MATH 6262 Homework 3 Yixian Zhai

Maximum Likelihood Estimation Part

1. (a)

x	1	2	3	4
$\mathbb{P}(X = x)$	$\frac{1}{3+7\theta}$	$\frac{2}{3+7\theta}$	$\frac{3\theta}{3+7\theta}$	$\frac{4\theta}{3+7\theta}$

Table 3.1: 1a table

$$\begin{aligned}
1 &= c\theta + 2c\theta + 3c\theta^2 + 4c\theta^2 \\
&= 3c\theta + 7c\theta^2 \\
c(\theta) &= \frac{1}{3\theta + 7\theta^2}
\end{aligned}$$

(b)

$$\begin{aligned}
L &= \frac{X\theta^{\mathbb{1}\{X \in \{3,4\}\}}}{7\theta + 3} \\
l &= \ln(X) + \mathbb{1}\{X \in \{3,4\}\}\theta - \ln(7\theta + 3) \\
\frac{\mathrm{d}l}{\mathrm{d}\theta} &= \frac{\mathbb{1}\{X \in \{3,4\}\}}{\theta} - \frac{7}{7\theta + 3} \\
\frac{\mathrm{d}^2 l}{\mathrm{d}\theta^2} &= -\frac{\mathbb{1}\{X \in \{3,4\}\}}{\theta^2} + \frac{49}{(7\theta + 3)^2} \\
I_1(\theta) &= -\mathbb{E}\left[\frac{\mathrm{d}^2 l}{\mathrm{d}\theta^2}\right] \\
&= \frac{21}{\theta(7\theta + 3)^2}
\end{aligned}$$

(c)

$$
\begin{aligned}
S &:= \sum_{i=1}^{100} \mathbb{1}\{X_i \in \{3,4\}\} \\
L &= \frac{\prod_{i=1}^{100} X_i \theta^S}{(3+7\theta)^{100}} \\
l &= \sum_{i=1}^{100} \ln(X_i) + S\ln(\theta) - 100\ln(7\theta+3) \\
0 := \frac{\mathrm{d}l}{\mathrm{d}\theta} &= \frac{S}{\theta} - \frac{700}{7\theta+3} \\
\hat{\theta} &= \frac{3S}{7(100-S)} \\
&= \frac{3\sum_{i=1}^{100} \mathbb{1}\{X_i \in \{3,4\}\}}{7\sum_{i=1}^{100} \mathbb{1}\{X_i \in \{1,2\}\}}
\end{aligned}
$$

(d) $I_{100}(\hat{\theta}) = \frac{49(100-S)^3}{900S}$. Thus the confidence interval is

$$
\frac{3S}{7(100-S)}\left(1 \pm 1.96\sqrt{\frac{1}{S(100-S)}}\right)
$$

2. (a)

$$
\begin{aligned}
L &= \lambda^n e^{-\lambda\sum_{i=1}^n x_i} \\
l &= n\lambda - \lambda\sum_{i=1}^n x_i \\
0 := \frac{\mathrm{d}l}{\mathrm{d}\lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n x_i \\
\hat{\theta} &= \frac{n}{\sum_{i=1}^n x_i}
\end{aligned}
$$

(b)

$$
\begin{aligned}
I(\lambda) &= -\mathbb{E}\left[\frac{\mathrm{d}^2 l}{\mathrm{d}\lambda^2}\right] \\
&= \frac{n}{\lambda^2}
\end{aligned}
$$

(c) Thus the 95% confidence interval would be:

$$\frac{n}{\sum_{i=1}^{n} X_i} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{n}{(\sum_{i=1}^{n} X_i)^2}}$$

Let $S = \sum_{i=1}^{n} X_i$ and given $n = 100, z_{\frac{\alpha}{2}} = 1.96$, the confidence interval is

$$\left[\frac{80.4}{S}, \frac{119.6}{S}\right]$$

3. (a) From the table, it is easy to observe, that the maximum likelihood estimator of θ, given the sample x, is as follows:

$$\begin{cases} x = 4 & \hat{\theta} = 3 \\ x = 5 & \hat{\theta} = 1 \\ x = 6 & \hat{\theta} = 1 \end{cases}$$

(b) Let $x = g(4),\ y = g(5),\ z = g(6)$ where g is the decision procedure.

$$\begin{cases} 1 = 0.2x + 0.3y + 0.5z \\ 2 = 0.6x + 0.2y + 0.2z \\ 3 = 0.8x + 0.1y + 0.1z \end{cases} \Rightarrow \begin{cases} x = 4 \\ y = -6 \\ z = 4 \end{cases}$$

(c) Let $x = g(4),\ y = g(5),\ z = g(6)$ where g is the decision procedure.

$$\begin{cases} 1 = 0.2x + 0.3y + 0.5z \\ 2 = 0.6x + 0.2y + 0.2z \\ 3 = 0.8x + 0.2001y + 0.1999z \end{cases} \Rightarrow \begin{cases} x = -997 \\ y = \frac{13001}{2} \\ z = -\frac{6999}{2} \end{cases}$$

Not a good estimator.

4. (a)

$$10\% + 17\% - 20\% = 7\%$$

(b)

$$20\% - 7\% = 13\%$$

5. (a) This is a multinomial model, with a case $30\% - 10\%$ of 100, which is 20; b case $40\% - 10\%$ of 100, which is 30; similarly 10 of $a \cap b$ and 40 of neither.

$$L = \binom{100}{20\ 10\ 30\ 40} p_a^{20} p_{a\cap b}^{10} p_b^{30} p_{\neg a\cap\neg b}^{40}$$

$$l = c + 20\ln(p_a) + 30\ln(p_b) + 10\ln(p_{a\cap b}) + 40\ln(p_{\neg a\cap\neg b})$$

$$
\begin{aligned}
\text{maximize} \quad & l + \lambda(\sum p - 1) \\
\text{subject to } & 1 = \sum p \\
& 0 := \frac{\partial l}{\partial p_a} = \frac{20}{p_a} + \lambda \\
& 0 := \frac{\partial l}{\partial p_b} = \frac{30}{p_b} + \lambda \\
& 0 := \frac{\partial l}{\partial p_{a\cap b}} = \frac{10}{p_{a\cap b}} + \lambda \\
& 0 := \frac{\partial l}{\partial p_{a\cap b}} = \frac{10}{p_{\neg a\cap\neg b}} + \lambda
\end{aligned}
$$

$$
\begin{cases}
\hat{p}_a & = \frac{1}{5} \\
\hat{p}_b & = \frac{3}{10} \\
\hat{p}_{a\cap b} & = \frac{1}{10} \\
\hat{p}_{\neg a\cap\neg b} & = \frac{2}{5} \\
\lambda & = -100
\end{cases}
$$

Agree to their frequencies. This is consistent to binomial case, where intuitively we know based on the information we have, maximum likelihood estimator of proportion should be the frequencies of each category.

One can easily check that all second partial derivative are negative, hence convex.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial p_a^2} &= -\frac{20}{p_a^2} - \frac{40}{(1-p_a-p_b-p_{a\cap b})^2} < 0 \\
\frac{\partial^2 l}{\partial p_b^2} &= -\frac{30}{p_b^2} - \frac{40}{(1-p_a-p_b-p_{a\cap b})^2} < 0 \\
\frac{\partial^2 l}{\partial p_{a\cap b}^2} &= -\frac{10}{p_{a\cap b}^2} - \frac{40}{(1-p_a-p_b-p_{a\cap b})^2} < 0 \\
\frac{\partial^2 l}{\partial p_a \partial p_b} &= -\frac{40}{(1-p_a-p_b-p_{a\cap b})^2} < 0 \\
\frac{\partial^2 l}{\partial p_a \partial p_{a\cap b}} &= -\frac{40}{(1-p_a-p_b-p_{a\cap b})^2} < 0 \\
\frac{\partial^2 l}{\partial p_b \partial p_{a\cap b}} &= -\frac{40}{(1-p_a-p_b-p_{a\cap b})^2} < 0
\end{aligned}
$$

(b)

Testing Theory Part

1. (a)

x	1	2	3	4	5
ratio(x)	$\frac{2}{3}$	1	2	$\frac{4}{5}$	$\frac{4}{3}$
$\mathbb{P}(X = x \vert H_0)$	0.2	0.2	0.2	0.2	0.2
$\mathbb{P}(X = x \vert H_1)$	0.3	0.2	0.1	0.25	0.15
$c \in [0, \frac{2}{3}]$	I	I	I	I	I
$c \in \left(\frac{2}{3}, \frac{4}{5}\right]$	II	I	I	I	I
$c \in \left(\frac{4}{5}, 1\right]$	II	I	I	II	I
$c \in \left(1, \frac{4}{3}\right]$	II	II	I	II	I
$c \in \left(\frac{4}{3}, 2\right]$	II	II	I	II	II
$c \in (2, \infty)$	II	II	II	II	II

Table 3.2: 1a table

(b)

x	1	2	3	4	5
$\mathbb{P}(\text{reject} \vert H_0, X = x)$	1	0	0	$\frac{1}{2}$	0
$\mathbb{P}(X = x \vert H_0)$	0.2	0.2	0.2	0.2	0.2
$\mathbb{P}(X = x \vert H_1)$	0.3	0.2	0.1	0.25	0.15

Table 3.3: 1b table

According to the table ((a)), once we want to design a most powerful test, it has to be the case where $c \in \left(\frac{4}{5}, 1\right]$, with the significance level could be between 0.2 and 0.4 under manipulation. We randomize the possibility of rejection at $x = 4$ to be 0.5 so then the significance is 0.3.

$$0.3 = 0.2 + \mathbb{P}(\text{reject}|H_0, X = 4)0.2$$
$$\mathbb{P}(\text{reject}|H_0, X = 4) = \frac{1}{2}$$

Then similarly, power of such a test would be

$$0.3 + \frac{1}{2}0.25 = 0.425$$

(c)

x	1	2	3	4	5
$\mathbb{P}(X = x \mid H_0)$	**0.1**	**0.1**	0.1	**0.1**	0.1
$\mathbb{P}(X = x \mid H_1)$	0.15	0.1	**0.05**	0.125	**0.075**

Table 3.4: 1c table

Pick minimum in each column in (c), i.e. make the decision of the other case, then probability of making an error is

$$0.1 + 0.1 + 0.05 + 0.1 + 0.075 = 0.425$$

(d)

x	1	2	3	4	5
$\mathbb{P}(X = x \mid H_0)$	**0.08**	**0.08**	0.08	**0.08**	**0.08**
$\mathbb{P}(X = x \mid H_1)$	0.18	0.12	**0.06**	0.15	0.09

Table 3.5: 1d table

Pick minimum in each column in (d), i.e. make the decision of the other case, then probability of making an error is

$$0.08 + 0.08 + 0.06 + 0.08 + 0.008 = 0.38$$

(e)

x	1	2	3	4	5
$\mathbb{P}(X = x \mid H_0) \cdot L_0$	**0.16**	**0.16**	**0.16**	**0.16**	**0.16**
$\mathbb{P}(X = x \mid H_1) \cdot L_1$	0.72	0.48	0.24	0.60	0.36

Table 3.6: 1e table

It seems that to minimize the expected cost, we only need to stick on the alternative hypothesis. As any error would lead a uniformly smaller cost by null hypothesis. So far there is no evidence indicating a randomized test.

2.

x	1	2	3	4
$\text{ratio}_1(x)$	1	2	$\underline{1}$	$\underline{0.5}$
$\text{ratio}_2(x)$	0.2	4	$\underline{1}$	$\underline{2}$
$\mathbb{P}(X = x \mid H_0)$	0.1	0.4	0.3	0.2
$\mathbb{P}(X = x \mid H_1)$	0.1	0.2	0.3	0.4
$\mathbb{P}(X = x \mid H_2)$	0.5	0.1	0.3	0.1

Table 3.7: p2 table

The underlined two groups of ratios in 2., are of different directions. According to Lemma 3.1, we know there couldn't exist UMP. For example at significance level less than 0.6, most powerful tests for single alternative hypothesis couldn't agree, hence no UMP.

8. Technically there doesn't exist. If the goal is to test whether the die is fair, then the alternative hypothesis would contain different hypotheses. Under such circumstances, the ratios of any two would violate the necessary and sufficient condition.

x	1	2	3	4	5	6
$\text{ratio}_1(x)$	$\frac{1}{6}p_1^{-1}$	$\frac{1}{6}p_2^{-1}$	$\frac{1}{6}p_3^{-1}$	$\frac{1}{6}p_4^{-1}$	$\frac{1}{6}p_5^{-1}$	$\frac{1}{6}p_6^{-1}$
$\text{ratio}_2(x)$	$\frac{1}{6}(p'_1)^{-1}$	$\frac{1}{6}(p'_2)^{-1}$	$\frac{1}{6}(p'_3)^{-1}$	$\frac{1}{6}(p'_4)^{-1}$	$\frac{1}{6}(p'_5)^{-1}$	$\frac{1}{6}(p'_6)^{-1}$
$\mathbb{P}(X = x \mid H_0)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\mathbb{P}(X = x \mid H_1)$	p_1	p_2	p_3	p_4	p_5	p_6
$\mathbb{P}(X = x \mid H_1)$	p'_1	p'_2	p'_3	p'_4	p'_5	p'_6

Table 3.8: p8 table

Since the hull hypothesis is uniform, all ratios are proportional to the reciprocal of the probability.

For example let's consider a simple case where the composite alternative only includes two. Without loss of generality, we may assume

$$p_1 \geqslant p_2 \geqslant p_3 \geqslant p_4 \geqslant p_5 \geqslant p_6$$

To satisfy the sufficient and necessary condition we need the order of ratio 2 is the same as order of ratio 1. That is

$$p_1^{-1} \leqslant p_2^{-1} \leqslant p_3^{-1} \leqslant p_4^{-1} \leqslant p_5^{-1} \leqslant p_6^{-1}$$

$$(p_1')^{-1} \leqslant (p_2')^{-1} \leqslant (p_3')^{-1} \leqslant (p_4')^{-1} \leqslant (p_5')^{-1} \leqslant (p_6')^{-1}$$

$$p_1' \geqslant p_2' \geqslant p_3' \geqslant p_4' \geqslant p_5' \geqslant p_6'$$

Therefore consequently in general such UMP exists if and only if all alternative hypotheses are made of ordered probability distribution in the same direction.

This could be the case if we restrict the alternative hypothesis to be testing whether the die has all faces within a specific probability order. Otherwise by symmetry there should be other orders of each combination, hence the Lemma is violated.