Math 4317 (Prof. Swiech, S'18): HW #2

Peter Williams

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Section 8

D. If w_1 and w_2 are strictly positive, show that the definition, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1w_1 + x_2y_2w_2$, yields an inner product on $\mathbb{R}^2$, generalize this for $\mathbb{R}^p$.

Checking the properties of inner products, we have, based on definition above, (i) $x \cdot x \geq 0$, since $(x_1, x_2)(x_1, x_2) = w_1x_1^2 + w_2x_2^2 \geq 0$, since $w_1, w_2 > 0$, and $x_i^2 > 0$, $i = 1, 2$. For $x \in \mathbb{R}^p$, we have $x \cdot x = \sum_{j=1}^{p} w_jx_j^2 \geq 0$, since each element in the summation $w_i, x_i^2 > 0$. For property (ii), we have $x \cdot x = 0$, if and only if $x = 0$. In this case, since $w_1, w_2 > 0$, $w_1x_2^2 + w_2x_2^2 = 0$, when x_1^2 and x_2^2 equal zero, that is when $x = 0$. This holds for $x \in \mathbb{R}^p$, since for $w_i > 0$, $\ i = 1, ..., p$ we have $\sum_{j=1}^{p} w_jx_j^2 = 0$, only when each element $w_ix_i2 = 0$, since each element is greater than or equal to zero. For property (iii), we show $x \cdot y = y \cdot x$ since $x \cdot y = w_1x_1y_1 + w_2x_2y_2 = w_1x_1y_1 + w_2x_2y_2 = w_1y_1x_2 + w_2y_2x_2 = y \cdot x$. Extending to $x \in \mathbb{R}^p$, we have again, by commutative property, $x \cdot y = \sum_{j=1}^{p} w_jx_jy_j = \sum_{j=1}^{p} w_jy_jx_j = y \cdot x$. Property (iv), $x \cdot (y + z) = x \cdot y + x \cdot z$, $x, y, z \in \mathbb{R}^p$. In this case we have $\sum_{j=1}^{p} w_jx_j(y_j + z_j) = \sum_{j=1}^{p} w_jx_jy_j + w_jx_jz_j = \sum_{j=1}^{p} w_jx_jy_j + \sum_{j=1}^{p} w_jx_jz_j = x \cdot y + x \cdot z$, which clearly holds for base case, $p = 2$ as well. For property (v), we have $(ax) \cdot y = x \cdot (ay)$, $a \in \mathbb{R}$. We have $(ax) \cdot y = \sum_{j=1}^{p} w_jax_jy_j = a\sum_{j=1}^{p} w_jx_jy_j = a(x \cdot y) = \sum_{j=1}^{p} w_jx_jay_j = x \cdot (ay)$. Since all five properties are satisfied, an inner product is yielded here.

E. $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1$ is not an inner product on $\mathbb{R}^2$. Why?

By property (ii), i.e. $x \cdot x = 0$ if and only if $x = 0$, the definition above, $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1 = 0 \Leftrightarrow x = 0$, however, we can't say $x = 0$, since in this case if $x_1y_1 = 0 \implies x_1 = 0$, but we don't have information about x_2, or $x_i, i = 3, ..., p$, for $x \in \mathbb{R}^p$. Thus for this operation $x \cdot x = 0$ does not necessarily mean $x = 0$.

F. If $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, define $||x||_1$ by $||x||_1 = |x_1| + |x_2| + ... + |x_p|$. Prove that $x \to ||x||_1$ is a norm on $\mathbb{R}^p$.

(i) $||x||_1 \geq 0$?. Since $|x_j| \geq 0 \ \forall j \implies ||x|| = \sum_{j=1}^{p} |x_j| \geq 0$ by definition of the absolute value.

(ii) $||x||_1 = 0$ if and only if $x = 0$?. $||x|| = \sum_{j=1}^{p} |x_j| = 0 \implies x_j = 0 \ \forall j \implies x = 0$.

(iii) $||ax||_1 = |a|||x|| \ \forall a \in \mathbb{R}, \ x \in V$? When $a \geq 0$, and $x_j \geq 0$ or $a < 0$ and $x_j < 0$, $||ax_j|| = ax_j = |a||x_j|$. For the case $a < 0$ and $x_j \geq 0$ or $a \geq 0$ and $x_j < 0$, we have $||(a_x)j|| = |ax_j| = (-1)ax_j$ or $a(-1)x_j = a|x_j| = |a||x_j|$.

(iv) $||x + y||_1 \leq ||x|| + ||y||$ for $x, y \in \mathbb{R}^p$?. $||x + y|| = |x_1 + y_1| + |x_2 + y_2| + ... + |x_p + y_p|$. By the triangle inequality, $|x_j + y_j| \leq |x_j| + |y_j|$ for all j. Therefore $|x_1 + y_1| + |x_2 + y_2| + ... + |x_p + y_p| \leq |x_1| + |x_2| + ... + |x_p| + |y_1| + |y_2| + ... + |y_p| = ||x|| + ||y||$. Thus $||x||_1$ is a norm on $\mathbb{R}^p$.

G.If $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, define $||x||_\infty$ by $||x||_\infty = sup\{|x_1| + |x_2| + ... + |x_p|\}$. Prove that $x \to ||x||_\infty$ is a norm on $\mathbb{R}^p$.

(i) $||x||_\infty \geq 0$? Since $|x_j| \geq 0 \ \forall j \implies ||x||_\infty = \sup\{|x_1| + |x_2| + ... + |x_p|\} \geq 0$ since each element in the set is greater than zero.

(ii) $||x||_\infty = 0$ if and only if $x = 0$?. Since each element in the set $\{|x_1| + |x_2| + ... + |x_p|\}$ is greater than or equal to zero, $||x||_\infty = 0$ if and only if $x_j = 0$ for all j, which implies $x = 0$.

(iii) $||ax||_\infty = |a|||x||_\infty \ \forall a \in \mathbb{R}, \ x \in V$? $||ax||_\infty = \sup\{|ax_1| + |ax_2| + ... + |ax_p|\}$, and as shown in 8.F $|ax_j| = |a||x_j|$, which implies $||ax||_\infty = \sup\{|a||x_1| + |a||x_2| + ... + |a||x_p|\} = |a|\sup\{|x_1| + |x_2| + ... + |x_p|\} = |a|||x||_\infty$, since $|a|, |x_j| > 0$. (iv)$||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$ for $x, y \in \mathbb{R}^p$?. Again, by the triangle inequality, $|x_j + y_j| \leq |x_j| + |y_j|$ for all j. Therefore $\sup\{|x_1 + y_1|, |x_2 + y_2|, ..., |x_p + y_p|\} \leq \sup\{|x_1| + |y_1|, |x_2| + |y_2|, ..., |x_p| + |y_p|\}$. If we take $u_x = \sup\{|x_j|\}$,$u_y = \sup\{|y_j|\}$. $u_x + u_y \geq |x_j| + |y_j|$ for all $j \implies \sup\{|x_j|\} + \sup\{|y_j|\} = \sup\{|x_j| + |y_j|\} \implies ||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$. Thus, $||x||_\infty$ is a norm on $\mathbb{R}^p$.

H. In the set $\mathbb{R}^2$, describe the sets:

$S_1 = \{x \in \mathbb{R}^2 : ||x||_1 < 1\}$. $||x||_1 = \sqrt{x_1^2 + x_2^2} < 1$ describes and open circle consisting of points less than 1 in all directions from the origin, satisfying the inequality, $\sqrt{x_1^2} < \sqrt{1 - x_2^2}$. $S_\infty = \{x \in \mathbb{R}^2 : ||x||_\infty < 1\}$, where $||x||_\infty = \sup\{|x_1|, |x_2|\}$, is a dense open box with vertices vertices at $(1,1), (-1,1), (-1,-1), (1,-1)$ with $-1 < x_1 < 1$, and $-1 < x_2 < 1$.

P. If x, y belongs to $\mathbb{R}^p$, show that $||x+y||^2 = ||x||^2 + ||y||^2$ if and only if $x \cdot y = 0$.

$||x+y||^2 = (x+y) \cdot (x+y) = x \cdot x + y \cdot x + x \cdot +y + y \cdot y = ||x||^2 + 2x \cdot y + ||y||^2$, and $2x \cdot y = 0$ if and only if $x \cdot y = 0$, thus, in order for $||x+y||^2 = ||x||^2 + ||y||^2$ to hold, $x \cdot y$ must equal zero.

Q. A subset K of $\mathbb{R}^p$ is said to be convex if, whenever, $x, y \in K$, and t is a real number such that $0 \le t \le 1$, then the point $tx + (1-t)y$ also belongs to K. Show that K_1, K_2, K_3 are convex, but that K_4 is not.

1) $K_1 = \{x \in \mathbb{R}^2 : ||x|| < 1\}$. Let $x, y \in K_1$, then $||tx + (1-t)y|| \le ||tx|| + ||(1-t)y|| = |t|||x|| + |(1-t)|||y||$, and since $||x|| \le 1$ and $||y|| \le 1$, it implies $|t|||x|| + |(1-t)|||y|| \le |t|(1) + |(1-t)|(1) = t + 1 - t = 1 \implies tx + (1-t)y \in K_1$.
2) For $K_2 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < \eta\}$. Let $x = (x_1, x_2), y = (y_1, y_2) \in K_2 \implies 0 < x_1 < x_2$ and $0 < y_2 < y_2$, for the point $tx + (1-t)y$ to belong in K_2 it implies for $t \in [0,1] \implies 0 < tx_1 < tx_2$, and $0 < (1-t)y_1 < (1-t)y2$. Adding these inequalities, we have for $tx + (1-t)y$, $0 < tx_1 + (1-t)y_1 < tx_2 + (1-t)y_2 \implies tx + (1-t)y \in K_2$.
3) Similarly for $K_3 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 \le \xi \le \eta \le 1\}$, $x, y \in K_3$, $t \in [0,1]$, we have $0 \le x_1 \le x_2 \le 1$ and $0 \le y_1 \le y_2 \le 1 \implies 0 \le tx_1 \le tx_2 \le t$ and $0 \le (1-t)y_1 \le (1-t)y_2 \le (1-t)$, again adding the inequalities, we have $0 \le tx_1 + (1-t)y_1 \le tx_2 + (1-t)y_2 \le t + (1-t) = 1 \implies tx + (1-t)y \in K_3$.
4) For $K_4 = \{x \in \mathbb{R}^2 : ||x|| = 1\}$. Like in K_1, $x, y \in K_4$, then $||tx + (1-t)y|| = ||tx|| + ||(1-t)y|| = |t|||x|| + |(1-t)|||y||$, and since $||x|| \le 1$ and $||y|| \le 1$, it implies $|t|||x|| + |(1-t)|||y|| \le |t|(1) + |(1-t)|(1) = 1$. This equality could hold in some cases where $||x|| = 1$, e.g. $(1,0), (0,1)$, but does not hold for all points, and thus K_4 is not convex.

Section 9

Section 10

Section 11

Section 12