Maximizing the Time of Invariance for Large Collections of Switched Systems

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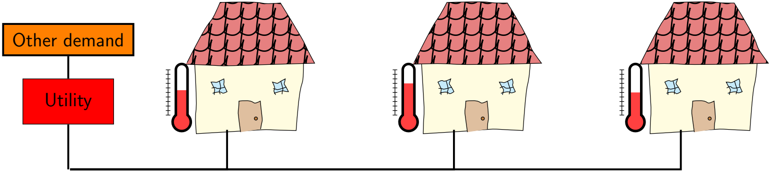
Formal methods in control

- Fundamental problem in synthesis and verification: curse of dimensionality
- Need to exploit structure and symmetry
- Find an appropriate problem **abstraction**
- Common abstraction: state-space discretization
- This work: control large collections of heterogeneous subsystems
- Time domain abstraction to “abstract away” heterogeneity

Thermostatically Controlled Load (TCL) Scheduling

- A TCL can be in mode `on` or `off`
- (Local) state constraints: Each TCL should maintain temperature within a desired temperature range
- (Global) counting constraint: Aggregate electricity consumption should be controlled over time
- The flexibility in individual specifications can be leveraged to control aggregate demand to for instance mitigate fluctuations



- How to schedule on/off cycles to meet both local and global constraints?

TCL Mode-Counting Problem

- N (heterogeneous) subsystems $x^i \in \mathbb{R}$ with switched dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} x^i(t) = \begin{cases} f^i_{\texttt{off}}\left(x^i(t)\right) & \text{if } \sigma^i(t) = \texttt{off}, \\ f^i_{\texttt{on}}\left(x^i(t)\right) & \text{if } \sigma^i(t) = \texttt{on}, \end{cases}$$

- Local (heterogeneous) state constraints:

$$\underline{a}^i \leq x^i(t) \leq \overline{a}^i \quad \forall t \geq 0 \tag{1}$$

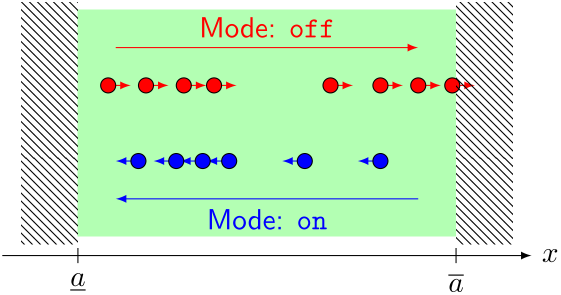
- Global counting constraint:

$$\underline{K}_{\texttt{on}} \leq \sum_{i:\ \sigma^i(t) = \texttt{on}} 1 \leq \overline{K}_{\texttt{on}} \quad \forall t \geq 0 \tag{2}$$

- Objective: find switching strategy $\{\sigma^i(\cdot)\}_{i=1}^N$ that enforces (1)-(2).

Problem Illustration

- Assumption: $f^i_{\texttt{on}}$ strictly negative, $f^i_{\texttt{off}}$ strictly positive



Previous work on counting problems

- HSCC'17: complete analytic solution of **infinite-horizon TCL mode-counting problem**
  - Main result: problem has solution if and (almost) only if

$$\sum_{i=1}^{N} \frac{\mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}(\underline{a}^i)}{1 + \mathcal{L}^i_{\mathtt{off}} T^i_{\mathtt{on}}(\underline{a}^i)} > \underline{K}_{\mathtt{on}}, \quad \sum_{i=1}^{N} \frac{\mathcal{L}^i_{\mathtt{on}} T^i_{\mathtt{off}}(\overline{a}^i)}{1 + \mathcal{L}^i_{\mathtt{on}} T^i_{\mathtt{off}}(\overline{a}^i)} > N - \overline{K}_{\mathtt{on}}$$

  - Resulted in aggregate flexibility $(323 - 250)/1000 \approx 7\%$ in case study. In practice aggregate flexibility upwards of 60% may be required but for short durations
- Here: solution to **finite-horizon TCL mode-counting problem**

Finite-horizon TCL Mode-Counting Problem

- Local (heterogeneous) state constraints:

$$\underline{a}^i \leq x^i(t) \leq \overline{a}^i \quad \forall t \in [0, T_I] \tag{3}$$

- Global counting constraint:

$$\underline{K}_{\text{on}} \leq \sum_{i:\ \sigma^i(t)=\text{on}} 1 \leq \overline{K}_{\text{on}} \quad \forall t \in [0, T_I] \tag{4}$$

- Objective: find switching strategy $\{\sigma^i(\cdot)\}_{i=1}^N$ that enforces (3)-(4) and s.t. T_I is maximized

- **Time of Invariance** T_I will be a function of initial condition
- Previous results [HSCC'17] identify when $T_I = +\infty$
- **Only one bound relevant at a time**, we consider $\leq \overline{K}_{\text{on}}$

Outline

Reformulation as Optimal Control Problem I

- "Convexified" subsystem dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}x^i(t) = (1-\alpha^i(t))f^i_{\mathtt{off}}\left(x^i(t)\right) + \alpha^i(t)f^i_{\mathtt{on}}\left(x^i(t)\right).$$

- $\alpha^i(t) \in [0,1]$ instead of hybrid on/off system
- Aggregate dynamics in vector notation:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = (\mathbf{1}_N - \alpha(t)) \odot \mathbf{f}_{\mathtt{off}}(\mathbf{x}(t)) + \alpha(t) \odot \mathbf{f}_{\mathtt{on}}(\mathbf{x}(t)),\\
&\mathbf{x}(0) = \mathbf{x}_0,\\
&\alpha : \mathbb{R}^+ \to \Delta_N\left(\overline{K}_{\mathtt{on}}\right),
\end{aligned}$$

- Aggregate safe set: $S = \prod_i [\underline{a}^i, \overline{a}^i]$

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\end{aligned}$$

Elementwise multiplication

$\{\alpha : \sum_i \alpha_i = \overline{K}_{\mathtt{on}}, 0 \le \alpha_i \le 1\}$

- Aggregate safe set: $S = \prod_i [\underline{a}^i, \overline{a}^i]$

Reformulation as Optimal Control Problem II

- Want to maximize time of constraint satisfaction:

$$V^*(\mathbf{x}_0) = \sup_{\alpha(\cdot)} \left\{ \int_0^\infty \mathbb{1}_S(\mathbf{x}(s))\,\mathrm{d}s \text{ s.t. aggregate dynamics} \right\}.$$

- Obvious that optimal TOI $T_I^*(\mathbf{x}_0) \leq V^*(\mathbf{x}_0)$. **We conjecture equality.**
- $V^*(\mathbf{x}_0)$ viscosity solution to HJB PDE

$$\max_{\alpha \in \Delta_N(\overline{K}_{\texttt{on}})} \sum_{i \in [N]} \frac{\partial V(\mathbf{x})}{\partial x^i} \begin{bmatrix} (1-\alpha^i) f^i_{\texttt{off}}(x^i) \\ +\alpha^i f^i_{\texttt{on}}(x^i) \end{bmatrix} = -\mathbb{1}_S(\mathbf{x}).$$

- Control extraction:

$$\alpha^*(\mathbf{x}) \in \underset{\alpha \in \Delta_N(\overline{K}_{\texttt{on}})}{\arg\max} \sum_{i \in [N]} \alpha^i \frac{\partial V^*(\mathbf{x})}{\partial x^i} (f^i_{\texttt{on}}(x^i) - f^i_{\texttt{off}}(x^i)).$$

Method Outline

- Control extraction:

$$\alpha^*(\mathbf{x}) \in \underset{\alpha \in \Delta_N\left(\overline{K}_{\texttt{on}}\right)}{\arg\max} \sum_{i \in [N]} \alpha^i \frac{\partial V^*(\mathbf{x})}{\partial x^i} \left(f^i_{\texttt{on}}(x^i) - f^i_{\texttt{off}}(x^i)\right). \quad (5)$$

1. Construct an **analytical under-approximation** $T_I^-(\mathbf{x})$ of the optimal time of invariance $T_I^*(\mathbf{x})$.
2. Substitute $T_I^-(\mathbf{x})$ for $V^*(\mathbf{x})$ in (5) to construct a strategy C which achieves an unknown TOI $T_I^C(\mathbf{x}) > T_I^-(\mathbf{x})$.
3. Analytically bound the optimality gap between $V^*(\mathbf{x})$ and $T_I^-(\mathbf{x})$.

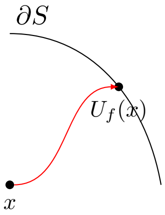
Time to Exit

Definition

Given a set S, for $x \in S$ and some f, the **time to exit** $T_f(x)$ is the time it takes for the flow of f starting in x to reach ∂S:

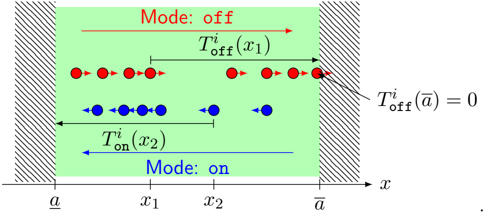
$$T_f\left(x\right) = \inf\left\{\tau : \phi_f(x, \tau) \in \partial S\right\}.$$

For $x \in S$ and f, the **exit point** $U_f(x)$ is $U_f(x) = \phi_f\left(x, T_f(x)\right)$.



Time to Exit: Linear Case

- Dynamics: $\dot{x}^i = (1-\alpha^i)\gamma^i(-x^i + \overline{b}^i) + \alpha^i\gamma^i(-x^i + \underline{b}^i)$.



- Exact time to exit for $\tilde{b}^i(\alpha^i) = \underline{b}^i\alpha^i + (1-\alpha^i)\overline{b}^i$:

$$T^i_{\texttt{off}}(x^i_0, \alpha^i) = \frac{1}{\gamma^i}\log\left(1 + \frac{\overline{a}^i - x^i_0}{\tilde{b}^i(\alpha^i) - \overline{a}^i}\right),$$

- Lower bound based on inequality $\log(1+x) \geq 2x/(2+x)$:

$$T^{i-}_{\texttt{off}}(x^i_0, \alpha^i) = \frac{1}{\gamma^i}\left(\frac{2(\overline{a}^i - x^i_0)}{2(\overline{b}^i - \alpha^i(\overline{b}^i - \underline{b}^i)) - \overline{a}^i + x^i_0}\right).$$

Analytic Approximate Value Function

- **Idea**: select α^i such that

$$T^{j-}_{\texttt{off}}(x^j_0, \alpha^j) = T^{i-}_{\texttt{off}}(x^i_0, \alpha^i), \quad \forall i, j \in J$$

- This choice gives

$$\forall j, \quad T^{j-}_{\texttt{off}}(x^j_0, \alpha^j) = T^-_I(\mathbf{x}_0, J) := \frac{\displaystyle\sum_{i\in J} \frac{\overline{a}^i - x^i_0}{\gamma^i \left(\overline{b}^i - \underline{b}^i\right)}}{\displaystyle\sum_{i\in J} \frac{\overline{b}^i - \frac{1}{2}\left(\overline{a}^i + x^i_0\right)}{\overline{b}^i - \underline{b}^i} - \overline{K}_{\texttt{on}}}.$$

- Index set $J^*(\mathbf{x_0})$ recursively selected s.t. $\alpha^i \in (0, 1]$ for $i \in J^*$ and $\alpha^i = 0$ for $i \notin J^*$:

$$T^-_I(\mathbf{x}_0) := T^-_I(\mathbf{x}_0, J^*(\mathbf{x}_0)).$$

Results

- Let $T_I^C(\mathbf{x})$ be the TOI achieved by the control strategy induced by $T_I^-(\mathbf{x})$.

Theorem

$$\frac{1 - e^{-\bar{\epsilon} T_I^*(\mathbf{x})}}{\bar{\epsilon}} \leq T_I^-(\mathbf{x}) \leq T_I^C(\mathbf{x}) \leq T_I^*(\mathbf{x}),$$

- Practical issue: achieving TOI $T_I^C(\mathbf{x})$ might require excessive switching

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- Practical issue: achieving TOI $T_I^C(\mathbf{x})$ might require excessive switching
- "Lazy" strategy L achieves TOI $T_I^L(\mathbf{x})$

Theorem

$$T_I^L(\mathbf{x}) \geq \frac{1}{|\underline{\epsilon}|} \log(1 + |\underline{\epsilon}| T_I^-(\mathbf{x})).$$

Expressions for bounds

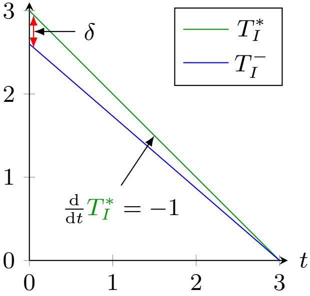
$$\overline{\epsilon} = \frac{\frac{1}{2}\left(\sum_{i\in J^*}\gamma^i \frac{\overline{b}^i - \overline{a}^i}{\overline{b}^i - \underline{b}^i}\frac{2+\gamma^i T_I^-(\mathbf{x}_0)}{2-\gamma^i T_I^-(\mathbf{x}_0)} - \min_{\alpha\in\Delta_N(\overline{K}_{\texttt{on}})}\sum_{i\in J^*}\alpha^i\gamma^i\right)}{\sum_{i\in J^*}\frac{\overline{b}^i - \overline{a}^i}{\overline{b}^i - \underline{b}^i} - \overline{K}_{\texttt{on}}}$$

$$\underline{\epsilon} = \frac{\frac{1}{2}\left(\sum_{i\in J^*}\gamma^i \frac{\overline{b}^i - \overline{a}^i}{\overline{b}^i - \underline{b}^i}\frac{-\gamma^i T_i^-(\mathbf{x}_0)}{2-\gamma^i T_I^-(\mathbf{x}_0)} - \max_{\alpha\in\Delta_N(\overline{K}_{\texttt{on}})}\sum_{i\in J^*}\alpha^i\gamma^i\right)}{\sum_{i\in J^*}\frac{\overline{b}^i - \overline{a}^i}{\overline{b}^i - \underline{b}^i} - \overline{K}_{\texttt{on}}}.$$

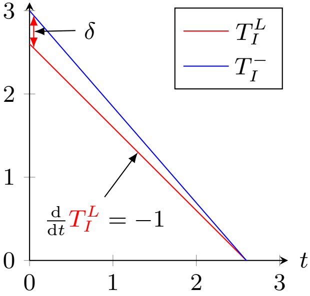
- Bounds tighter when infinite-horizon problem is far from feasible

Proof idea

- Study derivative of value function $\frac{\mathrm{d}}{\mathrm{d}t}T_I^-(\mathbf{x})$ along different trajectories
  - Close to -1 along optimal trajectories $\Longrightarrow$ $T_I^-(\mathbf{x})$ close to optimal TOI $T_I^*(\mathbf{x})$
  - Close to -1 along L-trajectory $\Longrightarrow$ achieved TOI $T_I^L(\mathbf{x})$ close to $T_I^-(\mathbf{x})$



Along optimal trajectory



Along L-trajectory

Myopic Strategy for Nonlinear Case

Proposition [HSCC'17]

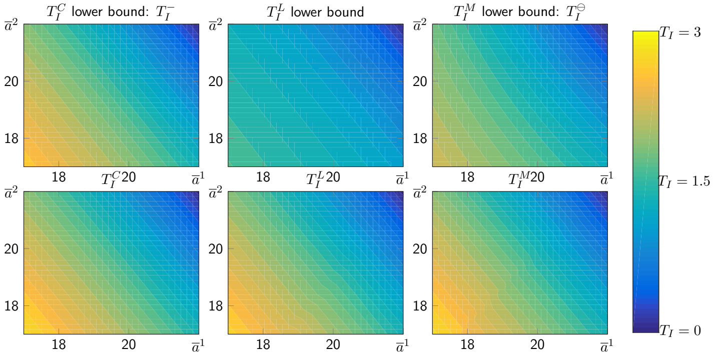
$$\mathcal{L}_{f^i_{\texttt{off}}} T_{f^i_{\texttt{on}}}(x) = -\frac{f^i_{\texttt{off}}(x)}{f^i_{\texttt{on}}(x)}, \quad \mathcal{L}_{f^i_{\texttt{on}}} T_{f^i_{\texttt{off}}}(x) = -\frac{f^i_{\texttt{on}}(x)}{f^i_{\texttt{off}}(x)}.$$

$$T_I^{\ominus}(\mathbf{x}) = \frac{\displaystyle\sum_{i \in J^*} \frac{T_{f^i_{\texttt{off}}}(x^i)}{1 + \mathcal{L}_{f^i_{\texttt{on}}} T_{f^i_{\texttt{off}}}(x^i)}}{\displaystyle\sum_{i \in J^*} \frac{1}{1 + \mathcal{L}_{f^i_{\texttt{on}}} T_{f^i_{\texttt{off}}}(x^i)} - \overline{K}_{\texttt{on}}}.$$

- $T_I^{\ominus}(\mathbf{x})$ equalizes derivatives of times to exit across subsystems in J^* rather than actual times to exit

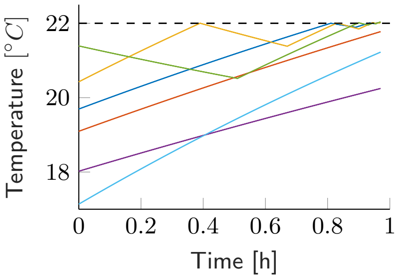
2D Numerical Example

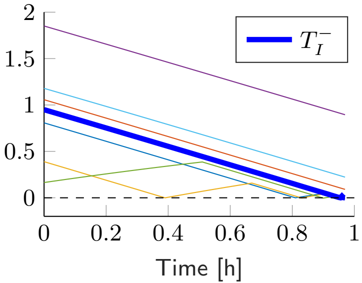
- Example with two subsystems
- Top: guarantee, bottom: performance
- M "myopic" strategy for general nonlinear case.



Large-Scale Numerical Example

- 10,000 subsystems with randomized parameters, **lazy strategy**





- Subsystems in J^* left unattended
- Guaranteed TOI: $0.86h$, achieved TOI: $0.96h$
- TOI achieved with temperature-driven switching (IEEE Power Syst. 30.1): $0.67h$

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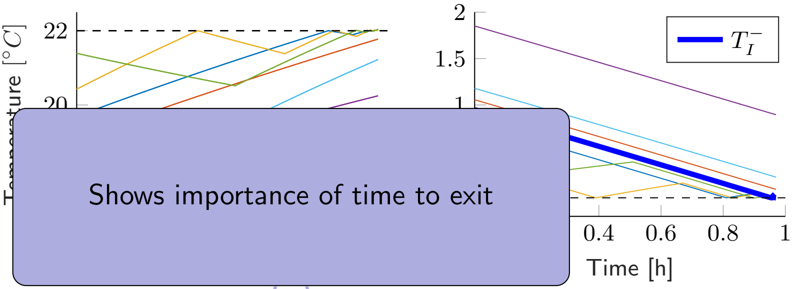

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Outline

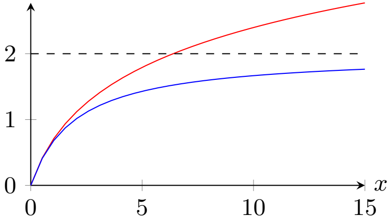
Summary

- Studied the TCL mode-counting problem
  - Previous work: solved infinite-horizon case
  - This work: analyzed finite-horizon case when infinite-horizon problem infeasible
- Approximate analytical value function gives **easy-to-implement policy with performance guarantees**
- **Time to exit** as abstraction for decision-making under heterogeneity

Future work

- Use guarantees to make informed decisions in high-level load distribution algorithm, tight conditions for problem generalizations
- Obtain bound that are tight for "large" $T_I^*(\mathbf{x})$
  - Performance loss for large $T_I^*(\mathbf{x})$ comes from use of inequality

$$\log(1+x) \geq \frac{2x}{2+x}$$



- Trade-off between tight bound and computability of value function

Thank you for your attention

Time to exit

Proposition

The Lie derivative of $T_f(x)$ with respect to g, $\mathcal{L}_g T_f(x)$, is

$$-\frac{\left(\hat{n}^S_{U_f(x)}\right)^T \left(\nabla_x \phi_f(x, T_f(x))\right) g(x)}{\left(\hat{n}^S_{U_f(x)}\right)^T f(U_f(x))}.$$