Math 4317 (Prof. Swiech, S'18): HW #2

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Section 8

D. If w_1 and w_2 are strictly positive, show that the definition, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1w_1 + x_2y_2w_2$, yields an inner product on $\mathbb{R}^2$, generalize this for $\mathbb{R}^p$.

Checking the properties of inner products, we have, based on definition above, (i) $x \cdot x \geq 0$, since $(x_1, x_2)(x_1, x_2) = w_1x_1^2 + w_2x_2^2 \geq 0$, since $w_1, w_2 > 0$, and $x_i^2 > 0$, $i = 1, 2$. For $x \in \mathbb{R}^p$, we have $x \cdot x = \sum_{j=1}^{p} w_jx_j^2 \geq 0$, since each element in the summation $w_i, x_i^2 > 0$. For property (ii), we have $x \cdot x = 0$, if and only if $x = 0$. In this case, since $w_1, w_2 > 0$, $w_1x_2^2 + w_2x_2^2 = 0$, when x_1^2 and x_2^2 equal zero, that is when $x = 0$. This holds for $x \in \mathbb{R}^p$, since for $w_i > 0, \quad i = 1, ..., p$ we have $\sum_{j=1}^{p} w_jx_j^2 = 0$, only when each element $w_ix_i2 = 0$, since each element is greater than or equal to zero. For property (iii), we show $x \cdot y = y \cdot x$ since $x \cdot y = w_1x_1y_1 + w_2x_2y_2 = w_1x_1y_1 + w_2x_2y_2 = w_1y_1x_2 + w_2y_2x_2 = y \cdot x$. Extending to $x \in \mathbb{R}^p$, we have again, by commutative property, $x \cdot y = \sum_{j=1}^{p} w_jx_jy_j = \sum_{j=1}^{p} w_jy_jx_j = y \cdot x$. Property (iv), $x \cdot (y+z) = x \cdot y + x \cdot z$, $x, y, z \in \mathbb{R}^p$. In this case we have $\sum_{j=1}^{p} w_jx_j(y_j + z_j) = \sum_{j=1}^{p} w_jx_jy_j + w_jx_jz_j = \sum_{j=1}^{p} w_jx_jy_j + \sum_{j=1}^{p} w_jx_jz_j = x \cdot y + x \cdot z$, which clearly holds for base case, $p = 2$ as well. For property (v), we have $(ax) \cdot y = x \cdot (ay)$, $a \in \mathbb{R}$. We have $(ax) \cdot y = \sum_{j=1}^{p} w_jax_jy_j = a \sum_{j=1}^{p} w_jx_jy_j = a(x \cdot y) = \sum_{j=1}^{p} w_jx_jay_j = x \cdot (ay)$. Since all five properties are satisfied, an inner product is yielded here.

E. $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1$ is not an inner product on $\mathbb{R}^2$. Why?

By property (ii), i.e. $x \cdot x = 0$ if and only if $x = 0$, the definition above, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1 = 0 \Leftrightarrow x = 0$, however, we can't say $x = 0$, since in this case if $x_1y_1 = 0 \implies x_1 = 0$, but we don't have information about x_2, or $x_i, i = 3, ..., p$, for $x \in \mathbb{R}^p$. Thus for this operation $x \cdot x = 0$ does not necessarily mean $x = 0$.

F. If $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, define $||x||_1$ by $||x||_1 = |x_1| + |x_2| + ... + |x_p|$. Prove that $x \to ||x||_1$ is a norm on $\mathbb{R}^p$.

(i) $||x||_1 \geq 0$?. Since $|x_j| \geq 0 \ \forall j \implies ||x|| = \sum_{j=1}^{p} |x_j| \geq 0$ by definition of the absolute value.

(ii) $||x||_1 = 0$ if and only if $x = 0$? $||x|| = \sum_{j=1}^{p} |x_j| = 0 \implies x_j = 0 \ \forall j \implies x = 0$.

(iii) $||ax||_1 = |a|||x|| \ \forall a \in \mathbb{R}, \ x \in V$? When $a \geq 0$, and $x_j \geq 0$ or $a < 0$ and $x_j < 0$, $||ax_j|| = ax_j = |a||x_j|$. For the case $a < 0$ and $x_j \geq 0$ or $a \geq 0$ and $x_j < 0$, we have $||a_x)j|| = |ax_j| = (-1)ax_j$ or $a(-1)x_j = a|x_j| = |a||x_j|$.

(iv) $||x + y||_1 \leq ||x|| + ||y||$ for $x, y \in \mathbb{R}^p$?. $||x + y|| = |x_1 + y_1| + |x_2 + y_2| + ... + |x_p + y_p|$. By the triangle inequality, $|x_j + y_j| \leq |x_j| + |y_j|$ for all j. Therefore $|x_1 + y_1| + |x_2 + y_2| + ... + |x_p + y_p| \leq |x_1| + |x_2| + ... + |x_p| + |y_1| + |y_2| + ... + |y_p| = ||x|| + ||y||$. Thus $||x||_1$ is a norm on $\mathbb{R}^p$.

G.If $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, define $||x||_\infty$ by $||x||_\infty = sup\{|x_1| + |x_2| + ... + |x_p|\}$. Prove that $x \to ||x||_\infty$ is a norm on $\mathbb{R}^p$.

(i) $||x||_\infty \geq 0$? Since $|x_j| \geq 0 \ \forall j \implies ||x||_\infty = \sup\{|x_1| + |x_2| + ... + |x_p|\} \geq 0$ since each element in the set is greater than zero.

(ii) $||x||_\infty = 0$ if and only if $x = 0$?. Since each element in the set $\{|x_1| + |x_2| + ... + |x_p|\}$ is greater than or equal to zero, $||x||_\infty = 0$ if and only if $x_j = 0$ for all j, which implies $x = 0$.

(iii) $||ax||_\infty = |a|||x||_\infty \ \forall a \in \mathbb{R}, \ x \in V$? $||ax||_\infty = \sup\{|ax_1| + |ax_2| + ... + |ax_p|\}$, and as shown in 8.F $|ax_j| = |a||x_j|$, which implies $||ax||_\infty = \sup\{|a||x_1| + |a||x_2| + ... + |a||x_p|\} = |a|\sup\{|x_1| + |x_2| + ... + |x_p|\} = |a|||x||_\infty$, since $|a|, |x_j| > 0$. (iv)$||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$ for $x, y \in \mathbb{R}^p$?. Again, by the triangle inequality, $|x_j + y_j| \leq |x_j| + |y_j|$ for all j. Therefore $\sup\{|x_1 + y_1|, |x_2 + y_2|, ..., |x_p + y_p|\} \leq \sup\{|x_1| + |y_1|, |x_2| + |y_2|, ..., |x_p| + |y_p|\}$. If we take $u_x = \sup\{|x_j|\}$,$u_y = \sup\{|y_j|\}$. $u_x + u_y \geq |x_j| + |y_j|$ for all $j \implies \sup\{|x_j|\} + \sup\{|y_j|\} = \sup\{|x_j| + |y_j|\} \implies ||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$. Thus, $||x||_\infty$ is a norm on $\mathbb{R}^p$.

H. In the set $\mathbb{R}^2$, describe the sets:

$S_1 = \{x \in \mathbb{R}^2 : ||x||_1 < 1\}$. $||x||_1 = \sqrt{x_1^2 + x_2^2} < 1$ describes and open circle consisting of points less than 1 in all directions from the origin, satisfying the inequality, $\sqrt{x_1^2} < \sqrt{1 - x_2^2}$. $S_\infty = \{x \in \mathbb{R}^2 : ||x||_\infty < 1\}$, where $||x||_\infty = \sup\{|x_1|, |x_2|\}$, is a dense open box with vertices at $(1,1), (-1,1), (-1,-1), (1,-1)$ with $-1 < x_1 < 1$, and $-1 < x_2 < 1$.

P. If x, y belongs to $\mathbb{R}^p$, show that $||x + y||^2 = ||x||^2 + ||y||^2$ if and only if $x \cdot y = 0$.

$||x + y||^2 = (x + y) \cdot (x + y) = x \cdot x + y \cdot x + x \cdot +y + y \cdot y = ||x||^2 + 2x \cdot y + ||y||^2$, and $2x \cdot y = 0$ if and only if $x \cdot y = 0$, thus, in order for $||x + y||^2 = ||x||^2 + ||y||^2$ to hold, $x \cdot y$ must equal zero.

Q. A subset K of $\mathbb{R}^p$ is said to be convex if, whenever, $x, y \in K$, and t is a real number such that $0 \leq t \leq 1$, then the point $tx + (1 - t)y$ also belongs to K. Show that K_1, K_2, K_3 are convex, but that K_4 is not.

1) $K_1 = \{x \in \mathbb{R}^2 : ||x|| < 1\}$. Let $x, y \in K_1$, then $||tx+(1-t)y|| \leq ||tx||+||(1-t)y|| = |t|||x||+|(1-t)|||y||$, and since $||x|| \leq 1$ and $||y|| \leq 1$, it implies $|t|||x|| + |(1-t)|||y|| \leq |t|(1) + |(1-t)|(1) = t + 1 - t = 1 \implies tx + (1 - t)y \in K_1$.
2) For $K_2 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < \eta\}$. Let $x = (x_1, x_2), y = (y_1, y_2) \in K_2 \implies 0 < x_1 < x_2$ and $0 < y_2 < y_2$, for the point $tx + (1 - t)y$ to belong in K_2 it implies for $t \in [0, 1] \implies 0 < tx_1 < tx_2$, and $0 < (1 - t)y_1 < (1 - t)y2$. Adding these inequalities, we have for $tx + (1 - t)y$, $0 < tx_1 + (1 - t)y_1 < tx_2 + (1 - t)y_2 \implies tx + (1 - t)y \in K_2$.
3) Similarly for $K_3 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 \leq \xi \leq \eta \leq 1\}$, $x, y \in K_3$, $t \in [0, 1]$, we have $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1 \implies 0 \leq tx_1 \leq tx_2 \leq t$ and $0 \leq (1 - t)y_1 \leq (1 - t)y_2 \leq (1 - t)$, again adding the inequalities, we have $0 \leq tx_1 + (1 - t)y_1 \leq tx_2 + (1 - t)y_2 \leq t + (1 - t) = 1 \implies tx + (1 - t)y \in K_3$.
4) For $K_4 = \{x \in \mathbb{R}^2 : ||x|| = 1\}$. Like in K_1, $x, y \in K_4$, then $||tx + (1 - t)y|| = ||tx|| + ||(1 - t)y|| = |t|||x||+|(1-t)|||y||$, and since $||x|| \leq 1$ and $||y|| \leq 1$, it implies $|t|||x||+|(1-t)|||y|| \leq |t|(1)+|(1-t)|(1) = 1$. This equality could hold in some cases where $||x|| = 1$, e.g. $(1, 0), (0, 1)$, but does not hold for all points, and thus K_4 is not convex.

Section 9

B. Justify assertions from 9.2(c):

(i) Denote $x = (x_1, x_2)$ the set $G = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ which is equivalent to $G = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} = ||x|| < 1\}$. Let $\varepsilon = 1 - ||x|| > 0$. Take $y \in \mathbb{R}^2$ such that $||y - x|| < 1$, then, by triangle inequality $||y|| = ||y - x + x|| \leq ||y - x|| + ||x|| < \varepsilon + ||x|| = 1 - ||x|| + ||x|| = 1 \implies y \in G$, and thus G is open.

(ii) Take $x = (x_1, x_2)$, and $H = \{x \in \mathbb{R}^2 : 0 < ||x||^2 < 1\}$. Take $y \in \mathbb{R}^2$ such that $||y - x|| < \varepsilon$, where $\varepsilon = \inf\{||x||, 1 - ||x||\}$. Again $||y|| = ||y - x + x|| \leq ||y - x|| + ||x|| < \varepsilon + ||x|| = 1 - ||x|| + ||x|| = 1 \implies ||y|| < 1$. With $||x - y|| < \varepsilon \implies ||x|| - ||y|| < \varepsilon \implies ||y|| > ||x|| - \varepsilon \implies ||y|| > ||x|| - ||x|| \implies ||y|| > 0 \implies y \in H$, and H is open.

(iii) $F = \{x \in \mathbb{R}^2 : ||x||^2 \leq 1\}$. The complement of F, $F^c = \{x \in \mathbb{R}^2 : ||x||^2 > 1\}$ is open, since for $\varepsilon = ||x|| - 1 > 0$, $y \in \mathbb{R}^2$, $||x - y|| > ||x|| - ||y|| < 1 \implies ||x|| - \varepsilon < ||y|| \implies 1 < ||y|| \implies y \in F^c \implies F^c$ is open, and its complement F must be closed as a result.

D. What are the interior, boundary, and exterior points in $\mathbb{R}$ of the set $[0, 1)$. Conclude that it is neither open nor closed.

Let $A = [0, 1)$. The interior points of A consist of points in the open interval $(0, 1)$ which is entirely contained in A. The boundary points of A are the points 0 and 1. Since neighborhoods around the point 1 and 0 contain both points in A and in its complement A^c. The exterior points of A are points in the set consisting of the union of the intervals $(-\infty, 0) \cup [1, \infty)$. A is not closed, since it does not contain the boundary point, 1. A is not open, by construction, since it is the union of an open and closed set or interval.

G. Show that a subset of $\mathbb{R}^p$ is open if and only if it is the union of a countable collection of open balls.

Let $U \subseteq \mathbb{R}^p$ be open, and $\{x_n : n \in \mathbb{N}\}$ be the set of all rational points in U. Since U is open $\implies$ there exists $r > 0$, such that each point x_n can be contained in the open ball $B_r(x_n) = \{y \in \mathbb{R}^p : |y - x_n| < r\}$, such that $B_r(x_n) \subseteq U \implies \cup_{n\in\mathbb{N}} B_r(x_n) \subseteq U$ if we choose r large enough.

Let $U \subseteq \mathbb{R}^p$ be a countable collection of open balls $\implies$ for every rational point x_n, there exists an open ball $B_r(x_n),\ r > 0$, where $x_n \in B_r(x_n) \implies U \subseteq \cup_{n\in\mathbb{N}} B_r(x_n)$. Which implies $U =\subseteq \cup_{n\in\mathbb{N}} B_r(x_n)$.

I. Show every closed subset of $\mathbb{R}^p$ is the intersection of a countable collection of open sets.

If $U \subseteq \mathbb{R}^p$ is a closed subset, i.e. for $y \in \mathbb{R}^p,\ x \in U,\ r_c > 0,\ U = \{y : ||x - y|| \leq r_c\}$, take the open set $\{y : ||x - y|| > r_c + 1/n\},\ n \in \mathbb{N} \implies x \in U \subseteq \cap_{n\in N}\{y : ||x - y|| < r_c + 1/n\}$.

If $x \notin U \implies x \in \mathbb{R}^p \setminus U \implies x \in \{y : ||x - y|| > r_c\} \implies x \notin \{y : ||x - y|| > r_c + 1/n\},\ n \in \mathbb{N} \implies x \in \mathbb{R}^p \setminus \cap_{n\in N}\{y : ||x - y|| > r_c + 1/n\} \implies \mathbb{R}^p \setminus U \subseteq \cap_{n\in N}\{y : ||x - y|| > r_c + 1/n\} \implies \cap_{n\in N}\{y : ||x - y|| > r_c + 1/n\} \subseteq U$. Thus $U = \cap_{n\in N}\{y : ||x - y|| > r_c + 1/n\}$.

J. If A is any subset of $\mathbb{R}^p$, let A^0 denote the union of all open sets which are contained in A; the set A^0 is called the interior of A Note that A^0 is an open set; (i) prove that it is the largest open set contained in A, also prove: (ii) $A^0 \subseteq A$, (iii) $(A^0)^0 = A^0$, (iv) $(A \cap B)^0 = A^0 \cap B^0$, and (v) $(\mathbb{R}^p)^0 = \mathbb{R}^p$. Also give and example to show $(A \cup B)^0 = A^0 \cup B^0$ may not hold.

(i) Take U as any open set contained in A. A^0 by definition is a union of all these sets, thus each $U \subseteq A^0 \implies A^0 \subseteq A$.

(ii) By definition $(A^0)^0 \subseteq A^0$, and since $(A^0)^0$ is by definition, the union of all open sets in $A^0 \implies A^0 \subseteq (A^0)^0 \implies A^0 = (A^0)^0$.

(iii) $(A \cap B)^0$ is the union of all open sets in $A \cap B \implies (A \cap B)^0 \subseteq A \cap B \implies (A \cap B)^0 \subseteq A$ and $(A \cap B)^0 \subseteq B$. Since A^0, B^0 contain all their open sets $\implies (A \cap B)^0 \subseteq A^0$ and that $(A\cap B)^0 \subseteq B^0 \implies (A\cap B)^0 \subseteq A^0 \cap B^0$. In the other direction, $A^0 \subseteq A, B^0 \subseteq B \implies A^0 \cap B^0 \subseteq (A\cap B)$, and since $A^0 \cap B^0$ is the intersection of two open sets, it follows that $A^0 \cap B^0 \subseteq (A \cap B)^0$. This implies $(A \cap B)^0 = A^0 \cap B^0$.

(iv) $\mathbb{R}^p$ is an open set, and equals the collection of all open sets in it, which implies $\mathbb{R}^p = (\mathbb{R}^p)^0$. *Give an example that $(A \cup B)^0 = A^0 \cup B^0$ may not hold.*
If we take $A = [0, 1], B = [1, 2] \implies A^0 = (0, 1), B^0 = (1, 2) \implies A^0 \cup B^0 = (0, 1) \cup (1, 2),\ (A \cup B)^0 = (0, 2) \implies \{1\} \in (A \cup B)^0,\ \{1\} \notin A^0 \cup B^0$.

K.Prove that a point belongs to A^0 if and only if it is an interior point of A.
Let x be an interior point of $A \implies x$ can be contained in an open set in A, and since A^0 is the union of all open sets in $A \implies x \in A^0$. Let x belong to $A^0 \implies$ belongs to an open set that is contained in $A^0 \implies x$ is an interior point in $A^0 implies$ x in an interior point of A.

L. If A is any subset of $\mathbb{R}^p$, let A^0 denote the intersection of all closed sets which are containing A; the set A^- is called the closure of A Note that A^- is an closed set; (i) prove that it is the smallest closed set containing A, prove that : (ii) $A \subseteq A^-$, (iii) $(A^-)^- = A^-$, (iv) $(A \cup B)^- = A^- \cup B^-$, and (v) $\emptyset^- = \emptyset$

(i) Since A^- is an intersection of all closed sets containing A, including the smallest closed set containing A, A^- must be the smallest closed set containing A. This implies that a closed set $A \subseteq A^-$.

(ii) Since A^- is closed the smallest closed set that contains A^- is $A^- \implies A^- \supseteq (A^-)^-$ and $A^- \subseteq (A^-)^- \implies A^- = (A^-)^-$.

(iii) Let point $x \in (A \cup B)^- = A^- \cup B^- \implies x$ belongs to the smallest closed set containing A or B $\implies x \in A^-$ or $x \in B^- \implies x \in A^- \cup B^-$.

(iv) Since $\emptyset$ is closed and contains no elements, the smallest losed set containing $\emptyset$ is $\emptyset^- \implies \emptyset^- = \emptyset$.
Give an example to show that $(A \cap B)^- = A^- \cap B^-$ may not hold.
Take $A = [0,1], B = (1,2]$, thus $(A \cap B) = \emptyset = (A \cap B)^-, A^- = [0,1], B^- = [1,2]$ and $A^- \cap B^- = \{1\} \neq (A \cap B)^-$.

M. Prove that a point belongs to A^- if and only if it is either and interior or boundary point of A.

Let $x \in A^- \implies x$ belongs to the smallest closed set that contains $A \implies$ a neighborhood of x is either entirely in A or partly in A and $A^c \implies x$ is either an interior or boundary point.

Let x be an interior or boundary point of $A \implies$ any neighborhood of x is either contained in A and $A^c \implies x$ is either in A or in a closed set containing A, i.e. $x \in A^-$.

Section 10

C. A point x is a cluster point of a set $A \subseteq \mathbb{R}^p$ if and only if every neighborhood of x contains infinitely many points of A.
Let x be a cluster point of $A \subseteq \mathbb{R}^p \implies$ there exists and element $a_n \in A$ such that $a_n \neq x$, $0 < ||x - a_n|| < \frac{1}{n}, n \in \mathbb{N} \implies$ there exists an element $a_{n+1} \in A$, such that $0 < ||x - a_{n+1}|| < \frac{1}{n+1}$ such that $a_n \neq a_{n+1}$ etc. which implies that is there is always an element of A that satisfy this property that implies every neighborhood of a point x contains infinitely many points.

D. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that every point of A is a boundary point in $\mathbb{R}$, but that 0 is the only clustter point of A in $\mathbb{R}^p$.

Take $z > 0, z \in \mathbb{R}$. By the completeness of $\mathbb{R}$, and properties of rational numbers, we have a number $\frac{1}{n}$ such that $0 < \frac{1}{n} < z, n \in \mathbb{N}$. Then for each point $x = \frac{1}{n}, n \in \mathbb{N}$, the neighborhood of x consists of only the point $x \in A$, and points in the set $\{y \in \mathbb{R} : \frac{1}{n+1} < y < \frac{1}{n}\} \cup \{y \in \mathbb{R} : \frac{1}{n} < y < \frac{1}{n-1}\}$, but this implies $y \notin A \implies y \in A^c \implies x$ is a boundary point.

Since for $n \in \mathbb{N}$, the point 0, is the only point in A for which the property $0 < ||0 - \frac{1}{n+1}|| < \frac{1}{n} \implies 0 < ||0 - \frac{1}{n+2}|| < \frac{1}{n+1}$ and so on holds, which implies 0 is the only cluster point in A.

E. Let A, B be subsets of $\mathbb{R}^p$ and let x be a cluster point in $A \cap B \in \mathbb{R}^p$. Prove that x is a cluster point of both A and B.

Let x be a cluster point in $A \cap B \subseteq B, A \cap B \subseteq A \implies$ there exists and open set in $A \cap B$ that contains x and a point distinct from $x \implies$ there exists and open set in A that contains x and a point distinct from it, and the same holds for $B \implies x$ is a cluster point of A and B.

F.Let A, B be subsets of $\mathbb{R}^p$ and let x be a cluster point in $A \cup B \in \mathbb{R}^p$. Prove that x is a cluster point of either A or B.

Let x be a cluster point in $A \cup B \subseteq B \implies$ there exists an open set in A or B that contains x and a point distinct from $x \implies$ either A contains x and its neighborhood containing at least another point distinct from x, or B contains x and its neighborhood containing at least one point distinct from $x \implies x$ is a cluster point of either A or B.

G. Show that every point in the Cantor set F is a cluster point of both F and the complement of F,F^c.

The Cantor set, F by definition, is constructed by the intersection of sets $F_n, n \in \mathbb{N}$, where each set F_n is constructed by the union of closed intervals, of the form $[\frac{k}{3^n}, \frac{k+1}{3^n}] \implies$ points in F belonging to all intervals $F_n, n \in \mathbb{N} \implies$ these points are all boundary points of F, examples including $0, \frac{1}{3}, \frac{2}{3}, 1$. Neighborhoods around these boundary points include a point in F and its complement $F^c \implies$ for $n \in \mathbb{N}$, and then the Cantor set F consists of only boundary points which implies every point of F is a cluster point of both F and F^c.

Section 11

A. Show directly from the definition (i.e. with using the Heine-Borel Theorem) that the open ball given by $\{(x,y) : x^2 + y^2 < 1\}$ is not compact in $\mathbb{R}^2$.
Let $H = \{(x,y) : x^2 + y^2 < 1\}$ and let $G_n = \{(x,y) : x^2 + y^2 < 1 - \frac{1}{n}\}$ so that $G^{'} = \{G_n : n \in \mathbb{N}\}$ be a collection collection of these open sets in $\mathbb{R}^2$ whose union contains H. If $\{G_{n_1}, ..., G_{n_k}\}$ is a finite subcollection of $G^{'}$, and $M = \sup\{n_1, ..., n_k\} \implies G_{n_j} \subseteq G_M,\ j = 1, ..., k \implies G_M = \cup_{j=1}^{k} G_{n_j}$, but the point (x,y) satisfying $x^2 + y^2 < 1 - \frac{1}{M}$ does not belong to $G_M \implies (x,y) \notin \cup_{j=1}^{k} G_{n_j} \implies$ no finite union of the sets $G^{'}$ contain $H \implies H$ is not compact.

B. Show directly that the entire space $\mathbb{R}^2$ is not compact.
Let $H = \{(x,y) \in \mathbb{R}^2\}$, $G_n = \{(x,y) : x^2 + y^2 < n^2\}$, and $G^{'} = \{G_n : n \in \mathbb{N}\}$ be a collection of these open sets in $\mathbb{R}^2$ whose union contains H. If $\{G_{n_1}, ..., G_{n_k}\}$ is a finite subcollection of $G^{'}$, and $M = \sup\{n_1, ..., n_k\} \implies G_{n_j} \subseteq G_M,\ j = 1, ..., k \implies G_M = \cup_{j=1}^{k} G_{n_j}$, but the point (x,y) satisfying $x^2 + y^2 < M^2$ does not belong to $G_M \implies (x,y) \notin \cup_{j=1}^{k} G_{n_j} \implies$ no finite union of the sets in $G^{'}$ can contain $\mathbb{R}^2$.

C. Prove directly that if K is compact in $\mathbb{R}^p$ and $F \subseteq K$ is a closed set, then F is compact in $\mathbb{R}^p$
If K is compact in $\mathbb{R}^p$ and F is a closed subset of $K \implies$ there exists a finite collection of open sets $G^{'} = \{G_\alpha\}$ whose union covers K, and further, contains F. Since the complement of closed F, namely, F^c must be open $\implies$, the union of the open set F^c and collection of open sets $G^{'}$ is a finite collection of sets that form a covering for K. Since K is compact, and $F^c \cup G^{'}$ is finite $\implies G^{'}$ is a union of a finite collection of open sets containing $F \implies F$ is compact.

D. Prove that if K is a compact subset of $\mathbb{R}$, then K is compact when regarded as a subset of $\mathbb{R}^p$.
If K is compact $\implies$ that is K is covered by a collection of open sets, G, then it is contained by a finite number of the sets in G. Let $G^{'}$ be an open subset of $\mathbb{R}^2$ such that $G = G^{'} \cap \mathbb{R} \implies G^{'} \subseteq \mathbb{R}^2$ is a union of finite open sets, thus K is compact in regards to being a subset of $\mathbb{R}^2$.

G. Prove the Canton Intersection Theorem by selecting a point x_n from F_n and then applying the Bolzano-Weierstrass Theorem 10.6 to the set $\{x_n : n \in \mathbb{N}\}$.
If $x_n \in F_n, n \in \mathbb{N} \implies$ there exists at least one point in the set of possible x_n that is a common point among the sets F_n, and by construction that each set F_n is bounded and closed. By Bolzano-Weierstrass, every bounded infinite subset of $\mathbb{R}^{'}$ has a cluster point. This implies that if there is at least one x_n common among, these sets, and that there is a cluster point $x \in F_n$ which belongs to all sets $F_k, k \in \mathbb{N}$.

H. If F is closed in $\mathbb{R}^p$ and if $d(x,F) = inf\{||x - z|| : z \in F\} = 0$, then x belongs to F.
$d(x,F) = \inf\{||x - z|| : z \in F\} = 0 \implies x = z,\ z \in F$ or there exists $n \in \mathbb{N}$ such that $0 < ||x - z|| = ||z - x|| < \frac{1}{n} \implies x, z$ are cluster points of $F \in \mathbb{R}^p \implies x \in F$.

J. If F is a non-empty closed set in $\mathbb{R}^p$ and if $x \notin F$, is there is a unique point of F that is nearest to x?

Let $F = \{y \in \mathbb{R}^2 : ||y - x|| = r\} \implies$ we can define a non-empty set where every element contained in the set is the same distance from $x \implies$ there is not a unique element nearest to x.

Section 12

A. If A and B are connected subsets of $\mathbb{R}^p$, give examples to show that $A \cup B, A \cap B, A \setminus B$ can be either connected or disconnected.

Example 1: Take $A = \{x \in \mathbb{R}^p : ||x|| < 1\}, B = \{x \in \mathbb{R}^p : ||x|| = 1\}$, this yields: $A \cup B = \{x \in \mathbb{R}^p : ||x|| \leq 1\}$ which is a connected subset of $\mathbb{R}^p$. $A \cap B = \emptyset$ which could be considered connected since it can't be written as the union of two non-empty sets by lemma 12.6. $A \setminus B$ is connected since $A \setminus B = \{x \in \mathbb{R}^p : ||x|| < 1\}$.

Example 2: Take $A = \{x \in \mathbb{R}^p : ||x|| < 1\}, B = \{x \in \mathbb{R}^p : ||x|| > 1\}$, this yields: $A \cup B = \{x \in \mathbb{R}^p : ||x|| < 1\} \cup \{x \in \mathbb{R}^p : ||x|| > 1\}$ which is disconnected since there is not path through $\{x \in \mathbb{R}^p : ||x|| = 1\}$. $A \cap B = \emptyset$ again, which could be considered connected since it can't be written as the union of two non-empty sets by lemma 12.6. $A \setminus B$ is connected since $A \setminus B = \{x \in \mathbb{R}^p : ||x|| < 1\}$ which is connected.

Example 3: Take $A = \{x \in \mathbb{R}^p : 0 \leq ||x|| \leq 1\}, B = \{x \in \mathbb{R}^p : 0 < ||x|| < 1\}$, this yields: $A \cup B = A$ which is a connected. $A \cap B = B$ which is also connected. $A \setminus B$ is disconnected since $A \setminus B = \{x \in \mathbb{R}^p : ||x|| = 1\} \cup \{x \in \mathbb{R}^p : ||x|| = 0\}$ is disconnected since it can be formed by a union of two open, disjoint, non-empty sets in $\mathbb{R}^p$.

Example 4: Take $A = \{x \in \mathbb{R}^2 : (x-1)^2 + y^2 = 1\}$, a circle of radius 1, centered at the point $(1, 0)$ and $B = \{x \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, a circle of radius 1 centered at the origin. $A \cup B$ yields a connected set since the intersection of these two sets is non-empty. $A \cap B$ is disconnected since the intersection of these two circles consists of two distinct separated points. $A \setminus B$ is disconnected since it consists of the connected set A less the two distinct points where the circles intersect, meaning the set is not pathwise connected.

B. If $C \subseteq \mathbb{R}^p$ is connected and x is a cluster point of C, then $C \cup \{x\}$ is connected.

Assume $C' = C \cup \{x\}$ is disconnected $\implies$ there exists open sets A, B such that $A \cap C'$ and $B \cap C'$ are disjoint, non-empty, and $A \cup B = C'$. Since $x \in C' \implies x \in A$ or $x \in B$, and since x is a cluster point, and A, B are open $\implies$ there is a neighborhood around x with at least one other distinct point *implies* if $x \in A \implies B \cap C' = \emptyset$, if $x \in B \implies A \cap C' = \emptyset \implies C'$ must be connected, otherwise we would have a contradiction.

C. $C \subseteq \mathbb{R}^p$ is connected, show that its closure C^- is also connected.

Suppose $C^- \subseteq A \cup B$, where A, B are open disjoint sets. By the property of the closure, $C \subseteq A \cup B$. Since C is connected, this implies $C \subseteq A$ or $C \subseteq B$. If we take $C \subseteq A \implies C \subseteq B^c$, where B^c is the complement of B. Since A is open, B^c must be closed, and then $C^- \subseteq B^c \implies C^- \cap B = \emptyset \implies C^- \subseteq A \implies C^- \subseteq A^- \implies C^-$ is connected in A.

E. If $K \subseteq \mathbb{R}^p$ is convex, then K is connected.

Since K be convex $\implies$ there exists for $t \in [0, 1],\ x, y \in K$, the point $tx + (1-t)y \in K$.

If we assume that K is not connected $\implies$ there exists open sets A, B such that $A \cup B = K, A \cap B = \emptyset$. If $x, y \in A \cup B \implies tx + (1-t)y \in A \cup B$. But if we take $x \in A,\ y \in B, tx + (1-t)y$ cannot belong to $A \cap B$, since $A \cap B = \emptyset$ by construction. This implies that if $x, y \in K$, that $tx + (1-t)y \in K \implies K$ must be connected.

F. The Cantor set F is wildly disconnected. Show that if $x, y \in F, x \neq y$, than there is a disconnection A, B of F such that $x \in B, y \in B$.

By construction the Cantor set F, with $F_n, n \in \mathbb{N}$ each set consisting of the union of closed intervals $[\frac{k}{3^n}, \frac{k+1}{2^n}]$, which are separate, disjoint.

If we take $x \neq y$ where x and y belong to different closed intervals in $F_n \implies$ we can take sets $A, B \subseteq [0, 1]$ with $x \in A$, $y \in B$ such that $x \in A \cap F_n$, $y \in B \cap F_n$ such that $A \cup B$ consists of the union of two disjoint sets covering all of F.

H. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : 0 < y \leq x^2, x \neq 0\} \cup \{(0, 0)\}$ is connected in $\mathbb{R}^2$. However there does not exist a polygonal curve lying entirely in A joining $(0, 0)$ to other points in the set.

Assume that $A \cup \{(0, 0)\}$ is disconnected $\implies$ there exists non-empty, open, disjoint sets $B, C \subseteq \mathbb{R}^2$ such that $B \cup C = A$ and $B \cap C = \emptyset$. If we take any pair of coordinates $x \neq 0, y > 0$ such that $(x, y) \in B \implies (x, y) \notin C, (x, y) \notin C \cap A \implies C$ consists of the point $(0, 0)$. However, the set consisting of the single point $(0, 0)$ is not open implying a contradiction. Therefore, A must be connected.

If we assume that A is disconnected $\implies$ there exists open sets B, C such that $B \cup C = A$ and $B \cap C = \emptyset$. If we take the first coordinate of $(x, y) \in A$ where $x \neq 0$, that is $x > 0$ or $x < 0$, and $y > 0 \implies$ for points $(x, y) \in A$ there isn't a path connection along $t = [0, 1]$ connecting point $(0, 0)$ to any point $(x, y) \in A$.