Assignment

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Contents

***Abstract*—This manual provides solutions to the Assignment of Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/1−1.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and compile and execute above files using following commands

```
$ gcc 1−1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1}$$

Solution: The following code plots Fig. 1.2

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/1−2.py
```

to get above fig compile above code using following commands

```
$ python3 1−2.py
```

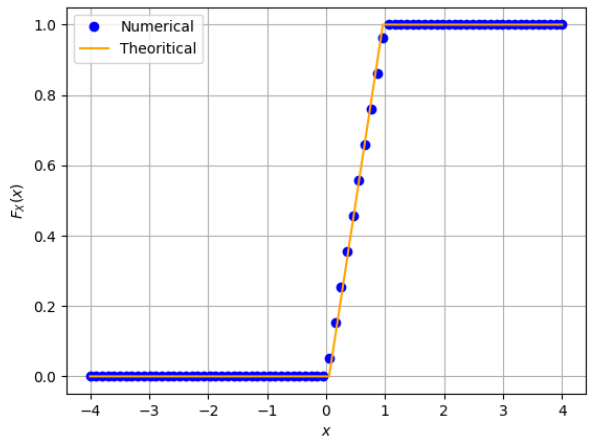

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du \tag{2}$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \le x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{3}$$

c) for $x > 1$ $F_U(x) = 1$

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{6}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/1−4.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

execute them using following files

```
$ gcc 1−4.c
$ ./a.out
```

The value of mean is 0.500007
the value of variance is 0.083301

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{7}$$

Solution: we know that

$$\text{var}[U] = E[U - E[U]]^2 \tag{8}$$
$$= E\left[U^2\right] - (E[U])^2 \tag{9}$$
$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{10}$$
$$= \int_{-\infty}^{\infty} x^2 p_U(x)dx \tag{11}$$
$$= \int_{0}^{1} x^2 dx = \frac{1}{3} \tag{12}$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{13}$$
$$= \int_{-\infty}^{\infty} x p_U(x)dx \tag{14}$$
$$= \int_{0}^{1} x dx = \frac{1}{2} \tag{15}$$

we know mean $= E[U] = 0.5$

$$\text{var}[U] = E[U - E[U]]^2 \tag{16}$$
$$= E\left[U^2\right] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{17}$$
$$\tag{18}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{19}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: download following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−1.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and execute them using below commands

```
$ gcc 2−1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 by downloading below file

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−2.py
```

and executed using below commands

```
$ python3 2−2.py
```

the properties of CDF are
1)The CDF is non decreasing function
2)It is symmetric about one point

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{20}$$

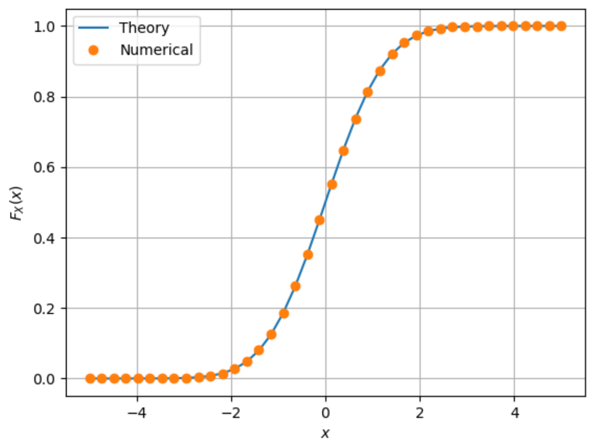

Fig. 2.2: The CDF of X

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 by downloading following codes

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−3.py
```

and executed by using following commands

```
$ python3 2−3.py
```

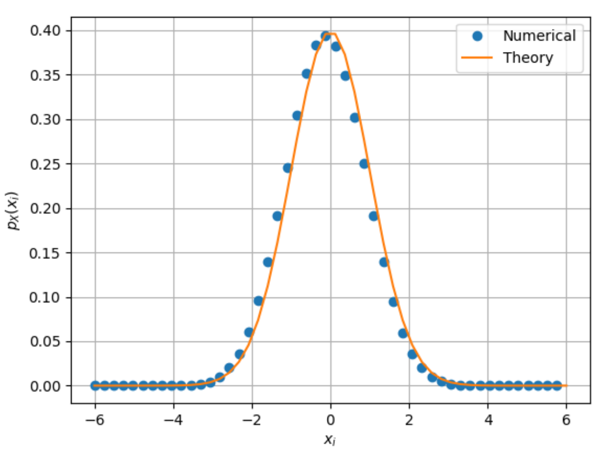

Fig. 2.3: The PDF of X

some properties are
1)area under curve is 1

2)it is symmetric about $x = \mu$

3)it is increasing for $x < \mu$ and decreasing for $x > \mu$

2.4 Find the mean and variance of X by writing a C program.
Solution: to find mean and variance download following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−4.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and execute them using following commands

```
$ gcc 2−4.c
$ ./a.out
```

the value of mean is 0.000326
the value of variance is 1.000906

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (21)$$

repeat the above exercise theoretically.
Solution:
1)CDF is calculated as follows :

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (22)$$

$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \quad (23)$$

$$= 1 \quad (24)$$

MEAN can be calculated as follows :

$$\int_{-\infty}^{\infty} x \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (25)$$

$$= 0 \qquad + \quad (26)$$

2)mean is

$$E(x) = \int_{-\infty}^{\infty} x p_x(x).dx \quad (27)$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).dx \quad (28)$$

$$= 0 \quad (29)$$

$$(30)$$

E(x)=0

3)VARIANCE can be calculated as follows : $var[x] = E[X^2] - E[X]^2$
$var[X] = E[X^2]$ The variance is given by

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = 0 \tag{31}$$

$$= \int_{-\infty}^{\infty} x \times x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{32}$$

$$= x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \tag{33}$$

$$+ \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{34}$$

$$= x \times 0 + \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{35}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \tag{36}$$

$$= 1 \tag{37}$$

$$\tag{38}$$

$\therefore var[X] = 1$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{39}$$

and plot its CDF.
Solution: download the following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/3−1.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and execute them using following commands

```
$ gcc 3−1.c
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using following codes

```
https://github.com/pettugadipranav/Randomvar
    /blob/main/codes/3−1.py
```

and execute them by following commands

```
$ python3 3−1.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: for$Y = G(X)$ then $F_Y(y) = F_X(G^{-1}(y))$
$V = G(U)$

$$U = G^{-1}(V) \tag{40}$$

$$= 1 - e^{-\frac{V}{2}} \tag{41}$$

then

$$F_V(v) = F_U(G^{-1}(v)) \tag{42}$$

$$= F_U(1 - e^{-\frac{V}{2}}) \tag{43}$$

$$\implies F_V(v) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{V}{2}} & x \geq 0 \end{cases} \tag{44}$$

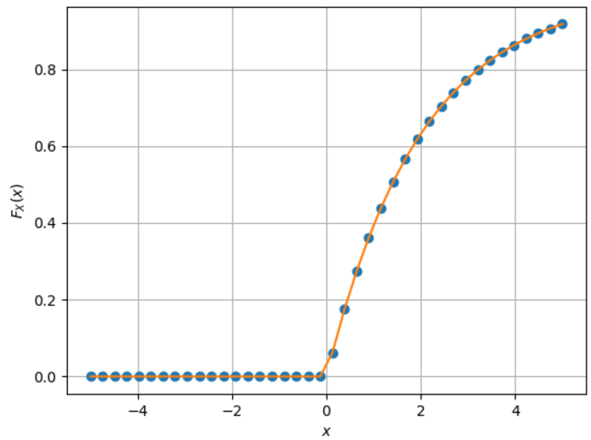

Fig. 3.1: The CDF of V

Solution: Note that the function

$$v = f(u) = -2\ln(1 - u) \tag{45}$$

is monotonically increasing in [0, 1] and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \tag{46}$$

Therefore, from the monotonicity of v, and using (4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \tag{47}$$

$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \tag{48}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{49}$$

Solution: Download the following files and execute the C program.

```
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/4−1.c
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

Execute the above C program files using the following commands

```
$ gcc 4−1.c
$ ./a.out
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. 4.2 using the code below

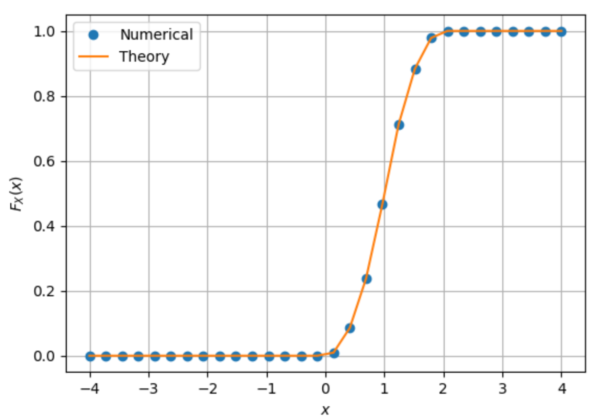

Fig. 4.2: The CDF of T

```
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/4−2.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4−2.py
```

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/4−3.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4−3.py
```

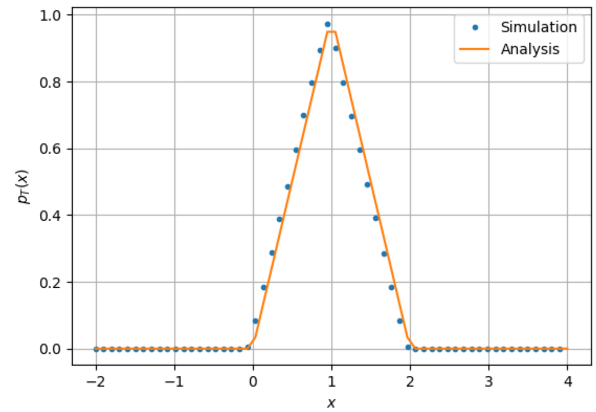

Fig. 4.3: The PDF of T

4.4 Find the Theoreotical Expression for the PDF and CDF of T
Solution:

$$T = U_1 + U_2 \tag{50}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_{U1}(x)p_{U2}(y)dx \tag{51}$$

$$\text{As,} p_{U1}(x) = p_{U1}(y) = p_U(u) \tag{52}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_U(u)p_U(t-u)du \tag{53}$$

a) Theoretical PDF

i) For $0 < t \leq 1$

$$p_T(t) = \int_{0}^{t} p_U(t-u)du \tag{54}$$

$$\implies p_T(t) = \int_{0}^{t} du = t \tag{55}$$

ii) For $1 < t \leq 2$

$$p_T(t) = \int_{0}^{1} p_U(t-u)du \tag{56}$$

$$\implies p_T(t) = \int_{t-1}^{1} du = 2 - t \tag{57}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^{t} p_T(u)du \tag{58}$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot
Solution: The above theoretical results are verified through the plots in Fig 4.2 and Fig 4.3

5 Guasssian to Other

5.1 Generate equiprobable $X \in \{1, -1\}$. **Solution:** Download the following files and execute the C program.

Download the above files and execute the following commands

```
$ gcc 5-1.c
$ ./a.out
```

5.2 Generate

$$Y = AX + N, \tag{59}$$

Where $A = 5$ dB, and $N \sim \aleph(0, 1)$.
Solution: :

Download the above files and execute the following commands

```
$ gcc 5-2.c
$ ./a.out
```

5.3 Plot Y using a scatter plot.
Solution: The CDF of V is plotted in Fig. 5.3 using the code below

Download the above files and execute the following commands to produce Fig.5.3

```
$ python3 5-3.py
```

5.4 Guess how to estimate X from Y.
Solution: :

a) If $Y < 0$ then probably $X = -1$
b) If $Y > 0$ then probably $X = 1$

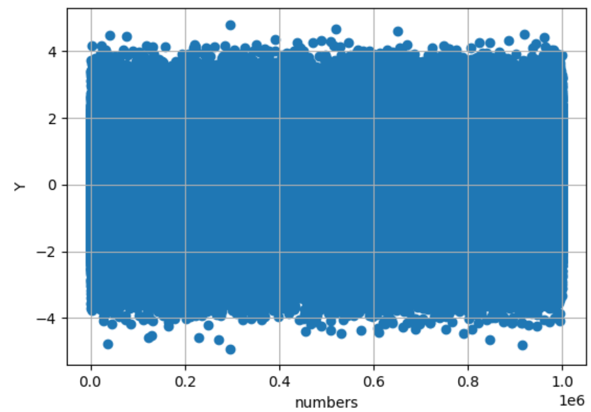

Fig. 5.3: The Scatter Plot

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{60}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{61}$$

Solution: :

Download the above files and execute the following commands to get the result

```
$ gcc 5-5.c
$ ./a.out
```

$$P_{e|0} = 0.499033 \tag{62}$$
$$P_{e|1} = 0.500138 \tag{63}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = P(X = 1)P_{e|0} + P(X = -1)P_{e|1} \tag{64}$$

Since X is equiprobable

$$P(X = 1) = P(X = -1) = 0.5 \tag{65}$$
$$\implies P_e = \frac{P_{e|0} + P_{e|1}}{2} \tag{66}$$
$$\implies \boxed{P_e = 0.499585} \tag{67}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

Solution: :

$$
\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) && (68)\\
P_{e|0} &= \Pr\left(AX + N < 0 | X = 1\right) && (69)\\
P_{e|0} &= \Pr\left(N < -A\right) && (70)\\
P_{e|0} &= \int_{-\infty}^{-A} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} && (71)\\
P_{e|0} &= \int_{A}^{\infty} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} && (72)\\
P_{e|0} &= Q_N(A) && (73)\\
\text{Similarly, } P_{e|1} &= Q_N(A) && (74)
\end{aligned}
$$

Download the above files and execute the following commands to produce Fig.5.7

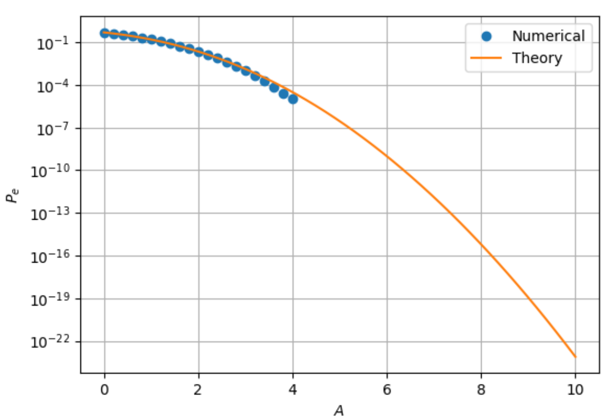

Fig. 5.7: $P_e(A)$ with semilog-y axis

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: To estimate X from Y, we now consider the following:

$$
X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \quad (75)
$$

Therefore,

$$
\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) && (76)\\
&= \Pr\left(AX + N < \delta | X = 1\right) && (77)\\
\implies P_{e|0} &= \Pr\left(N < \delta - A\right) && (78)\\
&= \int_{-\infty}^{\delta - A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx && (79)\\
&= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx && (80)\\
\implies P_{e|0} &= Q_N(A - \delta) && (81)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
P_{e|1} &= Q_N(A + \delta) && (82)\\
P_e &= P_{e|0} \Pr\left(X = 1\right) + P_{e|1} \Pr\left(X = -1\right) && (83)\\
&= \frac{Q_N(A - \delta) + Q_N(A + \delta)}{2} && (84)
\end{aligned}
$$

Differentiating the above equation wrt δ:

$$
\begin{aligned}
0 &= \frac{d}{d\delta}\left(\frac{Q_N(A - \delta) + Q_N(A + \delta)}{2}\right) && (85)\\
&= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}\right) && (86)\\
\implies (\delta - A)^2 &= (\delta + A)^2 && (87)\\
\implies & \boxed{\delta = 0} && (88)
\end{aligned}
$$

5.9 Repeat the above exercise when

$$
p_X(0) = p \quad (89)
$$

Solution: Using Eq. (83), we have:

$$
\begin{aligned}
P_e &= P_{e|0} p + P_{e|1}(1 - p) && (90)\\
&= pQ_N(A - \delta) + (1 - p)Q_N(A + \delta) && (91)
\end{aligned}
$$

Differentiating as before, we get:

$$
0 = p\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - (1 - p)\frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \quad (92)
$$

$$
e^{\frac{(\delta + A)^2 - ((\delta - A))^2}{2}} = \frac{1 - p}{p} \quad (93)
$$

$$
\implies \boxed{\delta = 0} \quad (94)
$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: Assume that

$$\Pr(X = -1) = p \tag{95}$$
$$\Pr(X = 1) = (1 - p) \tag{96}$$

From Total Probability Theorem, we have:

$$p_Y(y) = p_{Y|X=-1}(y|-1)\Pr(X = -1) + p_{Y|X=1}(y|1)\Pr(X = 1) \tag{97}$$
$$p_Y(y) = p \times p_{(-A+N)}(y) \tag{98}$$
$$+ (1 - p) \times p_{(A+N)}(y) \tag{99}$$

$$p_Y(y) = p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p)\,\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}} \tag{100}$$

To use the MAP criterion, we must find $p_{X|Y}(x|y)$. To do this, we use the Theorem of Conditional Probability:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \tag{101}$$

When $X = 1$, we have:

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)} \tag{102}$$
$$= \frac{(1-p)\,\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p)\,\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}} \tag{103}$$
$$= \frac{(1-p)\,e^{2yA}}{p + (1-p)\,e^{2yA}} \tag{104}$$

Similarly, when $X = -1$, we get:

$$p_{X|Y}(-1|y) = \frac{p}{p + (1-p)\,e^{2yA}} \tag{105}$$

Therefore, when $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$, we have:

$$\frac{(1-p)\,e^{2yA}}{p + (1-p)\,e^{2yA}} > \frac{p}{p + (1-p)\,e^{2yA}} \tag{106}$$
$$e^{2yA} > \frac{p}{(1-p)} \tag{107}$$
$$\implies y > \frac{1}{2A}\ln\frac{p}{(1-p)} \tag{108}$$

Therefore, when Eq. (**??**), we can assert that $X = 1$, and $X = -1$ otherwise. Now, consider when $p = \frac{1}{2}$. We have:

$$y > \frac{1}{2A}\ln\frac{p}{(1-p)} \tag{109}$$
$$= \frac{1}{2A}\ln 1 \tag{110}$$
$$= 0 \tag{111}$$

Therefore, when $y > 0$, we choose $X = 1$, and we choose $X = -1$ otherwise.