

Math 4317 (Prof. Swiech, S'18): HW #1

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Section 1

F. Show that the symmetric difference D , defined in the preceding exercise is also given by $D = (A \cup B) \setminus (A \cap B)$. Show $D = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$:

First, $x \in (A \setminus B) \cup (B \setminus A) \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies$, x is in A but not B , or, x is in B but not $A \implies x$ is in A or B but not in A and $B \implies x \in (A \cup B) \setminus (A \cap B)$.

In the other direction, $x \in (A \cup B) \setminus (A \cap B) \implies x \in (A \cup B)$ but not in $(A \cap B) \implies x$ is in A but not B , or, x is in B but not $A \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies x \in (A \setminus B) \cup (B \setminus A) \implies (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

I. If $\{A_1, A_2, \dots, A_n\}$ is a collection of sets, and if E is any set, show that:

$$(i) \ E \cap \bigcup_{j=1}^n A_j = \bigcup_{j=1}^n (E \cap A_j), \text{ and } (ii), \ E \cup \bigcup_{j=1}^n A_j = \bigcup_{j=1}^n (E \cup A_j)$$

- (i) $x \in E \cap \bigcup_{j=1}^n A_j \implies x \in E$ and $x \in \{A_1 \text{ or } A_2 \dots \text{or } A_n\} \implies x \in E$ and that there exists for some $j = 1, 2, \dots, n$ an A_j such that $x \in A_j$ and $x \in E \implies (x \in E \text{ and } A_1) \text{ or } (x \in E \text{ and } A_2) \dots \text{or } (x \in E \text{ and } A_n) \implies x \in \bigcup_{j=1}^n (E \cap A_j)$.

In the other direction, $x \in \bigcup_{j=1}^n (E \cap A_j) \Leftrightarrow x \in (E \cap A_1) \cup (E \cap A_2) \dots \cup (E \cap A_n) \implies x \in E$ and A_1 or E and $A_2 \dots \implies$ there exists a $j = 1, \dots, n$ such that $x \in (E \cap A_j) \implies x \in E$ and $x \in A_1$ or A_2, \dots , or $A_n \implies x \in E$ and $\bigcup_{j=1}^n A_j \implies x \in E \cap \bigcup_{j=1}^n A_j$.

- (ii) $x \in E \cup \bigcup_{j=1}^n A_j \implies x \in E$ or $x \in A_1$ or $A_2 \dots$ or $A_n \implies$ for some $j = 1, \dots, n$ that $x \in E \cup A_j \implies x \in E \cup A_1$ or $x \in E \cup A_2 \dots$ or $x \in E \cup A_n \implies x \in \bigcup_{j=1}^n (E \cup A_j)$. In the other direction, $x \in \bigcup_{j=1}^n (E \cup A_j) \Leftrightarrow x \in E \cup A_1$ or $x \in E \cup A_2 \dots$ or $x \in E \cup A_n \implies$ there exists some $j = 1, \dots, n$ such that $x \in E \cup A_j \implies (x \in E \text{ or } x \in A_1) \text{ or } (x \in E \text{ or } x \in A_2) \dots \text{or } (x \in E \text{ or } x \in A_n) \implies x \in E$ or $x \in \bigcup_{j=1}^n A_j \implies x \in E \cup \bigcup_{j=1}^n A_j$.

J. If $\{A_1, A_2, \dots, A_n\}$ is a collection of sets, and if E is any set, show that:

$$(i) \ E \cap \bigcap_{j=1}^n A_j = \bigcap_{j=1}^n (E \cap A_j), \text{ and } (ii), \ E \cup \bigcap_{j=1}^n A_j = \bigcap_{j=1}^n (E \cup A_j)$$

- (i) $x \in E \cap \bigcap_{j=1}^n A_j \implies x \in E$ and $x \in \bigcap_{j=1}^n A_j \implies x \in E$ and $x \in A_j$ for all $j = 1, \dots, n \implies x \in E$ and $[x \in A_1 \text{ and } x \in A_2 \dots \text{and } x \in A_n] \implies [x \in E \text{ and } A_1] \text{ and } \dots \text{and } [x \in E \text{ and } A_n] \implies x \in \bigcap_{j=1}^n (E \cap A_j)$. In the other direction, $x \in \bigcap_{j=1}^n (E \cap A_j) \implies x \in (E \cap A_1)$ and $x \in (E \cap A_2) \dots$ and $x \in (E \cap A_n) \implies x \in (E \cap A_j)$ for all $j = 1, \dots, n \implies x \in E$ and $x \in A_1$ and $x \in A_2 \dots$ and $x \in A_n \implies x \in E$ and $x \in \bigcap_{j=1}^n A_j \implies x \in E \cap \bigcap_{j=1}^n A_j$.

- (ii) $x \in E \cup \bigcap_{j=1}^n A_j \implies x \in E$ or $x \in \bigcap_{j=1}^n A_j \implies x \in E$ or $[x \in A_1 \text{ and } x \in A_2 \dots \text{and } x \in A_n] \implies x \in E \text{ or } A_1 \text{ and } x \in E \text{ or } A_2 \dots \text{and } x \in E \text{ or } A_n \implies x \in \bigcap_{j=1}^n (E \cup A_j)$. In the other direction, $x \in \bigcap_{j=1}^n (E \cup A_j) \implies x \in (E \cup A_1)$ and $x \in (E \cup A_2) \dots$ and $x \in (E \cup A_n) \implies$ that for all $j = 1, \dots, n$, $x \in (E \cup A_j) \implies x \in E$ or $(x \in A_1 \text{ and } x \in A_2 \dots \text{and } x \in A_n) \implies x \in \bigcap_{j=1}^n A_j$ or $x \in E \implies x \in E \cup \bigcap_{j=1}^n A_j$.

K. Let E be a set and $\{A_1, A_2, \dots, A_n\}$ be a collection of sets. Establish the De Morgan laws:

$$(i) E \setminus \bigcap_{j=1}^n A_j = \bigcup_{j=1}^n (E \setminus A_j), \text{ and, } (ii) E \setminus \bigcup_{j=1}^n A_j = \bigcap_{j=1}^n (E \setminus A_j)$$

- (i) $x \in E \setminus \bigcap_{j=1}^n A_j \implies x \in E$ but not $(A_1 \text{ and } A_2 \dots \text{ and } A_n) \implies$ there exists a $j = 1, \dots, n$ such that $x \in E$ but not $A_j \implies x \in E$ but not A_1 , or $x \in E$ but not A_2, \dots , or $x \in E$ but not $A_n \implies x \in E \setminus A_1$ or $x \in E \setminus A_2 \dots$ or $x \in E \setminus A_n \implies x \in \bigcup_{j=1}^n (E \setminus A_j)$. In the other direction, $x \in \bigcup_{j=1}^n (E \setminus A_j) \implies x \in (E \text{ but not } A_1)$ or $(E \text{ but not } A_2)$ or $(E \text{ but not } A_n) \implies$ there exists $j = 1, \dots, n$, $x \in E$ but not $A_j \implies x \in E$ but not $(A_1 \text{ and } A_2 \dots \text{ and } A_n) \implies x \in E \setminus \bigcap_{j=1}^n A_j$.
- (ii) $x \in E \setminus \bigcup_{j=1}^n A_j \implies x \in E$ but A_1 or $A_2 \dots$ or $A_n \implies x \in E$ and $x \notin A_j$ for all $j = 1, \dots, n \implies x \in E$ but not A_1 , and $x \in E$ but not A_2, \dots , and $x \in E$ but not $A_n \implies x \in (E \setminus A_1)$ and $x \in (E \setminus A_2) \dots$ and $x \in (E \setminus A_n) \implies x \in \bigcap_{j=1}^n (E \setminus A_j)$. In the other direction, $x \in \bigcap_{j=1}^n (E \setminus A_j) \implies x \in (E \setminus A_1 \text{ and } E \setminus A_2 \dots \text{ and } E \setminus A_n) \implies x \in E$ but not A_j for all $j = 1, \dots, n \implies x \in E$ but A_1 or $A_2 \dots$ or $A_n \implies x \in E$ but not $\bigcup_{j=1}^n A_j \implies x \in E \setminus \bigcup_{j=1}^n A_j$.

Section 2

C. Consider the subset of $\mathbb{R} \times \mathbb{R}$ defined by $D = \{(x, y) : |x| + |y| = 1\}$. Describe this set in words. Is it a function?

This set consists of points on the line segments connecting a rotated square in the (x, y) plane with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. If we attempt to define a function, with the elements (x, y) from the set D , i.e. $y = f(x)$, $f : x \rightarrow y$, we have $|x| + |y| = 1 \implies \sqrt{y^2} = 1 - |x| \implies y = \pm\sqrt{(1 - |x|)^2}$. $f(x) = y = \pm\sqrt{(1 - |x|)^2}$ does not fit the definition of a function, since, as an example, the set D includes the elements $(0, 1)$ and $(0, -1)$, which if, f is a function, $f : x \rightarrow y \implies -1 = 1$, which is clearly not true.

E. Prove that if f is an injection from A to B , then $f^{-1} = \{(b, a) : (a, b) \in f\}$ is a function. Then prove it is an injection.

If f is an injection, and $(a, b) \in f$, and $(a', b) \in f$, then $a = a'$. $f^{-1} = \{(b, a) : (a, b) \in f\}$ contains the pair (b, a) and (b, a') , and we know that $a = a'$ from the definition of f , so we can assume that f^{-1} is a function. Since f is injective, each unique element $b = f(a)$, is mapped to by a unique element a , and by definition $f^{-1} = \{(b, a) : (a, b) \in f\}$ maps the unique element b back to a , meaning $f^{-1}(b) = a$ and $f^{-1}(b') = a$ if and only if $b = b'$, thus f^{-1} is also injective.

H. Let f, g be functions such that

$$g \circ f(x) = x, \text{ for all } x \text{ in } D(f)$$

$$f \circ g(y) = y, \text{ for all } y \text{ in } D(g)$$

Prove that $g = f^{-1}$

For two elements $x, x' \in D(f)$, if $f(x) = f(x') \implies g \circ f(x) = g(f(x)) = g(f(x')) \implies g(f(x)) = x = g(f(x')) = x'$, that is $x = x' \implies g \circ f$ is an injection. For two elements $y, y' \in D(g)$, if $g(y) = g(y') \implies f \circ g(y) = f(g(y)) = f(g(y')) \implies f(g(y)) = y = f(g(y')) = y'$, that is $y = y' \implies f \circ g$ is an injection, and implies f and g are injections as well.

This implies g can be defined $g = \{(f(x), x) : (x, f(x)) \in f\}$, which is the definition for f^{-1} , implying $g = f^{-1}$.

J. Let f be the function on \mathbb{R} to \mathbb{R} given by $f(x) = x^2$, and let $E = \{x \in \mathbb{R} : -1 \leq x \leq 0\}$ and $F = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Then $E \cap F = \{0\}$ and $f(E \cap F) = \{0\}$ while $f(E) = f(F) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. Now delete 0 from E and F .

The sets E and F with 0 deleted are denoted $E' = \{x \in \mathbb{R} : -1 \leq x < 0\}$ and $F' = \{x \in \mathbb{R} : 0 < x \leq 1\}$, respectively. We still have the equality $f(E') = f(F') = \{y \in \mathbb{R} : 0 < y \leq 1\} = f(E') \cap f(F')$. We also have $E' \cap F' = \emptyset$, and thus $f(E' \cap F') = \emptyset$, and $\emptyset = f(E' \cap F') \subseteq f(E') \cap f(F')$, since the empty set is a subset of all sets.

Section 3

B. Exhibit a one-to-one correspondence between the set O of odd natural numbers and \mathbb{N}

The function $f(x) = \frac{x+1}{2}, x \in \mathbb{N}$ maps the set of odd natural numbers, $O = \{2k - 1 : k \in \mathbb{N}\} \rightarrow \mathbb{N}$.

D. If A is contained in some initial segment of \mathbb{N} , use the well-ordering property of \mathbb{N} to define a bijection of A onto some initial segment of \mathbb{N} .

If $A \neq \emptyset$ is a subset of some initial segment \mathbb{N} , by the well-ordering principle, there exists an $m \in A$ such that $m \leq k$ for all $k \in A$. A bijection f can be defined by the mapping from set A consisting of elements $\{a_1, a_2, \dots, a_k\}$ to elements of some initial segment $S_k = \{1, 2, \dots, k\}$ as a set of ordered pairs $\{(a_1, 1), (a_2, 2), \dots, (a_k, k)\}$, such that $a_1 \leq a_2 \leq \dots \leq a_k$ and clearly the corresponding elements in the pairs from set S_k , $1 \leq 2 \leq \dots \leq k$. Here the number of elements in A and S_k are the same, which has a one-one correspondence $f : A \rightarrow S_k$ and the $R(f) = S_k$.

F. Use the fact that every infinite set has a denumerable subset to show that every infinite set can be put into one-one correspondence with a proper subset of itself.

By definition, having a denumerable subset \implies there exists a bijective function that maps a proper subset of an infinite set, B , onto \mathbb{N} . If we take infinite set $B = \{b_1, b_2, \dots, b_n, \dots\}$ and $B_1 = \{b_2, b_3, \dots, b_n, b_{n+1}, \dots\}$, $B_1 \subseteq B$, we can create a one-one correspondence $f : B \rightarrow B_1$ defined by the set or ordered pairs $\{(b_n, b_{n+1}) : n \in \mathbb{N}\}$ which maps B to $B_1 = B \setminus \{b_1\}$.

H. Show that if the set A can be put into one-one correspondence with a set B , and if B can be put into one-one correspondence with a set C , then A can be put into one-one correspondence with C .

If A can be put into one-one correspondence with a set $B \implies$ there exists an injective function, f from $A \rightarrow B$. This means that for $a, a' \in A$, and $b \in B$, $f(a) = f(a') = b \implies a = a'$. Similarly, if B can be put into one-one correspondence with a set $C \implies$ there exists an injective function, g from $B \rightarrow C$, and with $b, b' \in B$, $g(b) = g(b') = c \in C \implies b = b'$. By these properties, the composition of these two injective functions, $g \circ f(a) = g \circ f(a') \implies f(a) = f(a') \implies a = a'$ putting A and C in one-one correspondence.

I. Using induction on $n \in \mathbb{N}$, show that the initial segment determined by n cannot be put into one-one correspondence with the initial segment determined by $m \in \mathbb{N}$, if $m < n$.

Let $S_n = \{1, 2, 3, \dots, n\}$ be the initial segment determined by $n \in \mathbb{N}$ and S_m be the initial segment determined by $m \in \mathbb{N}, m < n$. If S_n can be put into one-one correspondence with $S_m \implies$ there exists an injection $f : S_n \rightarrow S_m$. For $n = 1$ we have $f : \{1\} \rightarrow S_m$, $m < 1$, but S_m does not exist by definition for $m < 1$ implying the function is not valid for the case $n = 1, m < n$. For, the case $n = k + 1$, we again have a map $f : \{1, 2, \dots, k + 1\} \rightarrow \{1, \dots, m\}$, $m < k + 1$ which implies a mapping of $k + 1$ elements to $m < k + 1$ elements \implies there exists at least two elements $x, x' \in S_{k+1}$ for which $f(x) = f(x')$ and $x \neq x' \implies$ an injection does not exist between these sets.

Section 4

C. Prove part (b) of Theorem 4.4, that is, Let $a \neq 0$ and b be arbitrary elements of \mathbb{R} . Then the equation $a \cdot x = b$ has the unique solution $x = \frac{1}{a}b$

Let x_1 be any solution to the equation, that is, $a \cdot x_1 = b$. By (M4) we have that there is exists for each element $a \neq 0$ in \mathbb{R} there exists an element $\frac{1}{a}$ such that $a(\frac{1}{a}) = 1$. Thus we have $(\frac{1}{a})ax_1 = b(\frac{1}{a}) \implies 1 \cdot x_1 = b(\frac{1}{a}) \implies a \cdot x_1 = b$ has the unique solution $x_1 = \frac{b}{a}$.

F. Use the argument in Theorem 4.7 to show that there does not exist a rational number s such that $s^2 = 6$.

If we assume that $s^2 = (\frac{p}{q})^2 = 6$, where $p, q \in \mathbb{Z}, q \neq 0$ and assume that p and q have no common integral factors, since $p^2 = 2(3q^2) \implies$ that p^2 , and p is even for some $p = 2k, k \in \mathbb{N} \implies p^2 = 4k^2 = 2(3q^2) \implies 2k^2 = 3q^2 \implies q^2$, and q must be even, which is a contradiction of the assumption that p and q have no common integral factors, and thus a rational number s such that $s^2 = 6$ does not exist.

G. Modify the argument in Theorem 4.7 to show there does not exist a rational number t such that $t^2 = 3$.

If we assume that $t^2 = (\frac{p}{q})^2 = 3$, where $p, q \in \mathbb{Z}, q \neq 0$ and assume that p and q have no common integral factors, we have $p^2 = 3q^2$ which implies that p^2 and p are divisible by 3 \implies there exists $k \in \mathbb{N}$ such that $p = 3k \implies p^2 = 9k^2 = 3q^2 \implies 3k^2 = q^2$. This implies that q^2 is also divisible by 3 $\implies q$ is divisible by 3. This is again a contradiction of assumption p and q have no common integral factors, and thus a rational number t such that $t^2 = 3$ does not exist.

H. If $\xi \in \mathbb{R}$ is irrational and $r \in \mathbb{R}, r \neq 0$, is rational, show that $r + \xi$ and $r\xi$ are irrational.

If we take another rational number $c = \frac{a}{b}$, $a, b \in \mathbb{Z}, b \neq 0$, and assume the contradiction that $r + \xi, r = \frac{p}{q}$, $p, q \in \mathbb{Z}, q \neq 0$ is rational, that is $r + \xi = c$, we have $\xi = c - r = \frac{a}{b} - \frac{p}{q} = \frac{aq - bp}{bq}$ where $\frac{aq - bp}{bq}$ is a rational number, but clearly ξ cannot not be equal to a rational number. Similarly for $r\xi = c \implies \xi = \frac{c}{r} = \frac{aq}{bp}$ where $\frac{aq}{bp}$ is clearly a rational number, again implying the contradiction that ξ is equal to a rational number. Thus, by contradiction, $r + \xi$ and $r\xi$ must be irrational.

Section 5

B. If $n \in \mathbb{N}$, show that $n^2 \geq n$ and hence $\frac{1}{n^2} \leq \frac{1}{n}$.

If $n \in \mathbb{N}$, then $n \geq 1 \implies n^2 \geq n$, since $n^2 = n \cdot n \cdot 1 \geq n \cdot 1 \implies n \geq \frac{n \cdot 1}{n \cdot 1} \implies n \geq 1$, a condition of n being a natural number.

C. If $a \geq -1$, $a \in \mathbb{R}$, show that $(1 + a)^n \geq 1 + na$ for all $n \in \mathbb{N}$.

Let S be the set of all $n \in \mathbb{N}$ for which $(1 + a)^n \geq 1 + na$ is true. For $n = 1$ we have $(1 + a)^1 \geq 1 + (1)a = 1 + a$. For $k \in S$, we assume $(1 + a)^k \geq 1 + ka$ is true. For case $n = k + 1$, we have, using the binomial theorem,

$$(1+a)^{k+1} = (1+a)(1+a)^k = (1+a) \sum_{j=0}^k \binom{k}{j} a^j = (1+a) \left(\binom{k}{0} a^0 + \binom{k}{1} a^1 + \dots + \binom{k}{k} a^k \right) = (1+a)(1 + ka + \dots + a^k)$$

This implies, $(1 + a)^{k+1} \geq (1 + a)(1 + ka) = 1 + ka + a + ka^2 = 1 + (k + 1)a + ka^2 \geq 1 + (k + 1)a$, since $ka^2 \geq 0$. Thus, $(1 + a)^{k+1} \geq 1 + (k + 1)a$ holds, for $k + 1 \in S$.

F. Suppose that $0 < c < 1$. If $m \geq n$, $m, n \in \mathbb{N}$, show that $0 < c^m \leq c^n < 1$.

By property 5.6(c), for $a, b, c \in \mathbb{R}$, if $a > b$ and $c > 0$, then $ac > bc$. Applying this property here we have, $0 < c < 1 \implies 1 > c$ and $c > 0 \implies c = 1 \cdot c > c \cdot c = c^2$, thus $0 < c^2 < c < 1 \implies 1 > c$ and $c^2 > 0$, and $c^2 > c^3$, up to $c^k > c^{k+1}$, $k \in \mathbb{N}$. Thus for $m, n \in \mathbb{N}$, $m \geq n$, we have $0 < c^m \leq c^n < 1$.

G. Show that $n < 2^n$ for all $n \in \mathbb{N}$. Hence $(1/2)^n < 1/n$ for all $n \in \mathbb{N}$.

Applying induction, for case $n = 1$ we have true statement $1 < 2^1$. We assume the inequality is valid for $k \in \mathbb{N}$, and for case $n = k + 1$, we have $k + 1 < 2^{k+1} = 2 \cdot 2^k$. For all $k \geq 1$ we have first, $k + 1 \leq k + k = 2k$, and since $2k \leq 2^{k+1}$, i.e. $k \leq 2^k \implies k + 1 \leq 2^{k+1}$. Since the inequality holds for $n = k + 1$, we assume it holds for all $n \in \mathbb{N}$.

K. If $a, b \in \mathbb{R}$ and $b \neq 0$, show that $|a/b| = |a|/|b|$

(i) For the case, $a \geq 0$, $b > 0$, $a \cdot 1/b \geq 0$, and we thus have $|a/b| = |a \cdot 1/b| = a/b = |a| \cdot |1/b|$, thus $a/b = |a|/|b|$.

(ii) For the case, $a \geq 0$, $b < 0$, we have $a/b \leq 0 \forall a, b$, thus $|a/b| = |a \cdot 1/b| = -(a/b) = a \cdot 1/-b$, and $a, -b \in \mathbb{P} \implies a \cdot 1/-b \geq 0$, thus $a/-b = |a|/|b|$.

(iii) For the case, $a \leq 0$, $b < 0$, we have $a/b \geq 0$, $\forall a, b$, thus, $|a/b| = |a \cdot 1/b| = (a/b) = -a \cdot 1/-b$, thus $-a/-b = a/b = |a|/|b|$.

(iv) For the case, $a \leq 0$, $b > 0$ we have $a/b \leq 0 \forall a, b$, thus, $|a/b| = -(a/b) = -a/b = -a/|b| = |a|/|b|$.

L. If $a, b \in \mathbb{R}$, then $|a + b| = |a| + |b|$ if and only if $ab \geq 0$.

$ab \geq 0 \implies a, b \in \mathbb{P}$ or $-a, -b \in \mathbb{P}$. For the case, $a, b \in \mathbb{P}$, we have $|a + b| = a + b = |a| + |b| \quad \forall a, b \in \mathbb{P}$. For the case, $-a, -b \in \mathbb{P}$, we have, $|a + b| = -(a + b) = -a - b = |a| + |b|$.

Section 6

B. C. G. H. J. K.

Section 7

F. G. K.