Math 4317 (Prof. Swiech, S'18): HW #4

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Section 20

A. Prove that if f is defined for $x \geq 0$ by $f(x) = \sqrt{x}$, then f is continuous at every point of its domain.

For $f(x) = \sqrt{x},\ \mathcal{D}(f) = \{x \in \mathbb{R} : x \geq 0\}$, let $a \in \mathcal{D}(f)$.

When $a = 0,\ |f(x) - f(a)| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon$. If we let $\delta(\varepsilon) = \varepsilon^2$, when $x < \varepsilon^2,\ |f(x)| < \varepsilon$.

When $a \neq 0,\ |f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = \frac{|\sqrt{x}-\sqrt{a}|}{|\sqrt{x}+\sqrt{a}|}|\sqrt{x} + \sqrt{a}| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} < \frac{|x-a|}{\sqrt{a}} < \varepsilon \implies$ when $|x - a| < \varepsilon\sqrt{a}$, then, $|f(x) - f(a)| < \varepsilon$, thus we can choose $\delta(\varepsilon) = \varepsilon\sqrt{a} \implies f$ is continuous at every point in its domain.

B. Show that a "polynomial function"; that is, a function f with the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,\ x \in \mathbb{R}$ is continuous at every point of $\mathbb{R}$.

Relying on the properties of algebraic combinations of continuous of functions, we construct f as a combination of continuous functions to show its continuity. Considering the last term of the polynomial function, denoted here, $f_0(x) = a_0,\ f_0(x)$ is a continuous, constant function, since, for any $a \in \mathbb{R}$ we have $|f_0(x) - f_0(a)| = |a_0 - a_0| < \varepsilon = \delta(\varepsilon),\ \varepsilon > 0$. We consider the second to last term of f, $a_1 x$, as a constant, a_1 multiplied by the identity function, denoted, $f_1(x) = x$. Since $f_1(x) = x$, for any real number $a \in \mathbb{R}$, we have $|f_1(x) - f_1(a)| = |x - a| < \varepsilon = \delta(\varepsilon),\ \varepsilon > 0 \implies a_1 f_1(x) = a_1 x$ is continuous.

Relying on the continuity of $f_1(x) = x$ multiplied by any constant, we can construct higher order terms of f through repeated multiplication of $f_1(x)$, e.g. $a_2 \cdot f_1(x) \cdot f_1(x) = a_2 x^2$ and $a_n \prod_{j=1}^{n} f_1(x) = a_n \cdot f_1(x) \cdot f_1(x) \cdot ... \cdot f_1(x) = a_n x^n$, and so on, where each term constructed $a_n x^n$ is continuous on $\mathbb{R}$ since it is constructed via algebraic combinations of continuous functions $\implies f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, is continuous at every point $x \in \mathbb{R}$.

E. Let f be the function on $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = x,\ x$ irrational, $f(x) = 1 - x,\ x$ rational. Show that f is continuous at $x = \frac{1}{2}$ and discontinuous elsewhere.

Considering the point $a = \frac{1}{2}$, we have $f(a) = \frac{1}{2}$, and $|f(x) - f(a)| = |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = |x - a| < \varepsilon = \delta(\varepsilon)$. So if $|f(x) - f(a)| < \varepsilon = \delta(\varepsilon) > 0 \implies |x - a| < \delta(\varepsilon)$, and then we have f continuous at the point $a = \frac{1}{2}$. For the case $a \neq \frac{1}{2},\ a$ irrational, take a sequence $X = (x_n)$ of rational numbers converging to a. Since the sequence $(f(x_n))$ converges to $1 - a$, and we have $f(a) = a$, f is not continuous at irrational points by the Discontinuity Criterion. For the case $a \neq \frac{1}{2},\ a$ rational, take a sequence $Y = (Y_n)$ of irrational numbers converging to a, the sequence (f(y_n)) converges to a, but $f(a) = 1 - a$, which equation is only satisfied when $a = \frac{1}{2}$, thus f is not continuous for rational numbers at any point other than $\frac{1}{2}$.

F.Let f be continuous on $\mathbb{R} \to \mathbb{R}$. Show that if $f(x) = 0$ for rational x, then $f(x) = 0$ for all $x \in \mathbb{R}$.

Every real point, $x \in \mathbb{R}$ is the limit of a sequence of rational numbers. If f is continuous $\implies$ for a sequence of rational numbers $X = (x_n) \to x$, we have $(f(x_n)) = 0$, for all $n \in \mathbb{N}$. Since f is continuous at each rational point $x \in \mathbb{R}$, we can find $|f(x_n) - f(x)| < \varepsilon,\ \varepsilon > 0$, and $|x_n - a| < \delta(\varepsilon) \implies (f(x_n)) \to f(x) = 0, \forall x \in \mathbb{R}$.

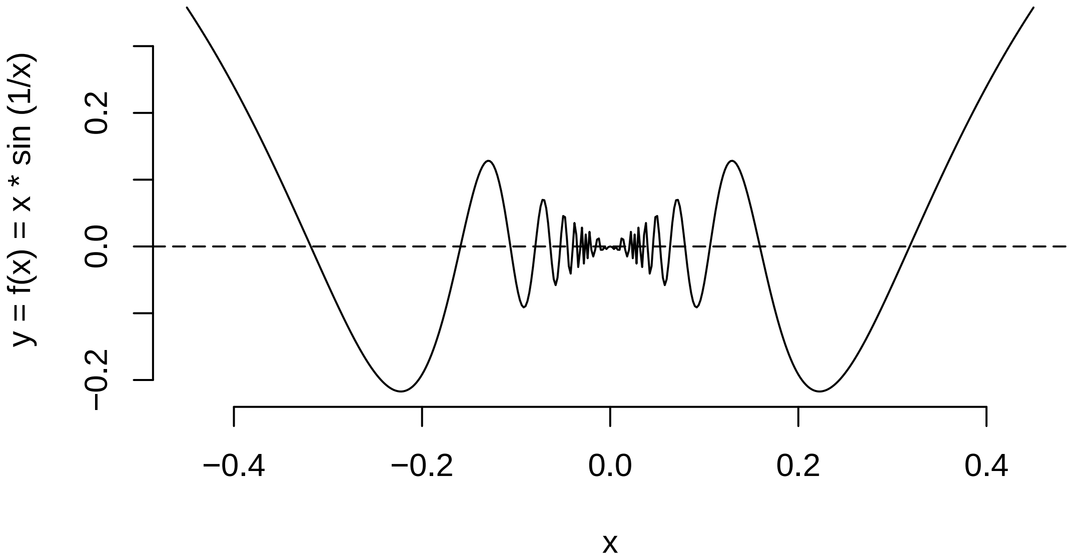
I. Using the results of the preceding exercise, show that the function g, defined on $\mathbb{R} \to \mathbb{R}$ by $g(x) = xsin(\frac{1}{x}),\ x \neq 0,\ g(x) = 0,\ x = 0$ is continuous at every point. Sketch a graph of this function.

For the case $a = 0$, we have $|g(x) - g(a)| = |x \sin\frac{1}{x} - 0| = |x||\sin\frac{1}{x}| \leq |x| \cdot 1 < \varepsilon,\ \varepsilon > 0$, since $-1 \leq \sin\frac{1}{x} \leq 1$. So when $|g(x) - g(0)| < \varepsilon = \delta(\varepsilon)$, we then have $|x| = |x - 0| < \delta(\varepsilon) \implies g$ continuous at 0.

For the case $a \neq 0$, we have $|g(x) - g(a)| = |x \sin\frac{1}{x} - a \sin\frac{1}{a}| = |x \sin\frac{1}{x} - a \sin\frac{1}{a} - a \sin\frac{1}{x} + a \sin\frac{1}{x}| = |(x - a)(\sin\frac{1}{x}) + a(\sin\frac{1}{x} - \sin\frac{1}{a})| \leq |x - a||\sin\frac{1}{x}| + |a||\sin\frac{1}{x} - \sin\frac{1}{a}|$, by Triangle Inequality. Since both $|\sin\frac{1}{x}| \leq 1$ and $|\sin\frac{1}{x} - \sin\frac{1}{a}| \leq 1$, we have $|x - a||\sin\frac{1}{x}| + |a||\sin\frac{1}{x} - \sin\frac{1}{a}| \leq |x - a| \cdot 1 + |a| \cdot 1 = |x - a| + |a| < \varepsilon$.

It then follows that if $\delta(\varepsilon) = \varepsilon - |a|$, i.e. $\varepsilon > \delta(\varepsilon) + |a|$, when $|g(x) - g(a)| < \varepsilon$, then $|x - a| < \delta(\varepsilon) \implies$ g continuous at every point in $\mathbb{R}$.

Sketch of function below:



N. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x + y) = g(x)g(y),\ x, y \in \mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point. Also if $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

If g is continuous at $x = 0 \implies g(x + y) = g(y) = g(0) \cdot g(y)$. This implies also that $g(0)g(y) = g(y) \implies g(0)g(y) - g(y) = 0 = g(y)(g(0) - 1) = 0 \implies g(0) = 1$, or that $g(0) = 0$.
If $g(0) = 0 \implies -g(y) = 0 = g(y)$. In this case then $g(y) = 0,\ \forall y \in \mathbb{R} \implies g(x) = 0,\ \forall x \in \mathbb{R}$.
On the other hand if $g(0) = 1, \implies g(0) \cdot g(y) = g(y)$ continuous for every point $y \in \mathbb{R}$.

Section 21

I. Let g be a linear function from $\mathbb{R}^p \to \mathbb{R}^q$. Show that g is one-one and only if $g(x) = 0$ implies that $x = 0$.

J. If h is a one-one linear function from $\mathbb{R}^p \to \mathbb{R}^p$, show that the inverse function h^{-1} is a linear function from $\mathbb{R}^p \to \mathbb{R}^p$.

K. Show that the sum and the composition of two linear functions are linear functions.

L. If f is a linear map on $\mathbb{R}^p \to \mathbb{R}^q$, define $||f||_{pq} = \sup\{||f(x)|| : x \in \mathbb{R}^p,\ ||x|| \leq 1\}$. Show that the mapping $f \to ||f||_{pq}$ defines a norm on the vector space $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ of all linear functions on $\mathbb{R}^p \to \mathbb{R}^q$. Show that $||f(x)|| \leq ||f||_{pq}||x||$ for all $x \in \mathbb{R}^p$.

Section 22

B.
C.
F.
H.
K.
O.