Assignment

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CS21BTECH11063

Contents

***Abstract*—This manual provides solutions to the Assignment of Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/1−1.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and compile and execute above files using following commands

```
$ gcc 1−1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1}$$

Solution: The following code plots Fig. 1.2

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/1−2.py
```

to get above fig compile above code using following commands

```
$ python3 1−2.py
```

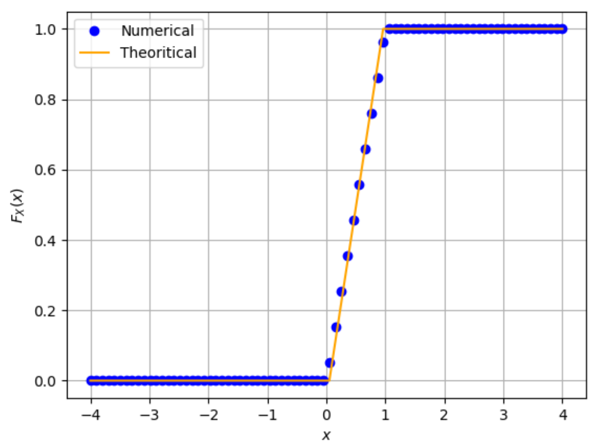

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{2}$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{3}$$

c) for $x > 1$ $F_U(x) = 1$

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{6}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/1−4.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

execute them using following files

```
$ gcc 1−4.c
$ ./a.out
```

The value of mean is 0.500007
the value of variance is 0.083301

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{7}$$

Solution: we know that

$$\text{var}[U] = E[U - E[U]]^2 \tag{8}$$
$$= E\left[U^2\right] - (E[U])^2 \tag{9}$$
$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{10}$$
$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \tag{11}$$
$$= \int_{0}^{1} x^2 dx = \frac{1}{3} \tag{12}$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{13}$$
$$= \int_{-\infty}^{\infty} x p_U(x) dx \tag{14}$$
$$= \int_{0}^{1} x dx = \frac{1}{2} \tag{15}$$

we know mean $= E[U] = 0.5$

$$\text{var}[U] = E[U - E[U]]^2 \tag{16}$$
$$= E\left[U^2\right] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{17}$$
$$\tag{18}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{19}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: download following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−1.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and execute them using below commands

```
$ gcc 2−1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 by downloading below file

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−2.py
```

and executed using below commands

```
$ python3 2−2.py
```

the properties of CDF are
1)The CDF is non decreasing function
2)It is symmetric about one point

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{20}$$

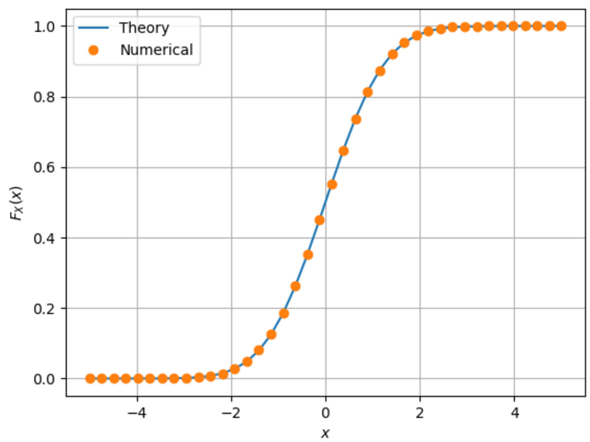

Fig. 2.2: The CDF of X

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 by downloading following codes

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−3.py
```

and executed by using following commands

```
$ python3 2−3.py
```

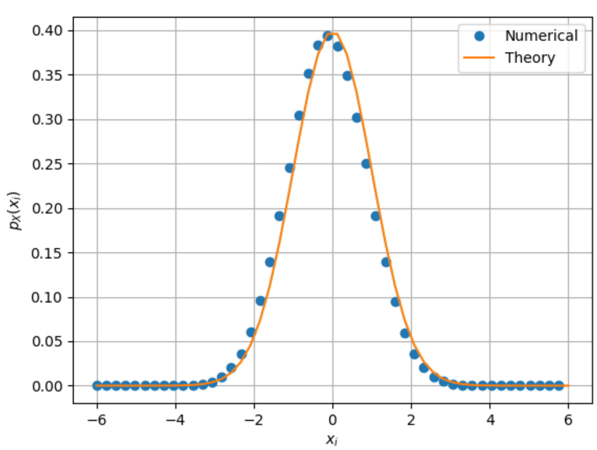

Fig. 2.3: The PDF of X

some properties are
1)area under curve is 1

2)it is symmetric about $x = \mu$

3)it is increasing for $x < \mu$ and decreasing for $x > \mu$

2.4 Find the mean and variance of X by writing a C program.
Solution: to find mean and variance download following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/2−4.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and execute them using following commands

```
$ gcc 2−4.c
$ ./a.out
```

the value of mean is 0.000326
the value of variance is 1.000906

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{21}$$

repeat the above exercise theoretically.
Solution:
1)CDF is calculated as follows :

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{22}$$

$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{23}$$

$$= 1 \tag{24}$$

MEAN can be calculated as follows :

$$\int_{-\infty}^{\infty} x \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{25}$$

$$= 0 \qquad + \tag{26}$$

2)mean is

$$E(x) = \int_{-\infty}^{\infty} x p_x(x).dx \tag{27}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).dx \tag{28}$$

$$= 0 \tag{29}$$

$$\tag{30}$$

E(x)=0

3)VARIANCE can be calculated as follows : $var[x] = E[X^2] - E[X]^2$
$var[X] = E[X^2]$ The variance is given by

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = 0 \tag{31}$$
$$= \int_{-\infty}^{\infty} x \times x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{32}$$
$$= x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Big|_{-\infty}^{\infty} \tag{33}$$
$$+ \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{34}$$
$$= x \times 0 + \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{35}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \tag{36}$$
$$= 1 \tag{37}$$
$$\tag{38}$$

$\therefore$var[X]=1

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{39}$$

and plot its CDF.
Solution: download the following files

```
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/3−1.c
we get https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

and execute them using following commands

```
$ gcc 3−1.c
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using following codes

```
https://github.com/pettugadipranav/Randomvar
    /blob/main/codes/3−1.py
```

and execute them by following commands

```
$ python3 3−1.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: for$Y = G(X)$ then $F_Y(y) =$

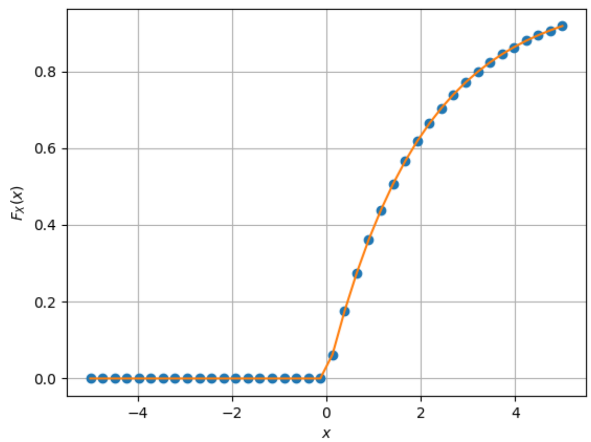

Fig. 3.1: The CDF of V

$F_X(G^{-1}(y))$
$V = G(U)$

$$U = G^{-1}(V) \tag{40}$$
$$= 1 - e^{-\frac{V}{2}} \tag{41}$$

then

$$F_V(v) = F_U(G^{-1}(v)) \tag{42}$$
$$= F_U(1 - e^{-\frac{V}{2}}) \tag{43}$$

$$\implies F_V(v) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{V}{2}} & x \geq 0 \end{cases} \tag{44}$$

Solution: Note that the function

$$v = f(u) = -2\ln(1 - u) \tag{45}$$

is monotonically increasing in [0, 1] and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \tag{46}$$

Therefore, from the monotonicity of v, and using (4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \tag{47}$$
$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \tag{48}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{49}$$

Solution: Download the following files and execute the C program.

```
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/4−1.c
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/cfunc.h
```

Execute the above C program files using the following commands

```
$ gcc 4−1.c
$ ./a.out
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. 4.2 using the code below

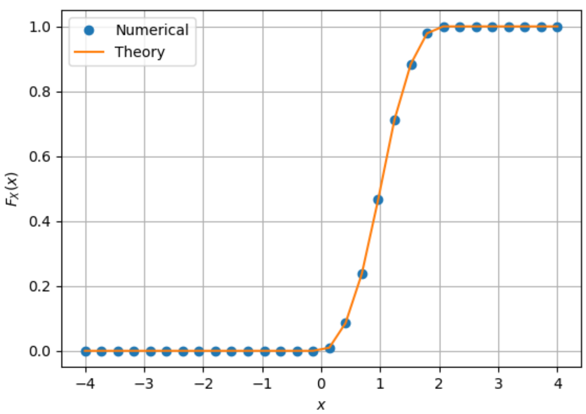

Fig. 4.2: The CDF of T

```
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/4−2.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4−2.py
```

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
$ https://github.com/pettugadipranav/
    Randomvar/blob/main/codes/4−3.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4−3.py
```

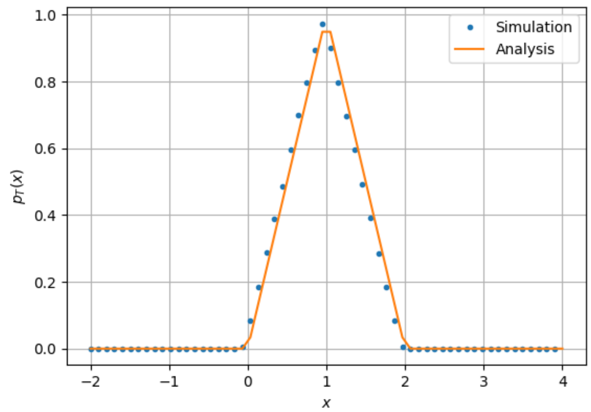

Fig. 4.3: The PDF of T

4.4 Find the Theoretical Expression for the PDF and CDF of T
Solution:

$$T = U_1 + U_2 \tag{50}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_{U1}(x)p_{U2}(y)dx \tag{51}$$

$$\text{As,} p_{U1}(x) = p_{U1}(y) = p_U(u) \tag{52}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_U(u)p_U(t-u)du \tag{53}$$

a) Theoretical PDF
   i) For $0 < t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \tag{54}$$

$$\implies p_T(t) = \int_0^t du = t \tag{55}$$

   ii) For $1 < t \leq 2$

$$p_T(t) = \int_0^1 p_U(t-u)du \tag{56}$$

$$\implies p_T(t) = \int_{t-1}^1 du = 2 - t \tag{57}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^{t} p_T(u)du \tag{58}$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t^2}{2} & 0 \le t \le 1 \\ 2t - 1 - \dfrac{t^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot
Solution: The above theoretical results are verified through the plots in Fig 4.2 and Fig 4.3