MATH 6262, TEST 1 SOLUTIONS

Problem 1. (1) *Assume* $X \sim Exp(1)$. *Find* $\mathbb{E}[X^2 - X + 1|X^3]$.

(2) *If* $X \sim N(0,1)$, *find* $\mathbb{E}[X^2 + X^3|X^4]$.

(3) *If* (X, Y) *have density given by*

$$f_{X,Y}(x,y) = \begin{cases} x+y, & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise}, \end{cases}$$

find $\mathbb{E}[X|Y]$.

(4) *Assume* (X, Y) *take the values* $(0,0), (3,1), (1,2), (0,2)$ *with probabilities* $1/2, 1/4, 1/8, 1/8$. *Compute* $\mathbb{E}[X|Y]$.

Solution. (1) Because $X^2 - X + 1 = \phi(X^3)$, where the function $\phi(x) = x^{2/3} - x^{1/3} + 1$ is defined for $x > 0$. Thus we get that

$$\mathbb{E}[X^2 - X + 1|X^3] = X^2 - X + 1.$$

(2) In this case we get that $\mathbb{E}[X^2 + X^3|X^4] = \mathbb{E}[X^2|X^4] + \mathbb{E}[X^3|X^4]$. For the first one, $\mathbb{E}[X^2|X^4] = X^2$ because X^2 is a function of X^4. Indeed $X^2 = (X^4)^{1/2}$ so the conditional expectation $\mathbb{E}[X^2|X^4] = X^2$. On the other hand, $\mathbb{E}[X^3|X^4] = 0$ because intuitively, knowing X^4, we determine X up to a sign and these signs balance each other. The formal argument needs a little work out. One way is to mimick the argument we had for the case of $\mathbb{E}[X|X^2]$. Thus let's assume that $\mathbb{E}[X^3|X^4] = \phi(X^4)$ for some function ϕ. Then according to the characterization of the conditional expectation we get

$$\mathbb{E}[X^3\psi(X^4)] = \mathbb{E}[\phi(X^4)\psi(X^4)]$$

for any choice of the function ψ. The key is that

$$\mathbb{E}[X^3\psi(X^4)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 \psi(x^4) e^{-x^2/2} dx$$

This last integral is 0 because changing x into $-x$ changes the integral into it's negative, thus it's only possible value is 0. Therefore we get

$$\begin{aligned} 0 = \mathbb{E}[\phi(X^4)\psi(X^4)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x)\psi(x^4) e^{-x^2/2} dx \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \phi(x^4)\psi(x^4) e^{-x^2/2} dx \\ &\underset{x=t^{1/4}}{=} \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \phi(t)\psi(t) e^{-t^{1/2}/2} dx \end{aligned}$$

Therefore, because this is true for any choice of ψ, it follows that ϕ is 0 on the positive line.

Another way of arguing that $\mathbb{E}[X^3|X^4]$ is the following. If we set $Y = X^3$, then $\mathbb{E}[X^3|X^4] = \mathbb{E}[Y|Y^{4/3}]$ and because Y has now a symmetric distribution, we can use one of the homework problems to finish the argument that $\mathbb{E}[Y|Y^{4/3}] = 0$.

(3) This is based on the formula for the conditional distribution. Essentially we need to compute $f_{X|Y}(x|y)$ and then integrate x against this density. Thus

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Now, $f_Y(y) = \int_0^1 (x+y)dx = y + 1/2$ for $y \in [0,1]$ and thus

$$f_{X|Y}(x|y) = \frac{x+y}{y+1/2} = \frac{2(x+y)}{2y+1},$$

and finally, if we take

$$\phi(y) = \int_0^1 x f_{X|Y}(x|y)dx = \int_0^1 \frac{2x(x+y)}{2y+1}dx = \frac{2/3+y}{2y+1} = \frac{3y+2}{6y+3}$$

And thus $\mathbb{E}[X|Y] = \phi(Y)$.

(4) For this we use the same formula as

$$\mathbb{E}[X|Y] = \phi(Y)$$

where $\phi(y) = \sum_x x f_{X|Y}(x|y)$ and

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

where $f_Y(y)$ is given by

y	0	1	2
$f_Y(y)$	1/2	1/4	1/4

Thus, we get that

$$\phi(y) = \begin{cases} 0 & y = 0 \\ 3 & y = 1 \\ 1/2 & y = 2 \end{cases}.$$

and thus $\mathbb{E}[X|Y] = \phi(Y)$.

□

Problem 2. *If X_1, X_2, X_3 are iid Bernoulli with parameter θ, show that $2X_1 + 3X_2 + 4X_3$ is not a sufficient statistic. Is this complete? Can you find a sufficient and complete statistic?*

Solution. This problem is not correct as it is formulated. The statistic I had on paper was $Z = 2X_1 + 3X_2 + 5X_3$ not $Y = 2X_1 + 3X_2 + 4X_3$. Essentially Y *is* sufficient, while Z *is not* sufficient. However, neither Y nor Z are complete.

I would like to thank Yixian Zhai for bringing this to my attention.

Now, to the problem. We are going to check that Y *is* sufficient. We can use the factorization theorem. The likelihood function is

$$L(x_1, x_2, x_3; p) = p^{x_1}(1-p)^{1-x_1}p^{x_2}(1-p)^{1-x_2}p^{x_3}(1-p)^{1-x_3} = p^{\sum_{i=1}^3 x_i}(1-p)^{3-\sum_{i=1}^3 x_i}$$

For Y to be a sufficient statistic, we want this likelihood function to be written as

$$L(x_1, x_2, x_3; p) = k_1(2x_1 + 3x_2 + 4x_3; p)k_2(x_1, x_2, x_3)$$

This is possible because the values of $2x_1+3x_2+4x_3$ are different as x_1, x_2, x_3 run in $\{0,1\}$. More precisely, we have

(1)

(x_1,x_2,x_3)	$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(0,1,1)$	$(1,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
$2x_1+3x_2+4x_3$	0	4	3	7	2	6	5	9

As one can see we have a one-to-one assignment in this and thus we can define

$$k_1(y;p) = \begin{cases} (1-p)^3 & y = 0 \\ p(1-p)^2 & y = 2,3,4 \\ p^2(1-p) & y = 5,6,7 \\ p^3 & y = 9 \end{cases}$$

and we can simply take $k_2(x_1, x_2, x_3) = 1$.

This shows that Y is a sufficient statistic.

To show that Z is not a sufficient statistic, we argue again using the factorization theorem. To do this, we argue by contradiction, assuming that we can write

$$L(x_1, x_2, x_3; p) = p^{\sum_{i=1}^3 x_i}(1-p)^{3-\sum_{i=1}^3 x_i} = k_1(2x_1 + 3x_2 + 5x_3; p)k_2(x_1, x_2, x_3) \tag{2}$$

for some functions, k_1 and k_2. However, we have now that

(3)

(x_1, x_2, x_3)	$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(0,1,1)$	$(1,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
$2x_1 + 3x_2 + 5x_3$	0	5	3	8	2	7	5	10

Thus we observe here that 5 is taken for two different sequences of x_1, x_2, x_3. Thus applying this to (2), we get for $(x_1, x_2, x_3) = (0,0,1)$,

$$p(1-p)^2 = k_1(5;p)k_2(0,0,1)$$

and also for $(1,1,0)$ that

$$p^2(1-p) = k_1(5;p)k_2(1,1,0).$$

Well we can not have both of these true for all choices of p in $[0,1]$. This is proves that Z is not a sufficient statistic.

Now we move to the completeness and show that neither Y, Z are complete. Indeed, if

$$\mathbb{E}[u(Y)] = 0 \text{ for all } p \in [0,1]$$

then, using (1) we have

$$(1-p)^3u(0) + p(1-p)^2(u(2) + u(3) + u(4)) + p^2(1-p)(u(5) + u(6) + u(7)) + p^3u(9) = 0$$

for any choice of p. This means that $u(0) = 0$, $u(2) + u(3) + u(4) = 0$ and $u(5) + u(6) + u(7) = 0$ and $u(9) = 0$ which means that we can for instance choose $u(2) = -1$, $u(3) = 1$, while take all the other values of u to be 0. This shows that u is not identically and $u(Y)$ is not identically 0.

In a similar vein we also have for Z, that if

$$\mathbb{E}[u(Z)] = 0 \text{ for all } p \in [0,1]$$

then using (3) we get that

$$(1-p)^3u(0) + p(1-p)^2(u(2) + u(3) + u(5)) + p^2(1-p)(u(5) + u(7) + u(8)) + p^3u(9) = 0$$

and in a similar vein to the above we can take $u(2) = -1$, $u(3) = 1$ and all the other values to be 0, which shows that $u(Z)$ is not identically 0, thus showing that Z is not complete.

The obvious choice for a sufficient and complete statistic is $W = \sum_{i=1}^3 X_i$. $\square$

Problem 3. *Let $X_1, X_2, \ldots, X_n$ be a sample from a distribution with pdf given by $f(x;\theta) = \frac{1}{\theta^2}xe^{-x/\theta}$ for $x > 0$ and $\theta > 0$. Find a complete and sufficient statistic and then the MVUE.*

Solution. This is a regular exponential family, with

$$f(x;\theta) = e^{-x/\theta+\ln(x)-2\ln(\theta)}$$

Thus a complete and sufficient statistic is $T = \sum_{i=1}^{n} X_i$. The MVUE is a function of this statistic which is unbiased. In fact, the obvious thing to di it to start trying to try and see if T itself is an unbiased estimator. To this end,

$$\mathbb{E}[T] = n\mathbb{E}[X_1] = n\int xf(x:\theta)dx = \frac{n}{\theta^2}\int_0^\infty x^2 e^{-x/\theta}dx \overset{x=\theta y}{=} n\theta\int_0^\infty x^2 e^{-x}dx = 2n\theta.$$

The last integral above is computed using twice integration by parts. Even if you do not know how to compute that, it is a constant and we can get the MVUE by simply taking $T/(cn)$, where $c = \int_0^\infty x^2 e^{-x}dx = 2$.

□

Problem 4. *Let $X_1, X_2, \ldots, X_n$ be a sample from the uniform distribution on $[\theta, 1]$ for $\theta < 1$.*

(1) *Is this family a regular exponential family?*
(2) *Show that the statistic $T = \min\{X_1, X_2, \ldots, X_n\}$ is a sufficient statistic for θ.*
(3) *Is T also complete?*

Solution. (1) This is not a regular exponential family because the support of the density is $[\theta, 1]$ and thus depends on $\theta/$

(2) To show that this is a sufficient statistic, we use the factorization theorem to write

$$\begin{aligned}
f(x_1;\theta)f(x_2;\theta)\ldots f(x_n;\theta) &= \frac{1}{(1-\theta)^n}\mathbb{1}_{[\theta,1]}(x_1)\mathbb{1}_{[\theta,1]}(x_2)\ldots\mathbb{1}_{[\theta,1]}(x_n) \\
&= \frac{1}{(1-\theta)^n}\mathbb{1}_{[\theta,\infty)}(\min(x_1,x_2,\ldots,x_n))\mathbb{1}_{(-\infty,1]}(\max(x_1,x_2,\ldots,x_n))
\end{aligned}$$

which shows that T is a sufficient statistic.

(3) To see if T is complete, we need to look at

$$\mathbb{E}[u(T)] = 0$$

for any θ. To figure this out, we need to find the distribution of T. This is, we first find the cdf of T and then the pdf. To find the cdf we take

$$\begin{aligned}
F_T(t) &= \mathbb{P}(\min(X_1,X_2,\ldots,X_n) \le t) = 1 - \mathbb{P}(\min(X_1,X_2,\ldots,X_n) > t) \\
&= 1 - \mathbb{P}(X_1 > t)\mathbb{P}(X_2 > t)\ldots\mathbb{P}(X_n > t) = 1 - \mathbb{P}(X_1 > t)^n = 1 - \left(\frac{1-t}{1-\theta}\right)^n.
\end{aligned}$$

Thus the density is

$$f_T(t) = \frac{n(1-t)^{n-1}}{(1-\theta)^n}$$

which then gives

$$\mathbb{E}[u(T)] = \frac{n}{(1-\theta)^n}\int_\theta^1 u(t)(1-t)^{n-1}dt = 0$$

for any θ. This implies that $\int_\theta^1 u(t)(1-t)^{n-1}dt = 0$ for all choices of θ and differentiating this w.r.t θ leads to $u(\theta)(1-\theta)^{n-1} = 0$ for all θ. Thus $u(t) = 0$ for all $t \in [0, 1]$, which means that $u(T) = 0$. This shows that T is a complete statistic.

□

Problem 5. *Take a sample $X_1, X_2, \ldots, X_n$ from the geometric distribution $Geom(1/\theta)$ (i.e. $f(x;\theta) = 1/\theta(1-1/\theta)^x$ for $x = 0, 1, 2, \ldots$ and $\theta > 1$). Find a sufficient statistic for θ. Is this complete? Find the MVUE for θ.*

Solution. This is pretty straightforward. We have that $f(x;\theta)$ is a regular exponential family with

$$f(x;\theta) = e^{\ln(1-1/\theta)x - \ln(\theta)}.$$

Thus, this yields a complete and sufficient statistic given by $T = \sum_{k=1}^n X_i$.

To find an MVUE, we need to find an unbiased estimator of θ. If we integrate T we obtain that

$$\mathbb{E}[T] = n\mathbb{E}[X_1] = n(\theta - 1)$$

and thus $1 + T/n$ is the MVUE.

□