Math. 4317, Practice Test 2

1. Let $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$ be a nested sequence of closed connected subsets of $\mathbb{R}^2$. Is it true that $\bigcap_{n\geq 1} A_n$ must be connected? Prove or give a counterexample.

2. Let $0 < x_1 \leq 3$ and let $x_{n+1} = \sqrt{2x_n + 3}$. Show that the sequence (x_n) is convergent and find its limit.

3. Let $f_n \to f$ and $g_n \to g$ uniformly on some set $E \subseteq \mathbb{R}$. Does it follow that $(f_n g_n)$ converge uniformly to fg on E?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at a point $b \in \mathbb{R}$ and let $f(b) < M$ for some $M \in \mathbb{R}$. Show that there is an open interval I containing b such that $f(x) < M$ for all $x \in I$.

5. Let J be an interval and $f : J \to \mathbb{R}$ be an increasing function (i.e. if $x \leq y$ for $x, y \in J$ then $f(x) \leq f(y)$) such that $f(J)$ is an interval. Show that f must be continuous on J.

1. Let $A_n = \mathbb{R}^2 \setminus \{(x,y) : 0 < x < 1, -n < y < n\}$. The A_n are nested, closed, and connected as every two points in A_n can be connected trivially by a polygonal curve that consists of at most three line segments. But

$$\bigcap_{n\geq 1} A_n = \{(x,y) : x \leq 0\} \cup \{(x,y) : x \geq 1\}$$

which is not connected.

2. We first show that $0 < x_n \leq 3$ for all $n \in \mathbb{N}$. We will prove it by induction. The inequality is true for $k = 1$. Suppose that $0 < x_k \leq 3$ for some $k \in \mathbb{N}$. Then $0 < 2x_k + 3 \leq 9$ which implies that $0 \leq x_{k+1} = \sqrt{2x_k + 3} \leq \sqrt{9} = 3$ and we are done.

We will now show that $x_n \leq x_{n+1}$. For every $n \in \mathbb{N}$. Since we know that $0 \leq x_n \leq 3$ it is enough to show that $x \leq \sqrt{2x+3}$ for $0 \leq x \leq 3$. Since $x \geq 0$ this inequality is equivalent to $x^2 \leq 2x + 3$ which is equivalent to $(x-3)(x+1) \leq 0$, which holds for $-1 \leq x \leq 3$. This shows that we must have $x_n \leq x_{n+1}$. We also notice that if $x \geq 0$ then the equality $x = \sqrt{2x+3}$ holds only if $x = 3$.

Therefore (x_n) is bounded and monotone increasing and so it has a limit. If we denote the limit by x then obviously $0 \leq x \leq 3$. Moreover, passing to the limit in the expression $x_{n+1} = \sqrt{2x_n + 3}$, we get $x = \sqrt{2x+3}$ which, as we noted above, implies that $x = 3$.

3. The answer is no. For instance, let $E = (0, +\infty)$, $f_n(x) = f(x) = x$, $g_n(x) = 1/n$. Then obviously $g(x) = 0$ but $f_n(x)g_n(x) = x/n$ does not converge uniformly to $f(x)g(x) = 0$. The result is true if we assume additionally that the functions f and g are bounded. Prove it.

4. Let $\epsilon = M - f(b)$. By the definition of continuity there exists $\delta(\epsilon) > 0$ such that if $|x - b| < \delta(\epsilon)$ then $|f(x) - f(b)| < \epsilon$. Take $I = (b - \delta(\epsilon), b + \delta(\epsilon))$. Then if $x \in I$ we have $|x - b| < \delta(\epsilon)$ and so $|f(x) - f(b)| < \epsilon = M - f(b)$. Therefore we have

$$f(x) - f(b) < M - f(b)$$

which implies $f(x) < M$.

5. Let $a \in J$ and let (x_n) be a sequence in J such that $x_n \to a$. We notice that, since f is increasing, f is bounded in $J \cap [a - \epsilon, a + \epsilon]$ for some $\epsilon > 0$. If $(f(x_n))$ does not converge to $f(a)$ then, since it is bounded, it must have a convergent subsequence $(f(x_{n_k}))$ which converges to a number $z \neq f(a)$. By choosing a further subsequence we can assume without loss of generality that $x_{n_k} < a, k = 1, 2, \ldots.$ (The proof is similar if we assume $x_{n_k} > a, k = 1, 2, \ldots.$) If $x \in J, x < a$ then there exists a natural number k_0 such that for $k \geq n_0$, $x < x_{n_k}$, which implies $f(x) \leq f(x_{n_k})$ and thus $f(x) \leq z$. Since for $x \in J, x \geq a$ we have $f(x) \geq f(a)$, we thus obtained that $(z, f(a)) \cap f(J) = \emptyset$ but $f(x_{n_k}) \in f(J), f(x_{n_k}) \leq z$ and $f(a) \in f(J)$. This contradicts that $f(J)$ is an interval. Thus we must have $\lim f(x_n) = f(a)$ for every sequence $x_n \to a$ and hence f is continuous at a.