Math 4317 (Prof. Swiech, S'18): HW #2

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Section 8

D. If w_1 and w_2 are strictly positive, show that the definition, $(x_1, x_2)\cdot(y_1, y_2) = x_1y_1w_1 + x_2y_2w_2$, yields an inner product on $\mathbb{R}^2$, generalize this for $\mathbb{R}^p$.

Checking the properties of inner products, we have, based on definition above, (i) $x \cdot x \geq 0$, since $(x_1, x_2)(x_1, x_2) = w_1x_1^2 + w_2x_2^2 \geq 0$, since $w_1, w_2 > 0$, and $x_i^2 > 0$, $i = 1, 2$. For $x \in \mathbb{R}^p$, we have $x \cdot x = \sum_{j=1}^{p} w_jx_j^2 \geq 0$, since each element in the summation $w_i, x_i^2 > 0$. For property (ii), we have $x \cdot x = 0$, if and only if $x = 0$. In this case, since $w_1, w_2 > 0$, $w_1x_2^2 + w_2x_2^2 = 0$, when x_1^2 and x_2^2 equal zero, that is when $x = 0$. This holds for $x \in \mathbb{R}^p$, since for $w_i > 0,\ \ i = 1, ..., p$ we have $\sum_{j=1}^{p} w_jx_j^2 = 0$, only when each element $w_ix_i2 = 0$, since each element is greater than or equal to zero. For property (iii), we show $x\cdot y = y\cdot x$ since $x\cdot y = w_1x_1y_1 + w_2x_2y_2 = w_1x_1y_1 + w_2x_2y_2 = w_1y_1x_2 + w_2y_2x_2 = y\cdot x$. Extending to $x \in \mathbb{R}^p$, we have again, by commutative property, $x\cdot y = \sum_{j=1}^{p} w_jx_jy_j = \sum_{j=1}^{p} w_jy_jx_j = y\cdot x$. Property (iv), $x\cdot(y+z) = x\cdot y + x\cdot z,\ x, y, z \in \mathbb{R}^p$. In this case we have $\sum_{j=1}^{p} w_jx_j(y_j+z_j) = \sum_{j=1}^{p} w_jx_jy_j + w_jx_jz_j = \sum_{j=1}^{p} w_jx_jy_j + \sum_{j=1}^{p} w_jx_jz_j = x\cdot y + x\cdot z$, which clearly holds for base case, $p = 2$ as well. For property (v), we have $(ax)\cdot y = x\cdot(ay),\ a \in \mathbb{R}$. We have $(ax)\cdot y = \sum_{j=1}^{p} w_jax_jy_j = a\sum_{j=1}^{p} w_jx_jy_j = a(x\cdot y) = \sum_{j=1}^{p} w_jx_jay_j = x\cdot(ay)$. Since all five properties are satisfied, an inner product is yielded here.

E. $(x_1, x_2)\cdot(y_1, y_2) = x_1y_1$ is not an inner product on $\mathbb{R}^2$. Why?

By property (ii), i.e. $x\cdot x = 0$ if and only if $x = 0$, the definition above, $(x_1, x_2)\cdot(y_1, y_2) = x_1y_1 = 0 \Leftrightarrow x = 0$, however, we can't say $x = 0$, since in this case if $x_1y_1 = 0 \implies x_1 = 0$, but we don't have information about x_2, or $x_i, i = 3, ..., p$, for $x \in \mathbb{R}^p$. Thus for this operation $x\cdot x = 0$ does not necessarily mean $x = 0$.

F. If $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, define $||x||_1$ by $||x||_1 = |x_1| + |x_2| + ... + |x_p|$. Prove that $x \to ||x||_1$ is a norm on $\mathbb{R}^p$.

(i) $||x||_1 \geq 0$?. Since $|x_j| \geq 0\ \forall j \implies ||x|| = \sum_{j=1}^{p} |x_j| \geq 0$ by definition of the absolute value.

(ii) $||x||_1 = 0$ if and only if $x = 0$?. $||x|| = \sum_{j=1}^{p} |x_j| = 0 \implies x_j = 0\ \forall j \implies x = 0$.

(iii) $||ax||_1 = |a|||x||\ \forall a \in \mathbb{R},\ x \in V$? When $a \geq 0$, and $x_j \geq 0$ or $a < 0$ and $x_j < 0$, $||ax_j|| = ax_j = |a||x_j|$. For the case $a < 0$ and $x_j \geq 0$ or $a \geq 0$ and $x_j < 0$, we have $||a_x)j|| = |ax_j| = (-1)ax_j$ or $a(-1)x_j = a|x_j| = |a||x_j|$.

(iv) $||x + y||_1 \leq ||x|| + ||y||$ for $x, y \in \mathbb{R}^p$?. $||x + y|| = |x_1 + y_1| + |x_2 + y_2| + ... + |x_p + y_p|$. By the triangle inequality, $|x_j + y_j| \leq |x_j| + |y_j|$ for all j. Therefore $|x_1 + y_1| + |x_2 + y_2| + ... + |x_p + y_p| \leq |x_1| + |x_2| + ... + |x_p| + |y_1| + |y_2| + ... + |y_p| = ||x|| + ||y||$. Thus $||x||_1$ is a norm on $\mathbb{R}^p$.

G.If $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$, define $||x||_\infty$ by $||x||_\infty = sup\{|x_1| + |x_2| + ... + |x_p|\}$. Prove that $x \to ||x||_\infty$ is a norm on $\mathbb{R}^p$.

(i) $||x||_\infty \geq 0$? Since $|x_j| \geq 0\ \forall j \implies ||x||_\infty = \sup\{|x_1| + |x_2| + ... + |x_p|\} \geq 0$ since each element in the set is greater than zero.

(ii) $||x||_\infty = 0$ if and only if $x = 0$?. Since each element in the set $\{|x_1| + |x_2| + ... + |x_p|\}$ is greater than or equal to zero, $||x||_\infty = 0$ if and only if $x_j = 0$ for all j, which implies $x = 0$.

(iii) $||ax||_\infty = |a|||x||_\infty\ \forall a \in \mathbb{R},\ x \in V$? $||ax||_\infty = \sup\{|ax_1| + |ax_2| + ... + |ax_p|\}$, and as shown in 8.F $|ax_j| = |a||x_j|$, which implies $||ax||_\infty = \sup\{|a||x_1| + |a||x_2| + ... + |a||x_p|\} = |a|\sup\{|x_1| + |x_2| + ... + |x_p|\} = |a|||x||_\infty$, since $|a|, |x_j| > 0$. (iv)$||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$ for $x, y \in \mathbb{R}^p$?. Again, by the triangle inequality, $|x_j + y_j| \leq |x_j| + |y_j|$ for all j. Therefore $\sup\{|x_1 + y_1|, |x_2 + y_2|, ..., |x_p + y_p|\} \leq \sup\{|x_1| + |y_1|, |x_2| + |y_2|, ..., |x_p| + |y_p|\}$. If we take $u_x = \sup\{|x_j|\}$,$u_y = \sup\{|y_j|\}$. $u_x + u_y \geq |x_j| + |y_j|$ for all $j \implies \sup\{|x_j|\} + \sup\{|y_j|\} = \sup\{|x_j| + |y_j|\} \implies ||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$. Thus, $||x||_\infty$ is a norm on $\mathbb{R}^p$.

H. In the set $\mathbb{R}^2$, describe the sets:

$S_1 = \{x \in \mathbb{R}^2 : ||x||_1 < 1\}$. $||x||_1 = \sqrt{x_1^2 + x_2^2} < 1$ describes and open circle consisting of points less than 1 in all directions from the origin, satisfying the inequality, $\sqrt{x_1^2} < \sqrt{1 - x_2^2}$. $S_\infty = \{x \in \mathbb{R}^2 : ||x||_\infty < 1\}$, where $||x||_\infty = \sup\{|x_1|, |x_2|\}$, is a dense open box with vertices vertices at $(1, 1), (-1, 1), (-1, -1), (1, -1)$ with $-1 < x_1 < 1$, and $-1 < x_2 < 1$.

P. If x, y belongs to $\mathbb{R}^p$, show that $||x + y||^2 = ||x||^2 + ||y||^2$ if and only if $x \cdot y = 0$.

$||x+y||^2 = (x+y)\cdot(x+y) = x \cdot x + y \cdot x + x \cdot +y + y \cdot y = ||x||^2 + 2x \cdot y + ||y||^2$, and $2x \cdot y = 0$ if and only if $x \cdot y = 0$, thus, in order for $||x + y||^2 = ||x||^2 + ||y||^2$ to hold, $x \cdot y$ must equal zero.

Q. A subset K of $\mathbb{R}^p$ is said to be convex if, whenever, $x, y \in K$, and t is a real number such that $0 \le t \le 1$, then the point $tx + (1 - t)y$ also belongs to K. Show that K_1, K_2, K_3 are convex, but that K_4 is not.

1) $K_1 = \{x \in \mathbb{R}^2 : ||x|| < 1\}$. Let $x, y \in K_1$, then $||tx+(1-t)y|| \le ||tx||+||(1-t)y|| = |t|||x||+|(1-t)|||y||$, and since $||x|| \le 1$ and $||y|| \le 1$, it implies $|t|||x|| + |(1 - t)|||y|| \le |t|(1) + |(1 - t)|(1) = t + 1 - t = 1 \implies tx + (1 - t)y \in K_1$.
2) For $K_2 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < \eta\}$. Let $x = (x_1, x_2), y = (y_1, y_2) \in K_2 \implies 0 < x_1 < x_2$ and $0 < y_2 < y_2$, for the point $tx + (1 - t)y$ to belong in K_2 it implies for $t \in [0, 1] \implies 0 < tx_1 < tx_2$, and $0 < (1 - t)y_1 < (1 - t)y2$. Adding these inequalities, we have for $tx + (1 - t)y$, $0 < tx_1 + (1 - t)y_1 < tx_2 + (1 - t)y_2 \implies tx + (1 - t)y \in K_2$.
3) Similarly for $K_3 = \{(\xi, \eta) \in \mathbb{R}^2 : 0 \le \xi \le \eta \le 1\}$, $x, y \in K_3$, $t \in [0, 1]$, we have $0 \le x_1 \le x_2 \le 1$ and $0 \le y_1 \le y_2 \le 1 \implies 0 \le tx_1 \le tx_2 \le t$ and $0 \le (1 - t)y_1 \le (1 - t)y_2 \le (1 - t)$, again adding the inequalities, we have $0 \le tx_1 + (1 - t)y_1 \le tx_2 + (1 - t)y_2 \le t + (1 - t) = 1 \implies tx + (1 - t)y \in K_3$.
4) For $K_4 = \{x \in \mathbb{R}^2 : ||x|| = 1\}$. Like in K_1, $x, y \in K_4$, then $||tx + (1 - t)y|| = ||tx|| + ||(1 - t)y|| = |t|||x||+|(1-t)|||y||$, and since $||x|| \le 1$ and $||y|| \le 1$, it implies $|t|||x||+|(1-t)|||y|| \le |t|(1)+|(1-t)|(1) = 1$. This equality could hold in some cases where $||x|| = 1$, e.g. $(1, 0), (0, 1)$, but does not hold for all points, and thus K_4 is not convex.

Section 9

B. Justify assertions from 9.2(c):

(i) Denote $x = (x_1, x_2)$ the set $G = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ which is equivalent to $G = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} = ||x|| < 1\}$. Let $\varepsilon = 1 - ||x|| > 0$. Take $y \in \mathbb{R}^2$ such that $||y - x|| < 1$, then, by triangle inequality $||y|| = ||y - x + x|| \le ||y - x|| + ||x|| < \varepsilon + ||x|| = 1 - ||x|| + ||x|| = 1 \implies y \in G$, and thus G is open.

(ii) Take $x = (x_1, x_2)$, and $H = \{x \in \mathbb{R}^2 : 0 < ||x||^2 < 1\}$. Take $y \in \mathbb{R}^2$ such that $||y - x|| < \varepsilon$, where $\varepsilon = \inf\{||x||, 1 - ||x||\}$. Again $||y|| = ||y - x + x|| \le ||y - x|| + ||x|| < \varepsilon + ||x|| = 1 - ||x|| + ||x|| = 1 \implies ||y|| < 1$. With $||x - y|| < \varepsilon \implies ||x|| - ||y|| < \varepsilon \implies ||y|| > ||x|| - \varepsilon \implies ||y|| > ||x|| - ||x|| \implies ||y|| > 0 \implies y \in H$, and H is open.

(iii) $F = \{x \in \mathbb{R}^2 : ||x||^2 \le 1\}$. The complement of F, $F^c = \{x \in \mathbb{R}^2 : ||x||^2 > 1\}$ is open, since for $\varepsilon = ||x|| - 1 > 0$, $y \in \mathbb{R}^2$, $||x - y|| > ||x|| - ||y|| < 1 \implies ||x|| - \varepsilon < ||y|| \implies 1 < ||y|| \implies y \in F^c \implies F^c$ is open, and its complement F must be closed as a result.

D. What are the interior, boundary, and exterior points in $\mathbb{R}$ of the set $[0, 1)$. Conclude that it is neither open nor closed.

Let $A = [0, 1)$. The interior points of A consist of points in the open interval $(0, 1)$ which is entirely contained in A. The boundary points of A are the points 0 and 1. Since neighborhoods around the point 1 and 0 contain both points in A and in its complement A^c. The exterior points of A are points in the set consisting of the union of the intervals $(-\infty, 0) \cup [1, \infty)$. A is not closed, since it does not contain the boundary point, 1. A is not open, by construction, since it is the union of an open and closed set or interval.

G. Show that a subset of $\mathbb{R}^p$ is open if and only if it is the union of a countable collection of open balls.

Let $U \subseteq \mathbb{R}^p$ be open, and $\{x_n : n \in \mathbb{N}\}$ be the set of all rational points in U. Since U is open $\implies$ there exists $r > 0$, such that each point x_n can be contained in the open ball $B_r(x_n) = \{y \in \mathbb{R}^p : |y - x_n| < r\}$, such that $B_r(x_n) \subseteq U \implies \cup_{n\in\mathbb{N}} B_r(x_n) \subseteq U$ if we choose r large enough.

Let $U \subseteq \mathbb{R}^p$ be a countable collection of open balls $\implies$ for every rational point x_n, there exists an open ball $B_r(x_n),\ r > 0$, where $x_n \in B_r(x_n) \implies U \subseteq \cup_{n\in\mathbb{N}} B_r(x_n)$. Which implies $U =\subseteq \cup_{n\in\mathbb{N}} B_r(x_n)$.

I. Show every closed subset of $\mathbb{R}^p$ is the intersection of a countable collection of open sets.

If $U \subseteq \mathbb{R}^p$ is a closed subset, i.e. for $y \in \mathbb{R}^p,\ x \in U,\ r_c > 0,\ U = \{y : ||x-y|| \leq r_c\}$, take the open set $\{y : ||x-y|| > r_c + 1/n\},\ n \in \mathbb{N} \implies x \in U \subseteq \cap_{n\in N}\{y : ||x-y|| < r_c + 1/n\}$.

If $x \notin U \implies x \in \mathbb{R}^p \setminus U \implies x \in \{y : ||x-y|| > r_c\} \implies x \notin \{y : ||x-y|| > r_c + 1/n\},\ n \in \mathbb{N} \implies x \in \mathbb{R}^p \setminus \cap_{n\in N}\{y : ||x-y|| > r_c + 1/n\} \implies \mathbb{R}^p \setminus U \subseteq \cap_{n\in N}\{y : ||x-y|| > r_c + 1/n\} \implies \cap_{n\in N}\{y : ||x-y|| > r_c + 1/n\} \subseteq U$. Thus $U = \cap_{n\in N}\{y : ||x-y|| > r_c + 1/n\}$.

J. If A is any subset of $\mathbb{R}^p$, let A^0 denote the union of all open sets which are contained in A; the set A^0 is called the interior of A Note that A^0 is an open set; (i) prove that it is the largest open set contained in A, also prove: (ii) $A^0 \subseteq A$, (iii) $(A^0)^0 = A^0$, (iv) $(A \cap B)^0 = A^0 \cap B^0$, and (v) $(\mathbb{R}^p)^0 = \mathbb{R}^p$. Also give and example to show $(A \cup B)^0 = A^0 \cup B^0$ may not hold.

(i) Take U as any open set contained in A. A^0 by definition is a union of all these sets, thus each $U \subseteq A^0 \implies A^0 \subseteq A$.

(ii) By definition $(A^0)^0 \subseteq A^0$, and since $(A^0)^0$ is by definition, the union of all open sets in $A^0 \implies A^0 \subseteq (A^0)^0 \implies A^0 = (A^0)^0$.

(iii) $(A \cap B)^0$ is the union of all open sets in $A \cap B \implies (A \cap B)^0 \subseteq A \cap B \implies (A \cap B)^0 \subseteq A$ and $(A \cap B)^0 \subseteq B$. Since A^0, B^0 contain all their open sets $\implies (A \cap B)^0 \subseteq A^0$ and that $(A\cap B)^0 \subseteq B^0 \implies (A\cap B)^0 \subseteq A^0 \cap B^0$. In the other direction, $A^0 \subseteq A, B^0 \subseteq B \implies A^0 \cap B^0 \subseteq (A \cap B)$, and since $A^0 \cap B^0$ is the intersection of two open sets, it follows that $A^0 \cap B^0 \subseteq (A \cap B)^0$. This implies $(A \cap B)^0 = A^0 \cap B^0$.

(iv) $\mathbb{R}^p$ is an open set, and equals the collection of all open sets in it, which implies $\mathbb{R}^p = (\mathbb{R}^p)^0$.

(v) Example that $(A \cup B)^0 = A^0 \cup B^0$ may not hold. If we take $A = [0,1], B = [1,2] \implies A^0 = (0,1), B^0 = (1,2) \implies A^0 \cup B^0 = (0,1) \cup (1,2),\ (A \cup B)^0 = (0,2) \implies \{1\} \in (A \cup B)^0,\ \{1\} \notin A^0 \cup B^0$.

K.Prove that a point belongs to A^0 if and only if it is an interior point of A.
Let x be an interior point of $A \implies x$ can be contained in an open set in A, and since A^0 is the union of all open sets in $A \implies x \in A^0$. Let x belong to $A^0 \implies$ belongs to an open set that is contained in $A^0 \implies x$ is an interior point in $A^0 implies$ x in an interior point of A.

L. If A is any subset of $\mathbb{R}^p$, let A^0 denote the intersection of all closed sets which are containing A; the set A^- is called the closure of A Note that A^- is an closed set; (i) prove that it is the smallest closed set containing A, prove that : (ii) $A \subseteq A^-$, (iii) $(A^-)^- = A^-$, (iv) $(A \cup B)^- = A^- \cup B^-$, and (v) $\emptyset^- = \emptyset$
(i) Since A^- is an intersection of all closed sets containing A, including the smallest closed set containing A, A^- must be the smallest closed set containing A. This implies that a closed set $A \subseteq A^-$.
(ii) Since A^- is closed the smallest closed set that contains A^- is $A^- \implies A^- \supseteq (A^-)^-$ and $A^- \subseteq (A^-)^- \implies A^- = (A^-)^-$.
(iii) Let point $x \in (A \cup B)^- = A^- \cup B^- \implies x$ belongs to the smallest closed set containing A or B $\implies x \in A^-$ or $x \in B^- \implies x \in A^- \cup B^-$.
(iv) Since $\emptyset$ is closed and contains no elements, the smallest losed set containing $\emptyset$ is $\emptyset^- \implies \emptyset^- = \emptyset$.

M. Prove that a point belongs to A^- if and only if it is either and interior or boundary point of A.

Section 10

Section 11

Section 12