Math 4317 (Prof. Swiech, S'18): HW #1

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Section 1

F. Show that the symmetric difference D, defined in the preceding exercise is also given by $D = (A \cup B) \setminus (A \cap B)$
Show $D = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$:

First, $x \in (A \setminus B) \cup (B \setminus A) \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies$, x is in A but not B, or, x is in B but not $A \implies x$ is in A or B but not in A and $B \implies x \in (A \cup B) \setminus (A \cap B)$.

Second, $x \in (A \cup B) \setminus (A \cap B) \implies x \in (A \cup B)$ but not in $(A \cap B) \implies x$ is in A but not B, or, x is in B but not $A \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies x \in (A \setminus B) \cup (B \setminus A) \implies (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

I. If $\{A_1, A_2, ..., A_n\}$ is a collection of sets, and if E is any set, show that:

$$(i) \quad E \cap \cup_{j=1}^{n} A_j = \cup_{j=1}^{n} (E \cap A_j), \text{ and } (ii), \quad E \cup \cup_{j=1}^{n} A_j = \cup_{j=1}^{n} (E \cup A_j)$$

(i) First, $x \in E \cap \cup_{j=1}^{n} A_j \implies x \in E$ *and* $x \in \{A_1$ *or* $A2$ *...or* $A_n\} \implies x \in E$ and that there exists for some $j = 1, 2, ..., n$ an A_j such that $x \in A_j$ *and* $x \in E \implies (x \in E$ *and* $A_1)$ *or* $(x \in E$ *and* $A_2)$ ... *or* $(x \in E$ *and* $A_n) \implies x \in \cup_{j=1}^{n} (E \cap A_j)$.
In the other direction, $x \in \cup_{j=1}^{n} (E \cap A_j) \Leftrightarrow x \in (E \cap A_1) \cup (E \cap A_2)$ ... $\cup (E \cap A_n) \implies x \in E$ and A_1 or E and $A_2 \ldots \implies$ there exists a $j = 1, ..., n$ such that $x \in (E \cap A_j) \implies x \in E$ and $x \in A_1$ or $A_2, \ldots$, or $A_n \implies x \in E$ and $\cup_{j=1}^{n} A_j \implies x \in E \cap \cup_{j=1}^{n} A_j$.

(ii) First, $x \in E \cup \cup_{j=1}^{n} A_j \implies x \in E$ or $x \in A_1$ or $A_2 \ldots$ or $A_n \implies$ for some $j = 1, .., n$ that $x \in E \cup A_j \implies x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies x \in \cup_{j=1}^{n} (E \cup A_j)$. In the other direction, $x \in \cup_{j=1}^{n} (E \cup A_j) \Leftrightarrow x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies$ there exists some $j = 1, ..., n$ such that $x \in E \cup A_j \implies (x \in E$ *or* $x \in A_1)$ or $(x \in E$ *or* $x \in A_2) \ldots$ or $(x \in E$ *or* $x \in A_n) \implies x \in E$ or $x \in \cup_{j=1}^{n} A_j \implies x \in E \cup \cup_{j=1}^{n} A_j$.

J. If $\{A_1, A_2, ..., A_n\}$ is a collection of sets, and if E is any set, show that:

$$(i) \; E \cap \cap_{j=1}^{n} A_j = \cap_{j=1}^{n} (E \cap A_j), \text{ and } (ii), \quad E \cup \cap_{j=1}^{n} A_j = \cap_{j=1}^{n} (E \cup A_j)$$

(i) First, $x \in \cap \cap_{j=1}^{n} A_j \implies x \in E$ and $x \in \cap_{j=1}^{n} A_j \implies x \in E$ *and* $x \in A_j$ for all $j = 1, .., n \implies x \in E$ and $[x \in A_1$ *and* $x \in A_2$ ... *and* $x \in A_n] \implies [x \in E$ *and* $A_1]$ and $\ldots$ and $[x \in E$ *and* $A_n] \implies x \in \cap_{j=1}^{n} (E \cap A_j)$. In the other direction, $x \in \cap_{j=1}^{n} (E \cap A_j) \implies x \in (E \cap A_1)$ and $a \in (E \cap A_2) \ldots$ and $x \in (E \cap A_n) \implies x \in (E \cap A_j)$ for all $j = 1, ..., n \implies x \in E$ and $x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n \implies x \in E$ and $x \in \cap_{j=1}^{nA_j} \implies x \in E \cap \cap_{j=1}^{nA_j}$.

(ii) First, $x \in E \cup \cap_{j=1}^{n} A_j \implies x \in E$ or $x \in \cap_{j=1}^{n} A_j \implies x \in E$ or $[x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n] \implies x \in E$ *or* A_1 *and* $x \in E$ *or* A_2 ... *and* $x \in E$ *or* $A_n \implies x \in \cap_{j=1}^{n} (E \cup A_j)$. In the other direction,

K. Let E be a set and $\{A_1, A_2, ..., A_n\}$ be a collection of sets. Establish the De Morgan laws:

$$E \setminus \cap_{j=1}^{n} A_j = \cup_{j=1}^{n} (E \setminus A_j), \quad E \setminus \cup_{j=1}^{n} A_j = \cap_{j=1}^{n} (E \setminus A_j)$$

Section 2

C. Consider the subset of $\mathbb{R} \times \mathbb{R}$ defined by $D = \{(x, y) \; : |x| + |y| = 1\}$. Describe this set in words. Is it a function?

E. Prove that if f is an injection from A to B, than $f^{-1} = \{(b,a) : (a,b) \in f\}$ is a function. Then prove it is an injection.

H. Let f, g be functions such that

$$g \circ f(x) = x, \quad for\ all\ x\ in\ D(f)$$

$$f \circ g(y) = y, \quad for\ all\ y\ in\ D(g)$$

Prove that $g = f^{-1}$

J. Let f be the function on $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = x^2$, and let $E = \{x \in \mathbb{R} -1 \leq x \leq 0\}$ and $F = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.Then $E \cap F = \{0\}$ and $f(E \cap F) = \{0\}$ while $f(E) = f(F) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. Now delete 0 from E and F.