Math 4317 (Prof. Swiech, S'18): HW #1

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Section 1

F. Show that the symmetric difference D, defined in the preceding exercise is also given by $D = (A \cup B) \setminus (A \cap B)$
Show $D = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$:

First, $x \in (A \setminus B) \cup (B \setminus A) \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies$, x is in A but not B, or, x is in B but not $A \implies x$ is in A or B but not in A and $B \implies x \in (A \cup B) \setminus (A \cap B)$.

In the other direction, $x \in (A \cup B) \setminus (A \cap B) \implies x \in (A \cup B)$ but not in $(A \cap B) \implies x$ is in A but not B, or, x is in B but not $A \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies x \in (A \setminus B) \cup (B \setminus A) \implies (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

I. If $\{A_1, A_2, ..., A_n\}$ *is a collection of sets, and if E is any set, show that:*

$$(i) \quad E \cap \cup_{j=1}^{n} A_j = \cup_{j=1}^{n} (E \cap A_j), \; and \; (ii), \quad E \cup \cup_{j=1}^{n} A_j = \cup_{j=1}^{n} (E \cup A_j)$$

(i) $x \in E \cap \cup_{j=1}^{n} A_j \implies x \in E \; and \; x \in \{A_1 \; or \; A2 \; ...or \; A_n\} \implies x \in E$ and that there exists for some $j = 1, 2, ..., n$ an A_j such that $x \in A_j \; and \; x \in E \implies (x \in E \; and \; A_1) \; or \; (x \in E \; and \; A_2) \; ... \; or \; (x \in E \; and \; A_n) \implies x \in \cup_{j=1}^{n} (E \cap A_j)$.
In the other direction, $x \in \cup_{j=1}^{n} (E \cap A_j) \Leftrightarrow x \in (E \cap A_1) \cup (E \cap A_2) \; ... \; \cup (E \cap A_n) \implies x \in E$ and A_1 or E and $A_2 \ldots \implies$ there exists a $j = 1, ..., n$ such that $x \in (E \cap A_j) \implies x \in E$ and $x \in A_1$ or $A_2, \ldots,$ or $A_n \implies x \in E$ and $\cup_{j=1}^{n} A_j \implies x \in E \cap \cup_{j=1}^{n} A_j$.

(ii) $x \in E \cup \cup_{j=1}^{n} A_j \implies x \in E$ or $x \in A_1$ or $A_2 \ldots$ or $A_n \implies$ for some $j = 1, .., n$ that $x \in E \cup A_j \implies x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies x \in \cup_{j=1}^{n} (E \cup A_j)$. In the other direction, $x \in \cup_{j=1}^{n} (E \cup A_j) \Leftrightarrow x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies$ there exists some $j = 1, ..., n$ such that $x \in E \cup A_j \implies (x \in E \; or \; x \in A_1)$ or $(x \in E \; or \; x \in A_2) \ldots$ or $(x \in E \; or \; x \in A_n) \implies x \in E$ or $x \in \cup_{j=1}^{n} A_j \implies x \in E \cup \cup_{j=1}^{n} A_j$.

J. If $\{A_1, A_2, ..., A_n\}$ *is a collection of sets, and if E is any set, show that:*

$$(i) \; E \cap \cap_{j=1}^{n} A_j = \cap_{j=1}^{n} (E \cap A_j), \; and \; (ii), \quad E \cup \cap_{j=1}^{n} A_j = \cap_{j=1}^{n} (E \cup A_j)$$

(i) $x \in \cap \cap_{j=1}^{n} A_j \implies x \in E$ and $x \in \cap_{j=1}^{n} A_j \implies x \in E \; and \; x \in A_j$ for all $j = 1, .., n \implies x \in E$ and $[x \in A_1 \; and \; x \in A_2 \; ... \; and \; x \in A_n] \implies [x \in E \; and \; A_1]$ and $\ldots$ and $[x \in E \; and \; A_n] \implies x \in \cap_{j=1}^{n} (E \cap A_j)$. In the other direction, $x \in \cap_{j=1}^{n} (E \cap A_j) \implies x \in (E \cap A_1)$ and $a \in (E \cap A_2) \ldots$ and $x \in (E \cap A_n) \implies x \in (E \cap A_j)$ for all $j = 1, ..., n \implies x \in E$ and $x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n \implies x \in E$ and $x \in \cap_{j=1}^{nA_j} \implies x \in E \cap \cap_{j=1}^{nA_j}$.

(ii) $x \in E \cup \cap_{j=1}^{n} A_j \implies x \in E$ or $x \in \cap_{j=1}^{n} A_j \implies x \in E$ or $[x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n] \implies x \in E \; or \; A_1 \; and \; x \in E \; or \; A_2 \; ... \; and \; x \in E \; or \; A_n \implies x \in \cap_{j=1}^{n} (E \cup A_j)$. In the other direction, $x \in \cap_{j=1}^{n} (E \cup A_j) \implies x \in (E \; or \; A_1) \; and \; x \in (E \; or \; A_2) \; ...and \; x \in (E \; or \; A_n) \implies$ that for all $j = 1, ..., n$, $x \in (E \; or \; A_j) \implies x \in E \; or \; (x \in A_1 \; and \; x \in A_2 ... \; and \; x \in A_n) \implies x \in \cap_{j=1}^{n} A_j \; or \; x \in E \implies x \in E \cup \cap_{j=1}^{n} A_j$.

K. Let E be a set and $\{A_1, A_2, ..., A_n\}$ be a collection of sets. Establish the De Morgan laws:

$$(i)\ E \setminus \cap_{j=1}^{n} A_j = \cup_{j=1}^{n}(E \setminus A_j),\ and,\ (ii)\ \ E \setminus \cup_{j=1}^{n} A_j = \cap_{j=1}^{n}(E \setminus A_j)$$

(i) $x \in E \setminus \cap_{j=1}^{n} A_j \implies x \in E$ but not (A_1 and $A_2 \ldots$ and A_n) $\implies$ there exists a $j = 1, ..., n$ such that $x \in E$ but not $A_j \implies x \in E$ but not A_1, or $x \in E$ but not A_2,. . . ,or $x \in E$ but not $A_n \implies x \in E \setminus A_1$ or $x \in E \setminus A_2 \ldots$ or $x \in E \setminus A_n \implies x \in \cup_{j=1}^{n}(E \setminus A_j)$. In the other direction, $x \in \cup_{j=1}^{n}(E \setminus A_j) \implies x \in (E\ but\ not\ A_1)$ or $(E\ but\ not\ A_2)$ or $(E\ but\ not\ A_n) \implies$ there exists $j = 1, ..., n$, $x \in E$ but not $A_j \implies x \in E$ but not $(A_1\ and\ A_2\ ...\ and\ A_n) \implies x \in E \setminus \cap_{j=1}^{n} A_j$.

(ii) $x \in E \setminus \cup_{j=1}^{n} \implies x \in E$ but A_1 or $A_2 \ldots$ or $A_n \implies x \in E$ and $x \notin A_j$ for all $j = 1, ..., n \implies x \in E$ but not A_1, and $x \in E$ but not A_2, . . . , and $x \in E$ but not $A_n \implies x \in (E \setminus A_1)\ and\ x \in (E \setminus A_2)\ ...\ and\ x \in (E \setminus A_n) \implies x \in \cap_{j=1}^{n}(E \setminus A_j)$. In the other direction, $x \in \cap_{j=1}^{n}(E \setminus A_j) \implies x \in (E \setminus A_1\ and\ E \setminus A_2\ ...\ and\ E \setminus A_n) \implies x \in E$ but not A_j for all $j = 1, ..., n \implies x \in E$ but A_1 or $A_2 \ldots$ or $A_n \implies x \in E$ but not $\cup_{j=1}^{n} A_j \implies x \in E \setminus \cup_{j=1}^{n} A_j$

Section 2

C. Consider the subset of $\mathbb{R} \times \mathbb{R}$ defined by $D = \{(x, y)\ : |x| + |y| = 1\}$. Describe this set in words. Is it a function?

This set consists of points on the line segments connecting a rotated square in the (x, y) plane with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. If we attempt to define a function, with the elements (x, y) from the set D, i.e. $y = f(x), f : x \to y$, we have $|x| + |y| = 1 \implies \sqrt{y^2} = 1 - |x| \implies y = \pm\sqrt{(1 - |x|)^2}$. $f(x) = y = \pm\sqrt{(1 - |x|)^2}$ does not fit the defintion of a function, since, as an example, the set D includes the elements $(0, 1)$ and $(0, -1)$, which if, f is a function, $f : x \to y \implies -1 = 1$, which is clearly not true.

E. Prove that if f is an injection from A to B, then $f^{-1} = \{(b, a) : (a, b) \in f\}$ is a function. Then prove it is an injection.

If f is an injection, and $(a, b) \in f$, and $(a', b) \in f$, then $a = a'$. $f^{-1} = \{(b, a) : (a, b) \in f\}$ contains the pair (b, a) and (b, a'), and we know that $a = a'$ from the definition of f, so we can assume that f^{-1} is a function. Since f is injective, each unique element $b = f(a)$, is mapped to by a unique element a, and by definition $f^{-1} = \{(b, a) : (a, b) \in f\}$ maps the unique element b back to a, meaning $f^{-1}(b) = a$ and $f^{-1}(b') = a$ if and only if $b = b'$, thus f^{-1} is also injective.

H. Let f, g be functions such that

$$g \circ f(x) = x, \quad for\ all\ x\ in\ D(f)$$

$$f \circ g(y) = y, \quad for\ all\ y\ in\ D(g)$$

Prove that $g = f^{-1}$

For two elements $x, x' \in D(f)$, if $f(x) = f(x') \implies g \circ f(x) = g(f(x)) = g(f(x')) \implies g(f(x)) = x = g(f(x')) = x'$, that is $x = x' \implies g \circ f$ is an injection. For two elements $y, y' \in D(g)$, if $g(y) = g(y') \implies f \circ g(y) = f(g(y)) = f(g(y')) \implies f(g(y)) = y = f(g(y')) = y'$, that is $y = y' \implies f \circ g$ is an injection, and implies f and g are injections as well.

This implies g can be defined $g = \{(f(x), x) : (x, f(x)) \in f\}$, which is the definition for f^{-1}, implying $g = f^{-1}$.

J. Let f be the function on $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = x^2$, and let $E = \{x \in \mathbb{R}\ -1 \leq x \leq 0\}$ and $F = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.Then $E \cap F = \{0\}$ and $f(E \cap F) = \{0\}$ while $f(E) = f(F) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. Now delete 0 from E and F.

The sets E and F with 0 deleted are denoted $E' = \{x \in \mathbb{R} : -1 \leq x < 0\}$ and $F' = \{x \in \mathbb{R} : 0 < x \leq 1\}$, respectively. We still have the equality $f(E') = f(F') = \{y \in \mathbb{R} : 0 < y \leq 1\} = f(E') \cap f(F')$. We also have $E' \cap F' = \emptyset$, and thus $f(E' \cap F') = \emptyset$, and $\emptyset = f(E' \cap F') \subseteq F(E') \cap f(F')$, since the empty set is a subset of all sets.

Section 3

B. Exhibit a one-to-one correspondence between the set O of odd natural numbers and $\mathbb{N}$

The function $f(x) = \frac{x+1}{2}, x \in \mathbb{N}$ maps the set of odd natural numbers, $O = \{2k-1 : k \in N\} \to \mathbb{N}$.

D. If A is contained in some initial segment of $\mathbb{N}$, use the well-ordering property of $\mathbb{N}$ to define a bijection of A onto some initial segment of $\mathbb{N}$.

If $A \neq \emptyset$ is a subset of some initial segment $\mathbb{N}$, by the well-ordering principle, there exists an $m \in A$ such that $m \leq k$ for all $k \in A$. A bijection f can be defined by the mapping from set A consisting of elements $\{a_1, a_2, ..., a_k\}$ to elements of some initial segment $S_k = \{1, 2, ..., k\}$ as a set of ordered pairs $\{(a_1, 1), (a_2, 2), ..., (a_k, k)\}$, such that $a_1 \leq a_2 \leq ... \leq a_k$ and clearly the corresponding elements in the pairs from set S_k, $1 \leq 2 \leq ... \leq k$. Here the number of elements in A and S_k are the same, which has a one-one correspondence $f : A \to S_k$ and the $R(f) = S_k$.

F. Use the fact that every infinite set has a denumerable subset to show that every infinite set can be put into one-one correspondence with a proper subset of itself.

By defintion, having a denumberable subset $\implies$ there exists a bijective function that maps a proper subset of an infinite set, B, onto $\mathbb{N}$. If we take infinite set $B = \{b_1, b_2, ..., b_n, ...\}$ and $B_1 = \{b_2, b_3, ..., b_n, b_{n+1}, ...\}$, $B_1 \subseteq B$, we can create a one-one correspondence $f : B \to B_1$ defined by the set or ordered pairs $\{(b_n, b_{n+1}) : n \in N\}$ which maps B to $B_1 = B \setminus \{b_1\}$.

H. Show that if the set A can be put into one-one correspondence with a set B, and if B can be put into one-one correspondence with a set C, then A can be put into one-one correspondence with C.

If A can be put into one-one correspondence with a set $B \implies$ there exists an injective function, f from $A \to B$. This means that for $a, a' \in A$, and $b \in B$, $f(a) = f(a') = b \implies a = a'$. Similarly, if B can be put into one-one correspondence with a set $B \implies$ there exists an injective function, g from $B \to C$, and with $b, b' \in A$, $g(b) = g(b') = c \in C \implies b = b'$. By these properties, the composition of these two injective functions, $g \circ f(a) = g \circ f(a') \implies f(a) = f(a') \implies a = a'$ putting A and C in one-one correspondence.

I. Using induction on $n \in \mathbb{N}$, show that the initial segment determined by n cannot be put into one-one correspondence with the initial segment determined by $m \in \mathbb{N}$, if $m < n$.

Let $S_n = \{1, 2, 3, ..., n\}$ be the initial segment determined by $n \in N$ and S_m be the initial segment determined by $m \in N, m < n$. If S_n can be put into one-one correspondence with $S_m \implies$ there exists and injection $f : S_n \to S_m$. For $n = 1$ we have $f : \{1\} \to S_m,\ m < 1$, but S_m does not exist by definition for $m < 1$ implying the function is not valid for the case $n = 1$, $m < n$. For, the case $n = k+1$, we again have a map $f : \{1, 2, ..., k+1\} \to \{1, ..., m\},\ m < k+1$ which implies a mapping of $k+1$ elements to $m < k+1$ elements $\implies$ there exists at least two elements $x, x' \in S_{k+1}$ for which $f(x) = f(x')$ and $x \neq x' \implies$ an injection does not exist between these sets.

Section 4

C. Prove part (b) of Theorem 4.4, that is, Let $a \neq 0$ and b be arbitrary elements of $\mathbb{R}$. Then the equation $a \cdot x = b$ has the unique solution $x = \frac{1}{a}b$

Let x_1 be any solution to the equation, that is, $a \cdot x_1 = b$. By $(M4)$ we have that there is exists for each element $a \neq 0$ in $\mathbb{R}$ there exists an element $\frac{1}{a}$ such that $a(\frac{1}{a}) = 1$. Thus we have $(\frac{1}{a})ax_1 = b(\frac{1}{a}) \implies 1 \cdot x_1 = b(\frac{1}{a}) \implies a \cdot x_1 = b$ has the unique solution $x_1 = \frac{b}{a}$.

F. Use the argument in Theorem 4.7 to show that there does not exist a rational number s such that $s^2 = 6$.

If we assume that $s^2 = (\frac{p}{q})^2 = 6$, where $p, q \in \mathbb{Z}, q \neq 0$ and assume that p and q have no common integral factors, since $p^2 = 2(3q^2) \implies$ that p^2, and p is even for some $p = 2k$, $k \in \mathbb{N} \implies p^2 = 4k^2 = 2(3q^2) \implies 2k^2 = 3q^2 \implies q^2$, and q must be even, which is a contradiction of the assumption that p and q have no common integral factors, and thus a rational number s such that $s^2 = 6$ does not exist.

G. Modify the argument in Theorem 4.7 to shwo there there does not exists a ration number t such that $t^2 = 3$.

If we assume that $t^2 = (\frac{p}{q})^2 = 3$, where $p, q \in \mathbb{Z}, q \neq 0$ and assume that p and q have no common integral factors, we have $p^2 = 3q^3$ which implies that p^2 and p are divisible by 3 $\implies$ there exists $k \in \mathbb{N}$ such that $p = 3k \implies p^2 = 9k^2 = 3q^2 \implies 3k^2 = q^2$. This implies that q^2 is also divisible by 3 $\implies$ q is divisible by 3. This is again a contradiction of assumption p and q have no common integral factors, and thus a rational number t such that $t^2 = 3$ does not exist.

H. If $\xi \in \mathbb{R}$ is irrational and $r \in \mathbb{R}$, $r \neq 0$, is rational, show that $r + \xi$ and $r\xi$ are irrational.

If we take another rational number $c = \frac{a}{b},\ a, b \in \mathbb{Z}, b \neq 0$, and assume the contradiction that $r + \xi, r = \frac{p}{q},\ p, q \in \mathbb{Z}, q \neq 0$ is rational, that is $r + \xi = c$, we have $\xi = c - r = \frac{a}{b} - \frac{p}{q} = \frac{aq-bp}{bq}$ where $\frac{aq-bp}{bq}$ is a rational number, but clearly ξ cannot not be equal to a rational number. Similarly for $r\xi = c \implies \xi = \frac{c}{r} = \frac{aq}{bp}$ where $\frac{aq}{bp}$ is clearly a rational number, again implying the contradiction that ξ is equal to a rational number. Thus, by contradiction, $r + \xi$ and $r\xi$ must be irrational.

Section 5

B. If $n \in \mathbb{N}$, show that $n^2 \geq n$ and hence $\frac{1}{n^2} \leq \frac{1}{n}$.

If $n \in \mathbb{N}$, then $n \geq 1 \implies n^2 \geq n$, since $n^2 = n \cdot n \cdot 1 \geq n \cdot 1 \implies n \geq \frac{n \cdot 1}{n \cdot 1} \implies n \geq 1$, a condition of n being a natural number.

C. If $a \geq -1$, $a \in \mathbb{R}$, show that $(1 + a)^n \geq 1 + na$ for all $n \in \mathbb{N}$.

Let S be the set of all $n \in \mathbb{N}$ for which $(1+a)^n \geq 1+na$ is true. For $n = 1$ we have $(1+a)^1 \geq 1+(1)a = 1+a$. For $k \in S$, we assume $(1 + a)^k \geq 1 + ka$ is true. For case $n = k + 1$, we have, using the binomial theorem,

$$(1+a)^{k+1} = (1+a)(1+a)^k = (1+a)\sum_{j=0}^{k}\binom{k}{j}a^j = (1+a)(\binom{k}{0}a^0+\binom{k}{1}a^1+...+\binom{k}{k}a^k) = (1+a)(1+ka+...+a^k)$$

This implies, $(1 + a)^{k+1} \geq (1 + a)(1 + ka) = 1 + ka + a + ka^2 = 1 + (k + 1)a + ka^2 \geq 1 + (k + 1)a$, since $ka^2 \geq 0$. Thus, $(1 + a)^{k+1} \geq 1 + (k + 1)a$ holds, for $k + 1 \in S$.

F. Suppose that $0 < c < 1$. If $m \geq n,\ m, n \in \mathbb{N}$, show that $0 < c^m \leq c^n < 1$.

By property 5.6(c), for $a, b, c \in \mathbb{R}$, if $a > b$ and $c > 0$, then $ac > bc$. Applying this property here we have, $0 < c < 1 \implies 1 > c$ and $c > 0 \implies c = 1 \cdot c > c \cdot c = c^2$, thus $0 < c^2 < c < 1 \implies 1 > c$ and $c^2 > 0$, and $c^2 > c^3$, up to $c^k > c^{k+1}$, $k \in \mathbb{N}$. Thus for $m, n \in \mathbb{N}$, $m \geq n$, we have $0 < c^m \leq c^n < 1$.

G. Show that $n < 2^n$ for all $n \in \mathbb{N}$. Hence $(1/2)^n < 1/n$ for all $n \in \mathbb{N}$.

Applying induction, for case $n = 1$ we have true statement $1 < 2^1$. We assume the inequality is valid for $k \in \mathbb{N}$, and for case $n = k + 1$, we have $k + 1 < 2^{k+1} = 2 \cdot 2^k$. For all $k \geq 1$ we have first, $k + 1 \leq k + k = 2k$, and since $2k \leq 2^{k+1}$, i.e. $k \leq 2^k \implies k + 1 \leq 2^{k+1}$. Since the inequality holds for $n = k + 1$, we assume it holds for all $n \in \mathbb{N}$.

K. If $a, b \in \mathbb{R}$ and $b \neq 0$, show that $|a/b| = |a|/|b|$

(i) For the case, $a \geq 0,\ b > 0,\ a \cdot 1/b \geq 0$, and we thus have $|a/b| = |a \cdot 1/b| = a/b = |a| \cdot |1/b|$, thus $a/b = |a|/|b|$.

(ii) For the case, $a \geq 0,\ b < 0$, we have $a/b \leq 0\ \ \forall\ a, b$, thus $|a/b| = |a \cdot 1/b| = -(a/b) = a \cdot 1/-b$, and $a, -b \in \mathbb{P} \implies a \cdot 1/-b \geq 0$, thus $a/-b = |a|/|b|$.

(iii) For the case, $a \leq 0,\ b < 0$, we have $a/b \geq 0,\ \forall\ a, b$, thus, $|a/b| = |a \cdot 1/b| = (a/b) = -a \cdot 1/-b$, thus $-a/-b = a/b = |a|/|b|$.

(iv) For the case, $a \leq 0,\ b > 0$ we have $a/b \leq 0\ \ \forall\ a, b$, thus, $|a/b| = -(a/b) = -a/b = -a/|b| = |a|/|b|$.

L. If $a, b \in \mathbb{R}$, then $|a+b| = |a| + |b|$ if and only if $ab \geq 0$.

$ab \geq 0 \implies a, b \in \mathbb{P}$ or $-a, -b \in \mathbb{P}$. For the case, $a, b \in \mathbb{P}$, we have $|a+b| = a+b = |a|+|b| \;\; \forall\, a, b \in \mathbb{P}$. For the case, $-a, -b \in \mathbb{P}$, we have, $|a+b| = -(a+b) = -a-b = |a|+|b|$.

Section 6

B. Show that if a subset S of $\mathbb{R}$ contains an upper bound, then this upper bound is the supremum of S.

Let the upper bound of $S \subseteq \mathbb{R}$ be $u \in \mathbb{R}$, then assume for all $s \in S$, $u \geq s$. If $s \leq v \;\; \forall s \in S$, then $u \leq v$, then there is another number that satisfies the supremum and u is not a supremum of S.

C. Give an example of a set of rational numbers which is bounded but does not have a rational supremum.

Take the set $S = \{x \in \mathbb{Q} : x^2 < 3\}$, bounded above by the irrational $\sqrt(3)$, where $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$.

G. If S is a bounded set in $\mathbb{R}$ and if S_0 is a non-empty subset of S, then show that inf $S \leq$ inf $S_0 \leq$ sup $S_0 \leq$ sup S

By definition, $S_0 \subseteq S \implies$ there exists either, an element in S that is not in S_0 or S_0 exhausts all of S (i.e. they are equal). Let $u = \inf S \implies u \leq s \;\; \forall s \in S$ and $s \in S_0$. Let $u_0 = \inf S_0 \implies u_0 \leq s \;\; \forall s \in S_0 \subseteq S \implies u \leq u_0 \implies \inf\; S \leq \inf S_0$. Let $w = \sup S \implies w \geq s \forall s \in S$ and $s \in S_0$. Let $w_0 = \sup S_0 \implies w_0 \geq s \;\; \forall s \in S_0$, but not necessarily for all $s \in S$. This implies $w \geq w_0 \;\; \forall s \in S$. Since by definition $\sup S_0 \geq \inf S$, and since $w \geq w_0 \implies u \leq u_0 \leq w_0 \leq w \Leftrightarrow \inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$.

H. Let X and Y be non-empty sets and let $f : X \times Y \to \mathbb{R}$ have a bounded range in $\mathbb{R}$. Let, $f_1(x) = sup\{f(x,y) : y \in Y\}$, and $f_2(y) = sup\{f(x,y) : x \in X\}$. Establish the Principle of Iterated Suprema: $sup\{f(x,y) : x \in X, y \in Y\} = sup\{f(x,y) : y \in Y\} = sup\{f(x,y) : x \in X\}$.

Let $u = \sup\; \{f(x,y) : x \in X, y \in Y\} \implies u \geq f(x,y) \;\; \forall f(x,y)$ where $x \in X, y \in Y$. This implies that $f_1(x) \leq u \, \forall y \in Y$. Conversely, let $u_0 = \sup f_1(x) = \sup\; \{f(x,y) : y \in Y\}$. This implies $u_0 \geq u \;\; \forall x \in X, y \in Y$. This implies that $u = u_0$, and thus $\sup\; \{f(x,y) : x \in X, y \in Y\} = f_1(x) = \sup\; \{f(x,y) : y \in Y\}$. By extension the same argument hold for $\sup\; \{f(x,y) : x \in X, y \in Y\} = \sup\; f_2(y) = \sup\{f(x,y) : x \in X\}$.

J. Let X be a non-empty set and let $f : X \to \mathbb{R}$ have a bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that: $sup\{a + f(x) : x \in X\} = a + sup\{f(x) : x \in X\}$, and *$inf\{a + f(x) : x \in X\} = a + inf\{f(x) : x \in X\}$.*

Let $u = \sup\{a + f(x) : x \in X\} \implies u \geq a + f(x) \;\; \forall\, x \in X \implies u - a \geq f(x) \;\; \forall x \in X \implies \sup\{f(x) : x \in X\} = u - a$. This implies that $u = a + \sup\{f(x) : x \in X\}$, and thus $\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}$.

Using the same argument, let $w = \inf\{a + f(x) : x \in X\} \implies w \leq a + f(x) \;\; \forall\, x \in X \implies w - a \leq f(x) \;\; \forall x \in X \implies \inf\{f(x) : x \in X\} = w - a$. This implies that $w = a + \inf\{f(x) : x \in X\}$, and thus $\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}$.

K. Let X be a non-empty set and let f and g be defined on X have a bounded ranges in $\mathbb{R}$. Show that: $inf\; \{f(x) : x \in X\} + inf\; \{g(x) : x \in X\} \leq inf\; \{f(x) + g(x) : x \in X\} \leq inf\; \{f(x) : x \in X\} + sup\; \{g(x) : x \in X\} \leq sup\; \{f(x) + g(x) : x \in X\} \leq sup\; \{f(x) : x \in X\} + sup\; \{g(x) : x \in X\}$

(i) Let $l = \inf\; \{f(x) : x \in X\}$ and $l_0 = \inf\; \{g(x) : x \in X\}$, thus, $l \leq f(x) \; \forall x \in X$ and $l_0 \leq g(x) \; \forall x \in X$, summing these inequalities we have $l + l_0 \leq f(x) + g(x) \; \forall x \in X \implies l + l_0 = \inf\; \{f(x) : x \in X\} + \inf\; \{g(x) : x \in X\} \leq \inf\; \{f(x) + g(x) : x \in X\}$.

(ii) Since $l + l_0 \leq \inf\; \{f(x) + g(x) : x \in X\} \leq \inf\; \{f(x) : x \in X\} + \sup\; \{g(x) : x \in X\} \implies l + l_0 \leq l + \sup\{g(x) : x \in X\} \implies l_0 \leq \sup\{g(x) : x \in X\}$, which must be true, since $\inf\{g(x) : x \in X\} \leq \sup\{g(x) : x \in X\}$ by definition.

(iii) Let $w = \sup\; \{f(x) + g(x) : x \in X\}$, $\inf\; \{f(x) : x \in X\} + \sup\; \{g(x) : x \in X\} \leq w \implies w \geq f(x) + g(x) \; \forall x \in X \implies w \geq u_0 + l$, where again $u_0 \geq g(x) \; \forall x \in X$, thus $w - u_0 \geq f(x) \; \forall x \in X$, implying $w - u_0$ is an upper bound for $f(x)$. Thus $w - u_0$, must be greater than $\inf\{f(x) : x \in X\} \implies \inf\; \{f(x) : x \in X\} + \sup\; \{g(x) : x \in X\} \leq \sup\; \{f(x) + g(x) : x \in X\}$.

(iv) Let $u = \sup\{f(x) : x \in X\}$ and $u_0 = \sup\{g(x) : x \in X\}$, thus, $u \geq f(x)\ \forall x \in X$ and $u_0 \geq g(x)\ \forall x \in X$, summing these inequalities we have $u + u_0 \geq f(x) + g(x)\ \forall x \in X \implies u + u_0 = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} \geq \sup\{f(x) + g(x) : x \in X\}$.

An example of a strict inequality: the functions f, g, on the set $X = \{x \in \mathbb{R} : 0 < x < 1\}$ for $f(x) = g(x) = x$. Clearly $\inf\{x : 0 < x < 1\} = 0$, thus $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} = 0$ which is less than $\inf\{f(x) + g(x) : 0 < x < 1\} > 0$, since, $f(x) > 0$ and $g(x) > 0\ \ \forall x \in X$. $\inf\{f(x) + g(x) : x \in X\} \leq \inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$, holds, since $\sup\{x : 0 < x < 1\} = 1$, which is clearly greater than $\inf\{f(x) + g(x) : x \in X\}$, since the bound $\inf\{f(x) + g(x) : x \in X\}$ is close to zero and is clearly less than 1.

For the inequality $\inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} \leq \sup\{f(x) + g(x) : x \in X\}$, clearly the left hand side is 1, since $\sup\{x : x \in X\} = 1$, and the right hand must be greater than one since $f(x) + g(x)$ can clearly equate to a number greater than 1 given range and domain.

Lastly , $\sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} = 2$, clearly, which is greater than $\sup\{f(x) + g(x) : 0 < x < 1\}$, since $f(x) < 1$, and $g(x) < 1\ \forall x \in X$.

Section 7

F. Let $J_n = (0, \frac{1}{n})$, for $n \in \mathbb{N}$. Show that this sequence of intervals is nested, but that there is no common point.

First, $J_1 \supseteq J_2 \supseteq ... \supseteq J_n \supseteq ...$, clearly, since for $n = 1$, $(0,1) \supseteq (0, \frac{1}{2})$, and for $(0, \frac{1}{n}) \supseteq (0, \frac{1}{n+1})$, for $n \in \mathbb{N}$. Using corollary $6.7(b)$, there exists a natural number $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < z,\ z \in \mathbb{R},\ z > 0$, which implies there are arbitrarily small rational numbers of the form $1/n$. Therefore the sequence J_n has no common point, i.e. $\cap_{j=1}^{n} J_n = \emptyset$, because for each open interval $(0, \frac{1}{n})$, there is a narrower open cell for $n \in \mathbb{N}$, such that the elements in that cell are always less than $1/n$.

G. If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested sequence of closed cells, show that $a_1 \leq a_2 \leq ... \leq a_n \leq ... \leq b_m \leq ... \leq b_2 \leq b_1$.

If $I_n = [a_n, b_n]$ is a nested sequence of closed cells, that is $I_1 \supseteq I_2 \supseteq ... \supseteq I_n \supseteq ... \implies [a_1, b_1] \supseteq [a_n, b_n],\ n \in \mathbb{N}$. For the case $n = 2$, we have $I_1 \supseteq I_2 \Leftrightarrow [a_1, b_1] \supseteq [a_2, b_2] \implies a_1 \leq a_2,\ b_2 \leq b_1$ and $a_2 \leq b_1$. For case $n = k+1, k \in \mathbb{N}$, we have $I_k \supseteq I_{k+1} \Leftrightarrow [a_k, b_k] \supseteq [a_{k+1}, b_{k+1}] \implies a_k \leq a_{k+1},\ b_{k+1} \leq b_k$ and $a_{k+1} \leq b_{k+1} \leq b_k ... \implies a_1 \leq a_2 \leq ... \leq a_n \leq ... \leq b_m \leq ... \leq b_2 \leq b_1$.

If we put $\xi = sup\{a_n : n \in \mathbb{N}\}$ and $\eta = inf\{b_n : n \in \mathbb{N}$, show that $[\xi, \eta] = \cap_{n \in \mathbb{N}} I_n$.

With $\xi = \sup\{a_n : n \in \mathbb{N}\},\ \eta = \inf\{b_n : n \in \mathbb{N} \implies \xi \geq a_n\ \ \forall n \in \mathbb{N}, \eta \leq b_n\ \ \forall n \in \mathbb{N}$. By the nested cell property, $\xi \leq \eta\ \forall\ n \in \mathbb{N} \implies [a_n, b_n] \supseteq [\eta, \xi],\ n \in \mathbb{N} \implies [a_1, b_1] \cap [a_2, b_2] \cap ... \cap [a_n, b_n] \cap ... = \cap_{n \in \mathbb{N}} I_n = [\eta, \xi]$.

K. By removing sets with ever decreasing length, show that we can construct a "Cantor-like" set which has positive length. How large can we make the length of this set?

If, as an example, we construct a "Cantor-like" set where instead of removing the "middle-third" intervals, we remove "middle-fourths", i.e. for $F_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$ we remove the middle $\frac{2^0}{4^1} = \frac{1}{4}$, for F_2 we remove the middle $\frac{2^1}{4^2} = \frac{2}{16}$ from preceding intervals, $F_2 = [0, \frac{5}{32}] \cup [\frac{7}{32}, \frac{12}{32}] \cup [\frac{20}{32}, \frac{25}{32}] \cup [\frac{27}{32}, 1]$, and so on. This construction leads to removing 1 section of length, $\frac{1}{4}$, at the first cut, 2 sections of of length $\frac{2}{16}$ at the second cut, and on. That is, the sum of the length removed from the interval $[0,1]$ at the n^{th} cut, can be represented by the sum $\frac{2^0}{4^1} + \frac{2^1}{4^2} + ... + \frac{2^{n-1}}{4^n} = \sum_{i=1}^{n} \frac{2^{i-1}}{4^i} = \frac{1}{2} \sum_{i=1}^{n} (\frac{1}{2})^i = \frac{1}{2} \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{1}{2}$. By this logic, for an arbitrarily large n, if we remove the "middle n^{ths}", we can remove an amount approaching 0, that is, create a set with a length approaching 1.