Math 4317 (Prof. Swiech, S'18): HW #2

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Section 8

D. If w_1 and w_2 are strictly positive, show that the definition, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1w_1 + x_2y_2w_2$, yields an inner product on $\mathbb{R}^2$, generalize this for $\mathbb{R}^p$.

Checking the properties of inner products, we have, based on definition above, (i) $x \cdot x \geq 0$, since $(x_1, x_2)(x_1, x_2) = w_1x_1^2 + w_2x_2^2 \geq 0$, since $w_1, w_2 > 0$, and $x_i^2 > 0$, $i = 1, 2$. For $x \in \mathbb{R}^p$, we have $x \cdot x = \sum_{j=1}^{p} w_jx_j^2 \geq 0$, since each element in the summation $w_i, x_i^2 > 0$. For property (ii), we have $x \cdot x = 0$, if and only if $x = 0$. In this case, since $w_1, w_2 > 0$, $w_1x_2^2 + w_2x_2^2 = 0$, when x_1^2 and x_2^2 equal zero, that is when $x = 0$. This holds for $x \in \mathbb{R}^p$, since for $w_i > 0, \quad i = 1, ..., p$ we have $\sum_{j=1}^{p} w_jx_j^2 = 0$, only when each element $w_ix_i2 = 0$, since each element is greater than or equal to zero. For property (iii), we show $x \cdot y = y \cdot x$ since $x \cdot y = w_1x_1y_1 + w_2x_2y_2 = w_1x_1y_1 + w_2x_2y_2 = w_1y_1x_2 + w_2y_2x_2 = y \cdot x$. Extending to $x \in \mathbb{R}^p$, we have again, by commutative property, $x \cdot y = \sum_{j=1}^{p} w_jx_jy_j = \sum_{j=1}^{p} w_jy_jx_j = y \cdot x$. Property (iv), $x \cdot (y + z) = x \cdot y + x \cdot z,\ x, y, z \in \mathbb{R}^p$. In this case we have $\sum_{j=1}^{p} w_jx_j(y_j + z_j) = \sum_{j=1}^{p} w_jx_jy_j + w_jx_jz_j = \sum_{j=1}^{p} w_jx_jy_j + \sum_{j=1}^{p} w_jx_jz_j = x \cdot y + x \cdot z$, which clearly holds for base case, $p = 2$ as well. For property (v), we have $(ax) \cdot y = x \cdot (ay),\ a \in \mathbb{R}$. We have $(ax) \cdot y = \sum_{j=1}^{p} w_jax_jy_j = a\sum_{j=1}^{p} w_jx_jy_j = a(x \cdot y) = \sum_{j=1}^{p} w_jx_jay_j = x \cdot (ay)$. Since all five properties are satisfied, an inner product is yielded here.

E. $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1$ is not an inner product on $\mathbb{R}^2$. Why?

By property (ii), i.e. $x \cdot x = 0$ if and only if $x = 0$, the definition above, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1 = 0 \Leftrightarrow x = 0$, however, we can't say $x = 0$, since in this case if $x_1y_1 = 0 \implies x_1 = 0$, but we don't have information about x_2, or $x_i, i = 3, ..., p$, for $xin\mathbb{R}^p$. Thus for this operation $x \cdot x = 0$ does not necessarily mean $x = 0$.

F.
G.
H.
P.
Q.

Section 9

Section 10

Section 11