Math 4317 (Prof. Swiech, S'18): HW #3

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Section 14

A. Let $b \in \mathbb{R}$, show $\lim \frac{b}{n} = 0$.

Take $\varepsilon > 0$, if $|\frac{b}{n} - 0| < \varepsilon$, there exists natural number $K(\varepsilon)$ such that $\frac{b}{n} < \frac{n}{K(\varepsilon)} < \varepsilon$. If $n \geq K(\varepsilon)$, and we choose $K(\varepsilon)$ such that $K(\varepsilon) > \frac{b}{\varepsilon} \implies \frac{b}{n} < \varepsilon \implies \lim \frac{b}{n} = 0$.

B. Show that $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

Take $\varepsilon > 0$, note that for $n \in \mathbb{N}$, $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} < \frac{1}{n}$. So we choose natural number $K(\varepsilon)$ such that $\frac{1}{K(\varepsilon)} < \varepsilon$. Therefore if $n \geq K(\varepsilon) \implies \frac{1}{n} < \varepsilon$. Therefore $|\frac{1}{n} - \frac{1}{n+1} - 0| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \varepsilon \implies \lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

D. Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ which is convergent to x. Show that $\lim ||x_n|| = ||x||$. *(Hint: use the Triangle Inequality.)*

Let $||x|| = \lim(||x_n||)$, $\varepsilon > 0$, which implies there exists natural number $K(\varepsilon)$ such that for $n \geq K(\varepsilon), ||x_n - x|| < \varepsilon$. If $n \geq K(\varepsilon)$, $||x_n|| = ||x_n - x + x|| \leq ||x_n - x|| + ||x|| < \varepsilon + ||x|| \implies ||x_n|| - ||x|| \leq ||x_n - x|| < \varepsilon \implies \lim ||x_n|| = ||x||$.

G. Let $d \in \mathbb{R}$ satisfy $d > 1$. Use Bernoulli's inequality to show that the sequence (d_n) is not bounded in $\mathbb{R}$. Hence it is not convergent.$

We have the sequence $D = (d_n)$, where $d_n = d^n$. Let $d = 1 + a$ for some $a > 0 \implies d^n = (1+a)^n > 1 + na$ by Bernoulli's inequality. For any $a > b > 0$, $(1+a)^n > (1+b)^n$ which implies the sequence d_n is increasing. Take $M > 0$, we have $d^n > 1 + na > M > 0$, if $n > \frac{M}{a} \implies 1 + na > M$. Thus (d_n) is not bounded.

H. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$; show that $\lim(nb^n) = 0$. (Hint: use the Binomial theorem as in Example 14.8(e).)

Let $b = \frac{1}{1+a}, a > 0$, we have $b^n = \frac{1}{(1+a)^n}$. By Binomial theorem, $(1+a)^n > \frac{n(n-1)}{2}a^2 \implies \frac{1}{(1+a)^n} < \frac{2}{n(n-1)a^2}$, therefore $nb^n = \frac{n}{(1+a)^n} < \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2}$. Take $\varepsilon > 0$, natural number $K(\varepsilon)$, if $n \geq K(\varepsilon)$ we have $nb^n = \frac{n}{(1+a)^n} < \frac{2}{(n-1)a^2} < \frac{2}{(K(\varepsilon)-1)a^2} < \varepsilon$. Then $|nb^n - 0| < \varepsilon \implies nb^n < \varepsilon \implies \lim nb^n = 0$.

I. Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) < 1$. Show that for some r with $0 < r < 1$ and some $C > 0$, then we have $0 < x_n < Cr^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $lim(x_n) = 0$

Since $L = \lim(\frac{x_{n+1}}{x_n}) < 1,\ 0 < r < 1 \implies |\frac{x_{n+1}}{x_n} - L| < r$ or $0 < \frac{x_{n+1}}{x_n} < r$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Since $\frac{x_{n+1}}{x_n} < r < 1 \implies x_{n+1} < rx_n < x_n \implies x_n < \frac{x_n}{r}$. If we set $C = \frac{x_n}{r^{n+1}} > 0$, we have $x_n < Cr^n$. Since $\lim_{n\to\infty} r^n = 0 \implies lim(x_n) = 0$.

J. Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) > 1$. Show that X is not a bounded sequence and hence is not convergent.

Take $\varepsilon > 0$, since $L = \lim(\frac{x_{n+1}}{x_n}) > 1 \implies |\frac{x_{n+1}}{x_n} - L| = |L - \frac{x_{n+1}}{x_n}| < \varepsilon \implies L - \varepsilon < \frac{x_{n+1}}{x_n}$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Take $L - \varepsilon = r > 1$ when ε is small. This implies $x_{n+1} > rx_n$. Take $C = \frac{x_n}{r^{n-1}} > 0 \implies x_{n+1} > Cr^n$. Since $r > 1$, r^n diverges which implies the sequence x_{n+1} is not bounded or convergent.

K. Give and example of a convergent sequence (x_n) of strictly positive real numbers such that $\lim(\frac{x_n+1}{x_n}) = 1$. Give an example of a divergent sequence with this property.

Consider convergent sequence $X = (x_n) = (\frac{1}{n})$. $\lim(\frac{x_n+1}{x_n}) = 1 \implies |\frac{\frac{1}{n+1}}{\frac{1}{n}} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1} < \varepsilon,\ \varepsilon > 0.$

If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n+1} < \frac{1}{K(\varepsilon)+1} < \varepsilon$, indicating $(\frac{x_n+1}{x_n})$ is a convergent sequence with limit 1.

Consider divergent sequence $X = (x_n) = n$. $\lim(\frac{x_n+1}{x_n}) = 1 \implies |\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} < \varepsilon,\ \varepsilon > 0$. If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n} < \frac{1}{K(\varepsilon)} < \varepsilon$, indicating $(\frac{x_n+1}{x_n})$ is a convergent sequence with limit 1.

L. Apply the results of Exercises 14.I and 14.J to the following sequences. (Here $0 < a < 1, 1 < b, c > 0$)

(a) (a^n)
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} = a < 1 \implies a^n$ is convergent, bounded.

(b) (na^n)
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{(n+1)a^{n+1}}{na^n} = (\frac{n+1}{n})a$ which tends to $1 \cdot a < 1 \implies na^n$ is convergent, bounded.

(c) (b^n)
$\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{b^n} = b > 1 \implies b^n$ is divergent, not bounded.

(d) $(\frac{b^n}{n})$ In this case $\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{b^{n+1}}{n+1}}{\frac{b^n}{n}} = (\frac{n}{n+1})b$ which tends to $1 \cdot b > 1 \implies \frac{b^n}{n}$ diverges, not bounded.

(e) $(\frac{c^n}{n!})$
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}} = \frac{c}{n+1}$ which tends to $0 < 1$ implying $(\frac{c^n}{n!})$ converges, bounded.

(f) $(\frac{2^{3n}}{3^{2n}})$
converge

Section 15

C(a-e).
E.
F.
L.
N.

Section 16

A,B,E,G,J,M(a)(c)(d),N

Section 17

A,B,D,E,L,M

Section 18

A(a-c),D,F,I