1 Normal Form Games

A normal form game is $(I, (A^i)_{i=1,...,n}, (u^i)_{i=1,...,n})$, where $\forall i$ A_i is an action set, $A = \times_{i=1}^n A^i$, and $u_i : A \to \mathbb{R}$ $\forall i$.

$I = \{1, ..., n\}$ is the set of players.

Assume A^i is finite for all i.

Examples: Coordination, Matching pennies, Prisoner's dilemma, Battle of the Sexes.

1.1 Dominance

Let $S^i = \Delta(A^i) = \{(s(a_1^i), ..., s(a_{k_i}^i)) : \forall i,\ s(a_i) \geq 0, \sum_{A^i} s(a^i) = 1\}$.

A mixed extension of a normal form game is $(I, (S^i)_{i=1,...,n}, (u^i)_{i=1,...,n})$, where $\forall$ $S^i = \Delta(A^i)$, $S = \prod_{i=1}^n S^i$ and $u^i : S \to \mathbb{R}$ is defined by

$$u^i(s^1, ..., s^n) = \sum_{a \in A} u^i(a) \prod_{i=1}^n s^i(a^i).$$

We write $Pr_s(a) = \prod_{i=1}^n s^i(a^i) \in \Delta A$.

Example: Show that in the game below, the player can get a better payoff by mixing T and M than by playing B, no matter what his belief is about what his partner is doing.

	L	R
T	3	0
M	0	3
B	1	1

We say $s^i \in S^i$ strictly dominates $a^i \in A^i$ iff for all a^{-i}

$$u^i(s^i, a^{-i}) > u^i(a^i, a^{-i}).$$

Alternatively,

$$s^i D_2 a^i \Leftrightarrow \forall s^{-i} \in S^{-i} \quad u^i(s^i, s^{-i}) > u_i(a^i, s^{-i})$$

or

$$s^i D_3 a^i \Leftrightarrow \forall \mu \in \Delta(A^{-i}) \quad u^i(s^i, \mu) > u_i(a^i, \mu)$$

Exercise: $s^i D_3 a^i \Leftrightarrow s^i D_2 a^i \Leftrightarrow s^i D_1 a^i$.

Example: Note that T and L are both dominated in the game below.

	L	R
T	-2,-2	-10,-1
B	-1,-10	-5,-5

This leads to the counter-intuitive prediction of playing (B,R). Of course this doesn't happen in real life.

Example:

	L	R
T	3	0
M	0	3
B	x	x

Consider a belief p for Player 1 that Player 2 chooses L. Note that if $x < \frac{3}{2}$, B is never a best response. For every belief, Player 1 is better of playing T or M. Dually, $\exists s^1 \in S^1$ that dominates B.

If $x = \frac{3}{2}$, there exists a belief ($p = 0.5$) for which B is a best response. Dually, B is not strictly dominated.

This example suggests that an action is never a best response if and only if it is strictly dominated by a strategy.

Definition: An action $a^i \in A^i$ is never a best response if there is no $\mu \in \Delta(A^{-i})$ such that $u^i(a^i, \mu) \geq u^i(b^i, \mu)$ for all b^i.

Theorem: An action $a^i \in A^i$ is strictly dominated if and only if it is never a best response.

One direction is easy to prove (see your class notes). The proof for the other direction can be found in Osborne and Rubinstein.

1.2 IESDA

We illustrate this with examples:

	L	R
T	0,-2	-10,-1
B	-1,-10	-5,-5

	L	R
T	3,0	0,1
M	0,0	3,1
B	1,1	1,0

Example (Cournot Duopoly):

Consider a two player game with two firms $i = 1, 2$. Each firm faces the demand curve $p = a - b(q_1 + q_2)$ and per-unit costs of production c. Show that iterated elimination of strictly dominated actions yields a unique outcome in which each firm produces $\frac{a-c}{3b}$.

1.3 Rationalizability

An action $a^i \in A^i$ is rationalizable if there exist sets $(R^1, ..., R^N)$ such that:

1. $a^i \in R^i$
2. For all j, $R^j \subset A^j$
3. $\forall j, b_j \in R^j, \quad \exists \mu(b^j) \in \Delta(A^{-j})$ (with support R^{-j}) s.t. $u^j(b^j, \mu) \geq u^j(a^j, \mu) \quad \forall a^j \in A^j$.

We will also sometimes talk of sets as being rationalizable. In this case, we will talk of $(R^1, ..., R^N)$ as being rationalizable if conditions 2 and 3 above are satisfied.

Example:

	L	R
T	3,0	0,1
M	0,0	3,1
B	1,1	1,0

$(R^1, R^2) = (\{M\}, \{R\})$.

Example 2:

	L	R
T	3,1	0,0
B	0,0	1,3

$(R^1, R^2) = (\{T\}, \{L\})$. $(T^1, T^2) = (\{B\}, \{R\})$.

So these sets are not unique, unless maximality is required. The maximal set of rationalizable actions in this example is $\{\{T, B\}, \{L, R\}\}$.

Proposition: If $(R^1, ..., R^N)$ and $(T^1, ..., T^N)$ are rationalizable, then $(R^1 \cup T^1, ..., R^N \cup T^N)$ is rationalizable, as well.

As we discussed before, $D \Leftrightarrow NBR$. D is related to iterated dominance, while NBR is related to rationalizability. As we will show below, IESDA and rationalizability are in some sense equivalent:

$$IESDA \Leftrightarrow Rationalizability$$

We can prove this using the following propositions (you don't need to know the details or the proofs):

Proposition 1: Let R be a set of rationalizable actions. Let $(A_1, ..., A_T)$ be an iterated elimination of strictly dominated actions. Then, $R^i \subset A_T^i \quad \forall i$.

Proposition 2: Let R denote the maximal set of rationalizable actions (in terms of set inclusion), and let $(A_1, ..., A_T)$ be a complete elimination of strictly dominated strategies. Then $A_T^i \subset R^i$ for every i.

What these propositions show is that the set of actions that survives *complete* IESDA is unique and equal to the maximal set of rationalizable actions.

Example:

	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	0,-2	0,0	10,-1

It's easy to show that the maximal set of rationalizable actions is $(R^1, R^2) = (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$

(eliminate b_4 in step 1, and a_4 in step 2).

1.4 Nash Equilibrium

Definition (Pure Best Reply): $PBR^i(s) = \{a^i \in A^i : u^i(a^i, s^{-i}) \geq u^i(b^i, s^{-i}) \quad \forall b^i \in A^i\}$.

Note that this set is nonempty for a finite game.

Definition (Best Reply): $BR^i(s) = \{s^i \in S^i : u^i(s^i, s^{-i}) \geq u^i(b^i, s^{-i}) \quad \forall b^i \in A^i\}$.

Note that for every s, $BR^i(s)$ is closed, convex, nonempty, and equal to the mixed strategies concentrated on $PBR^i(s)$.

Example: Find $PBR^i(s)$ and $BR^i(s)$ in the following game:

	L	R
T	3,1	0,0
B	0,0	1,3

Definition:

$$BR(s) = \times_{i=1}^{N} BR^i(s).$$

Note that $BR : S \twoheadrightarrow S$ is a closed, convex, and nomempty valued correspondence.

Definition (Nash Equilibrium): A Nash Equilibrium of a NFG is a strategy profile $\hat{s}$ s.t. $\hat{s} \in BR(\hat{s})$.

Theorem (Nash, 1950): The set of Nash Equilibrium strategy profiles is nonempty.

Philosophical point: Keep in mind that Nash equilibrium is *not* an implication of rationality. Rationalizability is an implication of rationality. Nash has stronger "epistemic" assumptions. E.g., it assumes that every player knows what every other player is playing.