6262 HOMEWORK 2

Problem 1. *Assume we have a parameter θ taking only two values, $\{1, 2\}$ and for each of these we have distributions given by*

x	-1	0	1	2
$p(x;1)$	$1/4$	$1/4$	$1/4$	$1/4$
$p(x;2)$	$1/8$	$1/8$	$1/4$	$1/2$

Now we observe the sample $x_1 = -1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 2$, $x_5 = 0$.

(1) *For the case of square loss $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, find the risk function associated to any estimator $\hat{\theta}$.*
(2) *Find the minimax estimator.*

Problem 2. *Assume we take a sample $X_1, X_2, \dots, X_n$ from $N(\theta, 1)$. Assume that a prior distribution on θ is $N(a, b^2)$.*

(1) *Find the posterior distribution of θ.*
(2) *Compute the expectation of this posterior distribution and denote it $\hat{\theta}$.*
(3) *Find the risk function associated to this estimator with respect to the square loss function $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$.*
(4) *Find the maximum of this risk over all θ. For which values of a, b is this finite?*
(5) *Try the same thing, this time with respect to the loss function $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^4$.*

Problem 3. *Assume that $X \sim N(\theta, 1)$ and take the estimator $\hat{\theta} = cX$ for some constant. Given a loss function L, we say that the estimator $\hat{\theta}$ is better than $\bar{\theta}$ if $R(\hat{\theta}, \theta) \le R(\bar{\theta}, \theta)$ for all choices of $\theta \in \mathbb{R}$.*

(1) *Find the risk associated to $\hat{\theta}$ for the square loss function. Show that θ_1 is better than θ_c for any $c > 1$.*
(2) *Is $\hat{\theta}_{1/2}$ better than θ_1? Comment?*

Problem 4.

(1) *Assume $X_1, X_2, \dots, X_n$ is a sample from a Bernoulli random variable with parameter p. Under the assumption that the prior of p is a $Beta(\alpha, \beta)$, find the posteriori distribution of p. Show that this estimator is a Bayes estimator for the square loss function.*
(2) *Under the same assumptions, assume that the loss function is $L(\hat{p}, p) = \frac{(\hat{p}-p)^2}{p(1-p)}$. Find a Bayes rule for this loss function.*
(3) *Can you do the same for the loss function $L(\hat{\theta}, \theta) = \frac{(\hat{p}-p)^2}{p^\alpha(1-p)^\beta}$?*

Problem 5. *Let X be a Bernoulli with parameter p and assume the prior is uniform on $[0, 1]$.*

(1) *Find the posterior distribution of p and then the Bayes estimator for the square loss function.*
(2) *If in addition, we know that $p \notin (1/3, 2/3)$ and use the prior to be the uniform on the $(0, 1/3) \cup (2/3, 1)$, find the Bayes estimator now, again with respect to the square loss function.*

Problem 6. *This problem is about the Γ and $Beta$ distributions. For $\alpha, \beta > 0$, we set*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

while

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

(1) $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ *and* $\Gamma(1) = 1$, $\Gamma(n) = (n-1)!$.

(2) *Show that*

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx, \text{ for } x > 0 \\ 0, \text{ for } x < 0 \end{cases}$$

is a density. It is the density of a Gamma distribution with parameters $\alpha, \beta > 0$*, in short* $\Gamma(\alpha, \beta)$*.*

(3) *Compute the mean and the variance of* $\Gamma(\alpha, \beta)$*.*

(4) *Show that the following*

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{x^\alpha (1-x)^\beta}{B(\alpha,\beta)}, 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$$

is a density. This is called the $Beta(\alpha, \beta)$*. Compute the mean and the variance of* $Beta(\alpha, \beta)$*.*