Solution for Homework 1 for math6263

1) Assume that we have a fair five- sided die and another skewed die with the same number of sides. Matzinger throws one of the two dies and you have to decide which die was used based on the number he obtains. The fact that the fair die was used is called H_0-hypothesis, the fact that the other die is used is called H_1-hypothesis. The probabilities for the two dice are given in the table below

x	1	2	3	4	5
$P(X=x\|H_0)$	0.2	0.2	0.2	0.2	0.2
$P(X=x\|H_1)$	0.3	0.2	0.1	0.25	0.15

a) Find all the most powerful tests if randomization is not allowed. That is tests that are most powerful among all non-randomized tests of same significance. **Answer:** We simply build the likelihood ratio. the most powerful tests are then those that include all points with likelihood ratio above a certain level in the acceptance region. The likelihood ratio is given by

x	1	2	3	4	5
$P(X=x\|H_0)/P(X=x\|H_1)$	$0.6\bar{6}$	1	2	0.8	1.3

So, there are 6 non-randomized optimal tests

	$accept\ regio$	$significance$	$power$
$TEST0$	$\{\}$	1	0
$TEST1$	$\{3\}$	0.8	0.9
$TEST2$	$\{3,5\}$	0.6	0.75
$TEST3$	$\{3,5,2\}$	0.4	0.55
$TEST4$	$\{3,5,2,4\}$	0.2	0.3
$TEST5$	$\{3,5,2,4,1\}$	0	1

We see that for those optimal tests as the significance becomes worse (goes up), the power gets better (increases). which of these tests you prefer, depends on what significance you need.

b) How would you proceed if you had to construct a most powerful test with significance 0.3? This time randomizing is allowed. What will the power of such a test be? **Answer:** The optimal test still "follows the order given by the likelihood ratio" except that now one point will not be entirely included into the acceptance region, but split by randomization. So, we need a test which is between TEST4 and TEST3. This means that we take as acceptance region $\{3,5,2\}$

and if we get $X = 4$ we flip a fair coin to decide if we accept or reject. Let us represent this randomized test by giving the conditional rejection probability given $X = x$ for all x:

x	1	2	3	4	5
$P(\texttt{reject}\vert\texttt{X} = \texttt{x})$	1	0	0	0.5	0

c) Assume now that Matzinger chooses the die at random (Bayesian case). The following (prior) probabilities are used:

$$P(H_0) = 0.5, P(H_1) = 0.5.$$

Which test is best in this case, in the sense that its probability to make an error is smallest? (Error of first and second type combined). **Answer:** Let α designate the significance and β the probability of an error of second type. We want to minimize the overall probability of an error:

$$P(H_0)\alpha + P(H_1)\beta. \tag{1}$$

Now, note that if with the same α, we get a better (smaller) β, then we can also improve the ratio above. So, this means that the test we are looking for has maximum power for its level of significance! In other words: the best test for the Bayesian case is to be found among our optimal (most powerful) tests for the non-Bayesian case. Also, we can write $\beta = \beta(\alpha)$, since for every significance level there is a best (smallest) β. Now for a point x let us look at what happens if we include that point into the acceptance region vs not including it. If x is included in the acceptance region, then the error is if we have H_1 and x comes, so this gives a contribution to the quantity **??** of $P(X = x|H_1)0.5$. If, x is not included into the acceptance region, then the contribution to **??** is $P(X = x|H_0)0.5$. So, we have to go for whichever is smaller. We clearly can do this for every point and this then gives us the best test for our Bayesian-case. Here since both prior probabilities are 0.5, we simply look at the ration $P(X = x|H_0)/P(X = x|H_1)$ and where it is bigger than 1 we have our acceptance region. Note that due to taking the priors 0.5 and 0.5, we get here the maximum-likelihood estimate.....In the present case, the acceptance region contains 3 and 5 and then for 2 you can decide either-way or randomize since the ratio there is 1. With 1 and 4 you decide to reject the hypothesis. Note that this procedure simply consists in deciding which of the two hypothesis is likelier under the observation $X = x$.

d) Same question then under c, but this time with the following prior probabilities: $P(H_0) = 0.4$ and $P(H_1) = 0.6$. **Answer:** Again, we calculate for each point x, which of the two hypothesis is likelier under the fact that we saw the number x. That is we compare

$$P(H_0|X = x) = \frac{P(X = x|H_0)P(H_0)}{P(X = x)}$$

to

$$P(H_1|X = x) = \frac{P(X = x|H_1)P(H_1)}{P(X = x)}$$

Both expressions above contain $P(X = x)$ so that repression does not matter. Hence, we compare

$$P(X = x|H_0)P(H_0)$$

to

$$P(X = x|H_1)P(H_1)$$

If $P(X=x|H_0)$ is bigger, then we decide to accept H_0. Otherwise we reject it. If the two are equal we can go either way or randomize. So, the acceptance region is given by

$$P(X=x|H_0)P(H_0) \geq P(X=x|H_1)P(H_1)$$

which is equivalent to

$$\frac{P(X=x|H_0)}{P(X=x|H_1)} \geq \frac{P(H_1)}{P(H_0)}.$$

Again, when we have equality we can go either way. But the interesting thing is that for the Bayesian case, we find optimal tests which are also based on the likelihood ratio $P(X=x|H_0)/P(X=x|H_1)$ just as the non-Bayesian tests are. (Another way of seeing the the optimal tests for the Bayesian case are to be found among the optimal tests for the non-Bayesian case). In the present case we have

x	1	2	3	4	5
$0.4 \cdot P(X=x\|H_0)$	0.08	0.08	0.08	0.08	0.08
$0.6 \cdot P(X=x\|H_1)$	0.18	0.12	0.06	0.15	0.09

So, $0.4P(X=x|H_0)$ is larger for $x=3$. So, our optimal way of testing for this Bayesian-case, is to accept the H_0-hypothesis when we have a 3 and reject otherwise.
e) Same question then under d), but this time there is different cost associated with an error of first type and an error of second type. Say, the error of first type has cost $L_0 = 2$ and the cost for an error of second type is $L_1 = 4$. What is the best test if we want to minimize the expected cost? Do we need to consider randomized algorithms if we want to minimized overall cost? **Answer:** When including x into the acceptance region, the contribution to the expected cost is

$$L_1 P(X=x, H_1) = L_1 P(X=x|H_1)P(H_1).$$

Otherwise, when x is in the rejection region, then the contribution to expected cost is

$$L_0 P(X=x, H_0) = L_0 P(X=x|H_0)P(H_0).$$

So, we compare $L_1 P(X=x|H_1)P(H_1)$ to $L_0 P(X=x|H_0)P(H_0)$ and chose the one which is smaller. In a table we can now compare these two cost functions:

x	1	2	3	4	5
$L_0 P(X=x\|H_0)P(H_0)$	0.16	0.16	0.16	0.16	0.16
$L_1 P(X=x\|H_1)P(H_1)$	0.78	0.48	0.24	0.6	0.36

Hence, the optimal test consists in always rejecting H_0.
Note that again this is a test based on our likelihood ratio since the acceptance region is given by:

$$L_0 P(X=x|H_0)P(H_0) \geq L_1 P(X=x|H_1)P(H_1)$$

which is equivalent to

$$\frac{P(X=x|H_0)}{P(X=x|H_1)} \geq \frac{L_1 P(H_1)}{L_0 P(H_0)}.$$

Hence, same type of likelihood-ratio test as with non-Bayesian case.

2) Assume this time that we have three dice. the first one constitutes the null-hypothesis. The alternative hypothesis is denoted by K and is that either the second or the third die was used. The fact that the second die is used is denoted by H_1,whilst the fact that the third die is used is denoted by H_2. So, K consists of H_1 and H_2 and is a composite hypothesis. The probabilities of the three dice are summarized in the table below:

x	1	2	3	4
$P(X = x \mid H_0)$	0.1	0.4	0.3	0.2
$P(X = x \mid H_1)$	0.1	0.2	0.3	0.4
$P(X = x \mid H_2)$	0.5	0.1	0.3	0.1

a) is there for every significance level a UMP-test for testing H_0 against K? (We assume that we allow for randomized tests). If not indicate why not and for which significance level there is not. **Answer:** Let us build the ratios

$$ratio_1 := \frac{P(X = x|H_0)}{P(X = x|H_1)}, ratio_2 := \frac{P(X = x|H_0)}{P(X = x|H_2)}.$$

In order to have best tests which are optimal for both H_1 and H_2, we need the two ratios to give the same ordering on the numbers from 1 to 4. Let us look at a table with the ratios:

x	1	2	3	4
$ratio_1$	1	2	1	0.5
$ratio_2$	0.2	4	1	2

So the order according to $ratio_1$ is

$$ratio_1(4) < ratio_1(1) = ratio_1(3) < ratio_1(2),$$

whilst for $ratio_2$ we get

$$ratio_2(1) < ratio_2(3) < ratio_2(4) < ratio_2(2).$$

The only place where the two ratios agree is for $x = 2$. So, the test which accepts H_0 when $x = 2$ and rejects otherwise is optimal when testing against H_1 as well as when testing against H_2. So, that test is UMP and has significance of 0.6. Its power for H_1 is 0.8 and for H_2 the power is 0.9. Also, any randomized test which accepts with a given probability γ when $x = 2$ and rejects in all other cases, would be optimal for both: testing against H_1 as well as testing against H_2. Any such a randomized test is UMP. There is not other possible UMP test since whenever we take a set of the type $ratio_1 > c_1$, this gives a set which is not of type $ratio_2 > c_2$. The reason being that the ordering of the numbers x for $ratio_1$ and $ratio_2$ do not coincide.....

3) Assume that X is obtained from a uniform random variable on a unit interval (an interval of length 1). Test the null-hypothesis that the unit interval is $[0, 1]$ vs the alternative composite hypothesis that the unit interval of the uniform distribution which was used to generate X is not $[0, 1]$. Are there UMP-tests in this case and if yes how would they look? **Answer:** There is no UMP test except for significance 0. (And significance 1 but that is not interesting). Let us see why. Take H_1 the hypothesis that the uniform variable is uniform on $[0.5, 1.5]$. Let H_2 be the hypothesis that the variable is uniform on $[-0.5, 0.5]$. When we build the ratio $P(X =$

$x|H_0)/P(X = x|H_1)$ we find something which is infinite from 0 to 0.5. Then 1 from 0.5 to 1 and finally 0 for $x > 1$. So, any optimal test again H_1 should be of the form: reject when above 1, accept between 0 and 0.5 and do whatever you want between 0.5 and 1. That is select a set A in $[0.5, 1]$, so that when $x \in A$ you reject H_0. Now, when we take the ratio for testing against H_2, we find that an optimal test should accept when X falls between 0.5 and 1. So a test which should be optimal against both H_1 and H_2 would have to accept the hypothesis H_0 on $[0, 0.5]$ and $[0.5, 1]$. So, this test would accept H_0 on all of $[0, 1]$ and reject otherwise always. This test has significance 0 and its power varies according to the which possibility in the alternative bag we look at. But, for significance 0 it is clearly the most powerful test, so this is an UMP-test and the only one that there is. (Except for the crazy test with significance 1).

4) We assume that X and Y are two random variables which both can take value 0 or 1. We assume that $E[X] = E[Y] = 0.5$. We test hypotheses concerning the joint probability distribution of X and Y. Let H_0 be the hypothesis that X and Y are independent. Let K be the hypothesis that X and Y are not independent. a) Are there UMP-tests to test H_0 against H_1 for every significance? Why or why not. **Answer:** if X and Y are independent then the joint probability would be given as in the table below:

$Y = 1$	0.25	0.25
$Y = 0$	0.25	0.25
	$X = 0$	$X = 1$

Now with our condition that $P(X = 0) = P(X = 1) = P(Y = 0) = P(Y = 1) = 0.5$ we find that any joint probability we consider in this problem can be written as

$Y = 1$	$0.25 - q$	$0.25 + q$
$Y = 0$	$0.25 + q$	$0.25 - q$
	$X = 0$	$X = 1$

where q is a parameter which can vary between -0.25 and $+0.25$. Now, when we compute the likelihood ratio, that is the probability under H_0, in our case $q = 0$, over the probability under the alternative, that is $q \neq 0$, we find

$Y = 1$	$\frac{P(X=0,Y=1\|H_0)}{P(X=0,Y=1\|q)}$	$\frac{P(X=1,Y=1\|H_0)}{P(X=1,Y=1\|q)}$
$Y = 0$	$\frac{P(X=0,Y=0\|H_0)}{P(X=0,Y=0\|q)}$	$\frac{P(X=1,Y=0\|H_0)}{P(X=1,Y=0\|q)}$
	$X = 0$	$X = 1$

which is equal to

$Y = 1$	$\frac{0.25}{0.25-q}$	$\frac{0.25}{0.25+q}$
$Y = 0$	$\frac{0.25}{0.25+q}$	$\frac{0.25}{0.25-q}$
	$X = 0$	$X = 1$

So, here we have a probability space with four possibilities: $(x, y) = (0, 0)$,$(x, y) = (1, 0)$,$(x, y) = (0, 1)$ and $(x, y) = (1, 1)$. We calculate the likelihood ratio of H_0 against the alternative $q \neq 0$ for each of these four points. Now on the point $x = 1, y = 1$, we get that the ratio is bigger than for the point $x = 0, y = 1$ when $q < 0$. But the opposite holds true when $q > 0$. So, there can not be UMP-tests at all levels of significance against the general hypothesis $q \neq 0$. Indeed

against $q < 0$, there would be a most powerful test where the acceptance region would include $(1,1)$, but not $(0,1)$. For $q > 0$ such a test could not be optimal, since under $q > 0$ if it would include $(1,1)$ it would also need to include every point with strictly higher likelihood ratio, so it would also need to include $(0,1)$ which it does not. Hence contradiction.
b) What restriction could we add which would help for getting UMP-tests? **Answer:** if we also request $q > 0$ for the alternative (or $q < 0$), then the ordering of any likelihood ratio function of the four points would be the same and hence we would get UMP-tests at any level of significance. This condition $q > 0$ or $q < 0$ corresponds to positive or negative correlation....

5) Same question as in problem 4, but this time the die can take any of three values in the set $\{1,2,3\}$. We assume known that

$$P(X=1) = P(X=2) = P(X=3) = P(Y=1) = P(Y=2) = P(Y=3) = \frac{1}{3}.$$

We want to test the hypothesis H_0 that X and Y are independent against the hypothesis K that they are not independent.
a) Is there a UMP test for all significance levels?

6) Let X be a positive random variable (hence $P(X \geq 0) = 1$) having density function given by

$$f_\theta(x) = c(\theta)x^2\theta^x$$

for $x > 0$, where θ is a positive parameter which is strictly less than 1. Here $c(.)$ is a function of θ which is used as normalizing constant.
a) Assume that H_0 is the hypothesis that $\theta = 0.2$ and H_1 is the hypothesis that $1 > \theta > 0.2$. Are there UMP-tests at all levels of significance. Explain why and if yes describe them. **Answer:** The densities we consider here are from a exponential family since we can write the density as

$$f_\theta(x) = c(\theta)x^2 \exp(\ln(\theta) \cdot x)$$

So, here the statistic in which f_θ has the monotone likelihood ratio is x. Let H_1 be the hypothesis that the parameter is equal to θ. Let H_0 be the hypothesis that $\theta = 0.2$. When you take the likelihood ratio for testing H_0 against H_1, you get

$$\frac{f(x|H_0)}{f(x|H_1)} = \frac{c(0.2)}{c(\theta)} \exp((\ln(0.2) - \ln(\theta))x). \tag{2}$$

Putting the above ratio bigger than a constant yields all the optimal tests. Now, $c(0.2)/c(\theta)$ is positive and can be included into that constant. So, putting expression **??** bigger than $c \cdot c(0.2)/c(\theta))$ yields

$$(\ln(0.2) - \ln(\theta)) \cdot x \geq \ln c.$$

Now $\ln(0.2) - \ln(\theta)$ is negative since we assume $\theta > 0.2$. Hence, we get optimal test with acceptance region of type

$$x \leq k$$

where k is $k := \ln(c)/(\ln(0.2) - \ln(\theta))$. So, all optimal tests when testing $\theta = 0.2$ against specific θ, where $\theta > 0.2$ and $\theta < 1$ are of type: take acceptance region as $x \leq k$, where $k \geq 0$ is a constant.So, since they work for any $\theta > 0.2$ in our bag of alternative hypotheses, they are UMP. And yes by varying $k > 0$, we get any significance we want.

b) Let H_0 be like in a. But this time let H_1 be the hypothesis that $\theta \neq 0.2$ and θ strictly between 0 and 1. Are there UMP-tests at all levels? **Answer:** Except with significance 0 and 1 there are no UMP-tests here. The reason that is that for $\theta > 0.2$, the tests are of the type accept when $X \leq k$. On the opposite, when $\theta < 0.25$, the optimal tests would be of the type: accept when $X \geq k$. Both things are not possible at the same time, so no UMP tests which work for both alternative $\theta > 0.2$ and alternative $\theta < 0.2$.

7) Let X be a positive random variable with density given by

$$f_\theta(x) = c(\theta) \cdot (x\theta^x + x^2\theta^{3x})$$

for $x > 0$ where $\theta > 0$ is a parameter strictly smaller than 1. Assume that H_0 is the hypothesis that $\theta = 0.2$ and H_1 is the hypothesis that $1 > \theta > 0.2$. Are there UMP-tests at all levels of significance. Explain why and if yes describe them.

8) You own a six sided die. (Real life situation, not the ridiculous little game which Matzinger used to explain statistics!) You want to know if the die is fair. you throw it a few times independently. is there a UMP-test to figure out if the die is fair (all sides have same probability) against the alternative that it is not fair? What do you say?????????????????