Math 4317 (Prof. Swiech, S'18): HW #1

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Section 1

F. Show that the symmetric difference D, defined in the preceding exercise is also given by $D = (A \cup B) \setminus (A \cap B)$
Show $D = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$:

First, $x \in (A \setminus B) \cup (B \setminus A) \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies$, x is in A but not B, or, x is in B but not $A \implies x$ is in A or B but not in A and $B \implies x \in (A \cup B) \setminus (A \cap B)$.

In the other direction, $x \in (A \cup B) \setminus (A \cap B) \implies x \in (A \cup B)$ but not in $(A \cap B) \implies x$ is in A but not B, or, x is in B but not $A \implies x \in (A \setminus B)$ or $x \in (B \setminus A) \implies x \in (A \setminus B) \cup (B \setminus A) \implies (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

I. If $\{A_1, A_2, ..., A_n\}$ is a collection of sets, and if E is any set, show that:

$$(i)\ \ E \cap \cup_{j=1}^{n} A_j = \cup_{j=1}^{n}(E \cap A_j),\ \textit{and (ii)},\ \ E \cup \cup_{j=1}^{n} A_j = \cup_{j=1}^{n}(E \cup A_j)$$

(i) $x \in E \cap \cup_{j=1}^{n} A_j \implies x \in E$ *and* $x \in \{A_1$ *or* $A2$ *...or* $A_n\} \implies x \in E$ and that there exists for some $j = 1, 2, ..., n$ an A_j such that $x \in A_j$ *and* $x \in E \implies (x \in E$ *and* $A_1)$ *or* $(x \in E$ *and* $A_2)$ *... or* $(x \in E$ *and* $A_n) \implies x \in \cup_{j=1}^{n}(E \cap A_j)$.
In the other direction, $x \in \cup_{j=1}^{n}(E \cap A_j) \Leftrightarrow x \in (E \cap A_1) \cup (E \cap A_2)\ ...\ \cup (E \cap A_n) \implies x \in E$ and A_1 or E and $A_2 \ldots \implies$ there exists a $j = 1, ..., n$ such that $x \in (E \cap A_j) \implies x \in E$ and $x \in A_1$ or $A_2, \ldots,$ or $A_n \implies x \in E$ and $\cup_{j=1}^{n} A_j \implies x \in E \cap \cup_{j=1}^{n} A_j$.

(ii) $x \in E \cup \cup_{j=1}^{n} A_j \implies x \in E$ or $x \in A_1$ or $A_2 \ldots$ or $A_n \implies$ for some $j = 1, .., n$ that $x \in E \cup A_j \implies x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies x \in \cup_{j=1}^{n}(E \cup A_j)$. In the other direction, $x \in \cup_{j=1}^{n}(E \cup A_j) \Leftrightarrow x \in E \cup A_1$ or $x \in E \cup A_2 \ldots$ or $x \in E \cup A_n \implies$ there exists some $j = 1, ..., n$ such that $x \in E \cup A_j \implies (x \in E$ *or* $x \in A_1)$ or $(x \in E$ *or* $x \in A_2) \ldots$ or $(x \in E$ *or* $x \in A_n) \implies x \in E$ or $x \in \cup_{j=1}^{n} A_j \implies x \in E \cup \cup_{j=1}^{n} A_j$.

J. If $\{A_1, A_2, ..., A_n\}$ is a collection of sets, and if E is any set, show that:

$$(i)\ E \cap \cap_{j=1}^{n} A_j = \cap_{j=1}^{n}(E \cap A_j),\ \textit{and (ii)},\ \ E \cup \cap_{j=1}^{n} A_j = \cap_{j=1}^{n}(E \cup A_j)$$

(i) $x \in \cap \cap_{j=1}^{n} A_j \implies x \in E$ and $x \in \cap_{j=1}^{n} A_j \implies x \in E$ *and* $x \in A_j$ for all $j = 1, .., n \implies x \in E$ and $[x \in A_1$ *and* $x \in A_2$ *... and* $x \in A_n] \implies [x \in E$ *and* $A_1]$ and $\ldots$ and $[x \in E$ *and* $A_n] \implies x \in \cap_{j=1}^{n}(E \cap A_j)$. In the other direction, $x \in \cap_{j=1}^{n}(E \cap A_j) \implies x \in (E \cap A_1)$ and $a \in (E \cap A_2) \ldots$ and $x \in (E \cap A_n) \implies x \in (E \cap A_j)$ for all $j = 1, ..., n \implies x \in E$ and $x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n \implies x \in E$ and $x \in \cap_{j=1}^{nA_j} \implies x \in E \cap \cap_{j=1}^{nA_j}$.

(ii) $x \in E \cup \cap_{j=1}^{n} A_j \implies x \in E$ or $x \in \cap_{j=1}^{n} A_j \implies x \in E$ or $[x \in A_1$ and $x \in A_2 \ldots$ and $x \in A_n] \implies x \in E$ *or* A_1 *and* $x \in E$ *or* A_2 *... and* $x \in E$ *or* $A_n \implies x \in \cap_{j=1}^{n}(E \cup A_j)$. In the other direction, $x \in \cap_{j=1}^{n}(E \cup A_j) \implies x \in (E$ *or* $A_1)$ *and* $x \in (E$ *or* $A_2)$ *...and* $x \in (E$ *or* $A_n) \implies$ that for all $j = 1, ..., n$, $x \in (E$ *or* $A_j) \implies x \in E$ *or* $(x \in A_1$ *and* $x \in A_2$*... and* $x \in A_n) \implies x \in \cap_{j=1}^{n} A_j$ *or* $x \in E \implies x \in E \cup \cap_{j=1}^{n} A_j$.

K. Let E be a set and $\{A_1, A_2, ..., A_n\}$ be a collection of sets. Establish the De Morgan laws:

$$(i)\ E \setminus \cap_{j=1}^{n} A_j = \cup_{j=1}^{n}(E \setminus A_j),\ \textit{and},\ \ (ii)\ \ E \setminus \cup_{j=1}^{n} A_j = \cap_{j=1}^{n}(E \setminus A_j)$$

(i) $x \in E \setminus \cap_{j=1}^{n} A_j \implies x \in E$ but not $(A_1$ and $A_2 \ldots$ and $A_n) \implies$ there exists a $j = 1, ..., n$ such that $x \in E$ but not $A_j \implies x \in E$ but not A_1, or $x \in E$ but not A_2,. . . ,or $x \in E$ but not

$A_n \implies x \in E \setminus A_1$ or $x \in E \setminus A_2 \ldots$ or $x \in E \setminus A_n \implies x \in \cup_{j=1}^{n}(E \setminus A_j)$. In the other direction, $x \in \cup_{j=1}^{n}(E \setminus A_j) \implies x \in (E$ *but not* $A_1)$ or $(E$ *but not* $A_2)$ or $(E$ *but not* $A_n) \implies$ there exists $j = 1, ..., n,\ x \in E$ but not $A_j \implies x \in E$ but not $(A_1$ *and* A_2 ... *and* $A_n) \implies x \in E \setminus \cap_{j=1}^{n} A_j$.

(ii) $x \in E \setminus \cup_{j=1}^{n} \implies x \in E$ but A_1 or $A_2 \ldots$ or $A_n \implies x \in E$ and $x \notin A_j$ for all $j = 1, ..., n \implies x \in E$ but not A_1, and $x \in E$ but not A_2, $\ldots$, and $x \in E$ but not $A_n \implies x \in (E \setminus A_1)$ *and* $x \in (E \setminus A_2)$ ... *and* $x \in (E \setminus A_n) \implies x \in \cap_{j=1}^{n}(E \setminus A_j)$. In the other direction, $x \in \cap_{j=1}^{n}(E \setminus A_j) \implies x \in (E \setminus A_1$ *and* $E \setminus A_2$ ... *and* $E \setminus A_n) \implies x \in E$ but not A_j for all $j = 1, ..., n \implies x \in E$ but A_1 or $A_2 \ldots$ or $A_n \implies x \in E$ but not $\cup_{j=1}^{n} A_j \implies x \in E \setminus \cup_{j=1}^{n} A_j$

Section 2

C. Consider the subset of $\mathbb{R} \times \mathbb{R}$ *defined by* $D = \{(x, y)\ : |x| + |y| = 1\}$. *Describe this set in words. Is it a function?*

This set consists of points on the line segments connecting a rotated square in the (x, y) plane with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. If we attempt to define a function, with the elements (x, y) from the set D, i.e. $y = f(x), f : x \to y$, we have $|x| + |y| = 1 \implies \sqrt{y^2} = 1 - |x| \implies y = \pm\sqrt{(1 - |x|)^2}$. $f(x) = y = \pm\sqrt{(1 - |x|)^2}$ does not fit the defintion of a function, since, as an example, the set D includes the elements $(0, 1)$ and $(0, -1)$, which if, f is a function, $f : x \to y \implies -1 = 1$, which is clearly not true.

E. Prove that if f *is an injection from* A *to* B, *then* $f^{-1} = \{(b, a) : (a, b) \in f\}$ *is a function. Then prove it is an injection.*

If f is an injection, and $(a, b) \in f$, and $(a', b) \in f$, then $a = a'$. $f^{-1} = \{(b, a) : (a, b) \in f\}$ contains the pair (b, a) and (b, a'), and we know that $a = a'$ from the definition of f, so we can assume that f^{-1} is a function. Since f is injective, each unique element $b = f(a)$, is mapped to by a unique element a, and by definition $f^{-1} = \{(b, a) : (a, b) \in f\}$ maps the unique element b back to a, meaning $f^{-1}(b) = a$ and $f^{-1}(b') = a$ if and only if $b = b'$, thus f^{-1} is also injective.

H. Let f, g *be functions such that*

$$g \circ f(x) = x, \quad \textit{for all } x \textit{ in } D(f)$$

$$f \circ g(y) = y, \quad \textit{for all } y \textit{ in } D(g)$$

Prove that $g = f^{-1}$

For two elements $x, x' \in D(f)$, if $f(x) = f(x') \implies g \circ f(x) = g(f(x)) = g(f(x')) \implies g(f(x)) = x = g(f(x')) = x'$, that is $x = x' \implies g \circ f$ is an injection. For two elements $y, y' \in D(g)$, if $g(y) = g(y') \implies f \circ g(y) = f(g(y)) = f(g(y')) \implies f(g(y)) = y = f(g(y')) = y'$, that is $y = y' \implies f \circ g$ is an injection, and implies f and g are injections as well.

This implies g can be defined $g = \{(f(x), x) : (x, f(x)) \in f\}$, which is the definition for f^{-1}, implying $g = f^{-1}$.

J. Let f *be the function on* $\mathbb{R}$ *to* $\mathbb{R}$ *given by* $f(x) = x^2$, *and let* $E = \{x \in \mathbb{R}\ -1 \le x \le 0\}$ *and* $F = \{x \in \mathbb{R} : 0 \le x \le 1\}$.*Then* $E \cap F = \{0\}$ *and* $f(E \cap F) = \{0\}$ *while* $f(E) = f(F) = \{y \in \mathbb{R} : 0 \le y \le 1\}$. *Hence* $f(E \cap F)$ *is a proper subset of* $f(E) \cap f(F)$. *Now delete* 0 *from* E *and* F.

The sets E and F with 0 deleted are denoted $E' = \{x \in \mathbb{R} : -1 \le x < 0\}$ and $F' = \{x \in \mathbb{R} : 0 < x \le 1\}$, respectively. We still have the equality $f(E') = f(F') = \{y \in \mathbb{R} : 0 < y \le 1\} = f(E') \cap f(F')$. We also have $E' \cap F' = \emptyset$, and thus $f(E' \cap F') = \emptyset$, and $\emptyset = f(E' \cap F') \subseteq F(E') \cap f(F')$, since the empty set is a subset of all sets.

Section 3

B. Exhibit a one-to-one correspondence between the set O *of odd natural numbers and* $\mathbb{N}$

The function $f(x) = \frac{x+1}{2}, x \in \mathbb{N}$ maps the set of odd natural numbers, $O = \{2k - 1 : k \in N\} \to \mathbb{N}$.

D. If A is contained in some initial segment of $\mathbb{N}$, use the well-ordering property of $\mathbb{N}$ to define a bijection of A onto some initial segment of $\mathbb{N}$.

If $A \neq \emptyset$ is a subset of some initial segment $\mathbb{N}$, by the well-ordering principle, there exists an $m \in A$ such that $m \leq k$ for all $k \in A$. A bijection f can be defined by the mapping from set A consisting of elements $\{a_1, a_2, ..., a_k\}$ to elements of some initial segment $S_k = \{1, 2, ..., k\}$ as a set of ordered pairs $\{(a_1, 1), (a_2, 2), ..., (a_k, k)\}$, such that $a_1 \leq a_2 \leq ... \leq a_k$ and clearly the corresponding elements in the pairs from set S_k, $1 \leq 2 \leq ... \leq k$. Here the number of elements in A and S_k are the same, which has a one-one correspondence $f : A \to S_k$ and the $R(f) = S_k$.

F. Use the fact that every infinite set has a denumerable subset to show that every infinite set can be put into one-one correspondence with a proper subset of itself.

By defintion, having a denumberable subset $\implies$ there exists a bijective function that maps a proper subset of an infinite set, B, onto $\mathbb{N}$. If we take infinite set $B = \{b_1, b_2, ..., b_n, ...\}$ and $B_1 = \{b_2, b_3, ..., b_n, b_{n+1}, ...\}$, $B_1 \subseteq B$, we can create a one-one correspondence $f : B \to B_1$ defined by the set or ordered pairs $\{(b_n, b_{n+1}) : n \in N\}$ which maps B to $B_1 = B \setminus \{b_1\}$.

H. Show that if the set A can be put into one-one correspondence with a set B, and if B can be put into one-one correspondence with a set C, then A can be put into one-one correspondence with C.

If A can be put into one-one correspondence with a set $B \implies$ there exists an injective function, f from $A \to B$. This means that for $a, a' \in A$, and $b \in B$, $f(a) = f(a') = b \implies a = a'$. Similarly, if B can be put into one-one correspondence with a set $B \implies$ there exists an injective function, g from $B \to C$, and with $b, b' \in A$, $g(b) = g(b') = c \in C \implies b = b'$. By these properties, the composition of these two injective functions, $g \circ f(a) = g \circ f(a') \implies f(a) = f(a') \implies a = a'$ putting A and C in one-one correspondence.

I. Using induction on $n \in \mathbb{N}$, show that the initial segment determined by n cannot be put into one-one correspondence with the initial segment determined by $m \in \mathbb{N}$, if $m < n$.

Let $S_n = \{1, 2, 3, ..., n\}$ be the initial segment determined by $n \in N$ and S_m be the initial segment determined by $m \in N, m < n$. If S_n can be put into one-one correspondence with $S_m \implies$ there exists and injection $f : S_n \to S_m$. For $n = 1$ we have $f : \{1\} \to S_m,\ m < 1$, but S_m does not exist by definition for $m < 1$ implying the function is not valid for the case $n = 1$, $m < n$. For, the case $n = k + 1$, we again have a map $f : \{1, 2, ..., k + 1\} \to \{1, ..., m\},\ m < k + 1$ which implies a mapping of $k + 1$ elements to $m < k + 1$ elements $\implies$ there exists at least two elements $x, x' \in S_{k+1}$ for which $f(x) = f(x')$ and $x \neq x' \implies$ an injection does not exist between these sets.

Section 4 (C, F, G, H)

Section 5 (B, C, F, G, K, L)

Section 6 (B, C, G, H, J, K)

Section 7 (F, G, K)