(a) If $\sigma_c^2 = 9$, use (6.2.34) to fill in the following table:

ϵ	0	0.05	0.10	0.15
$e(Q_2, \overline{X})$				

(b) Notice from the table that the sample median becomes the "better" estimator when ϵ increases from 0.10 to 0.15. Determine the value for ϵ where this occurs [this involves a third-degree polynomial in ϵ, so one way of obtaining the root is to use the Newton algorithm discussed around expression (6.2.32)].

6.2.7. Let X have a gamma distribution with $\alpha = 4$ and $\beta = \theta > 0$.

(a) Find the Fisher information $I(\theta)$.

(b) If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ.

(c) What is the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta)$?

6.2.8. Let X be $N(0, \theta)$, $0 < \theta < \infty$.

(a) Find the Fisher information $I(\theta)$.

(b) If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ.

(c) What is the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta)$?

6.2.9. If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pdf

$$f(x;\theta) = \begin{cases} \frac{3\theta^3}{(x+\theta)^4} & 0 < x < \infty, 0 < \theta < \infty \\ 0 & \text{elsewhere,} \end{cases}$$

show that $Y = 2\overline{X}$ is an unbiased estimator of θ and determine its efficiency.

6.2.10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(0, \theta)$ distribution. We want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c\sum_{i=1}^n |X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine its efficiency.

6.2.11. Let $\overline{X}$ be the mean of a random sample of size n from a $N(\theta, \sigma^2)$ distribution, $-\infty < \theta < \infty, \sigma^2 > 0$. Assume that σ^2 is known. Show that $\overline{X}^2 - \frac{\sigma^2}{n}$ is an unbiased estimator of θ^2 and find its efficiency.

6.2.12. Recall that $\widehat{\theta} = -n/\sum_{i=1}^n \log X_i$ is the mle of θ for a beta$(\theta, 1)$ distribution. Also, $W = -\sum_{i=1}^n \log X_i$ has the gamma distribution $\Gamma(n, 1/\theta)$.

(a) Show that $2\theta W$ has a $\chi^2(2n)$ distribution.

(b) Using part (a), find c_1 and c_2 so that

$$P\left(c_1 < \frac{2\theta n}{\widehat{\theta}} < c_2\right) = 1 - \alpha,$$

for $0 < \alpha < 1$. Next, obtain a $(1-\alpha)100\%$ confidence interval for θ.

(c) For $\alpha = 0.05$ and $n = 10$, compare the length of this interval with the length of the interval found in Example 6.2.6.

6.2.13. By using expressions (6.2.21) and (6.2.22), obtain the result for the one-step estimate discussed at the end of this section.

6.2.14. Let S^2 be the sample variance of a random sample of size $n > 1$ from $N(\mu, \theta)$, $0 < \theta < \infty$, where μ is known. We know $E(S^2) = \theta$.

(a) What is the efficiency of S^2?

(b) Under these conditions, what is the mle $\widehat{\theta}$ of θ?

(c) What is the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta)$?

6.3 Maximum Likelihood Tests

The last section presented an inference for pointwise estimation and confidence intervals based on likelihood theory. In this section, we present a corresponding inference for testing hypotheses.

As in the last section, let $X_1, \ldots, X_n$ be iid with pdf $f(x;\theta)$ for $\theta \in \Omega$. In this section, θ is a scalar, but in Sections 6.4 and 6.5 extensions to the vector-valued case are discussed. Consider the two-sided hypotheses

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta \neq \theta_0, \tag{6.3.1}$$

where θ_0 is a specified value.

Recall that the likelihood function and its log are given by

$$L(\theta) = \prod_{i=1}^n f(X_i;\theta)$$
$$l(\theta) = \sum_{i=1}^n \log f(X_i;\theta).$$

Let $\widehat{\theta}$ denote the maximum likelihood estimate of θ.

To motivate the test, consider Theorem 6.1.1, which says that if θ_0 is the true value of θ, then, asymptotically, $L(\theta_0)$ is the maximum value of $L(\theta)$. Consider the ratio of two likelihood functions, namely,

$$\Lambda = \frac{L(\theta_0)}{L(\widehat{\theta})}. \tag{6.3.2}$$

Note that $\Lambda \leq 1$, but if H_0 is true, Λ should be large (close to 1), while if H_1 is true, Λ should be smaller. For a specified significance level α, this leads to the intuitive decision rule

$$\text{Reject } H_0 \text{ in favor of } H_1 \text{ if } \Lambda \leq c, \tag{6.3.3}$$

where c is such that $\alpha = P_{\theta_0}[\Lambda \leq c]$. We call it the **likelihood ratio test** (LRT). Theorem 6.3.1 derives the asymptotic distribution of Λ under H_0, but first we look at two examples.