

Math 4317 (Prof. Swiech, S'18): HW #2

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Section 8

D. If w_1 and w_2 are strictly positive, show that the definition, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1w_1 + x_2y_2w_2$, yields an inner product on \mathbb{R}^2 , generalize this for \mathbb{R}^p .

Checking the properties of inner products, we have, based on definition above, (i) $x \cdot x \geq 0$, since $(x_1, x_2)(x_1, x_2) = w_1x_1^2 + w_2x_2^2 \geq 0$, since $w_1, w_2 > 0$, and $x_i^2 \geq 0$, $i = 1, 2$. For $x \in \mathbb{R}^p$, we have $x \cdot x = \sum_{j=1}^p w_jx_j^2 \geq 0$, since each element in the summation $w_i, x_i^2 \geq 0$. For property (ii), we have $x \cdot x = 0$, if and only if $x = 0$. In this case, since $w_1, w_2 > 0$, $w_1x_1^2 + w_2x_2^2 = 0$, when x_1^2 and x_2^2 equal zero, that is when $x = 0$. This holds for $x \in \mathbb{R}^p$, since for $w_i > 0$, $i = 1, \dots, p$ we have $\sum_{j=1}^p w_jx_j^2 = 0$, only when each element $w_ix_i^2 = 0$, since each element is greater than or equal to zero. For property (iii), we show $x \cdot y = y \cdot x$ since $x \cdot y = w_1x_1y_1 + w_2x_2y_2 = w_1x_1y_1 + w_2x_2y_2 = w_1y_1x_2 + w_2y_2x_2 = y \cdot x$. Extending to $x \in \mathbb{R}^p$, we have again, by commutative property, $x \cdot y = \sum_{j=1}^p w_jx_jy_j = \sum_{j=1}^p w_jy_jx_j = y \cdot x$. Property (iv), $x \cdot (y+z) = x \cdot y + x \cdot z$, $x, y, z \in \mathbb{R}^p$. In this case we have $\sum_{j=1}^p w_jx_j(y_j+z_j) = \sum_{j=1}^p w_jx_jy_j + \sum_{j=1}^p w_jx_jz_j = \sum_{j=1}^p w_jx_jy_j + \sum_{j=1}^p w_jx_jz_j = x \cdot y + x \cdot z$, which clearly holds for base case, $p = 2$ as well. For property (v), we have $(ax) \cdot y = x \cdot (ay)$, $a \in \mathbb{R}$. We have $(ax) \cdot y = \sum_{j=1}^p w_jax_jy_j = a \sum_{j=1}^p w_jx_jy_j = a(x \cdot y) = \sum_{j=1}^p w_jx_jay_j = x \cdot (ay)$. Since all five properties are satisfied, an inner product is yielded here.

E. $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1$ is not an inner product on \mathbb{R}^2 . Why?

By property (ii), i.e. $x \cdot x = 0$ if and only if $x = 0$, the definition above, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1 = 0 \Leftrightarrow x = 0$, however, we can't say $x = 0$, since in this case if $x_1y_1 = 0 \Rightarrow x_1 = 0$, but we don't have information about x_2 , or $x_i, i = 3, \dots, p$, for $x \in \mathbb{R}^p$. Thus for this operation $x \cdot x = 0$ does not necessarily mean $x = 0$.

F. If $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$, define $\|x\|_1$ by $\|x\|_1 = |x_1| + |x_2| + \dots + |x_p|$. Prove that $x \rightarrow \|x\|_1$ is a norm on \mathbb{R}^p .

- (i) $\|x\|_1 \geq 0$? Since $|x_j| \geq 0 \forall j \Rightarrow \|x\|_1 = \sum_{j=1}^p |x_j| \geq 0$ by definition of the absolute value.
- (ii) $\|x\|_1 = 0$ if and only if $x = 0$? $\|x\|_1 = \sum_{j=1}^p |x_j| = 0 \Rightarrow x_j = 0 \forall j \Rightarrow x = 0$.
- (iii) $\|ax\|_1 = |a|\|x\|_1 \forall a \in \mathbb{R}, x \in V$? When $a \geq 0$, and $x_j \geq 0$ or $a < 0$ and $x_j < 0$, $\|ax_j\|_1 = ax_j = |a||x_j|$. For the case $a < 0$ and $x_j \geq 0$ or $a \geq 0$ and $x_j < 0$, we have $\|a_xj\|_1 = |ax_j| = (-1)ax_j$ or $a(-1)x_j = a|x_j| = |a||x_j|$.
- (iv) $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$ for $x, y \in \mathbb{R}^p$? $\|x+y\|_1 = |x_1+y_1| + |x_2+y_2| + \dots + |x_p+y_p|$. By the triangle inequality, $|x_j+y_j| \leq |x_j| + |y_j|$ for all j . Therefore $|x_1+y_1| + |x_2+y_2| + \dots + |x_p+y_p| \leq |x_1| + |x_2| + \dots + |x_p| + |y_1| + |y_2| + \dots + |y_p| = \|x\|_1 + \|y\|_1$. Thus $\|x\|_1$ is a norm on \mathbb{R}^p .

G. If $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$, define $\|x\|_\infty$ by $\|x\|_\infty = \sup\{|x_1| + |x_2| + \dots + |x_p|\}$. Prove that $x \rightarrow \|x\|_\infty$ is a norm on \mathbb{R}^p .

- (i) $\|x\|_\infty \geq 0$? Since $|x_j| \geq 0 \forall j \Rightarrow \|x\|_\infty = \sup\{|x_1| + |x_2| + \dots + |x_p|\} \geq 0$ since each element in the set is greater than zero.
- (ii) $\|x\|_\infty = 0$ if and only if $x = 0$? Since each element in the set $\{|x_1| + |x_2| + \dots + |x_p|\}$ is greater than or equal to zero, $\|x\|_\infty = 0$ if and only if $x_j = 0$ for all j , which implies $x = 0$.
- (iii) $\|ax\|_\infty = |a|\|x\|_\infty \forall a \in \mathbb{R}, x \in V$? $\|ax\|_\infty = \sup\{|ax_1| + |ax_2| + \dots + |ax_p|\}$, and as shown in 8.F $|ax_j| = |a||x_j|$, which implies $\|ax\|_\infty = \sup\{|a||x_1| + |a||x_2| + \dots + |a||x_p|\} = |a|\sup\{|x_1| + |x_2| + \dots + |x_p|\} = |a|\|x\|_\infty$, since $|a|, |x_j| > 0$. (iv) $\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ for $x, y \in \mathbb{R}^p$? Again, by the triangle inequality, $|x_j+y_j| \leq |x_j| + |y_j|$ for all j . Therefore $\sup\{|x_1+y_1|, |x_2+y_2|, \dots, |x_p+y_p|\} \leq \sup\{|x_1| + |y_1|, |x_2| + |y_2|, \dots, |x_p| + |y_p|\}$. If we take $u_x = \sup\{|x_j|\}, u_y = \sup\{|y_j|\}$. $u_x + u_y \geq |x_j| + |y_j|$ for all $j \Rightarrow \sup\{|x_j| + |y_j|\} = \sup\{|x_j| + |y_j|\} \Rightarrow \|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$. Thus, $\|x\|_\infty$ is a norm on \mathbb{R}^p .

H. In the set \mathbb{R}^2 , describe the sets:

$S_1 = \{x \in \mathbb{R}^2 : \|x\|_1 < 1\}$. $\|x\|_1 = \sqrt{x_1^2 + x_2^2} < 1$ describes an open circle consisting of points less than 1 in all directions from the origin, satisfying the inequality, $\sqrt{x_1^2} < \sqrt{1 - x_2^2}$. $S_\infty = \{x \in \mathbb{R}^2 : \|x\|_\infty < 1\}$, where $\|x\|_\infty = \sup\{|x_1|, |x_2|\}$, is a dense open box with vertices at $(1, 1), (-1, 1), (-1, -1), (1, -1)$ with $-1 < x_1 < 1$, and $-1 < x_2 < 1$.

P. If x, y belongs to \mathbb{R}^p , show that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if and only if $x \cdot y = 0$.

$\|x + y\|^2 = (x + y) \cdot (x + y) = x \cdot x + y \cdot x + x \cdot y + y \cdot y = \|x\|^2 + 2x \cdot y + \|y\|^2$, and $2x \cdot y = 0$ if and only if $x \cdot y = 0$, thus, in order for $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ to hold, $x \cdot y$ must equal zero.

Q. A subset K of \mathbb{R}^p is said to be convex if, whenever, $x, y \in K$, and t is a real number such that $0 \leq t \leq 1$, then the point $tx + (1 - t)y$ also belongs to K . Show that K_1, K_2, K_3 are convex, but that K_4 is not.

- 1) $K_1 = \{x \in \mathbb{R}^2 : \|x\| < 1\}$. Let $x, y \in K_1$, then $\|tx + (1 - t)y\| \leq \|tx\| + \|(1 - t)y\| = |t|\|x\| + (1 - t)\|y\|$, and since $\|x\| \leq 1$ and $\|y\| \leq 1$, it implies $|t|\|x\| + (1 - t)\|y\| \leq |t|(1) + (1 - t)(1) = t + 1 - t = 1 \implies tx + (1 - t)y \in K_1$.
- 2) For $K_2 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < \eta\}$. Let $x = (x_1, x_2), y = (y_1, y_2) \in K_2 \implies 0 < x_1 < x_2$ and $0 < y_1 < y_2$, for the point $tx + (1 - t)y$ to belong in K_2 it implies for $t \in [0, 1] \implies 0 < tx_1 < tx_2$, and $0 < (1 - t)y_1 < (1 - t)y_2$. Adding these inequalities, we have for $tx + (1 - t)y$, $0 < tx_1 + (1 - t)y_1 < tx_2 + (1 - t)y_2 \implies tx + (1 - t)y \in K_2$.
- 3) Similarly for $K_3 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 \leq \xi \leq \eta \leq 1\}$, $x, y \in K_3$, $t \in [0, 1]$, we have $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1 \implies 0 \leq tx_1 \leq tx_2 \leq t$ and $0 \leq (1 - t)y_1 \leq (1 - t)y_2 \leq (1 - t)$, again adding the inequalities, we have $0 \leq tx_1 + (1 - t)y_1 \leq tx_2 + (1 - t)y_2 \leq t + (1 - t) = 1 \implies tx + (1 - t)y \in K_3$.
- 4) For $K_4 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Like in K_1 , $x, y \in K_4$, then $\|tx + (1 - t)y\| = |t|\|x\| + \|(1 - t)y\| = |t|\|x\| + (1 - t)\|y\|$, and since $\|x\| \leq 1$ and $\|y\| \leq 1$, it implies $|t|\|x\| + (1 - t)\|y\| \leq |t|(1) + (1 - t)(1) = 1$. This equality could hold in some cases where $\|x\| = 1$, e.g. $(1, 0), (0, 1)$, but does not hold for all points, and thus K_4 is not convex.

Section 9

B. Justify assertions from 9.2(c):

- (i) Denote $x = (x_1, x_2)$ the set $G = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ which is equivalent to $G = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} = \|x\| < 1\}$. Let $\varepsilon = 1 - \|x\| > 0$. Take $y \in \mathbb{R}^2$ such that $\|y - x\| < 1$, then, by triangle inequality $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| < \varepsilon + \|x\| = 1 - \|x\| + \|x\| = 1 \implies y \in G$, and thus G is open.
- (ii) Take $x = (x_1, x_2)$, and $H = \{x \in \mathbb{R}^2 : 0 < \|x\|^2 < 1\}$. Take $y \in \mathbb{R}^2$ such that $\|y - x\| < \varepsilon$, where $\varepsilon = \inf\{\|x\|, 1 - \|x\|\}$. Again $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| < \varepsilon + \|x\| = 1 - \|x\| + \|x\| = 1 \implies \|y\| < 1$. With $\|x - y\| < \varepsilon \implies \|x\| - \|y\| < \varepsilon \implies \|y\| > \|x\| - \varepsilon \implies \|y\| > \|x\| - \|x\| \implies \|y\| > 0 \implies y \in H$, and H is open.
- (iii) $F = \{x \in \mathbb{R}^2 : \|x\|^2 \leq 1\}$. The complement of F , $F^c = \{x \in \mathbb{R}^2 : \|x\|^2 > 1\}$ is open, since for $\varepsilon = \|x\| - 1 > 0$, $y \in \mathbb{R}^2$, $\|x - y\| > \|x\| - \|y\| < 1 \implies \|x\| - \varepsilon < \|y\| \implies 1 < \|y\| \implies y \in F^c \implies F^c$ is open, and its complement F must be closed as a result.

D. What are the interior, boundary, and exterior points in \mathbb{R} of the set $[0, 1)$. Conclude that it is neither open nor closed.

Let $A = [0, 1)$. The interior points of A consist of points in the open interval $(0, 1)$ which is entirely contained in A . The boundary points of A are the points 0 and 1. Since neighborhoods around the point 1 and 0 contain both points in A and in its complement A^c . The exterior points of A are points in the set consisting of the union of the intervals $(-\infty, 0) \cup [1, \infty)$. A is not closed, since it does not contain the boundary point, 1. A is not open, by construction, since it is the union of an open and closed set or interval.

G. Show that a subset of \mathbb{R}^p is open if and only if it is the union of a countable collection of open balls.

Let $U \subseteq \mathbb{R}^p$ be open, and $\{x_n : n \in \mathbb{N}\}$ be the set of all rational points in U . Since U is open \implies there exists $r > 0$, such that each point x_n can be contained in the open ball $B_r(x_n) = \{y \in \mathbb{R}^p : |y - x_n| < r\}$, such that $B_r(x_n) \subseteq U \implies \cup_{n \in \mathbb{N}} B_r(x_n) \subseteq U$ if we choose r large enough.

Let $U \subseteq \mathbb{R}^p$ be a countable collection of open balls \implies for every rational point x_n , there exists an open ball $B_r(x_n)$, $r > 0$, where $x_n \in B_r(x_n) \implies U \subseteq \cup_{n \in \mathbb{N}} B_r(x_n)$. Which implies $U = \subseteq \cup_{n \in \mathbb{N}} B_r(x_n)$.

I. Show every closed subset of \mathbb{R}^p is the intersection of a countable collection of open sets.

If $U \subseteq \mathbb{R}^p$ is a closed subset, i.e. for $y \in \mathbb{R}^p$, $x \in U$, $r_c > 0$, $U = \{y : \|x - y\| \leq r_c\}$, take the open set $\{y : \|x - y\| > r_c + 1/n\}$, $n \in \mathbb{N} \implies x \in U \subseteq \cap_{n \in \mathbb{N}} \{y : \|x - y\| < r_c + 1/n\}$.

If $x \notin U \implies x \in \mathbb{R}^p \setminus U \implies x \in \{y : \|x - y\| > r_c\} \implies x \notin \{y : \|x - y\| > r_c + 1/n\}$, $n \in \mathbb{N} \implies x \in \mathbb{R}^p \setminus \cap_{n \in \mathbb{N}} \{y : \|x - y\| > r_c + 1/n\} \implies \mathbb{R}^p \setminus U \subseteq \cap_{n \in \mathbb{N}} \{y : \|x - y\| > r_c + 1/n\} \implies \cap_{n \in \mathbb{N}} \{y : \|x - y\| > r_c + 1/n\} \subseteq U$. Thus $U = \cap_{n \in \mathbb{N}} \{y : \|x - y\| > r_c + 1/n\}$.

J. If A is any subset of \mathbb{R}^p , let A^0 denote the union of all open sets which are contained in A ; the set A^0 is called the interior of A . Note that A^0 is an open set; (i) prove that it is the largest open set contained in A , also prove: (ii) $A^0 \subseteq A$, (iii) $(A^0)^0 = A^0$, (iv) $(A \cap B)^0 = A^0 \cap B^0$, and (v) $(\mathbb{R}^p)^0 = \mathbb{R}^p$. Also give and example to show $(A \cup B)^0 = A^0 \cup B^0$ may not hold.

(i) Take U as any open set contained in A . A^0 by definition is a union of all these sets, thus each $U \subseteq A^0 \implies A^0 \subseteq A$.

(ii) By definition $(A^0)^0 \subseteq A^0$, and since $(A^0)^0$ is by definition, the union of all open sets in $A^0 \implies A^0 \subseteq (A^0)^0 \implies A^0 = (A^0)^0$.

(iii) $(A \cap B)^0$ is the union of all open sets in $A \cap B \implies (A \cap B)^0 \subseteq A \cap B \implies (A \cap B)^0 \subseteq A$ and $(A \cap B)^0 \subseteq B$. Since A^0, B^0 contain all their open sets $\implies (A \cap B)^0 \subseteq A^0$ and that $(A \cap B)^0 \subseteq B^0 \implies (A \cap B)^0 \subseteq A^0 \cap B^0$. In the other direction, $A^0 \subseteq A, B^0 \subseteq B \implies A^0 \cap B^0 \subseteq (A \cap B)$, and since $A^0 \cap B^0$ is the intersection of two open sets, it follows that $A^0 \cap B^0 \subseteq (A \cap B)^0$. This implies $(A \cap B)^0 = A^0 \cap B^0$.

(iv) \mathbb{R}^p is an open set, and equals the collection of all open sets in it, which implies $\mathbb{R}^p = (\mathbb{R}^p)^0$.

(v) Example that $(A \cup B)^0 = A^0 \cup B^0$ may not hold. If we take $A = [0, 1], B = [1, 2] \implies A^0 = (0, 1), B^0 = (1, 2) \implies A^0 \cup B^0 = (0, 1) \cup (1, 2)$, $(A \cup B)^0 = (0, 2) \implies \{1\} \in (A \cup B)^0, \{1\} \notin A^0 \cup B^0$.

K. Prove that a point belongs to A^0 if and only if it is an interior point of A .

Let x be an interior point of $A \implies x$ can be contained in an open set in A , and since A^0 is the union of all open sets in $A \implies x \in A^0$. Let x belong to $A^0 \implies$ belongs to an open set that is contained in $A^0 \implies x$ is an interior point in A^0 implies x in an interior point of A .

L. If A is any subset of \mathbb{R}^p , let A^0 denote the intersection of all closed sets which are containing A ; the set A^- is called the closure of A . Note that A^- is an closed set; (i) prove that it is the smallest closed set containing A , prove that : (ii) $A \subseteq A^-$, (iii) $(A^-)^- = A^-$, (iv) $(A \cup B)^- = A^- \cup B^-$, and (v) $\emptyset^- = \emptyset$

(i) Since A^- is an intersection of all closed sets containing A , including the smallest closed set containing A , A^- must be the smallest closed set containing A . This implies that a closed set $A \subseteq A^-$.

(ii) Since A^- is closed the smallest closed set that contains A^- is $A^- \implies A^- \supseteq (A^-)^-$ and $A^- \subseteq (A^-)^- \implies A^- = (A^-)^-$.

(iii) Let point $x \in (A \cup B)^- = A^- \cup B^- \implies x$ belongs to the smallest closed set containing A or $B \implies x \in A^-$ or $x \in B^- \implies x \in A^- \cup B^-$.

(iv) Since \emptyset is closed and contains no elements, the smallest closed set containing \emptyset is $\emptyset^- \implies \emptyset^- = \emptyset$.

Give an example to show that $(A \cap B)^- = A^- \cap B^-$ may not hold. Take $A = [0, 1], B = (1, 2]$, thus $(A \cap B) = \emptyset = (A \cap B)^-, A^- = [0, 1], B^- = [1, 2]$ and $A^- \cap B^- = \{1\} \neq (A \cap B)^-$.

M. Prove that a point belongs to A^- if and only if it is either an interior or boundary point of A .

Let $x \in A^- \implies x$ belongs to the smallest closed set that contains $A \implies$ a neighborhood of x is either entirely in A or partly in A and $A^c \implies x$ is either an interior or boundary point.

Let x be an interior or boundary point of $A \implies$ any neighborhood of x is either contained in A and $A^c \implies x$ is either in A or in a closed set containing A , i.e. $x \in A^-$.

Section 10

C. A point x is a cluster point of a set $A \subseteq \mathbb{R}^p$ if and only if every neighborhood of x contains infinitely many points of A . D. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that every point of A is a boundary point in \mathbb{R} , but that 0 is the only cluster point of A in \mathbb{R}^p . E. Let A, B be subsets of \mathbb{R}^p and let x be a cluster point in $A \cap B$ in \mathbb{R}^p . Prove that x is a cluster point of both A and B . F. Let A, B be subsets of \mathbb{R}^p and let x be a cluster point in $A \cup B$ in \mathbb{R}^p . Prove that x is a cluster point of either A or B . G. Show that every point in the Cantor set F is a cluster point of both F and the complement of F , F^c .

Section 11

Section 12