Homework 1 for math6263

1) Assume that we have a fair five- sided die and another skewed die with the same number of sides. Matzinger throws one of the two dies and you have to decide which die was used based on the number he obtains. The fact that the fair die was used is called H_0-hypothesis, the fact that the other die is used is called H_1-hypothesis. The probabilities for the two dice are given in the table below

x	1	2	3	4	5
$P(X = x\|H_0)$	0.2	0.2	0.2	0.2	0.2
$P(X = x\|H_1)$	0.3	0.2	0.1	0.25	0.15

a) Find all the most powerful tests if randomization is not allowed. That is tests that are most powerful among all non-randomized tests of same significance.
b) How would you proceed if you had to construct a most powerful test with significance 0.3? This time randomizing is allowed. What will the power of such a test be?
c) Assume now that Matzinger chooses the die at random (Bayesian case). The following (prior) probabilities are used:

$$P(H_0) = 0.5, P(H_1) = 0.5.$$

Which test is best in this case, in the sense that its probability to make an error is smallest? (Error of first and second type combined).
d) Same question then under c but this time with the following prior probabilities: $P(H_0) = 0.4$ and $P(H_1) = 0.6$.
e) Same question then under d), but this time there is different cost associated with an error of first type and an error of second type. Say, the error of first type has cost $L_0 = 2$ and the cost for an error of second type is $L_1 = 4$. What is the best test if we want to minimize the expected cost? Do we need to consider randomized algorithms if we want to minimized overall cost?

2) Assume this time that we have three dice. the first one constitutes the null-hypothesis. The alternative hypothesis is denoted by K and is that either the second or the third die was used. The fact that the second die is used is denoted by H_1,whilst the fact that the third die is used is denoted by H_2. So, K consists of H_1 and H_2 and is a composite hypothesis. The probabilities of the three dice are summarized in the table below:

x	1	2	3	4
$P(X = x\|H_0)$	0.1	0.4	0.3	0.2
$P(X = x\|H_1)$	0.1	0.2	0.3	0.4
$P(X = x\|H_2)$	0.5	0.1	0.3	0.1

a) is there for every significance level a UMP-test for testing H_0 against K? (We assume that we allow for randomized tests). If not indicate why not and for which significance level there is not.

3) Assume that X is obtained from a uniform random variable on a unit interval (an interval of length 1). Test the null-hypothesis that the unit interval is $[0,1]$ vs the alternative composite hypothesis that the unit interval of the uniform distribution which was used to generate X is not $[0,1]$. Are there UMP-tests in this case and if yes how would they look?

4) We assume that X and Y are two random variables which both can take value 0 or 1. We assume that $E[X] = E[Y] = 0.5$. We test hypotheses concerning the joint probability distribution of X and Y. Let H_0 be the hypothesis that X and Y are independent. Let K be the hypothesis that X and Y are not independent. a) Are there UMP-tests to test H_0 against H_1 for every significance? Why or why not. b) What restriction could we add which would help for getting UMP-tests?

5) Same question as in problem 4, but this time the die can take any of three values in the set $\{1,2,3\}$. We assume known that

$$P(X=1) = P(X=2) = P(X=3) = P(Y=1) = P(Y=2) = P(Y=3) = \frac{1}{3}.$$

We want to test the hypothesis H_0 that X and Y are independent against the hypothesis K that they are not independent.
a) Is there a UMP test for all significance levels?

6) Let X be a positive random variable (hence $P(X \geq 0) = 1$) having density function given by

$$f_\theta(x) = c(\theta)x^2\theta^x$$

for $x > 0$, where θ is a positive parameter which is strictly less than 1. Here $c(.)$ is a function of θ which is used as normalizing constant.
a) Assume that H_0 is the hypothesis that $\theta = 0.2$ and H_1 is the hypothesis that $1 > \theta > 0.2$. Are there UMP-tests at all levels of significance. Explain why and if yes describe them. b) Let H_0 be like in a. But this time let H_1 be the hypothesis that $\theta \neq 0.2$ and θ stricly between 0 and 1. Are there UMP-tests at all levels?

7) Let X be a positive random variable with density given by

$$f_\theta(x) = c(\theta) \cdot (x\theta^x + x^2\theta^{3x})$$

for $x > 0$ where $\theta > 0$ is a parameter strictly smaller than 1. Assume that H_0 is the hypothesis that $\theta = 0.2$ and H_1 is the hypothesis that $1 > \theta > 0.2$. Are there UMP-tests at all levels of significance. Explain why and if yes describe them.

8) You own a six sided die. (Real life situation, not the ridiculous little game which Matzinger used to explain statistics!) You want to know if the die is fair. you throw it a few times independently. is there a UMP-test to figure out if the die is fair (all sides have same probability) against the alternative that it is not fair? What do you say?????????????????