

# Math 4317 (Prof. Swiech, S'18): HW #4

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## Section 20

A. Prove that if  $f$  is defined for  $x \geq 0$  by  $f(x) = \sqrt{x}$ , then  $f$  is continuous at every point of its domain.

For  $f(x) = \sqrt{x}$ ,  $\mathcal{D}(f) = \{x \in \mathbb{R} : x \geq 0\}$ , let  $a \in \mathcal{D}(f)$ .

When  $a = 0$ ,  $|f(x) - f(a)| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon$ . If we let  $\delta(\varepsilon) = \varepsilon^2$ , when  $x < \varepsilon^2$ ,  $|f(x)| < \varepsilon$ .

When  $a \neq 0$ ,  $|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = \frac{|\sqrt{x} - \sqrt{a}|}{|\sqrt{x} + \sqrt{a}|} |\sqrt{x} + \sqrt{a}| = \frac{|x - a|}{|\sqrt{x} + \sqrt{a}|} < \frac{|x - a|}{\sqrt{a}} < \varepsilon \implies$  when  $|x - a| < \varepsilon\sqrt{a}$ , then,  $|f(x) - f(a)| < \varepsilon$ , thus we can choose  $\delta(\varepsilon) = \varepsilon\sqrt{a} \implies f$  is continuous at every point in its domain.

B. Show that a “polynomial function”; that is, a function  $f$  with the form  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ ,  $x \in \mathbb{R}$  is continuous at every point of  $\mathbb{R}$ .

Relying on the properties of algebraic combinations of continuous of functions, we construct  $f$  as a combination of continuous functions to show its continuity. Considering the last term of the polynomial function, denoted here,  $f_0(x) = a_0$ ,  $f_0(x)$  is a continuous, constant function, since, for any  $a \in \mathbb{R}$  we have  $|f_0(x) - f_0(a)| = |a_0 - a_0| < \varepsilon = \delta(\varepsilon)$ ,  $\varepsilon > 0$ . We consider the second to last term of  $f$ ,  $a_1 x$ , as a constant,  $a_1$  multiplied by the identity function, denoted,  $f_1(x) = x$ . Since  $f_1(x) = x$ , for any real number  $a \in \mathbb{R}$ , we have  $|f_1(x) - f_1(a)| = |x - a| < \varepsilon = \delta(\varepsilon)$ ,  $\varepsilon > 0 \implies a_1 f_1(x) = a_1 x$  is continuous.

Relying on the continuity of  $f_1(x) = x$  multiplied by any constant, we can construct higher order terms of  $f$  through repeated multiplication of  $f_1(x)$ , e.g.  $a_2 \cdot f_1(x) \cdot f_1(x) = a_2 x^2$  and  $a_n \prod_{j=1}^n f_1(x) = a_n \cdot f_1(x) \cdot f_1(x) \cdot \dots \cdot f_1(x) = a_n x^n$ , and so on, where each term constructed  $a_n x^n$  is continuous on  $\mathbb{R}$  since it is constructed via algebraic combinations of continuous functions  $\implies f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ , is continuous at every point  $x \in \mathbb{R}$ .

E. Let  $f$  be the function on  $\mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x$ ,  $x$  irrational,  $f(x) = 1 - x$ ,  $x$  rational. Show that  $f$  is continuous at  $x = \frac{1}{2}$  and discontinuous elsewhere.

Considering the point  $a = \frac{1}{2}$ , we have  $f(a) = \frac{1}{2}$ , and  $|f(x) - f(a)| = |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = |x - a| < \varepsilon = \delta(\varepsilon)$ . So if  $|f(x) - f(a)| < \varepsilon = \delta(\varepsilon) > 0 \implies |x - a| < \delta(\varepsilon)$ , and then we have  $f$  continuous at the point  $a = \frac{1}{2}$ . For the case  $a \neq \frac{1}{2}$ ,  $a$  irrational, take a sequence  $X = (x_n)$  of rational numbers converging to  $a$ . Since the sequence  $(f(x_n))$  converges to  $1 - a$ , and we have  $f(a) = a$ ,  $f$  is not continuous at irrational points by the Discontinuity Criterion. For the case  $a \neq \frac{1}{2}$ ,  $a$  rational, take a sequence  $Y = (Y_n)$  of irrational numbers converging to  $a$ , the sequence  $(f(y_n))$  converges to  $a$ , but  $f(a) = 1 - a$ , which equation is only satisfied when  $a = \frac{1}{2}$ , thus  $f$  is not continuous for rational numbers at any point other than  $\frac{1}{2}$ .

F. Let  $f$  be continuous on  $\mathbb{R} \rightarrow \mathbb{R}$ . Show that if  $f(x) = 0$  for rational  $x$ , then  $f(x) = 0$  for all  $x \in \mathbb{R}$ .

Every real point,  $x \in \mathbb{R}$  is the limit of a sequence of rational numbers. If  $f$  is continuous  $\implies$  for a sequence of rational numbers  $X = (x_n) \rightarrow x$ , we have  $(f(x_n)) = 0$ , for all  $n \in \mathbb{N}$ . Since  $f$  is continuous at each rational point  $x \in \mathbb{R}$ , we can find  $|f(x_n) - f(x)| < \varepsilon$ ,  $\varepsilon > 0$ , and  $|x_n - a| < \delta(\varepsilon) \implies (f(x_n)) \rightarrow f(x) = 0, \forall x \in \mathbb{R}$ .

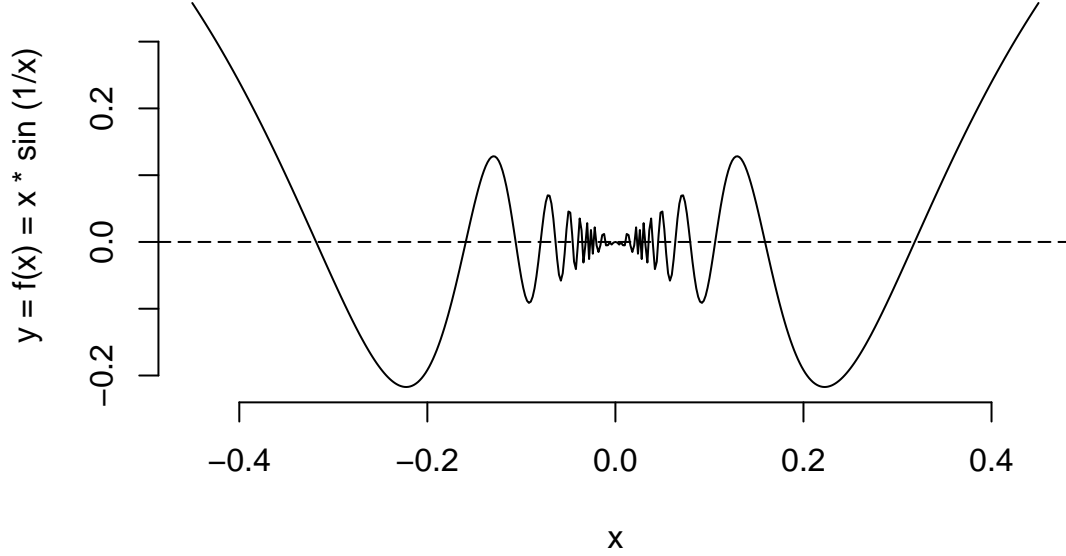
I. Using the results of the preceding exercise, show that the function  $g$ , defined on  $\mathbb{R} \rightarrow \mathbb{R}$  by  $g(x) = x \sin(\frac{1}{x})$ ,  $x \neq 0$ ,  $g(x) = 0$ ,  $x = 0$  is continuous at every point. Sketch a graph of this function.

For the case  $a = 0$ , we have  $|g(x) - g(a)| = |x \sin \frac{1}{x} - 0| = |x| |\sin \frac{1}{x}| \leq |x| \cdot 1 < \varepsilon$ ,  $\varepsilon > 0$ , since  $-1 \leq \sin \frac{1}{x} \leq 1$ . So when  $|g(x) - g(0)| < \varepsilon = \delta(\varepsilon)$ , we then have  $|x| = |x - 0| < \delta(\varepsilon) \implies g$  continuous at 0.

For the case  $a \neq 0$ , we have  $|g(x) - g(a)| = |x \sin \frac{1}{x} - a \sin \frac{1}{a}| = |x \sin \frac{1}{x} - a \sin \frac{1}{a} - a \sin \frac{1}{x} + a \sin \frac{1}{x}| = |(x - a)(\sin \frac{1}{x}) + a(\sin \frac{1}{x} - \sin \frac{1}{a})| \leq |x - a| |\sin \frac{1}{x}| + |a| |\sin \frac{1}{x} - \sin \frac{1}{a}|$ , by Triangle Inequality. Since both  $|\sin \frac{1}{x}| \leq 1$  and  $|\sin \frac{1}{x} - \sin \frac{1}{a}| \leq 1$ , we have  $|x - a| |\sin \frac{1}{x}| + |a| |\sin \frac{1}{x} - \sin \frac{1}{a}| \leq |x - a| \cdot 1 + |a| \cdot 1 = |x - a| + |a| < \varepsilon$ .

It then follows that if  $\delta(\varepsilon) = \varepsilon - |a|$ , i.e.  $\varepsilon > \delta(\varepsilon) + |a|$ , when  $|g(x) - g(a)| < \varepsilon$ , then  $|x - a| < \delta(\varepsilon) \implies g$  continuous at every point in  $\mathbb{R}$ .

Sketch of function below:



N. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  satisfy the relation  $g(x + y) = g(x)g(y)$ ,  $x, y \in \mathbb{R}$ . Show that if  $g$  is continuous at  $x = 0$ , then  $g$  is continuous at every point. Also if  $g(a) = 0$  for some  $a \in \mathbb{R}$ , then  $g(x) = 0$  for all  $x \in \mathbb{R}$ .

If  $g$  is continuous at  $x = 0 \implies g(x + y) = g(y) = g(0) \cdot g(y)$ . This implies also that  $g(0)g(y) = g(y) \implies g(0)g(y) - g(y) = 0 = g(y)(g(0) - 1) = 0 \implies g(0) = 1$ , or that  $g(0) = 0$ .

If  $g(0) = 0 \implies -g(y) = 0 = g(y)$ . In this case then  $g(y) = 0, \forall y \in \mathbb{R} \implies g(x) = 0, \forall x \in \mathbb{R}$ .

On the other hand if  $g(0) = 1, \implies g(0) \cdot g(y) = g(y)$  continuous for every point  $y \in \mathbb{R}$ .

## Section 21

I. Let  $g$  be a linear function from  $\mathbb{R}^p \rightarrow \mathbb{R}^q$ . Show that  $g$  is one-one and only if  $g(x) = 0$  implies that  $x = 0$ . Since  $g$  is linear  $\implies$  for  $x, y \in \mathbb{R}^p$ ,  $g(x + y) = g(x) + g(y)$ . Then if  $g(x) = 0 \implies g(x + y) = 0 + g(y) = g(y) \implies g(x + y) = g(y) \implies g(x + y) = g(0 + y) = g(y)$  which implies  $x = 0$ . If we assume that  $g$  is one-one, then for any  $g(x) = g(y) \implies x = y$ . So in the case  $g(x) = 0$ , and  $g(x + y) = g(x) + g(y) = 0 + g(y)$ . Since  $g(x) + g(y) = g(y) \implies g(y) - g(x) = g(y) \implies x + y = x - y$ , which is satisfied when  $x = 0$ .

J. If  $h$  is a one-one linear function from  $\mathbb{R}^p \rightarrow \mathbb{R}^p$ , show that the inverse function  $h^{-1}$  is a linear function from  $\mathbb{R}^p \rightarrow \mathbb{R}^p$ .

Since  $h$  is one-one  $\implies$  if  $h(x_1) = h(x_2)$ ,  $x_1 = x_2$ ,  $x_1, x_2 \in \mathbb{R}^p$ . Extending the linear case, we have if  $h(ax + by) = h(ax_1 + by_1) = ah(x) + bh(y) = ah(x_1) + bh(y_1)$  then  $ax_1 + by_1 = ax + by$ . By definition  $h^{-1} = \{ax + by : h(ax + by) \in \mathbb{R}^p\} = \{ax : h(ax) \in \mathbb{R}^p\} + \{by : h(by) \in \mathbb{R}^p\}$ . This implies  $h^{-1}(ax + by) = h^{-1}(h(ax)) + h^{-1}(h(by)) \implies h^{-1}$  is linear, and  $h^{-1} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ , since  $h^{-1}(h(ax)) + h^{-1}(h(by)) = ax + by \in \mathbb{R}^p$  by construction.

K. Show that the sum and the composition of two linear functions are linear functions.

By definition a function is linear if  $f(ax + by) = af(x) + bf(y)$ ,  $a, b \in \mathbb{R}$ ,  $x, y \in \mathbb{R}^p$ .

For the sum of two linear functions we then have  $(f + g)(ax + by) = f(ax + by) + g(ax + by) = af(x) + bf(y) + ag(x) + bg(y) = a(f(x) + g(x)) + b(f(y) + g(y)) = a(f + g)(x) + b(f + g)(y) \implies$  linearity. For the composition of two linear functions we have  $f \circ g(ax + by) = f(g(ax + by)) = f(ag(x) + bg(y)) = af(g(x)) + bf(g(y)) = a(f \circ g)(x) + b(f \circ g)(y) \implies$  composition of two linear functions is linear.

*L. If  $f$  is a linear map on  $\mathbb{R}^p \rightarrow \mathbb{R}^q$ , define  $\|f\|_{pq} = \sup\{\|f(x)\| : x \in \mathbb{R}^p, \|x\| \leq 1\}$ . Show that the mapping  $f \rightarrow \|f\|_{pq}$  defines a norm on the vector space  $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$  of all linear functions on  $\mathbb{R}^p \rightarrow \mathbb{R}^q$ . Show that  $\|f(x)\| \leq \|f\|_{pq}\|x\|$  for all  $x \in \mathbb{R}^p$ .*

We have  $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$ ,  $f(x) = y = (y_1, y_2, \dots, y_q) \in \mathbb{R}^q$ , and matrix  $A_{q \times p} = (c_{ij})$ ,  $1 \leq i \leq q$ ,  $1 \leq j \leq p$ , with

$$y_1 = c_{11}x_1 + c_{12}x_2 + \dots + c_{1p}x_p$$

...

$$y_q = c_{q1}x_1 + c_{q2}x_2 + \dots + c_{qp}x_p$$

We then have  $\|f(x)\| = \|(y_1, \dots, y_q)\| = \sqrt{y_1^2 + \dots + y_q^2}$ . To show  $\|f\|_{qp} = \sup\{\|f(x)\| : x \in \mathbb{R}^p, \|x\| \leq 1\}$  is a norm in  $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ , we have (i)  $\|f\|_{pq} \geq 0$ ,  $x \in \mathbb{R}^p$ ? Since each element in  $\|f(x)\| = \sqrt{y_1^2 + \dots + y_q^2}$ ,  $y_j^2 \geq 0$ ,  $\forall j = 1, \dots, q \implies \sup\{\|f(x)\|\} \geq 0 \forall x \in \mathbb{R}^p$  since by definition,  $\sup\{\|f(x)\|\} \geq \|f(x)\| \forall x \in \mathbb{R}^p \implies \|f\|_{pq} \geq 0$ .

(ii)  $\|f\|_{pq} = 0 \iff f(x) = 0$ ? Since  $\|f(x)\| = \|y\| = \sqrt{y_1^2 + \dots + y_q^2} = 0 \implies$  each  $y_j^2 = 0, \forall j = 1, \dots, q$

(iii)  $\sup\|af(x)\| = |a|\sup\|f(x)\| = |a|\|f\|_{qp}$ ,  $a \in \mathbb{R}$ ? We have  $\|af(x)\| = \|ay\| = \sqrt{a^2y_1^2 + \dots + a^2y_q^2} = \sqrt{a^2}\|y\| = |a|\|y\|$ , and  $|a| > 0 \implies \sup\{\|af(x)\|\} = \sup\{|a|\|f(x)\|\} = |a|\sup\{\|f(x)\|\}$ .

(iv)  $\sup\{\|f(x+x')\|\} \leq \sup\|f(x)\| + \sup\|f(x')\|$ ,  $x, x' \in \mathbb{R}^p$ ? Since  $f$  is linear  $\|f(x+x')\| = \|f(x) + f(x')\| \leq \|f(x)\| + \|f(x')\|$ ,  $\forall x, x' \in \mathbb{R}^p$  by Triangle Inequality, then  $\sup\{\|f(x) + f(x')\|\} \leq \sup\{\|f(x)\|\} + \sup\{\|f(x')\|\}$ . This implies  $\|f\|_{qp}$  is a norm.

To show  $\|f(x)\| \leq \|f\|_{pq}\|x\|$ , we use the earlier notation for a linear map,  $f(x) = Ax$ , where,  $A_{q \times p} = (c_{ij})$ . Thus  $\|f(x)\| = \|Ax\| \leq \|A\|\|x\|$  as shown in (21.5). This implies  $\sup\{\|f(x)\| : x \in \mathbb{R}^p, \|x\| \leq 1\} = \sup\{\|Ax\|\} \leq \sup\{\|A\|\|x\|\}$  which is achieved when  $x$  is the max value in its domain, i.e.  $\|x\| = 1$ . This implies  $\sup\{\|Ax\|\|x\|\} = \sup\{\|f(x)\|\|x\|\} = \sup\{\|f(x)\|\} \cdot 1$ . This implies  $\|f(x)\| \leq \sup\{\|f(x)\| : x \in \mathbb{R}^p, \|x\| \leq 1\}\|x\| \forall x \in \mathbb{R}^p$ .

## Section 22

*B. Let  $H : \mathbb{R} \rightarrow \mathbb{R}$  be defined by,  $h(x) = 1, 0 \leq x \leq 1$ .  $h(x) = 0$ , otherwise. Exhibit an open set  $G$  such that  $h^{-1}(G)$  is not open in  $\mathbb{R}$ , and a closed set  $F$ , such that  $h^{-1}(F)$  is not closed in  $\mathbb{R}$ .*

If we take  $G = (0, 2)$ , and open set,  $h^{-1}(G) = \{x \in \mathcal{D}(f) : h(x) \in G\} = [0, 1]$ , a closed set. If we take  $F = [-2, 0]$ , a closed set, the inverse image,  $h^{-1}(F) = \{x \in \mathcal{D}(f) : h(x) \in F\}$  is the union of two open sets  $(-\infty, 0) \cup (1, +\infty)$  which is open.

*C. If  $f$  is bounded and continuous on  $\mathbb{R}^p \rightarrow \mathbb{R}$  and if  $f(x_0) > 0$ , show that  $f$  is strictly positive on some neighborhood of  $x_0$ . Does the same conclusion hold if  $f$  is merely continuous at  $x_0$ ?*

$f$  is bounded and continuous which implies  $0 < f(x_0) < M$ , for some  $M > 0$ . Since  $f$  is continuous, for each point  $a \in \mathcal{D}(f)$ , there is a neighborhood  $V$  of  $f(a)$  and a neighborhood  $U(a) \cap D$  such that if  $f(a) \in V \implies a \in U(a)$ . Since  $f(a) > 0 \implies$  we can take a neighborhood  $V$  of  $f(a)$  that is also strictly positive, i.e.  $V = \{y \in \mathbb{R} : 0 < y < M\}$ . If  $f$  is not bounded the same argument can be made with  $V = \{y \in \mathbb{R} : y > 0\}$ .

*F. A subset  $D \subseteq \mathbb{R}^p$  is disconnected if and only if there exists a continuous function  $f : D \rightarrow \mathbb{R}$  such that  $f(D) = \{0, 1\}$ .*

$\rightarrow D$  disconnected implies there exists two open sets  $B, C$  such that  $B \cap D$  and  $C \cap D$  are disjoint and  $(B \cap D) \cup (C \cap D) = D$ . We can then construct a function  $f$  on  $D$ ,  $f(x) = 1, x \in (B \cap D)$ ,  $f(x) = 0, x \in (C \cap D)$ .  $\leftarrow$  Let  $f : D \rightarrow \mathbb{R}$  be such that  $f(D) = \{0, 1\} \implies$  the inverse image  $f^{-1}(\{0, 1\}) = \{x \in D \subseteq : f(x) \in \{0, 1\}\}$  could consist of two disjoint open sets such for  $f$  on  $D$ ,  $f(x) = 1, x \in (B \cap D)$ ,  $f(x) = 0, x \in (C \cap D)$ , where  $D = (B \cap D) \cup (C \cap D) \subseteq \mathcal{D}(f) \implies$  there exists a continuous function  $f : D \rightarrow \mathbb{R}$  such that  $f(D) = \{0, 1\}$ .

H. Let  $f, g_1, g_2$  be related by the formulas in the preceding exercise. Show that from the continuity of  $g_1$  and  $g_2$  at  $t = 0$  one cannot prove the continuity of  $f$  at  $(0, 0)$ .

Considering  $g_1, g_2$  which are valid restrictions of the domain of  $f$ , given  $x = (x_1, x_2) \in \mathbb{R}^2$ , we can construct  $f(x) = 0, x_1 \cdot x_2 = 0, f(x) = 1, x_1 \cdot x_2 \neq 1$ . With this  $f$  we have  $\lim_{x \rightarrow (0,0)} f(x) \neq 0$ , and  $f((0,0)) = 0 \implies$  discontinuity for  $f$  at  $(0,0)$ . Therefore continuity for  $g_1, g_2$  on restrictions of  $\mathcal{D}(f)$  does not imply continuity of  $f$ .

K. Give an example of a bounded and continuous function  $g$  on  $\mathbb{R} \rightarrow \mathbb{R}$  which does not take on either of the numbers  $\sup\{g(x) : x \in \mathbb{R}\}$  or  $\inf\{g(x) : x \in \mathbb{R}\}$

If we take  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x, x \in (0, 1) \subseteq \mathbb{R}$ , the function is bounded above by 1, below by 0, and continuous on  $(0, 1)$ , but  $f(x) \neq 1 = \sup\{f(x) : x \in (0, 1)\}$ , and  $f(x) \neq 0 = \inf\{f(x) : x \in (0, 1)\}$  for any  $x$  in interval  $(0, 1)$ .

O. Let  $f$  be a continuous function on  $\mathbb{R} \rightarrow \mathbb{R}$  which is strictly increasing (in the sense that if  $x' < x''$  then  $f(x') < f(x'')$ ). Prove that  $f$  is injective and that its inverse function is continuous and strictly increasing.

For points  $x, a, b \in \mathcal{D}(f)$ , by  $f$  be strictly increasing, we have  $a > b \implies f(a) > f(b), a = b \implies f(a) = f(b)$  and  $a < b \implies f(a) < f(b)$ . If we take point  $x$  to be  $a < x < b$ , we can define two neighborhoods  $(a, b) \subseteq \mathcal{D}(f)$ , and  $(f(a), f(b)) \subseteq \mathcal{R}(f)$ , such that  $x \in (a, b)$ , and  $f(x) \in (f(a), f(b))$ . This implies  $f^{-1}$  is continuous, and since  $f^{-1}(f(a)) = a > f^{-1}(f(b)) = b$  if  $f(a) > f(b)$ , implies  $f^{-1}$  is strictly increasing. Also since,  $f(a) = f(b) \implies a = b$ ,  $f$  is injective.

## Section 23

A. Examine each of the functions in Example 20.5 and either show that the function is uniformly continuous on its domain or not.

(a) The constant function,  $\mathcal{D}(f) \subseteq \mathbb{R}, f(x) = c, \forall x \in \mathcal{D}(f)$ , where  $c$  is a real number.

Let  $\varepsilon > 0$ , we have  $|f(x) - f(y)| = |0 - 0| = 0 < \varepsilon, \forall x, y \in \mathcal{D}(f)$ . Regardless of the choice of  $\delta(\varepsilon)$ , we have  $|f(x) - f(y)| < \varepsilon$  whenever  $|x - y| < \delta(\varepsilon) \implies$  uniform continuity.

(b) The identity function  $f(x) = x, x \in \mathbb{R}$ .

For all  $x, y \in \mathbb{R}$ , we have  $|f(x) - f(y)| < \varepsilon, \varepsilon > 0$ . Choose  $\delta(\varepsilon) = \varepsilon$ . Then whenever  $|f(x) - f(y)| = |x - y| < \varepsilon = \delta(\varepsilon)$  we have  $|x - y| < \delta(\varepsilon) \implies$  uniform continuity.

(c)  $f(x) = x^2, x \in \mathbb{R}$

(d)  $f(x) = \frac{1}{x}, x \in \{x \in \mathbb{R} : x \neq 0\}$

(e)  $f(x) = 0, x \geq 0, f(x) = 1, x > 1$

(f)  $f(x) = 1, x$  rational,  $f(x) = 0, x$  irrational

(g)  $\mathcal{D}(f) = \{x \in \mathbb{R} : x > 0\}, f(x) = 0, x$  irrational,  $x > 0$ . For rational numbers of the form  $\frac{m}{n}$ , with  $m, n \in \mathbb{N}$  that have no common factor but 1,  $f(\frac{m}{n}) = \frac{1}{n}$ .

(h)  $\mathcal{D}(f) = \mathbb{R}^2, f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, f(x, y) = (2x + y, x - 3y)$ .

(i)  $\mathcal{D}(f) = \mathbb{R}^2, f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, f(x, y) = (x^2 + y^2, 2xy)$ .

C. If  $B$  is bounded in  $\mathbb{R}^p$  and  $f : B \rightarrow \mathbb{R}^p$  is uniformly continuous, show that  $f$  is bounded on  $B$ . Show that this conclusion fails if  $B$  is not bounded in  $\mathbb{R}^p$ .

If we take two sequences in  $B, x_n, y_n, n \in \mathbb{N}$  by uniform continuity of  $f$ , whenever  $\|x_n - y_n\| \leq \frac{1}{n}$  we have  $\|f(x_n) - f(y_n)\| < \varepsilon$  for some  $\varepsilon > 0$ . If we consider the point  $x_0 \in B$ , for which  $f(x_0) = M = \sup\{\|f(x)\| : x \in B\}$ . By Bolzano-Weierstrass, we can find a subsequence of  $(x_n), (x_{n_1}, \dots, x_{n_k})$  that converges to  $x_0 \implies$  whenever  $\|x_0 - y_n\| \leq \frac{1}{n}, n \in \mathbb{N}$ , we have  $\|f(x_0) - f(y_n)\| \leq \varepsilon \implies f$  is bounded on  $B$ .

D. Show that  $f(x) = \frac{1}{1+x^2}$  for  $x \in \mathbb{R}$  is uniformly continuous.

Take  $\varepsilon > 0$ , for  $x, y \in \mathbb{R}$ , we have  $|f(x) - f(y)| = \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| = \left| \frac{(1+y^2)}{(1+x^2)(1+y^2)} - \frac{(1+x^2)}{(1+y^2)(1+x^2)} \right| =$

$|\frac{y^2-x^2}{(1+y^2)(1+x^2)}| = |x+y||x-y| |\frac{1}{(1+x^2)(1+y^2)}| \leq (|\frac{x}{(1+x^2)(1+y^2)}| + |\frac{y}{(1+x^2)(1+y^2)}|)|x-y| \leq (|\frac{y}{(1+y^2)}| + |\frac{x}{(1+x^2)}|)|x-y|$ .  
 Since  $\forall x \in \mathbb{R}$ , we have  $|\frac{x}{1+x^2}| < 1$ , we have  $(|\frac{y}{(1+y^2)}| + |\frac{x}{(1+x^2)}|)|x-y| < 2|x-y| = 2\delta(\varepsilon) \implies$  if we choose  $\delta(\varepsilon) = \frac{\varepsilon}{2}$  whenever  $|x-y| < \frac{\varepsilon}{2}$  we have  $|f(x) - f(y)| < \varepsilon$ , for all  $x, y \in \mathbb{R}$

F. Show that  $f(x) = \frac{1}{x^2}$ ,  $\mathcal{D}(f) = \{x \in \mathbb{R} : x > 0\}$  is not uniformly continuous on its domain.

If we take  $\delta(\varepsilon) = \varepsilon/2$ , and  $\varepsilon = 1$ , and consider points in a subset of  $\mathcal{D}(f)$ , namely  $x, y \in (0, 1)$ , and then take  $y = \frac{x}{2} \in (0, 1)$ , we have we have  $|x-y| = |x/2| < \varepsilon/2 = 1/2 \implies |f(x) - f(y)| = |\frac{1}{x^2} - \frac{4}{x^2}| = \frac{3}{x^2} < 1$ . But, for all  $x, y \in (0, 1)$ ,  $\frac{3}{x^2} > 1 \implies f(x)$  is not uniformly continuous on its domain.

G. A function  $g : \mathbb{R} \rightarrow \mathbb{R}^p$  is periodic if there exists a number  $p > 0$  such that  $g(x+p) = g(x)$  for all  $x \in \mathbb{R}$ . Show that a continuous periodic function is bounded and uniformly continuous on  $\mathbb{R}$ .

H. Let  $f$  be defined on  $D \subseteq \mathbb{R}^p$  to  $\mathbb{R}^q$ , and suppose that  $f$  is uniformly continuous on  $D$ . If  $(x_n)$  is a Cauchy sequence in  $D$ , show that  $(f(x_n))$  is a Cauchy sequence in  $\mathbb{R}^q$ .

A sequence is Cauchy

## Section 24

B. Give an example of a sequence of continuous functions which converges to a continuous function but where the convergence is not uniform.

D. Let  $(f_n)$  be a sequence of continuous functions on  $D \subseteq \mathbb{R}^p$  to  $\mathbb{R}^q$  such that  $(f_n)$  converges uniformly to  $f$  on  $D$ , and let  $(x_n)$  be a sequence of elements in  $D$  which converges to  $x \in D$ . Does it follow that  $(f_n(x_n))$  converges to  $f(x)$ ?

E. Consider the sequences  $(f_n)$  defined on  $D = \{x \in \mathbb{R} : x \geq 0\}$  to  $\mathbb{R}$  by the following formulas:

(b)  $\frac{x^n}{1+x^n}$ ,

(c)  $\frac{x^n}{n+x^n}$ ,

(d)  $\frac{x^{2n}}{1+x^n}$ ,

(e)  $\frac{x^n}{1+x^{2n}}$

J. Prove the following theorem of G. Polya. If for each  $n \in \mathbb{N}$  the function  $f_n$  on  $I \rightarrow \mathbb{R}$  is monotone increasing and if  $f(x) = \lim(f_n(x))$  is continuous on  $I$ , then the convergence is uniform on  $I$ . (Observe that it is not assumed that  $f_n$  is continuous.)

N. If  $f_3(x) = x^3$  for  $x \in \mathcal{I}$ , calculate the  $n^{\text{th}}$  Bernstein polynomial for  $f_3$ . Show directly that this sequence of polynomials converges uniformly to  $f_3$  on  $\mathcal{I}$ .

S. Show that the Weierstrass Approximation Theorem fails for bounded open intervals.

## Section 26

N. If  $K \subseteq \mathbb{R}^p$  is compact and  $(f_n)$  is a sequence of continuous functions on  $K$  to  $\mathbb{R}^q$  which is uniformly convergent on  $K$ , show that the family  $\{f_n\}$  is uniformly equicontinuous on  $K$  in the sense of Definition 26.6.

O. Let  $\mathcal{F}$  be a bounded and uniformly equicontinuous collection of functions on  $D \subseteq \mathbb{R}^p$  to  $\mathbb{R}$  and let  $f$  be defined on  $D \rightarrow \mathbb{R}$  by  $f = \sup\{f(x) : f \in \mathcal{F}\}$ . Show that  $f$  is continuous on  $D \rightarrow \mathbb{R}$ .

Q. Consider the following sequences of functions which show that the Arzela-Ascoli Theorem 26.7 may fail if the various hypotheses are dropped.

(a)  $f_n(x) = x + n$ ,  $x \in [0, 1]$ ;

(b)  $f_n(x) = x^n$ ,  $x \in [0, 1]$ ;

(c)  $f_n(x) = \frac{1}{1+(x-n)^2}$ ,  $x \in [0, +\infty)$ .