Midterm 1: Math 6266

Peter Williams

Section 1.1

Exercise 1. Consider the linear regression model with mean zero, uncorrelated, heteroscedastic noise:

$$Y_i = X_i^\intercal \theta + \varepsilon_i,\ for\ i = 1,..,n,\ E\varepsilon_i = 0,\ cov(\varepsilon_i, \varepsilon_j) = \begin{cases} \sigma_i^2, & \text{if } i = j \\ 0, & i \neq j \end{cases} \tag{1}$$

Find expressions for the LSE and response estimator in this model

To set up the problem, take $W^{-1} = diag\{\sigma_1^2, ..., \sigma_n^2\}$, $W = diag\{\frac{1}{\sigma_1^2}, ..., \frac{1}{\sigma_n^2}\}, W^{1/2} = diag\{\sqrt{\frac{1}{\sigma_1^2}}, ..., \sqrt{\frac{1}{\sigma_n^2}}\}$, with $W^\intercal = W$, and $W^{1/2}W^{1/2} = W$, since they are diagonal matrices. Also we will use $w_i = \frac{1}{\sigma_i^2} = W_{ii}$.

Under heteroscedastic noise assumptions, we define the least squares estimator, denoted $\hat{\theta}$, as:

$$\hat{\theta} = \underset{\theta}{argmin} \sum_{i=1}^n w_i (Y_i - X_i^\intercal \theta)^2 = \underset{\theta}{argmin} \sum_{i=1}^n (\sqrt{w_i} Y_i - \sqrt{w_i} X_i^\intercal \theta)^2 = \underset{\theta}{argmin} ||W^{1/2} Y - W^{1/2} X^\intercal \theta||^2$$

$G(\theta) = ||W^{1/2}Y - W^{1/2}X^\intercal\theta||^2 = (W^{1/2}Y - W^{1/2}X^\intercal\theta)^\intercal(W^{1/2}Y - W^{1/2}X^\intercal\theta) = Y^\intercal WY - 2\theta^\intercal XWY + \theta^\intercal XWX^\intercal\theta$

with gradient,

$$\nabla G(\theta) = -2XWY + 2XWX^\intercal\theta$$

Setting this expression equal to zero leads to estimator $\hat{\theta} = (XWX^\intercal)^{-1}XWY$, which leads to response estimator $\hat{Y} = X^\intercal\hat{\theta} = X^\intercal(XWX^\intercal)^{-1}XWY$.

Exercise 2. Assume that $\varepsilon_i \sim N(0, \sigma_i^2)$ in the previous problem. What is known about the distribution of $\hat{\theta}$ and $\hat{Y}$?

For $\hat{\theta}$, we have,

$$E[\hat{\theta}] = E[(XWX^\intercal)^{-1}XWY] = E[(XWX^\intercal)^{-1}XW(X^\intercal\theta^* + \varepsilon)] = E[\theta^*] + E[(XWX^\intercal)^{-1}XW\varepsilon] = \theta^*$$

indicating that $\hat{\theta}$ is unbiased. Further $\hat{\theta}$ is normally distributed, since is a linear transformation of $\varepsilon \sim N(0, W^{-1})$. Further we have,

$$Var(\hat{\theta}) = Var((XWX^\intercal)^{-1}XWY) = Var((XWX^\intercal)^{-1}XW(X^\intercal\theta^* + \varepsilon)) = Var((XWX^\intercal)^{-1}XW\varepsilon)) = ...$$

$$= (XWX^\intercal)^{-1}XWVar(\varepsilon)W^\intercal X^\intercal(XWX^\intercal)^{-1} = (XWX^\intercal)^{-1}XWX^\intercal(XWX^\intercal)^{-1} = (XWX^\intercal)^{-1} = Var(\hat{\theta})$$

For $\hat{Y}$ we have,

$$E[\hat{Y}] = E[X^\intercal(XWX^\intercal)^{-1}XWY] = E[X^\intercal(XWX^\intercal)^{-1}XW(X^\intercal\theta^* + \varepsilon)] = E[X^\intercal\theta^* + X^\intercal(XWX^\intercal)^{-1}XW\varepsilon] = E[X^\intercal\theta^*] = Y$$

and,

$$Var[\hat{Y}] = Var[X^\intercal(XWX^\intercal)^{-1}XWY] = Var[X^\intercal(XWX^\intercal)^{-1}XW(X^\intercal\theta^* + \varepsilon)] = Var[X^\intercal\theta^* + X^\intercal(XWX^\intercal)^{-1}XW\varepsilon] = \ ...$$

$$... = Var[X^\intercal(XWX^\intercal)^{-1}XW\varepsilon] = X^\intercal(XWX^\intercal)^{-1}XW\ Var(\varepsilon)\ W^\intercal X^\intercal(XWX^\intercal)^{-1}X = ...$$

$$= X^\intercal(XWX^\intercal)^{-1}XWX^\intercal(XWX^\intercal)^{-1}X = X^\intercal(XWX^\intercal)^{-1}X$$

Now suppose additionally that $\sigma_i^2 \equiv \sigma^2 > 0$. What can be said about distribution of the estimator $\hat{\sigma^2}$?

With $\sigma_i^2 \equiv \sigma^2 > 0$, we have $\hat{\sigma^2} = \frac{||Y - X^\intercal \hat{\theta}||^2}{n-p} = \frac{||\hat{\varepsilon}||^2}{n-p}$. Further denote, $||\hat{\varepsilon}|| = ||Y - \hat{Y}|| = ||Y - \Pi Y|| = ||(I_n - \Pi)Y||$, also noting that $(I_n - \Pi)X^\intercal = X^\intercal - \Pi X^\intercal = X^\intercal - X^\intercal (XX^\intercal)^{-1} XX^\intercal = X^\intercal - X^\intercal = 0$.

Then we have,

$$(n-p)E[\hat{\sigma^2}] = E||Y - X^\intercal \hat{\theta}||^2 = E||\hat{\varepsilon}||^2 = E[tr(\hat{\varepsilon}\hat{\varepsilon}^\intercal)] = E[tr((I_n - \Pi)YY^\intercal(I_n - \Pi))] = ...$$

$$= E[tr((I_n - \Pi)(X^\intercal\theta^* + \varepsilon)(X^\intercal\theta^* + \varepsilon)^\intercal(I_n - \Pi))] = E[tr((I_n - \Pi)\varepsilon\varepsilon^\intercal(I_n - \Pi))] = tr((I_n - \Pi)E[\varepsilon\varepsilon^\intercal]) = ...$$

Using the cylic property of the trace operator, the property that $(I_n - \Pi)(I_n - \Pi) = (I_n - \Pi)$, and the expectation $E[\varepsilon\varepsilon^\intercal] = \sigma^2 I_n$, leading to

$$... = \sigma^2 tr(I_n - \Pi) = \sigma^2(n-p) = (n-p)E[\hat{\sigma}^2]$$

Looking further at the distribution of $||Y - X^\intercal\hat{\theta}||^2 = \hat{\varepsilon}^\intercal\hat{\varepsilon}$, we have

$$\hat{\varepsilon}^\intercal\hat{\varepsilon} = ((I_n - \Pi)Y)^\intercal((I_n - \Pi)Y) = Y^\intercal(I_n - \Pi)Y = (X^\intercal\theta^* + \varepsilon)^\intercal(I_n - \Pi)(X^\intercal\theta^* + \varepsilon) = \varepsilon^\intercal(I_n - \Pi)\varepsilon$$

Since we know that $\varepsilon \sim N(0, \sigma^2 I_n)$, and further $\frac{\varepsilon^\intercal\varepsilon}{\sigma^2} \sim \chi^2(n)$, $(\frac{\varepsilon}{\sigma})^\intercal(I_n - \Pi)(\frac{\varepsilon}{\sigma}) \sim \chi^2(n-p)$, since we know from earlier that $(I_n - \Pi)$, is idempotent, with rank equal to $tr(I_n - \Pi) = tr(I_n) - tr(\Pi) = n - p$.

Section 1.3

Exercise 4. Let $A \in R^{n\times n}$ be a matrix (corresponding to a linear map in R^n). Show that A preserves length for all $x \in R^n$ iff it preserves the inner product. I.e. one needs to show the following:

$||Ax|| = ||x|| \ \forall \ x \in R^n \iff (Ax)^\intercal(Ay) \ \forall \ x, y \in R^n$.

Take,

$$||x|| = \sqrt{x \cdot x} = \sqrt{x^\intercal x} \implies ||Ax|| = \sqrt{Ax \cdot Ax} = \sqrt{x^\intercal A^\intercal Ax} \implies$$

,

$$A^\intercal A = I_n = A^{-1}, \ A^\intercal = A^{-1}, ||Ax|| = ||x||$$

this implies A is an orthogonal matrix, and further,

$$(Ax)^\intercal(Ay) = ||AxAy||^2 = x^\intercal A^\intercal Ay = x^\intercal y = ||xy||^2$$

Exercise 5. (a) Let $x_0 \in R^n$ be some fixed vector, find a projection map on the subspace $span(x_0)$. Compare your result with matrix Π (from section 1.3) for the case of $p = 1$.

Let $x = span(x_0) = span(x_1, x_2, .., x_n)$, denote the subspace of interest, and $x_1, x_2, ...$ are basis vectors and $y = (y_1, y_2, ..., y_n)^\intercal$. The projection map is,

$$Proj_x(y) = \frac{< y \cdot x >}{< y \cdot y >}x = \sum_{i=1}^{n} \frac{< y_i \cdot \ x_i >}{< y_i \cdot y_i >}x_i$$

For the case $p = 1$, and $\Pi = X^\intercal(XX^\intercal)^{-1}X, X^\intercal \in R^n$, we have,

$$\Pi y = \hat{y} = X^\intercal(XX^\intercal)^{-1}Xy = X^\intercal\frac{Xy}{XX^\intercal} = \frac{\sum_i^n x_i y_i}{\sum_i^n x_i^2}(x_1, x_2, ..., x_n)^\intercal = \frac{< X \cdot y >}{< y \cdot y >}X^\intercal = Proj_X(y)$$

(b) Prove part 3) of Lemma 1.1 for an arbitrary orthogonal projection in R^n. Show $\forall h \in R^n$, $||h||^2 = ||\Pi h||^2 + ||h - \Pi h||^2$.

Using the fact that $(I_n - \Pi)^\intercal(I_n - \Pi) = I_n - 2\Pi + \Pi = I_n - \Pi$, we have,

$||h||^2 = ||\Pi h||^2 + ||h - \Pi h||^2 = h^\intercal\Pi^\intercal\Pi h + h^\intercal(I_n - \Pi)^\intercal(I_n - \Pi)h = h^\intercal\Pi h + h^\intercal(I_n - \Pi)h = h^\intercal I_n h + h^\intercal\Pi h - h^\intercal\Pi h = ||h||^2$

Section 2.1

Exercise 8. Let $X \sim N(0, I_n)$, $Q = X^\intercal X$. Suppose that Q is decomposed into the sum of two quadratic forms: $Q = Q1 + Q2$, where $Qi = X^\intercal A_i X$, $i = 1, 2$ for some symmetric matrices $A1, A2$ with $rank(A1) = n1$ and $rank(A2) = n2$. Show that if $n1 + n2 = n$, then $Q1$ and $Q2$ are independent and $Q_i \sim \chi^2(n_i) for$ $i = 1, 2$.

First note that $X^\intercal X \sim \chi^2(n)$, since $X^\intercal X = \sum_{i=1}^n x_i^2$, which is the sum of iid squared normal random variables with variance 1.

Since $A1$ is a symmetric matrix, we can diagonalize it, $A_1 = U^\intercal \Lambda U$. We know the rank of A_1 is n_1. This implies that $U^\intercal A_1 U = \Lambda = diag\{\Lambda_1, ..., \Lambda_{n_1}, ..., \Lambda_n\}$, has n_1 non-zero, positive eigenvalues, and n_2 eigenvalues that equal zero.

Using the orthogonal matrix U from the decomposition of A_1, we set $X = UY$, so that $X^\intercal X = Y^\intercal U^\intercal UY = Y^\intercal I_n Y = Y^\intercal Y$. So $Q = X^\intercal X = Y^\intercal Y = \sum_{i=1}^n Y_i^2$.

We can write

$$Q = Q_1 + Q_2 = \sum_{i=1}^n Y_i^2 = Y^\intercal U^\intercal A_1 UY + Y^\intercal U^\intercal A_2 UY = Y^\intercal \Lambda Y + Y^\intercal U^\intercal A_2 UY = \sum_{i=1}^n \Lambda_i Y_i^2 + Y^\intercal U^\intercal A_2 UY$$

Since only n_1 eigenvalues in Λ are non-zero, we have

$$Q = \sum_{i=1}^{n_1} \Lambda_i Y_i^2 + \sum_{i=n_1+1}^{n} \Lambda_i Y_i^2 + Y^\intercal U^\intercal A_2 UY = Q = \sum_{i=1}^{n_1} \Lambda_i Y_i^2 + Y^\intercal U^\intercal A_2 UY$$

,

if we organize Λ in way such that the positive eigenvalues on the diagonal are present in the first n_1 diagonal elements. So we have $Q_1 = \sum_{i=1}^{n_1} \Lambda_i Y_i^2$

To solve for $Q_2 = X^\intercal X = Y^\intercal U^\intercal A_2 UY$, from above we have

$$Y^\intercal U^\intercal A_2 UY = Q - Q_1 = Q - \sum_{i=1}^{n_1} \Lambda_i Y_i^2 = \sum_{i=1}^{n_1} Y_i^2 + \sum_{i=n_1+1}^{n} Y_i^2 - \sum_{i=1}^{n_1} \Lambda_i Y_i^2 = \sum_{i=1}^{n_1} (1 - \Lambda_i) Y_i^2 + \sum_{i=n_1+1}^{n} Y_i^2$$

We know the rank of A_2 is $n_2 = n - n_1$. So the term $\sum_{i=1}^{n_1} (1 - \Lambda_i) Y_i^2$ must equal zero, implying that $\Lambda_1 = \Lambda_2 = ... = \Lambda_{n_1} = 1$. This also implies $Q = Q1 + Q2 = \sum_{i=1+1}^{n_1} Y_i^2 + \sum_{i=n_1+1}^{n} Y_i^2$.

Since each squared element $Y_i^2 = X_i^2 \sim \chi^2(1)$ in Q only occurs once in the summand, we can say that $Q1 = \sum_{i=1}^{n_1} Y_i^2 \sim \chi^2(n_1)$, and $Q2 = \sum_{i=n_1+1}^{n} Y_i^2 \sim \chi^2(n_2)$.