\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{bmatrix}\mid \boldsymbol{v}\in\mathbb{R}^3,\boldsymbol{\omega}\in ???\right\},\quad \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{bmatrix}^{\wedge}=\begin{bmatrix}\boldsymbol{\omega}^{\wedge} & \boldsymbol{v}\\ 0 & 0\end{bmatrix}\in\mathfrak{se}(3). \tag{13.16}$$

Adjoint

$$\mathrm{Ad}_X = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p}^\wedge \boldsymbol{R} \\ 0 & \boldsymbol{R} \end{bmatrix}. \tag{13.17}$$

Exponential and Logarithm

$$\exp_{\mathbb{SE}(3)} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \exp_{\mathbb{SO}(3)}(\boldsymbol{\omega}) & S(\boldsymbol{\omega})\boldsymbol{v} \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix}, \tag{13.18a}$$

$$\log_{\mathbb{SE}(3)} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} S(\boldsymbol{\alpha})^{-1}\boldsymbol{p} \\ \boldsymbol{\alpha} \end{bmatrix}, \tag{13.18b}$$

where $\boldsymbol{\alpha} = \log_{\mathbb{SO}(3)}(\boldsymbol{\omega})$ and $S(\boldsymbol{\alpha}) = \mathrm{d}^{\mathrm{l}} \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\alpha}}$.

Bracket and Lowercase adjoint

$$\left[\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}, \begin{bmatrix} \boldsymbol{v}' \\ \boldsymbol{\omega}' \end{bmatrix}\right] = \begin{bmatrix} \boldsymbol{\omega}^\wedge \boldsymbol{v}' - (\boldsymbol{\omega}')^\wedge \boldsymbol{v} \\ [\boldsymbol{\omega}, \boldsymbol{\omega}']_{\mathbb{SO}(3)} \end{bmatrix} \tag{13.19}$$

$$\mathrm{ad}_{\boldsymbol{a}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \hat{\boldsymbol{v}} \\ 0 & \hat{\boldsymbol{\omega}} \end{bmatrix}. \tag{13.20}$$

Note that in these formulas $\boldsymbol{\omega}^\wedge$ and $\boldsymbol{v}^\wedge$ denote the hat operator on $\mathbb{SO}(3)$.

Derivatives of the Exponential Let $Q^{l/r}$ be as in (13.15),

$$\mathrm{d}^r \exp_{\boldsymbol{a}} = \begin{bmatrix} J^r_{\mathbb{SO}(3)} & Q^l(-\boldsymbol{v}, -\boldsymbol{\omega}) \\ 0 & J^r_{\mathbb{SO}(3)} \end{bmatrix}, \tag{13.21a}$$

$$\mathrm{d}^l \exp_{\boldsymbol{a}} = \begin{bmatrix} J^l_{\mathbb{SO}(3)} & Q^l(\boldsymbol{v}, \boldsymbol{\omega}) \\ 0 & J^l_{\mathbb{SO}(3)} \end{bmatrix} \tag{13.21b}$$

$$\left(\mathrm{d}^r \exp_{\boldsymbol{a}}\right)^{-1} = \begin{bmatrix} \left(J^r_{\mathbb{SO}(3)}\right)^{-1} & -\left(J^r_{\mathbb{SO}(3)}\right)^{-1} Q^l(-\boldsymbol{v}, -\boldsymbol{\omega}) \left(J^r_{\mathbb{SO}(3)}\right)^{-1} \\ 0 & \left(J^r_{\mathbb{SO}(3)}\right)^{-1} \end{bmatrix}, \tag{13.21c}$$

$$\left(\mathrm{d}^l \exp_{\boldsymbol{a}}\right)^{-1} = \begin{bmatrix} \left(J^l_{\mathbb{SO}(3)}\right)^{-1} & -\left(J^l_{\mathbb{SO}(3)}\right)^{-1} Q^l(\boldsymbol{v}, \boldsymbol{\omega}) \left(J^l_{\mathbb{SO}(3)}\right)^{-1} \\ 0 & \left(J^l_{\mathbb{SO}(3)}\right)^{-1} \end{bmatrix}. \tag{13.21d}$$

where $J^{l/r}_{\mathbb{SO}(3)} = \mathrm{d}^{l/r} \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}}$ and $\left(J^{l/r}_{\mathbb{SO}(3)}\right)^{-1} = \left(\mathrm{d}^{l/r} \left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\omega}}\right)^{-1}$.

13.2. Parameterization via Isomorphism with $\mathbb{S}^3 \ltimes \mathbb{R}^3$

Similarly to $\mathbb{SE}(2) \cong \mathbb{U}(1) \ltimes \mathbb{R}^2$ we utilize that $\mathbb{SE}(3) \cong \mathbb{S}^3 \times \mathbb{R}^3$ to get a compact representation of $\mathbb{SE}(3)$ elements and with group operation

$$(\boldsymbol{q}, \boldsymbol{p}) \circ (\boldsymbol{q}, \boldsymbol{p}') = \left(\boldsymbol{q} \circ \boldsymbol{q}', \langle \boldsymbol{q}, \boldsymbol{p}' \rangle_{\mathbb{S}^3} + \boldsymbol{p}\right). \tag{13.22}$$

The hat and vee maps between $\mathbb{SE}(3)$ and $\mathbb{S}^3 \ltimes \mathbb{R}^3$ are

$$\begin{tikzcd} \mathbb{S}^3 \ltimes \mathbb{R}^3 \ni (\boldsymbol{q}, \boldsymbol{p}) \arrow[r, bend left, "\wedge"] & \begin{bmatrix} \boldsymbol{q}^\wedge & \boldsymbol{p} \\ 0 & 1 \end{bmatrix} \in \mathbb{SE}(3) \arrow[l, bend left, "\vee"] \end{tikzcd}$$

from where the exponential and log maps follow from (13.18).

$\mathbb{S}^3 \ltimes \mathbb{R}^3$ formula sheet

Group Definition The parameterization of $\mathbb{S}^3 \ltimes \mathbb{R}^3$ is

$$\left\{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{S}^3 \times \mathbb{R}^3\right\} \tag{13.23}$$

and the group operation is defined in (13.22). $\mathbb{S}^3 \ltimes \mathbb{R}^3$ is isomorphic to $\mathbb{SE}(3)$ and inherits its Lie algebra properties.

Frame Transformation Action on $\boldsymbol{u} \in \mathbb{R}^3$

$$\langle (\boldsymbol{q}, \boldsymbol{p}), \boldsymbol{u} \rangle_{\mathbb{S}^3 \ltimes \mathbb{R}^3} = \langle \boldsymbol{q}, \boldsymbol{u} \rangle_{\mathbb{S}^3} + \boldsymbol{p}. \tag{13.24}$$

Exponential and Logarithm

$$\exp \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \left(\exp_{\mathbb{S}^3}(\boldsymbol{\omega}), S(\boldsymbol{\omega})\boldsymbol{v}\right), \tag{13.25a}$$

$$\log(\boldsymbol{q}, \boldsymbol{p}) = \begin{bmatrix} S(\boldsymbol{\alpha})^{-1}\boldsymbol{p} \\ \boldsymbol{\alpha} \end{bmatrix}, \quad \boldsymbol{\alpha} = \log_{\mathbb{S}^3} \boldsymbol{q}. \tag{13.25b}$$

where $\boldsymbol{\alpha} = \log_{\mathbb{SO}(3)}(\boldsymbol{\omega})$ and $S(\boldsymbol{\alpha}) = \mathrm{d}^{\mathrm{l}}\left(\exp_{\mathbb{SO}(3)}\right)_{\boldsymbol{\alpha}}$.

Mention $\mathbb{SE}_2(3)$

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{v}_1 & \boldsymbol{v}_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13.26}$$

Mention Galilean group

[21, p. 321]

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{v}_1 & \boldsymbol{v}_2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{bmatrix} \tag{13.27}$$

Part III.

Applications

14. Geometric Numerical Integration

- Two methods:
  - use $\oplus_r$, $\ominus_r$ for regular RK schemes
  - magnus method schemes like in [6]

15. Control

Summary

- Extend PD theory to Lie Groups.
- Model-predictive control

15.1. A Stabilizing Lie Group Controller

Should be able to do this on group-linear systems

Consider the system

$$\begin{aligned} \mathrm{d}^r \boldsymbol{X}_t &= v \\ \mathrm{d}^r v_t &= u \end{aligned} \tag{15.1}$$

where u is a control input, and the objective of tracking a twice differentiable trajectory $\boldsymbol{X}_d(t)$ with first and second right-derivatives v_d and a_d. Consider the error

$$e_X := \boldsymbol{X}_d \ominus_r \boldsymbol{X}, \tag{15.2}$$

with derivative

$$\mathrm{d}^r(e_X)_t \overset{(5.50),(5.51)}{=} \left(\mathrm{d}^r \exp_{e_X}\right)^{-1} \mathrm{d}^r \boldsymbol{X}_d - \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} \mathrm{d}^r \boldsymbol{X} = \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} (\mathbf{Ad}_{\exp(e_X)} v_d - v). \tag{15.3}$$

Note that $\mathbf{Ad}_{\exp(e_X)} = \exp \mathrm{ad}_e = \sum_{k\geq 0} \frac{\mathrm{ad}_e}{k!}$ can typically be found on closed form via the usual expansion tricks. Let $e_v := \mathbf{Ad}_{\exp(e_X)} v_d - v$ be the velocity error in the body frame; we then have the double intergrator-like error system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e_X &= \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v, \\ \frac{\mathrm{d}}{\mathrm{d}t} e_v &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{Ad}_{\exp(e_X)} v_d\right) - u, \end{aligned} \tag{15.4}$$

Where we can further simplify

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{Ad}_{\exp(e_X)} v_d\right) \overset{(5.17c)}{=} \left[\mathrm{d}^l \exp_{e_X} \dot{e}_X, \mathbf{Ad}_{\exp(e_X)} v_d\right] + \mathbf{Ad}_{\exp(e_X)} \dot{v}_d = \left[e_v, \mathbf{Ad}_{\exp(e_X)} v_d\right] + \mathbf{Ad}_{\exp(e_X)} \dot{v}_d. \tag{15.5}$$

If we further consider an input on the form $u = \left[e_v, \mathbf{Ad}_{\exp(e_X)} v_d\right] + \mathbf{Ad}_{\exp(e_X)} \dot{v}_d + k_p \left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X + k_d e_v$ that cancels out the contribution from v_d and adds PD feedback terms the closed-loop dynamics become

$$\frac{\mathrm{d}}{\mathrm{d}t} e_X = \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} e_v = -k_p \left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X - k_d e_v. \tag{15.6}$$

Now consider a Lyapunov candidate function on the form

$$V = \frac{k_p}{2}\|e_X\|^2 + \frac{1}{2}\|e_v\|^2 + c\langle e_v, e_X\rangle \geq \frac{1}{2}\begin{bmatrix}\|e_X\| \\ \|e_v\|\end{bmatrix}^T \begin{bmatrix} k_p & -c \\ -c & 1\end{bmatrix}\begin{bmatrix}\|e_X\| \\ \|e_v\|\end{bmatrix}, \tag{15.7}$$

where c is s.t. $k_p - c^2 \geq 0$ so that the matrix is positive definite. Its derivative evaluates to

$$\begin{aligned}
\dot{V} &= k_p \left\langle e_X, \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v \right\rangle - k_p \left\langle e_v, \left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X \right\rangle - k_d\|e_v\|^2 + c\langle \dot{e}_v, e_X\rangle + c\langle e_v, \dot{e}_X\rangle \\
&= -k_d\|e_v\|^2 - c\left\langle k_p\left(\mathrm{d}^l \exp_{e_X}\right)^{-T} e_X + k_d e_v, e_X\right\rangle + c\left\langle e_v, \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v\right\rangle \\
&= -k_d\|e_v\|^2 - ck_p\|e_X\|^2 - ck_d\langle e_v, e_X\rangle + c\left\langle e_v, \left(\mathrm{d}^l \exp_{e_X}\right)^{-1} e_v\right\rangle - ck_p\left\langle ((\mathrm{d}^l \exp_{e_X})^{-T} - I)e_X, e_X\right\rangle \\
&\leq -k_d\|e_v\|^2 - ck_p\|e_X\|^2 + ck_d\|e_v\|\|e_X\| + c\lambda_{\max}\left(\left(\mathrm{d}^l \exp_{e_X}\right)^{-1}\right)\|e_v\|^2 + ck_p\lambda_{\max}\left(\left(\mathrm{d}^l \exp_{e_X}\right)^{-1} - I\right)\|e_X\|^2.
\end{aligned}$$

- Eigenvalues of $(\mathrm{d}^l \exp_{e_X})^{-1}$ can be shown to be on the form $\frac{\lambda}{e^\lambda - 1} = \sum_{k=0}^\infty \frac{B_n}{n!}\lambda^n$, where λ is an eigenvalue of ad_e.
- Zero is always an eigenvalue of ad_e since $\mathrm{ad}_e e = 0$ due to it being a commutator (the corresponding eigenvalue of $(\mathrm{d}^l \exp_e)^{-1}$ is 1
- Often, eigenvalues of ad_e are purely imaginary. The corresponding eigenvalues of $(\mathrm{d}^l \exp_{e_X})^{-1}$ are

$$\frac{i\lambda}{e^{i\lambda} - 1} = \frac{i\lambda e^{-i\lambda/2}}{e^{i\lambda/2} - e^{-i\lambda/2}} = \frac{i\lambda e^{-i\lambda/2}}{2i\sin\lambda/2} = \lambda\frac{\cos\lambda/2 - i\sin\lambda/2}{2\sin\lambda/2} = \frac{\lambda}{2}\cot\frac{\lambda}{2} - i\frac{\lambda}{2}. \tag{15.8}$$

  That is, the real part is equal to $\frac{\lambda}{2}\cot\frac{\lambda}{2}$.
- For angular groups we should throttle the angular part of $\|e_X\|$ at $\pm\pi/2$ in order to avoid the region where the eigenvalues approach zero which otherwise would lead to sluggish convergence

The maximal real part for $\lambda \in [-\pi, \pi]$ is attained at $\lambda = 0$ and is equal to 1, as shown in Figure 15.1. Thus, for lie groups s.t. $\mathrm{ad}_{\boldsymbol{a}}$ has purely imaginary eigenvalues in the range $[-\pi, \pi]$ for all $\boldsymbol{a}$, it holds that $\left(\mathrm{d}^l \exp_{e_X}\right)^{-1}$ has no eigenvalue with real absolute magnitude larger than 1.

Let $\epsilon = \lambda_{\max}\left(\left(\mathrm{d}^l \exp_{e_X}\right)^{-1} - I\right)$; then we have

$$\dot{V} \leq -\begin{bmatrix}\|e_X\| \\ \|e_v\|\end{bmatrix}^T \begin{bmatrix} ck_p(1-\epsilon) & -\frac{ck_d}{2} \\ -\frac{ck_d}{2} & k_d - c\end{bmatrix}\begin{bmatrix}\|e_X\| \\ \|e_v\|\end{bmatrix} \tag{15.9}$$

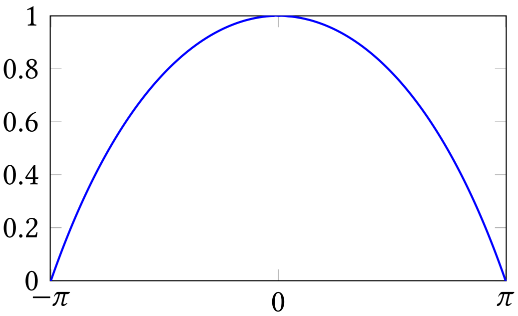


Figure 15.1.: Function $x \mapsto \frac{x}{2} \cot \frac{x}{2}$.

Therefore, if

$$\begin{aligned} ck_p(1-\epsilon) + k_d - c &\geq 0 \\ ck_p(1-\epsilon) - \frac{c^2 k_d^2}{4} &\geq 0 \end{aligned} \tag{15.10}$$

In the following we let $M = SO(3)$ be the Lie group consisting of rotation matrices with matrix multiplication being the group action. We also write $e = I_3$ for the identity element of the group.

Consider the rigid body dynamics

$$\dot{R} = R\, {}^R\hat{\boldsymbol{\omega}}, \tag{15.11a}$$

$$J\, {}^R\dot{\boldsymbol{\omega}} = -{}^R\hat{\boldsymbol{\omega}} J {}^R\boldsymbol{\omega} + u, \tag{15.11b}$$

where J is the moment of inertia, ${}^R\hat{\boldsymbol{\omega}} \in TM_R$ is the angular velocity in the body frame, and $R \in SO(3)$ is the attitude. We can see that the angular velocity ${}^e\hat{\boldsymbol{\omega}} \in TM_e$ in the inertial frame can be obtained as

$${}^e\boldsymbol{\omega} = \mathrm{Ad}_R\left({}^R\boldsymbol{\omega}\right) = R {}^R\boldsymbol{\omega}. \tag{15.12}$$

We also see that ${}^e\hat{\boldsymbol{\omega}} = \widehat{\mathrm{Ad}_R^R\,\boldsymbol{\omega}} = R\, {}^R\boldsymbol{\omega}\, R^T$, so it follows that (15.11a) can be written as

$$\dot{R} = {}^e\hat{\boldsymbol{\omega}} R. \tag{15.13}$$

We assume that a smooth trajectory in the inertial frame is given by R_d and ${}^e\boldsymbol{\omega}_d$ satisfying the dynamics

$$\dot{R}_d = {}^e\hat{\boldsymbol{\omega}}_d R_d, \tag{15.14}$$

and the goal is to control u in (15.11) so that R and ${}^e\boldsymbol{\omega}$ are close to R_d and ${}^e\boldsymbol{\omega}_d$.

15.2. Error Functions

In general we would like to pick for $\tilde{e}_r = R_d \ominus R$ the error function $\frac{1}{2}\|\tilde{e}_r\|^2$ with derivative $\langle \tilde{e}_r, \tilde{e}_\omega \rangle$ for

$$\tilde{e}_\omega = \dot{\tilde{e}}_r = J_{R_d}^{R_d \ominus R}\, {}^{R_d}\boldsymbol{\omega}_d + J_R^{R_d \ominus R}\, {}^R\boldsymbol{\omega}. \tag{15.15}$$

This is general for any Lie group, and we can pick u to stabilize a double integrator system in the tangent space. However, the derivative of $\tilde{e}_\omega$ is cumbersome to evaluate and it is possible to arrive at a simpler formulation in $SO(3)$. Consider the error functions

$$\Psi(R, R_d) = 1 - \cos(\theta) = \frac{1 - \mathrm{Tr}(RR_d^T)}{2} = -\frac{1 - \langle R_d, R\rangle_F}{2}, \tag{15.16a}$$

$$e_r = \frac{1}{2}(R_d^T R - R^T R_d)^\vee, \tag{15.16b}$$

$$e_\omega = \boldsymbol{\omega} - R^T \boldsymbol{\omega}^d \in TSO(3)_R. \tag{15.16c}$$

It can be seen by (11.16b) that e_r is a rescaling of $\tilde{e}_r$. The derivative of Ψ is $\langle e_r, e_\omega\rangle$ as above, indeed

$$\begin{aligned}
\dot{\Psi} &= -\frac{1}{2}\left(\langle R_d, \dot{R}\rangle_F + \langle \dot{R}_d, R\rangle_F\right) = -\frac{1}{2}\left(\langle R_d, R\hat{\boldsymbol{\omega}}\rangle_F + \langle \hat{\boldsymbol{\omega}}_d R_d, R\rangle_F\right) = \\
&= -\frac{1}{2}\left(\langle R^T R_d, \hat{\boldsymbol{\omega}}\rangle_F - \langle \hat{\boldsymbol{\omega}}_d^T R_d, R\rangle_F\right) = -\frac{1}{2}\left(\langle R^T R_d, \hat{\boldsymbol{\omega}}\rangle_F - \langle R_d, \hat{\boldsymbol{\omega}}_d R\rangle_F\right) \\
&= -\frac{1}{2}\left(\langle R^T R_d, \hat{\boldsymbol{\omega}}\rangle_F - \left\langle R_d, R\widehat{R^T\boldsymbol{\omega}_d}\right\rangle_F\right) = -\frac{1}{2}\langle R^T R_d, \hat{e}_\omega\rangle_F \\
&= \frac{1}{4}\langle R_d^T R - R^T R_d, \hat{e}_\omega\rangle_F = e_r \cdot e_\omega,
\end{aligned} \tag{15.17}$$

where we have used the property that the Frobenius product $\langle A, B\rangle_F = -\langle A^T, B\rangle_F$ for B skew-symmetric.

Do derivatives via jacobians instead

15.3. Lyapunov Stability

We let the input be

$$u = -k_r e_r - k_\omega e_\omega + \widehat{R^T\boldsymbol{\omega}_d} J R^T \boldsymbol{\omega}_d + J R^T \dot{\boldsymbol{\omega}}_d. \tag{15.18}$$

and consider a Lyapunov candidate on the form

$$V = \frac{1}{2} e_\omega \cdot J e_\omega + k_r \Psi + c e_r \cdot J e_\omega \tag{15.19}$$

The derivative of the Lyapunov candidate then

Proposition 15.1. *It holds that*

$$J\dot{e}_\omega = -k_r e_r - k_\omega e_\omega + \left(J e_\omega + \left(2 J R^T \boldsymbol{\omega}_d - \mathrm{trace}(J) I\right) R^T \boldsymbol{\omega}_d\right) \times e_\omega. \tag{15.20}$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} J e_\omega &\overset{(15.28b)}{=} J\dot{\boldsymbol{\omega}} - J\dot{R}^T\boldsymbol{\omega}_d - JR^T\dot{\boldsymbol{\omega}}_d \overset{(15.11)}{=} u - \hat{\boldsymbol{\omega}} J\boldsymbol{\omega} - J\left(R\hat{\boldsymbol{\omega}}\right)^T\boldsymbol{\omega}_d - JR^T\dot{\boldsymbol{\omega}}_d \\
&\overset{(15.29)}{=} -k_r e_r - k_\omega e_\omega + \widehat{R^T\ \boldsymbol{\omega}_d} JR^T\ \boldsymbol{\omega}_d - \hat{\boldsymbol{\omega}} J\boldsymbol{\omega} - J\hat{\boldsymbol{\omega}}^T R^T\boldsymbol{\omega}_d \\
&\overset{(15.28b)}{=} -k_r e_r - k_\omega e_\omega + \widehat{R^T\ \boldsymbol{\omega}_d} JR^T\ \boldsymbol{\omega}_d - (\hat{e}_\omega + \widehat{R^T\boldsymbol{\omega}_d}) J (e_\omega + R^T\boldsymbol{\omega}_d) \\
&\quad + J\left(\hat{e}_\omega + \cancelto{0}{\widehat{R^T\boldsymbol{\omega}_d}}\right) R^T\boldsymbol{\omega}_d \\
&= -k_r e_r - k_\omega e_\omega + \left(\widehat{Je_\omega} + \widehat{JR^T\boldsymbol{\omega}_d} - \widehat{R^T\boldsymbol{\omega}_d} J - J\widehat{R^T\boldsymbol{\omega}_d}\right) e_\omega \\
&= -k_r e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T\boldsymbol{\omega}_d - \operatorname{trace}(J)I\right) R^T\boldsymbol{\omega}_d\right)^\wedge e_\omega.
\end{aligned}$$

□

We then get

$$\dot{V} = -k_\omega \|e_\omega\|^2 + c\dot{e}_r \cdot Je_\omega + ce_r \cdot J\dot{e}_\omega \tag{15.21}$$

It remains to bound the terms involving c. We have that $\|\boldsymbol{\omega}\|_2^2 = \frac{1}{2}\|\hat{\boldsymbol{\omega}}\|_F^2$. We also have

$$\frac{\mathrm{d}}{\mathrm{d}t} R_d^T R = R_d^T R\hat{\boldsymbol{\omega}} + R_d^T\hat{\boldsymbol{\omega}}_d^T R = R_d^T R\hat{\boldsymbol{\omega}} - R_d^T\hat{\boldsymbol{\omega}}_d R = R_d^T R\hat{\boldsymbol{\omega}} - R_d^T R\ \widehat{R^T\boldsymbol{\omega}_d} = R_d^T R\hat{e}_\omega. \tag{15.22}$$

and therefore we get that $\|\dot{\hat{e}}_r\|_F = \left\|\frac{1}{2}\left(R_d^T R\hat{e}_\omega + \hat{e}_\omega R^T R_d\right)\right\|_F \leq \|\hat{e}_\omega\|_F$, so it follows that

$$\|\dot{e}_r\|_2 \leq \|e_\omega\|_2 \qquad \Longrightarrow\ \dot{e}_r \cdot Je_\omega \leq \lambda_M(J)\|e_\omega\|_2^2. \tag{15.23}$$

Finally, using that $\|e_r\| \leq 1$,

$$\begin{aligned}
J\dot{e}_\omega \cdot e_r &\overset{(15.20)}{=} \left(-k_r e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T\boldsymbol{\omega}_d - \operatorname{trace}(J)I\right) R^T\boldsymbol{\omega}_d\right) \times e_\omega\right) \cdot e_r \\
&\leq -k_r\|e_r\|^2 + k_\omega\|e_r\|\|e_\omega\| + \lambda_M(J)\|e_\omega\|^2 + B\|e_\omega\|\|e_r\|.
\end{aligned} \tag{15.24}$$

We can now bound the derivative as follows:

$$\dot{V} \leq -\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix}^T \begin{bmatrix} ck_r & -c(k_\omega + B)/2 \\ -c(k_\omega + B)/2 & k_\omega - 2c\lambda_M(J)\end{bmatrix}\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix}, \tag{15.25}$$

and it follows that if we choose c small enough then the matrix is positive definite and thus V decreases along trajectories of the closed-loop system.

15.4. Direction-driven Attitude Control on SO(3)

We pick two orthogonal unit-length directions b_1 and b_2 and define the following error function:

$$\Psi_i(R) = \frac{1}{2}\|Rb_i - R_d b_i\|^2 = 1 - (Rb_i) \cdot (R_d b_i). \tag{15.26}$$

The derivative of $\Psi_i(R)$ becomes

$$\begin{aligned}
\dot{\Psi}_i(R) &= -\dot{R}b_i \cdot R_d b_i - Rb_i \cdot \dot{R}_d b_i \overset{(15.11a),(15.14)}{=} -R^R\hat{\boldsymbol{\omega}} b_i \cdot R_d b_i - Rb_i \cdot {}^e\hat{\boldsymbol{\omega}}_d R_d b_i \\
&= -{}^R\hat{\boldsymbol{\omega}} b_i \cdot R^T R_d b_i - b_i \cdot R^T\, {}^e\hat{\boldsymbol{\omega}}_d R_d b_i = -{}^R\hat{\boldsymbol{\omega}} b_i \cdot R^T R_d b_i - b_i \cdot \widehat{R^T\, {}^e\boldsymbol{\omega}_d} R^T R_d b_i \\
&= -{}^R\boldsymbol{\omega} \cdot \left(\widehat{b_i}\, R^T R_d b_i\right) - R^T\, {}^e\boldsymbol{\omega}_d \cdot \widehat{R^T R_d b_i} b_i = \underbrace{\left({}^R\boldsymbol{\omega} - R^T\, {}^e\boldsymbol{\omega}_d\right)}_{e_\omega} \cdot \underbrace{\widehat{R^T R_d b_i} b_i}_{e_{r_i}},
\end{aligned} \tag{15.27}$$

where we have defined two error functions

$$e_{r_i} = \widehat{R^T R_d b_i} b_i, \tag{15.28a}$$

$$e_\omega = {}^R\boldsymbol{\omega} - R^T\, {}^e\boldsymbol{\omega}_d, \tag{15.28b}$$

that are small when $R \approx R_d$ and when $\mathrm{Ad}_R\, {}^R\hat{\boldsymbol{\omega}} = R^R\hat{\boldsymbol{\omega}} \approx {}^e\boldsymbol{\omega}_d$, respectively.

15.5. Feedback Control

Given these error functions we consider the feedback control

$$u = -e_r - k_\omega e_\omega + \widehat{R^T\, {}^e\boldsymbol{\omega}_d} J R^T\, {}^e\boldsymbol{\omega}_d + J R^{T\,e}\dot{\boldsymbol{\omega}}_d, \tag{15.29}$$

where

$$e_r = k_1 e_{r_1} + k_2 e_{r_2}, \tag{15.30}$$

and k_1, k_2, k_ω are positive gains. Take the candidate Lyapunov function

$$V = \frac{1}{2} e_\omega \cdot J e_\omega + k_1 \Psi_1(R) + k_2 \Psi_2(R) + c J e_\omega \cdot e_r. \tag{15.31}$$

In the following we drop the upper left superscripts and write $\boldsymbol{\omega} = {}^R\boldsymbol{\omega}$ and $\boldsymbol{\omega}_d = {}^e\boldsymbol{\omega}_d$.

15.6. Lyapunov lower bound

We would like to show that $V = 0$ implies that $\|e_r\|$ and $\|e_\omega\|$ are zero. The main challenge lies in bounding the terms containing Ψ_i. Note that

$$\|e_{r_i}\| = \|\widehat{R^T R_d b_i} b_i\| = \|R^T R_d b_i \times b_i\| = \sin\theta_i, \tag{15.32}$$

where θ_i is the angle between $R^T R_d b_i$ and b_i. Note that θ_i is always in the range $[0, \pi]$. Similarly,

$$\Psi_i(R) = 1 - R^T R_d b_i \cdot b_i = 1 - \cos\theta_i. \tag{15.33}$$

Utilizing this and $(a+b)^2 \le 2(a^2+b^2)$ we get:

$$
\begin{aligned}
\|e_r\|^2 &\overset{(15.30)}{=} \|k_1 e_{r_1} + k_2 e_{r_2}\|^2 \le (k_1\|e_{r_1}\| + k_2\|e_{r_2}\|)^2 \overset{(15.32)}{=} (k_1 \sin\theta_1 + k_2 \sin\theta_2)^2 \\
&= \left(k_1\sqrt{1-\cos^2\theta_1} + k_2\sqrt{1-\cos^2\theta_2}\right)^2 \le \left(k_1\sqrt{2(1-\cos\theta_1)} + k_2\sqrt{2(1-\cos\theta_2)}\right)^2 \\
&\overset{(15.33)}{=} 2\left(k_1\sqrt{\Psi_1(R)} + k_2\sqrt{\Psi_2(R)}\right)^2 \le 4\min(k_1,k_2)\left(k_1\Psi_1(R) + k_2\Psi_2(R)\right).
\end{aligned}
$$

We therefore get

$$
V \ge \frac{1}{2}\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix}^T \begin{bmatrix}\frac{1}{2\min(k_1,k_2)} & -c\lambda_M(J) \\ -c\lambda_M(J) & \lambda_m(J)\end{bmatrix}\begin{bmatrix}\|e_r\| \\ \|e_\omega\|\end{bmatrix} \tag{15.34}
$$

where the matrix is positive definite for small enough c.

15.7. Lyapunov derivative

We start with an intermediate result

Proposition 15.2. *It holds that*

$$
J\dot{e}_\omega = -e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T\omega_d - \mathrm{trace}(J)I\right)R^T\omega_d\right) \times e_\omega. \tag{15.35}
$$

Proof.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Je_\omega &\overset{(15.28b)}{=} J\dot\omega - J\dot{R}^T\omega_d - JR^T\dot\omega_d \overset{(15.11)}{=} u - \hat\omega J\omega - J\left(R\hat\omega\right)^T\omega_d - JR^T\dot\omega_d \\
&\overset{(15.29)}{=} -e_r - k_\omega e_\omega + \widehat{R^T\omega_d}JR^T\omega_d - \hat\omega J\omega - J\hat\omega^T R^T\omega_d \\
&\overset{(15.28b)}{=} -e_r - k_\omega e_\omega + \widehat{R^T\omega_d}JR^T\omega_d - (\hat{e}_\omega + \widehat{R^T\omega_d})J(e_\omega + R^T\omega_d) \\
&\quad + J\left(\hat{e}_\omega + \cancelto{0}{\widehat{R^T\omega_d}}\right)R^T\omega_d \\
&= -e_r - k_\omega e_\omega + \left(\widehat{Je_\omega} + \widehat{JR^T\omega_d} - \widehat{R^T\omega_d}J - J\widehat{R^T\omega_d}\right)e_\omega \\
&= -e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T\omega_d - \mathrm{trace}(J)I\right)R^T\omega_d\right)^\wedge e_\omega.
\end{aligned}
$$

□

Thus the derivative of V is

$$
\dot{V} \overset{(15.27)}{=} e_\omega \cdot J\dot{e}_\omega + e_r \cdot e_\omega + cJ\dot{e}_\omega \cdot e_r + cJe_\omega \cdot \dot{e}_r \overset{(15.35)}{=} -k_\omega\|e_\omega\|^2 + cJ\dot{e}_\omega \cdot e_r + cJe_\omega \cdot \dot{e}_r, \tag{15.36}
$$

so we would like to bound $J\dot{e}_\omega \cdot e_r$ and $Je_\omega \cdot \dot{e}_r$ in terms of $\|e_\omega\|$ and $\|e_r\|$. First we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}R_d^T R = R_d^T R\hat\omega + R_d^T\hat\omega_d^T R = R_d^T R\hat\omega - R_d^T\hat\omega_d R = R_d^T R\hat\omega - R_d^T R\,\widehat{R^T\omega_d} = R_d^T R\hat{e}_\omega. \tag{15.37}
$$

Now, $e_{r_i} = \widehat{R^T R_d b_i} b_i$, so by linearity of the hat mapping and that $\|\hat{b}_i\| = \|b_i\| = 1$ it follows that

$$\dot{e}_{r_i} = R_d^T \widehat{R\hat{e}_\omega b_i}\, b_i = -R_d^T \widehat{R\hat{b}_i e_\omega}\, b_i, \qquad \Longrightarrow \|\dot{e}_{r_i}\| \le \|R_d^T R\|\|\hat{b}_i\|\|e_\omega\|\|b_i\| = \|e_\omega\|. \tag{15.38}$$

Thus, for $\lambda_M(J)$ the maximal eigenvalue of J,

$$\|Je_\omega \cdot \dot{e}_r\| \le \lambda_M(J)(k_1 + k_2)\|e_\omega\|^2. \tag{15.39}$$

Finally, we bound the last term, utilizing that $\|e_r\| \le k_1 + k_2$:

$$\begin{aligned} J\dot{e}_\omega \cdot e_r &\overset{(15.35)}{=} \left(-e_r - k_\omega e_\omega + \left(Je_\omega + \left(2JR^T \boldsymbol{\omega}_d - \operatorname{trace}(J)I\right) R^T \boldsymbol{\omega}_d\right) \times e_\omega\right) \cdot e_r \\ &\le -\|e_r\|^2 + k_\omega\|e_r\|\|e_\omega\| + \lambda_M(J)(k_1 + k_2)\|e_\omega\|^2 + B\|e_\omega\|\|e_r\|, \end{aligned} \tag{15.40}$$

where B is some number that upper bounds $\|\left(2JR^T\boldsymbol{\omega}_d - \operatorname{trace}(J)I\right) R^T \boldsymbol{\omega}_d\|$.

We can now bound the derivative as follows:

$$\dot{V} \le -\begin{bmatrix} \|e_r\| \\ \|e_\omega\| \end{bmatrix}^T \begin{bmatrix} c & -c(k_\omega + B)/2 \\ -c(k_\omega + B)/2 & k_\omega - 2c\lambda_M(J)(k_1 + k_2) \end{bmatrix} \begin{bmatrix} \|e_r\| \\ \|e_\omega\| \end{bmatrix}, \tag{15.41}$$

and it follows that if we choose c small enough then the matrix is positive definite and thus V decreases along trajectories of the closed-loop system.

Remaining steps:

- Show that undesired equilibria are unstable

15.8. Model-Predictive Control

Consider a system $\boldsymbol{X}(t)$ evolving on a Matrix Lie group

$$\mathrm{d}^r \boldsymbol{X}_t = f(\boldsymbol{X}, u), \quad \boldsymbol{X} \in \mathbb{M}, \qquad f : \mathbb{M} \times U \to T\mathbb{M}. \tag{15.42}$$

We are interested in finding an approximate solution to the optimal control problem

$$\begin{cases} \min & \int_0^T \left\|\sqrt{Q(\tau)}(\boldsymbol{X}(\tau) \ominus_r \boldsymbol{X}_d(\tau))\right\|_2^2 + \left\|\sqrt{R(\tau)}(u(\tau) - u_d(\tau))\right\| \mathrm{d}\tau + \left\|\sqrt{Q(T)}(\boldsymbol{X}(T) \ominus_r x_d(T))\right\|_2^2 \\ \text{s.t.} & (15.42) \\ & \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases} \tag{15.43}$$

for positive semi-definite matrices Q and R.

We start by considering the dynamics around a nominal trajectory $(\boldsymbol{X}_l(t), u_l(t))$. Consider the error $\boldsymbol{a}_e(t) = \boldsymbol{X}(t) \ominus_r \boldsymbol{X}_l(t)$ and relative control $u_e(t) = u(t) - u_l(t)$. Changing coordinates allows us to rewrite (15.43) as

$$\begin{cases} \min & \int_0^T \left\|\sqrt{Q(\tau)}\left((\boldsymbol{X}_l(\tau) \oplus_r \boldsymbol{a}_e(\tau)) \ominus_r \boldsymbol{X}_d(\tau)\right)\right\|_2^2 + \left\|\sqrt{R(\tau)}(u_l(\tau) + u_e(\tau) - u_d(\tau))\right\| \mathrm{d}\tau, \\ \text{s.t.} & \dot{\boldsymbol{a}}_e = A(t)\boldsymbol{a}_e + B(t)u_e + E(t), \\ & \boldsymbol{a}_e(0) = \boldsymbol{X}_0 \ominus \boldsymbol{X}_l(0). \end{cases} \tag{15.44}$$

This is a regular optimal control problem for a linear time variant (LTV) system where the linearized dynamics are as in (6.37):

$$A(t) := -\frac{1}{2}\operatorname{ad}_{f(\boldsymbol{x}_l(t),u_l(t))} - \frac{1}{2}\operatorname{ad}_{\mathrm{d}^\mathrm{r}(\boldsymbol{x}_l)_t} + \mathrm{d}^\mathrm{r}\left(f(\boldsymbol{x}, u_l(t))\right)_{\boldsymbol{x}=\boldsymbol{x}_l(t)}, \tag{15.45}$$

$$B(t) := \mathrm{d}^r\left(f\left(\boldsymbol{X}_l(t), u\right)\right)_{u=u_l(t)}, \tag{15.46}$$

$$E(t) := f\left(\boldsymbol{X}_l(t), u_l(t)\right) - \mathrm{d}^r(\boldsymbol{X}_l)_t. \tag{15.47}$$

To facilitate evaluating the cost function we note that

$$(\boldsymbol{X}_l \oplus_r \boldsymbol{a}_e) \ominus_r \boldsymbol{X}_d = \log\left(\boldsymbol{X}_d^{-1} \circ \boldsymbol{X}_l \circ \exp(\boldsymbol{a}_e)\right) = \log(\exp(\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d) \circ \exp(\boldsymbol{a}_e)) \approx \boldsymbol{X}_l \ominus_r \boldsymbol{X}_d + \boldsymbol{a}_e(t), \tag{15.48}$$

where the last approximate step follows from the Baker-Campbell-Hausdorff formula (5.61). We can thus write it approximately as

$$\left\|\sqrt{Q}\left((\boldsymbol{X}_l \oplus_r \boldsymbol{a}_e) \ominus_r \boldsymbol{X}_d\right)\right\|_2^2 \approx (\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d + \boldsymbol{a}_e)^T Q\,(\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d + \boldsymbol{a}_e) = \boldsymbol{a}_e^T Q \boldsymbol{a}_e + 2\,(\boldsymbol{X}_l \ominus_r \boldsymbol{X}_d)^T Q \boldsymbol{a}_e. \tag{15.49}$$

In conclusion, Lie group model-predictive control consists of solving

$$\begin{cases} \min_{u_e(\cdot)} & \int_0^T \boldsymbol{a}_e^T Q \boldsymbol{a}_e + 2(\boldsymbol{x}_l \ominus_r \boldsymbol{x}_d)^T Q \boldsymbol{a}_e + \left\|\sqrt{R(\tau)}(u_l(\tau) + u_e(\tau) - u_d(\tau))\right\| \mathrm{d}\tau, \\ \text{s.t.} & \dot{\boldsymbol{a}}_e = A(t)\boldsymbol{a}_e + B(t)u_e + E(t), \\ & \boldsymbol{a}_e(0) = \boldsymbol{X}_0 \ominus \boldsymbol{X}_l(0), \end{cases} \tag{15.50}$$

and applying $u(t) = u_l(t) + u_e(t)$. This is an optimal control problem with linear time-varying (LTV) dynamics and quadratic cost that after time discretization can be efficiently solved as a quadratic program.

Remark 15.1. *If the system under consideration is group-linear in $\boldsymbol{x}$ and linear in u, and the nominal trajectory $(\boldsymbol{x}_l, u_l)$ is dynamically feasible (i.e. $\mathrm{d}^\mathrm{r}(\boldsymbol{x}_l)_t = f(\boldsymbol{x}_l, u_l)$), then it follows from (6.38) that the tangent space system is in fact linear time-invariant (LTI) with dynamics $\dot{\boldsymbol{a}}_e = A\boldsymbol{a}_e + Bu$ for*

$$\begin{aligned} A &= -\operatorname{ad}_{f(e,0)} + \mathrm{d}^\mathrm{r} f(\boldsymbol{x}, 0)_{\boldsymbol{x}=e}, \\ B &= \mathrm{d}^\mathrm{r} f(e, u)_{u=0}. \end{aligned} \tag{15.51}$$

16. State Estimation

Summary

- Complementary filtering for attitude estimation.
- Extended Kalman filter on Lie groups, and leveraging group linearity.
- Equivariant filtering.
- IMU pre-integration.

16.1. Complementary Filter for Attitude Estimation

Filter in [19] is of form

$$\dot{\hat{R}} = (\hat{R}\Omega + k_P \hat{R}\omega)\hat{R}, \quad \omega = vex\left(\frac{1}{2}\left(\tilde{R} - \tilde{R}^T\right)\right), \quad \tilde{R} = \hat{R}R_y.$$

That is, the natural dynamics are amended with a term $k_P \hat{R}\omega$ that induces stability of the observer. The quantitiy ω is a rotation quantity in body coordinates that corresponds to the anti-symmetric part of the empirical rotation error. It can be shown that for $R_y = R$ we have

$$\frac{d}{dt}\frac{1}{2}\mathrm{Tr}\left(I - \hat{R}^T R\right) = -\frac{k_P}{2}|\omega|^2.$$

Remark: Filter above is the *passive* form, if $R_y\Omega$ is used instead of $\hat{R}\Omega$ it is called *direct*. This can be amended with an integrator that estimates a bias term in the gyro estimates.

16.2. Extended Kalman Filter

Consider a system

$$\mathrm{d}^r \boldsymbol{x}_t = f\left(\boldsymbol{x}(t)\right), \quad w_t \in \mathcal{N}(0, t), \tag{16.1}$$

and assume that unbiased measurements $\boldsymbol{y} \sim \mathcal{N}_r(h(\boldsymbol{x}), H)$ are available. The objective is to estimate the true value of the system.

To this end we maintain an estimate $\hat{\boldsymbol{x}}$ and study the estimation error between $\hat{\boldsymbol{x}}$ and the true state $\boldsymbol{x}$ of the system:

$$\boldsymbol{a} \coloneqq \boldsymbol{x} \ominus_r \hat{\boldsymbol{x}}. \tag{16.2}$$

From Section 6.2.1 we can linearize the dynamcis around $\hat{\boldsymbol{x}}$ so that the linearized dynamics of $\boldsymbol{a}$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a} = \left(-\operatorname{ad}_{f(\hat{x}(t))} + \mathrm{d}^{\mathrm{r}} f_{\hat{x}(t)}\right)\boldsymbol{a}. \tag{16.3}$$

Add process noise in tangent space!

Take a small τ. From the Baker-Campbell-Hausdorff formula (5.61) and expanding $f(\boldsymbol{x}_0 \exp(\sqrt{P}v))$ around $v = 0$ it approximately holds that

$$\begin{aligned}\boldsymbol{x}(\tau) &= \boldsymbol{x}_0 \exp\left(\sqrt{P}v\right)\exp\left(\tau\left(f\left(\boldsymbol{x}_0 \exp\sqrt{P}v\right) + \sqrt{Q}w\right)\right)\\ &\approx \boldsymbol{x}_0 \exp\left(\sqrt{P}v\right)\exp\left(\tau\left[f(\boldsymbol{x}_0) + \mathrm{d}^r f_{x_0}\sqrt{P}v + \sqrt{Q}w\right]\right).\end{aligned} \tag{16.4}$$

The expectation and variance of $\boldsymbol{x}(\tau)$ are

Matrix from linearization, if system group-linear enough to evaluate once at identity

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{x}(\tau)\right] &= \boldsymbol{x}_0 \exp(\tau f(\boldsymbol{x}_0)),\\ \mathbb{V}\mathrm{ar}\left[\boldsymbol{x}(\tau)\right] &= \mathbb{E}\left[\left(\boldsymbol{x}(\tau)\ominus_r \mathbb{E}\left(x(\tau)\right)\right)\left(\boldsymbol{x}(\tau)\ominus_r \mathbb{E}\left(x(\tau)\right)\right)^T\right] = P + \tau\left(Q + \mathrm{d}^r f_{x_0}P + P\left(\mathrm{d}^r f_{x_0}\right)^T\right).\end{aligned} \tag{16.5}$$

This is not entirely true due to v appearing twice and resulting non-zero brackets. Do we need additional Lie terms?

Thus the forward propagation of $\hat{\boldsymbol{x}}\exp\left(\sqrt{P}v\right)$ is goverened by

$$\begin{aligned}\mathrm{d}^r\hat{\boldsymbol{x}}_t &= f(\hat{\boldsymbol{x}}(t)),\\ \frac{\mathrm{d}}{\mathrm{d}t}P(t) &= Q(t) + \mathrm{d}^r f_{\hat{x}(t)}P(t) + P(t)\left(\mathrm{d}^r f_{\hat{x}(t)}\right)^T.\end{aligned} \tag{16.6}$$

Lie group extended Kalman filter

Initialization. Intitialize $(\hat{\boldsymbol{x}}_{0|0}, P_{0|0})$ such that

$$\boldsymbol{x} \sim \mathcal{N}_r(\hat{\boldsymbol{x}}_{0|0}, P_{0|0}) \tag{16.7}$$

models the prior belief of $\boldsymbol{x}$.
Predict. Propagate the belief to $(\hat{\boldsymbol{x}}_{k|k-1} P_{k|k-1})$ by integrating over the relevant interval $[t_{k-1}, t_k]$.

$$\begin{aligned} \mathrm{d}^{\mathrm{r}} \hat{\boldsymbol{x}}_t &= f(\hat{\boldsymbol{x}}(t)), \\ \frac{\mathrm{d}}{\mathrm{d}t} P(t) &= \left(-\operatorname{ad}_{f(\hat{\boldsymbol{x}}(t))} + \mathrm{d}^{\mathrm{r}} f_{\hat{\boldsymbol{x}}(t)}\right) P(t) + P(t) \left(-\operatorname{ad}_{f(\hat{\boldsymbol{x}}(t))} + \mathrm{d}^{\mathrm{r}} f_{\hat{\boldsymbol{x}}(t)}\right)^T + Q(t), \\ \hat{\boldsymbol{x}}(t_{k-1}) &= \hat{\boldsymbol{x}}_{k-1|k-1}, \\ P(t_{k-1}) &= \hat{P}_{k-1|k-1}. \end{aligned} \tag{16.8}$$

Update. Given a measurement $y \sim \mathcal{N}(h(\boldsymbol{x}), R)$ update the belief as

$$\begin{aligned} \hat{\boldsymbol{x}}_{k|k} &= \hat{\boldsymbol{x}}_{k|k-1} \oplus_r K(y - h(\hat{\boldsymbol{x}})), \\ P_{k|k} &= (I - KH) P_{k|k-1} (I - KH)^T + KRK^T, \end{aligned} \tag{16.9}$$

where $H = \mathrm{d}^{\mathrm{r}} h_{\hat{\boldsymbol{x}}}$, and the Kalman gain is

$$K = P_{k|k-1} H^T (H P_{k|k-1} H^T + R)^{-1}. \tag{16.10}$$

16.2.1. EKF and group linearity

In the EKF above linearization was used both for the dynamics and for the measurement function. However, as pointed out in Section 6.2.1 the "linearization" is in fact exact if the system is group-linear. Below we point out how the predict step of Kalman filter simplifies for group-linear systems, and derive a similar property for group-linear measurements.

As it turns out, if both the dynamics and the measurements are group-linear there are in fact no linearization errors in the Lie group EKF, just like how an EKF on $\mathbb{R}^n$ specializes to a regular Kalman filter in the case of linear dynamics and linear measurements.

Group-linear dynamics If the system is known to be group-linear Section 6.2 implies that the linearized system is in fact time-invariant, so the covariance dynamics simplify to

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = \left(-\operatorname{ad}_{f(e)} + \mathrm{d}^{\mathrm{r}} f_e\right) P(t) + P(t) \left(-\operatorname{ad}_{f(e)} + \mathrm{d}^{\mathrm{r}} f_e\right)^T + Q(t). \tag{16.11}$$

Note that there is no difference in behavior between the EKF (16.8) and the linearized counterpart (16.11) if the system is group-linear. The only gain from using (16.11) is avoiding the cost of continuously evaluating the linearized system matrix (which anyway is constant for a group-linear system). Nevertheless, leveraging this simplification is known in the literature as the "Invariant extended Kalman filter" [5].

Group-linear measurements If $h(\boldsymbol{x})$ is group-linear then by Section 4.8 $y = h(\boldsymbol{x}) + r = h(\hat{\boldsymbol{x}}) + H(\boldsymbol{x} \ominus_r \hat{\boldsymbol{x}}) + r$ where $r \sim \mathcal{N}(0, R)$ and $H = \mathrm{d}^{\mathrm{r}}\, h_e$. This means that the measurement error is a function only of the estimate error:

$$y - h(\hat{\boldsymbol{x}}) = h(\boldsymbol{x}) - h(\hat{\boldsymbol{x}}) + r = H(\boldsymbol{x} \ominus_r \hat{\boldsymbol{x}}) + r. \tag{16.12}$$

It follows that $y - h(\hat{\boldsymbol{x}}) \sim \mathcal{N}\left(0, HPH^T + R\right)$ *exactly*, i.e. there is no error from linearization of the mesurement function.

Again, implementing (16.8) is sufficient to realize this benefit, but a modest computational gain can be had by pre-computing $H = \mathrm{d}^{\mathrm{r}}\, h_e$ for use in the update step.

16.3. Equivariant filtering

- Equivariant Filter [12, 13, 20]
  1. Assumes homogeneous space: $G \times M \to M$ must be transitive. Only makes sense as a symmetry when G is "lower-dimensional" compared to M.
  2. Dynamics must be equivariant w.r.t. action. This is always possible by extending input space
  3. Filter state is in G, dynamics on G obtained via a "lift".
  4. **Is it the case that for all symmetries we are better off by writing down a different dynamical system?**

16.4. IMU Pre-Integration

An imu typically consists of a gyro measuring angular velocity, an accelerometer, and a magnetometer that estimates the orientation with respect to the earth magnetic field. Consider a body moving in the world described by the IMU frame to world frame transform $P_{WI} = (\boldsymbol{q}_{WI}, \boldsymbol{p}_{WI}) \in \mathbb{SE}(3)$.

The gyro and accelerometer measurements of the IMU are then well modeled by the following:

$$\tilde{\boldsymbol{\omega}} = \mathrm{d}^{\mathrm{r}}(\boldsymbol{q}_{WI})_t + \boldsymbol{b}_{\boldsymbol{\omega}} + \eta_{\boldsymbol{\omega}}, \tag{16.13a}$$

$$\tilde{\boldsymbol{a}} = \left[\mathrm{d}^{2r} P_{WI}\right]_{3:6} + \boldsymbol{q}_{WI}^{-1} \boldsymbol{g}_W + \boldsymbol{b}_{\boldsymbol{a}} + \eta_{\boldsymbol{a}}. \tag{16.13b}$$

In (16.13a), the first term is the actual body angular velocity, $\boldsymbol{b}_{\boldsymbol{\omega}}$ is a gyro bias, and $\eta_{\boldsymbol{\omega}}$ is white noise.

For the accelerometer model (16.13b), $\left[\mathrm{d}^{2r}(P_{WI})_t\right]_{3:6}$ denotes the linear acceleration in the body frame (the last three components of the second derivative), $\boldsymbol{g}_W$ is the gravity in the world frame, and $\boldsymbol{b}_{\boldsymbol{a}}$ and $\eta_{\boldsymbol{a}}$ are bias and noise as above.

17. Nonlinear Least Squares

Like how Lie groups thread the line between linear and nonlinear manifolds, the same can be said for the role of nonlinear least squares in optimization, which is a type of optimization problem is rich enough for to model a wide variety of situations, yet structured enough to be amenable to practical algorithms.

A non-linear least squares problem has the general form

$$\min_{\boldsymbol{x} \in \mathbb{M}} \frac{1}{2} \sum_{i=1}^{N} \left\| r_i(\boldsymbol{x}) \right\|^2, \quad r_i : \mathscr{M} \to \mathbb{R}^{n_{r_i}}. \tag{17.1}$$

The manifold $\mathbb{M}$ can be a Lie group or a Lie group product $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) \in \mathbb{M}_1 \times \dots \times \mathbb{M}_k$. For the latter case, typically not every residual depends on each member of the bundle, i.e. $r_i(\boldsymbol{x}) = r_i\left(\{\boldsymbol{x}_j\}_{j \in I_i}\right)$ where $I_i \subset \{1, \dots, k\}$ is a subset of variables.

Remark 17.1. *An equivalent problem with a single residual is* $\min_{\boldsymbol{x} \in \mathbb{M}} \frac{1}{2} \|r(\boldsymbol{x})\|^2$ *for*

$$r(\boldsymbol{x}) = \begin{bmatrix} r_1(\boldsymbol{x}) \\ \vdots \\ r_k(\boldsymbol{x}) \end{bmatrix}. \tag{17.2}$$

Although the single residual formulation simplifies notation somewhat, in practice it is for large problems important to leverage the sparsity structure which is better exposed in (17.1).

17.1. Solution Sensitivity

In many applications the residuals $r_i(\boldsymbol{x})$ are obtained from data and are therefore associated with uncertainty. In this situation it is natural to ask how sensitive the optimal solution of the nonlinear least squares problem is to noise in the data. Assume that the noise associated with each residual is Gaussian and independent of other residuals, i.e. that

$$r_i(\boldsymbol{x}) \sim \mathcal{N}\left(\bar{r}_i(\boldsymbol{x}), I\right), \tag{17.3}$$

and consider a point $\bar{\boldsymbol{x}}$. We expand the objective using a Taylor approximation as

$$\min_{\bar{x} \in \mathbb{M}} \frac{1}{2} \sum_{i=1}^{N} \left\| r_i(\bar{\boldsymbol{x}}) \right\|^2 \approx \min_{\boldsymbol{a} \in T_{\bar{x}} \mathbb{M}} \frac{1}{2} \sum_{i=1}^{N} \left\| r_i(\bar{\boldsymbol{x}}) + \mathrm{d}^r (r_i)_{\bar{x}} \boldsymbol{a} \right\|^2. \tag{17.4}$$

The optimal solution $\boldsymbol{x}^*$ of the left problem can be approximately retrieved from $\boldsymbol{a}^*$ as $\boldsymbol{x}^* = \bar{\boldsymbol{x}} \oplus_r \boldsymbol{a}^*$ assuming that $\boldsymbol{a}^*$ is small.

Letting $r_i := r_i(\bar{x})$ and $J_i := \mathrm{d}^r(r_i)_{\bar{x}}$ expanding the square and ignoring the constant term yields

$$\min_{\boldsymbol{a}\in T_{\bar{x}}\mathbb{M}} \sum_{i=1}^{N} \frac{1}{2}\boldsymbol{a}^T J_i^T J_i \boldsymbol{a} + \boldsymbol{a}^T J_i^T r_i = \min_{\boldsymbol{a}\in T_{\bar{x}}\mathbb{M}} \frac{1}{2}\boldsymbol{a}^T \left(\sum_{i=1}^{N} J_i^T J_i\right)\boldsymbol{a} + \boldsymbol{a}^T \sum_{i=1}^{N} J_i^T r_i. \tag{17.5}$$

The optimal solution of this problem can be obtained by setting the gradient w.r.t. $\boldsymbol{a}$ to zero and is

$$\boldsymbol{a}^* = -\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\left(\sum_{i=1}^{N} J_i^T r_i\right). \tag{17.6}$$

From this we can infer the sensitivity of $\boldsymbol{a}^*$ to noise in r_i: recalling that $\mathrm{Var}(Ax+By) = A\mathrm{Var}(x)A^T + B\mathrm{Var}(y)B^T$ we get that

$$\boldsymbol{a}^* \sim \mathcal{N}\left(-\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\left(\sum_{i=1}^{N} J_i^T \bar{r}_i\right), \left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\left(\sum_{i=1}^{N} J_i^T \Sigma_i J_i\right)\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\right). \tag{17.7}$$

For the special case when all r_i's have unit covariance, i.e. $\Sigma_i = I$, the expression simplifies to

$$\boldsymbol{a}^* \sim \mathcal{N}\left(-\left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\left(\sum_{i=1}^{N} J_i^T \bar{r}_i\right), \left(\sum_{i=1}^{N} J_i^T J_i\right)^{\dagger}\right). \tag{17.8}$$

Given a residual $r(x) \sim \mathcal{N}(\bar{r}(x), \Sigma)$ a residual with unit covariance can be obtained by left-multiplying with the square root information matrix $\sqrt{I} := \Sigma^{-1/2}$:

$$\sqrt{I}r(x) \sim \mathcal{N}\left(\sqrt{I}\bar{r}(x), I\right). \tag{17.9}$$

Scaling with $\sqrt{I}$ makes sense in many applications since it in effect scales the residual by the inverse noise magnitude.

For the unit covariance case $\Sigma_i = I$ the tangent space covariance of the optimal solution x^* is

$$\left(\sum_{i=1}^{N} \left(\mathrm{d}^r(r_i)_{x^*}\right)^T \mathrm{d}^r(r_i)_{x^*}\right)^{\dagger}. \tag{17.10}$$

17.2. Levenberg-Marquardt

Resources

- Original MINPACK manual: https://www.netlib.org/minpack/

LM Implementations

- Original MINPACK in fortran
- cminpack (ported from fortran) https://devernay.github.io/cminpack/
- Eigen unsupported (ported from cminpack)

Consider a Lie group optimization problem

$$\min_{\boldsymbol{x}} \frac{1}{2} \|f(\boldsymbol{x})\|^2, \quad f : \mathbb{M} \to \mathbb{R}^m. \tag{17.11}$$

We are interested in devising an iterative algorithm for minimizing this function.

Given a point $\boldsymbol{x}$ we can solve a local optimization problem to find step $\boldsymbol{a} \in T\mathbb{M}_{\boldsymbol{x}}$ that leads to an improved estimate $\boldsymbol{x} \oplus_r \boldsymbol{a}$. The optimization problem can be re-formulated in terms of $\boldsymbol{a}$ as

$$\arg\min_{\boldsymbol{a}} \|f(\boldsymbol{x} \oplus_r \boldsymbol{a})\|^2. \tag{17.12}$$

Since the problem is nonlinear we resort to linearization. To avoid stepping outside the region where the linearization is accurate we also limit the stepsize and obtain the new problem

The diagonal scaling matrix $D = \mathrm{Diag}(d_1, \ldots, d_n)$ is typically chosen so that a component d_i is inversely proportional to the magnutide of the gradient in that direction, which has the effect of allowing larger steps in directions with low gradient. Common choices include $D = \sqrt{\mathrm{Diag}\left(\mathrm{diag}(J^T J)\right)}$ and $d_i = \left\|[\mathrm{d}^r f_{\boldsymbol{x}}]_{\cdot,i}\right\|$—the norm of the i:th column of the jacobian.

A complete Levenberg-Marquardt procedure is shown in Algorithm 1. The crucial step occurs on line 3 and is discussed further below.

Calculation of the actual to predicted reduction ratio can be rewritten as

$$\rho = \frac{1 - \left(\frac{\|f(\boldsymbol{x} \oplus \boldsymbol{a}_{\mathrm{LM}})\|}{\|r\|}\right)^2}{\left(\frac{\|J\boldsymbol{a}\|}{\|r\|}\right)^2 + 2\left(\frac{\sqrt{\lambda}\|D\boldsymbol{a}\|}{\|r\|}\right)^2} \tag{17.13}$$

where we have used that $\|r\|^2 - \|r + J\boldsymbol{a}\|^2 = -2\boldsymbol{a}^T J r - \boldsymbol{a}^T J^T J \boldsymbol{a} = \|J\boldsymbol{a}\|^2 + 2\lambda\|D\boldsymbol{a}\|^2$ which is a consequence of (17.16a). This formulation has the benefit of avoiding subtraction of numbers of large magnitude which may cause floating point roundoff errors.

17.2.1. Trust-Region Problem

We discuss how to solve a trust-region problem on the form

$$\arg\min_{\boldsymbol{a} : \|D\boldsymbol{a}\| \leq \Delta} \|J\boldsymbol{a} + r\|^2, \tag{17.14}$$

where D is a diagonal scaling matrix and Δ is a maximal step size. This constrained problem can be transformed into an unconstrained problem.

Algorithm 1: One iteration of the Levenberg-Marquardt algorithm.

Data: Iteration variables: point $\boldsymbol{x}^k$, trust region Δ^k, scaling parameters d_i^k as diagonal matrix D^k

Result: Updated iteration variables $\boldsymbol{x}^{k+1}$, Δ^{k+1}, d_i^{k+1}

$r = f(\boldsymbol{x})$

$J = \mathrm{d}^r f_{\boldsymbol{x}}$

$\boldsymbol{a}_{\mathrm{LM}} = \arg\min_{\boldsymbol{a}\,:\,\|D^k\boldsymbol{a}\|\leq\Delta^k} \|r + J\boldsymbol{a}\|^2$ `// calculate increment step`

$\rho = \frac{\|r\|^2 - \|f(\boldsymbol{x}\oplus_r\boldsymbol{a}_{\mathrm{LM}})\|^2}{\|r\|^2 - \|r + J\boldsymbol{a}_{\mathrm{LM}}\|^2}$ `// actual to predicted reduction ratio`

if $\rho \leq 0.25$ **then**

  $\Delta^{k+1} = \Delta^k/2$ `// decrease trust region`

else if $\rho \geq 0.75$ **then**

  $\Delta^{k+1} = 2\Delta^k$ `// increase trust region`

end

if $\rho \leq 0.0001$ **then**

  $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k$ `// reject step`

else

  $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k \oplus_r \boldsymbol{a}_{\mathrm{LM}}$ `// accept step`

  $d_i^{k+1} = \max\left(d_i^k, \left\|\left[\mathrm{d}^r f_{\boldsymbol{x}^{k+1}}\right]_{\cdot,i}\right\|\right)$ `// update scaling parameters`

end

Theorem 17.1. *A vector $\boldsymbol{a}^*$ is a global minimizer of*

$$\underset{\|D\boldsymbol{a}\|\leq\Delta}{\arg\min}\ \frac{1}{2}\|J\boldsymbol{a}+r\|^2. \tag{17.15}$$

if and only if there exists $\lambda \geq 0$ such that

$$(J^TJ+\lambda D^TD)\boldsymbol{a}^* = -J^Tr, \tag{17.16a}$$
$$\lambda\left(\|D\boldsymbol{a}^*\|-\Delta\right) = 0. \tag{17.16b}$$

We provide an argument based on duality to support this fact, see e.g. [23, Theorem 4.1] for a more rigorous proof.

Proof. Let the lagrangian of the problem be

$$L(\boldsymbol{a},\lambda) = \frac{1}{2}\|J\boldsymbol{a}+r\|^2 + \frac{\lambda}{2}\left(\|D\boldsymbol{a}\|^2-\Delta^2\right), \tag{17.17}$$

so that the optimization problem (17.14) equivalently can be written $\inf_{\boldsymbol{a}}\sup_{\lambda\geq 0} L(\boldsymbol{a},\lambda)$, since the value of the inner problem is $+\infty$ when the constraint $\|D\boldsymbol{a}\|\leq\Delta$ is not satisfied.

Assuming that strong duality holds, the dual problem $\sup_{\lambda\geq 0}\inf_{\boldsymbol{a}} L(\boldsymbol{a},\lambda)$ has the same optimal value. The inner infimum of the dual problem can be re-written as

$$\inf_{\boldsymbol{a}} L(\boldsymbol{a},\lambda) = \frac{1}{2}\boldsymbol{a}^T(J^TJ+\lambda D^TD)\boldsymbol{a} + r^TJ\boldsymbol{a} - \lambda\frac{\Delta^2}{2}, \tag{17.18}$$

where the square term is a (semi-)positive definite matrix. This inner has a finite optimal value attained for $\boldsymbol{a}^*$ such that $\left(J^TJ+\lambda D^TD\right)\boldsymbol{a}^* = -J^Tr$. The derivative this expression w.r.t. λ yields

$$0 = D^TD\boldsymbol{a}^* + \left(J^TJ+\lambda D^TD\right)\frac{\mathrm{d}\boldsymbol{a}^*}{\mathrm{d}\lambda}, \tag{17.19}$$

which gives

$$-(\boldsymbol{a}^*)^TD^TD\boldsymbol{a}^* = (\boldsymbol{a}^*)^T(J^TJ+\lambda D^TD)\frac{\mathrm{d}\boldsymbol{a}^*}{\mathrm{d}\lambda} = -r^TJ\frac{\mathrm{d}\boldsymbol{a}^*}{\mathrm{d}\lambda}. \tag{17.20}$$

Substituting the expression for $\boldsymbol{a}^*$ reduces the dual problem to

$$\sup_{\lambda\geq 0}\frac{1}{2}\left(r^TJ\boldsymbol{a}^*-\lambda\Delta^2\right) \tag{17.21}$$

This problem has a closed-form solution: either the optimal solution is attained at the boundary, i.e. $\lambda = 0$, or it is attained at zero derivative w.r.t. λ which necessitates

$$0 = r^TJ\frac{\mathrm{d}\boldsymbol{a}^*}{\mathrm{d}\lambda} - \Delta^2 \overset{(17.20)}{=} (\boldsymbol{a}^*)^TD^TD\boldsymbol{a}^* - \Delta^2 = \|D\boldsymbol{a}^*\| - \Delta^2. \tag{17.22}$$

These two latter conditions imply that the complementarity condition $\lambda(\|D\boldsymbol{a}^*\|-\Delta) = 0$ always holds. □

Equation (17.16a) represents the normal equations for the least-squares problem

$$\arg\min_{\boldsymbol{a}} \frac{1}{2}\boldsymbol{a}^T\left(J^TJ + \lambda D^TD\right)\boldsymbol{a} + r^TJ\boldsymbol{a}. \tag{17.23}$$

Equivalently, it can be written on the standard form

$$\arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} J \\ \sqrt{\lambda}D \end{bmatrix} \boldsymbol{a} + \begin{bmatrix} r \\ 0 \end{bmatrix} \right\|^2. \tag{17.24}$$

Theorem 17.1 suggests that the trust-region problem can be recast as a least squares problem, but doing so requires knowledge of the parameter λ. In the following we show how the least squares problem can be efficiently solved assuming knowledge of λ, and then discuss an algorithm for finding λ.

17.2.2. Solving the Least-Squares Problem

The least-squares problem

$$\arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} J \\ \sqrt{\lambda}D^k \end{bmatrix} \boldsymbol{a} + \begin{bmatrix} r \\ 0 \end{bmatrix} \right\|^2 \tag{17.25}$$

can be solved in multiple ways. It can be solved directly by factorizing $\begin{bmatrix} J \\ \sqrt{\lambda}D^k \end{bmatrix}$ and finding the least-squares solution, or the normal equations

$$(J^TJ + \lambda D^TD)\boldsymbol{a} = -J^Tr \tag{17.26}$$

can be solved, which have the advantage of having a positive (semi-)definite symmetric system matrix which enables e.g. Cholesky decomposition.

Next a third method that is competetive for dense matrices of moderate size is discussed. It further exploits problem structure and only requires a QR decomposition of J. Consider a QR decomposition with column pivoting of J s.t. $JP = QR$, where $P \in \mathbb{R}^{n\times n}$ is a permutation matrix, $Q \in \mathbb{R}^{n\times n}$ is orthogonal, and $R \in \mathbb{R}^{n\times n}$ is upper-diagonal. First note that if $\boldsymbol{a}$ is a minimizer of (17.25) then it is also a minimizer of

$$\arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} Q^TJP \\ \sqrt{\lambda}P^TD^kP \end{bmatrix} P^T\boldsymbol{a} + \begin{bmatrix} Q^Tr \\ 0 \end{bmatrix} \right\|^2 = \arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} R \\ \sqrt{\lambda}P^TD^kP \end{bmatrix} P^T\boldsymbol{a} + \begin{bmatrix} Q^Tr \\ 0 \end{bmatrix} \right\|^2. \tag{17.27}$$

Consider a second QR decomposition s.t.

$$\begin{bmatrix} R \\ \sqrt{\lambda}P^TD^kP \end{bmatrix} = \tilde{Q}\begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} \tag{17.28}$$

where $\tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ is orthogonal and $\tilde{R} \in \mathbb{R}^{n\times n}$ is upper-diagonal and has rank $\boldsymbol{n}$. Since there are only $\boldsymbol{n}$ non-zero variables in the lower triangular part this step can be efficiently computed via $n(n+1)/2$ Givens rotations. In these variables the least-squares problem takes the form

$$\arg\min_{\boldsymbol{a}} \left\| \tilde{Q} \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} P^T \boldsymbol{a} + \begin{bmatrix} Q^T r \\ 0 \end{bmatrix} \right\|^2 = \arg\min_{\boldsymbol{a}} \left\| \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} P^T \boldsymbol{a} + \tilde{Q}^T \begin{bmatrix} Q^T r \\ 0 \end{bmatrix} \right\|^2 = \arg\min_{\boldsymbol{a}} \left\| \tilde{R} P^T \boldsymbol{a} + \tilde{Q}_{11}^T Q^T r \right\|^2,$$

and it is now apparent that the optimal solution is

$$\boldsymbol{a}_{\mathrm{LM}} = -P\tilde{R}^{-1}\tilde{Q}_{11}^T Q^T r. \tag{17.29}$$

When solving (17.25) repeatedly for different values of λ only the second QR decomposition needs to be re-computed.

17.2.3. Finding the LM Parameter

To search for a parameter that satisfies the relations in Theorem 17.1 consider the function

$$\phi(\alpha) = \left\| D \left(J^T J + \alpha D^T D \right)^{-1} J^T r \right\| - \Delta. \tag{17.30}$$

Note that $\phi(\alpha)$ can be evaluated by solving the a structured least-squares problem as discussed in Section 17.2.2. With those variables we have

$$\phi(\alpha) = \left\| DP\tilde{R}^{-1}\tilde{Q}_{11}^T Q^T r \right\| - \Delta \tag{17.31}$$

where $\tilde{R}$ and $\tilde{Q}$ depend on α due to (17.28).

We are interested in finding a value of $\alpha > 0$ s.t. $\phi(\alpha) \approx 0$. If $\phi(0) \leq 0$ it must hold that $\lambda = 0$ in Theorem 17.1, so we disregard this case. We follow [22] to construct an algorithm for the case $\phi(0) > 0$.

The function ϕ is strictly decreasing in α, and approximately of the form $\phi(\alpha) \approx \tilde{\phi}(\alpha) = \frac{a}{b+\alpha} - \Delta$. Setting $\tilde{\phi}(\alpha)$ to zero then gives $\alpha = -b + a/\Delta$, and fitting a and b s.t. $\phi(\alpha_k) = \tilde{\phi}(\alpha_k)$ and $\phi'(\alpha_k) = \tilde{\phi}'(\alpha_k)$ gives the update rule

$$\alpha_{k+1} = \alpha_k - \frac{\phi(\alpha_k) + \Delta}{\Delta} \frac{\phi(\alpha_k)}{\phi'(\alpha_k)}. \tag{17.32}$$

To implement this algorithm not only ϕ needs to be calculated, but also its derivative.

Derivative of ϕ Introduce $q(\alpha) = D\left(J^T J + \alpha D^T D\right)^{-1} J^T r$ so that $\phi(\alpha) = \sqrt{q(\alpha)^T q(\alpha)} - \Delta$. The derivative of q becomes

$$q'(\alpha) = -D\left(J^T J + \alpha D^T D\right)^{-1} D^T D \left(J^T J + \alpha D^T D\right)^{-1} J^T r = -D\left(J^T J + \alpha D^T D\right) D^T q(\alpha). \tag{17.33}$$

where we have utilized that the derivative of $(A+\alpha B)^{-1}(A+\alpha B)$ is zero which gives

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(A+\alpha B)^{-1} = -(A+\alpha B)^{-1} B (A+\alpha B)^{-1}. \tag{17.34}$$

Therefore the derivative of ϕ becomes

$$\phi'(\alpha) = \frac{q(\alpha)^T q'(\alpha)}{\|q(\alpha)\|} = -\frac{\left(D^T q(\alpha)\right)^T \left(J^T J + \alpha D^T D\right)^{-1} \left(D^T q(\alpha)\right)}{\|q(\alpha)\|}. \tag{17.35}$$

The left-hand side of the normal equations for (17.25) reappear here which is good news—after solving a linear system $Ax = b$ for a value b the factorization of A can generally be re-used to solve a different system $Ax = c$.

To exemplify, consider utilizing the method in Section 17.2.2 involving the QR decompositions $PJ = QR$ of J. Then (17.28) can be leveraged to evaluate the expression efficiently.

$$\begin{aligned} J^T J + \alpha D^T D &= (QRP^T)^T (QRP^T) + \alpha D^T D = PR^T RP^T + \alpha D^T D \\ &= P \begin{bmatrix} R^T & \sqrt{\alpha}(P^T DP)^T \end{bmatrix} \begin{bmatrix} R \\ \sqrt{\alpha} P^T DP \end{bmatrix} P^T \overset{(17.28)}{=} P\tilde{R}^T \tilde{R} P^T. \end{aligned} \tag{17.36}$$

Therefore

$$\phi'(\alpha) = -\frac{\left\|\tilde{R}^{-T} P^T D^T q(\alpha)\right\|^2}{\|q(\alpha)\|} = -\|q(\alpha)\| \left\|\tilde{R}^{-T} \frac{P^T D^T q(\alpha)}{\|q(\alpha)\|}\right\|^2 \tag{17.37}$$

We conclude by writing down the algorithm from [22] for finding λ. To reduce the number of iterations the termination constraints are somewhat relaxed: $\lambda = 0$ is considered an acceptable solution if $\phi(0) < 0.1\Delta$ rather than $\phi(0) < 0$, etc. Not obtaining an exact solution is fine since an exact solution of (17.14) is not necessary to move effectively towards the function minimum.

Remark 17.2. *Ceres just uses $\lambda = 1/\Delta$.*

Sparse strategy: as ceres use $\lambda = 1/\Delta$ and use a sparse eigen solver for the step.

Algorithm 2: LM parameter algorithm

Data: Matrices J, D, vector r, scalar Δ

Result: λ s.t. $\lambda = 0, \phi(0) < 0.1\Delta$, or $\lambda > 0, |\phi(\lambda) - \Delta| \leq 0.1\Delta$

Calculate QR decomposition $JP = QR$

If $\phi(0) < 0.1\Delta$, return 0

$l_0 = \begin{cases} -\phi(0)/\phi'(0) & \text{if } J \text{ nonsingular} \\ 0 & \text{otherwise} \end{cases}$

$u_0 = \frac{\|(JD^{-1})^T r\|}{\Delta}$

$\alpha_0 = \sqrt{l_0 u_0}$

repeat

  If $\alpha_k \notin [l_k, u_k]$, set $\alpha_k = \max\left(0.001 u_k, \sqrt{l_k u_k}\right)$

  Calculate QR decomposition $\begin{bmatrix} R \\ \sqrt{\lambda} P^T D^k P \end{bmatrix} = \tilde{Q} \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}$

  $z = -P\tilde{R}^{-1}\tilde{Q}_{11}^T Q^T r$

  $\phi = \|Dz\| - \Delta$

  $\phi' = -\|Dz\| \left\| \tilde{R}^{-T} \frac{P^T D^T Dz}{\|Dz\|} \right\|^2$

  $l_{k+1} = \max\left(l_k, \alpha_k - \frac{\phi}{\phi'}\right)$

  $u_{k+1} = \begin{cases} \alpha_k & \text{if } \phi < 0 \\ u_k & \text{otherwise} \end{cases}$

  $\alpha_{k+1} = \alpha_k - \frac{\phi+\Delta}{\Delta} \frac{\phi}{\phi'}$

until $|\phi - \Delta| \leq 0.1\Delta$

18. Pose Graph Optimization

18.1. Maximum Likelihood Estimation as Nonlinear Least Squares

Notation :

1. $X = \{x_i\}$ set of variables
2. $\hat{y}_j \in \mathbb{R}^{d_j}$
3. h_j measurement function s.t. $y_j \sim \mathcal{N}(h_j(X_j), \Sigma_j)$ for a (often small) subset of variables $X_j \subset X$.

Given a collection of measurements $\hat{y}_j$ we are interested in finding the **maximum-likelihood estimate** of the variables X. In the Gaussian setting this becomes

$$
\begin{aligned}
X^* = \arg\max_x \prod_j p_j(\hat{y}_j \mid X_j) &= \arg\max_x \prod_j \frac{1}{\sqrt{(2\pi)^{d_j}|\Sigma_j|}} \exp\left(-\frac{(\hat{y}_j - h_j(X_j))^T \Sigma_j^{-1} (\hat{y}_j - h_j(X_j))}{2}\right) \\
&= \arg\max_x \prod_j \exp\left(-\frac{(\hat{y}_j - h_j(X_j))^T \Sigma_j^{-1} (\hat{y}_j - h_j(X_j))}{2}\right)
\end{aligned}
$$

Maximizing $f(x)$ is equivalent to minimizing $2\log(-f(x))$. Taking the negative log and multiplying by two yields

$$
X^* = \arg\min_x \sum_j \left(\hat{y}_j - h_j(X_j)\right)^T \Sigma_j^{-1} \left(\hat{y}_j - h_j(X_j)\right) = \arg\min_x \sum_j \left\| \sqrt{I}_j \left(\hat{y}_j - h_j(X_j)\right) \right\|^2,
$$

where $\sqrt{I}_j := \Sigma_j^{-1/2}$ is the **square root information matrix**. We have thus converted the maximum-likelihood estimation problem into a **least squares problem**. When the functions h_j involve 3D geometry the least squares problem is typically **nonlinear**.

The least-squares problem can be viewed as a **bipartite factor graph** where variables and measurements (factors) are nodes. By exploiting the graph structure updates can be made locally in the graph, but this requires sophisticated data structures and solvers [10].

18.2. Measurement functions

18.2.1. Absolute pose measurement

Let the measurement be $\hat{y}_j = \log(\hat{P})$ where $\log : SE(3) \to \mathfrak{se}(3)$ is the logarithm on $SE(3)$ and $\hat{P}$ a pose measurement, then

$$h(P) = \log(P) \in \mathbb{R}^6.$$

Since $SE(3) = SO(3) \times \mathbb{R}^3$ we can use the same formula for individual measurements of orientation ($SO(3)$) or position ($\mathbb{R}^3$).

18.2.2. Relative pose measurement

Let the measurement be $\hat{y}_j = \log(\hat{P}_{12})$ where $\log : SE(3) \to \mathfrak{se}(3)$ is the logarithm on $SE(3)$ and $\hat{P}_{12}$ an estimate of the relative pose. Then

$$h(P_1, P_2) = \log(P_1^{-1} P_2) \in \mathbb{R}^6.$$

18.2.3. Rectified stereo landmark measurement

* $P \in SE(3)$ is the pose of the left camera (variable) * $l \in \mathbb{R}^3$ world location of a landmark (variable) * $P_{rl} \in SE(3)$ the pose of the right camera w.r.t. the left camera (known) * CM_l, CM_r camera projection matrices

The landmark is projected to the left and right image pixel planes as

$$\lambda_l \tilde{\mathbf{x}}_l = CM_l P^{-1} l, \quad \lambda_r \tilde{\mathbf{x}}_r = CM_r P_{rl} P^{-1} l.$$

In a rectified system we have $y_l = y_r$, so we can let the 3-dimensional measurement be the pixel locations $\hat{x}_l, \hat{x}_r, \hat{y}_l$. The measurement function $h(P, l)$ is described by the equations above.

18.3. Marginalization of nonlinear least squares

Objective is to remove a variable from the problem in a way so that

* The optimal solution is not effected * The first derivative at the optimal solution remains the same

For a nonlinear problem this is not possible, so we do it around a linearization point.

18.4. Lifted information matrix

Consider a set of variables $X = \{x_1, \dots, x_k\}$ where $x_i \in M_i$ and a square form

$$S = \frac{1}{2} \sum_j \left(h_j(X_j) - y_j\right)^T I_j \left(h_j(X_j) - y_j\right),$$

where $X_j = \{x_{j_1}, \dots x_{j_{n_j}}\} \subset X$ is a set of variables for the j:th measurement, and $h_j : X_j \mapsto h_j(X_j) \in \mathbb{R}^{p_j}$ are nonlinear measurement functions. Also let $I_j = \{j_1, \dots, j_{n_j}\}$ be the variable indices for measurement number j.

We are interested in marginalizing the expression S around a point $\{x_k = \mu_k\}$. Via Taylor expansion we obtain with $\mu_j = \begin{bmatrix} \mu_{j_1} & \cdots & \mu_{j_{n_j}} \end{bmatrix}$ being the measurement mean:

$$2S \approx \sum_j \left(h_j(\mu_j) + \sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i - y_j \right)^T I_j \left(h_j(\mu_j) + \sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i - y_j \right).$$

Here $[\mathrm{d}_i h_j]_{\mu_j} : TM_i \to \mathbb{R}^{p_j}$ is the differential of the measurement $h_j : \prod_{i \in I_j} M_i \to \mathbb{R}^{p_j}$ with respect to x_i evaluated at μ_j. The error differentials e_i are such that

$$x_i = \mu_i \oplus e_i = \mu_i \mathrm{Exp}_i(e_i) \quad \iff \quad e_i = x_i \ominus \mu_i = \mathrm{Log}_i(\mu_i^{-1} x_i),$$

where $\mathrm{Exp} : \mathbb{R}^{n_i} \to M_i$ maps from coordinates in the tangent space to the manifold, and Log is the inverse mapping.

We now expand the sum

$$\begin{aligned} 2S \approx & \sum_j \left(h_j(\mu_j) - y_j\right)^T I_j \left(h_j(\mu_j) - y_j\right) + \sum_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right)^T I_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right) \\ & + 2 \sum_j \left(h_j(\mu_j) - y_j\right)^T I_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right) \end{aligned}$$

and consider the term quadratic in e. We can augment the middle matrix with the differentials and

$$\begin{aligned} & \sum_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right)^T I_j \left(\sum_{i \in I_j} [\mathrm{d}_i h_j]_{\mu_j} e_i \right) \\ &= \sum_j \begin{bmatrix} e_{j_1} & \cdots & e_{j_{n_j}} \end{bmatrix} \begin{bmatrix} [\mathrm{d}_{j_1} h_j]_{\mu_j}^T \\ \vdots \\ [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j}^T \end{bmatrix} I_j \begin{bmatrix} [\mathrm{d}_{j_1} h_j]_{\mu_j} & \cdots & [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j} \end{bmatrix} \begin{bmatrix} e_{j_1} \\ \vdots \\ e_{j_{n_j}} \end{bmatrix} \\ &= \sum_j \begin{bmatrix} e_{j_1} & \cdots & e_{j_{n_j}} \end{bmatrix} \begin{bmatrix} [\mathrm{d}_{j_1} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_1} h_j]_{\mu_j} & \dots & [\mathrm{d}_{j_1} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j} \\ \vdots & \ddots & \vdots \\ [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_1} h_j]_{\mu_j} & \cdots & [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j}^T I_j [\mathrm{d}_{j_{n_j}} h_j]_{\mu_j} \end{bmatrix} \begin{bmatrix} e_{j_1} \\ \vdots \\ e_{j_{n_j}} \end{bmatrix} \\ &= \sum_{i_1, i_2} e_{i_1} \left[\sum_{j : i_1, i_2 \in I_j} [\mathrm{d}_{i_1} h_j]_{\mu_j}^T I_j [\mathrm{d}_{i_2} h_j]_{\mu_j} \right] e_{i_2} = \begin{bmatrix} e_1 & \dots & e_k \end{bmatrix} \Lambda \begin{bmatrix} e_1 \\ \vdots \\ e_k \end{bmatrix}, \end{aligned}$$

where Λ is the sum of **block-lifted information matrices** obtained by placing the blocks from cost functions at the appropriate places.

It follows that we can write S on the information form

$$S \sim \eta^T \mathbf{e} + \frac{1}{2}\mathbf{e}^T \Lambda \mathbf{e}$$

with Λ as above and η a similarly block-lifted column vector such that

$$\eta^T \mathbf{e} = \sum_{i=1}^{k} \left[\sum_{j\,:\,i \in I_j} (h_j(\mu_j) - y_j)^T I_j [\mathrm{d}_i h_j]_{\mu_j} \right] e_i.$$

18.5. Marginalization

We group the variables into e_α and e_β, where e_β is the variable to be removed, and write S on the general information form

$$S \sim \begin{bmatrix} \eta_\alpha^T & \eta_\beta^T \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta \end{bmatrix} + \frac{1}{2} \begin{bmatrix} e_\alpha^T & e_\beta^T \end{bmatrix} \begin{bmatrix} \Lambda_{\alpha\alpha} & \Lambda_{\alpha\beta} \\ \Lambda_{\beta\alpha} & \Lambda_{\beta\beta} \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta \end{bmatrix}$$

We can then expand the information matrix in the same way as in Joplin normal distribution to obtain

$$S \sim \begin{bmatrix} \eta_\alpha^T - \eta_\beta^T \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha} & \eta_\beta^T \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta + \Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha} e_\alpha \end{bmatrix} + \frac{1}{2} \begin{bmatrix} e_\alpha \\ e_\beta + \Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha} e_\alpha \end{bmatrix}^T \begin{bmatrix} \Lambda/\Lambda_{\beta\beta} & 0 \\ 0 & \Lambda_{\beta\beta} \end{bmatrix} \begin{bmatrix} e_\alpha \\ e_\beta + \Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha} e_\alpha \end{bmatrix}.$$

That is, we have separated the expression into two quadratic expressions. A coordinate change

$$\kappa = e_\beta + \Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha} e_\alpha$$

reveals that they can be solved independently. The sub-problem for e_α reads

$$\left(\eta_\alpha^T - \eta_\beta^T \Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha}\right) e_a + \frac{1}{2} e_\alpha^T (\Lambda/\Lambda_{\beta\beta}) e_\alpha.$$

By completing the square we can write this as

$$\sim \frac{1}{2}\left(e_\alpha + (\Lambda/\Lambda_{\beta\beta})^{-1}\left(\eta_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\eta_\beta\right)\right)^T (\Lambda/\Lambda_{\beta\beta}) \left(e_\alpha + (\Lambda/\Lambda_{\beta\beta})^{-1}\left(\eta_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\eta_\beta\right)\right)$$

which is the marginalized form of the problem.

18.6. Algorithm

Suppose we want to remove a variable x_k. Consider the set of factors F such that $k \in I_f$ for all $f \in F$, and the resulting blanket set of variables $X_F = \bigcup_{f \in F} \bigcup_{j \in f_j} X_j$.

Ceres provides evaluations $E_j = \sqrt{I_j}(h_j(\mu_j) - y_j)$ and gradient blocks $G_{ji} = \sqrt{I_j}[\mathrm{d}_i h_j]_{\mu_j}$. **It's fine if residuals are defined as the negative since the signs will cancel in multiplication**.

1. Find the information matrix Λ by summing over all factors in F. Sum blocks in Λ are of the form $[\mathrm{d}_{i_1} h_j]^T_{\mu_j} I_j [\mathrm{d}_{i_2} h_j]_{\mu_j} = G^T_{ji_1} G_{ji_2}$. 2. Find the mean vector η by summing over all factors in F. Sum segments in η are of the form $(h_j(\mu_j) - y_j)^T I_j [\mathrm{d}_i h_j]_{\mu_j} = E^T_j G_{ji}$. 3. Partition Λ and η as

$$\Lambda = \begin{bmatrix} \Lambda_{\neg k \neg k} & \Lambda_{\neg k k} \\ \Lambda_{k \neg k} & \Lambda_{kk} \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_{\neg k} \\ \eta_l \end{bmatrix}.$$

3. Calculate $\tilde{\Lambda} = \Lambda / \Lambda_k$ and $\gamma = -(\Lambda/\Lambda_k)^{-1} \left(\eta_{\neg k} - \Lambda_{\neg kk} \Lambda^{-1}_{kk} \eta_k \right)$ 4. Remove factors F and instead insert a new factor with cost function

$$(e_{\neg k} - \gamma)^T \tilde{\Lambda} (e_{\neg k} - \gamma) = \begin{bmatrix} e_1 - \gamma_1 \\ \vdots \\ e_n - \gamma_n \end{bmatrix}^T \tilde{\Lambda} \begin{bmatrix} e_1 - \gamma_1 \\ \vdots \\ e_n - \gamma_n \end{bmatrix} = \begin{bmatrix} \mathrm{Log}(\mu_1^{-1} x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_n^{-1} x_n) - \gamma_n \end{bmatrix}^T \tilde{\Lambda} \begin{bmatrix} \mathrm{Log}(\mu_1^{-1} x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_n^{-1} x_n) - \gamma_n \end{bmatrix}$$

18.7. Correction for singular information matrix

In the event that $\tilde{\Lambda}$ is singular we can not calculate γ. Instead consider the decomposition $\tilde{\Lambda} = UDU^T$, where D is a square diagonal matrix with only non-zero diagonal entries. We can then let

$$\gamma = -UD^{-1}U^T \left(\eta_{\neg k} - \eta_k \Lambda^{-1}_{kk} \Lambda_{k \neg k} \right)$$

and consider the cost

$$\left\| \sqrt{D} U^T \begin{bmatrix} \mathrm{Log}(\mu_1^{-1} x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_n^{-1} x_n) - \gamma_n \end{bmatrix} \right\|^2$$

which has a non-zero information matrix.

18.8. Marginalization factor in local frame

The above linearizes around a world point $\{\mu\}$. This makes sense if marginalizing a node that has an absolute factor, but perhaps not when it is only connected via relative factors and there may be a lot of drift. We can transform the measurements into a local frame by instead introducing the cost

$$\left\| \sqrt{D} U^T \begin{bmatrix} \mathrm{Log}(\mu_{01}^{-1} x_0^{-1} x_1) - \gamma_1 \\ \vdots \\ \mathrm{Log}(\mu_{0n}^{-1} x_0^{-1} x_n) - \gamma_n \end{bmatrix} \right\|^2$$

where $\mu_{0i} = \mu_0^{-1} \mu_i$ are linearization points transformed into the local frame of x_0, which should be selected as a pose in the vicinity of the removed node.

* Only works if the measurements h_j are invariant to rigid transformations. This property holds for relative measurements such as relative poses and landmark triangulations.

* For a node with absolute factors the measurement h would have to be adjusted before building the *gamma* vector.

18.9. Example

Consider the least-squares problem

$$S = (x_1 - x_3 + 1)^2 + (x_2 - x_3 + 1)^2$$

where $h_1(\mathbf{x}) = x_1 - x_3$, $y_1 = -1$, $h_2(\mathbf{x}) = x_2 - x_3$, and $y_2 = -1$.

We expand in a Taylor form as above

$$\begin{aligned} S &\approx \left(h_1(\mathbf{x}_0) + \mathrm{d}h_1 \cdot (\mathbf{x} - \mathbf{x}_0) - y_1\right)^2 + \left(h_2(\mathbf{x}_0) + \mathrm{d}h_2 \cdot (\mathbf{x} - \mathbf{x}_0) - y_2\right)^2 \\ &= (\mathrm{d}h_1 \cdot (\mathbf{x} - \mathbf{x}_0))^2 + (\mathrm{d}h_2 \cdot (\mathbf{x} - \mathbf{x}_0))^2 \\ &\quad + 2\left[(h_1(\mathbf{x}_0) - y_1)\mathrm{d}h_1 \cdot (\mathbf{x} - \mathbf{x}_0) + (h_2(\mathbf{x}_0) - y_2)\mathrm{d}h_2 \cdot (\mathbf{x} - \mathbf{x}_0)\right] + C \end{aligned}$$

where C is a constant. Since both h are linear this expression is exact for any $\mathbf{x}_0$ (can be verified).

We let $e_i = x_i - x_0^i$ and get

$$\begin{aligned} S &= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \left(\mathrm{d}h_1^T \mathrm{d}h_1 + \mathrm{d}h_2^T \mathrm{d}h_2\right) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + 2\left[(h_1(\mathbf{x}_0) - y_1)\mathrm{d}h_1 + (h_2(\mathbf{x}_0) - y_2)\mathrm{d}h_2\right] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \\ &= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \\ &\quad + 2\left[(x_1^0 - x_3^0 + 1)\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + (x_2^0 - x_3^0 + 1)\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\right] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \\ &= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \\ &\quad + 2\begin{bmatrix} x_1^0 - x_3^0 + 1 & x_2^0 - x_3^0 + 1 & -x_1^0 - x_2^0 + 2x_3^0 - 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \end{aligned}$$

We now marginalize out x_3 and identify the marginalized covariance

$$\tilde{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

and the marginalized mean

$$\begin{aligned} \tilde{\eta}^T &= \eta_\alpha^T - \eta_\beta^T \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha} \\ &= \begin{bmatrix} x_1^0 - x_3^0 + 1 & x_2^0 - x_3^0 + 1 \end{bmatrix} - \begin{bmatrix} -x_1^0 - x_2^0 + 2x_3^0 - 2 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} x_1^0 - x_2^0 & -x_1^0 + x_2^0 \end{bmatrix}. \end{aligned}$$

The marginalized problem is now

$$\tilde{S} = \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix}^T \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix} + 2 \begin{bmatrix} \frac{x_1^0 - x_2^0}{2} & \frac{-x_1^0 + x_2^0}{2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{bmatrix}.$$

After expanding and removing constant terms this is equal to

$$\tilde{S} = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}(x_1 - x_2)^2.$$

That is, the problem is independent of the linearization point, as expected.

18.10. As part of least-squares problem

Now, for the least-squares problem that has this as a factor:

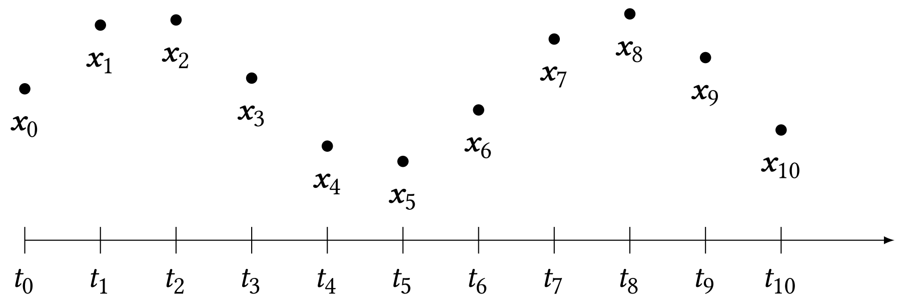
$$x_1^2 + (x_1 - x_2 + 1)^2 + (x_1 - x_3 + 1)^2 + (x_2 - x_3 + 1)^2$$

that the above as terms, marginalizing x_3 replaces the last two terms with the above, so the marginalized problem becomes

$$x_1^2 + (x_1 - x_2 + 1)^2 + \frac{1}{2}(x_1 - x_2)^2.$$

This has the same optimial solution $x_1 = 0, x_2 = 2/3$ and optimzal value $1/3$ as the full problem above.

19. Interpolation



It is often convenient to be able to interpolate sparsely defined data. Examples in robotics include representing trajectories as a sequence of $(t_i, \boldsymbol{x}_i)$ pairs, and interpolating sensor data for calibration.

A general form for a spline $p(x)$ constructed from *control points* $\boldsymbol{x}_i$ is

$$p(t) = \sum_{i=0}^{n} B_i(t)\boldsymbol{x}_i, \tag{19.1}$$

where $\boldsymbol{B}_i$ are some form of *basis functions* that form a partition of unity, i.e. $\sum_i B_i(t) = 1$ for all t. The weighted sum does not have an immediate analogue on Lie groups, but (19.1) can be re-arranged on *cumulative form* as

$$p(t) = \boldsymbol{x}_0 + \sum_{i=1}^{n} \tilde{B}_i(t)(\boldsymbol{x}_i - \boldsymbol{x}_{i-1}), \tag{19.2}$$

where $\tilde{B}_i(t) = \sum_{j=i}^{n} B_i(t)$ are *cumulative basis functions.* From this formulation we can write down a Lie group generalization as follows:

$$p(t) = \boldsymbol{x}_0 \circ \exp\left(\tilde{B}_1(t)(\boldsymbol{x}_1 \ominus \boldsymbol{x}_0)\right) \circ \exp\left(\tilde{B}_2(t)(\boldsymbol{x}_2 \ominus \boldsymbol{x}_1)\right) \circ \ldots \circ \exp\left(\tilde{B}_n(t)(\boldsymbol{x}_n \ominus \boldsymbol{x}_{n-1})\right) \tag{19.3}$$

We first discuss two choices for basis functions—Bezier curves and B-Splines—and then show how the value and derivatives of such splines can be calculated.

Below we will need the derivative with respect to t of (19.3). For this purpose it will be convenient to introduce the variables

$$\begin{aligned} \boldsymbol{v}_i &= \boldsymbol{x}_i \ominus \boldsymbol{x}_{i-1}, \\ \boldsymbol{s}_i &= \tilde{B}_i(t)\boldsymbol{v}_i. \end{aligned} \tag{19.4}$$

With this notation, $\mathrm{d}^r(\exp \boldsymbol{s}_i)_t = \tilde{B}_i'(t)\mathrm{d}^r \exp_{\boldsymbol{s}_i} \boldsymbol{v}_i = \tilde{B}_i'(t)\boldsymbol{v}_i$. We can now write down the derivative of (19.3) by utilizing (5.39) and (5.41):

$$\begin{aligned}\mathrm{d}^r p_t &= \mathbf{Ad}_{\exp(-\boldsymbol{s}_n)} \mathrm{d}^r(\boldsymbol{x}_0 \exp(\boldsymbol{s}_1)\dots\exp(\boldsymbol{s}_{n-1}))_t + \mathrm{d}^r \exp(\boldsymbol{s}_n)_t \\ &= \mathbf{Ad}_{\exp(-\boldsymbol{s}_n)\exp(-\boldsymbol{s}_{n-1})} \mathrm{d}^r(\boldsymbol{x}_0 \exp(\boldsymbol{s}_1)\dots\exp(\boldsymbol{s}_{n-2}))_t + \mathbf{Ad}_{\exp(-\boldsymbol{s}_n)} \mathrm{d}^r \exp(\boldsymbol{s}_{n-1})_t + \mathrm{d}^r \exp(\boldsymbol{s}_n)_t \\ &= \sum_{i=1}^{n} \mathbf{Ad}_{\exp(-\boldsymbol{s}_n)\dots\exp(-\boldsymbol{s}_{i+1})} \mathrm{d}^r \exp(\boldsymbol{s}_i)_t = \sum_{i=1}^{n} \tilde{B}_i'(t)\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_n)\dots\exp(-\boldsymbol{s}_{i+1})} \boldsymbol{v}_i .\end{aligned} \tag{19.5}$$

19.1. Bezier Curves

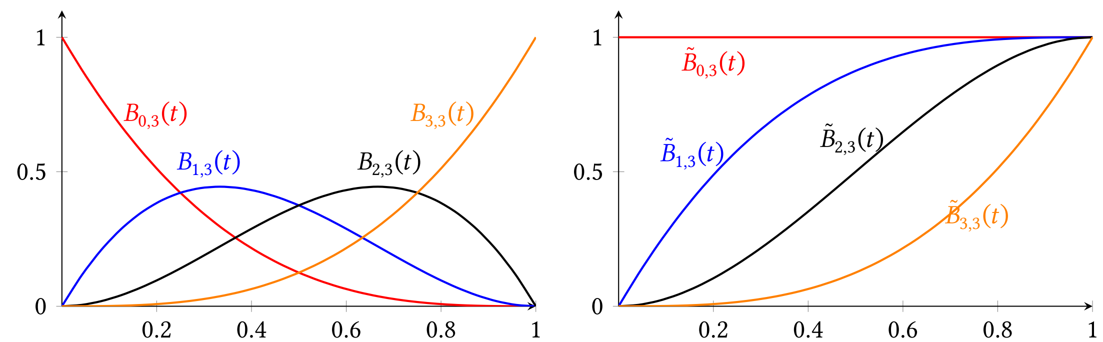


Figure 19.1.: Bernstein cubic (order 3) basis functions (left) and cumulative basis functions (right).

A Bezier curve of order k is defined on the interval $[0, 1]$ via the *Bernstein basis functions* defined as

$$B_{i,k}(t) = \binom{k}{i} t^i (1-t)^{k-i}, \qquad i = 0, \dots, k. \tag{19.6}$$

The binomial formula $1 = (t + 1 - t)^n = \sum_{i=0}^{n} B_{i,k}(t)$ shows that the Berstein basis is indeed a partition of unity. A Bezier spline is a curve that consists of Bezier curve pieces.

The Bernstein basis functions have the following properties

- They satisfy the recurrence relation

$$B_{i,k}(t) = tB_{i-1,k-1}(t) + (1-t)B_{i,k-1}(t). \tag{19.7}$$

- They are symmetric in the sense that $B_{i,k}(t) = B_{k-i,k}(1-t)$.
- The derivative can be expressed in terms of lower-order polynomials

$$\begin{aligned}B_{i,k}'(t) &= i\binom{k}{i} t^{i-1}(1-t)^{k-i} - (k-i)\binom{k}{i} t^i (1-t)^{k-i-1} \\ &= \frac{k!}{(i-1)!(k-i)!} t^{i-1}(1-t)^{k-i} + \frac{k!}{i!(k-i-1)!} t^i (1-t)^{k-i+1} = k\left(B_{i-1,k-1}(t) - B_{i,k-1}(t)\right).\end{aligned}$$

- At $t = 0$ only the first spline is nonzero: $B_{0,k}(0) = 1$, and $B_{1,k}(0) = \ldots = B_{k,k}(0) = 0$. It follows that the non-zero derivatives at $t = 0$ are $B'_{0,k}(0) = -k$ and $B'_{1,k}(0) = k$. The converse holds for $t = 1$ due to symmetry.
- At $t = 0$ only the $i = 0$ cumulative spline is nonzero: $\tilde{B}_{0,k}(0) = 1$, and only the $i = 1$ cumulative derivative is non-zero: $\tilde{B}'_{1,k}(0) = k$.
- At $t = 1$ only the $i = k$ cumulative spline is nonzero: $\tilde{B}_{k,k}(1) = 1$, and only the $i = k$ cumulative derivative is non-zero: $\tilde{B}'_{k,k}(0) = k$.

Utilizing these fact and that $\mathrm{d}^r \exp_{\boldsymbol{a}} \boldsymbol{a} = \boldsymbol{a}$ we can evaluate (19.3) and (19.5)

$$p(0) = \boldsymbol{x}_0, \quad p(1) = \boldsymbol{x}_n, \quad \mathrm{d}^r p_t\big|_{t=0} = \tilde{B}'_{1,k}(0)\boldsymbol{v}_1 = k\boldsymbol{v}_1, \quad \mathrm{d}^r p_t\big|_{t=1} = \tilde{B}'_{k,k}(0)\boldsymbol{v}_k = k\boldsymbol{v}_k. \tag{19.8}$$

These formulas can be used to fit low-degree splines to given boundary conditions.

19.1.1. Quadratic Bezier curve

A $k = 2$ spline $\boldsymbol{x}(t) = \boldsymbol{x}_0 \exp\left(\tilde{B}_{1,2}(t)\boldsymbol{v}_1\right) \exp\left(\tilde{B}_{2,2}(t)\boldsymbol{v}_2\right)$ such that $\boldsymbol{x}(0) = \boldsymbol{x}_a$, $\boldsymbol{x}(1) = \boldsymbol{x}_b$, and $\mathrm{d}^r \boldsymbol{x}_t\big|_{t=0} = \boldsymbol{a}_a$ is defined by the coefficients

$$\boldsymbol{x}_0 = \boldsymbol{x}_a, \quad \boldsymbol{v}_1 = \frac{\boldsymbol{a}_a}{2}, \quad \boldsymbol{v}_2 = \log\left(\exp\left(-\frac{\boldsymbol{a}_a}{2}\right)\boldsymbol{x}_a^{-1}\boldsymbol{x}_b\right), \tag{19.9}$$

and has the property that $\mathrm{d}^r \boldsymbol{x}_t\big|_{t=1} = 2\boldsymbol{v}_2$.

19.1.2. Cubic Bezier curve

For the cubic case $k = 3$ a Bezier curve takes the form

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 \exp\left(\tilde{B}_{1,3}(t)\boldsymbol{v}_1\right) \exp\left(\tilde{B}_{2,3}(t)\boldsymbol{v}_2\right) \exp\left(\tilde{B}_{3,3}(t)\boldsymbol{v}_3\right) \tag{19.10}$$

and has first and second derivatives w.r.t. time

$$\begin{aligned}
\mathrm{d}^r \boldsymbol{x}_t &= \tilde{B}'_{1,3}(t)\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_3)}\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_2)}\, \boldsymbol{v}_1 \\
&\quad + \tilde{B}'_{2,3}(t)\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_3)}\, \boldsymbol{v}_2 \\
&\quad + \tilde{B}'_{3,3}(t)\boldsymbol{v}_3 \\
\mathrm{d}^{2r} \boldsymbol{x}_{tt} &= \tilde{B}''_{1,3}(t)\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_3)}\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_2)}\, \boldsymbol{v}_1 \\
&\qquad - \tilde{B}'_{1,3}(t)\tilde{B}'_{3,3}(t)\left[\boldsymbol{v}_3, \mathbf{Ad}_{\exp(-\boldsymbol{s}_3)}\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_2)}\, \boldsymbol{v}_1\right] \\
&\qquad - \tilde{B}'_{1,3}(t)\tilde{B}'_{2,3}(t)\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_3)}\left[\boldsymbol{v}_2, \mathbf{Ad}_{\exp(-\boldsymbol{s}_2)}\, \boldsymbol{v}_1\right] \\
&\quad + \tilde{B}''_{2,3}(t)\, \mathbf{Ad}_{\exp(-\boldsymbol{s}_3)}\, \boldsymbol{v}_2 - \tilde{B}'_{2,3}(t)\tilde{B}'_{3,3}(t)\left[\boldsymbol{v}_3, \mathbf{Ad}_{\exp(-\boldsymbol{s}_3)}\, \boldsymbol{v}_2\right] \\
&\quad + \tilde{B}''_{3,3}(t)\boldsymbol{v}_3.
\end{aligned} \tag{19.11}$$

The cumulative basis functions of order three are $\tilde{B}_{1,3}(t) = 3t - 3t^2 + t^3$, $\tilde{B}_{2,3}(t) = 3x^2 - 2x^3$, and $\tilde{B}_{3,3}(t) = t^3$. With the derivative formulas above it therefore follows that

$$\begin{aligned}
\mathrm{d}^r \boldsymbol{x}_t\big|_{t=0} &= 3\boldsymbol{v}_1, \quad \mathrm{d}^r \boldsymbol{x}_t\big|_{t=1} = 3\boldsymbol{v}_3, \\
\mathrm{d}^{2r} \boldsymbol{x}_{tt}\big|_{t=0} &= 6(\boldsymbol{v}_2 - \boldsymbol{v}_1), \quad \mathrm{d}^{2r} \boldsymbol{x}_{tt}\big|_{t=1} = 6\left(\boldsymbol{v}_3 - \mathbf{Ad}_{\exp(-\boldsymbol{v}_3)}\, \boldsymbol{v}_2\right)
\end{aligned} \tag{19.12}$$

This information can be used in two ways.

Bezier curve with given endpoint velocities First, if the values $\boldsymbol{x}_a, \boldsymbol{x}_b$ and first-order derivatives $\boldsymbol{a}_a, \boldsymbol{a}_b$ at the end points are given there is a unique Bezier curve $\boldsymbol{x}(t)$ such that $\boldsymbol{x}(0) = \boldsymbol{x}_a$, $\boldsymbol{x}(1) = \boldsymbol{x}_b$, $\mathrm{d}^r \boldsymbol{x}_t|_{t=0} = \boldsymbol{a}_a$, and $\mathrm{d}^r \boldsymbol{x}_t|_{t=1} = \boldsymbol{a}_b$. It is defined by the coefficients

$$\boldsymbol{x}_0 = \boldsymbol{x}_a, \quad \boldsymbol{v}_1 = \frac{\boldsymbol{a}_a}{3}, \quad \boldsymbol{v}_3 = \frac{\boldsymbol{a}_b}{3}, \quad \boldsymbol{v}_2 = \log\left(\exp\left(-\frac{\boldsymbol{a}_a}{3}\right)\boldsymbol{x}_a^{-1}\boldsymbol{x}_b \exp\left(-\frac{\boldsymbol{a}_b}{3}\right)\right). \tag{19.13}$$

Cubic interpolating Bezier spline Secondly, an interpolating spline for a dataset $\{(t_i, \boldsymbol{x}_i)\}_{i=0}^n$ can be defined as a collection of n Bezier curves

$$\boldsymbol{x}_i(u) = \boldsymbol{x}_{a,i} \exp(\tilde{B}_{1,3}(u)\boldsymbol{v}_{1,i}) \exp(\tilde{B}_{2,3}(u)\boldsymbol{v}_{2,i}) \exp(\tilde{B}_{3,3}(u)\boldsymbol{v}_{3,i})$$

where $u = (t-t_i)/(t_{i+1}-t_i) \in [0,1]$. To ensure a smooth interpolation consider the following constraints (for simplicity we assume that $t_{i+1}-t_i = 1$ for all i; if not the derivative constraints have to be rescaled):

- Interpolation: $\boldsymbol{x}_{a,i} = \boldsymbol{x}_i$ and $\boldsymbol{x}_{a,i} \exp(\boldsymbol{v}_1)\exp(\boldsymbol{v}_2)\exp(\boldsymbol{v}_3) = \boldsymbol{x}_{i+1}$, for $i = 0, \dots, n-1$,
- First derivative continuity: $\boldsymbol{v}_{3,i} = \boldsymbol{v}_{1,i+1}$ for $i = 0, \dots, n-2$,
- Second derivative continuity: $\boldsymbol{v}_{3,i} - \mathbf{Ad}_{\exp(-\boldsymbol{v}_{3,i})} \boldsymbol{v}_{2,i} = \boldsymbol{v}_{2,i+1} - \boldsymbol{v}_{1,i+1}$ for $i = 0, \dots, n-2$,
- Zero second derivative at start: $\boldsymbol{v}_{2,0} - \boldsymbol{v}_{1,0} = 0$, and at end: $\mathbf{Ad}_{\exp(-\boldsymbol{v}_{3,n-1})} \boldsymbol{v}_{2,n-1} - \boldsymbol{v}_{3,n-1} = 0$.

The variables $\boldsymbol{x}_{a,i}, \boldsymbol{x}_{b,i}$ can be eliminated to end up with the following system of equations for $\boldsymbol{v}_{j,i}$:

$$\begin{aligned}
&\boldsymbol{v}_{2,0} = \boldsymbol{v}_{1,0}, && \\
&\mathbf{Ad}_{\exp(-\boldsymbol{v}_{3,n-1})} \boldsymbol{v}_{2,n-1} = \boldsymbol{v}_{3,n-1}, && \\
&\exp(\boldsymbol{v}_{1,i})\exp(\boldsymbol{v}_{2,i})\exp(\boldsymbol{v}_{3,i}) = \boldsymbol{x}_i^{-1}\boldsymbol{x}_{i+1}, && i = 0, \dots, n-1, \\
&\boldsymbol{v}_{3,i} = \boldsymbol{v}_{1,i+1}, && i = 0, \dots, n-2, \\
&\boldsymbol{v}_{3,i} - \mathbf{Ad}_{\exp(-\boldsymbol{v}_{3,i})} \boldsymbol{v}_{2,i} = \boldsymbol{v}_{2,i+1} - \boldsymbol{v}_{1,i+1}, && i = 0, \dots, n-2.
\end{aligned} \tag{19.14}$$

For Euclidean space this is a linear system of equations and is therefore straightforward to solve, but on Lie groups this is no longer possible due to nonlinearities. To find an interpolating spline we therefore either have to solve a nonlinear system of equations, or give up the strict requirement on continuity of the second-order derivatives. The following simple algorithm is of the latter kind:

1. Solve the following linearized version of (19.14):

$$\begin{aligned}
&\boldsymbol{v}_{2,0} = \boldsymbol{v}_{1,0}, && \\
&\boldsymbol{v}_{2,n-1} = \boldsymbol{v}_{3,n-1}, && \\
&\boldsymbol{v}_{1,i} + \boldsymbol{v}_{2,i} + \boldsymbol{v}_{3,i} = \log\left(\boldsymbol{x}_i^{-1}\boldsymbol{x}_{i+1}\right), && i = 0, \dots, n-1, \\
&\boldsymbol{v}_{3,i} = \boldsymbol{v}_{1,i+1}, && i = 0, \dots, n-2, \\
&\boldsymbol{v}_{3,i} - \boldsymbol{v}_{2,i} = \boldsymbol{v}_{2,i+1} - \boldsymbol{v}_{1,i+1}, && i = 0, \dots, n-2.
\end{aligned} \tag{19.15}$$

2. Set $\boldsymbol{v}_{2,i} = \log\left(\exp(-\boldsymbol{v}_{1,i})x_i^{-1}x_{i+1}\exp(-\boldsymbol{v}_{3,i})\right)$ for $i = 0, \dots, n-1$.

The first step finds values for all coefficients that do not necessarily produce a continuous spline. This is however rectified in the second step. The resulting spline is guaranteed to be continuous and have a continuous first-order derivative, but in general does not have a continuous second-order derivative.

19.2. B-Splines

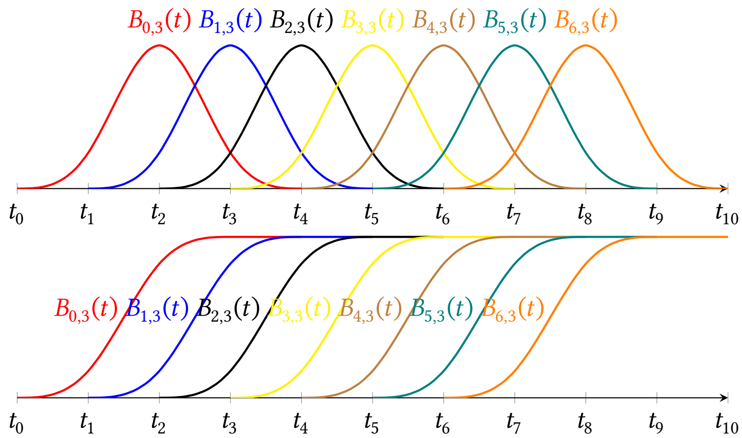


Figure 19.2.: B-spline basis function and cumulative basis functions. For $k = 3$ and $t \in [t_4, t_5)$ the non-zero basis functions are $B_{1,4}$, $B_{2,4}$, $B_{3,4}$ and $B_{4,4}$.

A B-spline interpolation of order k is a function $\boldsymbol{x}(t) = \sum_{i=0}^{N} B_{i,k}(t)\boldsymbol{x}_{\nu(i)}$ where $\boldsymbol{x}_{\nu(i)} \in \mathbb{R}^n$ are **control points** for **knots** t_i, and $B_{i,k}(t)$ are **basis functions** recursively defined as

$$B_{i,0}(t) = \begin{cases} 1 & t_i \leq t < t_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$
$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t). \tag{19.16}$$

The following are some well-known properties of B-splines:

- $B_{i,k}(t)$ has finite support and is zero outside the interval $[t_i, t_{i+k+1})$,
- Inside this interval it is a piecewise polynomial of degree k,
- It is centered in the middle of that interval, it therefore makes sense to select k odd and $\nu(i) = i + (k+1)/2$ so that $x_{\nu(i)}$ coincides with the maximum of $B_{i,k}(t)$ (c.f. Figure 19.2),
- $\sum_i B_{i,k}(t) = 1$ for all t,

We are interested in an expression for the coefficients of the polynomial $B_{i,k}(t)$. We pose that for a **fixed interval** $t \in [t_{i^*}, t_{i^*+1})$ we have scalar cofficients $\alpha^i_{i,k}$ such that

$$B_{i,k}(t) = \sum_{l=0}^{k} \alpha^l_{i,k}\, u^l(t), \quad u(t) = \frac{t - t_{i^*}}{t_{i^*+1} - t_{i^*}} \tag{19.17}$$

where $i \in \{i^* - k, i^* - k + 1, \dots, i^*\}$ are the indices for which $B_{i,k}(t)$ is non-zero on $[t_{i^*}, t_{i^*+1})$ (c.f. Figure 19.2). If we also introduce

$$\begin{aligned} N_{i,k} &:= \begin{bmatrix} \alpha_{i,k}^0 & \alpha_{i,k}^1 & \cdots & \alpha_{i,k}^k \end{bmatrix}^T \in \mathbb{R}^{k+1} \\ M_{i^*,k} &:= \begin{bmatrix} N_{i^*-k,k} & N_{i^*-k+1,k} & \cdots & N_{i^*,k} \end{bmatrix} \in \mathbb{R}^{k+1,k+1} \end{aligned} \tag{19.18}$$

we can write the value of a spline $x(t)$ for $t \in [t_{i^*}, t_{i^*+1}]$ as

$$\boldsymbol{x}(t) = \sum_{j=0}^{n} B_{j,k}(t)\boldsymbol{x}_{\nu(j)} = \sum_{j=i^*-k}^{i^*} B_{j,k}(t)\boldsymbol{x}_{\nu(j)} = \sum_{j=i^*-k}^{i^*} \sum_{l=0}^{k} \alpha_{j,k}^l u^l \boldsymbol{x}_{\nu(j)} = \begin{bmatrix} 1 & u & \cdots & u^k \end{bmatrix} M_{i^*,k} \begin{bmatrix} \boldsymbol{x}_{\nu(i^*-k)} \\ \vdots \\ \boldsymbol{x}_{\nu(i^*)} \end{bmatrix}. \tag{19.19}$$

19.3. Evaluating Cumulative Splines

Since for $t \in [t_{i^*}, t_{i^*+1})$ it holds that $\tilde{B}_{i,k}(t) = 1$ for $i \le i^* - k$ and $\tilde{B}_{i,k}(t) = 0$ for $i \ge i^* + 1$ we can simplify (19.2) into

$$\boldsymbol{x}(t) = \boldsymbol{x}_{i^*-k} \circ \prod_{j=i^*-k+1}^{i^*} \exp\left[\tilde{B}_{j,k}(t)\boldsymbol{v}_j\right], \qquad \boldsymbol{v}_j := \boldsymbol{x}_j \ominus_r \boldsymbol{x}_{j-1} = \log\left(\boldsymbol{x}_{j-1}^{-1} \circ \boldsymbol{x}_j\right). \tag{19.20}$$

Given the $N_{j,k}$:s we can evaluate $\tilde{B}_{j,k}(t)$ as

$$\begin{bmatrix} \tilde{B}_{i^*-k,k}(t) & \tilde{B}_{i^*-k+1,k}(t) & \dots & \tilde{B}_{i^*,k}(t) \end{bmatrix} = \begin{bmatrix} 1 & u & \dots & u^k \end{bmatrix} \tilde{M}_{i^*,k}, \tag{19.21}$$

where $\tilde{M}_{i^*,k} \in \mathbb{R}^{k+1,k+1}$ is the column-wise reverse cumulative sum of $M_{i^*,k}$:

$$\tilde{M}_{i^*,k} = \begin{bmatrix} \sum_{j=i^*-k}^{i^*} N_{j,k} & \sum_{j=i^*-k+1}^{i^*} N_{j,k} & \dots & N_{i^*,k} \end{bmatrix} \tag{19.22}$$

19.3.1. First order derivative

To evaluate the derivative of a spline consider the formula

$$\boldsymbol{x}(t) = \boldsymbol{y}(t) \circ \boldsymbol{z}(t), \quad \boldsymbol{z}(t) := \exp\left(\lambda(t)\boldsymbol{v}\right). \tag{19.23}$$

Since $t \in \mathbb{R}$ we can, as discussed in Remark 5.1, evaluate the derivative of x w.r.t. t as

$$\mathrm{d}^r \boldsymbol{x}_t = \mathrm{d}^r(\boldsymbol{y} \circ \boldsymbol{z})_{\boldsymbol{y}}\, \mathrm{d}^r \boldsymbol{y}_t + \mathrm{d}^r(\boldsymbol{y} \circ \boldsymbol{z})_{\boldsymbol{z}}\, \mathrm{d}^r \boldsymbol{z}_t. \tag{19.24}$$

From the right-jacobian derivative rules (5.39), (5.41) we know that $\mathrm{d}^r(\boldsymbol{y} \circ \boldsymbol{z})_{\boldsymbol{y}} = \mathbf{Ad}_{\boldsymbol{z}^{-1}}$ and $\mathrm{d}^r(\boldsymbol{y} \circ \boldsymbol{z})_{\boldsymbol{z}} = I$. It therefore follows that

$$\mathrm{d}^r \boldsymbol{x}_t = \mathbf{Ad}_{\exp(-\lambda(t)\boldsymbol{v})}\, \mathrm{d}^r \boldsymbol{y}_t + \lambda'(t)\boldsymbol{v}. \tag{19.25}$$

This gives us a recursive procedure to calculate the derivative of a form (19.20) where we instead of matrix elements consider tangent elements $\boldsymbol{w}_i, \boldsymbol{v}_j \in \mathbb{R}^n$:

$$\begin{aligned} \boldsymbol{w}_{i^*-k} &= \mathbf{0}, \\ \boldsymbol{w}_j &= \mathbf{Ad}_{\exp(-\tilde{B}_{j,k}(t)\boldsymbol{v}_j)}\, \boldsymbol{w}_{j-1} + \tilde{B}'_{j,k}(t)\boldsymbol{v}_j, \quad j = i^*+1, \dots, i^*+k, \\ \mathrm{d}^r \boldsymbol{x}_t &= \boldsymbol{w}_{i^*}. \end{aligned} \tag{19.26}$$

Due to using right jacobians this will result in a body velocity along the spline. If the world velocity is instead desired it can be obtained using

$$\mathrm{d}^l \boldsymbol{x}_t = \mathbf{Ad}_{x(t)}\, \mathrm{d}^r \boldsymbol{x}_t. \tag{19.27}$$

19.3.2. Second order derivative

The recursion in (19.26) can be differentiated a second time with respect to t to obtain the second order derivative. We use some properties of the adjoint to show

$$\begin{aligned} \mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{u}_1)}\, \boldsymbol{u}_2 &\overset{(5.22)}{=} \exp\left(\mathrm{ad}_{\lambda(t)\boldsymbol{u}_1}\right) \boldsymbol{u}_2 \overset{(5.20)}{=} \exp\left(\lambda(t)\, \mathrm{ad}_{\boldsymbol{u}_1}\right) \boldsymbol{u}_2 \overset{(5.21)}{=} \sum_{k=0}^{\infty} \frac{\lambda(t)^k\, \mathrm{ad}_{\boldsymbol{u}_1}^k}{k!} \boldsymbol{u}_2 \\ \implies \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{u}_1)}\, \boldsymbol{u}_2 &= \lambda'(t) \sum_{k=1}^{\infty} \frac{\lambda(t)^{k-1}\, \mathrm{ad}_{\boldsymbol{u}_1}^k}{(k-1)!} \boldsymbol{u}_2 = \lambda'(t)\, \mathrm{ad}_{\boldsymbol{u}_1} \sum_{k=1}^{\infty} \frac{\lambda(t)^{k-1}\, \mathrm{ad}_{\boldsymbol{u}_1}^{k-1}}{(k-1)!} \boldsymbol{u}_2 \\ &= \lambda'(t) \left[\boldsymbol{u}_1, \sum_{k=0}^{\infty} \frac{\lambda(t)^k\, \mathrm{ad}_{\boldsymbol{u}_1}^k}{k!} \boldsymbol{u}_2 \right] = \lambda'(t) \left[\boldsymbol{u}_1, \mathbf{Ad}_{\exp(\lambda(t)\boldsymbol{u}_1)}\, \boldsymbol{u}_2 \right]. \end{aligned} \tag{19.28}$$

With $\lambda(t) \to -\tilde{B}_{j,k}(t), \boldsymbol{u}_1 \to \boldsymbol{v}_j, \boldsymbol{u}_2 \to \boldsymbol{w}_{j-1}$ we get $\mathbf{Ad}_{\exp(-\tilde{B}_{j,k}(t)\boldsymbol{v}_j)}\, \boldsymbol{w}_{j-1} \overset{(19.26)}{=} \boldsymbol{w}_j - \tilde{B}'_{j,k}(t)\boldsymbol{v}_j$, and therefore by introducing $\boldsymbol{q}_j \coloneqq \frac{\mathrm{d}\boldsymbol{w}_j}{\mathrm{d}t}$:

$$\begin{aligned} \boldsymbol{q}_{i^*-k} &= \mathbf{0}, \\ \boldsymbol{q}_j &= \tilde{B}'_{j,k}(t) \left[\boldsymbol{w}_j, \boldsymbol{v}_j\right] + \mathbf{Ad}_{\exp(-\tilde{B}_{j,k}(t)\boldsymbol{v}_j)}\, \boldsymbol{q}_{j-1} + \tilde{B}^{(2)}_{j,k}(t)\boldsymbol{v}_j, \qquad j = i^*+1, \dots, i^*+k, \\ \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}^r \boldsymbol{x}_t &= \boldsymbol{q}_{i^*+k}. \end{aligned} \tag{19.29}$$

19.3.3. Derivatives w.r.t. Control Points

Finally it can be useful to express the derivative of $\boldsymbol{x}(t)$ with respect to the control point values $\boldsymbol{x}_j$. Recall that

$$\boldsymbol{x}(t) = \boldsymbol{x}_{i^*-k} \circ \prod_{j=i^*-k+1}^{i^*} \exp\left[\tilde{B}_{j,k}(t)\boldsymbol{v}_j\right], \tag{19.30}$$

so it is again just a matter of differentiating.

Let $\boldsymbol{s}_j = \tilde{B}_{j,k}(t)\boldsymbol{v}_j$, we then have that $\boldsymbol{x}(t) = x_{i^*-k} \circ \prod_{j=i^*-k+1}^{i^*} \exp(s_j)$. Derivatives with respect to the terms are

$$\mathrm{d}^r(\boldsymbol{s}_j)_{x_j} \overset{(5.50)}{=} \tilde{B}_{j,k}(t)\left[\mathrm{d}^r \exp_{\boldsymbol{v}_j}\right]^{-1} \implies \mathrm{d}^r\left(\exp(\boldsymbol{s}_j)\right)_{\boldsymbol{x}_j} = \tilde{B}_{j,k}(t)\left[\mathrm{d}^r \exp_{\boldsymbol{s}_j}\right]\left[\mathrm{d}^r \exp_{\boldsymbol{v}_j}\right]^{-1}, \tag{19.31a}$$

$$\mathrm{d}^r(\boldsymbol{s}_j)_{x_{j-1}} \overset{(5.51)}{=} -\tilde{B}_{j,k}(t)\left[\mathrm{d}^l \exp_{\boldsymbol{v}_j}\right]^{-1} \implies \mathrm{d}^r\left(\exp(\boldsymbol{s}_j)\right)_{\boldsymbol{x}_{j-1}} = -\tilde{B}_{j,k}(t)\left[\mathrm{d}^r \exp_{\boldsymbol{s}_j}\right]\left[\mathrm{d}^l \exp_{\boldsymbol{v}_j}\right]^{-1}. \tag{19.31b}$$

Thus the derivatives $\boldsymbol{r}_j \coloneqq \mathrm{d}^r\boldsymbol{x}(t)_{\boldsymbol{x}_j}$ of $\boldsymbol{x}$ become (where the $\bar{z}$'s are constant w.r.t. the differentiation variable)

$$\begin{aligned}
\boldsymbol{r}_{i^*} &= \mathrm{d}^r\left(\bar{\boldsymbol{z}} \circ \exp\left[\boldsymbol{s}_{i^*}\right]\right)_{\boldsymbol{x}_{i^*}} \overset{(5.12)}{=} \mathrm{d}^r \exp\left(\boldsymbol{s}_{i^*}\right)_{\boldsymbol{x}_{i^*}} \overset{(19.31a)}{=} \tilde{B}_{i^*,k}\mathrm{d}^r \exp_{\boldsymbol{s}_{i^*}}\left[\mathrm{d}^r \exp_{\boldsymbol{v}_{i^*}}\right]^{-1}, \\
\boldsymbol{r}_j &= \mathrm{d}^r\left(\bar{z}_1 \circ \exp\left[\boldsymbol{s}_j\right] \circ \exp\left[\boldsymbol{s}_{j+1}\right] \circ \bar{z}_2\right)_{\boldsymbol{x}_j} \overset{(5.12)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}}\, \mathrm{d}^r\left(\bar{z}_1 \circ \exp\left[\boldsymbol{s}_j\right] \circ \exp\left[\boldsymbol{s}_{j+1}\right]\right)_{\boldsymbol{x}_j} \\
&\overset{(5.12)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}}\left(\mathbf{Ad}_{\exp(-\boldsymbol{s}_{j+1})}\, \mathrm{d}^r(\bar{z}_1 \circ \exp(\boldsymbol{s}_j))_{\boldsymbol{x}_j} + \mathrm{d}^r(\exp(\boldsymbol{s}_{j+1}))_{\boldsymbol{x}^j}\right) \\
&\overset{(5.12)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}}\left(\mathbf{Ad}_{\exp(-\boldsymbol{s}_{j+1})}\, \mathrm{d}^r(\exp(\boldsymbol{s}_j))_{\boldsymbol{x}_j} + \mathrm{d}^r(\exp(\boldsymbol{s}_{j+1}))_{\boldsymbol{x}^j}\right) \\
&\overset{(19.31)}{=} \mathbf{Ad}_{\bar{z}_2^{-1}}\left(\tilde{B}_{j,k}(t)\, \mathbf{Ad}_{\exp(-s_{j+1})}\left[\mathrm{d}^r \exp_{\boldsymbol{s}_j}\right]\left[\mathrm{d}^r \exp_{\boldsymbol{v}_j}\right]^{-1} - \tilde{B}_{j+1,k}(t)\left[\mathrm{d}^r \exp_{\boldsymbol{s}_{j+1}}\right]\left[\mathrm{d}^l \exp_{\boldsymbol{v}_{j+1}}\right]^{-1}\right), \\
\boldsymbol{r}_{i^*-k} &= \mathrm{d}^r\left(\boldsymbol{x}_{i^*-k} \circ \exp\left[\boldsymbol{s}_{i^*-k+1}\right] \circ \bar{z}_2^{-1}\right) = \\
&= \mathbf{Ad}_{\bar{z}_2^{-1}}\left(\mathbf{Ad}_{\exp(-\boldsymbol{s}_{i^*-k+1})} - \tilde{B}_{i^*-k+1}(t)\mathrm{d}^r \exp_{\boldsymbol{s}_{i^*-k+1}}\left[\mathrm{d}^r \exp_{v_{i^*-k+1}}\right]^{-1}\right).
\end{aligned}$$

19.4. General Coefficient Recursion

We seek an expression for $M_{i^*,k}$ which via (19.22) immediately gives $\tilde{M}_{i^*,k}$ that allows easy evaluation of the basis functions $\tilde{B}_{i,j}$. Inserting the basis expansion (19.17) into the recursive definition (19.16) yields

$$
\begin{aligned}
\sum_{j=0}^{k} \alpha_{i,k}^j u^j &=: B_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t) \\
&= \left[\frac{t_{i^*} - t_i}{t_{i+k} - t_i} + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} u \right] B_{i,k-1}(t) + \left[\frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} u \right] B_{i+1,k-1}(t) \\
&= \frac{t_{i^*} - t_i}{t_{i+k} - t_i} \sum_{j=0}^{k-1} \alpha_{i,k-1}^j u^j + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} \sum_{j=0}^{k-1} \alpha_{i,k-1}^j u^{j+1} \\
&+ \frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=0}^{k-1} \alpha_{i+1,k-1}^j u^j - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=0}^{k-1} \alpha_{i+1,k-1}^j u^{j+1} \\
&= \frac{t_{i^*} - t_i}{t_{i+k} - t_i} \sum_{j=0}^{k-1} \alpha_{i,k-1}^j u^j + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} \sum_{j=1}^{k} \alpha_{i,k-1}^{j-1} u^j + \\
&\frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=0}^{k-1} \alpha_{i+1,k-1}^j u^j - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \sum_{j=1}^{k} \alpha_{i+1,k-1}^{j-1} u^j \\
&= \sum_{j=0}^{k} \left[\frac{t_{i^*} - t_i}{t_{i+k} - t_i} \alpha_{i,k-1}^j + \frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i} \alpha_{i,k-1}^{j-1} + \frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \alpha_{i+1,k-1}^j - \frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}} \alpha_{i+1,k-1}^{j-1} \right] u^j,
\end{aligned}
$$

with the convention that $\alpha_{i,k}^j = 0$ for $j < 0$ and for $j > k$. By matching coefficients we therefore have that

$$
\alpha_{i,k}^j = \underbrace{\frac{t_{i^*} - t_i}{t_{i+k} - t_i}}_{=:\tilde{\beta}_{i,i^*,k}} \alpha_{i,k-1}^j + \underbrace{\frac{t_{i^*+1} - t_{i^*}}{t_{i+k} - t_i}}_{=:\tilde{\gamma}_{i,i^*,k}} \alpha_{i,k-1}^{j-1} + \underbrace{\frac{t_{i+k+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}}}_{1-\tilde{\beta}_{i+1,i^*,k}} \alpha_{i+1,k-1}^j - \underbrace{\frac{t_{i^*+1} - t_{i^*}}{t_{i+k+1} - t_{i+1}}}_{\tilde{\gamma}_{i+1,iz*,k}} \alpha_{i+1,k-1}^{j-1}, \tag{19.32}
$$

or equivalently that for $N_{i,k}$ as in (19.18),

$$
\begin{aligned}
N_{i,k} &= \tilde{\beta}_i \begin{bmatrix} N_{i,k-1} \\ 0 \end{bmatrix} + \tilde{\gamma}_i \begin{bmatrix} 0 \\ N_{i,k-1} \end{bmatrix} + (1 - \tilde{\beta}_{i+1,i^*,k}) \begin{bmatrix} N_{i+1,k-1} \\ 0 \end{bmatrix} - \tilde{\gamma}_{i+1,i^*,k} \begin{bmatrix} 0 \\ N_{i+1,k-1} \end{bmatrix} \\
&= \begin{bmatrix} N_{ik-1} & N_{i+1,k-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\beta}_i \\ 1 - \tilde{\beta}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ N_{i,k-1} & N_{i+1,k-1} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{i,i^*,k} \\ -\tilde{\gamma}_{i+1,i^*,k} \end{bmatrix},
\end{aligned} \tag{19.33}
$$

For convenience we re-define β and γ as

$$
\beta_{j,i^*,k} := \tilde{\beta}_{i^*-j,i^*,k} = \frac{t_{i^*} - t_{i^*-j}}{t_{i^*-j+k} - t_{i^*-j}}, \quad \gamma_{j,i^*,k} := \tilde{\gamma}_{i^*-j,i^*,k} = \frac{t_{i^*+1} - t_{i^*}}{t_{i^*-j+k} - t_{i^*-j}} \tag{19.34}
$$

Now we can write down a recursive formula for $M_{i^*,k}$ as given in (19.18):

$$
\begin{aligned}
M_{i^*,0} &= \begin{bmatrix} 1 \end{bmatrix}, \\
M_{i^*,k} &= \begin{bmatrix} M_{i^*,k-1} \\ 0 \end{bmatrix} \begin{bmatrix} 1-\beta_{k-1,i^*,k} & \beta_{k-1,i^*,k} & 0 & \cdots & 0 \\ 0 & 1-\beta_{k-2,i^*,k} & \beta_{k-2,i^*,k} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1-\beta_{0,i^*,k} & \beta_{0,i^*,k} \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ M_{i^*,k-1} \end{bmatrix} \begin{bmatrix} -\gamma_{k-1,i^*,k} & \gamma_{k-1,i^*,k} & 0 & \cdots & 0 \\ 0 & -\gamma_{k-2,i^*,k} & \gamma_{k-2,i^*,k} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\gamma_{0,i^*,k} & \gamma_{0,i^*,k} \end{bmatrix}.
\end{aligned} \tag{19.35}
$$

However, close to the endpoints $\beta_{j,i^*,k}$ and $\gamma_{j,i^*,k}$ can no longer be evaluated. We can introduce artificial boundary knot points—$k-1$ to the left and $k-2$ on the right—to ensure that all splines have full support. Then β and γ can be computed using the expressions

$$
\beta_{j,i^*,k} = \frac{t_{i^*} - t_{\max(i^*-j,0)}}{t_{\min(i^*-j+k,n)} - t_{\max(i^*-j,0)}}, \quad \gamma_{j,i^*,k} = \frac{t_{i^*+1} - t_{i^*}}{t_{\min(i^*-j+k,n)} - t_{\max(i^*-j,0)}} \tag{19.36}
$$

that are valid for all indices $0 \leq i^* < n$.

19.5. Cardinal Cofficient Recursion

When all control points with indices $i^*-k+1, \ldots, i^*$ are equally spaced such that $t_{i+1} - t_i = \Delta t$ for all i the expression can be simplified and $M_{i^*,k}$ no longer depends on i^* for interior points. In this case we have that

$$
\beta_{j,i^*,k} = \frac{j\Delta t}{k\Delta t} = \frac{j}{k}, \quad \gamma_{j,i^*,k} = \frac{1}{k}. \tag{19.37}
$$

We can use this to retrieve the first couple of matrices:

$$
M_{i^*,0} = \begin{bmatrix} 1 \end{bmatrix}, \tag{19.38a}
$$

$$
M_{i^*,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1-\beta_{i^*,1} & \beta_{i^*,1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -\gamma_1 & \gamma_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \tag{19.38b}
$$

$$
M_{i^*,2} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1-\beta_{i^*-1,2} & \beta_{i^*-1,2} & 0 \\ 0 & 1-\beta_{i^*,2} & \beta_{i^*,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\gamma_2 & \gamma_2 & 0 \\ 0 & -\gamma_2 & \gamma_2 \end{bmatrix} = \frac{1}{2!} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \tag{19.38c}
$$

$$
M_{i^*,3} = \ldots = \frac{1}{3!} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}. \tag{19.38d}
$$

Close to the boundary the formulas (19.36) should instead be applied.

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