# Math 4317 (Prof. Swiech, S'18): HW #4

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**Section 20**

*A. Prove that if $f$ is defined for $x \geq 0$ by $f(x) = \sqrt{x}$, then $f$ is continuous at every point of its domain.*

For $f(x) = \sqrt{x}$, $\mathcal{D}(f) = \{x \in \mathbb{R} : x \geq 0\}$, let $a \in \mathcal{D}(f)$.

When $a = 0$, $|f(x) - f(a)| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon$. If we let $\delta(\varepsilon) = \varepsilon^2$, when $x < \varepsilon^2$, $|f(x)| < \varepsilon$.

When $a \neq 0$, $|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = \frac{|\sqrt{x} - \sqrt{a}|}{|\sqrt{x} + \sqrt{a}|}|\sqrt{x} + \sqrt{a}| = \frac{|x-a|}{|\sqrt{x} + \sqrt{a}|} < \frac{|x-a|}{\sqrt{a}} < \varepsilon \implies$ when $|x - a| < \varepsilon\sqrt{a}$, then, $|f(x) - f(a)| < \varepsilon$, thus we can choose $\delta(\varepsilon) = \varepsilon\sqrt{a} \implies f$ is continuous at every point in its domain.

*B. Show that a "polynomial function"; that is, a function $f$ with the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, $x \in \mathbb{R}$ is continuous at every point of $\mathbb{R}$.*

Relying on the properties of algebraic combinations of continuous of functions, we construct $f$ as a combination of continuous functions to show its continuity. Considering the last term of the polynomial function, denoted here, $f_0(x) = a_0$, $f_0(x)$ is a continuous, constant function, since, for any $a \in \mathbb{R}$ we have $|f_0(x) - f_0(a)| = |a_0 - a_0| < \varepsilon = \delta(\varepsilon)$, $\varepsilon > 0$. We consider the second to last term of $f$, $a_1 x$, as a constant, $a_1$ multiplied by the identity function, denoted, $f_1(x) = x$. Since $f_1(x) = x$, for any real number $a \in \mathbb{R}$, we have $|f_1(x) - f_1(a)| = |x - a| < \varepsilon = \delta(\varepsilon)$, $\varepsilon > 0 \implies a_1 f_1(x) = a_1 x$ is continuous.

Relying on the continuity of $f_1(x) = x$ multiplied by any constant, we can construct higher order terms of $f$ through repeated multiplication of $f_1(x)$, e.g. $a_2 \cdot f_1(x) \cdot f_1(x) = a_2 x^2$ and $a_n \prod_{j=1}^{n} f_1(x) = a_n \cdot f_1(x) \cdot f_1(x) \cdot ... \cdot f_1(x) = a_n x^n$, and so on, where each term constructed $a_n x^n$ is continuous on $\mathbb{R}$ since it is constructed via algebraic combinations of continuous functions $\implies f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, is continuous at every point $x \in \mathbb{R}$.

*E. Let $f$ be the function on $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$, $x$ irrational, $f(x) = 1 - x$, $x$ rational. Show that $f$ is continuous at $x = \frac{1}{2}$ and discontinuous elsewhere.*

Considering the point $a = \frac{1}{2}$, we have $f(a) = \frac{1}{2}$, and $|f(x) - f(a)| = |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = |x - a| < \varepsilon = \delta(\varepsilon)$. So if $|f(x) - f(a)| < \varepsilon = \delta(\varepsilon) > 0 \implies |x - a| < \delta(\varepsilon)$, and then we have $f$ continuous at the point $a = \frac{1}{2}$. For the case $a \neq \frac{1}{2}$, $a$ irrational, take a sequence $X = (x_n)$ of rational numbers converging to $a$. Since the sequence $(f(x_n))$ converges to $1 - a$, and we have $f(a) = a$, $f$ is not continuous at irrational points by the Discontinuity Criterion. For the case $a \neq \frac{1}{2}$, $a$ rational, take a sequence $Y = (Y_n)$ of irrational numbers converging to $a$, the sequence (f(y\_n)) converges to $a$, but $f(a) = 1 - a$, which equation is only satisfied when $a = \frac{1}{2}$, thus $f$ is not continuous for rational numbers at any point other than $\frac{1}{2}$.

*F. Let $f$ be continuous on $\mathbb{R} \to \mathbb{R}$. Show that if $f(x) = 0$ for rational $x$, then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Every real point, $x \in \mathbb{R}$ is the limit of a sequence of rational numbers. If $f$ is continuous $\implies$ for a sequence of rational numbers $X = (x_n) \to x$, we have $(f(x_n)) = 0$, for all $n \in \mathbb{N}$. Since $f$ is continuous at each rational point $x \in \mathbb{R}$, we can find $|f(x_n) - f(x)| < \varepsilon$, $\varepsilon > 0$, and $|x_n - a| < \delta(\varepsilon) \implies (f(x_n)) \to f(x) = 0, \forall x \in \mathbb{R}$.

*I. Using the results of the preceding exercise, show that the function $g$, defined on $\mathbb{R} \to \mathbb{R}$ by $g(x) = x\sin(\frac{1}{x})$, $x \neq 0$, $g(x) = 0$, $x = 0$ is continuous at every point. Sketch a graph of this function.*

For the case $a = 0$, we have $|g(x) - g(a)| = |x\sin\frac{1}{x} - 0| = |x||\sin\frac{1}{x}| \leq |x| \cdot 1 < \varepsilon$, $\varepsilon > 0$, since $-1 \leq \sin\frac{1}{x} \leq 1$. So when $|g(x) - g(0)| < \varepsilon = \delta(\varepsilon)$, we then have $|x| = |x - 0| < \delta(\varepsilon) \implies g$ continuous at 0.

For the case $a \neq 0$, we have $|g(x) - g(a)| = |x\sin\frac{1}{x} - a\sin\frac{1}{a}| = |x\sin\frac{1}{x} - a\sin\frac{1}{a} - a\sin\frac{1}{x} + a\sin\frac{1}{x}| = |(x - a)(\sin\frac{1}{x}) + a(\sin\frac{1}{x} - \sin\frac{1}{a})| \leq |x - a||\sin\frac{1}{x}| + |a||\sin\frac{1}{x} - \sin\frac{1}{a}|$, by Triangle Inequality. Since both $|\sin\frac{1}{x}| \leq 1$ and $|\sin\frac{1}{x} - \sin\frac{1}{a}| \leq 1$, we have $|x - a||\sin\frac{1}{x}| + |a||\sin\frac{1}{x} - \sin\frac{1}{a}| \leq |x - a| \cdot 1 + |a| \cdot 1 = |x - a| + |a| < \varepsilon$.

It then follows that if $\delta(\varepsilon) = \varepsilon - |a|$, i.e. $\varepsilon > \delta(\varepsilon) + |a|$, when $|g(x) - g(a)| < \varepsilon$, then $|x - a| < \delta(\varepsilon) \implies$ g continuous at every point in $\mathbb{R}$.

Sketch of function below:
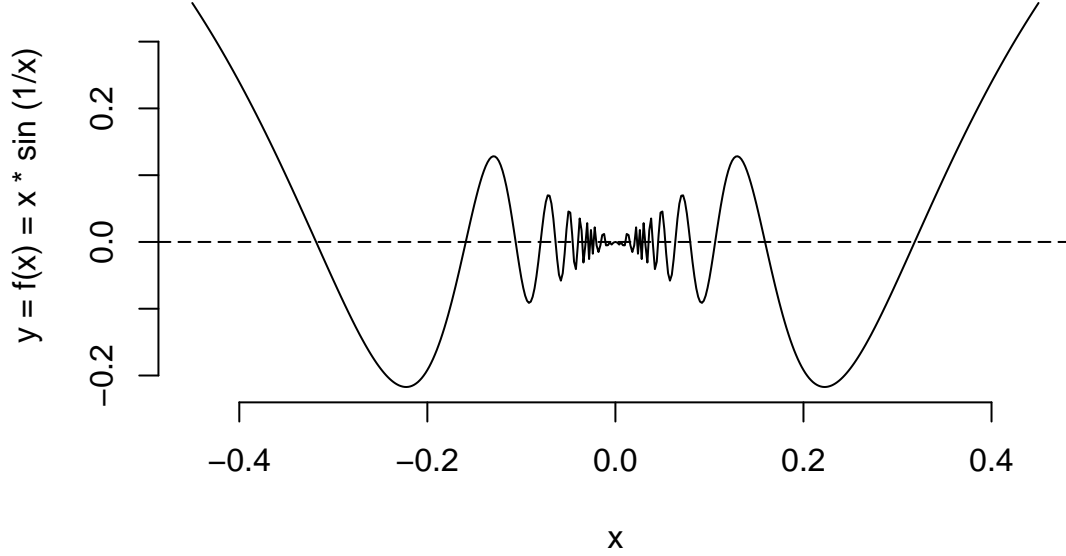


*N. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x + y) = g(x)g(y)$, $x, y \in \mathbb{R}$. Show that if $g$ is continuous at $x = 0$, then $g$ is continuous at every point. Also if $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.*

If $g$ is continuous at $x = 0 \implies g(x + y) = g(y) = g(0) \cdot g(y)$. This implies also that $g(0)g(y) = g(y) \implies$ $g(0)g(y) - g(y) = 0 = g(y)(g(0) - 1) = 0 \implies g(0) = 1$, or that $g(0) = 0$.
If $g(0) = 0 \implies -g(y) = 0 = g(y)$. In this case then $g(y) = 0$, $\forall y \in \mathbb{R} \implies g(x) = 0$, $\forall x \in \mathbb{R}$.
On the other hand if $g(0) = 1$, $\implies g(0) \cdot g(y) = g(y)$ continuous for every point $y \in \mathbb{R}$.

## Section 21

*I. Let $g$ be a linear function from $\mathbb{R}^p \to \mathbb{R}^q$. Show that $g$ is one-one and only if $g(x) = 0$ implies that $x = 0$.*
Since $g$ is linear $\implies$ for $x, y \in \mathbb{R}^p$, $g(x + y) = g(x) + g(y)$. Then if $g(x) = 0 \implies g(x + y) = 0 + g(y) =$ $g(y) \implies g(x + y) = g(y) \implies g(x + y) = g(0 + y) = g(y)$ which implies $x = 0$. If we assume that $g$ is one-one, then for any $g(x) = g(y) \implies x = y$. So in the case $g(x) = 0$, and $g(x + y) = g(x) + g(y) = 0 + g(y)$. Since $g(x) + g(y) = g(y) \implies g(y) - g(x) = g(y) \implies x + y = x - y$, which is satisfied when $x = 0$.

*J. If $h$ is a one-one linear function from $\mathbb{R}^p \to \mathbb{R}^p$, show that the inverse function $h^{-1}$ is a linear function from $\mathbb{R}^p \to \mathbb{R}^p$.*

Since $h$ is one-one $\implies$ if $h(x_1) = h(x_2)$, $x_1 = x_2$, $x_1, x_2 \in \mathbb{R}^p$. Extending the the linear case, we have if $h(ax + by) = h(ax_1 + by_1) = ah(x) + bh(y) = ah(x_1) + bh(y_1)$ then $ax_1 + by_1 = ax + by$. By definition $h^{-1} = \{ax + by : h(ax + by) \in \mathbb{R}^p\} = \{ax : h(ax) \in \mathbb{R}^p\} + \{by : h(by) \in \mathbb{R}^p\}$. This implies $h^{-1}(ax + by) =$ $h^{-1}(h(ax)) + h^{-1}(h(by)) \implies h^{-1}$ is linear, and $h^{-1} : \mathbb{R}^p \to \mathbb{R}^p$, since $h^{-1}(h(ax)) + h^{-1}(h(by)) = ax + by \in \mathbb{R}^p$ by construction.

*K. Show that the sum and the composition of two linear functions are linear functions.*

By definition a function is linear if $f(ax + by) = af(x) + bf(y)$, $a, b \in \mathbb{R}$, $x, y \in \mathbb{R}^p$.

For the sum of two linear functions we then have $(f + g)(ax + by) = f(ax + by) + g(ax + by) = af(x) + bf(y) +$ $ag(x) + bf(y) = a(f(x) + g(x)) + b(f(y) + g(y)) = a(f + g)(x) + b(f + g)(y) \implies$ linearity. For the composition of two linear functions we have $f \circ g(ax + bx) = f(g(ax + by)) = f(ag(x) + bg(y)) = af(g(x)) + bf(g(y)) =$ $a(f \circ g)(x) + b(f \circ g)(y) \implies$ composition of two linear functions is linear.

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*L. If $f$ is a linear map on $\mathbb{R}^p \to \mathbb{R}^q$, define $||f||_{pq} = \sup\{||f(x)|| : x \in \mathbb{R}^p, \; ||x|| \leq 1\}$. Show that the mapping $f \to ||f||_{pq}$ defines a norm on the vector space $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ of all linear functions on $\mathbb{R}^p \to \mathbb{R}^q$. Show that $||f(x)|| \leq ||f||_{pq}||x||$ for all $x \in \mathbb{R}^p$.*

We have $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, $f(x) = y = (y_1, y_2, ..., y_q) \in \mathbb{R}^q$, and matrix $A_{q \times p} = (c_{ij})$, $1 \leq i \leq q$, $1 \leq j \leq p$, with

$$y_1 = c_{11}x_1 + x_{12}x_2 + ... + c_{1p}x_p$$

. . .

$$y_q = c_{q1}x_1 + x_{q2}x_2 + ... + c_{qp}x_p$$

We then have $||f(x)|| = ||(y_1, ..., y_q)|| = \sqrt{y_1^2 + ... + y_q^2}$. To show $||f||_{qp} = \sup\{||f(x)|| : x \in \mathbb{R}^p, \; ||x|| \leq 1\}$ is a norm in $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, we have (i) $||f||_{pq} \geq 0$, $x \in \mathbb{R}^p$? Since each element in $||f(x)|| = \sqrt{y_1^2 + ... + y_q^2}$, $y_j^2 \geq 0$, $\forall j = 1, ..., q \implies \sup\{||f(x)||\} \geq 0 \forall x \in \mathbb{R}^p$ since by definition, $sup\{||f(x)||\} \geq ||f(x)|| \forall x \in \mathbb{R}^p \implies ||f||_{pq} \geq 0$.

(ii) $||f||_{pq} = 0 \iff f(x) = 0$? Since $||f(x)|| = ||y|| = \sqrt{y_1^2 + ... + y_q^2} = 0 \implies$ each $y_j^2 = 0, \forall j = 1, ..., q$

(iii) $\sup ||af(x)|| = |a| \sup ||f(x)|| = |a|||f||_{qp}$, $a \in \mathbb{R}$? We have $||af(x)|| = ||ay|| = \sqrt{a^2y_1^2 + ... + a^2y_1^2} = \sqrt{a^2}||y|| = |a|||y||$, and $|a| > 0 \implies \sup\{||af(x)||\} = \sup\{|a|||f(x)||\} = |a|sup\{||f(x)||\}$.

(iv) $\sup\{||f(x+x^{'})||\} \leq \sup ||f(x)|| + \sup ||f(x^{'})||$, $x, x^{'} \in \mathbb{R}^p$? Since $f$ is linear $||f(x+x^{'})|| = ||f(x)+f(x^{'})|| \leq ||f(x)||+||f(x^{'})||$, $\forall x, x^{'} \in \mathbb{R}^p$ by Triangle Inequality, then $\sup\{||f(x)+f(x^{'})||\} \leq \sup\{||f(x)||\}+\sup\{||f(x^{'})||\}$. This implies $||f||_{qp}$ is a norm.

To show $||f(x)|| \leq ||f||_{pq}||x||$, we use the earlier notation for a linear map, $f(x) = Ax$, where, $A_{q \times p} = (c_{ij})$. Thus $||f(x)|| = ||Ax|| \leq |A|||x||$ as shown in (21.5). This implies $\sup\{||f(x)|| : x \in \mathbb{R}^p, ||x|| \leq 1\} = \sup\{||Ax||\} \leq \sup\{|A|||x||\}$ which is achieved when $x$ is the max value in its domain, i.e. $||x|| = 1$. This implies $\sup\{||Ax||\}||x|| = \sup\{||f(x)||\}||x|| = \sup\{||f(x)||\} \cdot 1$. This implies $||f(x)|| \leq \sup\{||f(x)|| : x \in \mathbb{R}^p, ||x|| \leq 1\}||x||\forall x \in \mathbb{R}^p$.

## Section 22

*B. Let $H : \mathbb{R} \to \mathbb{R}$ be defined by, $h(x) = 1, 0 \leq x \leq 1$. $h(x) = 0,$ otherwise. Exhibit an open set $G$ such that $h^{-1}(G)$ is not open in $\mathbb{R}$, and a closed set $F$, such that $h^{-1}(F)$ is not closed in $\mathbb{R}$.*

If we take $G = (0, 2)$, and open set, $h^{-1}(G) = \{x \in \mathcal{D}(f) : h(x) \in G\} = [0, 1]$, a closed set. If we take $F = [-2, 0]$, a closed set, the inverse image, $h^{-1}(F) = \{x \in \mathcal{D}(f) : h(x) \in F\}$ is the union of two open sets $(-\infty, 0) \cup (1, +\infty)$ which is open.

*C. If $f$ is bounded and continuous on $\mathbb{R}^p \to \mathbb{R}$ and if $f(x_0) > 0$, show that $f$ is strictly positive on some neighborhood of $x_0$. Does the same conclusion hold if $f$ is merely continuous at $x_0$?*

$f$ is bounded and continuous which implies $0 < f(x_0) < M$, for some $M > 0$. Since $f$ is continuous, for each point $a \in \mathcal{D}(f)$, there is a neighborhood $V$ of $f(a)$ and a neigborhood $U(a) \cap D$ such that if $f(a) \in V \implies a \in U(a)$. Since $f(a) > 0 \implies$ we can take a neighborhood $V$ of $f(a)$ that is also strictly positive, i.e. $V = \{y \in \mathbb{R} : 0 < y < M\}$. If $f$ is not bounded the same argument can be made with $V = \{y \in \mathbb{R} : y > 0\}$.

*F. A subset $D \subseteq \mathbb{R}^p$ is disconnected if and only if there exists a continuous function $f : D \to \mathbb{R}$ such that $f(D) = \{0, 1\}$.*

$\to$ $D$ disconnected implies there exists two open sets $B, C$ such that $B \cap D$ and $C \cap D$ are disjoint and $(B \cap D) \cup (C \cap D) = D$. We can then construct a function $f$ on $D$, $f(x) = 1$, $x \in (B \cap D)$, $f(x) = 0$, $x \in (C \cap D)$. $\leftarrow$ Let $f : D \to \mathbb{R}$ be such that $f(D) = \{0, 1\} \implies$ the inverse image $f^{-1}(\{0, 1\}) = \{x \in D \subseteq: f(x) \in \{0, 1\}\}$ could consist of two disjoint open sets such for $f$ on $D$, $f(x) = 1$, $x \in (B \cap D)$, $f(x) = 0$, $x \in (C \cap D)$, where $D = (B \cap D) \cup (C \cap D) \subseteq \mathcal{D}(f) \implies$ there exists a continuous function $f : D \to \mathbb{R}$ such that $f(D) = \{0, 1\}$.

*H. Let $f, g_1, g_2$ be related by the formulas in the preceding exercise. Show that from the continuity of $g_1$ and $g_2$ at $t = 0$ one cannot prove the continuity of $f$ at $(0,0)$.*

Considering $g_1, g_2$ which are valid are restrictions of the domain of $f$, given $x = (x_1, x_2) \in \mathbb{R}^2$, we can construct $f(x) = 0$, $x_1 \cdot x_2 = 0$, $f(x) = 1$, $x_1 \cdot x_2 \neq 1$. With this $f$ we have $\lim_{x \to (0,0)} \neq 0$, and $f((0,0)) = 0 \implies$ discontinuity for $f$ at $(0,0)$. Therefore continuity for $g_1, g_2$ on restrictions of $\mathcal{D}(f)$ does not imply continuity of $f$.

*K. Give an example of a bounded and continuous function $g$ on $\mathbb{R} \to \mathbb{R}$ which does not take on either of the numbers $\sup\{g(x) : x \in \mathbb{R}\}$ or $\inf\{g(x) : x \in \mathbb{R}\}$*

If we take $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$, $x \in (0,1) \subseteq \mathbb{R}$, the function is bounded above by 1, below by 0, and continuous on $(0,1)$, but $f(x) \neq 1 = \sup\{f(x) : x \in (0,1)\}$, and $f(x) \neq 0 = \inf\{f(x) : x \in (0,1)\}$ for any x in interval $(0,1)$.

*O. Let $f$ be a continuous function on $\mathbb{R} \to \mathbb{R}$ which is strictly increasing (in the sense that if $x' < x''$ then $f(x') < f(x'')$). Prove that $f$ is injective and that its inverse function is continuous and strictly increasing.*

For points $x, a, b \in \mathcal{D}(f)$, by $f$ be strictly increasing, we have $a > b \implies f(a) > f(b)$, $a = b \implies f(a) = f(b)$ and $a < b \implies f(a) < f(b)$. If we take point $x$ to be $a < x < b$, we can define two neighborhoods $(a,b) \subseteq \mathcal{D}(f)$, and $(f(a), f(b)) \subseteq \mathcal{R}(f)$, such that $x \in (a,b)$, and $f(x) \in (f(a), f(b))$. This implies $f^{-1}$ in continuous, and since $f^{-1}(f(a)) = a > f^{-1}(f(b)) = b$ if $f(a) > f(b)$, implies $f^{-1}$ is strictly increasing. Also since, $f(a) = f(b) \implies a = b$, $f$ is injective.

## Section 23

*A. Examine each of the functions in Example 20.5 and either show that the function is uniformly continuous on its domain or not.*

*C. If $B$ is bounded in $\mathbb{R}^p$ and $f : B \to \mathbb{R}^p$ is uniformly continuous, show that $f$ is bounded on $B$. Show that this conclusion fails if $B$ is not bounded in $\mathbb{R}^p$.*

*D. Show that $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous.*

*F. Show that $f(x) = \frac{1}{x^2}$, $\mathcal{D}(f) = \{x \in \mathbb{R} : x > 0\}$ is on uniformly continuous on its domain. G. A function $g : \mathbb{R} \to \mathbb{R}^p$ is periodic if there exists a snumber $p > 0$ such that $g(x + p) = g(x)$ for all $x \in \mathbb{R}$. Show that a continuous perioddic function is bounded and uniformly continuous on $\mathbb{R}$.*

*H. Let $f$ be defined on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$, and suppose that $f$ is uniformly continuous on $D$. If $(x_n)$ is a Cauchy sequence in $D$, show that $(f(x\_n))$ is a Cauchy sequence in $\mathbb{R}^q$.*

## Section 24

*B. Give an example of a sequence of continuous functions which converges to a continuous function but where the convergence is not uniform.*

*D. Let $(f_n)$ be a sequence of continuous functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ such that $(f_n)$ converges uniformly to $f$ on $D$, and let $(x_n)$ be a sequence of elements in $D$ which converges to $x \in D$. Does it follow that $(f_n(x_n))$ converges to $f(x)$?*

*E. Consider the sequences $(f_n)$ defined on $D = \{x \in \mathbb{R} : x \geq 0\}$ to $\mathbb{R}$ by the following formulas:*

*(b) $\frac{x^n}{1+x^n}$,*

*(c) $\frac{x^n}{n+x^n}$,*

*(d) $\frac{x^{2n}}{1+x^n}$,*

*(e) $\frac{x^n}{1+x^{2n}}$*

*J. Prove the following theorem of G. Polya. If for each $n \in \mathbb{N}$ the function $f_n$ on $I \to \mathbb{R}$ is monotone increasing and if $f(x) = \lim(f_n(x))$ is continuous on $I$, then the convergence is uniform on $I$. (Observe that it is not assumed that $f_n$ is continuous.)*

*N. If $f_3(x) = x^3$ for $x \in \mathcal{I}$, calculate the $n^{th}$ Bernstein polynomial for $f_3$. Show directly that this sequence of polynomials converges uniformly to $f_3$ on $\mathcal{I}$.*

*S. Show that the Weierstrass Approximation Theorem fails for bounded open intervals.*

**Section 26**

*N. If $K \subseteq \mathbb{R}^p$ is compact and $(f_n)$ is a sequence of continuous functions on $K$ to $\mathbb{R}^q$ which is uniformly convergent on $K$, show that the family $\{f_n\}$ is uniformly equicontinuous on $K$ in the sense of Definition 26.6.*

*O. Let $\mathcal{F}$ be a bounded and uniformly equicontinuous collection of functions on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}$ and let $f$ be defined on $D \to \mathbb{R}$ by $f = \sup\{f(x) : f \in \mathcal{F}\}$. Show that $f$ is continuous on $D \to \mathbb{R}$.*

*Q. Consider the following sequences of functions which show that the Arzela-Ascoli Theorem 26.7 may fail if the various hypotheses are dropped.*

*(a) $f_n(x) = x + n, \ x \in [0, 1]$;*
*(b) $f_n(x) = x^n, \ x \in [0, 1]$;*
*(c) $f_n(x) = \frac{1}{1+(x-n)^2}, \ x \in [0, +\infty)$.*