Midterm 1: Math 6266

Peter Williams

Section 1.1

Exercise 1. Consider the linear regression model with mean zero, uncorrelated, heteroscedastic noise:

$$Y_i = X_i^\intercal \theta + \varepsilon_i,\ for\ i = 1, .., n,\ E\varepsilon_i = 0,\ cov(\varepsilon_j, \varepsilon_j) = \begin{cases} \sigma_i^2, & \text{if } i = j \\ 0, & i \neq j \end{cases} \tag{1}$$

Find expressions for the LSE and response estimator in this model:

Exercise 2. Assume that $\varepsilon_i \sim N(0, \sigma_i^2)$ in the previous problem. What is known about the distribution of $\hat{\theta}$ and $\hat{Y}$? Now suppose additionally that $\sigma_i^2 \equiv \sigma^2 > 0$. What can be said about distribution of the estimator $\hat{\sigma^2}$?

Exercise 3. Consider the linear regression model from exercise 1. Suppose, that the target of estimation is $h^\intercal \theta$ for some determinate non-zero vector $h \in R^p$. Find expression for the LSE of $h^\intercal \theta$. Is this estimate optimal in sense of Gauss-Markov theorem, i.e. does it have the smallest variance among all linear unbiased estimators? Study the same issue for the target $\eta = H^\intercal \theta$, where $H \in R^{q \times p}$ is some non-zero matrix with $q \leq p$.

Section 1.3

Exercise 4. Let $A \in R^{n \times n}$ be a matrix (corresponding to a linear map in R^n). Show that A preserves length for all $x \in R^n$ iff it preserves the inner product. I.e. one needs to show the following: $||Ax|| = ||x||\ \forall\ x \in R^n \iff (Ax)^\intercal (Ay)\ \forall\ x, y \in R^n$.

$$||x|| = \sqrt{x \cdot x} = \sqrt{x^\intercal x} \implies ||Ax|| = \sqrt{Ax \cdot Ax} = \sqrt{x^\intercal A^\intercal Ax} \implies$$
$$A^\intercal = A^{-1}, A^\intercal A = I_n = A^{-1}A\ and\ ||Ax|| = ||x||$$

this implies,

$$(Ax)^\intercal (Ay) = ||AxAy||^2 = x^\intercal A^\intercal Ay = x^\intercal y = ||xy||^2$$

Exercise 5. (a) Let $x_0 \in R^n$ be some fixed vector, find a projection map on the subspace $span(x_0)$. Compare your result with matrix Π (from section 1.3) for the case of $p = 1$. (b) Prove part 3) of Lemma 1.1 for an arbitrary orthogonal projection in R^n. Exercise 6. Let $L1, L2$ be some subspaces in R^n, and $L2 \subseteq L1 \subseteq R^n$. Let $PL1, PL2$ denote orthogonal projections on these subspaces. Prove the following properties: (a) $PL2 - PL1$ is an orthogonal projection, (b) (c) $PL2 \cdot PL1 = PL2$

Section 2.1

Exercise 7. (a) Using the notation from section 2.1, consider $X \sim N(\mu, I_n)$ for some $\mu \in R^n$. Find $EQ(X)$ and $VarQ(X)$. (b) Generalize the results from part (a) to the case $X \sim N(\mu, \Sigma)$ for some positive-definite covariance matrix $\Sigma \in R^{n \times n}$.

Exercise 8. Let $X \sim N(0, In)$, $Q = XX$. Suppose that Q is decomposed into the sum of two quadratic forms: $Q = Q1 + Q2$, where $Qi = X^\intercal A_i X$, $i = 1, 2$ for some symmetric matrices $A1, A2$ with $rank(A1) = n1$ and $rank(A2) = n2$. Show that if $n1 + n2 = n$, then $Q1$ and $Q2$ are independent and $Q_i \sim \chi^2(n_i) for\ i = 1, 2$.

Section 2.2

Exercise 9. In the Gaussian linear regression model 3, consider the target of estimation $\eta = H^{\intercal}\theta^*$, where $H \in R^{q\times p}$ is some non-zero matrix with $q \leq p$. Find an analogue of the quadratic form $S2$ (from (4)) for the new target η^*, and prove for the new quadratic form statements similar to (e) from Theorem 2.1, and Corollary 2.1.2.

Exercise 10. (a) Consider model (3) for $p = 2, X_i = (1, x_i)^{\intercal}, \theta^* = (\theta_1^*, \theta_2^*)^{\intercal}$ (similarly to section 1.5). Write explicit expressions for the confidence sets for $\theta^*, \theta_1^*, \theta_2^*$.

(b) Find a confidence interval for the expected response $E[Y_i]$ in the model in part (a).

Exercise 11. Find an elliptical confidence set for the expected response $E[Y]$ in model (3).

Exercise 12. Construct simultaneous confidence intervals (e.g., as in Corollary 2.2.1) for the expected responses $E[Y_1], ..., E[Y_n]$ in model (3).