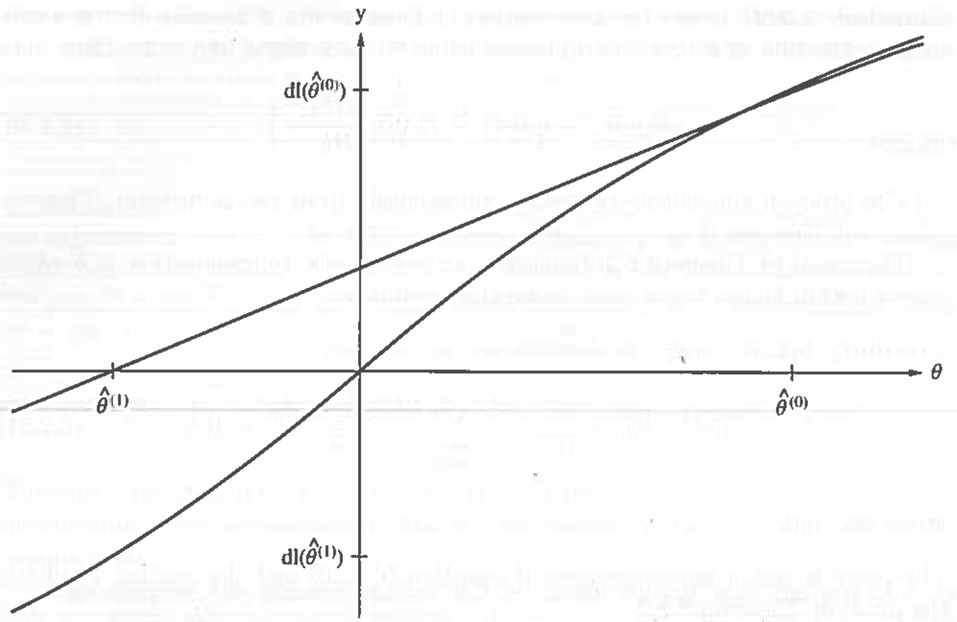

Figure 6.2.1: Beginning with the starting value $\widehat{\theta}^{(0)}$, the one-step estimate is $\widehat{\theta}^{(1)}$, which is the intersection of the tangent line to the curve $l'(\theta)$ at $\widehat{\theta}^{(0)}$ and the horizontal axis. In the figure, $dl(\theta) = l'(\theta)$.

Example 6.2.7 (Example 6.1.2, continued). Recall Example 6.1.2, where the random sample $X_1, \ldots, X_n$ has the common logisitic density

$$f(x;\theta) = \frac{\exp\{-(x-\theta)\}}{(1+\exp\{-(x-\theta)\})^2}, \quad -\infty < x < \infty, \; -\infty < \theta < \infty. \tag{6.2.33}$$

We showed that the likelihood equation has a unique solution, though it cannot be be obtained in closed form. To use formula (6.2.32), we need the first and second partial derivatives of $l(\theta)$ and an initial guess. Expression (6.1.9) of Example 6.1.2 gives the first partial derivative, from which the second partial is

$$l''(\theta) = -2\sum_{i=1}^{n} \frac{\exp\{-(x_i-\theta)\}}{(1+\exp\{-(x_i-\theta)\})^2}.$$

The logistic distribution is similar to the normal distribution; hence, we can use $\overline{X}$ as our initial guess of θ. The subroutine `mlelogistic` in Appendix B is an R routine which obtains the k-step estimates. ■

We close this section with a remarkable fact. The estimate $\widehat{\theta}^{(1)}$ in equation (6.2.32) is called the **one-step estimator**. As Exercise 6.2.13 shows, this estimator has the same asymptotic distribution as the mle, [i.e., (6.2.18)], provided that the

initial guess $\widehat{\theta}^{(0)}$ is a consistent estimator of θ. That is, the one-step estimate is an asymptotically efficient estimate of θ. This is also true of the other iterative steps.

EXERCISES

6.2.1. Prove that $\overline{X}$, the mean of a random sample of size n from a distribution that is $N(\theta, \sigma^2)$, $-\infty < \theta < \infty$, is, for every known $\sigma^2 > 0$, an efficient estimator of θ.

6.2.2. Given $f(x;\theta) = 1/\theta$, $0 < x < \theta$, zero elsewhere, with $\theta > 0$, formally compute the reciprocal of

$$nE\left\{\left[\frac{\partial \log f(X:\theta)}{\partial\theta}\right]^2\right\}.$$

Compare this with the variance of $(n+1)Y_n/n$, where Y_n is the largest observation of a random sample of size n from this distribution. Comment.

6.2.3. Given the pdf

$$f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty,$$

show that the Rao–Cramér lower bound is $2/n$, where n is the size of a random sample from this Cauchy distribution. What is the asymptotic distribution of $\sqrt{n}(\widehat{\theta}-\theta)$ if $\widehat{\theta}$ is the mle of θ?

6.2.4. Consider Example 6.2.2, where we discussed the location model.

(a) Write the location model when e_i has the logistic pdf given in expression (4.4.9).

(b) Using expression (6.2.8), show that the information $I(\theta) = 1/3$ for the model in part (a). *Hint:* In the integral of expression (6.2.8), use the substitution $u = (1+e^{-z})^{-1}$. Then $du = f(z)dz$, where $f(z)$ is the pdf (4.4.9).

6.2.5. Using the same location model as in part (a) Exercise 6.2.4, obtain the ARE of the sample median to mle of the model.
Hint: The mle of θ for this model is discussed in Example 6.2.7. Furthermore, as shown in Theorem 10.2.3 of Chapter 10, Q_2 is asymptotically normal with asymptotic mean θ and asymptotic variance $1/(4f^2(0)n)$.

6.2.6. Consider a location model (Example 6.2.2) when the error pdf is the contaminated normal (3.4.14) with ϵ as the proportion of contamination and with σ_c^2 as the variance of the contaminated part. Show that the ARE of the sample median to the sample mean is given by

$$e(Q_2, \overline{X}) = \frac{2[1+\epsilon(\sigma_c^2-1)][1-\epsilon+(\epsilon/\sigma_c)]^2}{\pi}. \tag{6.2.34}$$

Use the hint in Exercise 6.2.5 for the median.