

# Math 4317 (Prof. Swiech, S'18): HW #2

Peter Williams

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## Section 8

D. If  $w_1$  and  $w_2$  are strictly positive, show that the definition,  $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1w_1 + x_2y_2w_2$ , yields an inner product on  $\mathbb{R}^2$ , generalize this for  $\mathbb{R}^p$ .

Checking the properties of inner products, we have, based on definition above, (i)  $x \cdot x \geq 0$ , since  $(x_1, x_2)(x_1, x_2) = w_1x_1^2 + w_2x_2^2 \geq 0$ , since  $w_1, w_2 > 0$ , and  $x_i^2 \geq 0$ ,  $i = 1, 2$ . For  $x \in \mathbb{R}^p$ , we have  $x \cdot x = \sum_{j=1}^p w_jx_j^2 \geq 0$ , since each element in the summation  $w_i, x_i^2 \geq 0$ . For property (ii), we have  $x \cdot x = 0$ , if and only if  $x = 0$ . In this case, since  $w_1, w_2 > 0$ ,  $w_1x_1^2 + w_2x_2^2 = 0$ , when  $x_1^2$  and  $x_2^2$  equal zero, that is when  $x = 0$ . This holds for  $x \in \mathbb{R}^p$ , since for  $w_i > 0$ ,  $i = 1, \dots, p$  we have  $\sum_{j=1}^p w_jx_j^2 = 0$ , only when each element  $w_ix_i^2 = 0$ , since each element is greater than or equal to zero. For property (iii), we show  $x \cdot y = y \cdot x$  since  $x \cdot y = w_1x_1y_1 + w_2x_2y_2 = w_1x_1y_1 + w_2x_2y_2 = w_1y_1x_2 + w_2y_2x_2 = y \cdot x$ . Extending to  $x \in \mathbb{R}^p$ , we have again, by commutative property,  $x \cdot y = \sum_{j=1}^p w_jx_jy_j = \sum_{j=1}^p w_jy_jx_j = y \cdot x$ . Property (iv),  $x \cdot (y+z) = x \cdot y + x \cdot z$ ,  $x, y, z \in \mathbb{R}^p$ . In this case we have  $\sum_{j=1}^p w_jx_j(y_j+z_j) = \sum_{j=1}^p w_jx_jy_j + \sum_{j=1}^p w_jx_jz_j = \sum_{j=1}^p w_jx_jy_j + \sum_{j=1}^p w_jx_jz_j = x \cdot y + x \cdot z$ , which clearly holds for base case,  $p = 2$  as well. For property (v), we have  $(ax) \cdot y = x \cdot (ay)$ ,  $a \in \mathbb{R}$ . We have  $(ax) \cdot y = \sum_{j=1}^p w_jax_jy_j = a \sum_{j=1}^p w_jx_jy_j = a(x \cdot y) = \sum_{j=1}^p w_jx_jay_j = x \cdot (ay)$ . Since all five properties are satisfied, an inner product is yielded here.

E.  $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1$  is not an inner product on  $\mathbb{R}^2$ . Why?

By property (ii), i.e.  $x \cdot x = 0$  if and only if  $x = 0$ , the definition above,  $(x_1, x_2) \cdot (y_1, y_2) = x_1y_1 = 0 \Leftrightarrow x = 0$ , however, we can't say  $x = 0$ , since in this case if  $x_1y_1 = 0 \Rightarrow x_1 = 0$ , but we don't have information about  $x_2$ , or  $x_i, i = 3, \dots, p$ , for  $x \in \mathbb{R}^p$ . Thus for this operation  $x \cdot x = 0$  does not necessarily mean  $x = 0$ .

F. If  $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$ , define  $\|x\|_1$  by  $\|x\|_1 = |x_1| + |x_2| + \dots + |x_p|$ . Prove that  $x \rightarrow \|x\|_1$  is a norm on  $\mathbb{R}^p$ .

- (i)  $\|x\|_1 \geq 0$ ? Since  $|x_j| \geq 0 \forall j \Rightarrow \|x\|_1 = \sum_{j=1}^p |x_j| \geq 0$  by definition of the absolute value.
- (ii)  $\|x\|_1 = 0$  if and only if  $x = 0$ ?  $\|x\|_1 = \sum_{j=1}^p |x_j| = 0 \Rightarrow x_j = 0 \forall j \Rightarrow x = 0$ .
- (iii)  $\|ax\|_1 = |a|\|x\|_1 \forall a \in \mathbb{R}, x \in V$ ? When  $a \geq 0$ , and  $x_j \geq 0$  or  $a < 0$  and  $x_j < 0$ ,  $\|ax_j\|_1 = ax_j = |a||x_j|$ . For the case  $a < 0$  and  $x_j \geq 0$  or  $a \geq 0$  and  $x_j < 0$ , we have  $\|a_xj\|_1 = |ax_j| = (-1)ax_j$  or  $a(-1)x_j = a|x_j| = |a||x_j|$ .
- (iv)  $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$  for  $x, y \in \mathbb{R}^p$ ?  $\|x+y\|_1 = |x_1+y_1| + |x_2+y_2| + \dots + |x_p+y_p|$ . By the triangle inequality,  $|x_j+y_j| \leq |x_j| + |y_j|$  for all  $j$ . Therefore  $|x_1+y_1| + |x_2+y_2| + \dots + |x_p+y_p| \leq |x_1| + |x_2| + \dots + |x_p| + |y_1| + |y_2| + \dots + |y_p| = \|x\|_1 + \|y\|_1$ . Thus  $\|x\|_1$  is a norm on  $\mathbb{R}^p$ .

G. If  $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$ , define  $\|x\|_\infty$  by  $\|x\|_\infty = \sup\{|x_1| + |x_2| + \dots + |x_p|\}$ . Prove that  $x \rightarrow \|x\|_\infty$  is a norm on  $\mathbb{R}^p$ .

- (i)  $\|x\|_\infty \geq 0$ ? Since  $|x_j| \geq 0 \forall j \Rightarrow \|x\|_\infty = \sup\{|x_1| + |x_2| + \dots + |x_p|\} \geq 0$  since each element in the set is greater than zero.
- (ii)  $\|x\|_\infty = 0$  if and only if  $x = 0$ ?. Since each element in the set  $\{|x_1| + |x_2| + \dots + |x_p|\}$  is greater than or equal to zero,  $\|x\|_\infty = 0$  if and only if  $x_j = 0$  for all  $j$ , which implies  $x = 0$ .
- (iii)  $\|ax\|_\infty = |a|\|x\|_\infty \forall a \in \mathbb{R}, x \in V$ ?  $\|ax\|_\infty = \sup\{|ax_1| + |ax_2| + \dots + |ax_p|\}$ , and as shown in 8.F  $|ax_j| = |a||x_j|$ , which implies  $\|ax\|_\infty = \sup\{|a||x_1| + |a||x_2| + \dots + |a||x_p|\} = |a|\sup\{|x_1| + |x_2| + \dots + |x_p|\} = |a|\|x\|_\infty$ , since  $|a|, |x_j| > 0$ . (iv)  $\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$  for  $x, y \in \mathbb{R}^p$ ?. Again, by the triangle inequality,  $|x_j+y_j| \leq |x_j| + |y_j|$  for all  $j$ . Therefore  $\sup\{|x_1+y_1|, |x_2+y_2|, \dots, |x_p+y_p|\} \leq \sup\{|x_1| + |y_1|, |x_2| + |y_2|, \dots, |x_p| + |y_p|\}$ . If we take  $u_x = \sup\{|x_j|\}, u_y = \sup\{|y_j|\}$ .  $u_x + u_y \geq |x_j| + |y_j|$  for all  $j \Rightarrow \sup\{|x_j| + |y_j|\} = \sup\{|x_j| + |y_j|\} \Rightarrow \|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ . Thus,  $\|x\|_\infty$  is a norm on  $\mathbb{R}^p$ .

H. In the set  $\mathbb{R}^2$ , describe the sets:

$S_1 = \{x \in \mathbb{R}^2 : \|x\|_1 < 1\}$ .  $\|x\|_1 = \sqrt{x_1^2 + x_2^2} < 1$  describes an open circle consisting of points less than 1 in all directions from the origin, satisfying the inequality,  $\sqrt{x_1^2} < \sqrt{1 - x_2^2}$ .  $S_\infty = \{x \in \mathbb{R}^2 : \|x\|_\infty < 1\}$ , where  $\|x\|_\infty = \sup\{|x_1|, |x_2|\}$ , is a dense open box with vertices at  $(1, 1), (-1, 1), (-1, -1), (1, -1)$  with  $-1 < x_1 < 1$ , and  $-1 < x_2 < 1$ .

P. If  $x, y$  belongs to  $\mathbb{R}^p$ , show that  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$  if and only if  $x \cdot y = 0$ .

$\|x + y\|^2 = (x + y) \cdot (x + y) = x \cdot x + y \cdot x + x \cdot y + y \cdot y = \|x\|^2 + 2x \cdot y + \|y\|^2$ , and  $2x \cdot y = 0$  if and only if  $x \cdot y = 0$ , thus, in order for  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$  to hold,  $x \cdot y$  must equal zero.

Q. A subset  $K$  of  $\mathbb{R}^p$  is said to be convex if, whenever,  $x, y \in K$ , and  $t$  is a real number such that  $0 \leq t \leq 1$ , then the point  $tx + (1 - t)y$  also belongs to  $K$ . Show that  $K_1, K_2, K_3$  are convex, but that  $K_4$  is not.

- 1)  $K_1 = \{x \in \mathbb{R}^2 : \|x\| < 1\}$ . Let  $x, y \in K_1$ , then  $\|tx + (1 - t)y\| \leq \|tx\| + \|(1 - t)y\| = |t|\|x\| + (1 - t)\|y\|$ , and since  $\|x\| \leq 1$  and  $\|y\| \leq 1$ , it implies  $|t|\|x\| + (1 - t)\|y\| \leq |t|(1) + (1 - t)(1) = t + 1 - t = 1 \implies tx + (1 - t)y \in K_1$ .
- 2) For  $K_2 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < \eta\}$ . Let  $x = (x_1, x_2), y = (y_1, y_2) \in K_2 \implies 0 < x_1 < x_2$  and  $0 < y_1 < y_2$ , for the point  $tx + (1 - t)y$  to belong in  $K_2$  it implies for  $t \in [0, 1] \implies 0 < tx_1 < tx_2$ , and  $0 < (1 - t)y_1 < (1 - t)y_2$ . Adding these inequalities, we have for  $tx + (1 - t)y$ ,  $0 < tx_1 + (1 - t)y_1 < tx_2 + (1 - t)y_2 \implies tx + (1 - t)y \in K_2$ .
- 3) Similarly for  $K_3 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 \leq \xi \leq \eta \leq 1\}$ ,  $x, y \in K_3$ ,  $t \in [0, 1]$ , we have  $0 \leq x_1 \leq x_2 \leq 1$  and  $0 \leq y_1 \leq y_2 \leq 1 \implies 0 \leq tx_1 \leq tx_2 \leq t$  and  $0 \leq (1 - t)y_1 \leq (1 - t)y_2 \leq (1 - t)$ , again adding the inequalities, we have  $0 \leq tx_1 + (1 - t)y_1 \leq tx_2 + (1 - t)y_2 \leq t + (1 - t) = 1 \implies tx + (1 - t)y \in K_3$ .
- 4) For  $K_4 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ . Like in  $K_1$ ,  $x, y \in K_4$ , then  $\|tx + (1 - t)y\| = |t|\|x\| + \|(1 - t)y\| = |t|\|x\| + (1 - t)\|y\|$ , and since  $\|x\| \leq 1$  and  $\|y\| \leq 1$ , it implies  $|t|\|x\| + (1 - t)\|y\| \leq |t|(1) + (1 - t)(1) = 1$ . This equality could hold in some cases where  $\|x\| = 1$ , e.g.  $(1, 0), (0, 1)$ , but does not hold for all points, and thus  $K_4$  is not convex.

## Section 9

B. Justify assertions from 9.2(c):

- (i) Denote  $x = (x_1, x_2)$  the set  $G = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$  which is equivalent to  $G = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} = \|x\| < 1\}$ . Let  $\varepsilon = 1 - \|x\| > 0$ . Take  $y \in \mathbb{R}^2$  such that  $\|y - x\| < 1$ , then, by triangle inequality  $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| < \varepsilon + \|x\| = 1 - \|x\| + \|x\| = 1 \implies y \in G$ , and thus  $G$  is open.
- (ii) Take  $x = (x_1, x_2)$ , and  $H = \{x \in \mathbb{R}^2 : 0 < \|x\|^2 < 1\}$ . Take  $y \in \mathbb{R}^2$  such that  $\|y - x\| < \varepsilon$ , where  $\varepsilon = \inf\{\|x\|, 1 - \|x\|\}$ . Again  $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| < \varepsilon + \|x\| = 1 - \|x\| + \|x\| = 1 \implies \|y\| < 1$ . With  $\|x - y\| < \varepsilon \implies \|x\| - \|y\| < \varepsilon \implies \|y\| > \|x\| - \varepsilon \implies \|y\| > \|x\| - \|x\| \implies \|y\| > 0 \implies y \in H$ , and  $H$  is open.
- (iii)  $F = \{x \in \mathbb{R}^2 : \|x\|^2 \leq 1\}$ . The complement of  $F$ ,  $F^c = \{x \in \mathbb{R}^2 : \|x\|^2 > 1\}$  is open, since for  $\varepsilon = \|x\| - 1 > 0$ ,  $y \in \mathbb{R}^2$ ,  $\|x - y\| > \|x\| - \|y\| < 1 \implies \|x\| - \varepsilon < \|y\| \implies 1 < \|y\| \implies y \in F^c \implies F^c$  is open, and its complement  $F$  must be closed as a result.

D. What are the interior, boundary, and exterior points in  $\mathbb{R}$  of the set  $[0, 1]$ . Conclude that it is neither open nor closed. Let  $A = [0, 1]$ . The interior points of  $A$  consist of points in the open interval  $(0, 1)$  which is entirely contained in  $A$ . The boundary points of  $A$  are the points 0 and 1. Since neighborhoods around the point 1 and 0 contain both points in  $A$  and in its complement  $A^c$ . The exterior points of  $A$  are points in the set consisting of the union of the intervals  $(-\infty, 0) \cup [1, \infty)$ .  $A$  is not closed, since it does not contain the boundary point, 1.  $A$  is not open, by construction, since it is the union of an open and closed set or interval.

G. Show that a subset of  $\mathbb{R}^p$  is open if and only if it is the union of a countable collection of open balls.

I. Show every closed subset of  $\mathbb{R}^p$  is the intersection of a countable collection of open sets.

J. If  $A$  is any subset of  $\mathbb{R}^p$ , let  $A^0$  denote the union of all open sets which are contained in  $A$ ; the set  $A^0$

is called the interior of  $A$ . Note that  $A^0$  is an open set; (i) prove that it is the largest open set contained in  $A$ , also prove: (ii)  $A^0 \subseteq A$ , (iii)  $(A^0)^0 = A^0$ , (iv)  $(A \cap B)^0 = A^0 \cap B^0$ , and (v)  $(\mathbb{R}^p)^0 = \mathbb{R}^p$ . Also give an example to show  $(A \cup B)^0 = A^0 \cup B^0$  may not hold.

K. Prove that a point belongs to  $A^0$  if and only if it is an interior point of  $A$ . L. If  $A$  is any subset of  $\mathbb{R}^p$ , let  $A^-$  denote the intersection of all closed sets which contain  $A$ ; the set  $A^-$  is called the closure of  $A$ . Note that  $A^-$  is a closed set; (i) prove that it is the smallest closed set containing  $A$ , prove that: (ii)  $A \subseteq A^-$ , (iii)  $(A^-)^- = A^-$ , (iv)  $(A \cup B)^- = A^- \cup B^-$ , and (v)  $\emptyset^- = \emptyset$ . M. Prove that a point belongs to  $A^-$  if and only if it is either an interior or boundary point of  $A$ . ##### Section 10 ##### Section 11 ##### Section 12