# 1.4 RGB

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## Algorithm

Reduce the problem to an ordinary shortest path instance by first filter the graph so that it only contains edges that join vertices of different colours.

Then a plain BFS gives the required path.

(a) Build a new graph $G' = (V, E')$ with

$$E' = \{\{x, y\} \in E : \text{colour}(x) \neq \text{colour}(y)\}.$$

Checking each original edge once costs $O(|E|)$.

**Justification (validity of G')**

- Every colour-alternating path in the original graph consists only of edges connecting vertices of different colours, so these edges are preserved in $G'$. Conversely, any path in $G'$ uses only edges joining different colours, so any path found in $G'$ is a valid colour-alternating path in $G$.

(b) Run a standard BFS from $u$ to $v$ on $G'$ and output the path it finds.

$$\text{Time complexity of BFS: } O(|V| + |E'|)$$

**Justification (Choice of BFS)**

- BFS finds a path with the minimum number of edges in an unweighted graph. Since $G'$ preserves all legal alternating edges from $G$, this path is a shortest colour-alternating path.

## Complexity

Total time is

$$O(|E|) + O(|V| + |E'|) = O(|V| + |E| + |E'|).$$

Since $E' \subseteq E$, we have $|E'| \leq |E|$, so

$$O(|V| + |E| + |E'|) = O(|V| + 2|E|) = O(|V| + |E|).$$

## Correctness Proof

### Forward direction (forward correctness)

- We need to prove that when BFS is applied to the filtered graph $G'$, the shortest path it finds does not use edges with both vertices of the same colour.
  - By construction, $G' = (V, E')$ where
    $$E' = \{\{x, y\} \in E : \text{colour}(x) \neq \text{colour}(y)\}.$$
  - BFS only traverses edges in $G'$.
  - Therefore, every edge in any path returned by BFS connects vertices of different colours.
  - Consequently, any path found by BFS is a valid colour-alternating path in the original graph $G$.

### Backward direction (backward completeness)

- We need to prove that BFS on $G'$ will consider every path that does not use edges with vertices of the same colour.
  - Let $p$ be any colour-alternating path in the original graph $G$.
  - By definition, all edges of $p$ connect vertices of different colours.
  - Since $G'$ contains all edges connecting vertices of different colours, $p$ exists as a path in $G'$.
  - BFS explores all paths in $G'$, so every colour-alternating path in $G$ is considered by the algorithm.

## Conclusion

By reducing the graph first and then calling BFS, the shortest colour-alternating path is found in:

$$\boxed{O(|V| + |E|)}$$