

# Math 4317 (Prof. Swiech, S'18): HW #3

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## Section 14

A. Let  $b \in \mathbb{R}$ , show  $\lim \frac{b}{n} = 0$ .

Take  $\varepsilon > 0$ , if  $|\frac{b}{n} - 0| < \varepsilon$ , there exists natural number  $K(\varepsilon)$  such that  $\frac{b}{n} < \frac{b}{K(\varepsilon)} < \varepsilon$ . If  $n \geq K(\varepsilon)$ , and we choose  $K(\varepsilon)$  such that  $K(\varepsilon) > \frac{b}{\varepsilon} \implies \frac{b}{n} < \varepsilon \implies \lim \frac{b}{n} = 0$ .

B. Show that  $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$ .

Take  $\varepsilon > 0$ , note that for  $n \in \mathbb{N}$ ,  $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} < \frac{1}{n}$ . So we choose natural number  $K(\varepsilon)$  such that  $\frac{1}{K(\varepsilon)} < \varepsilon$ . Therefore if  $n \geq K(\varepsilon) \implies \frac{1}{n} < \varepsilon$ . Therefore  $|\frac{1}{n} - \frac{1}{n+1} - 0| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \varepsilon \implies \lim(\frac{1}{n} - \frac{1}{n+1}) = 0$ .

D. Let  $X = (x_n)$  be a sequence in  $\mathbb{R}^p$  which is convergent to  $x$ . Show that  $\lim \|x_n\| = \|x\|$ . (Hint: use the Triangle Inequality.)

$(x_n)$  convergent with limit  $x \implies$  there exists natural number  $K(\varepsilon)$  such that for  $n \geq K(\varepsilon)$ ,  $\|x_n - x\| < \varepsilon$ . If  $n \geq K(\varepsilon)$ . Since by triangle inequality,  $|\|x_n\| - \|x\|| \leq \|x_n - x\| < \varepsilon \implies \lim \|x_n\| = \|x\|$ .

G. Let  $d \in \mathbb{R}$  satisfy  $d > 1$ . Use Bernoulli's inequality to show that the sequence  $(d_n)$  is not bounded in  $\mathbb{R}$ . Hence it is not convergent.

We have the sequence  $D = (d_n)$ , where  $d_n = d^n$ . Let  $d = 1 + a$  for some  $a > 0 \implies d^n = (1 + a)^n > 1 + na$  by Bernoulli's inequality. For any  $a > b > 0$ ,  $(1 + a)^n > (1 + b)^n$  which implies the sequence  $d_n$  is increasing. Take  $M > 0$ , we have  $d^n > 1 + na > M > 0$ , if  $n > \frac{M}{a} \implies 1 + na > M$ . Thus  $(d_n)$  is not bounded.

H. Let  $b \in \mathbb{R}$  satisfy  $0 < b < 1$ ; show that  $\lim(nb^n) = 0$ . (Hint: use the Binomial theorem as in Example 14.8(e).)

Let  $b = \frac{1}{1+a}$ ,  $a > 0$ , we have  $b^n = \frac{1}{(1+a)^n}$ . By Binomial theorem,  $(1+a)^n > \frac{n(n-1)}{2}a^2 \implies \frac{1}{(1+a)^n} < \frac{2}{n(n-1)a^2}$ , therefore  $nb^n = \frac{n}{(1+a)^n} < \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2}$ . Take  $\varepsilon > 0$ , natural number  $K(\varepsilon)$ , if  $n \geq K(\varepsilon)$  we have  $nb^n = \frac{n}{(1+a)^n} < \frac{2}{(n-1)a^2} < \frac{2}{(K(\varepsilon)-1)a^2} < \varepsilon$ . Then  $|nb^n - 0| < \varepsilon \implies nb^n < \varepsilon \implies \lim nb^n = 0$ .

I. Let  $X = (x_n)$  be a sequence of strictly positive real numbers such that  $\lim(\frac{x_{n+1}}{x_n}) < 1$ . Show that for some  $r$  with  $0 < r < 1$  and some  $C > 0$ , then we have  $0 < x_n < Cr^n$  for all sufficiently large  $n \in \mathbb{N}$ . Use this to show that  $\lim(x_n) = 0$ .

Since  $L = \lim(\frac{x_{n+1}}{x_n}) < 1$ ,  $0 < r < 1 \implies |\frac{x_{n+1}}{x_n} - L| < r$  or  $0 < \frac{x_{n+1}}{x_n} < r$  for all  $n \geq K(\varepsilon) \in \mathbb{N}$ . Since  $\frac{x_{n+1}}{x_n} < r < 1 \implies x_{n+1} < rx_n < x_n \implies x_n < \frac{x_n}{r}$ . If we set  $C = \frac{x_n}{r^{n+1}} > 0$ , we have  $x_n < Cr^n$ . Since  $\lim_{n \rightarrow \infty} r^n = 0 \implies \lim(x_n) = 0$ .

J. Let  $X = (x_n)$  be a sequence of strictly positive real numbers such that  $\lim(\frac{x_{n+1}}{x_n}) > 1$ . Show that  $X$  is not a bounded sequence and hence is not convergent.

Take  $\varepsilon > 0$ , since  $L = \lim(\frac{x_{n+1}}{x_n}) > 1 \implies |\frac{x_{n+1}}{x_n} - L| < \varepsilon \implies L - \varepsilon < \frac{x_{n+1}}{x_n}$  for all  $n \geq K(\varepsilon) \in \mathbb{N}$ . Take  $L - \varepsilon = r > 1$  when  $\varepsilon$  is small. This implies  $x_{n+1} > rx_n$ . Take  $C = \frac{x_n}{r^{n+1}} > 0 \implies x_{n+1} > Cr^n$ . Since  $r > 1$ ,  $r^n$  diverges which implies the sequence  $x_{n+1}$  is not bounded or convergent.

K. Give an example of a convergent sequence  $(x_n)$  of strictly positive real numbers such that  $\lim(\frac{x_{n+1}}{x_n}) = 1$ . Give an example of a divergent sequence with this property.

Consider convergent sequence  $X = (x_n) = (\frac{1}{n})$ .  $\lim(\frac{x_{n+1}}{x_n}) = 1 \implies |\frac{\frac{1}{n+1}}{\frac{1}{n}} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1} < \varepsilon$ ,  $\varepsilon > 0$ .

If we choose natural number  $K(\varepsilon), n \geq K(\varepsilon)$  we have  $\frac{1}{n+1} < \frac{1}{K(\varepsilon)+1} < \varepsilon$ , indicating  $(\frac{x_{n+1}}{x_n})$  is a convergent sequence with limit 1.

Consider divergent sequence  $X = (x_n) = n$ .  $\lim(\frac{x_{n+1}}{x_n}) = 1 \implies |\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} < \varepsilon, \varepsilon > 0$ . If we choose natural number  $K(\varepsilon), n \geq K(\varepsilon)$  we have  $\frac{1}{n} < \frac{1}{K(\varepsilon)} < \varepsilon$ , indicating  $(\frac{x_{n+1}}{x_n})$  is a convergent sequence with limit 1.

*L. Apply the results of Exercises 14.I and 14.J to the following sequences. (Here  $0 < a < 1, 1 < b, c > 0$ )*

- (a)  $(a^n)$   
 $\lim(\frac{x_{n+1}}{x_n}) < 1$ , since  $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} = a < 1 \implies a^n$  is convergent, bounded.
- (b)  $(na^n)$   
 $\lim(\frac{x_{n+1}}{x_n}) < 1$ , since  $\frac{x_{n+1}}{x_n} = \frac{(n+1)a^{n+1}}{na^n} = (\frac{n+1}{n})a$  which tends to  $1 \cdot a < 1 \implies na^n$  is convergent, bounded.
- (c)  $(b^n)$   
 $\lim(\frac{x_{n+1}}{x_n}) > 1$ , since  $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{b^n} = b > 1 \implies b^n$  is divergent, not bounded.
- (d)  $(\frac{b^n}{n})$   
 In this case  $\lim(\frac{x_{n+1}}{x_n}) > 1$ , since  $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{\frac{n+1}{n}} = (\frac{n}{n+1})b$  which tends to  $1 \cdot b > 1 \implies \frac{b^n}{n}$  diverges, not bounded.
- (e)  $(\frac{c^n}{n!})$   
 $\lim(\frac{x_{n+1}}{x_n}) < 1$ , since  $\frac{x_{n+1}}{x_n} = \frac{c^{n+1}}{\frac{(n+1)!}{n!}} = \frac{c}{n+1}$  which tends to  $0 < 1$  implying  $(\frac{c^n}{n!})$  converges, bounded.
- (f)  $(\frac{2^{3n}}{3^{2n}})$   
 $\lim(\frac{x_{n+1}}{x_n}) < 1$ , since  $\frac{x_{n+1}}{x_n} = \frac{2^{3(n+1)}}{\frac{3^{2(n+1)}}{3^{2n}}} = \frac{2^3}{3} \frac{1}{3^2} = \frac{8}{9} < 1$  implying  $(\frac{2^{3n}}{3^{2n}})$  converges, bounded.

## Section 15

*C(a-e). For  $x_n$  given by the following formulas, either establish the convergence or the divergence of the sequence  $X = (x_n)$ :*

- (a)  $x_n = \frac{n}{n+1}$   
 $x_n = \frac{n}{n+1} = \frac{1/n}{1/n + 1/n} = \frac{1}{1 + \frac{1}{n}}$ . The limit of the sequence  $Y = (y_n) = (1 + \frac{1}{n})$  clearly has limit 1  $\implies \lim(x_n) = \lim \frac{1}{1 + \frac{1}{n}} = \frac{\lim 1}{\lim(1 + 1/n)} = 1 \implies$  this sequence converges to 1.
- (b)  $x_n = \frac{(-1)^n n}{n+1}$  Let  $X = (x_n) = (-1)^n, Y = (y_n) = \frac{n}{n+1}$ . Using theorem 15.6.a, if  $X$  converges to  $x$ , and  $Y$  converges to  $y$ .  $X \cdot Y$  converges to  $x \cdot y$ . In our case the series  $(x_n) = (-1)^n$  diverges, and  $(y_n) = \frac{n}{n+1}$  converges to 1  $\implies \lim X \cdot Y = \lim X \cdot 1 = \lim X$  which diverges.
- (c)  $x_n = \frac{2n}{3n^2+1}$   $x_n = \frac{2n}{3n^2+1} = \frac{1/n}{1/n + 3n} = \frac{2}{3n + \frac{1}{n}}$ . We estimate the limit to be 0  $\implies$  for  $n \geq K(\varepsilon), |\frac{2}{3n + \frac{1}{n}} - 0| = \frac{2}{3n + 1/n} < \frac{2}{3K(\varepsilon) + 1/K(\varepsilon)} < \varepsilon, \varepsilon > 0 \implies (x_n) \rightarrow 0$ . Converges.
- (d)  $x_n = \frac{2n^2+3}{3n^2+1}$   
 $x_n = \frac{2n^2+3}{3n^2+1} = \frac{1/n^2}{1/n^2 + \frac{3}{n^2}} = \frac{2+3/n^2}{3+1/n^2} \rightarrow \frac{2}{3}$ . Converges.
- (e)  $x_n = n^2 - n = n(n-1)$   
 The sequence  $(x_n) = n(n-1)$  is clearly divergent, since for all  $M > 0, n \geq M, n(n-1) > M(M-1) > 0$ . Diverges.

*E. If  $X$  and  $Y$  are sequences in  $\mathbb{R}^p$  and if  $X \cdot Y$  converges, do  $X$  and  $Y$  converge and have  $\lim(X \cdot Y) = \lim(X) \cdot \lim(Y)$*

As an example, if we take sequences  $X = (x_n) = (-1)^n = (-1, 1, -1, \dots)$  and  $Y = (y_n) = (-1)^{n+1} = (1, -1, 1, \dots)$ , then their product  $X \cdot Y = (-1, -1, -1, \dots)$  converges to  $-1 \implies$  that the product  $X \cdot Y$  converges, but each sequence  $X$  and  $Y$  does not have a limit, diverges.

As another example, in the case of the constant sequences  $X = (x_n) = (1, 1, \dots)$ , and  $Y = (y_n) = (2, 2, \dots)$ ,  $X \cdot Y$  is the constant sequence  $(2, 2, \dots)$  which converges to 2 which equals  $\lim X \cdot \lim Y$ . Therefore the convergence of  $X \cdot Y$  converges does not necessarily mean that each sequence converges, as there are examples of both cases.

F. If  $X = (x_n)$  is a positive sequence which converges to  $x$ , then  $(\sqrt{x_n})$  converges to  $\sqrt{x}$ . (Hint:  $\sqrt{x_n} - \sqrt{x} = \frac{(x_n - x)}{(\sqrt{x_n} + \sqrt{x})}$  when  $x \neq 0$ ).

In the case that  $\lim(x_n) = x = 0$  we have  $|x_n - x| = |x_n - 0| = x_n < \varepsilon^2$ ,  $\varepsilon^2 > 0$ ,  $n \geq K(\varepsilon)$ , for natural number  $K(\varepsilon)$ . This implies  $0 \leq x_n < \varepsilon^2$  for all  $n \geq K(\varepsilon) \implies 0 \leq \sqrt{x_n} < \varepsilon$ ,  $\varepsilon > 0 \implies \sqrt{x_n} - 0 < \varepsilon \implies |\sqrt{x_n} - \sqrt{x}| < \varepsilon$ ,  $n \geq K(\varepsilon) \implies \sqrt{x}$  is limit of  $\sqrt{x_n}$  when  $x = 0$ .

For case  $x > 0$ ,  $x > 0 \implies \sqrt{x} > 0$ . Since  $|\sqrt{x_n} - \sqrt{x}| = \sqrt{x_n} - \sqrt{x} = \sqrt{x_n} - \sqrt{x} \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$ . Since  $\sqrt{x} > 0$ , also implies  $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0 \implies \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \leq \frac{x_n - x}{\sqrt{x}} \implies |\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}}(x_n - x) = \frac{1}{\sqrt{x}}|x_n - x| < \varepsilon$ ,  $\varepsilon > 0$ . So if  $x_n \rightarrow x \implies \sqrt{x_n} \rightarrow \sqrt{x}$  for  $x > 0$ .

L. If  $0 < a \leq b$  and if  $x_n = (a^n + b^n)^{\frac{1}{n}}$ , then  $\lim(x_n) = b$ .

Since  $0 < a \leq b \implies b^n \leq a^n + b^n \leq b^n + b^n = 2b^n \implies (b^n)^{1/n} \leq (a^n + b^n)^{1/n} \leq (2b^n)^{1/n}$ , therefore,  $b \leq x_n \leq 2^{1/n}b$ . Since  $2^{1/n} \rightarrow 1$  as  $n \rightarrow \infty \implies b \leq x_n \leq b \implies \lim(x_n) = b$ .

N. Let  $A \subseteq \mathbb{R}^p$  and  $x \in \mathbb{R}^p$ . Then  $x$  is a boundary point of  $A$  if and only if there is a sequence  $(a_n)$  of elements in  $A$  and a sequence  $(b_n)$  of elements in  $C(A)$  such that  $\lim(a_n) = x = \lim(b_n)$ .

$\rightarrow$  Let  $x$  be a boundary point of  $A \implies$  there is a neighborhood  $V = \{y \in \mathbb{R}^p : \|y - x\| < r\}$ ,  $r > 0$ , that includes points in  $A$  and complement  $A^c$ . Since  $V$  is a neighborhood of  $x$ , by definition of the limit, there is a natural number  $K_v$  such that for all  $n \geq K_v$ ,  $a_n \in V$  and  $b_n \in V \implies (a_n)$  converges to  $x$  and  $(b_n)$  converges to  $x \implies \lim(a_n) = x = \lim(b_n)$ .

$\leftarrow$  Let  $x$  be limit of sequences  $(a_n)$ ,  $(b_n) \implies$  there is a neighborhood  $V = \{y \in \mathbb{R}^p : \|y - x\| < r\}$ ,  $r > 0$  for natural number  $K_v$ , such that  $n \geq K_v$ ,  $a_n \in V$ ,  $b_n \in V \implies V$  includes points from  $(a_n) \in A$  and  $(b_n) \in A^c \implies x$  is a boundary point of  $A$ .

## Section 16

A. Let  $x_1 \in \mathbb{R}$  satisfy  $x_1 > 1$  and let  $x_{n+1} = 2 - \frac{1}{x_n}$  for  $n \in \mathbb{N}$ . Show that the sequence  $(x_n)$  is monotone and bounded. What is its limit?

We have  $x_1 > 1$  and  $x_2 = 2 - \frac{1}{x_1}$ . We then have  $x_1 > 2 - \frac{1}{x_1} = x_2 > 1$  since  $1 > \frac{1}{x_1} > 0$ . This implies  $x_1 > x_2 > x_3 = 2 - \frac{1}{x_2} > 1$ . Using induction we have  $x_1 > x_2 = 2 - \frac{1}{x_1} > 1$ . We then assume  $x_{n-1} > x_n > 1$ . For case  $n + 1$  we have  $x_n > x_{n+1} > 1$ . Since  $x_n = 2 - \frac{1}{x_{n-1}} > x_{n+1} = 2 - \frac{1}{x_n} > 1$ , and since we assume  $x_{n-1} > x_n > 1$  this implies  $2 - \frac{1}{x_{n-1}} > 2 - \frac{1}{x_n} > 1$ ,  $n \in \mathbb{N}$ . This shows  $(x_n)$  is a monotone decreasing sequence bounded below by 1. Knowing that this sequence has a limit  $x$  that must satisfy the relation  $x = 2 - \frac{1}{x} = x \implies 2 = x + \frac{1}{x}$  which is satisfied when  $x = 1 \implies$  the limit of this sequence is 1.

B. Let  $y_1 = 1$  and  $y_{n+1} = (2 + y_n)^{1/2}$  for  $n \in \mathbb{N}$ . Show that  $(y_n)$  is monotone and bounded. What is its limit?

We have  $y_1 = 1$ ,  $y_2 = \sqrt{2+1} = \sqrt{3} < 2 \implies y_1 < y_2 < 2$ . Using induction we assume  $y_{n-1} < y_n < 2$ . For case  $n + 1$ , we have  $y_n < y_{n+1} < 2 \iff \sqrt{2+y_{n-1}} < \sqrt{2+y_n} < 2 \implies 2 + y_{n-1} < 2 + y_n < 4 \implies$  directly  $y_{n-1} < y_n < 2$ . This shows that  $(y_n)$  is a monotone increasing sequence bounded above by 2. If a limit of  $\lim(y_n) = y$  exists it must satisfy the relation,  $y = \sqrt{2+y} \implies y^2 = 2 + y$ , and we have  $y^2 - y - 2 = (y - 2)(y + 1) = 0$ , which has roots 2,  $-1$ . Since  $(y_n)$  is positive increasing, its limit must be 2.

E. Show that every sequence in  $\mathbb{R}$  either has a monotone increasing subsequence or a monotone decreasing subsequence.

Take an element of the sequence  $X = (x_n)$ ,  $x_k$ , such that  $x_k \geq x_n$ ,  $n > k$ . This implies for each  $k_1 < k_2 < \dots < k_j < \dots$  we have  $x_{k_1} > x_{k_2} > \dots > x_{k_j}$  which is a decreasing subsequence of  $X$ .

Relying on the decreasing subsequence  $x_{k_1} > x_{k_2} > \dots > x_{k_j}$ ,  $D = (x_{k_j})$ , if we take an index  $m_1 > k_j$ , such that  $x_{m_1} \notin D$ , we can construct  $x_{m_1} < x_{m_2} < \dots < x_{m_i}$  since there exists  $m_2 > m_1$  such that  $x_{m_1} < x_{m_2}$  for all  $m$ , which is an increasing subsequence of  $X$ .

G. Determine the convergence or divergence of the sequence  $(x_n)$  where,  $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$  for  $n \in \mathbb{N}$ .

We have  $x_1 = \frac{1}{2}$ ,  $x_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2} \implies x_1 < x_2 < 1$ . Using induction we assume  $x_{n-1} < x_n < 1$ . For the case  $n+1$ , we have  $x_n < x_{n+1} < 1 \iff \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} < \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n+2} < 1$ . Adding  $\frac{1}{n+1} > 1$  to each element we have  $x_n + \frac{1}{n+1} < x_n + \frac{1}{2n+1} + \frac{1}{2n+2} < 1 + \frac{1}{n+1}$ . Since  $\frac{1}{2n+2} + \frac{1}{2n+1} > \frac{1}{n+1} \forall n \in \mathbb{N}$ , because  $\frac{n+1}{2n+2} + \frac{n+1}{2n+1} > 1 \implies x_n < x_{n+1} \implies x_n + (\frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}) < 1 \implies x_n < x_{n+1} < 1 \forall n \in \mathbb{N}$ . This implies this sequence converges and is bounded above by 1.

J. Show directly that the following are not Cauchy sequences.

(a)  $((-1)^n)$

If we take  $\varepsilon = 1 > 0$ , for  $m, n$  greater than natural number  $M(\varepsilon)$ , we have  $|x_m - x_n| = 2 > \varepsilon$  for case  $m$  odd,  $n$  even, or case  $m$  even,  $n$  odd. For the cases  $m$  odd and  $n$  odd, or  $m$  even and  $n$  even we have  $|x_m - x_n| = 0 < \varepsilon \implies$  there exists  $m, n > M(\varepsilon)$  such that  $|x_m - x_n| \geq \varepsilon > 0 \implies X = (x_n) = ((-1)^n)$  is not Cauchy.

(b)  $(n + \frac{(-1)^n}{n})$

If we consider just the case  $m, n > M(\varepsilon) \in \mathbb{N}$ ,  $\varepsilon > 0$ . For the case  $m = n$  we have  $|x_m - x_n| = 0$ , but for the case  $m, n$  even,  $m > n$  we have  $|x_m - x_n| = |m + \frac{-1^m}{m} - n - \frac{-1^n}{n}| = |m - n + (\frac{1}{m} - \frac{1}{n})| > 1 > 0$ . This implies we can find a positive value of  $\varepsilon$  such that  $|x_m - x_n| \geq \varepsilon \implies X = (x_n) = (n + \frac{(-1)^n}{n})$  is not Cauchy.

(c)  $(n^2)$

For  $m, n \in \mathbb{N}$  greater than natural number  $M(\varepsilon)$ ,  $\varepsilon > 0$ , we have  $|x_m - x_n| = |m^2 - n^2| = 0$  for the case  $m = n$ . For the cases  $m > n > 1$ , or  $1 < m < n$  have  $|x_m - x_n| = |m^2 - n^2| \geq 5$ , since, for example, case  $m = 3, n = 2$ ,  $|m^2 - n^2| = 3^2 - 2^2 = 5$ . This implies there exists  $m, n > M(\varepsilon)$  such that  $|x_m - x_n| \geq \varepsilon > 0 \implies X = (x_n) = n^2$  is not Cauchy.

M. Establish the convergence and limits of the following sequences:

(a)  $((1 + \frac{1}{n})^{n+1})$

We have bound on  $x_n = (1 + \frac{1}{n})^{n+1} \geq (1 + (n+1)\frac{1}{n}) = 1 + 1 + \frac{1}{n} > 2$ ,  $\forall n \in \mathbb{N}$  by Bernoulli's Inequality, implying the sequence is bounded below by 2. For  $X = (x_n) = ((1 + \frac{1}{n})^{n+1})$ , we also have  $\forall n \in \mathbb{N}$ ,  $\frac{x_{n-1}}{x_n} = (\frac{\frac{n}{n-1}}{\frac{n}{n-1} + \frac{1}{n}})^n (\frac{1}{1 + \frac{1}{n}}) = (\frac{n}{n-1} \cdot \frac{n}{n+1})^n (\frac{n}{n+1}) = (\frac{n^2}{n^2-1})^n (\frac{n}{n+1}) > 1 \implies (x_n)$  is decreasing. So the sequence is bounded and decreasing. Applying the algebraic property of limits we then have  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^{n+1} = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n \cdot \lim_{n \rightarrow \infty} (1 + \frac{1}{n}) = e \cdot 1$

(c)  $((1 + \frac{2}{n})^n)$

We can write  $((1 + \frac{2}{n})^n) = ((1 + \frac{1}{n})^n) = ((1 + \frac{1}{\frac{n}{2}})^{\frac{n}{2}})^2$ . If we consider the subsequence of even numbers,  $n = 2k$ ,  $k \in \mathbb{N}$ , we have  $((1 + \frac{1}{\frac{n}{2}})^{\frac{n}{2}})^2 = (1 + \frac{1}{k})^k \cdot (1 + \frac{1}{k})^k$ , and using the algebraic property of limits, we have  $\lim_{k \rightarrow \infty} (1 + \frac{1}{k})^k \cdot (1 + \frac{1}{k})^k = e \cdot e = e^2$ , since the sequence has a limit, is convergent to  $e^2$ .

(d)  $((1 + \frac{1}{(n+1)})^{3n})$

We can write  $((1 + \frac{1}{(n+1)})^{3n}) = ((1 + \frac{1}{(n+1)})^n)^3 = (1 + \frac{1}{(n+1)})^n \cdot (1 + \frac{1}{(n+1)})^n \cdot (1 + \frac{1}{(n+1)})^n$ , the product of three convergent sequences, where the limit of each sequence  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n+1})^n = e \implies \lim_{n \rightarrow \infty} ((1 + \frac{1}{(n+1)})^{3n}) = e \cdot e \cdot e = e^3$ .

N. Let  $0 < a_1 < b_1$  and define, for  $n \in \mathbb{N}$ ,  $a_{n+1} = (a_n b_n)^{1/2}$ ,  $b_{n+1} = \frac{1}{2}(a_n + b_n)$ . By induction show that  $a_n < b_n$ , and show that  $a_n$  and  $b_n$  converge to the same limit.

Using induction, we are given  $0 < a_1 < b_1$ , and we assume  $0 < a_n < b_n$ . For the case  $n+1$  we have  $0 < (a_n b_n)^{1/2} < \frac{1}{2}(a_n + b_n) \leftrightarrow 0 < 2\sqrt{a_n b_n} < a_n + b_n \implies 0 < b_n + a_n - 2\sqrt{a_n b_n} = (\sqrt{b_n} - \sqrt{a_n})^2$ . Since we assumed  $b_n > a_n$ ,  $0 < (\sqrt{b_n} - \sqrt{a_n})^2 \leftrightarrow 0 < \sqrt{b_n} - \sqrt{a_n} \implies 0 < \sqrt{a_n} < \sqrt{b_n} \implies 0 < a_n < b_n \implies 0 < a_{n+1} < b_{n+1}$ .

We then take  $a$  to be the limit of  $(a_n)$ , and  $b$  of  $(b_n) \implies a$  satisfies  $a = \sqrt{ab}$ , and  $b$  satisfies  $b = \frac{a+b}{2}$ . This implies  $b = \frac{\sqrt{ab}+b}{2} \implies b$  satisfies  $b = \sqrt{ab} = a \implies (a_n)$  and  $(b_n)$  converge to the same limit.

## Section 17

A. For each  $n \in \mathbb{N}$ , let  $f_n$  be defined for  $x > 0$  by  $f_n(x) = \frac{1}{nx}$ . For what values of  $x$  does limit  $f_n(x)$  exist?

Since  $x > 0$ ,  $\frac{1}{nx}$  is defined for all  $n \in \mathbb{N}$ , and for fixed  $x$  is decreasing in  $n$  is indicative of  $\lim(f_n(x))$  existing for all  $x$

B. For each  $n \in \mathbb{N}$ , let  $g_n$  be defined for  $x \geq 0$  by the formula  $g_n(x) = nx$ ,  $0 \leq x \leq \frac{1}{n}$ ,  $g_n(x) = \frac{1}{nx}$ ,  $\frac{1}{n} < x$ . Show that  $\lim(g_n(x)) = 0$  for all  $x > 0$ .

For case  $x > \frac{1}{n}$ ,  $|g_n(x) - g(x)| = |\frac{1}{nx} - 0| = \frac{1}{nx}$ . For  $n \geq K(\varepsilon, x)$ ,  $nx \geq K(\varepsilon, x)x \implies \frac{1}{nx} \leq \frac{1}{K(\varepsilon, x)x} < \varepsilon$ ,  $\varepsilon > 0 \implies g_n(x) \rightarrow 0 = g(x)$ .

For case,  $0 \leq x \leq \frac{1}{n}$  if we assume  $\lim(g_n(x)) = g(x) = 0$ .

For case  $x = 0$ ,  $g_n(0) = n \cdot 0 = 0$  everywhere implying  $\lim(g_n(x)) = 0$  in this case.

For  $0 < x \leq \frac{1}{n}$ ,  $|g_n(x) - g(x)| = |nx - 0| = nx$ . As  $n$  grows in this case, the region from 0 to  $\frac{1}{n}$  shrinks as the region of valid  $x$  converges to 0  $\implies \lim g_n(x) = 0 = h(x)$ .

D. Show that, if we define  $f_n$  on  $\mathbb{R}$  by  $f_n(x) = \frac{nx}{1+n^2x^2}$ , then  $(f_n)$  converges on  $\mathbb{R}$ .

We have  $f_n(x) = x \frac{n}{1+n^2x^2}$  which can be separated into two functions  $g_n(x) = x$ ,  $h_n(x) = \frac{n}{1+n^2x^2}$ . Clearly  $g_n(x) = x \rightarrow x = g(x)$ ,  $h_n(x) = \frac{n}{1+n^2x^2} = \frac{1}{\frac{1}{n}+nx^2} < \frac{1}{nx^2}$ , since  $x^2 > 0 \implies h(x) = 0$ , and we have  $|h_n(x) - h(x)| = \frac{1}{\frac{1}{n}+nx^2} \leq \frac{1}{\frac{1}{K}+Kx^2} < \varepsilon$ ,  $\varepsilon > 0$ ,  $n \geq K \in \mathbb{N}$ . Using algebraic properties of limits we have,  $\lim f_n(x) = \lim g_n(x) \cdot \lim h_n(x) = 0 \cdot x \implies \lim f_n(x) = 0$ ,  $\implies$  convergence.

E. Let  $h_n$  be defined on the interval  $\mathbb{I} = [0, 1]$  by the formula  $h_n(x) = 1 - nx$ ,  $0 \leq x \leq \frac{1}{n}$ ,  $h_n(x) = 0$ ,  $\frac{1}{n} < x \leq 1$ . Show that  $\lim h_n(x)$  exists on  $[0, 1]$ .

For case  $x = 0$ , we have  $h_n(x) = 1 - nx \rightarrow_{n \rightarrow \infty} 1 = h(x)$ .

For case  $0 < x \leq \frac{1}{n}$ ,  $h_n(x) = 1 - nx \rightarrow 0 = h(x)$ , because as  $n$  grows, the region from 0 to  $\frac{1}{n}$  shrinks  $\implies nx \rightarrow 1 \implies h_n(x) \rightarrow 1 - 1 = 0$ .

For case  $\frac{1}{n} < x \leq 1$  as  $n$  grows we have  $h_n(x) = \frac{1}{nx} \rightarrow 0 = h(x) \implies \lim h_n(x)$  exists on the interval  $[0, 1]$ .

L. Show that the convergence in Exercise 17.B is not uniform on the domain  $x \geq 0$ , but that it is uniform on a set  $x \geq c$ , where  $c > 0$ .

We have

$$g_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ \frac{1}{nx}, & x > \frac{1}{n} \end{cases}$$

But if we take  $\sup_{x \in [0, \infty]} = \sup_{x \in [0, \infty]} |f_n(x) - f(x)|$ , where  $\lim f_n(x) = 0$ , we have  $\sup_{x \in [0, \infty]} |f_n(x) - f(x)| = 1 \implies f_n(x)$  is only pointwise convergent based on results from exercise 17.B.

M. Is the convergence in Exercise 17.D uniform on  $\mathbb{R}$ ?

We have  $\lim_{n \rightarrow \infty} \frac{nx}{1+n^2x^2}$ , which for large  $n$  is like  $\lim_{n \rightarrow \infty} \frac{nx}{n^2x^2} = \frac{1}{x} \lim_{n \rightarrow \infty} \frac{1}{n} \rightarrow 0$ ,  $x \neq 0$ . But if we take  $x = \frac{1}{n}$ , we have  $|f_n(x) - f(x)| = |f_n(\frac{1}{n}) - f(\frac{1}{n})| = |\frac{1}{\frac{1}{n} + n\frac{1}{n}} - 0| = \frac{1}{2} - 0 > \varepsilon$ ,  $0 < \varepsilon < 1/2 \implies f_n(x)$  does not converge uniformly.

## Section 18

A. Determine the limit superior and the limit inferior of the following bounded sequences in  $\mathbb{R}$ .

(a)  $((-1)^n)$

Considering two subsequences of  $X = (x_n)$ , we have  $(x_{2n}) = (1, 1, \dots, 1, \dots)$ , and  $(x_{2n-1}) = (-1, -1, \dots, -1, \dots) \implies \lim(x_{2n}) = 1$ ,  $\lim(x_{2n-1}) = -1 \implies \limsup(x_n) = 1$ ,  $\liminf(x_n) = -1$ .

(b)  $((-1)^n/n)$

Using the same approach,  $(x_n) = ((-1)^n/n) \implies (x_{2n}) = (1/2, 1/4, \dots, 1/2n, \dots)$ , and  $\lim(x_{2n}) = 0$ ,  $(x_{2n-1}) = (-1/1, -1/3, -1/5, \dots, -1/(2n-1), \dots)$ ,  $\implies \lim(x_{2n-1}) = 0 \implies \limsup(x_n) = \liminf(x_n) = 0$ .

(c)  $((-1)^n + 1/n)$

$((-1)^n + 1/n) = (x_n) \implies (x_{2n}) = (1 + 1/2, 1 + 1/4, 1 + 1/6, \dots, 1 + 1/2n, \dots)$ ,  $\implies \lim(x_{2n}) = 1$ .  $(x_{2n-1}) = (-1 + 1/1, -1 + 1/3, -1 + 1/5, \dots, -1 + 1/(2n-1), \dots)$ ,  $\implies \lim(x_{2n-1}) = -1 \implies \limsup(x_n) = 1$ ,  $\liminf(x_n) = -1$ .

D. Give a direct proof of Theorem 18.3(c).

$\liminf(x_n) + \liminf(y_n) \leq \liminf(x_n + y_n)$ . By definition  $\liminf(x_n)$  is the supremum of set  $V$  such that there are at most a finite number of  $n \in \mathbb{N}$  such that  $x_n < v$ , and denote  $\liminf(y_n)$  as the supremum of set  $U$  of  $u \in \mathbb{R}$  such that there are at most a finite number of  $n \in \mathbb{N}$  such that  $y_n < u$ .

Let  $v < \liminf(x_n)$ ,  $u < \liminf(y_n) \implies$  there are only finite  $n \in \mathbb{N}$  such that  $x_n < v$  and  $y_n < u \implies$  only finite  $n \in \mathbb{N}$  such that  $x_n + y_n < v + u \implies \liminf(x_n) + \liminf(y_n) \leq v + u \implies \liminf(x_n) + \liminf(y_n) \leq \liminf(x_n + y_n)$ .

F. If  $X = (x_n)$  is a bounded sequence of strictly positive elements in  $\mathbb{R}$ , show that  $\limsup(x_n^{1/n}) \leq \limsup(x_{n+1}/x_n)$ .

Because  $X = (x_n)$  is bounded, we have  $x^* = \limsup(\frac{x_{n+1}}{x_n})$ ,  $x^* < \infty \implies$  for  $\varepsilon > 0$ ,  $n, K \in \mathbb{N}$ , we have  $\frac{x_{n+1}}{x_n} \leq x^* + \varepsilon$ , for  $n \geq K$ , then  $\frac{x_n}{x_K} = \frac{x_{K+1}}{x_K} \cdot \frac{x_{K+2}}{x_{K+1}} \cdot \dots \cdot \frac{x_{n-1}}{x_{n-2}} \cdot \frac{x_n}{x_{n-1}} \leq (x^* + \varepsilon)^{n-K} \implies \frac{x_n}{x_K} \leq (x^* + \varepsilon)^{n-K} = \frac{(x^* + \varepsilon)^n}{(x^* + \varepsilon)^K} \implies x_n \leq \frac{x_K}{(x^* + \varepsilon)^K} \cdot (x^* + \varepsilon)^n \implies x_n^{1/n} \leq (\frac{x_K}{(x^* + \varepsilon)^K})^{1/n} \cdot (x^* + \varepsilon) \xrightarrow{n \rightarrow \infty} x_n \leq 1 \cdot (x^* + \varepsilon) \implies \limsup(x_n^{1/n}) \leq \limsup(x_{n+1}/x_n) \leq x^* + \varepsilon$ .

I. Show that  $\limsup X = +\infty$  if and only if there is a subsequence  $X'$  of  $X$  such that  $\lim X' = +\infty$ .

$\rightarrow$ . Let  $\limsup X = +\infty \implies \sup\{x_n : n \geq m\} = +\infty$ , for all  $m \in \mathbb{N} \implies$  there is a subsequence  $X'$  such that  $X' = (x_m, x_{m+1}, \dots, x_n)$ , has  $\lim X' = +\infty \implies$  if  $\limsup X = +\infty$  there is a subsequence  $X'$  of  $X$  such that  $\lim X' = +\infty$ .

$\leftarrow$ . Let there be a subsequence of  $X, X'$ , such that  $X'$  has  $\lim X' = +\infty \implies +\infty$  is in the set  $E$  which consists of the limits of all subsequences of  $X$ . This implies that  $\sup E = +\infty \implies \limsup X = +\infty$ .