

# Math 4317 (Prof. Swiech, S'18): HW #1

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## Section 1

*F. Show that the symmetric difference  $D$ , defined in the preceding exercise is also given by  $D = (A \cup B) \setminus (A \cap B)$ . Show  $D = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ :*

First,  $x \in (A \setminus B) \cup (B \setminus A) \implies x \in (A \setminus B)$  or  $x \in (B \setminus A) \implies$ ,  $x$  is in  $A$  but not  $B$ , or,  $x$  is in  $B$  but not  $A \implies x$  is in  $A$  or  $B$  but not in  $A$  and  $B \implies x \in (A \cup B) \setminus (A \cap B)$ .

In the other direction,  $x \in (A \cup B) \setminus (A \cap B) \implies x \in (A \cup B)$  but not in  $(A \cap B) \implies x$  is in  $A$  but not  $B$ , or,  $x$  is in  $B$  but not  $A \implies x \in (A \setminus B)$  or  $x \in (B \setminus A) \implies x \in (A \setminus B) \cup (B \setminus A) \implies (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

*I. If  $\{A_1, A_2, \dots, A_n\}$  is a collection of sets, and if  $E$  is any set, show that:*

$$(i) \ E \cap \bigcup_{j=1}^n A_j = \bigcup_{j=1}^n (E \cap A_j), \text{ and } (ii), \ E \cup \bigcup_{j=1}^n A_j = \bigcup_{j=1}^n (E \cup A_j)$$

- (i)  $x \in E \cap \bigcup_{j=1}^n A_j \implies x \in E$  and  $x \in \{A_1 \text{ or } A_2 \dots \text{or } A_n\} \implies x \in E$  and that there exists for some  $j = 1, 2, \dots, n$  an  $A_j$  such that  $x \in A_j$  and  $x \in E \implies (x \in E \text{ and } A_1) \text{ or } (x \in E \text{ and } A_2) \dots \text{ or } (x \in E \text{ and } A_n) \implies x \in \bigcup_{j=1}^n (E \cap A_j)$ .

In the other direction,  $x \in \bigcup_{j=1}^n (E \cap A_j) \Leftrightarrow x \in (E \cap A_1) \cup (E \cap A_2) \dots \cup (E \cap A_n) \implies x \in E$  and  $A_1$  or  $E$  and  $A_2 \dots \implies$  there exists a  $j = 1, \dots, n$  such that  $x \in (E \cap A_j) \implies x \in E$  and  $x \in A_1$  or  $A_2, \dots$ , or  $A_n \implies x \in E$  and  $\bigcup_{j=1}^n A_j \implies x \in E \cap \bigcup_{j=1}^n A_j$ .

- (ii)  $x \in E \cup \bigcup_{j=1}^n A_j \implies x \in E$  or  $x \in A_1$  or  $A_2 \dots$  or  $A_n \implies$  for some  $j = 1, \dots, n$  that  $x \in E \cup A_j \implies x \in E \cup A_1$  or  $x \in E \cup A_2 \dots$  or  $x \in E \cup A_n \implies x \in \bigcup_{j=1}^n (E \cup A_j)$ . In the other direction,  $x \in \bigcup_{j=1}^n (E \cup A_j) \Leftrightarrow x \in E \cup A_1$  or  $x \in E \cup A_2 \dots$  or  $x \in E \cup A_n \implies$  there exists some  $j = 1, \dots, n$  such that  $x \in E \cup A_j \implies (x \in E \text{ or } x \in A_1) \text{ or } (x \in E \text{ or } x \in A_2) \dots \text{ or } (x \in E \text{ or } x \in A_n) \implies x \in E$  or  $x \in \bigcup_{j=1}^n A_j \implies x \in E \cup \bigcup_{j=1}^n A_j$ .

*J. If  $\{A_1, A_2, \dots, A_n\}$  is a collection of sets, and if  $E$  is any set, show that:*

$$(i) \ E \cap \bigcap_{j=1}^n A_j = \bigcap_{j=1}^n (E \cap A_j), \text{ and } (ii), \ E \cup \bigcap_{j=1}^n A_j = \bigcap_{j=1}^n (E \cup A_j)$$

- (i)  $x \in E \cap \bigcap_{j=1}^n A_j \implies x \in E$  and  $x \in \bigcap_{j=1}^n A_j \implies x \in E$  and  $x \in A_j$  for all  $j = 1, \dots, n \implies x \in E$  and  $[x \in A_1 \text{ and } x \in A_2 \dots \text{ and } x \in A_n] \implies [x \in E \text{ and } A_1] \text{ and } \dots \text{ and } [x \in E \text{ and } A_n] \implies x \in \bigcap_{j=1}^n (E \cap A_j)$ . In the other direction,  $x \in \bigcap_{j=1}^n (E \cap A_j) \implies x \in (E \cap A_1)$  and  $x \in (E \cap A_2) \dots$  and  $x \in (E \cap A_n) \implies x \in (E \cap A_j)$  for all  $j = 1, \dots, n \implies x \in E$  and  $x \in A_1$  and  $x \in A_2 \dots$  and  $x \in A_n \implies x \in E$  and  $x \in \bigcap_{j=1}^n A_j \implies x \in E \cap \bigcap_{j=1}^n A_j$ .

- (ii)  $x \in E \cup \bigcap_{j=1}^n A_j \implies x \in E$  or  $x \in \bigcap_{j=1}^n A_j \implies x \in E$  or  $[x \in A_1 \text{ and } x \in A_2 \dots \text{ and } x \in A_n] \implies x \in E$  or  $A_1$  and  $x \in E$  or  $A_2 \dots$  and  $x \in E$  or  $A_n \implies x \in \bigcap_{j=1}^n (E \cup A_j)$ . In the other direction,  $x \in \bigcap_{j=1}^n (E \cup A_j) \implies x \in (E \cup A_1)$  and  $x \in (E \cup A_2) \dots$  and  $x \in (E \cup A_n) \implies$  that for all  $j = 1, \dots, n$ ,  $x \in (E \cup A_j) \implies x \in E$  or  $(x \in A_1 \text{ and } x \in A_2 \dots \text{ and } x \in A_n) \implies x \in \bigcap_{j=1}^n A_j$  or  $x \in E \implies x \in E \cup \bigcap_{j=1}^n A_j$ .

*K. Let  $E$  be a set and  $\{A_1, A_2, \dots, A_n\}$  be a collection of sets. Establish the De Morgan laws:*

$$(i) \ E \setminus \bigcap_{j=1}^n A_j = \bigcup_{j=1}^n (E \setminus A_j), \text{ and, } (ii) \ E \setminus \bigcup_{j=1}^n A_j = \bigcap_{j=1}^n (E \setminus A_j)$$

- (i)  $x \in E \setminus \bigcap_{j=1}^n A_j \implies x \in E$  but not  $(A_1 \text{ and } A_2 \dots \text{ and } A_n) \implies$  there exists a  $j = 1, \dots, n$  such that  $x \in E$  but not  $A_j \implies x \in E$  but not  $A_1$ , or  $x \in E$  but not  $A_2, \dots$ , or  $x \in E$  but not

$A_n \implies x \in E \setminus A_1$  or  $x \in E \setminus A_2 \dots$  or  $x \in E \setminus A_n \implies x \in \cup_{j=1}^n (E \setminus A_j)$ . In the other direction,  $x \in \cup_{j=1}^n (E \setminus A_j) \implies x \in (E \text{ but not } A_1)$  or  $(E \text{ but not } A_2)$  or  $(E \text{ but not } A_n) \implies$  there exists  $j = 1, \dots, n$ ,  $x \in E$  but not  $A_j \implies x \in E$  but not  $(A_1 \text{ and } A_2 \dots \text{ and } A_n) \implies x \in E \setminus \cap_{j=1}^n A_j$ .

- (ii)  $x \in E \setminus \cup_{j=1}^n A_j \implies x \in E$  but  $A_1$  or  $A_2 \dots$  or  $A_n \implies x \in E$  and  $x \notin A_j$  for all  $j = 1, \dots, n \implies x \in E$  but not  $A_1$ , and  $x \in E$  but not  $A_2, \dots$ , and  $x \in E$  but not  $A_n \implies x \in (E \setminus A_1)$  and  $x \in (E \setminus A_2) \dots$  and  $x \in (E \setminus A_n) \implies x \in \cap_{j=1}^n (E \setminus A_j)$ . In the other direction,  $x \in \cap_{j=1}^n (E \setminus A_j) \implies x \in (E \setminus A_1 \text{ and } E \setminus A_2 \dots \text{ and } E \setminus A_n) \implies x \in E$  but not  $A_j$  for all  $j = 1, \dots, n \implies x \in E$  but  $A_1$  or  $A_2 \dots$  or  $A_n \implies x \in E$  but not  $\cup_{j=1}^n A_j \implies x \in E \setminus \cup_{j=1}^n A_j$

## Section 2

C. Consider the subset of  $\mathbb{R} \times \mathbb{R}$  defined by  $D = \{(x, y) : |x| + |y| = 1\}$ . Describe this set in words. Is it a function?

This set consists of points on the line segments connecting a rotated square in the  $(x, y)$  plane with vertices  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$ , and  $(0, -1)$ . If we attempt to define a function, with the elements  $(x, y)$  from the set  $D$ , i.e.  $y = f(x)$ ,  $f : x \rightarrow y$ , we have  $|x| + |y| = 1 \implies \sqrt{y^2} = 1 - |x| \implies y = \pm\sqrt{(1 - |x|)^2}$ .  $f(x) = y = \pm\sqrt{(1 - |x|)^2}$  does not fit the definition of a function, since, as an example, the set  $D$  includes the elements  $(0, 1)$  and  $(0, -1)$ , which if,  $f$  is a function,  $f : x \rightarrow y \implies -1 = 1$ , which is clearly not true.

E. Prove that if  $f$  is an injection from  $A$  to  $B$ , then  $f^{-1} = \{(b, a) : (a, b) \in f\}$  is a function. Then prove it is an injection.

If  $f$  is an injection, and  $(a, b) \in f$ , and  $(a', b) \in f$ , then  $a = a'$ .  $f^{-1} = \{(b, a) : (a, b) \in f\}$  contains the pair  $(b, a)$  and  $(b, a')$ , and we know that  $a = a'$  from the definition of  $f$ , so we can assume that  $f^{-1}$  is a function. Since  $f$  is injective, each unique element  $b = f(a)$ , is mapped to by a unique element  $a$ , and by definition  $f^{-1} = \{(b, a) : (a, b) \in f\}$  maps the unique element  $b$  back to  $a$ , meaning  $f^{-1}(b) = a$  and  $f^{-1}(b') = a$  if and only if  $b = b'$ , thus  $f^{-1}$  is also injective.

H. Let  $f, g$  be functions such that

$$g \circ f(x) = x, \text{ for all } x \text{ in } D(f)$$

$$f \circ g(y) = y, \text{ for all } y \text{ in } D(g)$$

Prove that  $g = f^{-1}$

For two elements  $x, x' \in D(f)$ , if  $f(x) = f(x') \implies g \circ f(x) = g(f(x)) = g(f(x')) \implies g(f(x)) = x = g(f(x')) = x'$ , that is  $x = x' \implies g \circ f$  is an injection. For two elements  $y, y' \in D(g)$ , if  $g(y) = g(y') \implies f \circ g(y) = f(g(y)) = f(g(y')) \implies f(g(y)) = y = f(g(y')) = y'$ , that is  $y = y' \implies f \circ g$  is an injection, and implies  $f$  and  $g$  are injections as well.

This implies  $g$  can be defined  $g = \{(f(x), x) : (x, f(x)) \in f\}$ , which is the definition for  $f^{-1}$ , implying  $g = f^{-1}$ .

J. Let  $f$  be the function on  $\mathbb{R}$  to  $\mathbb{R}$  given by  $f(x) = x^2$ , and let  $E = \{x \in \mathbb{R} : -1 \leq x \leq 0\}$  and  $F = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ . Then  $E \cap F = \{0\}$  and  $f(E \cap F) = \{0\}$  while  $f(E) = f(F) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$ . Hence  $f(E \cap F)$  is a proper subset of  $f(E) \cap f(F)$ . Now delete 0 from  $E$  and  $F$ .

The sets  $E$  and  $F$  with 0 deleted are denoted  $E' = \{x \in \mathbb{R} : -1 \leq x < 0\}$  and  $F' = \{x \in \mathbb{R} : 0 < x \leq 1\}$ , respectively. We still have the equality  $f(E') = f(F') = \{y \in \mathbb{R} : 0 < y \leq 1\} = f(E') \cap f(F')$ . We also have  $E' \cap F' = \emptyset$ , and thus  $f(E' \cap F') = \emptyset$ , and  $\emptyset = f(E' \cap F') \subseteq f(E') \cap f(F')$ , since the empty set is a subset of all sets.

## Section 3