Math 4317 (Prof. Swiech, S'18): HW #3

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Section 14

A. Let $b \in \mathbb{R}$*, show* $\lim \frac{b}{n} = 0$.

Take $\varepsilon > 0$, if $|\frac{b}{n} - 0| < \varepsilon$, there exists natural number $K(\varepsilon)$ such that $\frac{b}{n} < \frac{n}{K(\varepsilon)} < \varepsilon$. If $n \geq K(\varepsilon)$, and we choose $K(\varepsilon)$ such that $K(\varepsilon) > \frac{b}{\varepsilon} \implies \frac{b}{n} < \varepsilon \implies \lim \frac{b}{n} = 0$.

B. Show that $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

Take $\varepsilon > 0$, note that for $n \in \mathbb{N}, \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} < \frac{1}{n}$. So we choose natural number $K(\varepsilon)$ such that $\frac{1}{K(\varepsilon)} < \varepsilon$. Therefore if $n \geq K(\varepsilon) \implies \frac{1}{n} < \varepsilon$. Therefore $|\frac{1}{n} - \frac{1}{n+1} - 0| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \varepsilon \implies \lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

D. Let $X = (x_n)$ *be a sequence in* $\mathbb{R}^p$ *which is convergent to* x*. Show that* $\lim ||x_n|| = ||x||$*. (Hint: use the Triangle Inequality.)*

Let $||x|| = \lim(||x_n||)$, $\varepsilon > 0$, which implies there exists natural number $K(\varepsilon)$ such that for $n \geq K(\varepsilon), ||x_n - x|| < \varepsilon$. If $n \geq K(\varepsilon)$, $||x_n|| = ||x_n - x + x|| \leq ||x_n - x|| + ||x|| < \varepsilon + ||x|| \implies ||x_n|| - ||x|| \leq ||x_n - x|| < \varepsilon \implies \lim ||x_n|| = ||x||$.

G. Let $d \in \mathbb{R}$ *satisfy* $d > 1$*. Use Bernoulli's Inequality to show that the sequence* (d_n) *is not bounded in* $\mathbb{R}$*. Hence it is not convergent.$*

We have the sequence $D = (d_n)$, where $d_n = d^n$. Let $d = 1 + a$ for some $a > 0 \implies d^n = (1+a)^n > 1 + na$ by Bernoulli's inequality. For any $a > b > 0$, $(1+a)^n > (1+b)^n$ which implies the sequence d_n is increasing. Take $M > 0$, we have $d^n > 1 + na > M > 0$, if $n > \frac{M}{a} \implies 1 + na > M$. Thus (d_n) is not bounded and its limit tends to ∞.

H. Let $b \in \mathbb{R}$ *satisfy* $0 < b < 1$*; show that* $\lim(nb^n) = 0$*. (Hint: use the Binomial Theorem as in Example 14.8(e).)*

Let $b = \frac{1}{1+a}, a > 0$, we have $b^n = \frac{1}{(1+a)^n}$. By binomial theorem, $(1+a)^n > \frac{n(n-1)}{2}a^2 \implies \frac{1}{(1+a)^n} < \frac{2}{n(n-1)a^2}$, therefore $nb^n = \frac{n}{(1+a)^n} < \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2}$. Take $\varepsilon > 0$, natural number $K(\varepsilon)$, if $n \geq K(\varepsilon)$ we have $nb^n = \frac{n}{(1+a)^n} < \frac{2}{(n-1)a^2} < \frac{2}{(K(\varepsilon)-1)a^2} < \varepsilon$. Then $|nb^n - 0| < \varepsilon \implies nb^n < \varepsilon \implies \lim nb^n = 0$.

I. Let $X = (x_n)$ *be a sequence of strictly positive real numbers such that* $\lim(\frac{x_{n+1}}{x_n}) < 1$*. Show that for some* r *with* $0 < r < 1$ *and some* $C > 0$*, then we have* $0 < x_n < Cr^n$ *for all sufficiently large* $n \in \mathbb{N}$*. Use this to show that* $lim(x_n) = 0$

J. Let $X = (x_n)$ *be a sequence of strictly positive real numbers such that* $\lim(\frac{x_{n+1}}{x_n}) > 1$*. Show that* X *is not a bounded sequence and hence is not convergent.*

K. Give and example of a convergent sequence (x_n) *of strictly positice real numbers such that* $\lim(\frac{x_n+1}{x_n}) = 1$*. Give an example of a divergent sequence with this property.*

L. Apply the results of Exercises 14.I and 14.J to the following sequences. (Here $0 < a < 1, 1 < b, c > 0$*)*

(a) (a^n)

(b) (na^n)

(c) (b^n)

(d) $(\frac{b^n}{n})$

(e) $(\frac{c^n}{n!})$

(f) $(\frac{2^{3n}}{3^{2n}})$

Section 15

C(a-e),E,F,L,N

Section 16

A,B,E,G,J,M(a)(c)(d),N

Section 17

A,B,D,E,L,M

Section 18

A(a-c),D,F,I