Math 4317 (Prof. Swiech, S'18): HW #3

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Section 14

A. Let $b \in \mathbb{R}$*, show* $\lim \frac{b}{n} = 0$.

Take $\varepsilon > 0$, if $|\frac{b}{n} - 0| < \varepsilon$, there exists natural number $K(\varepsilon)$ such that $\frac{b}{n} < \frac{b}{K(\varepsilon)} < \varepsilon$. If $n \geq K(\varepsilon)$, and we choose $K(\varepsilon)$ such that $K(\varepsilon) > \frac{b}{\varepsilon} \implies \frac{b}{n} < \varepsilon \implies \lim \frac{b}{n} = 0$.

B. Show that $\lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

Take $\varepsilon > 0$, note that for $n \in \mathbb{N}$, $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} < \frac{1}{n}$. So we choose natural number $K(\varepsilon)$ such that $\frac{1}{K(\varepsilon)} < \varepsilon$. Therefore if $n \geq K(\varepsilon) \implies \frac{1}{n} < \varepsilon$. Therefore $|\frac{1}{n} - \frac{1}{n+1} - 0| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \varepsilon \implies \lim(\frac{1}{n} - \frac{1}{n+1}) = 0$.

D. Let $X = (x_n)$ *be a sequence in* $\mathbb{R}^p$ *which is convergent to* x*. Show that* $\lim ||x_n|| = ||x||$*. (Hint: use the Triangle Inequality.)*

(x_n) convergent with limit $x \implies$ there exists natural number $K(\varepsilon)$ such that for $n \geq K(\varepsilon)$, $||x_n - x|| < \varepsilon$. If $n \geq K(\varepsilon)$. Since by triangle inequality, $|||x_n|| - ||x||| \leq ||x_n - x|| < \varepsilon \implies \lim ||x_n|| = ||x||$.

G. Let $d \in \mathbb{R}$ *satisfy* $d > 1$*. Use Bernoulli's inequality to show that the sequence* (d_n) *is not bounded in* $\mathbb{R}$*. Hence it is not convergent.$*

We have the sequence $D = (d_n)$, where $d_n = d^n$. Let $d = 1 + a$ for some $a > 0 \implies d^n = (1+a)^n > 1 + na$ by Bernoulli's inequality. For any $a > b > 0$, $(1+a)^n > (1+b)^n$ which implies the sequence d_n is increasing. Take $M > 0$, we have $d^n > 1 + na > M > 0$, if $n > \frac{M}{a} \implies 1 + na > M$. Thus (d_n) is not bounded.

H. Let $b \in \mathbb{R}$ *satisfy* $0 < b < 1$*; show that* $\lim(nb^n) = 0$*. (Hint: use the Binomial theorem as in Example 14.8(e).)*

Let $b = \frac{1}{1+a}, a > 0$, we have $b^n = \frac{1}{(1+a)^n}$. By Binomial theorem, $(1+a)^n > \frac{n(n-1)}{2}a^2 \implies \frac{1}{(1+a)^n} < \frac{2}{n(n-1)a^2}$, therefore $nb^n = \frac{n}{(1+a)^n} < \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2}$. Take $\varepsilon > 0$, natural number $K(\varepsilon)$, if $n \geq K(\varepsilon)$ we have $nb^n = \frac{n}{(1+a)^n} < \frac{2}{(n-1)a^2} < \frac{2}{(K(\varepsilon)-1)a^2} < \varepsilon$. Then $|nb^n - 0| < \varepsilon \implies nb^n < \varepsilon \implies \lim nb^n = 0$.

I. Let $X = (x_n)$ *be a sequence of strictly positive real numbers such that* $\lim(\frac{x_{n+1}}{x_n}) < 1$*. Show that for some* r *with* $0 < r < 1$ *and some* $C > 0$*, then we have* $0 < x_n < Cr^n$ *for all sufficiently large* $n \in \mathbb{N}$*. Use this to show that* $lim(x_n) = 0$

Since $L = \lim(\frac{x_{n+1}}{x_n}) < 1,\ 0 < r < 1 \implies |\frac{x_{n+1}}{x_n} - L| < r$ or $0 < \frac{x_{n+1}}{x_n} < r$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Since $\frac{x_{n+1}}{x_n} < r < 1 \implies x_{n+1} < rx_n < x_n \implies x_n < \frac{x_n}{r}$. If we set $C = \frac{x_n}{r^{n+1}} > 0$, we have $x_n < Cr^n$. Since $\lim_{n\to\infty} r^n = 0 \implies lim(x_n) = 0$.

J. Let $X = (x_n)$ *be a sequence of strictly positive real numbers such that* $\lim(\frac{x_{n+1}}{x_n}) > 1$*. Show that* X *is not a bounded sequence and hence is not convergent.*

Take $\varepsilon > 0$, since $L = \lim(\frac{x_{n+1}}{x_n}) > 1 \implies |\frac{x_{n+1}}{x_n} - L| = |L - \frac{x_{n+1}}{x_n}| < \varepsilon \implies L - \varepsilon < \frac{x_{n+1}}{x_n}$ for all $n \geq K(\varepsilon) \in \mathbb{N}$. Take $L - \varepsilon = r > 1$ when ε is small. This implies $x_{n+1} > rx_n$. Take $C = \frac{x_n}{r^{n-1}} > 0 \implies x_{n+1} > Cr^n$. Since $r > 1$, r^n diverges which implies the sequence x_{n+1} is not bounded or convergent.

K. Give and example of a convergent sequence (x_n) *of strictly positive real numbers such that* $\lim(\frac{x_n+1}{x_n}) = 1$*. Give an example of a divergent sequence with this property.*

Consider convergent sequence $X = (x_n) = (\frac{1}{n})$. $\lim(\frac{x_n+1}{x_n}) = 1 \implies |\frac{\frac{1}{n+1}}{\frac{1}{n}} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1} < \varepsilon,\ \varepsilon > 0$.

If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n+1} < \frac{1}{K(\varepsilon)+1} < \varepsilon$, indicating $(\frac{x_n+1}{x_n})$ is a convergent sequence with limit 1.

Consider divergent sequence $X = (x_n) = n$. $\lim(\frac{x_n+1}{x_n}) = 1 \implies |\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} < \varepsilon,\ \varepsilon > 0$. If we choose natural number $K(\varepsilon), n \geq K(\varepsilon)$ we have $\frac{1}{n} < \frac{1}{K(\varepsilon)} < \varepsilon$, indicating $(\frac{x_n+1}{x_n})$ is a convergent sequence with limit 1.

L. Apply the results of Exercises 14.I and 14.J to the following sequences. (Here $0 < a < 1, 1 < b, c > 0$)

(a) (a^n)
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} = a < 1 \implies a^n$ is convergent, bounded.

(b) (na^n)
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{(n+1)a^{n+1}}{na^n} = (\frac{n+1}{n})a$ which tends to $1 \cdot a < 1 \implies na^n$ is convergent, bounded.

(c) (b^n)
$\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{b^n} = b > 1 \implies b^n$ is divergent, not bounded.

(d) $(\frac{b^n}{n})$
In this case $\lim(\frac{x_{n+1}}{x_n}) > 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{b^{n+1}}{n+1}}{\frac{b^n}{n}} = (\frac{n}{n+1})b$ which tends to $1 \cdot b > 1 \implies \frac{b^n}{n}$ diverges, not bounded.

(e) $(\frac{c^n}{n!})$
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}} = \frac{c}{n+1}$ which tends to $0 < 1$ implying $(\frac{c^n}{n!})$ converges, bounded.

(f) $(\frac{2^{3n}}{3^{2n}})$
$\lim(\frac{x_{n+1}}{x_n}) < 1$, since $\frac{x_{n+1}}{x_n} = \frac{\frac{2^{3(n+1)}}{3^{2(n+1)}}}{\frac{2^{3n}}{3^{2n}}} = \frac{2^3}{1}\frac{1}{3^2} = \frac{8}{9} < 1$ implying $(\frac{2^{3n}}{3^{2n}})$ converges, bounded.

Section 15

C(a-e). For x_n given by the following formulas, either establish the convergence of the divergence of the sequence $X = (x_n)$:
(a) $x_n = \frac{n}{n+1}$
$x_n = \frac{n}{n+1} = \frac{1/n}{1/n}\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$. The limit of the sequence $Y = (y_n) = (1 + \frac{1}{n})$ clearly has limit 1 $\implies \lim(x_n) = \lim \frac{1}{1+\frac{1}{n}} = \frac{\lim 1}{\lim(1+1/n)} = 1 \implies$ this sequence converges to 1.

(b) $x_n = \frac{(-1)^n n}{n+1}$ Let $X = (x_n) = (-1)^n, Y = (y_n) = \frac{n}{n+1}$. Using theorem 15.6.a, if X converges to x, and Y converges to y. $X \cdot Y$ converges to $x \cdot y$. In our case the series $(x_n) = (-1)^n$ diverges, and $(y_n) = \frac{n}{n+1}$ converges to 1 $\implies \lim X \cdot Y = \lim X \cdot 1 = \lim X$ which diverges.

(c) $x_n = \frac{2n}{3n^2+1}$ $x_n = \frac{2n}{3n^2+1} = \frac{1/n}{1/n}\frac{2n}{3n^2+1} = \frac{2}{3n+\frac{1}{n}}$. We estimate the limit to be 0 $\implies$ for $n \geq K(\varepsilon)$, $|\frac{2}{3n+1/n} - 0| = \frac{2}{3n+1/n} < \frac{2}{3K(\varepsilon)+1/K(\varepsilon)} < \varepsilon,\ \varepsilon > 0 \implies (x_n) \to 0$. Converges.

(d) $x_n = \frac{2n^2+3}{3n^2+1}$
$x_n = \frac{2n^2+3}{3n^2+1} = \frac{1/n^2}{1/n^2}\frac{2n^2+3}{3n^2+1} = \frac{2+3/n^2}{3+1/n^2} \to \frac{2}{3}$. Converges.

(e) $x_n = n^2 - n = n(n-1)$
The sequence $(x_n) = n(n-1)$ is clearly divergent, since for all $M > 0,\ n \geq M$, $n(n-1) > M(M-1) > 0$. Diverges.

E. If X and Y are sequences in $\mathbb{R}^p$ and if $X \cdot Y$ converges, do X and Y converge and have $\lim(X \cdot Y) = \lim(X) \cdot \lim(Y)$

As an example, if we take sequences $X = (x_n) = (-1)^n = (-1, 1, -1, ...)$ and $Y = (y_n) = (-1)^{n+1} = (1, -1, 1, ...)$ then their product $X \cdot Y = (-1, -1, -1, ...)$ which converges to $-1 \implies$ that if the product $X \cdot Y$ converges, but each sequence X and Y does not have a limit, divergea.

As another example, in the case of the constant sequences $X = (x_N) = (1, 1, ...)$, and $Y = (y_n) = (2, 2, ...)$, $X \cdot Y$ is the constant sequence $(2, 2, ...)$ which converges to 2 which equals $\lim X \cdot \lim Y$. Therefore the convergence of $X \cdot Y$ converges does not necessarily mean that each sequence converges.

F. If $X = (x_n)$ is a positive sequence which converges to x, then $(\sqrt{x_n})$ converges to $\sqrt{x}$. (Hint: $\sqrt{x_n} - \sqrt{x} = \frac{(x_n - x)}{(\sqrt{x_n} + \sqrt{x})}$ when $x \neq 0$).

In the case that $\lim(x_n) = x = 0$ we have $|x_n - x| = |x_n - 0| = x_n < \varepsilon^2,\ \varepsilon^2 > 0,\ n \geq K(\varepsilon)$, for natural number $K(\varepsilon)$. This implies $0 \leq x_n < \varepsilon^2$ for all $n \geq K(\varepsilon) \implies 0 \leq \sqrt{(x_n)} < \varepsilon, \varepsilon > 0 \implies \sqrt{x_n} - 0 < \varepsilon \implies |\sqrt{x_n} - \sqrt{x}| < \varepsilon,\ n \geq K(\varepsilon) \implies \sqrt{x}$ is limit of $sqrt x_n$ when $x = 0$.

For case $x > 0$, $x > 0 \implies \sqrt{x} > 0$. Since $|\sqrt{x_n} - \sqrt{x}| = \sqrt{x_n} - \sqrt{x} = \sqrt{x_n} - \sqrt{x} \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$. Since $\sqrt{x} > 0$, also implies $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0 \implies \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \leq \frac{x_n - x}{\sqrt{x}} \implies |\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}}(x_n - x) = \frac{1}{\sqrt{x}}|x_n - x| < \varepsilon, \varepsilon > 0$. So if $x_n \to x \implies \sqrt{x_n} \to \sqrt{x}$ for $x > 0$.

L. If $0 < a \leq b$ and if $x_n = (a^n + b^n)^{\frac{1}{n}}$, then $\lim(x_n) = b$.

Since $0 < a \leq b \implies b^n \leq a^n + b^n \leq b^n + b^n = 2b^n \implies (b^n)^{1/n} \leq (a^n + b^n)^{1/n} \leq (2b^n)^{1/n}$, therefore, $b \leq x_n \leq 2^{1/n} b$. Since $2^{1/n} \to 1$ as $n \to \infty \implies b \leq x_n \leq b \implies \lim(x_n) = b$.

N.Let $A \subseteq \mathbb{R}^p$ and $x \in \mathbb{R}^p$. Then x is a boundary point of A if and only if there is a sequence (a_n) of elements in A and a sequence (b_n) of elements in $\mathcal{C}(A)$ such that $\lim(a_n) = x = lim(b_n)$.

$\to$ Let x be a boundary point of $A \implies$ there is a neighborhood $V = \{y \in \mathbb{R}^p : ||y - x|| < r\},\ r > 0$, that includes points in A and complement A^c. Since V is a neighborhood of x, by definition of the limit, there is a natural number K_v such that for all $n \geq K_v,\ a_n \in V$ and $b_n \in V \implies (a_n)$ converges to x and (b_n) converges to $x \implies \lim(a_n) = x = \lim(b_n)$.

$\leftarrow$ Let x be limit of sequences $(a_n),\ (b_n) \implies$ there is a neigborhood $V = \{y \in \mathbb{R}^p : ||y - x|| < r\},\ r > 0$ for natural number K_v, such that $n \geq K_v,\ a_n \in V,\ b_n \in V \implies V$ includes points from $(a_n) \in A$ and $(b_n) \in A^c \implies x$ is a boundary point of A.

Section 16

A,B,E,G,J,M(a)(c)(d),N

Section 17

A,B,D,E,L,M

Section 18

A(a-c),D,F,I