Math 4317 (Prof. Swiech, S'18): HW #3

Peter Williams

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Section 14

A. Let $b \in \mathbb{R}$, show $\lim \frac{b}{n} = 0$.

Take $\varepsilon > 0$, if $|\frac{b}{n} - 0| < \varepsilon$, there exists natural number $K(\varepsilon)$ such that $\frac{b}{n} < \frac{b}{K(\varepsilon)} < \varepsilon$. If $n \geq K(\varepsilon)$, and we choose $K(\varepsilon)$ such that $K(\varepsilon) > \frac{b}{\varepsilon} \implies \frac{b}{n} < \varepsilon \implies \lim \frac{b}{n} = 0$.

B. Show that $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

Take $\varepsilon > 0$, note that for $n \in \mathbb{N}$, $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} < \frac{1}{n}$. So we choose natural number $K(\varepsilon)$ such that $\frac{1}{K(\varepsilon)} < \varepsilon$. Therefore if $n \geq K(\varepsilon) \implies \frac{1}{n} < \varepsilon$. Therefore $|\frac{1}{n} - \frac{1}{n+1} - 0| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \varepsilon \implies \lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

D. Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ which is convergent to x. Show that $\lim ||x_n|| = ||x||$. (Hint: use the Triangle Inequality.)

(x_n) convergent with limit $x \implies$ there exists natural number $K(\varepsilon)$ such that for $n \geq K(\varepsilon)$, $||x_n - x|| < \varepsilon$. If $n \geq K(\varepsilon)$. Since by triangle inequality, $|||x_n|| - ||x||| \leq ||x_n - x|| < \varepsilon \implies \lim ||x_n|| = ||x||$.

G. Let $d \in \mathbb{R}$ satisfy $d > 1$. Use Bernoulli's inequality to show that the sequence (d_n) is not bounded in $\mathbb{R}$. Hence it is not convergent.$

We have the sequence $D = (d_n)$, where $d_n = d^n$. Let $d = 1 + a$ for some $a > 0 \implies d^n = (1+a)^n > 1 + na$ by Bernoulli's inequality. For any $a > b > 0$, $(1+a)^n > (1+b)^n$ which implies the sequence d_n is increasing. Take $M > 0$, we have $d^n > 1 + na > M > 0$, if $n > \frac{M}{a} \implies 1 + na > M$. Thus (d_n) is not bounded.

H. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$; show that $\lim(nb^n) = 0$. (Hint: use the Binomial theorem as in Example 14.8(e).)

Let $b = \frac{1}{1+a}, a > 0$, we have $b^n = \frac{1}{(1+a)^n}$. By Binomial theorem, $(1+a)^n > \frac{n(n-1)}{2}a^2 \implies \frac{1}{(1+a)^n} < \frac{2}{n(n-1)a^2}$, therefore $nb^n = \frac{n}{(1+a)^n} < \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2}$. Take $\varepsilon > 0$, natural number $K(\varepsilon)$, if $n \geq K(\varepsilon)$ we have $nb^n = \frac{n}{(1+a)^n} < \frac{2}{(n-1)a^2} < \frac{2}{(K(\varepsilon)-1)a^2} < \varepsilon$. Then $|nb^n - 0| < \varepsilon \implies nb^n < \varepsilon \implies \lim nb^n = 0$.

I. Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) < 1$. Show that for some r with $0 < r < 1$ and some $C > 0$, then we have $0 < x_n < Cr^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $lim(x_n) = 0$

Since $L = \lim(\frac{x_{n+1}}{x_n}) < 1,\ 0 < r < 1 \implies |\frac{x_{n+1}}{x_n} - L| < r$ or $0 < \frac{x_{n+1}}{x_n} < r$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Since $\frac{x_{n+1}}{x_n} < r < 1 \implies x_{n+1} < rx_n < x_n \implies x_n < \frac{x_n}{r}$. If we set $C = \frac{x_n}{r^{n+1}} > 0$, we have $x_n < Cr^n$. Since $\lim_{n\to\infty} r^n = 0 \implies lim(x_n) = 0$.

J. Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) > 1$. Show that X is not a bounded sequence and hence is not convergent.

Take $\varepsilon > 0$, since $L = \lim(\frac{x_{n+1}}{x_n}) > 1 \implies |\frac{x_{n+1}}{x_n} - L| = |L - \frac{x_{n+1}}{x_n}| < \varepsilon \implies L - \varepsilon < \frac{x_{n+1}}{x_n}$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Take $L - \varepsilon = r > 1$ when ε is small. This implies $x_{n+1} > rx_n$. Take $C = \frac{x_n}{r^{n-1}} > 0 \implies x_{n+1} > Cr^n$. Since $r > 1$, r^n diverges which implies the sequence x_{n+1} is not bounded or convergent.

K. Give and example of a convergent sequence (x_n) of strictly positive real numbers such that $\lim(\frac{x_n+1}{x_n}) = 1$. Give an example of a divergent sequence with this property.

Consider convergent sequence $X = (x_n) = (\frac{1}{n})$. $\lim(\frac{x_n+1}{x_n}) = 1 \implies |\frac{\frac{1}{n+1}}{\frac{1}{n}} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1} < \varepsilon,\ \varepsilon > 0$.

If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n+1} < \frac{1}{K(\varepsilon)+1} < \varepsilon$, indicating $(\frac{x_n+1}{x_n})$ is a convergent sequence with limit 1.

Consider divergent sequence $X = (x_n) = n$. $\lim(\frac{x_n+1}{x_n}) = 1 \implies |\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} < \varepsilon,\ \varepsilon > 0$. If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n} < \frac{1}{K(\varepsilon)} < \varepsilon$, indicating $(\frac{x_n+1}{x_n})$ is a convergent sequence with limit 1.

L. Apply the results of Exercises 14.I and 14.J to the following sequences. (Here $0 < a < 1, 1 < b, c > 0$)

(a) (a^n)
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} = a < 1 \implies a^n$ is convergent, bounded.

(b) (na^n)
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{(n+1)a^{n+1}}{na^n} = (\frac{n+1}{n})a$ which tends to $1 \cdot a < 1 \implies na^n$ is convergent, bounded.

(c) (b^n)
$\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{b^n} = b > 1 \implies b^n$ is divergent, not bounded.

(d) $(\frac{b^n}{n})$
In this case $\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{b^{n+1}}{n+1}}{\frac{b^n}{n}} = (\frac{n}{n+1})b$ which tends to $1 \cdot b > 1 \implies \frac{b^n}{n}$ diverges, not bounded.

(e) $(\frac{c^n}{n!})$
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}} = \frac{c}{n+1}$ which tends to $0 < 1$ implying $(\frac{c^n}{n!})$ converges, bounded.

(f) $(\frac{2^{3n}}{3^{2n}})$
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{2^{3(n+1)}}{3^{2(n+1)}}}{\frac{2^{3n}}{3^{2n}}} = \frac{2^3}{1}\frac{1}{3^2} = \frac{8}{9} < 1$ implying $(\frac{2^{3n}}{3^{2n}})$ converges, bounded.

Section 15

C(a-e). For x_n given by the following formulas, either establish the convergence of the divergence of the sequence $X = (x_n)$:

(a) $x_n = \frac{n}{n+1}$
$x_n = \frac{n}{n+1} = \frac{1/n}{1/n}\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$. The limit of the sequence $Y = (y_n) = (1 + \frac{1}{n})$ clearly has limit 1 $\implies \lim(x_n) = \lim \frac{1}{1+\frac{1}{n}} = \frac{\lim 1}{\lim(1+1/n)} = 1 \implies$ this sequence converges to 1.

(b) $x_n = \frac{(-1)^n n}{n+1}$ Let $X = (x_n) = (-1)^n, Y = (y_n) = \frac{n}{n+1}$. Using theorem 15.6.a, if X converges to x, and Y converges to y. $X \cdot Y$ converges to $x \cdot y$. In our case the series $(x_n) = (-1)^n$ diverges, and $(y_n) = \frac{n}{n+1}$ converges to 1 $\implies \lim X \cdot Y = \lim X \cdot 1 = \lim X$ which diverges.

(c) $x_n = \frac{2n}{3n^2+1}$ $x_n = \frac{2n}{3n^2+1} = \frac{1/n}{1/n}\frac{2n}{3n^2+1} = \frac{2}{3n+\frac{1}{n}}$. We estimate the limit to be 0 $\implies$ for $n \geq K(\varepsilon)$, $|\frac{2}{3n+1/n} - 0| = \frac{2}{3n+1/n} < \frac{2}{3K(\varepsilon)+1/K(\varepsilon)} < \varepsilon,\ \varepsilon > 0 \implies (x_n) \to 0$. Converges.

(d) $x_n = \frac{2n^2+3}{3n^2+1}$
$x_n = \frac{2n^2+3}{3n^2+1} = \frac{1/n^2}{1/n^2}\frac{2n^2+3}{3n^2+1} = \frac{2+3/n^2}{3+1/n^2} \to \frac{2}{3}$. Converges.

(e) $x_n = n^2 - n = n(n-1)$
The sequence $(x_n) = n(n-1)$ is clearly divergent, since for all $M > 0,\ n \geq M$, $n(n-1) > M(M-1) > 0$. Diverges.

E. If X and Y are sequences in $\mathbb{R}^p$ and if $X \cdot Y$ converges, do X and Y converge and have $\lim(X \cdot Y) = \lim(X) \cdot \lim(Y)$

As an example, if we take sequences $X = (x_n) = (-1)^n = (-1, 1, -1, ...)$ and $Y = (y_n) = (-1)^{n+1} = (1, -1, 1, ...)$, then their product $X \cdot Y = (-1, -1, -1, ...)$ converges to $-1 \implies$ that the product $X \cdot Y$ converges, but each sequence X and Y does not have a limit, diverges.

As another example, in the case of the constant sequences $X = (x_N) = (1, 1, ...)$, and $Y = (y_n) = (2, 2, ...)$, $X \cdot Y$ is the constant sequence $(2, 2, ...)$ which converges to 2 which equals $\lim X \cdot \lim Y$. Therefore the convergence of $X \cdot Y$ converges does not necessarily mean that each sequence converges, as there are examples of both cases.

F. If $X = (x_n)$ is a positive sequence which converges to x, then $(\sqrt{x_n})$ converges to $\sqrt{x}$. (Hint: $\sqrt{x_n} - \sqrt{x} = \frac{(x_n - x)}{(\sqrt{x_n} + \sqrt{x})}$ when $x \neq 0$).

In the case that $\lim(x_n) = x = 0$ we have $|x_n - x| = |x_n - 0| = x_n < \varepsilon^2,\ \varepsilon^2 > 0,\ n \geq K(\varepsilon)$, for natural number $K(\varepsilon)$. This implies $0 \leq x_n < \varepsilon^2$ for all $n \geq K(\varepsilon) \implies 0 \leq \sqrt{(x_n)} < \varepsilon$, $\varepsilon > 0 \implies \sqrt{x_n} - 0 < \varepsilon \implies |\sqrt{x_n} - \sqrt{x}| < \varepsilon,\ n \geq K(\varepsilon) \implies \sqrt{x}$ is limit of $sqrtx_n$ when $x = 0$.

For case $x > 0$, $x > 0 \implies \sqrt{x} > 0$. Since $|\sqrt{x_n} - \sqrt{x}| = \sqrt{x_n} - \sqrt{x} = \sqrt{x_n} - \sqrt{x} \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$. Since $\sqrt{x} > 0$, also implies $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0 \implies \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \leq \frac{x_n - x}{\sqrt{x}} \implies |\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}}(x_n - x) = \frac{1}{\sqrt{x}}|x_n - x| < \varepsilon$, $\varepsilon > 0$. So if $x_n \to x \implies \sqrt{x_n} \to \sqrt{x}$ for $x > 0$.

L. If $0 < a \leq b$ and if $x_n = (a^n + b^n)^{\frac{1}{n}}$, then $\lim(x_n) = b$.

Since $0 < a \leq b \implies b^n \leq a^n + b^n \leq b^n + b^n = 2b^n \implies (b^n)^{1/n} \leq (a^n + b^n)^{1/n} \leq (2b^n)^{1/n}$, therefore, $b \leq x_n \leq 2^{1/n} b$. Since $2^{1/n} \to 1$ as $n \to \infty \implies b \leq x_n \leq b \implies \lim(x_n) = b$.

N.Let $A \subseteq \mathbb{R}^p$ and $x \in \mathbb{R}^p$. Then x is a boundary point of A if and only if there is a sequence (a_n) of elements in A and a sequence (b_n) of elements in $\mathcal{C}(A)$ such that $\lim(a_n) = x = lim(b_n)$.

$\to$ Let x be a boundary point of $A \implies$ there is a neighborhood $V = \{y \in \mathbb{R}^p : ||y - x|| < r\},\ r > 0$, that includes points in A and complement A^c. Since V is a neighborhood of x, by definition of the limit, there is a natural number K_v such that for all $n \geq K_v,\ a_n \in V$ and $b_n \in V \implies (a_n)$ converges to x and (b_n) converges to $x \implies \lim(a_n) = x = \lim(b_n)$.

$\leftarrow$ Let x be limit of sequences $(a_n),\ (b_n) \implies$ there is a neigborhood $V = \{y \in \mathbb{R}^p : ||y - x|| < r\},\ r > 0$ for natural number K_v, such that $n \geq K_v,\ a_n \in V,\ b_n \in V \implies V$ includes points from $(a_n) \in A$ and $(b_n) \in A^c \implies x$ is a boundary point of A.

Section 16

A. Let $x_1 \in \mathbb{R}$ satisfy $x_1 > 1$ and let $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$. Show that the sequence (x_n) is monotone and bounded. What is its limit?

We have $x_1 > 1$ and $x_2 = 2 - \frac{1}{x_1}$. We then have $x_1 > 2 - \frac{1}{x_1} = x_2 > 1$ since since $1 > \frac{1}{x_1} > 0$. This implies $x_1 > x_2 > x_3 = 2 - \frac{1}{2 - \frac{1}{x_1}} > 1$. Using induction we have $x_1 > x_2 = 2 - \frac{1}{x_1} > 1$, . We then assume $x_{n-1} > x_n > 1$. For case $n+1$ we have $x_n > x_{n+1} > 1$. Since $x_n = 2 - \frac{1}{x_{n-1}} > x_{n+1} = 2 - \frac{1}{x_n} > 1$, and since we assume $x_{n-1} > x_n > 1$ this implies $2 - \frac{1}{x_{n-1}} > 2 - \frac{1}{x_n} > 1,\ n \in \mathbb{N}$. This shows (x_n) is a monotone decreasing sequence bounded below by 1. Knowing that this sequence has a limit x that must satisfy the relation $x = 2 - \frac{1}{x} = x \implies 2 = x + \frac{1}{x}$ which is satisfied when $x = 1 \implies$ the limit of this sequence is 1.

B. Let $y_1 = 1$ and $y_{n+1} = (2 + y_n)^{1/2}$ for $n \in \mathbb{N}$. Show that (y_n) is monotone and bounded. What is its limit?

We have $y_1 = 1,\ y_2 = \sqrt{2+1} = \sqrt{3} < 2 \implies y_1 < y_2 < 2$. Using induction we assume $y_{n-1} < y_n < 2$. For case $n+1$, we have $y_n < y_{n+1} < 2 \leftrightarrow \sqrt{2 + y_{n-1}} < \sqrt{2 + y_n} < 2 \implies 2 + y_{n-1} < 2 + y_n < 4 \implies$ directly $y_{n-1} < y_n < 2$. This shows that (y_n) is a monotone increasing sequence bounded above by 2. If a limit of $\lim(y_n) = y$ exists it must satisfy the relation, $y = \sqrt{2 + y} \implies y^2 = 2 + y$, and we have $y^2 - y - 2 = (y - 2)(y + 1) = 0$, which has roots $2,\ -1$. Since (y_n) is positive increasing, its limit must be 2.

E. Show that every sequence in $\mathbb{R}$ either has a monotone increasing subsequence or a monotone decreasing subseqence.

Take an element of the sequence $X = (x_n)$, x_k, such that $x_k \geq x_n,\ n > k$. This implies for each $k_1 < k_2 < ... < k_j < ...$ we have $x_{k_1} > x_{k_2} > ... > x_{k_j}$ which is a decreasing subsequence of X.

Relying the on the decreasing subsequence $x_{k_1} > x_{k_2} > ... > x_{k_j},\ D = (x_{k_j})$, if we take an index $m_1 > k_j$, such that $x_{m_1} \notin D$, we can construct $x_{m_1} < x_{m_2} < ... < x_{m_i}$ since there exists $m_2 > m_1$ such that $x_{m_1} < x_{m_2}$ for all m, which is an increasing subsequence of X.

G. Determine the convergence or divergence of the sequence (x_n) where, $x_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$ for $n \in \mathbb{N}$.

We have $x_1 = \frac{1}{2},\ x_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2} \implies x_1 < x_2 < 1$. Using induction we assume $x_{n-1} < x_n < 1$. For the case $n+1$, we have $x_n < x_{n+1} < 1 \leftrightarrow \frac{1}{n+1}\frac{1}{n+2} + ... + \frac{1}{2n} < \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n+2} < 1$. Adding $\frac{1}{n+1} > 1$ to each element we have $x_n + \frac{1}{n+1} < x_n + \frac{1}{2n+1} + \frac{1}{2n+2} < 1 + \frac{1}{n+1}$. Since $\frac{1}{2n+2} + \frac{1}{2n+1} > \frac{1}{n+1} \forall n \in \mathbb{N}$, because $\frac{n+1}{2n+2} + \frac{n+1}{2n+1} > 1 \implies x_n < x_{n+1} \implies x_n + (\frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}) < 1 \implies x_n < x_{n+1} < 1 \quad \forall n \in \mathbb{N}$. This implies this sequence converges and is bounded above by 1.

J. Show directly that the following are not Cauchy sequences.

(a) $((-1)^n)$
If we take $\varepsilon = 1 > 0$, for m, n greater than natural number $M(\varepsilon)$, we have $|x_m - x_n| = 2 > \varepsilon$ for case m odd, n even, or case m even, n odd. For the cases m odd and n odd, or m even and n even we have $|x_m - x_n| = 0 < \varepsilon \implies$ there exists $m, n > M(\varepsilon)$ such that $|x_m - x_n| \geq \varepsilon > 0 \implies X = (x_n) = ((-1)^n)$ is not Cauchy.

(b) $(n + \frac{(-1)^n}{n})$
If we consider just the case $m, n > M(\varepsilon) \in \mathbb{N},\ \varepsilon > 0$. For the case $m = n$ we have $|x_m - x_n| = 0$, but for the case m, n even, $m > n$ we have $|x_m = x_n| = |m + \frac{-1^m}{m} - n - \frac{-1^n}{n}| = |m - n + (\frac{1}{m} - \frac{1}{n})| > 1 > 0$. This implies we can find a positive value of ε such that $|x_m - x_n| \geq \varepsilon \implies X = (x_n) = (n + \frac{(-1)^n}{n})$ is not Cauchy.

(c) (n^2)
For $m, n \in \mathbb{N}$ greater than natural number $M(\varepsilon)$, $\varepsilon > 0$, we have $|x_m - x_n| = |m^2 - n^2| = 0$ for the case $m = n$. For the cases $m > n > 1$, or $1 < m < n$ have $|x_m - x_n| = |m^2 - n^2| \geq 5$, since, for example, case $m = 3, n = 2$, |m^2 - n^2| = 3^2 - 2^2 = 5. This implies there exists $m, n > M(\varepsilon)$ such that $|x_m - x_n| \geq \varepsilon > 0 \implies X = (x_n) = n^2$ is not Cauchy.

M. Establish the convergence and limits of the following sequences:
(a) $((1 + \frac{1}{n})^{n+1})$
We have bound on $x_n = (1 + \frac{1}{n})^{n+1} \geq (1 + (n+1)\frac{1}{n}) = 1 + 1 + \frac{1}{n} > 2,\ \forall n \in \mathbb{N}$ by Bernoulli's Inequality, implying the sequence is bounded below by 2. For $X = (x_n) = ((1 + \frac{1}{n})^{n+1})$, we also have $\forall n \in \mathbb{N}$, $\frac{x_{n-1}}{x_n} = (\frac{\frac{n}{n-1}}{\frac{n+1}{n}})^n (\frac{1}{1+\frac{1}{n}}) = (\frac{n}{n-1}\frac{n}{n+1})^n (\frac{n}{n+1}) = (\frac{n^2}{n^2-1})^n (\frac{n}{n+1}) > 1 \implies (x_n)$ is decreasing. So the sequence is bounded and decreasing. Applying the algebraic property of limits we then have $\lim_{n\to\infty}(1 + \frac{1}{n})^{n+1} = \lim_{n\to\infty}(1 + \frac{1}{n})^n \cdot \lim_{n\to\infty}(1 + \frac{1}{n}) = e * 1$

(c) $((1 + \frac{2}{n})^n)$
We can write $((1 + \frac{2}{n})^n) = ((1 + \frac{1}{\frac{n}{2}})^n) = ((1 + \frac{1}{\frac{n}{2}})^{\frac{n}{2}})^2$. If we consider the subsequence of even numbers, $n = 2k,\ k \in \mathbb{N}$, we have $((1 + \frac{1}{\frac{n}{2}})^{\frac{n}{2}})^2 = (1 + \frac{1}{k})^k \cdot (1 + \frac{1}{k})^k$, and using the algebraic property of limits, we have $\lim_{k\to\infty}(1 + \frac{1}{k})^k \cdot (1 + \frac{1}{k})^k = e \cdot e = e^2$, since the sequence has a limit, is convergent to e^2.

(d) $((1 + \frac{1}{(n+1)})^{3n})$
We can write $((1 + \frac{1}{(n+1)})^{3n}) = ((1 + \frac{1}{(n+1)})^n)^3 = (1 + \frac{1}{(n+1)})^n) \cdot (1 + \frac{1}{(n+1)})^n) \cdot (1 + \frac{1}{(n+1)})^n)$, the product of three convergent sequences, where the limit of each sequence $\lim_{n\to\infty}(1 + \frac{1}{n+1})^n = e \implies \lim_{n\to\infty}((1 + \frac{1}{(n+1)})^{3n}) = e \cdot e \cdot e = e^3$.

N. Let $0 < a_1 < b_1$ and define, for $n \in \mathbb{N}$, $a_{n+1} = (a_n b_n)^{1/2}$, $\quad b_{n+1} = \frac{1}{2}(a_n + b_n)$. By induction show that $a_n < b_n$, and show that a_n and b_n converge to the same limit.

Using induction, we are given $0 < a_1 < b_1$, and we assume $0 < a_n < b_n$. For the case $n+1$ we have $0 < (a_n b_n)^{1/2} < \frac{1}{2}(an + bn) \leftrightarrow 0 < 2\sqrt{a_n b_n} < a_n + b_n \implies 0 < b_n + a_n - 2\sqrt{a_n b_n} = (\sqrt{b_n} - \sqrt{a_n})^2$. Since we assumed $b_n > a_n$, $0 < (\sqrt{b_n} - \sqrt{a_n})^2 \leftrightarrow 0 < \sqrt{b_n} - \sqrt{a_n} \implies 0 < \sqrt{a_n} < \sqrt{b_n} \implies 0 < a_n < b_n \implies 0 < a_{n+1} < b_{n+1}$.

We then take a to the be the limit of (a_n), and b of $(b_n) \implies a$ satisfies $a = \sqrt{ab}$, and b satisfies $b = \frac{a+b}{2}$. This implies $b = \frac{\sqrt{ab}+b}{2} \implies b$ satisfies $b = \sqrt{ab} = a \implies (a_n)$ and (b_n) converge to the same limit.

Section 17

A. For each $n \in \mathbb{N}$, let f_n be defined for $x > 0$ by $f_n(x) = \frac{1}{nx}$. For what values of x does limit $f_n(x)$ exist?

Since $x > 0$, $\frac{1}{nx}$ is defined for all $n \in \mathbb{N}$, and for fixed x is decreasing in n is indicative of $\lim(f_n(x))$ existing for all x

B. For each $n \in \mathbb{N}$, let g_n be defined for $x \geq 0$ by the formula $g_n(x) = nx$, $0 \leq x \leq \frac{1}{n}, g_n(x) = \frac{1}{nx}, \frac{1}{n} < x$. Show that $\lim(g_n(x)) = 0$ for all $x > 0$.

For case $x > \frac{1}{n}$, $|g_n(x) - g(x)| = |\frac{1}{nx} - 0| = \frac{1}{nx}$. For $n \geq K(\varepsilon, x)$, $nx \geq K(\varepsilon, x)x \implies \frac{1}{nx} \leq \frac{1}{K(\varepsilon,x)x} < \varepsilon$, $\varepsilon > 0 \implies g_n(x) \to 0 = g(x)$.

For case, $0 \leq x \leq \frac{1}{n}$ if we assume $\lim(g_n(x)) = g(x) = 0$.

For case $x = 0$, $g_n(0) = n \cdot 0 = 0$ everywhere implying $\lim(g_n(x)) = 0$ in this case.

For $0 < x \leq \frac{1}{n}$, $|g_n(x) - g(x)| = |nx - 0| = nx$. As n grows in this case, the region from 0 to $\frac{1}{n}$ shrinks as the region of valid x converges to $0 \implies \lim g_n(x) = 0 = h(x)$.

D. Show that, if we define f_n on $\mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$, then (f_n) converges on $\mathbb{R}$.

We have $f_n(x) = x\frac{n}{1+n^2x^2}$ which can be separated into two functions $g_n(x) = x$, $h_n(x) = \frac{n}{1+n^2x^2}$. Clearly $g_n(x) = x \to x = g(x)$, $h_n(x) = \frac{n}{1+n^2x^2} = \frac{1}{\frac{1}{n}+nx^2} < \frac{1}{nx^2}$, since $x^2 > 0 \implies h(x) = 0$, and we have $|h_n(x) - h(x)| = \frac{1}{\frac{1}{n}+nx^2} \leq \frac{1}{\frac{1}{K}+Kx^2} < \varepsilon$, $\varepsilon > 0$, $n \geq K \in \mathbb{N}$. Using algebraic properties of limits we have, $\lim f_n(x) = \lim g_n(x) \cdot \lim h_n(x) = 0 \cdot x \implies \lim f_n(x) = 0, \implies$ convergence.

E. Let h_n be defined on the interval $\mathbb{I} = [0,1]$ by the formula $h_n(x) = 1 - nx$, $0 \leq x \leq \frac{1}{n}, h_n(x) = 0$, $\frac{1}{n} < x \leq 1$. Show that $\lim h_n(x)$ exists on $[0,1]$.

For case $x = 0$, we have $h_n(x) = 1 - nx \to_{n\to\infty} 1 = h(x)$.
For case $0 < x \leq \frac{1}{n}$, $h_n(x) = 1 - nx \to 0 = h(x)$, because as n grows, the region from 0 to $\frac{1}{n}$ shrinks $\implies nx \to 1 \implies h_n(x) \to 1 - 1 = 0$.
For case $\frac{1}{n} < x \leq 1$ as n grows we have $h_n(x) = \frac{1}{nx} \to 0 = h(x) \implies \lim h_n(x)$ exists on the interval $[0,1]$.

L. Show that the convergence in Exercise 17.B is not uniform on the domain $x \geq 0$, but that it is uniform on a set $x \geq c$, where $c > 0$.

We have

$$g_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ \frac{1}{nx}, & x > \frac{1}{n} \end{cases}$$

But if we take $\sup_{x\in[0,\infty]} = \sup_{x\in[0,\infty]} |f_n(x) - f(x)|$, where $\lim f_n(x) = 0$, we have $\sup_{x\in[0,\infty]} |f_n(x) - f(x)| = 1 \implies f_n(x)$ is only pointwise convergent based on results from exercise 17.B.

M. Is the convergence in Exercise 17.D uniform on $\mathbb{R}$?

We have $\lim_{n\to\infty} \frac{nx}{1+n^2x^2}$, which for large n is like $\lim_{n\to\infty} \frac{nx}{n^2x^2} = \frac{1}{x}\lim_{n\to\infty}\frac{1}{n} \to 0,\ x \neq 0$.. But if we take $x = \frac{1}{n}$, we have $|f_n(x) - f(x)| = |f_n(\frac{1}{n}) - f(\frac{1}{n})| = |\frac{1}{\frac{1}{n\frac{1}{n}} + n\frac{1}{n}} - 0| = \frac{1}{2} - 0 > \varepsilon,\ 0 < \varepsilon < 1/2 \implies f_n(x)$ does not converge uniformly.

Section 18

A. Determine the limit superior and the limit inferior of the following bounded sequences in $\mathbb{R}$.
(a) $((-1)^n)$

Considering two subsequences of $X = (x_n)$, we have $(x_{2n}) = (1, 1, ..., 1, ..)$, and $(x_{2n-1}) = (-1, -1, ..., -1, ..) \implies \lim(x_{2n}) = 1,\ \lim(x_{2n-1}) = -1 \implies \limsup(x_n) = 1,\ \liminf(x_n) = -1$.
(b) $((-1)^n/n)$

Using the same approach, $(x_n) = ((-1)^n/n) \implies (x_{2n}) = (1/2, 1/4, ..., 1/2n, ...)$, and $\lim(x_{2n}) = 0$, $(x_{2n-1}) = (-1/1, -1/3, -1/4, ..., -1/(2n-1), ...), \implies \lim(x_{2n-1}) = 0 \implies \limsup(x_n) = \lim \liminf(x_n) = 0$.

(c) $((-1)^n + 1/n)$

$((-1)^n + 1/n) = (x_n) \implies (x_{2n}) = (1 + 1/2, 1 + 1/4, 1 + 1/6, ..., 1 + 1/2n, ..), \implies \lim(x_{2n}) = 1$. $(x_{2n-1}) = (-1 + 1/1, -1 + 1/3, -1 + 1/5, ..., -1 + 1/(2n-1), ..), \implies \lim(x_{2n-1}) = -1 \implies \limsup(x_n) = 1,\ \liminf(x_n) = -11$.

D. Give a direct proof of Theorem 18.3(c).

$\liminf(x_n) + \liminf(y_n) \leq \liminf(x_n + y_n)$. By definition $\liminf(x_n)$ is the supremum of set V such that there are at most a finite number of $n \in \mathbb{N}$ such that $x_n < v$, and denote $\liminf(y_n)$ as the supremum of set U of $u \in \mathbb{R}$ such that there are at most a finite number of $n \in \mathbb{N}$ such that $y_n < u$.

Let $v < \liminf(x_n),\ u < \liminf(y_n) \implies$ there are only finite $n \in \mathbb{N}$ such that $x_n < v$ and $y_n < u \implies$ only finite $n \in \mathbb{N}$ such that $x_n + y_n < v + u \implies \liminf(x_n) + \liminf(y_n) \leq v + u \implies \liminf(x_n) + \liminf(y_n) \leq \liminf(x_n + y_n)$.

F. If $X = (x_n)$ is a bounded sequence of strictly positive elements in $\mathbb{R}$, show that $\limsup(x_n^{1/n}) \leq \limsup(x_{n+1}/x_n)$.

Because $X = (x_n)$ is bounded, we have $x^* = \limsup(\frac{x_{n+1}}{x_n}),\ x^* < \infty \implies$ for $\varepsilon > 0,\ n, K \in \mathbb{N}$, we have $\frac{x_{n+1}}{x_n} \leq x^* + \varepsilon$, for $n \geq K$, then $\frac{x_n}{x_K} = \frac{x_{K+1}}{x_K} \cdot \frac{x_{K+2}}{x_{K+1}} \cdot ... \cdot \frac{x_{n-1}}{x_{n-2}} \cdot \frac{x_n}{x_{n-1}} \leq (x^* + \varepsilon)^{n-K} \implies \frac{x_n}{x_K} \leq (x^* + \varepsilon)^{n-K} = \frac{(x^*+\varepsilon)^n}{(x^*+\varepsilon)^K} \implies x_n \leq \frac{x_K}{(x^*+\varepsilon)^K} \cdot (x^* + \varepsilon)^n \implies x_n^{1/n} \leq (\frac{x_K}{(x^*+\varepsilon)^K})^{1/n} \cdot (x^* + \varepsilon) \to_{n\to\infty} x_n \leq 1 \cdot (x^* + \varepsilon) \implies \limsup(x_n^{1/n}) \leq \limsup(x_{n+1}/x_n) \leq x^* + \varepsilon$.

I. Show that $\limsup X = +\infty$ *if and only if there is a subsequence X' of X such that* $\lim X' = +\infty$.

$\to$. Let $\limsup X = +\infty \implies \sup\{x_n : n \geq m\} = +\infty$, for all $m \in \mathbb{N} \implies$ there is a subsequence X' such that $X' = (x_m, x_{m+1}, ..., x_n)$, has $\lim X' = +\infty \implies$ if $\limsup X = +\infty$ there is a subsequence X' of X such that $\lim X' = +\infty$.

$\leftarrow$. Let there be a subsequence of X, X', such that X' has $\lim X' = +\infty \implies +\infty$ is in the set E which consists of the limits of all subsequences of X. This implies that $\sup E = +\infty \implies \limsup X = +\infty$.