

Assignment

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Abstract—This manual provides solutions to the Assignment of Random Numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
we get https://github.com/pettugadipranav/
Randomvar/blob/main/codes/1-1.c
we get https://github.com/pettugadipranav/
Randomvar/blob/main/codes/cfunc.h
```

and compile and execute above files using following commands

```
$ gcc 1-1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1)$$

Solution: The following code plots Fig. 1.2

```
we get https://github.com/pettugadipranav/
Randomvar/blob/main/codes/1-2.py
```

to get above fig compile above code using following commands

```
$ python3 1-2.py
```

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.

Solution:

The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(u) du \quad (2)$$

We now have three cases:

- a) $x < 0$: $p_U(x) = 0$, and hence $F_U(x) = 0$.
- b) $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \quad (3)$$

- c) for $x > 1$ $F_U(x) = 1$

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (4)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (6)$$

Write a C program to find the mean and variance of U .

Solution: Download the following files

```
we get https://github.com/pettugadipranav/
Randomvar/blob/main/codes/1-4.c
we get https://github.com/pettugadipranav/
Randomvar/blob/main/codes/cfunc.h
```

execute them using following files

```
$ gcc 1-4.c
$ ./a.out
```

The value of mean is 0.500007
the value of variance is 0.083301

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (7)$$

Solution: we know that

$$\text{var}[U] = E[U - E[U]]^2 \quad (8)$$

$$= E[U^2] - (E[U])^2 \quad (9)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (10)$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \quad (11)$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \quad (12)$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (13)$$

$$= \int_{-\infty}^{\infty} x p_U(x) dx \quad (14)$$

$$= \int_0^1 x dx = \frac{1}{2} \quad (15)$$

we know mean = $E[U] = 0.5$

$$\text{var}[U] = E[U - E[U]]^2 \quad (16)$$

$$= E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (17)$$

$$(18)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (19)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called

gau.dat

Solution: download following files

we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/2-1.c>
we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/cfunc.h>

and execute them using below commands

```
$ gcc 2-1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 by downloading below file

we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/2-2.py>

and executed using below commands

```
$ python3 2-2.py
```

Fig. 2.2: The CDF of X

the properties of CDF are

- 1) The CDF is non decreasing function
- 2) It is symmetric about one point

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (20)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 by downloading following codes

we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/2-3.py>

and executed by using following commands

```
$ python3 2-3.py
```

Fig. 2.3: The PDF of X

some properties are

- 1) area under curve is 1

2)it is symmetric about $x = \mu$

3)it is increasing for $x < \mu$ and decreasing for $x > \mu$

2.4 Find the mean and variance of X by writing a C program.

Solution: to find mean and variance download following files

we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/2-4.c>
we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/cfunc.h>

and execute them using following commands

```
$ gcc 2-4.c
$ ./a.out
```

the value of mean is 0.000326
the value of variance is 1.000906

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (21) \quad \therefore \text{var}[X]=1$$

repeat the above exercise theoretically.

Solution:

1)CDF is calculated as follows :

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (22)$$

$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \quad (23)$$

$$= 1 \quad (24)$$

MEAN can be calculated as follows :

$$\int_{-\infty}^{\infty} x \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (25)$$

$$= 0 \quad + \quad (26)$$

2)mean is

$$E(x) = \int_{-\infty}^{\infty} x p_x(x) dx \quad (27)$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (28)$$

$$= 0 \quad (29)$$

$$(30)$$

$$E(x)=0$$

3)VARIANCE can be calculated as follows :

$$\text{var}[x] = E[X^2] - E[X]^2$$

$\text{var}[X] = E[X^2]$ The variance is given by

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = 0 \quad (31)$$

$$= \int_{-\infty}^{\infty} x \times x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (32)$$

$$= x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \quad (33)$$

$$+ \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (34)$$

$$= x \times 0 + \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (35)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \quad (36)$$

$$= 1 \quad (37)$$

$$(38)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (39)$$

and plot its CDF.

Solution: download the following files

we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/3-1.c>
we get <https://github.com/pettugadipranav/Randomvar/blob/main/codes/cfunc.h>

and execute them using following commands

```
$ gcc 3-1.c
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using following codes

```
https://github.com/pettugadipranav/Randomvar/
/blob/main/codes/3-1.py
```

and execute them by following commands

```
$ python3 3-1.py
```

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution: for $Y = G(X)$ then $F_Y(y) = F_X(G^{-1}(y))$
 $V = G(U)$

$$U = G^{-1}(V) \quad (40)$$

$$= 1 - e^{-\frac{v}{2}} \quad (41)$$

then

$$F_V(v) = F_U(G^{-1}(v)) \quad (42)$$

$$= F_U(1 - e^{-\frac{v}{2}}) \quad (43)$$

$$\Rightarrow F_V(v) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{v}{2}} & x \geq 0 \end{cases} \quad (44)$$

Solution: Note that the function

$$v = f(u) = -2 \ln(1 - u) \quad (45)$$

is monotonically increasing in $[0, 1]$ and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \quad (46)$$

Therefore, from the monotonicity of v , and using (4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \quad (47)$$

$$\Rightarrow F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \quad (48)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (49)$$

Solution: Download the following files and execute the C program.

```
$ https://github.com/pettugadipranav/
Randomvar/blob/main/codes/4-1.c
$ https://github.com/pettugadipranav/
Randomvar/blob/main/codes/cfunc.h
```

Execute the above C program files using the following commands

```
$ gcc 4-1.c
$ ./a.out
```

4.2 Find the CDF of T .

Solution: The CDF of T is plotted in Fig. 4.2 using the code below

```
./figs/4-2.png
```

Fig. 4.2: The CDF of T

```
$ https://github.com/pettugadipranav/
Randomvar/blob/main/codes/4-2.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4-2.py
```

4.3 Find the PDF of T .

Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
$ https://github.com/pettugadipranav/
Randomvar/blob/main/codes/4-3.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4-3.py
```

./figs/4-3.png

Fig. 4.3: The PDF of T

4.4 Find the Theoretical Expression for the PDF and CDF of T

Solution:

$$T = U_1 + U_2 \quad (50)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_{U_1}(x)p_{U_2}(y)dx \quad (51)$$

$$\text{As, } p_{U_1}(x) = p_{U_1}(y) = p_U(u) \quad (52)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_U(u)p_U(t-u)du \quad (53)$$

a) Theoretical PDF

i) For $0 < t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \quad (54)$$

$$\Rightarrow p_T(t) = \int_0^t du = t \quad (55)$$

ii) For $1 < t \leq 2$

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (56)$$

$$\Rightarrow p_T(t) = \int_{t-1}^1 du = 2 - t \quad (57)$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^t p_T(u)du \quad (58)$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot

Solution: The above theoretical results are verified through the plots in Fig 4.2 and Fig 4.3