Example 6.2.5 (ARE of the Sample Median to the Sample Mean). We obtain this ARE under the Laplace and normal distributions. Consider first the Laplace location model as given in expression (6.2.9); i.e.,

$$X_i = \theta + e_i, \quad i = 1, \ldots, n. \tag{6.2.28}$$

By Example 6.1.1, we know that the mle of θ is the sample median, Q_2. By (6.2.10), the information $I(\theta_0) = 1$ for this distribution; hence, Q_2 is asymptotically normal with mean θ and variance $1/n$. On the other hand, by the Central Limit Theorem, the sample mean $\overline{X}$ is asymptotically normal with mean θ and variance σ^2/n, where $\sigma^2 = \mathrm{Var}(X_i) = \mathrm{Var}(e_i + \theta) = \mathrm{Var}(e_i) = E(e_i^2)$. But

$$E(e_i^2) = \int_{-\infty}^{\infty} z^2 2^{-1} \exp\{-|z|\}\, dz = \int_0^{\infty} z^{3-1} \exp\{-z\}\, dz = \Gamma(3) = 2.$$

Therefore, the $\mathrm{ARE}(Q_2, \overline{X}) = \frac{2}{1} = 2$. Thus, if the sample comes from a Laplace distribution, then asymptotically the sample median is twice as efficient as the sample mean.

Next suppose the location model (6.2.28) holds, except now the pdf of e_i is $N(0,1)$. Under this model, by Theorem 10.2.3, Q_2 is asymptotically normal with mean θ and variance $(\pi/2)/n$. Because the variance of $\overline{X}$ is $1/n$, in this case, the $\mathrm{ARE}(Q_2, \overline{X}) = \frac{1}{\pi/2} = 2/\pi = 0.636$. Since $\pi/2 = 1.57$, asymptotically, $\overline{X}$ is 1.57 times more efficient than Q_2 if the sample arises from the normal distribution. ■

Theorem 6.2.2 is also a practical result for it gives us a way of doing inference. The asymptotic standard deviation of the mle $\widehat{\theta}$ is $[nI(\theta_0)]^{-1/2}$. Because $I(\theta)$ is a continuous function of θ, it follows from Theorems 5.1.4 and 6.1.2 that

$$I(\widehat{\theta}_n) \xrightarrow{P} I(\theta_0).$$

Thus we have a consistent estimate of the asymptotic standard deviation of the mle. Based on this result and the discussion of confidence intervals in Chapter 4, for a specified $0 < \alpha < 1$, the following interval is an approximate $(1-\alpha)100\%$ confidence interval for θ,

$$\left(\widehat{\theta}_n - z_{\alpha/2}\frac{1}{\sqrt{nI(\widehat{\theta}_n)}}, \widehat{\theta}_n + z_{\alpha/2}\frac{1}{\sqrt{nI(\widehat{\theta}_n)}}\right). \tag{6.2.29}$$

Remark 6.2.2. If we use the asymptotic distributions to construct confidence intervals for θ, the fact that the $ARE(Q_2, \overline{X}) = 2$ when the underlying distribution is the Laplace means that n would need to be twice as large for $\overline{X}$ to get the same length confidence interval as we would if we used Q_2. ■

A simple corollary to Theorem 6.2.2 yields the asymptotic distribution of a function $g(\widehat{\theta}_n)$ of the mle.

Corollary 6.2.2. *Under the assumptions of Theorem 6.2.2, suppose $g(x)$ is a continuous function of x which is differentiable at θ_0 such that $g'(\theta_0) \neq 0$. Then*

$$\sqrt{n}(g(\widehat{\theta}_n) - g(\theta_0)) \xrightarrow{D} N\left(0, \frac{g'(\theta_0)^2}{I(\theta_0)}\right). \tag{6.2.30}$$

The proof of this corollary follows immediately from the Δ-method, Theorem 5.2.9, and Theorem 6.2.2.

The proof of Theorem 6.2.2 contains an asymptotic representation of $\widehat{\theta}$ which proves useful; hence, we state it as another corollary.

Corollary 6.2.3. *Under the assumptions of Theorem 6.2.2,*

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) = \frac{1}{I(\theta_0)}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial \log f(X_i;\theta_0)}{\partial\theta} + R_n, \tag{6.2.31}$$

where $R_n \xrightarrow{P} 0$.

The proof is just a rearrangement of equation (6.2.20) and the ensuing results in the proof of Theorem 6.2.2.

Example 6.2.6 (Example 6.2.4, Continued). Let $X_1, \ldots, X_n$ be a random sample having the common pdf (6.2.14). Recall that $I(\theta) = \theta^{-2}$ and that the mle is $\widehat{\theta} = -n/\sum_{i=1}^{n} \log X_i$. Hence, $\widehat{\theta}$ is approximately normally distributed with mean θ and variance θ^2/n. Based on this, an approximate $(1-\alpha)100\%$ confidence interval for θ is

$$\widehat{\theta} \pm z_{\alpha/2}\frac{\widehat{\theta}}{\sqrt{n}}.$$

Recall that we were able to obtain the exact distribution of $\widehat{\theta}$ in this case. As Exercise 6.2.12 shows, based on this distribution of $\widehat{\theta}$, an exact confidence interval for θ can be constructed. ■

In obtaining the mle of θ, we are often in the situation of Example 6.1.2; that is, we can verify the existence of the mle, but the solution of the equation $l'(\widehat{\theta}) = 0$ cannot be obtained in closed form. In such situations, numerical methods are used. One iterative method that exhibits rapid (quadratic) convergence is Newton's method. The sketch in Figure 6.2.1 helps recall this method. Suppose $\widehat{\theta}^{(0)}$ is an initial guess at the solution. The next guess (one-step estimate) is the point $\widehat{\theta}^{(1)}$, which is the horizontal intercept of the tangent line to the curve $l'(\theta)$ at the point $(\widehat{\theta}^{(0)}, l'(\widehat{\theta}^{(0)}))$. A little algebra finds

$$\widehat{\theta}^{(1)} = \widehat{\theta}^{(0)} - \frac{l'(\widehat{\theta}^{(0)})}{l''(\widehat{\theta}^{(0)})}. \tag{6.2.32}$$

We then substitute $\widehat{\theta}^{(1)}$ for $\widehat{\theta}^{(0)}$ and repeat the process. On the figure, trace the second step estimate $\widehat{\theta}^{(2)}$; the process is continued until convergence.