2.6 Caffeine

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Problem Summary

A cafe offers n drinks, where the i-th drink contains c_i milligrams of caffeine per serving. Buzz has n vouchers, each with value v_i, which can be used to redeem v_i servings of a single, unique drink (no two vouchers can be used on the same drink).

The goal is to assign each voucher to a different drink in such a way that the **total caffeine consumed** (i.e., $\sum v_i \cdot c_j$ for some assignment of drinks j) is maximized.

For example, suppose the drinks (say cappuccino, latte, mocha and espresso) have caffeine contents of $600, 200, 400,$ and 100 milligrams and Buzz's vouchers have values of $6, 3, 8$ and 6.

i	1	2	3	4
name	cappuccino	latte	mocha	espresso
c_i	600	200	400	100
v_i	6	3	8	6

One possibility is to use voucher 1 for six mochas, voucher 2 for three cappuccinos, voucher 3 for eight espressos and voucher 4 for six lattes. The total caffeine content would be

$$\begin{aligned} v_1c_3 + v_2c_1 + v_3c_2 + v_4c_4 &= (6 \times 400) + (3 \times 600) + (8 \times 200) + (6 \times 100) \\ &= 6400 \text{ milligrams.} \end{aligned}$$

Note that this is *not* the optimal allocation!

Question A)

We are given the task of proving that

- We should sort the drinks in ascending order
- We should sort the vouchers in ascending order

Intuitively this makes sense, but to prove it we need to show that our ordering G is as good or better than any alternative ordering A.

Firstly we define the cost function, and as we want to sort by

(a) Drinks with *highest* caffeine contents

(b) Vouchers with *highest* highest redemptions

If the drinks are ordered $c_1, c_2, \ldots, c_3$, and drink vouchers $v_1, v_2, \ldots, v_3$, then the expected cost function is:

$$E = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_j v_i$$

Claim

The maximized value is achieved when both drinks and vouchers are sorted in ascending order, e.g. lowest cost drink to lowest voucher value

$$C \in c_1 \leq c_2 \leq c_3 \leq \cdots \leq c_j$$
$$V \in v_1 \leq v_2 \leq v_3 \leq \cdots \leq v_i$$

Proof by Contradiction

Suppose there exists ordering A, wherein C nor V is in ascending order, and our pairing G where it is.

We assume A to be maximised, that is, to have the highest possible net value. For the sake of contradiction, there is pairing:

$$c_i > c_{i+1}, \quad v_i < v_{i+1}$$

In this scenario, we have total value:

$$T = c_i v_i + c_{i+1} v_{i+1}$$

If we were to switch the pairing, our resultant sum is:

$$T' = c_i v_{i+1} + c_{i+1} v_i$$

Now,

$$T - T' = c_i v_i + c_{i+1} v_{i+1} - (c_i v_{i+1} + c_{i+1} v_i)$$

$$= (c_i - c_{i+1}) v_i + (c_{i+1} - c_i) v_{i+1}$$

$$= (c_i - c_{i+1})(v_i - v_{i+1})$$

Since $c_i > c_{i+1}$ and $v_i < v_{i+1}$, this product is negative:

This means that any pairing that is not ordered identically,

$$T - T' \leq 0$$

$$\boxed{T \leq T'}$$

Therefore implying a contradiction, as T isn't the maximized sum.

Question b)

By the above argument, any inversion in the order (i.e., pairing a higher caffeine drink with a lower-value voucher or vice versa) can be swapped to increase the total caffeine.

Therefore, if we continue eliminating all such inversions, we will eventually arrive at a configuration where no such beneficial swap exists, which is precisely when both sequences are sorted in the same order.

Hence, the greedy matching G, where both vouchers and drinks are sorted in ascending order, yields the maximum total caffeine and is thus:

$$\boxed{\text{Optimal}}$$