

Math 4317 (Prof. Swiech, S'18): HW #3

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Section 14

A. Let $b \in \mathbb{R}$, show $\lim \frac{b}{n} = 0$.

Take $\varepsilon > 0$, if $|\frac{b}{n} - 0| < \varepsilon$, there exists natural number $K(\varepsilon)$ such that $\frac{b}{n} < \frac{b}{K(\varepsilon)} < \varepsilon$. If $n \geq K(\varepsilon)$, and we choose $K(\varepsilon)$ such that $K(\varepsilon) > \frac{b}{\varepsilon} \implies \frac{b}{n} < \varepsilon \implies \lim \frac{b}{n} = 0$.

B. Show that $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

Take $\varepsilon > 0$, note that for $n \in \mathbb{N}$, $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} < \frac{1}{n}$. So we choose natural number $K(\varepsilon)$ such that $\frac{1}{K(\varepsilon)} < \varepsilon$. Therefore if $n \geq K(\varepsilon) \implies \frac{1}{n} < \varepsilon$. Therefore $|\frac{1}{n} - \frac{1}{n+1} - 0| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \varepsilon \implies \lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

D. Let $X = (x_n)$ be a sequence in \mathbb{R}^p which is convergent to x . Show that $\lim \|x_n\| = \|x\|$. (Hint: use the Triangle Inequality.)

(x_n) convergent with limit $x \implies$ there exists natural number $K(\varepsilon)$ such that for $n \geq K(\varepsilon)$, $\|x_n - x\| < \varepsilon$. If $n \geq K(\varepsilon)$. Since by triangle inequality, $|\|x_n\| - \|x\|| \leq \|x_n - x\| < \varepsilon \implies \lim \|x_n\| = \|x\|$.

G. Let $d \in \mathbb{R}$ satisfy $d > 1$. Use Bernoulli's inequality to show that the sequence (d_n) is not bounded in \mathbb{R} . Hence it is not convergent.

We have the sequence $D = (d_n)$, where $d_n = d^n$. Let $d = 1 + a$ for some $a > 0 \implies d^n = (1 + a)^n > 1 + na$ by Bernoulli's inequality. For any $a > b > 0$, $(1 + a)^n > (1 + b)^n$ which implies the sequence d_n is increasing. Take $M > 0$, we have $d^n > 1 + na > M > 0$, if $n > \frac{M}{a} \implies 1 + na > M$. Thus (d_n) is not bounded.

H. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$; show that $\lim(nb^n) = 0$. (Hint: use the Binomial theorem as in Example 14.8(e).)

Let $b = \frac{1}{1+a}$, $a > 0$, we have $b^n = \frac{1}{(1+a)^n}$. By Binomial theorem, $(1+a)^n > \frac{n(n-1)}{2}a^2 \implies \frac{1}{(1+a)^n} < \frac{2}{n(n-1)a^2}$, therefore $nb^n = \frac{n}{(1+a)^n} < \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2}$. Take $\varepsilon > 0$, natural number $K(\varepsilon)$, if $n \geq K(\varepsilon)$ we have $nb^n = \frac{n}{(1+a)^n} < \frac{2}{(n-1)a^2} < \frac{2}{(K(\varepsilon)-1)a^2} < \varepsilon$. Then $|nb^n - 0| < \varepsilon \implies nb^n < \varepsilon \implies \lim nb^n = 0$.

I. Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) < 1$. Show that for some r with $0 < r < 1$ and some $C > 0$, then we have $0 < x_n < Cr^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $\lim(x_n) = 0$.

Since $L = \lim(\frac{x_{n+1}}{x_n}) < 1$, $0 < r < 1 \implies |\frac{x_{n+1}}{x_n} - L| < r$ or $0 < \frac{x_{n+1}}{x_n} < r$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Since $\frac{x_{n+1}}{x_n} < r < 1 \implies x_{n+1} < rx_n < x_n \implies x_n < \frac{x_n}{r}$. If we set $C = \frac{x_n}{r^{n+1}} > 0$, we have $x_n < Cr^n$. Since $\lim_{n \rightarrow \infty} r^n = 0 \implies \lim(x_n) = 0$.

J. Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) > 1$. Show that X is not a bounded sequence and hence is not convergent.

Take $\varepsilon > 0$, since $L = \lim(\frac{x_{n+1}}{x_n}) > 1 \implies |\frac{x_{n+1}}{x_n} - L| < \varepsilon \implies L - \varepsilon < \frac{x_{n+1}}{x_n}$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Take $L - \varepsilon = r > 1$ when ε is small. This implies $x_{n+1} > rx_n$. Take $C = \frac{x_n}{r^{n+1}} > 0 \implies x_{n+1} > Cr^n$. Since $r > 1$, r^n diverges which implies the sequence x_{n+1} is not bounded or convergent.

K. Give an example of a convergent sequence (x_n) of strictly positive real numbers such that $\lim(\frac{x_{n+1}}{x_n}) = 1$. Give an example of a divergent sequence with this property.

Consider convergent sequence $X = (x_n) = (\frac{1}{n})$. $\lim(\frac{x_{n+1}}{x_n}) = 1 \implies |\frac{\frac{1}{n+1}}{\frac{1}{n}} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1} < \varepsilon$, $\varepsilon > 0$.

If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n+1} < \frac{1}{K(\varepsilon)+1} < \varepsilon$, indicating $(\frac{x_{n+1}}{x_n})$ is a convergent sequence with limit 1.

Consider divergent sequence $X = (x_n) = n$. $\lim(\frac{x_{n+1}}{x_n}) = 1 \implies |\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} < \varepsilon, \varepsilon > 0$. If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n} < \frac{1}{K(\varepsilon)} < \varepsilon$, indicating $(\frac{x_{n+1}}{x_n})$ is a convergent sequence with limit 1.

L. Apply the results of Exercises 14.I and 14.J to the following sequences. (Here $0 < a < 1, 1 < b, c > 0$)

(a) (a^n)

$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} = a < 1 \implies a^n$ is convergent, bounded.

(b) (na^n)

$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{(n+1)a^{n+1}}{na^n} = (\frac{n+1}{n})a$ which tends to $1 \cdot a < 1 \implies na^n$ is convergent, bounded.

(c) (b^n)

$\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{b^n} = b > 1 \implies b^n$ is divergent, not bounded.

(d) $(\frac{b^n}{n})$

In this case $\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{\frac{n+1}{n}} = (\frac{n}{n+1})b$ which tends to $1 \cdot b > 1 \implies \frac{b^n}{n}$ diverges, not bounded.

(e) $(\frac{c^n}{n!})$

$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}} = \frac{c}{n+1}$ which tends to $0 < 1$ implying $(\frac{c^n}{n!})$ converges, bounded.

(f) $(\frac{2^{3n}}{3^{2n}})$

$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{2^{3(n+1)}}{3^{2(n+1)}} = \frac{2^3}{3^2} \cdot \frac{1}{3^2} = \frac{8}{9} < 1$ implying $(\frac{2^{3n}}{3^{2n}})$ converges, bounded.

Section 15

C(a-e). For x_n given by the following formulas, either establish the convergence or the divergence of the sequence $X = (x_n)$:

(a) $x_n = \frac{n}{n+1}$

$x_n = \frac{n}{n+1} = \frac{1/n}{1/n + 1/n} = \frac{1}{1 + \frac{1}{n}}$. The limit of the sequence $Y = (y_n) = (1 + \frac{1}{n})$ clearly has limit 1 $\implies \lim(x_n) = \lim \frac{1}{1 + \frac{1}{n}} = \frac{\lim 1}{\lim(1 + 1/n)} = 1 \implies$ this sequence converges to 1.

(b) $x_n = \frac{(-1)^n n}{n+1}$ Let $X = (x_n) = (-1)^n, Y = (y_n) = \frac{n}{n+1}$. Using theorem 15.6.a, if X converges to x , and Y converges to y . $X \cdot Y$ converges to $x \cdot y$. In our case the series $(x_n) = (-1)^n$ diverges, and $(y_n) = \frac{n}{n+1}$ converges to 1 $\implies \lim X \cdot Y = \lim X \cdot 1 = \lim X$ which diverges.

(c) $x_n = \frac{2n}{3n^2+1}$ $x_n = \frac{2n}{3n^2+1} = \frac{1/n}{1/n + 3n} = \frac{2}{3n + \frac{1}{n}}$. We estimate the limit to be 0 \implies for $n \geq K(\varepsilon), |\frac{2}{3n + \frac{1}{n}} - 0| = \frac{2}{3n + 1/n} < \frac{2}{3K(\varepsilon) + 1/K(\varepsilon)} < \varepsilon, \varepsilon > 0 \implies (x_n) \rightarrow 0$. Converges.

(d) $x_n = \frac{2n^2+3}{3n^2+1}$

$x_n = \frac{2n^2+3}{3n^2+1} = \frac{1/n^2}{1/n^2 + 3/n^2} = \frac{2+3/n^2}{3+1/n^2} \rightarrow \frac{2}{3}$. Converges.

(e) $x_n = n^2 - n = n(n-1)$

The sequence $(x_n) = n(n-1)$ is clearly divergent, since for all $M > 0, n \geq M, n(n-1) > M(M-1) > 0$. Diverges.

E. If X and Y are sequences in \mathbb{R}^p and if $X \cdot Y$ converges, do X and Y converge and have $\lim(X \cdot Y) = \lim(X) \cdot \lim(Y)$

As an example, if we take sequences $X = (x_n) = (-1)^n = (-1, 1, -1, \dots)$ and $Y = (y_n) = (-1)^{n+1} = (1, -1, 1, \dots)$, then their product $X \cdot Y = (-1, -1, -1, \dots)$ converges to $-1 \implies$ that the product $X \cdot Y$ converges, but each sequence X and Y does not have a limit, diverges.

As another example, in the case of the constant sequences $X = (x_n) = (1, 1, \dots)$, and $Y = (y_n) = (2, 2, \dots)$, $X \cdot Y$ is the constant sequence $(2, 2, \dots)$ which converges to 2 which equals $\lim X \cdot \lim Y$. Therefore the convergence of $X \cdot Y$ converges does not necessarily mean that each sequence converges, as there are examples of both cases.

F. If $X = (x_n)$ is a positive sequence which converges to x , then $(\sqrt{x_n})$ converges to \sqrt{x} . (Hint: $\sqrt{x_n} - \sqrt{x} = \frac{(x_n - x)}{(\sqrt{x_n} + \sqrt{x})}$ when $x \neq 0$).

In the case that $\lim(x_n) = x = 0$ we have $|x_n - x| = |x_n - 0| = x_n < \varepsilon^2$, $\varepsilon^2 > 0$, $n \geq K(\varepsilon)$, for natural number $K(\varepsilon)$. This implies $0 \leq x_n < \varepsilon^2$ for all $n \geq K(\varepsilon) \implies 0 \leq \sqrt{x_n} < \varepsilon$, $\varepsilon > 0 \implies \sqrt{x_n} - 0 < \varepsilon \implies |\sqrt{x_n} - \sqrt{x}| < \varepsilon$, $n \geq K(\varepsilon) \implies \sqrt{x}$ is limit of $\sqrt{x_n}$ when $x = 0$.

For case $x > 0$, $x > 0 \implies \sqrt{x} > 0$. Since $|\sqrt{x_n} - \sqrt{x}| = \sqrt{x_n} - \sqrt{x} = \sqrt{x_n} - \sqrt{x} \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$. Since $\sqrt{x} > 0$, also implies $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0 \implies \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \leq \frac{x_n - x}{\sqrt{x}} \implies |\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}}(x_n - x) = \frac{1}{\sqrt{x}}|x_n - x| < \varepsilon$, $\varepsilon > 0$. So if $x_n \rightarrow x \implies \sqrt{x_n} \rightarrow \sqrt{x}$ for $x > 0$.

L. If $0 < a \leq b$ and if $x_n = (a^n + b^n)^{\frac{1}{n}}$, then $\lim(x_n) = b$.

Since $0 < a \leq b \implies b^n \leq a^n + b^n \leq b^n + b^n = 2b^n \implies (b^n)^{1/n} \leq (a^n + b^n)^{1/n} \leq (2b^n)^{1/n}$, therefore, $b \leq x_n \leq 2^{1/n}b$. Since $2^{1/n} \rightarrow 1$ as $n \rightarrow \infty \implies b \leq x_n \leq b \implies \lim(x_n) = b$.

N. Let $A \subseteq \mathbb{R}^p$ and $x \in \mathbb{R}^p$. Then x is a boundary point of A if and only if there is a sequence (a_n) of elements in A and a sequence (b_n) of elements in $C(A)$ such that $\lim(a_n) = x = \lim(b_n)$.

\rightarrow Let x be a boundary point of $A \implies$ there is a neighborhood $V = \{y \in \mathbb{R}^p : \|y - x\| < r\}$, $r > 0$, that includes points in A and complement A^c . Since V is a neighborhood of x , by definition of the limit, there is a natural number K_v such that for all $n \geq K_v$, $a_n \in V$ and $b_n \in V \implies (a_n)$ converges to x and (b_n) converges to $x \implies \lim(a_n) = x = \lim(b_n)$.

\leftarrow Let x be limit of sequences (a_n) , $(b_n) \implies$ there is a neighborhood $V = \{y \in \mathbb{R}^p : \|y - x\| < r\}$, $r > 0$ for natural number K_v , such that $n \geq K_v$, $a_n \in V$, $b_n \in V \implies V$ includes points from $(a_n) \in A$ and $(b_n) \in A^c \implies x$ is a boundary point of A .

Section 16

A. Let $x_1 \in \mathbb{R}$ satisfy $x_1 > 1$ and let $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$. Show that the sequence (x_n) is monotone and bounded. What is its limit?

We have $x_1 > 1$ and $x_2 = 2 - \frac{1}{x_1}$. We then have $x_1 > 2 - \frac{1}{x_1} = x_2 > 1$ since $1 > \frac{1}{x_1} > 0$. This implies $x_1 > x_2 > x_3 = 2 - \frac{1}{x_2} > 1$. Using induction we have $x_1 > x_2 = 2 - \frac{1}{x_1} > 1$. We then assume $x_{n-1} > x_n > 1$. For case $n + 1$ we have $x_n > x_{n+1} > 1$. Since $x_n = 2 - \frac{1}{x_{n-1}} > x_{n+1} = 2 - \frac{1}{x_n} > 1$, and since we assume $x_{n-1} > x_n > 1$ this implies $2 - \frac{1}{x_{n-1}} > 2 - \frac{1}{x_n} > 1$, $n \in \mathbb{N}$. This shows (x_n) is a monotone decreasing sequence bounded below by 1. Knowing that this sequence has a limit x that must satisfy the relation $x = 2 - \frac{1}{x} = x \implies 2 = x + \frac{1}{x}$ which is satisfied when $x = 1 \implies$ the limit of this sequence is 1.

B. Let $y_1 = 1$ and $y_{n+1} = (2 + y_n)^{1/2}$ for $n \in \mathbb{N}$. Show that (y_n) is monotone and bounded. What is its limit?

We have $y_1 = 1$, $y_2 = \sqrt{2+1} = \sqrt{3} < 2 \implies y_1 < y_2 < 2$. Using induction we assume $y_{n-1} < y_n < 2$. For case $n + 1$, we have $y_n < y_{n+1} < 2 \iff \sqrt{2+y_{n-1}} < \sqrt{2+y_n} < 2 \implies 2 + y_{n-1} < 2 + y_n < 4 \implies$ directly $y_{n-1} < y_n < 2$. This shows that (y_n) is a monotone increasing sequence bounded above by 2. If a limit of $\lim(y_n) = y$ exists it must satisfy the relation, $y = \sqrt{2+y} \implies y^2 = 2 + y$, and we have $y^2 - y - 2 = (y - 2)(y + 1) = 0$, which has roots 2, -1 . Since (y_n) is positive increasing, its limit must be 2.

E. Show that every sequence in \mathbb{R} either has a monotone increasing subsequence or a monotone decreasing subsequence.

Take an element of the sequence $X = (x_n)$, x_k , such that $x_k \geq x_n$, $n > k$. This implies for each $k_1 < k_2 < \dots < k_j < \dots$ we have $x_{k_1} > x_{k_2} > \dots > x_{k_j}$ which is a decreasing subsequence of X .

Relying on the decreasing subsequence $x_{k_1} > x_{k_2} > \dots > x_{k_j}$, $D = (x_{k_j})$, if we take an index $m_1 > k_j$, such that $x_{m_1} \notin D$, we can construct $x_{m_1} < x_{m_2} < \dots < x_{m_i}$ since there exists $m_2 > m_1$ such that $x_{m_1} < x_{m_2}$ for all m , which is an increasing subsequence of X .

G. Determine the convergence or divergence of the sequence (x_n) where, $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$ for $n \in \mathbb{N}$.

We have $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2} \implies x_1 < x_2 < 1$. Using induction we assume $x_{n-1} < x_n < 1$. For the case $n+1$, we have $x_n < x_{n+1} < 1 \iff \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} < \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n+2} < 1$. Adding $\frac{1}{n+1} > 1$ to each element we have $x_n + \frac{1}{n+1} < x_n + \frac{1}{2n+1} + \frac{1}{2n+2} < 1 + \frac{1}{n+1}$. Since $\frac{1}{2n+2} + \frac{1}{2n+1} > \frac{1}{n+1} \forall n \in \mathbb{N}$, because $\frac{n+1}{2n+2} + \frac{n+1}{2n+1} > 1 \implies x_n < x_{n+1} \implies x_n + (\frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}) < 1 \implies x_n < x_{n+1} < 1 \forall n \in \mathbb{N}$. This implies this sequence converges and is bounded above by 1.

J. Show directly that the following are not Cauchy sequences.

(a) $((-1)^n)$

If we take $\varepsilon = 1 > 0$, for m, n greater than natural number $M(\varepsilon)$, we have $|x_m - x_n| = 2 > \varepsilon$ for case m odd, n even, or case m even, n odd. For the cases m odd and n odd, or m even and n even we have $|x_m - x_n| = 0 < \varepsilon \implies$ there exists $m, n > M(\varepsilon)$ such that $|x_m - x_n| \geq \varepsilon > 0 \implies X = (x_n) = ((-1)^n)$ is not Cauchy.

(b) $(n + \frac{(-1)^n}{n})$

If we consider just the case $m, n > M(\varepsilon) \in \mathbb{N}$, $\varepsilon > 0$. For the case $m = n$ we have $|x_m - x_n| = 0$, but for the case m, n even, $m > n$ we have $|x_m - x_n| = |m + \frac{-1^m}{m} - n - \frac{-1^n}{n}| = |m - n + (\frac{1}{m} - \frac{1}{n})| > 1 > 0$. This implies we can find a positive value of ε such that $|x_m - x_n| \geq \varepsilon \implies X = (x_n) = (n + \frac{(-1)^n}{n})$ is not Cauchy.

(c) (n^2)

For $m, n \in \mathbb{N}$ greater than natural number $M(\varepsilon)$, $\varepsilon > 0$, we have $|x_m - x_n| = |m^2 - n^2| = 0$ for the case $m = n$. For the cases $m > n > 1$, or $1 < m < n$ have $|x_m - x_n| = |m^2 - n^2| \geq 5$, since, for example, case $m = 3, n = 2$, $|m^2 - n^2| = 3^2 - 2^2 = 5$. This implies there exists $m, n > M(\varepsilon)$ such that $|x_m - x_n| \geq \varepsilon > 0 \implies X = (x_n) = n^2$ is not Cauchy.

M. Establish the convergence and limits of the following sequences:

(a) $((1 + \frac{1}{n})^{n+1})$

(c) $((1 + \frac{1}{n})^n)$

(d) $((1 + \frac{1}{n+1})^{3n})$

N. Let $0 < a_1 < b_1$ and define, for $n \in \mathbb{N}$, $a_{n+1} = (a_n b_n)^{1/2}$, $b_{n+1} = \frac{1}{2}(a_n + b_n)$. By induction show that $a_n < b_n$, and show that a_n and b_n converge to the same limit.

Section 17

A. For each $n \in \mathbb{N}$, let f_n be defined for $x > 0$ by $f_n(x) = \frac{1}{nx}$. For what values of x does limit $f_n(x)$ exist?

B. For each $n \in \mathbb{N}$, let g_n be defined for $x \geq 0$ by the formula $g_n(x) = nx$, $0 \leq x \leq \frac{1}{n}$, $g_n(x) = \frac{1}{nx}$, $\frac{1}{n} < x$. Show that $\lim(g_n(x)) = 0$ for all $x > 0$.

D. Show that, if we define f_n on \mathbb{R} by $f_n(x) = \frac{nx}{1+n^2x^2}$, then (f_n) converges on \mathbb{R} .

E. Let h_n be defined on the interval $\mathbb{I} = [0, 1]$ by the formula $h_n(x) = 1 - nx$, $0 \leq x \leq \frac{1}{n}$, $h_n(x) = 0$, $\frac{1}{n} < x \leq 1$.

L. Show that the convergence in Exercise 17.B is not uniform on the domain $x \geq 0$, but that it is uniform on a set $x \geq c$, where $c > 0$.

M. Is the convergence in Exercise 17.D uniform on \mathbb{R} ?

Section 18

A. Determine the limit superior and the limit inferior of the following bounded sequences in \mathbb{R} .

(a) $((-1)^n)$ (b) $((-1)^n/n)$ (c) $((-1)^n + 1/n)$

D. Give a direct proof of Theorem 18.3(c).

F. If $X = (x_n)$ is a bounded sequence of strictly positive elements in \mathbb{R} , show that $\limsup(x_n^{1/n}) \leq \limsup(x_{n+1}/x_n)$.

I. Show that $\limsup X = +\infty$ if and only if there is a subsequence X' of X such that $\lim X' = +\infty$.