Math 4317 (Prof. Swiech, S'18): HW #4

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Section 20

A. Prove that if f is defined for $x \geq 0$ by $f(x) = \sqrt{x}$, then f is continuous at every point of its domain.

For $f(x) = \sqrt{x},\ \mathcal{D}(f) = \{x \in \mathbb{R} : x \geq 0\}$, let $a \in \mathcal{D}(f)$.

When $a = 0,\ |f(x) - f(a)| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon$. If we let $\delta(\varepsilon) = \varepsilon^2$, when $x < \varepsilon^2,\ |f(x)| < \varepsilon$.

When $a \neq 0,\ |f(x)-f(a)| = |\sqrt{x}-\sqrt{a}| = \frac{|\sqrt{x}-\sqrt{a}|}{|\sqrt{x}+\sqrt{a}|}|\sqrt{x}+\sqrt{a}| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} < \frac{|x-a|}{\sqrt{a}} < \varepsilon \implies$ when $|x-a| < \varepsilon\sqrt{a}$, then, $|f(x) - f(a)| < \varepsilon$, thus we can choose $\delta(\varepsilon) = \varepsilon\sqrt{a} \implies f$ is continuous at every point in its domain.

B. Show that a "polynomial function"; that is, a function f with the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,\ x \in \mathbb{R}$ is continuous at every point of $\mathbb{R}$.

Relying on the properties of algebraic combinations of continuous of functions, we construct f as a combination of continuous functions to show its continuity. Considering the last term of the polynomial function, denoted here, $f_0(x) = a_0,\ f_0(x)$ is a continuous, constant function, since, for any $a \in \mathbb{R}$ we have $|f_0(x) - f_0(a)| = |a_0 - a_0| < \varepsilon = \delta(\varepsilon),\ \varepsilon > 0$. We consider the second to last term of f, $a_1 x$, as a constant, a_1 multiplied by the identity function, denoted, $f_1(x) = x$. Since $f_1(x) = x$, for any real number $a \in \mathbb{R}$, we have $|f_1(x) - f_1(a)| = |x - a| < \varepsilon = \delta(\varepsilon),\ \varepsilon > 0 \implies a_1 f_1(x) = a_1 x$ is continuous.

Relying on the continuity of $f_1(x) = x$ multiplied by any constant, we can construct higher order terms of f through repeated multiplication of $f_1(x)$, e.g. $a_2 \cdot f_1(x) \cdot f_1(x) = a_2 x^2$ and $a_n \prod_{j=1}^{n} f_1(x) = a_n \cdot f_1(x) \cdot f_1(x) \cdot ... \cdot f_1(x) = a_n x^n$, and so on, where each term constructed $a_n x^n$ is continuous on $\mathbb{R}$ since it is constructed via algebraic combinations of continuous functions $\implies f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, is continuous at every point $x \in \mathbb{R}$.

E. Let f be the function on $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = x,\ x$ irrational, $f(x) = 1 - x,\ x$ rational. Show that f is continuous at $x = \frac{1}{2}$ and discontinuous elsewhere.

Considering the point $a = \frac{1}{2}$, we have $f(a) = \frac{1}{2}$, and $|f(x) - f(a)| = |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = |x - a| < \varepsilon = \delta(\varepsilon)$. So if $|f(x) - f(a)| < \varepsilon = \delta(\varepsilon) > 0 \implies |x - a| < \delta(\varepsilon)$, and then we have f continuous at the point $a = \frac{1}{2}$. For the case $a \neq \frac{1}{2},\ a$ irrational, take a sequence $X = (x_n)$ of rational numbers converging to a. Since the sequence $(f(x_n))$ converges to $1 - a$, and we have $f(a) = a$, f is not continuous at irrational points by the Discontinuity Criterion. For the case $a \neq \frac{1}{2},\ a$ rational, take a sequence $Y = (Y_n)$ of irrational numbers converging to a, the sequence (f(y__n)) converges to a, but $f(a) = 1 - a$, which equation is only satisfied when $a = \frac{1}{2}$, thus f is not continuous for rational numbers at any point other than $\frac{1}{2}$.

F.Let f be continuous on $\mathbb{R} \to \mathbb{R}$. Show that if $f(x) = 0$ for rational x, then $f(x) = 0$ for all $x \in \mathbb{R}$.

Every real point, $x \in \mathbb{R}$ is the limit of a sequence of rational numbers. If f is continuous $\implies$ for a sequence of rational numbers $X = (x_n) \to x$, we have $(f(x_n)) = 0$, for all $n \in \mathbb{N}$. Since f is continuous at each rational point $x \in \mathbb{R}$, we can find $|f(x_n) - f(x)| < \varepsilon,\ \varepsilon > 0$, and $|x_n - a| < \delta(\varepsilon) \implies (f(x_n)) \to f(x) = 0, \forall x \in \mathbb{R}$.

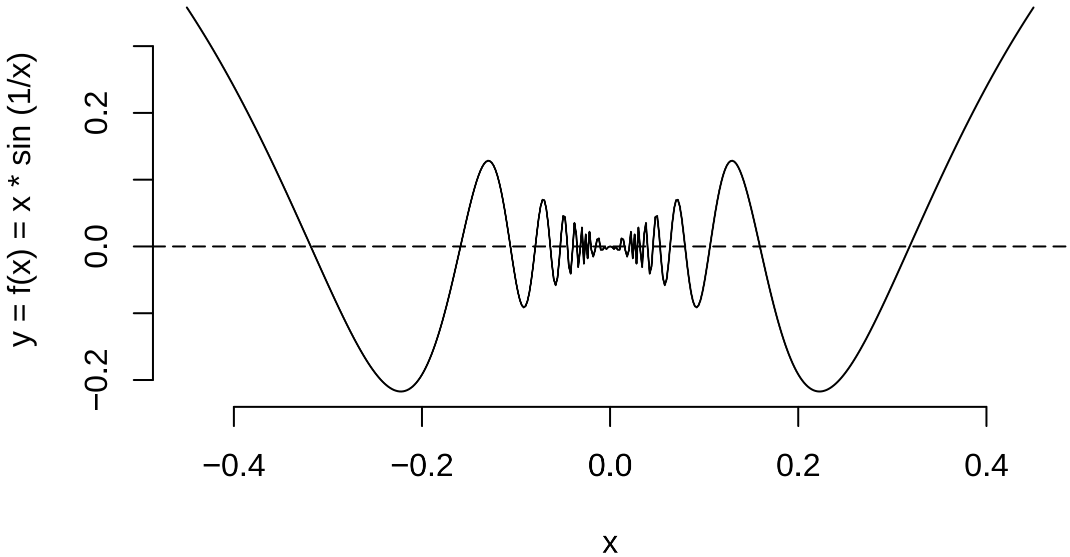
I. Using the results of the preceding exercise, show that the function g, defined on $\mathbb{R} \to \mathbb{R}$ by $g(x) = xsin(\frac{1}{x}),\ x \neq 0,\ g(x) = 0,\ x = 0$ is continuous at every point. Sketch a graph of this function.

For the case $a = 0$, we have $|g(x) - g(a)| = |x \sin \frac{1}{x} - 0| = |x||\sin \frac{1}{x}| \leq |x| \cdot 1 < \varepsilon,\ \varepsilon > 0$, since $-1 \leq \sin \frac{1}{x} \leq 1$. So when $|g(x) - g(0)| < \varepsilon = \delta(\varepsilon)$, we then have $|x| = |x - 0| < \delta(\varepsilon) \implies g$ continuous at 0.

For the case $a \neq 0$, we have $|g(x) - g(a)| = |x \sin \frac{1}{x} - a \sin \frac{1}{a}| = |x \sin \frac{1}{x} - a \sin \frac{1}{a} - a \sin \frac{1}{x} + a \sin \frac{1}{x}| = |(x - a)(\sin \frac{1}{x}) + a(\sin \frac{1}{x} - \sin \frac{1}{a})| \leq |x - a||\sin \frac{1}{x}| + |a||\sin \frac{1}{x} - \sin \frac{1}{a}|$, by Triangle Inequality. Since both $|\sin \frac{1}{x}| \leq 1$ and $|\sin \frac{1}{x} - \sin \frac{1}{a}| \leq 1$, we have $|x-a||\sin \frac{1}{x}| + |a||\sin \frac{1}{x} - \sin \frac{1}{a}| \leq |x-a| \cdot 1 + |a| \cdot 1 = |x-a| + |a| < \varepsilon$.

It then follows that if $\delta(\varepsilon) = \varepsilon - |a|$, i.e. $\varepsilon > \delta(\varepsilon) + |a|$, when $|g(x) - g(a)| < \varepsilon$, then $|x - a| < \delta(\varepsilon) \implies$ g continuous at every point in $\mathbb{R}$.

Sketch of function below:



N. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x + y) = g(x)g(y),\ x, y \in \mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point. Also if $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

If g is continuous at $x = 0 \implies g(x + y) = g(y) = g(0) \cdot g(y)$. This implies also that $g(0)g(y) = g(y) \implies g(0)g(y) - g(y) = 0 = g(y)(g(0) - 1) = 0 \implies g(0) = 1$, or that $g(0) = 0$.
If $g(0) = 0 \implies -g(y) = 0 = g(y)$. In this case then $g(y) = 0,\ \forall y \in \mathbb{R} \implies g(x) = 0,\ \forall x \in \mathbb{R}$.
On the other hand if $g(0) = 1, \implies g(0) \cdot g(y) = g(y)$ continuous for every point $y \in \mathbb{R}$.

Section 21

I. Let g be a linear function from $\mathbb{R}^p \to \mathbb{R}^q$. Show that g is one-one and only if $g(x) = 0$ implies that $x = 0$.
Since g is linear $\implies$ for $x, y \in \mathbb{R}^p$, $g(x + y) = g(x) + g(y)$. Then if $g(x) = 0 \implies g(x + y) = 0 + g(y) = g(y) \implies g(x + y) = g(y) \implies g(x + y) = g(0 + y) = g(y)$ which implies $x = 0$. If we assume that g is one-one, then for any $g(x) = g(y) \implies x = y$. So in the case $g(x) = 0$, and $g(x+y) = g(x)+g(y) = 0+g(y)$. Since $g(x) + g(y) = g(y) \implies g(y) - g(x) = g(y) \implies x + y = x - y$, which is satisfied when $x = 0$.

J. If h is a one-one linear function from $\mathbb{R}^p \to \mathbb{R}^p$, show that the inverse function h^{-1} is a linear function from $\mathbb{R}^p \to \mathbb{R}^p$.

Since h is one-one $\implies$ if $h(x_1) = h(x_2),\ x_1 = x_2,\ x_1, x_2 \in \mathbb{R}^p$. Extending the the linear case, we have if $h(ax + by) = h(ax_1 + by_1) = ah(x) + bh(y) = ah(x_1) + bh(y_1)$ then $ax_1 + by_1 = ax + by$. By definition $h^{-1} = \{ax + by : h(ax + by) \in \mathbb{R}^p\} = \{ax : h(ax) \in \mathbb{R}^p\} + \{by : h(by) \in \mathbb{R}^p\}$. This implies $h^{-1}(ax + by) = h^{-1}(h(ax)) + h^{-1}(h(by)) \implies h^{-1}$ is linear, and $h^{-}1 : \mathbb{R}^p \to \mathbb{R}^p$, since $h^{-1}(h(ax)) + h^{-1}(h(by)) = ax+by \in \mathbb{R}^p$ by construction.

K. Show that the sum and the composition of two linear functions are linear functions.

By definition a function is linear if $f(ax + by) = af(x) + bf(y),\ a, b \in \mathbb{R},\ x, y \in \mathbb{R}^p$.

For the sum of two linear functions we then have $(f+g)(ax+by) = f(ax+by)+g(ax+by) = af(x)+bf(y)+ag(x)+bf(y) = a(f(x)+g(x))+b(f(y)+g(y)) = a(f+g)(x)+b(f+g)(y) \implies$ linearity. For the composition of two linear functions we have $f \circ g(ax + bx) = f(g(ax + by)) = f(ag(x) + bg(y)) = af(g(x)) + bf(g(y)) = a(f \circ g)(x) + b(f \circ g)(y) \implies$ composition of two linear functions is linear.

L. If f is a linear map on $\mathbb{R}^p \to \mathbb{R}^q$, define $||f||_{pq} = \sup\{||f(x)|| : x \in \mathbb{R}^p,\ ||x|| \leq 1\}$. Show that the mapping $f \to ||f||_{pq}$ defines a norm on the vector space $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ of all linear functions on $\mathbb{R}^p \to \mathbb{R}^q$. Show that $||f(x)|| \leq ||f||_{pq}||x||$ for all $x \in \mathbb{R}^p$.

We have $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p,\ f(x) = y = (y_1, y_2, ..., y_q) \in \mathbb{R}^q$, and matrix $A_{q\times p} = (c_{ij}),\ 1 \leq i \leq q,\ 1 \leq j \leq p$, with

$$y_1 = c_{11}x_1 + x_{12}x_2 + ... + c_{1p}x_p$$

...

$$y_q = c_{q1}x_1 + x_{q2}x_2 + ... + c_{qp}x_p$$

We then have $||f(x)|| = ||(y_1, ..., y_q)|| = \sqrt{y_1^2 + ... + y_q^2}$. To show $||f||_{qp} = \sup\{||f(x)|| : x \in \mathbb{R}^p,\ ||x|| \leq 1\}$ is a norm in $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, we have (i) $||f||_{pq} \geq 0,\ x \in \mathbb{R}^p$? Since each element in $||f(x)|| = \sqrt{y_1^2 + ... + y_q^2}$, $y_j^2 \geq 0,\ \forall j = 1, ..., q \implies \sup\{||f(x)||\} \geq 0 \forall x \in \mathbb{R}^p$ since by definition, $sup\{||f(x)||\} \geq ||f(x)|| \forall x \in \mathbb{R}^p \implies ||f||_{pq} \geq 0$.
(ii) $||f||_{pq} = 0 \iff f(x) = 0$? Since $||f(x)|| = ||y|| = \sqrt{y_1^2 + ... + y_q^2} = 0 \implies$ each $y_j^2 = 0, \forall j = 1, ..., q$
(iii) $\sup ||af(x)|| = |a| \sup ||f(x)|| = |a|||f||_{qp},\ a \in \mathbb{R}$? We have $||af(x)|| = ||ay|| = \sqrt{a^2y_1^2 + ... + a^2y_1^2} = \sqrt{a^2}||y|| = |a|||y||$, and $|a| > 0 \implies \sup\{||af(x)||\} = \sup\{|a|||f(x)||\} = |a|sup\{||f(x)||\}$.
(iv) $\sup\{||f(x+x^{'})||\} \leq \sup ||f(x)|| + \sup ||f(x^{'})||,\ x, x^{'} \in \mathbb{R}^p$? Since f is linear $||f(x+x^{'})|| = ||f(x)+f(x^{'})|| \leq ||f(x)|| + ||f(x^{'})||,\ \forall x, x^{'} \in \mathbb{R}^p$ by Triangle Inequality, then $\sup\{||f(x)+f(x^{'})||\} \leq \sup\{||f(x)||\} + \sup\{||f(x^{'})||\}$. This implies $||f||_{qp}$ is a norm.

To show $||f(x)|| \leq ||f||_{pq}||x||$, we use the earlier notation for a linear map, $f(x) = Ax$, where, $A_{q\times p} = (c_{ij})$. Thus $||f(x)|| = ||Ax|| \leq |A|||x||$ as shown in (21.5). This implies $\sup\{||f(x)|| : x \in \mathbb{R}^p, ||x|| \leq 1\} = \sup\{||Ax||\} \leq \sup\{|A|||x||\}$ which is achieved when x is the max value in its domain, i.e. $||x|| = 1$. This implies $\sup\{|||Ax||\}||x|| = \sup\{||f(x)||\}||x|| = \sup\{||f(x)||\} \cdot 1$. This implies $||f(x)|| \leq \sup\{||f(x)|| : x \in \mathbb{R}^p, ||x|| \leq 1\}||x|| \forall x \in \mathbb{R}^p$.

Section 22

B. Let $H : \mathbb{R} \to \mathbb{R}$ be defined by, $h(x) = 1, 0 \leq x \leq 1$. $h(x) = 0$, otherwise. Exhibit an open set G such that $h^{-1}(G)$ is not open in $\mathbb{R}$, and a closed set F, such that $h^{-1}(F)$ is not closed in $\mathbb{R}$.

If we take $G = (0, 2)$, and open set, $h^{-1}(G) = \{x \in \mathcal{D}(f) : h(x) \in G\} = [0, 1]$, a closed set. If we take $F = [-2, -1]$, a closed set, the inverse image, $h^{-1}(F) = \{x \in \mathcal{D}(f) : h(x) \in F\} = [0, 1]$ is the union of two open sets $(-\infty, 0) \cup (1, +\infty)$ which is open.

C. If f is bounded and continuous on $\mathbb{R}^p \to \mathbb{R}$ and if $f(x_0) > 0$, show that f is strictly positive on some neighborhood of x_0. Does the same conclusion hold if f is merely continuous at x_0?

f is bounded and continuous which implies $0 < f(x_0) < M$, for some $M > 0$. Since f is continuous, for each point $a \in \mathcal{D}(f)$, there is a neighborhood V of $f(a)$ and a neigborhood $U(a) \cap D$ such that if $f(a) \in V \implies a \in U(a)$. Since $f(a) > 0 \implies$ we can take a neighborhood V of $f(a)$ that is also strictly positive, i.e. $V = \{y \in \mathbb{R} : 0 < y < M\}$. If f is not bounded the same argument can be made with $V = \{y \in \mathbb{R} : y > 0\}$.

F. A subset $D \subseteq \mathbb{R}^p$ is disconnected if and only if there exists a continuous function $f : D \to \mathbb{R}$ such that $f(D) = \{0, 1\}$.

$\to D$ disconnected implies there exists two open sets B, C such that $B \cap D$ and $C \cap D$ are disjoint and $(B \cap D) \cup (C \cap D) = D$. We can then construct a function f on D, $f(x) = 1,\ x \in (B \cap D)$, $f(x) = 0,\ x \in (C \cap D)$.
$\leftarrow$ Let $f : D \to \mathbb{R}$ be such that $f(D) = \{0, 1\} \implies$ the inverse image $f^{-1}(\{0, 1\}) = \{x \in D \subseteq: f(x) \in \{0, 1\}\}$ could consist of two disjoint open sets such for f on D, $f(x) = 1,\ x \in (B \cap D)$, $f(x) = 0,\ x \in (C \cap D)$, where $D = (B \cap D) \cup (C \cap D) \subseteq \mathcal{D}(f) \implies$ there exists a continuous function $f : D \to \mathbb{R}$ such that $f(D) = \{0, 1\}$.

H. Let f, g_1, g_2 be related by the formulas in the preceding exercise. Show that from the continuity of g_1 and g_2 at $t = 0$ one cannot prove the continuity of f at $(0, 0)$.

K. Give an example of a bounded and continuous function g on $\mathbb{R} \to \mathbb{R}$ which does not take on either of the numbers $\sup\{g(x) : x \in \mathbb{R}\}$ *or* $\inf\{g(x) : x \in \mathbb{R}\}$

If we take $f : \mathbb{R} \to \mathbb{R},\ f(x) = x,\ x \in (0, 1) \subseteq \mathbb{R}$, the function is bounded above by 1, below by 0, and continuous on $(0, 1)$, but $f(x) \neq 1 = \sup\{f(x) : x \in (0, 1)\}$, and $f(x) \neq 0 = \inf\{f(x) : x \in (0, 1)\}$ for any x in interval $(0, 1)$.

O. Let f be a continuous function on $\mathbb{R} \to \mathbb{R}$ which is strictly increasing (in the sense that if $x^{'} < x^{''}$ then $f(x^{'}) < f(x^{''})$). Prove that f is injective and that its inverse function is continuous and strictly increasing.