Math 4317 (Prof. Swiech, S'18): HW #1

Peter Williams

1/25/2018

Section 1

F. Show that the symmetric difference D, defined in the preceding exercise is also given by $D = (A \cup B) \setminus (A \cap B)$

Show $D = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$:

First, $x \in (A \setminus B) \cup (B \setminus A) \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies$, x is in A but not B, or, x is in B but not $A \implies x$ is in A or B but not in A and $B \implies x \in (A \cup B) \setminus (A \cap B)$.

In the other direction, $x \in (A \cup B) \setminus (A \cap B) \implies x \in (A \cup B)$ but not in $(A \cap B) \implies x$ is in A but not B, or, x is in B but not $A \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies x \in (A \setminus B) \cup (B \setminus A) \implies (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

I. If $\{A_1, A_2, ..., A_n\}$ *is a collection of sets, and if E is any set, show that:*

$$(i)\quad E \cap \cup_{j=1}^{n} A_j = \cup_{j=1}^{n}(E \cap A_j),\ \textit{and}\ (ii),\quad E \cup \cup_{j=1}^{n} A_j = \cup_{j=1}^{n}(E \cup A_j)$$

(i) $x \in E \cap \cup_{j=1}^{n} A_j \implies x \in E\ \textit{and}\ x \in \{A_1\ \textit{or}\ A2\ ...\textit{or}\ A_n\} \implies x \in E$ and that there exists for some $j = 1, 2, ..., n$ an A_j such that $x \in A_j\ \textit{and}\ x \in E \implies (x \in E\ \textit{and}\ A_1)\ \textit{or}\ (x \in E\ \textit{and}\ A_2)\ ...\ \textit{or}\ (x \in E\ \textit{and}\ A_n) \implies x \in \cup_{j=1}^{n}(E \cap A_j)$.
In the other direction, $x \in \cup_{j=1}^{n}(E \cap A_j) \Leftrightarrow x \in (E \cap A_1) \cup (E \cap A_2)\ ...\ \cup (E \cap A_n) \implies x \in E$ and A_1 or E and $A_2 \ldots \implies$ there exists a $j = 1, ..., n$ such that $x \in (E \cap A_j) \implies x \in E$ and $x \in A_1$ or $A_2, \ldots$, or $A_n \implies x \in E$ and $\cup_{j=1}^{n} A_j \implies x \in E \cap \cup_{j=1}^{n} A_j$.

(ii) $x \in E \cup \cup_{j=1}^{n} A_j \implies x \in E$ or $x \in A_1$ or $A_2 \ldots$ or $A_n \implies$ for some $j = 1, .., n$ that $x \in E \cup A_j \implies x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies x \in \cup_{j=1}^{n}(E \cup A_j)$. In the other direction, $x \in \cup_{j=1}^{n}(E \cup A_j) \Leftrightarrow x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies$ there exists some $j = 1, ..., n$ such that $x \in E \cup A_j \implies (x \in E\ \textit{or}\ x \in A_1)$ or $(x \in E\ \textit{or}\ x \in A_2) \ldots$ or $(x \in E\ \textit{or}\ x \in A_n) \implies x \in E$ or $x \in \cup_{j=1}^{n} A_j \implies x \in E \cup \cup_{j=1}^{n} A_j$.

J. If $\{A_1, A_2, ..., A_n\}$ *is a collection of sets, and if E is any set, show that:*

$$(i)\ E \cap \cap_{j=1}^{n} A_j = \cap_{j=1}^{n}(E \cap A_j),\ \textit{and}\ (ii),\quad E \cup \cap_{j=1}^{n} A_j = \cap_{j=1}^{n}(E \cup A_j)$$

(i) $x \in \cap \cap_{j=1}^{n} A_j \implies x \in E$ and $x \in \cap_{j=1}^{n} A_j \implies x \in E\ \textit{and}\ x \in A_j$ for all $j = 1, .., n \implies x \in E$ and $[x \in A_1\ \textit{and}\ x \in A_2\ ...\ \textit{and}\ x \in A_n] \implies [x \in E\ \textit{and}\ A_1]$ and $\ldots$ and $[x \in E\ \textit{and}\ A_n] \implies x \in \cap_{j=1}^{n}(E \cap A_j)$. In the other direction, $x \in \cap_{j=1}^{n}(E \cap A_j) \implies x \in (E \cap A_1)$ and $a \in (E \cap A_2) \ldots$ and $x \in (E \cap A_n) \implies x \in (E \cap A_j)$ for all $j = 1, ..., n \implies x \in E$ and $x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n \implies x \in E$ and $x \in \cap_{j=1}^{nA_j} \implies x \in E \cap \cap_{j=1}^{nA_j}$.

(ii) $x \in E \cup \cap_{j=1}^{n} A_j \implies x \in E$ or $x \in \cap_{j=1}^{n} A_j \implies x \in E$ or $[x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n] \implies x \in E\ \textit{or}\ A_1\ \textit{and}\ x \in E\ \textit{or}\ A_2\ ...\ \textit{and}\ x \in E\ \textit{or}\ A_n \implies x \in \cap_{j=1}^{n}(E \cup A_j)$. In the other direction, $x \in \cap_{j=1}^{n}(E \cup A_j) \implies x \in (E\ \textit{or}\ A_1)\ \textit{and}\ x \in (E\ \textit{or}\ A_2)\ ...\textit{and}\ x \in (E\ \textit{or}\ A_n) \implies$ that for all $j = 1, ..., n$, $x \in (E\ \textit{or}\ A_j) \implies x \in E\ \textit{or}\ (x \in A_1\ \textit{and}\ x \in A_2 ...\ \textit{and}\ x \in A_n) \implies x \in \cap_{j=1}^{n} A_j\ \textit{or}\ x \in E \implies x \in E \cup \cap_{j=1}^{n} A_j$.

K. Let E be a set and $\{A_1, A_2, ..., A_n\}$ *be a collection of sets. Establish the De Morgan laws:*

$$(i)\ E \setminus \cap_{j=1}^{n} A_j = \cup_{j=1}^{n}(E \setminus A_j),\ \textit{and},\ (ii)\quad E \setminus \cup_{j=1}^{n} A_j = \cap_{j=1}^{n}(E \setminus A_j)$$

(i) $x \in E \setminus \cap_{j=1}^{n} A_j \implies x \in E$ but not (A_1 and $A_2 \ldots$ and A_n) $\implies$ there exists a $j = 1, ..., n$ such that $x \in E$ but not $A_j \implies x \in E$ but not A_1, or $x \in E$ but not A_2,. . . ,or $x \in E$ but not

$A_n \implies x \in E \setminus A_1$ or $x \in E \setminus A_2 \ldots$ or $x \in E \setminus A_n \implies x \in \cup_{j=1}^{n}(E \setminus A_j)$. In the other direction, $x \in \cup_{j=1}^{n}(E \setminus A_j) \implies x \in (E \textit{ but not } A_1)$ or $(E \textit{ but not } A_2)$ or $(E \textit{ but not } A_n) \implies$ there exists $j = 1, ..., n,\ x \in E$ but not $A_j \implies x \in E$ but not $(A_1 \textit{ and } A_2 \textit{ ... and } A_n) \implies x \in E \setminus \cap_{j=1}^{n} A_j$.

(ii) $x \in E \setminus \cup_{j=1}^{n} \implies x \in E$ but A_1 or $A_2 \ldots$ or $A_n \implies x \in E$ and $x \notin A_j$ for all $j = 1, ..., n \implies x \in E$ but not A_1, and $x \in E$ but not A_2, $\ldots$, and $x \in E$ but not $A_n \implies x \in (E \setminus A_1)$ *and* $x \in (E \setminus A_2)$ ... *and* $x \in (E \setminus A_n) \implies x \in \cap_{j=1}^{n}(E \setminus A_j)$. In the other direction, $x \in \cap_{j=1}^{n}(E \setminus A_j) \implies x \in (E \setminus A_1 \textit{ and } E \setminus A_2 \textit{ ... and } E \setminus A_n) \implies x \in E$ but not A_j for all $j = 1, ..., n \implies x \in E$ but A_1 or $A_2 \ldots$ or $A_n \implies x \in E$ but not $\cup_{j=1}^{n} A_j \implies x \in E \setminus \cup_{j=1}^{n} A_j$

Section 2

C. Consider the subset of $\mathbb{R} \times \mathbb{R}$ *defined by* $D = \{(x, y)\ : |x| + |y| = 1\}$. *Describe this set in words. Is it a function?*

E. Prove that if f *is an injection from* A *to* B, *than* $f^{-1} = \{(b, a) : (a, b) \in f\}$ *is a function. Then prove it is an injection.*

H. Let f, g *be functions such that*

$$g \circ f(x) = x, \quad for\ all\ x\ in\ D(f)$$

$$f \circ g(y) = y, \quad for\ all\ y\ in\ D(g)$$

Prove that $g = f^{-1}$

J. Let f *be the function on* $\mathbb{R}$ *to* $\mathbb{R}$ *given by* $f(x) = x^2$, *and let* $E = \{x \in \mathbb{R}\ -1 \leq x \leq 0\}$ *and* $F = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. *Then* $E \cap F = \{0\}$ *and* $f(E \cap F) = \{0\}$ *while* $f(E) = f(F) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$. *Hence* $f(E \cap F)$ *is a proper subset of* $f(E) \cap f(F)$. *Now delete* 0 *from* E *and* F.