Synthesis of separable controlled invariant sets with tunable complexity

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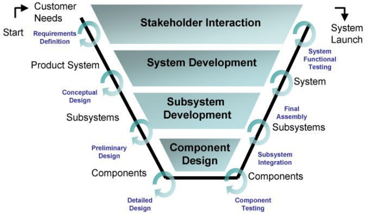
Why formal methods?



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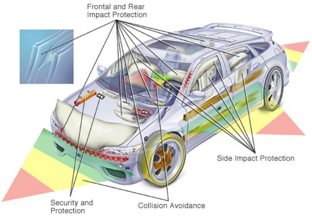

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- Vision: “Specify-and-compile” instead of “test-and-tune”

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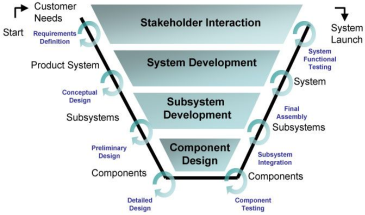

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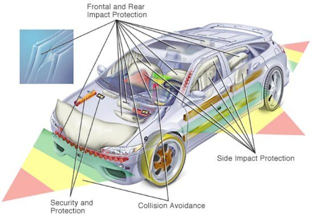

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- Vision: "Specify-and-compile" instead of "test-and-tune"
- Performance guarantees: explicit treatment of requirements and assumptions

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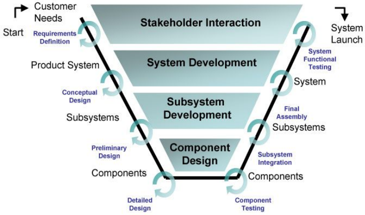

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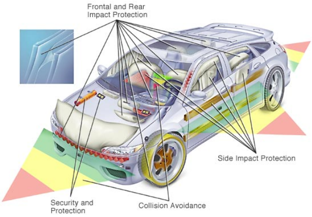
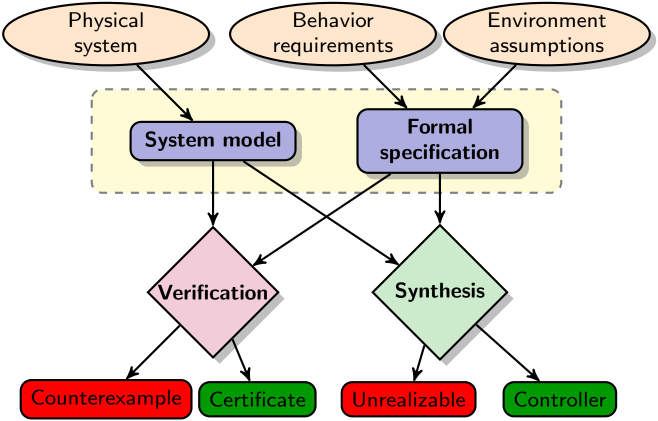


Image credit: MIT SDM

- Vision: "Specify-and-compile" instead of "test-and-tune"
- Performance guarantees: explicit treatment of requirements and assumptions
- Composition: formal contracts enable modular design

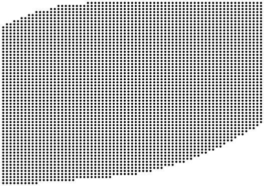
Verification and synthesis

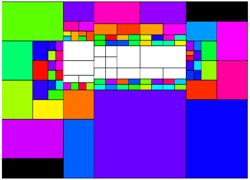


- Formal guarantees are essentially set membership guarantees
- Formal synthesis relies heavily on set computations

Issue: Curse of dimensionality

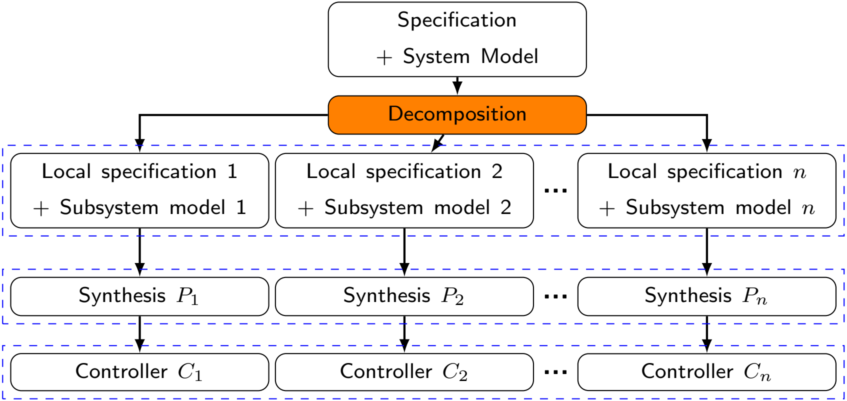
- State of the art in formal synthesis:
  - Bisimulation-based abstractions (e.g., Mazo Jr, Davitian, and Tabuada, 2010)
  - Partition-based abstractions (e.g., Liu et al., 2013)
  - Hamilton-Jacobi-Bellman methods + numerical PDE solving (e.g., Mitchell, Bayen, and Tomlin, 2005)
  - Occupation measures + SoS (e.g., Shia et al., 2014)
- With increasing dimensionality:
  - Abstraction size and partition complexity grow exponentially
  - Precise numerical PDE solving becomes intractable
  - SoS give large SDP's





Approach: Decomposition

- Solve smaller subproblems such that composition is correct



With compositional correctness guarantees

- Inspiration: ACC + Lane keeping + active steering + ...

Decomposition of dynamically coupled systems

$$\boxed{\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}$$

- Natural decomposition if A_{12} and A_{21} are “small”

Decomposition of dynamically coupled systems

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$$\boxed{\begin{aligned} x_1^+ &= A_{11}x_1 + B_1u_1 \\ &\underbrace{+A_{12}x_2 + E_2d_2}_{\text{Disturbance}} \end{aligned}} \quad \underset{\longrightarrow}{\overset{\longleftarrow}{\text{Disturbance}}} \quad \boxed{\begin{aligned} x_2^+ &= A_{22}x_2 + B_2u_2 \\ &\underbrace{+A_{21}x_1 + E_2d_2}_{\text{Disturbance}} \end{aligned}}$$

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- Each subsystem needs to be robust w.r.t. influence from other subsystems

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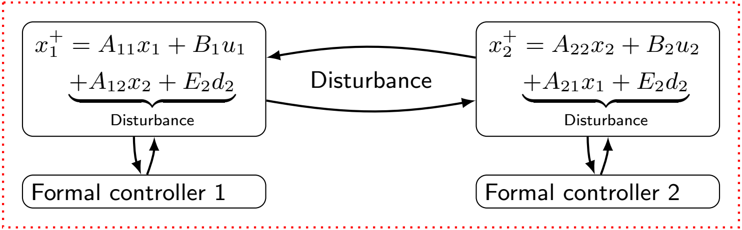

- Natural decomposition if A_{12} and A_{21} are "small"
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Outline

System description

Discrete time linear system model:

$$x_i^+ = A_{ii}x_i + \sum_{j \neq i} A_{ij}x_j + B_i u_i + E_i d_i$$

- Decomposition given a priori
- Coupling between subsystems
- Unique input and exogenous disturbance for each subsystem

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Constraints and assumptions:

- Linear input bounds: $H_u^i u_i \leq h_u^i$
- Disturbance assumptions: $-1 \leq H_d^i d_i \leq 1$

Problem statement

Problem (Separable controlled invariance)

Given d dynamically coupled linear subsystems with input, state, and disturbance constraints, find sets $\{\mathcal{X}_i\}_{i=1}^d$ such that

- States are constrained: $\mathcal{X}_i \subset \mathcal{S}_i := \{x_i : H_x^i x_i \leq h_x^i\}$
- $\mathcal{X}_i$ is robustly controlled invariant with respect both to coupling and exogenous disturbance:

$$\forall x_i \in \mathcal{X}_i \;\; \exists u_i \in \mathcal{U}_i \;\; \forall x_j \in \mathcal{X}_j \;\; \forall d_i \in \mathcal{D}_i, \; x_i^+ \in \mathcal{X}_i$$

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We restrict the search to $\mathcal{X}_i$'s that

- Are symmetric zonotopes: $\mathcal{X}_i = \left\{x_i : -\mathbf{1} \leq Z_i H_x^i x_i \leq \mathbf{1}\right\}$
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  - Z_i is a given matrix of (arbitrarily many) zonotope generators
- Can be rendered invariant by a local feedback controller $u_i = \sum_j K_{ij} x_j$
  - Non-zero K_{ij} indicates information availability

Problem features

- Controlled invariant sets $\{\mathcal{X}_i\}_{i=1}^d$ provide an assume guarantee protocol for invariance: system i stays in $\mathcal{X}_i$ as long as system j $(j \neq i)$ stays in $\mathcal{X}_j$
- Enables decoupled formal synthesis for more sophisticated control objectives can be performed within robust controlled invariant sets
- Trade-off: larger sets impose more disturbance on neighboring systems

Outline

Main result

Theorem

If there exist matrices H_x^{-1}, $\hat{K}$, $\Lambda \succ 0$, $D_x^j \succ 0$, Φ_j^{-1}, Γ_j, Ξ_j, Ψ_j, Ω_j^1, Ω_j^2 *with certain structures*[1] *for all* $j = 1, \ldots, \mathcal{N}_x$, *diagonal* $D_s^k \succ 0$ *for all* $k = 1, \ldots, \mathcal{N}_s$, *and diagonal* $D_u^l \succ 0$ *for all* $l = 1, \ldots, \mathcal{N}_u$ *such that a set of linear matrix inequalities hold, then the block diagonal pair* $(H_x, \hat{K} H_x)$ *constitutes a solution to the separable controlled invariance problem when appropriately decomposed.*

- LMI feasibility condition
- $\mathcal{N}_x, \mathcal{N}_s, \mathcal{N}_u$ are the total numbers of zonotope generators, state constraints, and input constraints, respectively
- Size of matrix variables linear in total system dimension

[1] such as block diagonal, symmetric, diagonal

LMI's

$$\left.\begin{aligned}
&\begin{bmatrix} \Xi_j - \Phi_j^{-1} & \Omega_j^1 - \begin{bmatrix} H_x^{-1} & 0 \\ 0 & H_x^{-1} \end{bmatrix} & \Omega_j^2 - \begin{bmatrix} H_x^{-1} & 0 \\ 0 & H_x^{-1} \end{bmatrix} & \Psi_j^T \begin{bmatrix} Z^T e_j \\ Z^T e_j \end{bmatrix} \\ * & 2\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} - \Gamma_j & \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} + (\Omega_j^1)^T - \Psi_j & 0 \\ * & * & \Omega_j^2 + (\Omega_j^2)^T - \Xi_j & 0 \\ * & * & * & \lambda_{i(j)} - \mathbf{1}^T D_x^j \mathbf{1} - \mathbf{1}^T D_d^j \mathbf{1} \end{bmatrix} \succ 0 \\
&\begin{bmatrix} \Gamma_j & \Psi_j \\ * & \Xi_j \end{bmatrix} \succ 0 \\
&\begin{bmatrix} Z^T D_x^j Z & 0 & -\frac{1}{2}(H_x^{-T} A^T + \hat{K}^T B^T) & 0, \\ * & D_d^j & 0 & -\frac{1}{2} H_d^{-T} E^T \\ * & & \left[\Phi_j^{-1}\right] & \end{bmatrix} \succ 0
\end{aligned}\right\} \text{(inv.)}$$

$$\begin{bmatrix} Z^T D_s^k Z & -\frac{1}{2} H_x^{-T} H_s^T e_k \\ * & e_k^T h_s - \mathbf{1}^T D_s^k \mathbf{1} \end{bmatrix} \succ 0 \quad \text{(state constraints)}$$

$$\begin{bmatrix} Z^T D_u^l Z & -\frac{1}{2} \hat{K}^T H_u^T e_l \\ * & e_l^T h_u - \mathbf{1}^T D_u^l \mathbf{1} \end{bmatrix} \succ 0 \quad \text{(input constraints)}$$

- Desired variables
- "S Procedure matrices"
- Slack variables

Proof outline

- Ideas borrowed from (Tahir and Jaimoukha, 2015, TAC)

Want to find $\mathcal{X}$ s.t. that $A_K \mathcal{X} \oplus E\mathcal{D} \subset \mathcal{X}$, i.e.,

$$e_j^T Z H_x (A_K x + Ed) - 1 \leq 0$$

for all $x \in \mathcal{X}$, $d \in \mathcal{D}$ and for all $j = 1, \ldots \mathcal{N}_x$.

Proof outline

- Express set membership as quadratic inequality

Lemma

$x \in \mathcal{X} = \{x : -\mathbb{1} \leq ZH_x x \leq \mathbb{1}\}$ *if and only if for all diagonal* $D_x \succ 0$

$$(\mathbb{1} - ZH_x x)^T D_x (\mathbb{1} + ZH_x x) \geq 0.$$

$d \in \mathcal{D} = \{d : -\mathbb{1} \leq H_d d \leq \mathbb{1}\}$ *if and only if for all diagonal* $D_d \succ 0$

$$(\mathbb{1} - H_d d)^T D_d (\mathbb{1} + H_d d) \geq 0.$$

Proof outline

- "For all" quantifiers can be expressed as matrix inequalities using the S Procedure

$$
\begin{aligned}
&e_j^T Z H_x (A_K x + Ed) - 1 \\
&\quad = -(\mathbb{1} - Z H_x x)^T \tilde{D}_x^j (\mathbb{1} + Z H_x x) \\
&\quad -(\mathbb{1} - Z H_d d)^T \tilde{D}_d^j (\mathbb{1} + Z H_d d) \\
&\quad - \begin{bmatrix} x^T & d^T & 1 \end{bmatrix} L_x^j(\tilde{D}_x^j, \tilde{D}_d^j) \begin{bmatrix} x^T & d^T & 1 \end{bmatrix}^T
\end{aligned}
$$

Lemma

$e_j^T Z H_x (A_K x + Ed) - 1 \leq 0$ *for all* $x \in \mathcal{X}$, $d \in \mathcal{D}$ *if and only if there are* $\tilde{D}_x^j \succ 0$, $\tilde{D}_d^j \succ 0$ *s.t.* $L_x^j(\tilde{D}_x^j, \tilde{D}_d^j) \succ 0$

Proof outline

- Use slack variable techniques to eliminate matrix products

Lemma

Let R (sym), Z (sym), A (full) and B (full) be arbitrary matrices. Then the following conditions are equivalent

1. $$\begin{bmatrix} R & AB \\ * & Z \end{bmatrix} \succ 0.$$

2. $$\exists X \text{ (sym)} : \begin{bmatrix} R & A \\ * & X^{-1} \end{bmatrix} \succ 0, \begin{bmatrix} X & B \\ * & Z \end{bmatrix} \succ 0$$

Proof outline

- Use slack variable techniques to eliminate matrix products

Lemma

If there exist Θ (full), Γ (sym) and Ξ (sym) such that

$$\Delta \doteq \begin{bmatrix} \Gamma & Y \\ * & \Xi \end{bmatrix} \succ 0, \quad \begin{bmatrix} Z+\Xi & [-X \ \ I]\Theta & V \\ * & \Theta+\Theta^T-\Delta & 0 \\ * & * & W \end{bmatrix} \succ 0,$$

then

$$\begin{bmatrix} Z+XY+Y^TX^T & V \\ * & W \end{bmatrix} \succ 0.$$

- LMI's are obtained by restricting shape of certain matrices
  - Introduces some conservatism

Outline

Ex 1: Advantage of Zonotopes

Subsystem 1

$$\begin{bmatrix} x_1^+ \\ y_1^+ \end{bmatrix} = 0.8R\left(\frac{\pi}{4}\right)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} u_1^x \\ u_2^y \end{bmatrix} + 0.1\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} d_1^x \\ d_1^y \end{bmatrix}$$

Subsystem 2

$$\begin{bmatrix} x_2^+ \\ y_2^+ \end{bmatrix} = 0.8R\left(\frac{\pi}{4}\right)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} u_2^x \\ u_2^y \end{bmatrix} + 0.1\begin{bmatrix} x_1 + x_3 \\ y_1 + y_3 \end{bmatrix} + \begin{bmatrix} d_2^x \\ d_2^y \end{bmatrix}$$

Subsystem 3

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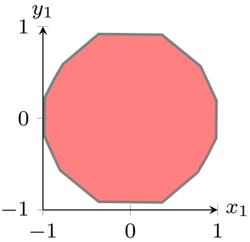

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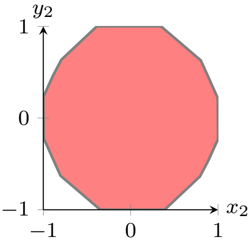

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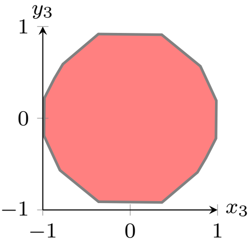

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Subsystem 1	Subsystem 2	Subsystem 3
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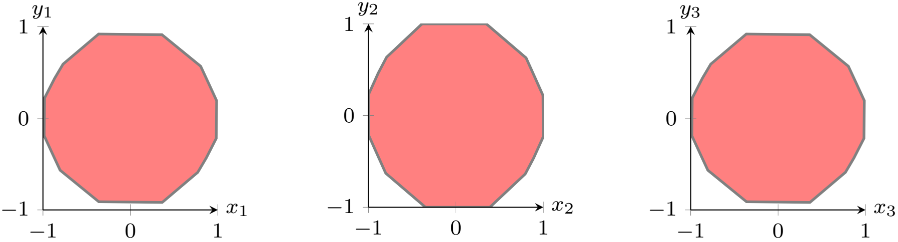
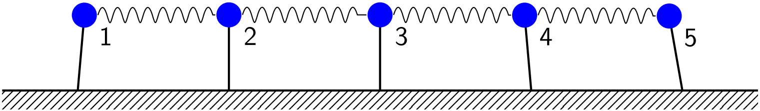


- Infeasible with hyperboxes as in (Tahir and Jaimoukha, 2015, TAC)

Ex 2: Connected inverted pendulums



- Appears in literature on decentralized stabilization
- Each pendulum has two states, 10-dimensional system
- Neighbor state sharing

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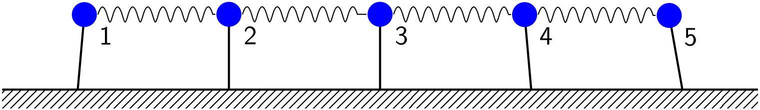

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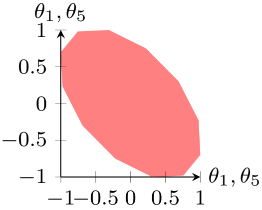
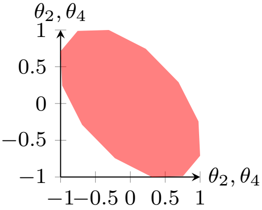
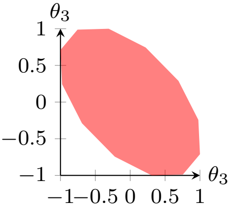
Robust controlled invariant sets:



Pendulums 1,5

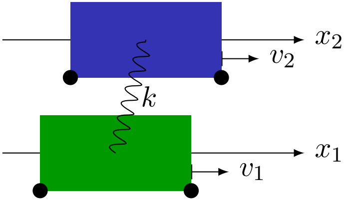


Pendulums 2,4



Pendulum 3

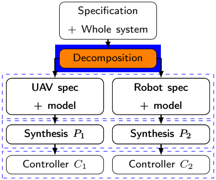
Ex 3: Interconnected ground robot and UAV



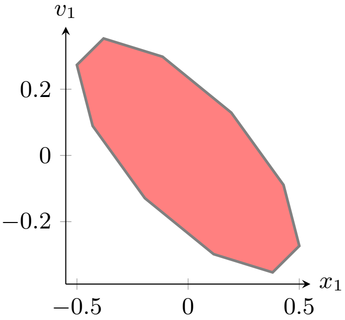
- Toy example of 1D-robots connected by a spring
- Integrator dynamics

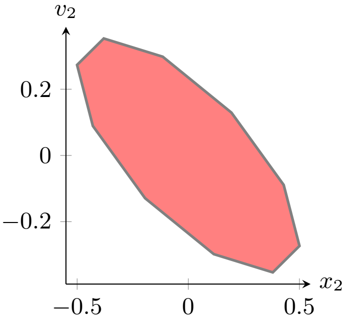
$$
\begin{aligned}
{x_1}^+ &= x_1 && +v_1 \\
{v_1}^+ &= kx_1 && +v_1 - kx_2 \\
{x_2}^+ &= && + x_2 \quad +v_2 \\
{v_2}^+ &= -kx_1 && + kx_2 \quad +v_2
\end{aligned}
$$

Ex 3: Invariant sets

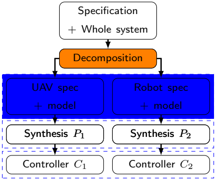


1. Find invariant sets



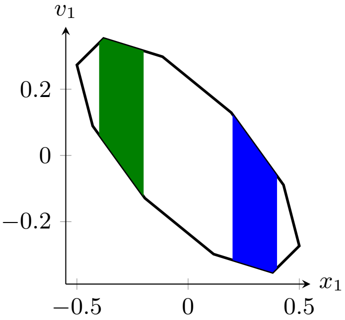


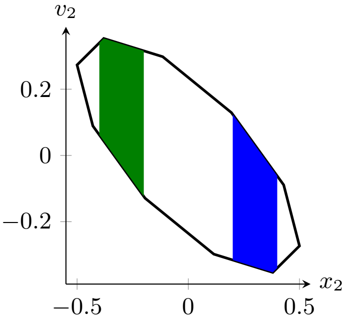
Ex 3: Invariant sets



1. Find invariant sets
2. Do local synthesis for

$$\bigwedge_{i=1}^{2} \Box\Diamond \left(x_i \in {\color{green}\blacksquare}\right) \wedge \Box\Diamond \left(x_i \in {\color{blue}\blacksquare}\right)$$

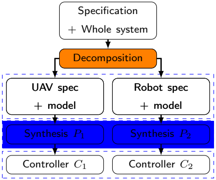


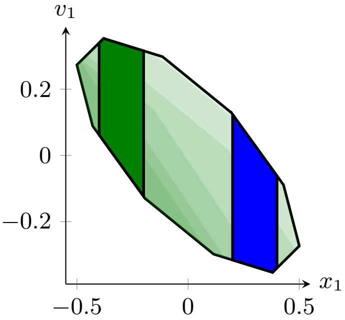


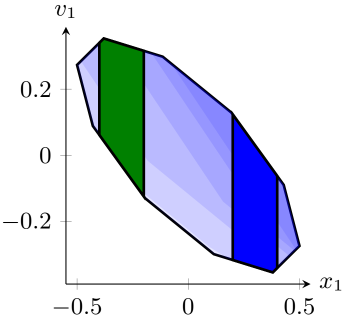
Ex 3: Invariant sets

1. Find invariant sets
2. Do local synthesis for

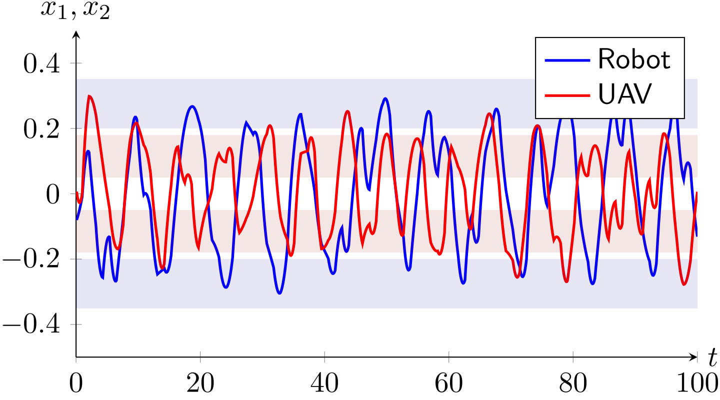
$$\bigwedge_{i=1}^{2} \Box\Diamond (x_i \in \text{green}) \wedge \Box\Diamond (x_i \in \text{blue})$$





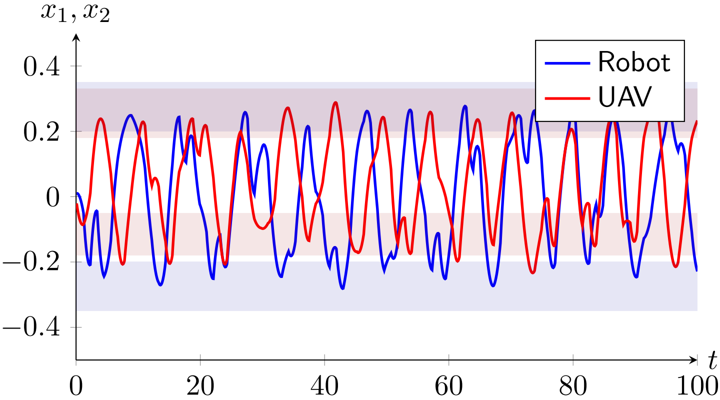


Ex 3: local synthesis



- Visit both green and blue areas infinitely often
- Solving two 2-dimensional problems instead of one 4-dimensional

Ex 3: local synthesis

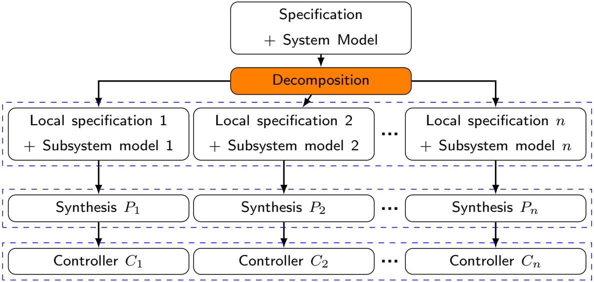


- Can re-synthesize for different specification without re-computing sets

Outline

Main takeaways:

- Decoupled formal synthesis through separable controlled invariant set computation
- Tunable complexity
- Handles input and state constraints, exogenous disturbance
- Flexible w.r.t. information sharing patterns



With compositional correctness guarantees

Main takeaways:

- Decoupled formal synthesis through separable controlled invariant set computation
- Tunable complexity
- Handles input and state constraints, exogenous disturbance
- Flexible w.r.t. information sharing patterns

Future work:

- Exploit sparsity to make LMI solving more efficient
- Application to more realistic systems

Thank you for your attention