way. For each n, the set of all solutions in the interval is bounded; hence, the infimum over solutions closest to θ_0 exists. ■

Note that this theorem is vague in that it discusses solutions of the equation. If, however, we know that the mle is the unique solution of the equation $l'(\theta) = 0$, then it is consistent. We state this as a corollary:

Corollary 6.1.1. *Assume that $X_1, \ldots, X_n$ satisfy the regularity conditions (R0) through (R2), where θ_0 is the true parameter, and that $f(x;\theta)$ is differentiable with respect to θ in Ω. Suppose the likelihood equation has the unique solution $\widehat{\theta}_n$. Then $\widehat{\theta}_n$ is a consistent estimator of θ_0.*

EXERCISES

6.1.1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\Gamma(\alpha = 3, \beta = \theta)$ distribution, $0 < \theta < \infty$. Determine the mle of θ.

6.1.2. Let $X_1, X_2, \ldots, X_n$ represent a random sample from each of the distributions having the following pdfs:

(a) $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, zero elsewhere.

(b) $f(x;\theta) = e^{-(x-\theta)}$, $\theta \le x < \infty$, $-\infty < \theta < \infty$, zero elsewhere. Note this is a nonregular case.

In each case find the mle $\hat{\theta}$ of θ.

6.1.3. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample from a distribution with pdf $f(x;\theta) = 1$, $\theta - \frac{1}{2} \le x \le \theta + \frac{1}{2}$, $-\infty < \theta < \infty$, zero elsewhere. Note this is a nonregular case. Show that every statistic $u(X_1, X_2, \ldots, X_n)$ such that

$$Y_n - \tfrac{1}{2} \le u(X_1, X_2, \ldots, X_n) \le Y_1 + \tfrac{1}{2}$$

is a mle of θ. In particular, $(4Y_1 + 2Y_n + 1)/6$, $(Y_1 + Y_n)/2$, and $(2Y_1 + 4Y_n - 1)/6$ are three such statistics. Thus, uniqueness is not, in general, a property of a mle.

6.1.4. Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x;\theta) = 2x/\theta^2$, $0 < x \le \theta$, zero elsewhere. Note this is a nonregular case. Find:

(a) The mle $\hat{\theta}$ for θ.

(b) The constant c so that $E(c\hat{\theta}) = \theta$.

(c) The mle for the median of the distribution.

6.1.5. Suppose $X_1, X_2, \ldots, X_n$ are iid with pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. Find the mle of $P(X \le 2)$.

6.1.6. Let the table

x	0	1	2	3	4	5
Frequency	6	10	14	13	6	1

represent a summary of a sample of size 50 from a binomial distribution having $n = 5$. Find the mle of $P(X \ge 3)$.

6.1.7. Let X_1, X_2, X_3, X_4, X_5 be a random sample from a Cauchy distribution with median θ, that is, with pdf

$$f(x;\theta) = \frac{1}{\pi}\frac{1}{1 + (x-\theta)^2}, \quad -\infty < x < \infty,$$

where $-\infty < \theta < \infty$. If $x_1 = -1.94$, $x_2 = 0.59$, $x_3 = -5.98$, $x_4 = -0.08$, and $x_5 = -0.77$, find by numerical methods the mle of θ.

6.1.8. Let the table

x	0	1	2	3	4	5
Frequency	7	14	12	13	6	3

represent a summary of a random sample of size 55 from a Poisson distribution. Find the maximum likelihood estimate of $P(X = 2)$.

6.1.9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter p. If p is restricted so that we know that $\frac{1}{2} \le p \le 1$, find the mle of this parameter.

6.1.10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ distribution, where σ^2 is fixed but $-\infty < \theta < \infty$.

(a) Show that the mle of θ is $\overline{X}$.

(b) If θ is restricted by $0 \le \theta < \infty$, show that the mle of θ is $\widehat{\theta} = \max\{0, \overline{X}\}$.

6.1.11. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson distribution with $0 < \theta \le 2$. Show that the mle of θ is $\widehat{\theta} = \min\{\overline{X}, 2\}$.

6.1.12. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with one of two pdfs. If $\theta = 1$, then $f(x;\theta = 1) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $-\infty < x < \infty$. If $\theta = 2$, then $f(x;\theta = 2) = 1/[\pi(1 + x^2)]$, $-\infty < x < \infty$. Find the mle of θ.

6.2 Rao–Cramér Lower Bound and Efficiency

In this section, we establish a remarkable inequality called the **Rao–Cramér** lower bound, which gives a lower bound on the variance of any unbiased estimate. We then show that, under regularity conditions, the variances of the maximum likelihood estimates achieve this lower bound asymptotically.

As in the last section, let X be a random variable with pdf $f(x;\theta)$, $\theta \in \Omega$, where the parameter space Ω is an open interval. In addition to the regularity conditions (6.1.1) of Section 6.1, for the following derivations, we require two more regularity conditions, namely,