midterm1WIP

Exercise 3.

Consider the linear regression model from exercise 1. Suppose, that the target of estimation is $h^\intercal \theta$ for some determinate non-zero vector $h \in R^p$. Find expression for the LSE of $h^\intercal \theta$. Is this estimate optimal in sense of Gauss-Markov theorem, i.e. does it have the smallest variance among all linear unbiased estimators?

Using our findings from exercise 2, we have an unbiased LSE estimator in $\gamma^\intercal \hat{\theta}$ since $E[\gamma^\intercal \hat{\theta} - \gamma^\intercal \theta] = \gamma^\intercal E[(XWX^\intercal)^{-1} XWY] - \gamma^\intercal \theta = \gamma^\intercal \theta - \gamma^\intercal \theta = 0$. Using another finding from exercise 2 we have, $Var(\gamma^\intercal \hat{\theta}) = \gamma^\intercal Var(\hat{\theta})\gamma = \gamma^\intercal (XWX^\intercal)^{-1}\gamma$.

To show that $\gamma^\intercal \hat{\theta}$ is BLUE, we compare its variance, to another estimator $\tilde{\theta} = ((XWX^\intercal)^{-1} XW + D)Y$, where D is another $p \times n$ matrix. The variance of $\gamma^\intercal \tilde{\theta}$ is then:

Section 2.1

Exercise 7. (a) Using the notation from section 2.1, consider $X \sim N(\mu, I_n)$ for some $\mu \in R^n$. Find $E(Q(X))$ and $Var(Q(X))$

For $Q(X) = \sum_i \sum_j a_{ij} X_i X_j = X^\intercal A X, X \sim N(\mu, I_n)$, we have, using the property of trace operator:

$$E(Q(X)) = tr(E(Q(X)) = E(tr(Q(X)) = E(tr(X^\intercal AX)) = E(tr(AXX^\intercal)) = tr(AE(XX^\intercal))$$

Since $E(XX^\intercal) = I_n + \mu\mu^\intercal$, we have,

$$tr(AE(XX^\intercal)) = tr(A(I_n + \mu\mu^\intercal)) = trA + tr(A\mu\mu^\intercal) = trA + \mu^\intercal A\mu$$

$Var(Q(X)) =$

(b) Generalize the results from part (a) to the case $X \sim N(\mu, \Sigma)$ for some positive-definite covariance matrix $\Sigma \in R^{n\times n}$. For $X \sim N(\mu, \Sigma)$ we have,

$$E(Q(X)) = tr(AE(XX^\intercal)) = tr(A(\Sigma + \mu\mu^\intercal)) = tr(A\Sigma) + tr(A\mu\mu^\intercal) = tr(A\Sigma) + \mu^\intercal A\mu$$

$Var(Q(X)) =$

Section 2.2

Exercise 11.

Find an elliptical confidence set for the expected response $E[Y]$ in model (3).

For the model $Y = X^\intercal \theta^* + \varepsilon$, $\varepsilon \sim N(0, \sigma^2 I_n)$, $\hat{Y} = X^\intercal \hat{\theta} = X^\intercal (XX^\intercal)^{-1} XY = \Pi Y$, we have

$$E(\hat{Y} - Y) = E(\hat{Y}) - Y = E[X^\intercal (XX^\intercal)^{-1} X(X^\intercal \theta^* + \varepsilon)] - Y = E[X^\intercal \theta^*] - Y = Y - Y = 0$$

and

$$Var(\hat{Y} - Y) = Var((\Pi - I_n)Y) = (\Pi - I_n)Var(X^\intercal \theta^* + \varepsilon)(\Pi - I_n)^\intercal = (\Pi - I_n)\sigma^2 I_n(\Pi - I_n)^\intercal = \sigma^2(I_n - \Pi)$$

and assume that $\hat{Y} - Y \sim N(0, \sigma^2(\Pi - I_n))$, and $\frac{(I_n - \Pi)^{-1/2}(\hat{Y} - Y)}{\sigma} \sim N(0, I_n)$ Using this information we can set up a confidence region for $\hat{Y}$,

Exercise 12. Construct simultaneous confidence intervals (e.g., as in Corollary 2.2.1) for the expected responses $E[Y_1], ..., E[Y_n]$ in model (3).