Math 4317 (Prof. Swiech, S'18): HW #4

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Section 20

A. Prove that if f is defined for $x \geq 0$ by $f(x) = \sqrt{x}$, then f is continuous at every point of its domain.

For $f(x) = \sqrt{x},\ \mathcal{D}(f) = \{x \in \mathbb{R} : x \geq 0\}$, let $a \in \mathcal{D}(f)$.

When $a = 0,\ |f(x) - f(a)| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon$. If we let $\delta(\varepsilon) = \varepsilon^2$, when $x < \varepsilon^2,\ |f(x)| < \varepsilon$.

When $a \neq 0,\ |f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = \frac{|\sqrt{x}-\sqrt{a}|}{|\sqrt{x}+\sqrt{a}|}|\sqrt{x}+\sqrt{a}| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} < \frac{|x-a|}{\sqrt{a}} < \varepsilon \implies$ when $|x-a| < \varepsilon\sqrt{a}$, then, $|f(x) - f(a)| < \varepsilon$, thus we can choose $\delta(\varepsilon) = \varepsilon\sqrt{a} \implies f$ is continuous at every point in its domain.

B. Show that a "polynomial function"; that is, a function f with the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,\ x \in \mathbb{R}$ is continuous at every point of $\mathbb{R}$.

Relying on the properties of algebraic combinations of continuous of functions, we construct f as a combination of continuous functions to show its continuity. Considering the last term of the polynomial function, denoted here, $f_0(x) = a_0,\ f_0(x)$ is a continuous, constant function, since, for any $a \in \mathbb{R}$ we have $|f_0(x) - f_0(a)| = |a_0 - a_0| < \varepsilon = \delta(\varepsilon),\ \varepsilon > 0$. We consider the second to last term of f, $a_1 x$, as a constant, a_1 multiplied by the identity function, denoted, $f_1(x) = x$. Since $f_1(x) = x$, for any real number $a \in \mathbb{R}$, we have $|f_1(x) - f_1(a)| = |x - a| < \varepsilon = \delta(\varepsilon),\ \varepsilon > 0 \implies a_1 f_1(x) = a_1 x$ is continuous.

Relying on the continuity of $f_1(x) = x$ multiplied by any constant, we can construct higher order terms of f through repeated multiplication of $f_1(x)$, e.g. $a_2 \cdot f_1(x) \cdot f_1(x) = a_2 x^2$ and $a_n \prod_{j=1}^n f_1(x) = a_n \cdot f_1(x) \cdot f_1(x) \cdot ... \cdot f_1(x) = a_n x^n$, and so on, where each term constructed $a_n x^n$ is continuous on $\mathbb{R}$ since it is constructed via algebraic combinations of continuous functions $\implies f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, is continuous at every point $x \in \mathbb{R}$.

E. Let f be the function on $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = x,\ x$ irrational, $f(x) = 1 - x,\ x$ rational. Show that f is continuous at $x = \frac{1}{2}$ and discontinuous elsewhere.

Considering the point $a = \frac{1}{2}$, we have $f(a) = \frac{1}{2}$, and $|f(x) - f(a)| = |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = |x - a| < \varepsilon = \delta(\varepsilon)$. So if $|f(x) - f(a)| < \varepsilon = \delta(\varepsilon) > 0 \implies |x - a| < \delta(\varepsilon)$, and then we have f continuous at the point $a = \frac{1}{2}$. For the case $a \neq \frac{1}{2},\ a$ irrational, take a sequence $X = (x_n)$ of rational numbers converging to a. Since the sequence $(f(x_n))$ converges to $1 - a$, and we have $f(a) = a$, f is not continuous at irrational points by the Discontinuity Criterion. For the case $a \neq \frac{1}{2},\ a$ rational, take a sequence $Y = (Y_n)$ of irrational numbers converging to a, the sequence $(f(y_n))$ converges to a, but $f(a) = 1 - a$, which equation is only satisfied when $a = \frac{1}{2}$, thus f is not continuous for rational numbers at any point other than $\frac{1}{2}$.

F.Let f be continuous on $\mathbb{R} \to \mathbb{R}$. Show that if $f(x) = 0$ for rational x, then $f(x) = 0$ for all $x \in \mathbb{R}$.

Every real point, $x \in \mathbb{R}$ is the limit of a sequence of rational numbers. If f is continuous $\implies$ for a sequence of rational numbers $X = (x_n) \to x$, we have $(f(x_n)) = 0$, for all $n \in \mathbb{N}$. Since f is continuous at each rational point $x \in \mathbb{R}$, we can find $|f(x_n) - f(x)| < \varepsilon,\ \varepsilon > 0$, and $|x_n - a| < \delta(\varepsilon) \implies (f(x_n)) \to f(x) = 0, \forall x \in \mathbb{R}$.

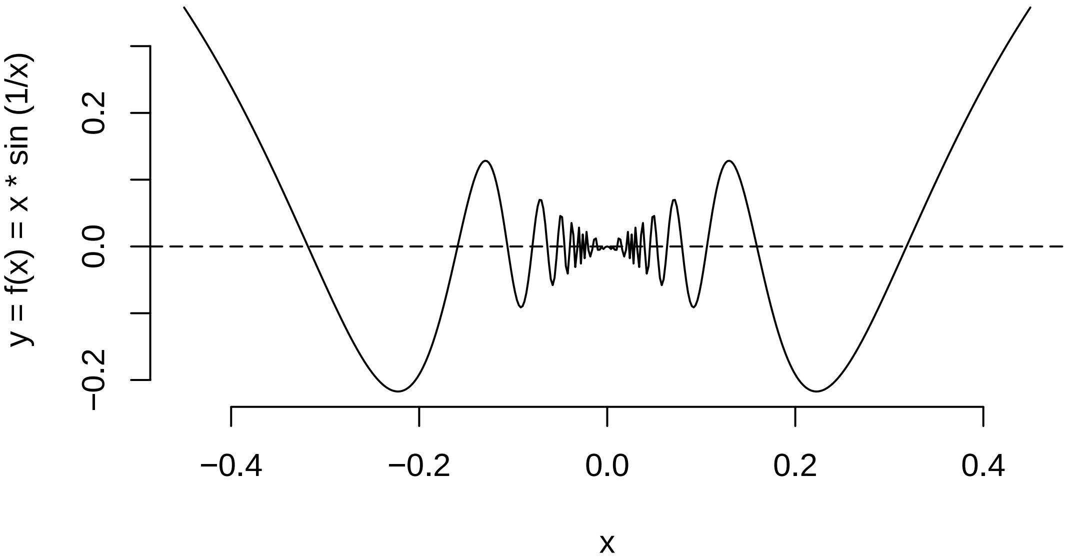
I. Using the results of the preceding exercise, show that the function g, defined on $\mathbb{R} \to \mathbb{R}$ by $g(x) = x sin(\frac{1}{x}),\ x \neq 0,\ g(x) = 0,\ x = 0$ is continuous at every point. Sketch a graph of this function.

For the case $a = 0$, we have $|g(x) - g(a)| = |x \sin \frac{1}{x} - 0| = |x||\sin \frac{1}{x}| \leq |x| \cdot 1 < \varepsilon,\ \varepsilon > 0$, since $-1 \leq \sin \frac{1}{x} \leq 1$. So when $|g(x) - g(0)| < \varepsilon = \delta(\varepsilon)$, we then have $|x| = |x - 0| < \delta(\varepsilon) \implies g$ continuous at 0.

For the case $a \neq 0$, we have $|g(x) - g(a)| = |x \sin \frac{1}{x} - a \sin \frac{1}{a}| = |x \sin \frac{1}{x} - a \sin \frac{1}{a} - a \sin \frac{1}{x} + a \sin \frac{1}{x}| = |(x - a)(\sin \frac{1}{x}) + a(\sin \frac{1}{x} - \sin \frac{1}{a})| \leq |x - a||\sin \frac{1}{x}| + |a||\sin \frac{1}{x} - \sin \frac{1}{a}|$, by Triangle Inequality. Since both $|\sin \frac{1}{x}| \leq 1$ and $|\sin \frac{1}{x} - \sin \frac{1}{a}| \leq 1$, we have $|x - a||\sin \frac{1}{x}| + |a||\sin \frac{1}{x} - \sin \frac{1}{a}| \leq |x - a| \cdot 1 + |a| \cdot 1 = |x - a| + |a| < \varepsilon$.

It then follows that if $\delta(\varepsilon) = \varepsilon - |a|$, i.e. $\varepsilon > \delta(\varepsilon) + |a|$, when $|g(x) - g(a)| < \varepsilon$, then $|x - a| < \delta(\varepsilon) \implies$ g continuous at every point in $\mathbb{R}$.

Sketch of function below:



N. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x + y) = g(x)g(y),\ x, y \in \mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point. Also if $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

If g is continuous at $x = 0 \implies g(x + y) = g(y) = g(0) \cdot g(y)$. This implies also that $g(0)g(y) = g(y) \implies g(0)g(y) - g(y) = 0 = g(y)(g(0) - 1) = 0 \implies g(0) = 1$, or that $g(0) = 0$.
If $g(0) = 0 \implies -g(y) = 0 = g(y)$. In this case then $g(y) = 0,\ \forall y \in \mathbb{R} \implies g(x) = 0,\ \forall x \in \mathbb{R}$.
On the other hand if $g(0) = 1, \implies g(0) \cdot g(y) = g(y)$ continuous for every point $y \in \mathbb{R}$.

Section 21

I. Let g be a linear function from $\mathbb{R}^p \to \mathbb{R}^q$. Show that g is one-one and only if $g(x) = 0$ implies that $x = 0$.
Since g is linear $\implies$ for $x, y \in \mathbb{R}^p$, $g(x + y) = g(x) + g(y)$. Then if $g(x) = 0 \implies g(x + y) = 0 + g(y) = g(y) \implies g(x + y) = g(y) \implies g(x + y) = g(0 + y) = g(y)$ which implies $x = 0$. If we assume that g is one-one, then for any $g(x) = g(y) \implies x = y$. So in the case $g(x) = 0$, and $g(x + y) = g(x) + g(y) = 0 + g(y)$. Since $g(x) + g(y) = g(y) \implies g(y) - g(x) = g(y) \implies x + y = x - y$, which is satisfied when $x = 0$.

J. If h is a one-one linear function from $\mathbb{R}^p \to \mathbb{R}^p$, show that the inverse function h^{-1} is a linear function from $\mathbb{R}^p \to \mathbb{R}^p$.

Since h is one-one $\implies$ if $h(x_1) = h(x_2),\ x_1 = x_2,\ x_1, x_2 \in \mathbb{R}^p$. Extending the the linear case, we have if $h(ax + by) = h(ax_1 + by_1) = ah(x) + bh(y) = ah(x_1) + bh(y_1)$ then $ax_1 + by_1 = ax + by$. By definition $h^{-1} = \{ax + by : h(ax + by) \in \mathbb{R}^p\} = \{ax : h(ax) \in \mathbb{R}^p\} + \{by : h(by) \in \mathbb{R}^p\}$. This implies $h^{-1}(ax + by) = h^{-1}(h(ax)) + h^{-1}(h(by)) \implies h^{-1}$ is linear, and $h^{-}1 : \mathbb{R}^p \to \mathbb{R}^p$, since $h^{-1}(h(ax)) + h^{-1}(h(by)) = ax + by \in \mathbb{R}^p$ by construction.

K. Show that the sum and the composition of two linear functions are linear functions.

By definition a function is linear if $f(ax + by) = af(x) + bf(y),\ a, b \in \mathbb{R},\ x, y \in \mathbb{R}^p$.

For the sum of two linear functions we then have $(f + g)(ax + by) = f(ax + by) + g(ax + by) = af(x) + bf(y) + ag(x) + bf(y) = a(f(x) + g(x)) + b(f(y) + g(y)) = a(f + g)(x) + b(f + g)(y) \implies$ linearity. For the composition of two linear functions we have $f \circ g(ax + bx) = f(g(ax + by)) = f(ag(x) + bg(y)) = af(g(x)) + bf(g(y)) = a(f \circ g)(x) + b(f \circ g)(y) \implies$ composition of two linear functions is linear.

L. If f is a linear map on $\mathbb{R}^p \to \mathbb{R}^q$, define $||f||_{pq} = \sup\{||f(x)|| : x \in \mathbb{R}^p,\ ||x|| \le 1\}$. Show that the mapping $f \to ||f||_{pq}$ defines a norm on the vector space $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ of all linear functions on $\mathbb{R}^p \to \mathbb{R}^q$. Show that $||f(x)|| \le ||f||_{pq}||x||$ for all $x \in \mathbb{R}^p$.

We have $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p,\ f(x) = y = (y_1, y_2, ..., y_q) \in \mathbb{R}^q$, and matrix $A_{q\times p} = (c_{ij}),\ 1 \le i \le q,\ 1 \le j \le p$, with

$$y_1 = c_{11}x_1 + x_{12}x_2 + ... + c_{1p}x_p$$

...

$$y_q = c_{q1}x_1 + x_{q2}x_2 + ... + c_{qp}x_p$$

We then have $||f(x)|| = ||(y_1, ..., y_q)|| = \sqrt{y_1^2 + ... + y_q^2}$. To show $||f||_{qp} = \sup\{||f(x)|| : x \in \mathbb{R}^p,\ ||x|| \le 1\}$ is a norm in $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, we have (i) $||f||_{pq} \ge 0,\ x \in \mathbb{R}^p$? Since each element in $||f(x)|| = \sqrt{y_1^2 + ... + y_q^2}$, $y_j^2 \ge 0,\ \forall j = 1, ..., q \implies \sup\{||f(x)||\} \ge 0 \forall x \in \mathbb{R}^p$ since by definition, $sup\{||f(x)||\} \ge ||f(x)|| \forall x \in \mathbb{R}^p \implies ||f||_{pq} \ge 0$.
(ii) $||f||_{pq} = 0 \iff f(x) = 0$? Since $||f(x)|| = ||y|| = \sqrt{y_1^2 + ... + y_q^2} = 0 \implies$ each $y_j^2 = 0, \forall j = 1, ..., q$
(iii) $\sup ||af(x)|| = |a| \sup ||f(x)|| = |a|||f||_{qp},\ a \in \mathbb{R}$? We have $||af(x)|| = ||ay|| = \sqrt{a^2y_1^2 + ... + a^2y_1^2} = \sqrt{a^2}||y|| = |a|||y||$, and $|a| > 0 \implies \sup\{||af(x)||\} = \sup\{|a|||f(x)||\} = |a|sup\{||f(x)||\}$.
(iv) $\sup\{||f(x+x^{'})||\} \le \sup ||f(x)|| + \sup ||f(x^{'})||,\ x, x^{'} \in \mathbb{R}^p$? Since f is linear $||f(x+x^{'})|| = ||f(x)+f(x^{'})|| \le ||f(x)|| + ||f(x^{'})||,\ \forall x, x^{'} \in \mathbb{R}^p$ by Triangle Inequality, then $\sup\{||f(x)+f(x^{'})||\} \le \sup\{||f(x)||\} + \sup\{||f(x^{'})||\}$. This implies $||f||_{qp}$ is a norm.

To show $||f(x)|| \le ||f||_{pq}||x||$, we use the earlier notation for a linear map, $f(x) = Ax$, where, $A_{q\times p} = (c_{ij})$. Thus $||f(x)|| = ||Ax|| \le |A|||x||$ as shown in (21.5). This implies $\sup\{||f(x)|| : x \in \mathbb{R}^p, ||x|| \le 1\} = \sup\{||Ax||\} \le \sup\{|A|||x||\}$ which is achieved when x is the max value in its domain, i.e. $||x|| = 1$. This implies $\sup\{|||Ax||\}||x|| = \sup\{||f(x)||\}||x|| = \sup\{||f(x)||\} \cdot 1$. This implies $||f(x)|| \le \sup\{||f(x)|| : x \in \mathbb{R}^p, ||x|| \le 1\}||x|| \forall x \in \mathbb{R}^p$.

Section 22

B. Let $H : \mathbb{R} \to \mathbb{R}$ be defined by, $h(x) = 1, 0 \le x \le 1$. $h(x) = 0$, otherwise. Exhibit an open set G such that $h^{-1}(G)$ is not open in $\mathbb{R}$, and a closed set F, such that $h^{-1}(F)$ is not closed in $\mathbb{R}$.

If we take $G = (0, 2)$, and open set, $h^{-1}(G) = \{x \in \mathcal{D}(f) : h(x) \in G\} = [0, 1]$, a closed set. If we take $F = [-2, 0]$, a closed set, the inverse image, $h^{-1}(F) = \{x \in \mathcal{D}(f) : h(x) \in F\}$ is the union of two open sets $(-\infty, 0) \cup (1, +\infty)$ which is open.

C. If f is bounded and continuous on $\mathbb{R}^p \to \mathbb{R}$ and if $f(x_0) > 0$, show that f is strictly positive on some neighborhood of x_0. Does the same conclusion hold if f is merely continuous at x_0?

f is bounded and continuous which implies $0 < f(x_0) < M$, for some $M > 0$. Since f is continuous, for each point $a \in \mathcal{D}(f)$, there is a neighborhood V of $f(a)$ and a neigborhood $U(a) \cap D$ such that if $f(a) \in V \implies a \in U(a)$. Since $f(a) > 0 \implies$ we can take a neighborhood V of $f(a)$ that is also strictly positive, i.e. $V = \{y \in \mathbb{R} : 0 < y < M\}$. If f is not bounded the same argument can be made with $V = \{y \in \mathbb{R} : y > 0\}$.

F. A subset $D \subseteq \mathbb{R}^p$ is disconnected if and only if there exists a continuous function $f : D \to \mathbb{R}$ such that $f(D) = \{0, 1\}$.

$\to D$ disconnected implies there exists two open sets B, C such that $B \cap D$ and $C \cap D$ are disjoint and $(B \cap D) \cup (C \cap D) = D$. We can then construct a function f on D, $f(x) = 1,\ x \in (B \cap D)$, $f(x) = 0,\ x \in (C \cap D)$.
$\leftarrow$ Let $f : D \to \mathbb{R}$ be such that $f(D) = \{0, 1\} \implies$ the inverse image $f^{-1}(\{0, 1\}) = \{x \in D \subseteq : f(x) \in \{0, 1\}\}$ could consist of two disjoint open sets such for f on D, $f(x) = 1,\ x \in (B \cap D)$, $f(x) = 0,\ x \in (C \cap D)$, where $D = (B \cap D) \cup (C \cap D) \subseteq \mathcal{D}(f) \implies$ there exists a continuous function $f : D \to \mathbb{R}$ such that $f(D) = \{0, 1\}$.

H. Let f, g_1, g_2 be related by the formulas in the preceding exercise. Show that from the continuity of g_1 and g_2 at $t = 0$ one cannot prove the continuity of f at $(0,0)$.

Considering g_1, g_2 which are valid are restrictions of the domain of f, given $x = (x_1, x_2) \in \mathbb{R}^2$, we can construct $f(x) = 0,\ x_1 \cdot x_2 = 0,\ f(x) = 1,\ x_1 \cdot x_2 \neq 1$. With this f we have $\lim_{x \to (0,0)} \neq 0$, and $f((0,0)) = 0 \implies$ discontinuity for f at $(0,0)$. Therefore continuity for g_1, g_2 on restrictions of $\mathcal{D}(f)$ does not imply continuity of f.

K. Give an example of a bounded and continuous function g on $\mathbb{R} \to \mathbb{R}$ which does not take on either of the numbers $\sup\{g(x) : x \in \mathbb{R}\}$ or $\inf\{g(x) : x \in \mathbb{R}\}$

If we take $f : \mathbb{R} \to \mathbb{R},\ f(x) = x,\ x \in (0,1) \subseteq \mathbb{R}$, the function is bounded above by 1, below by 0, and continuous on $(0,1)$, but $f(x) \neq 1 = \sup\{f(x) : x \in (0,1)\}$, and $f(x) \neq 0 = \inf\{f(x) : x \in (0,1)\}$ for any x in interval $(0,1)$.

O. Let f be a continuous function on $\mathbb{R} \to \mathbb{R}$ which is strictly increasing (in the sense that if $x^{'} < x^{''}$ then $f(x^{'}) < f(x^{''})$). Prove that f is injective and that its inverse function is continuous and strictly increasing.

For points $x, a, b \in \mathcal{D}(f)$, by f be strictly increasing, we have $a > b \implies f(a) > f(b)$, $a = b \implies f(a) = f(b)$ and $a < b \implies f(a) < f(b)$. If we take point x to be $a < x < b$, we can define two neighborhoods $(a,b) \subseteq \mathcal{D}(f)$, and $(f(a), f(b)) \subseteq \mathcal{R}(f)$, such that $x \in (a,b)$, and $f(x) \in (f(a), f(b))$. This implies f^{-1} in continuous, and since $f^{-1}(f(a)) = a > f^{-1}(f(b)) = b$ if $f(a) > f(b)$, implies f^{-1} is strictly increasing. Also since, $f(a) = f(b) \implies a = b$, f is injective.

Section 23

A. Examine each of the functions in Example 20.5 and either show that the function is uniformly continuous on its domain or not.
(a) The constant function, $\mathcal{D}(f) \subseteq \mathbb{R}, f(x) = c\ ,\ \forall x \in \mathcal{D}(f)$, where c is a real number.
Let $\varepsilon > 0$, we have $|f(x) - f(y)| = |0 - 0| = 0 < \varepsilon,\ \forall x, y \in \mathcal{D}(f)$. Regardless of the choice of $\delta(\varepsilon)$, we have $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta(\varepsilon) \implies$ uniform continuity.

(b) The identity function $f(x) = x,\ x \in \mathbb{R}$.
For all $x, y \in \mathbb{R}$, we have $|f(x) - f(y)| < \varepsilon,\ \varepsilon > 0$. Choose $\delta(\varepsilon) = \varepsilon$. Then whenever $|f(x) - |f(y)| = |x - y| < \varepsilon = \delta(\varepsilon)$ we have $|x - y| < \delta(\varepsilon) \implies$ uniform continuity.

(c) $f(x) = x^2,\ x \in \mathbb{R}$
If we take $\varepsilon = 1$,, and consider point positive real points $x, y = x + \frac{1}{2}$, then for $|f(x) - f(y)| = |x^2 - y^2| = |x^2 - (x + \frac{1}{2})^2| = |x^2 - x^2 - x - \frac{1}{4}| = |(-1)(x + \frac{1}{4})| = x + \frac{1}{4} < 1 = \varepsilon$, which is a contradiction, for example, for all $x > 1 \implies f(x)$ not uniformly continuous.

(d) $f(x) = \frac{1}{x}, x \in \{x \in \mathbb{R} : x \neq 0\}$
If we take $\varepsilon = 1$, consider points $x, y = \frac{x}{2} \in (0,1) \subseteq \mathbb{R}$ we have $|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{\frac{x}{2}}| = |\frac{-1}{x}| = \frac{1}{x}$. Since both $0 < x, y = \frac{x}{2} < 1 \implies \frac{1}{x} > 1$ for all $x, y \in (0,1)$ which implies $f(x) = \frac{1}{x}$ is not uniformly continuous on its domain.

(e) $f(x) = 0,\ x \geq 0,\ f(x) = 1,\ x > 1$
Since f is not continuous at point $a = 0$, and a is in the domain of f, f is not uniformly continuous, since we can find sequence $(f(x_n)) = (f(1/n)) = (1)$ which does not converge to $f(0)$.

(f) $f(x) = 1,\ x$ rational, $f(x) = 0,\ x$ irrational
f is discontinous at every point in its domain, therefore f cannot be uniformly continuous.

(g) $\mathcal{D}(f) = \{x \in \mathbb{R} : x > 0\},\ f(x) = 0,\ x$ irrational, $x > 0$. For rational numbers of the form $\frac{m}{n}$, with $m, n \in \mathbb{N}$ that have no common factor but 1, $f(\frac{m}{n}) = \frac{1}{n}$.
Since f is continuous at precisely irrational points, and not all points in its domain, f is not uniformly continuous.

(h) $\mathcal{D}(f) = \mathbb{R}^2,\ f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (2x+y, x-3y)$. For $(x, y), (a, b) \in \mathcal{D}(f)$ we have $||f(x, y)-f(a, b)|| = ||(2x+y-2a-b, x-3y-a+3b)|| \leq \sqrt{2}\sup\{||2x+y-2a-b||, ||x-3y-a+3b||\} \leq \sqrt{2}\cdot 4||(x, y)-(a, b)|| \leq \varepsilon$, since $|x-a| \leq \sqrt{(x-a)^2 + (y-b)^2} = ||(x, y)-(a, b)|| \implies ||2x+y-2a-b|| = ||2(x-a)+(y-b)|| \leq 3||(x, y)-(a, b)||$, and since $||x-3y-a+3b|| = ||(x-a)+3(b-y)|| \leq 4||(x, y)-(a, b)||$. Putting this together, we have for $\varepsilon > 0$, whenever $||f(x, y)-f(a, b)|| < \varepsilon$ we have $||(x, y)-(a, b)|| \leq \frac{\varepsilon}{4\sqrt{2}}$ which implies uniform continuity.

(i) $\mathcal{D}(f) = \mathbb{R}^2,\ f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (x^2 + y^2, 2xy)$.
Based on 20.j, if $||(x, y) - (a, b)|| < \delta(\varepsilon)$, then we have $||f(x, y) - f(a, b)|| < \varepsilon$ when $\delta(\varepsilon) = \inf\{1, \frac{\varepsilon}{2\sqrt{2}(|a|+|b|+1)}\}$, but since the choice of $\delta(\varepsilon)$ is not independent of points $(a, b) \in \mathbb{R}^2$, implying we can not use the number of all points $(a, b) \in \mathcal{D}(f)$.

C. If B is bounded in $\mathbb{R}^p$ and $f : B \to \mathbb{R}^p$ is uniformly continuous, show that f is bounded on B. Show that this conclusion fails if B is not bounded in $\mathbb{R}^p$.
If we take two sequences in B, $x_n, y_n,\ n \in \mathbb{N}$ by uniform continuity of f, whenever $||x_n - y_n|| \leq \frac{1}{n}$ we have $||f(x_n) - f(y_n)|| < \varepsilon$ for some $\varepsilon > 0$. If we consider the point $x_0 \in B$, for which $f(x_0) = M = \sup\{||f(x)|| : x \in B\}$. By Bolzano-Weierstrass, we can find a subsequence of (x_n), $(x_{n1}, ..., x_{nk})$that converges to $x_0 \implies$ whenever $||x_0 - y_n|| \leq \frac{1}{n}, n \in \mathbb{N}$, we have $||f(x_0) - f(y_n)|| \leq \varepsilon \implies f$ is bounded on B.

D. Show that $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous.
Take $\varepsilon > 0$, for $x, y \in \mathbb{R}$, we have $|f(x) - f(y)| = |\frac{1}{1+x^2} - \frac{1}{1+y^2}| = |\frac{(1+y^2)}{(1+x^2)(1+y^2)} - \frac{(1+x^2)}{(1+y^2)(1+x^2)}| = |\frac{y^2-x^2}{(1+y^2)(1+x^2)}| = |x+y||x-y||\frac{1}{(1+x^2)(1+y^2)}| \leq (|\frac{x}{(1+x^2)(1+y^2)}|+|\frac{y}{(1+x^2)(1+y^2)}|)|x-y| \leq (|\frac{y}{(1+y^2)}|+|\frac{x}{(1+x^2)}|)|x-y|$. Since $\forall x \in \mathbb{R}$, we have $|\frac{x}{1+x^2}| < 1$, we have $(|\frac{y}{(1+y^2)}| + |\frac{x}{(1+x^2)}|)|x-y| < 2|x-y| = 2\delta(\varepsilon) \implies$ if we choose $\delta(\varepsilon) = \frac{\varepsilon}{2}$ whenever $|x-y| < \frac{\varepsilon}{2}$ we have $|f(x) - f(y)| < \varepsilon$, for all $x, y \in \mathbb{R}$

F. Show that $f(x) = \frac{1}{x^2},\ \mathcal{D}(f) = \{x \in \mathbb{R} : x > 0\}$ is not uniformly continuous on its domain.
If we take $\delta(\varepsilon) = \varepsilon/2$, and $\varepsilon = 1$, and consider points in a subset of $\mathcal{D}(f)$, namely $x, y \in (0, 1)$,and then take $y = \frac{x}{2} \in (0, 1)$, we have we have $|x-y| = |x/2| < \varepsilon/2 = 1/2 \implies |f(x) - f(y)| = |\frac{1}{x^2} - \frac{4}{x^2}| = \frac{3}{x^2} < 1$. But, for all $x, y \in (0, 1)$, $\frac{3}{x^2} > 1 \implies f(x)$ is not uniformly continuous on its domain.

G. A function $g : \mathbb{R} \to \mathbb{R}^p$ is periodic if there exists a number $p > 0$ such that $g(x + p) = g(x)$ for all $x \in \mathbb{R}$. Show that a continuous periodic function is bounded and uniformly continuous on $\mathbb{R}$.
We assume g is continous, it implies for $x \in \mathbb{R}$, if we consider points x, y over the domain/interval $[x_0, x_0 + p]$ whenever $|x-y| \leq |x_0 - x_0 - p| = |-p| = p = \delta > 0$ we have $|g(x) - g(y)| < \varepsilon$. Considering points $x + np, y + np,\ n \in \mathbb{N}$, we have $|(x + np) - (y + np)| = |x-y| < \delta$ implying that $|g(x + np) - g(y + np)| = |g(x) - g(y)| < \varepsilon \implies g$ is bounded and uniformly continous on $\mathbb{R}$.

H. Let f be defined on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$, and suppose that f is uniformly continuous on D. If (x_n) is a Cauchy sequence in D, show that (f(x_n)) is a Cauchy sequence in $\mathbb{R}^q$.
A sequence is Cauchy if for some $\delta > 0$ $\exists M \in \mathbb{N}$ such that for all $m, n \geq M$ then we $||x_m - x_n|| < \delta$. Since f is uniform continuous, for $||f(x_m) - f(x_n)||$, for $x_m, x_n \in D$ for all $m, n \in \mathbb{N}$, whenever $||x_m - x_n|| < \delta$, we have $||f(x_m) - f(x_n)|| \leq \varepsilon$ for some $\varepsilon > 0 \implies$ there exists some $M \in \mathbb{N}$ such that for all $m, n \geq M$, $||f(x_m) - f(x_n)|| \leq \varepsilon \implies (f(x_n))$ is Cauchy.

Section 24

B. Give an example of a sequence of everywhere discontinuous functions which converges uniformly to a continuous function.
If we take the example:

$$f_n(x) = \begin{cases} \frac{1}{n} & x \text{ rational} \\ 0 & \text{otherwise} \end{cases}$$

We have discontinuity pointwise, but $\sup\{||f_n - f||\} = \frac{1}{n} \to_{n\to\infty} 0 \implies$ uniform continuity.

D. Let (f_n) be a sequence of continuous functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ such that (f_n) converges uniformly to f on D, and let (x_n) be a sequence of elements in D which converges to $x \in D$. Does it follow that $(f_n(x_n))$

converges to $f(x)$?

Since each $f_n, n \in N$ is continuous, f is continuous. Then whenever, $||x_n - x|| < \delta$, for some $n \geq K \in \mathbb{N}$ we can take $||f(x_n) - f(x)|| < \frac{\varepsilon}{2}$, for some $\varepsilon > 0$. Considering the sequence f_n, we have $||f(x_n) - f(x)|| = ||f(x_n) - f_n(x_n) + f_n(x_n) - f(x)|| \leq ||f(x_n) - f_n(x_n)|| + ||f_n(x_n) - f(x)|| = ||f_n(x_n) - f(x_n)|| + ||f_n(x_n) - f(x)||$. If we take $n \geq M \in \mathbb{N}$, $||f_n(x_n) - f(x_n)|| \leq \frac{\varepsilon}{2}$, by the uniform continuity of f_n. This implies that for $n \geq \sup\{K, M\}$ we have $||f_n(x_n) - f(x_n)|| + ||f_n(x_n) - f(x)|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \implies ||f_n(x_n) - f(x)|| \leq \varepsilon \implies f_n(x_n) \to f(x)$.

E. Consider the sequences (f_n) defined on $D = \{x \in \mathbb{R} : x \geq 0\}$ to $\mathbb{R}$ by the following formulas. Discuss the convergence and uniform convergence of these sequences and the continuity of the limit functions. In case of non-uniform convergence consider appropriate intervals in D.

(b) $\frac{x^n}{1+x^n}$,

For $0 \leq x < 1$, we have $f_n(x) = \frac{x^n}{1+x^n} \to_{n\to\infty} 0$ since $x^n \to 0$ for $0 \leq x < 1$. For $x = 1$, $f_n(x) = \frac{x^n}{1+x^n} = \frac{1}{2}$, $\forall n \in \mathbb{N}$. For $x > 1$, $f_n(x) = \frac{x^n}{1+x^n} \to_{n\to\infty} 1$ which implies (f_n) is pointwise convergent. To examine uniform convergence, we have limit function $f(x) = 0,\ 0 \leq x < 1$, $f(x) = \frac{1}{2},\ x = 1$, and then $f(x) = 1, x > 1 \implies$ uniform converges on closed intervals falling within the interval $x > 1$, or within the interval $0 \leq x < 1$, but not for closed intervals containing the point 1, since the limit function, for example, for x approaching 1 from below, $\lim f_n(x) = 0$, but $f_n(1) = 1/2, \forall n \in \mathbb{N}$. We then do not have uniform convergence over the entire domain, given discontinuous limit functions.

(c) $\frac{x^n}{n+x^n}$,

For $0 \leq x < 1$, we have $f_n(x) = \frac{x^n}{n+x^n} \to \frac{0}{n+0} \to 0$. For $x = 1$, we have $f_n(1) = \frac{1}{n+1} \to 0$. And for $x > 1$, we have $f_n(x) = \frac{x^n}{n+x^n} = \frac{\frac{x^n}{n}}{1+\frac{x^n}{n}} \to 1$, which implies pointwise convergence over $x \geq 0$. To examine uniform convergence we have limit function $f(x) = 0$ for $0 \leq x \leq 1$, and then $f(x) = 1$ for $x > 1$. For $x \in [0, 1]$, we have $||f_n - f||_D = \sup\{||x^n/(n + x^n)|| : x \in [0, 1]\} = \frac{1}{n+1} \to_{n\to\infty} 0 \implies$ uniform continuity on interval $[0, 1]$. For $x > 1$, we have $||f_n - f||_D = \sup\{||\frac{x^n}{n+x^n} - 1|| : x > 1\}$, and $||\frac{x^n}{n+x^n} - 1|| = ||\frac{x^n}{n+x^n} - \frac{n+x^n}{n+x^n}|| = ||\frac{-n}{n+x^n}|| = \frac{n}{n+x^n} = \frac{1/n}{(1/n)+(x^n/n)} \to 0$, since $x > 1$. This implies uniform convergence on the interval $x \in [a, \infty)$, such that $a > 1$.

(d) $\frac{x^{2n}}{1+x^n}$,

For $0 \leq x < 1$, we have $f_n(x) = \frac{x^{2n}}{1+x^n} \to \frac{0}{1+0} \to 0 = f(x)$. For $x = 1$, we have $f_n(1) = \frac{1}{1+1} \to \frac{1}{2} = f(1)$. And for $x > 1$, we have $f_n(x) = \frac{x^{2n}}{1+x^n} = \frac{x^{2n}/n}{1/n+x^n/n} \to \frac{x^{2n}/n}{x^n/n} \to x^n \to +\infty$, which implies pointwise convergence over the first two intervals, $0 \leq x < 1$, and $x = 1$.

To examine uniform convergence, we have limit function $f(x) = 0,\ 0 \leq x < 1$, $f(x) = \frac{1}{2},\ x = 1$, and then divergence for $x > 1$. This implies uniform converges on closed intervals falling within the interval $0 \leq x < 1$, but including the point 1, since the limit function, for example, for x approaching 1 from below, $\lim f_n(x) = 0$, but $f_n(1) = 1/2, \forall n \in \mathbb{N}$. For $x > 1$ we have a divergent sequence of functions. We then do not have uniform convergence given discontinuous limit functions.

(e) $\frac{x^n}{1+x^{2n}}$

For $0 \leq x < 1$, we have $f_n(x) = \frac{x^n}{1+x^{2n}} \to \frac{0}{1+0} \to 0 = f(x)$. For $x = 1$, we have $f_n(1) = \frac{1}{1+1} \to \frac{1}{2} = f(1)$. And for $x > 1$, we have $f_n(x) = \frac{x^n}{1+x^{2n}} = \frac{x^n/n}{1/n+x^{2n}/n} \to \frac{x^n/n}{x^{2n}/n} = \frac{1}{x^n} \to 0$, which implies pointwise convergence over the first two intervals, $0 \leq x < 1$, and $x = 1$.

To examine uniform convergence, we have limit function $f(x) = 0,\ 0 \leq x < 1$, $f(x) = \frac{1}{2},\ x = 1$, and then $f(x) = 0$ for $x > 1$. For $0 \leq x < 1$, we have $||f_n - f||_D = \sup\{\frac{x^n}{1+x^{2n}} : 0 \leq x < 1\} = 0 \implies$ uniform convergence on closed intervals contained in clopen interval $[0, 1)$. For $x > 1$, we have $||f_n - f||_D = \sup\{\frac{x^n}{1+x^{2n}} : x > 1\} = 0 \implies$ uniform convergence on closed intervals contained interval $[a, \infty)$, such that $a > 1$. For $x = 1$ we have $f_n(1) = 1/2$, and $f(1) = 1/2$, and thus have discontinuous limit functions.

J. Prove the following theorem of G. Polya. If for each $n \in \mathbb{N}$ the function f_n on $I \to \mathbb{R}$ is monotone increasing and if $f(x) = \lim(f_n(x))$ is continuous on I, then the convergence is uniform on I. (Observe that it is not assumed that f_n is continuous.)

We have f monotone increasing, and since f is uniformly continuous, if $\varepsilon > 0$, let $0 = x_0 < x_1 < ... < x_h = 1$

be such that $f(x_j) - f(x_{j-1}) < \varepsilon$ and let n_j be such that if $n \geq n_j$; then $|f(x_j) - f_n(x_j)| < \varepsilon$. If we take $||f(x_j) - f_n(x_j)|| \leq \varepsilon$, and take $x \in [x_j, x_{j+1}]$, since f is monotone, we have $f_n(x_j) \leq f_n(x) \leq f_n(x_{j+1})$, and also have $||f_n(x_j) - f(x_j)||$, $||f_n(x) - f(x)||$, $||f_n(x_{j+1}) - f(x_{j+1})||$ are all less than $\varepsilon > 0$. This implies $f(x_j) - \varepsilon \leq f_n(x) \leq f(x_{j+1}) + \varepsilon$, and $f(x_j) \geq f(x) - \varepsilon$, and $f(x_{j+1}) - \varepsilon \leq f(x)$. Putting this together, we then $f(x) - 3\varepsilon \leq f_n(x) \leq f(x) + 3\varepsilon$ which implies uniform convergence.

N. If $f_3(x) = x^3$ for $x \in \mathcal{I}$, calculate the n^{th} Bernstein polynomial for f_3. Show directly that this sequence of polynomials converges uniformly to f_3 on $\mathbb{I}$.
For $f_3 : [0,1] \to \mathbb{R}$, to calculate $B_n(x; f_3)$, for $n = n-3, k = j$, we have $1 = \sum_{j=0}^{n-3} \binom{n-3}{j} x^j (1-x)^{n-(j+3)}$ This $x^3 = \sum_{j=0}^{n-3} \binom{n-3}{j} x^{j+3}(1-x)^{n-(j+3)} = \sum_{j=0}^{n-3} \frac{(j+3)(j+2)(j+1)}{n(n-1)(n-2)} \binom{n}{j+3} x^{j+3}(1-x)^{n-(j+3)}$. If we let $k = j+3$, we then have $x^3 = \sum_{k=0}^{n} \frac{(k)(k-1)(k-2)}{n(n-1)(n-2)} \binom{n}{j+3} x^k (1-x)^{n-k}$, multiplying through by $\frac{1}{n^3}$, we have $\frac{1}{n^3} n(n-1)(n-2)x^3 = \sum_{k=0}^{n} \frac{k^3-3k^2+2k}{n^3} \binom{n}{j+3} x^k(1-x)^{n-k} = \sum_{k=0}^{n} \frac{k^3}{n^3} \binom{n}{j+3} x^k (1-x)^{n-k} - \frac{3}{n}[(1-\frac{1}{n})x^2 + \frac{1}{n}x] + \frac{2}{n^2}x$, since we have from (24.6), $x = \sum_{j=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k}$, and $(1-\frac{1}{n})x^2 + \frac{1}{n}x = \sum_{j=0}^{n} \frac{k^2}{n^2} \binom{n}{k} x^k (1-x)^{n-k}$. We then have $\sum_{k=0}^{n} \frac{k^3}{n^3} \binom{n}{j+3} x^k (1-x)^{n-k} = B_n(x; f_3) = \frac{n(n-1)(n-2)x^3}{n^3} + \frac{3x^2(n-1)}{n^2} + \frac{x^2}{n}$. By Bernstein approximation theorem, for $f_3(x) = x^3$, we have $\sup\{|f_3 - B_n(x; f_3)| : x \in [0,1]\} \leq 1 * (1 - \frac{n(n-1)(n-3)}{n^3}) + \frac{3}{n} + \frac{1}{n} \to_{n\to\infty} 0$, which implies uniform continuity on $[0,1]$.

S. Show that the Weierstrass Approximation Theorem fails for bounded open intervals.
Take (a,b) to be an open bounded interval, with $b > a$. The function $f(x) = \frac{1}{b-x}$, $x \in (a,b)$, we have $||f(x) - P_n(x)||_D = \sup\{||\frac{1}{b-x} - P_n(x)|| : x \in (a,b)\} = \infty$, since $P_n(x)$ must be bounded on (a,b), and $f(x)$ is unbounded as $x \to b$.

Section 26

N. If $K \subseteq \mathbb{R}^p$ is compact and (f_n) is a sequence of continuous functions on K to $\mathbb{R}^q$ which is uniformly convergent on K, show that the family $\{f_n\}$ is uniformly equicontinuous on K in the sense of Definition 26.6.
By uniform equicontinuity, whenever $|x-y| \leq \delta$, $\delta > 0$, then we can have $|f_n(x) - f_n(y)| \leq \frac{\varepsilon}{3}$, $\varepsilon > 0$, $\forall n \in \mathbb{N}$. Since each f_n is continuous it implies for $n \geq M(\varepsilon) \in \mathbb{N}$ we can find $||f_n(x) - f(x)|| \leq \frac{\varepsilon}{3}$, and $||f_n(x) - f_n(y)|| \leq \frac{\varepsilon}{3}$, and also $||f(x) - f(y)|| \leq \frac{\varepsilon}{3}$. This implies by triangle inequality, $||f_n(x) - f(x)|| \leq ||f_n(x) - f_n(y)|| + ||f_n(x) - f(y)|| + ||f(y) - f(x)|| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, which implies for $x, y \in K$, $n \geq M(\varepsilon)$, we have uniform equicontinuity on K.

O. Let $\mathcal{F}$ be a bounded and uniformly equicontinuous collection of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}$ and let f be defined on $D \to \mathbb{R}$ by $f = \sup\{f(x) : f \in \mathcal{F}\}$. Show that f is continuous on $D \to \mathbb{R}$.
We have for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that for $x, y \in D \subseteq \mathbb{R}^p$, whenever we have $||x-y|| < \delta(\varepsilon)$, and $f \in \mathcal{F} \implies ||f(x) - f(y)|| < \varepsilon$. Since functions in $\mathcal{F}$ are bounded and equicontinuous, by Arzela-Ascoli theorem, we have $f^* = \sup\{f(x) : f \in \mathcal{F}\}$, and for some $n \geq K(\varepsilon)$, we cna find a sequence of functions $(f_n) \to f^*$, that is $||f_n(x) - f^*(x)|| < \varepsilon \implies f^*$ is continuous on D.

Q. Consider the following sequences of functions which show that the Arzela-Ascoli Theorem 26.7 may fail if the various hypotheses are dropped.
(a) $f_n(x) = x + n$, $x \in [0,1]$;
$[0,1]$ is compact, and for $x, y \in [0,1]$ whenever $|x-y| < \delta$ have $|f(x) - f(y)| < \varepsilon = \delta$, however $f_n(x)$ is not bounded, since we can always find |f_n(x)| < |f_{n+1}(x)|$. (b) $f_n(x) = x^n$, $x \in [0,1]$;
$[0,1]$ is compact, and $0 \leq |f_n(x)| \leq 1$, $\forall n \in \mathbb{N}$, but not uniformly equicontinuos, example being problem $23.A(c)$, for case $n = 2$. (c) $f_n(x) = \frac{1}{1+(x-n)^2}$, $x \in [0, +\infty)$.
We have $0 < f_n(x) \leq 1$, bounded and uniformly continuous,i.e. $||f_n - f||_D \to_{n\to\infty} 0$, but the domain $[0,\infty)$ is not compact.