6262 HOMEWORK 1

Given two random variables X and Y which are square integrable, the definition of the conditional expectation of X given Y is interpreted as the function $\phi(Y)$ such that

$$\mathbb{E}[(X-\phi(Y))^2]$$

is minimized over all possible choices of the function ϕ such that $\phi(Y)$ remains square integrable. We write $\mathbb{E}[X|Y]=\phi(Y)$ or in more statistical slang $\mathbb{E}[X|Y=y]=\phi(y)$, though this could be a bit confusing.

We saw in class that the characterization of $\phi(Y)$ is given by the following equality

$$\mathbb{E}[X\phi(Y)]=\mathbb{E}[\phi(Y)\psi(Y)] \tag{0.1}$$

for any other choice of the function ψ.

As properties of the conditional expectation show the following.

Problem 1. (1) *If $Y=0$, then $E[X|Y]=\mathbb{E}[X]$. Interpret this.*
(2) *The function ϕ may not be unique, however $\phi(Y)$ is uniquely defined.*
(3) *If X and Y are independent, then $\mathbb{E}[X|Y]=\mathbb{E}[X]$. Interpret this.*
(4) *If $Z=h(Y)$ for another function h, then $\mathbb{E}[ZX|Y]=h(Y)\phi(Y)$. Written alternatively, $\mathbb{E}[ZX|Y]=Z\mathbb{E}[X|Y]$.*
(5) *If $X=g(Y)$, then $\mathbb{E}[X|Y]=g(Y)$.*
(6) *Argue that if $\mathbb{E}[X|Y]=\phi(Y)$ and $\mathbb{E}[Z|Y]=\psi(Y)$, then $\mathbb{E}[X+Z|Y]=\phi(Y)+\psi(Y)$.*

Problem 2. *If the pair (X,Y) has a joint pmf or pdf, show that*

$$\mathbb{E}[X|Y=y]=\phi(y) \text{ where } \phi(y)=\begin{cases}\sum_x x\frac{p_{X,Y}(x,y)}{p_Y(y)} & (X,Y) \text{ are discrete with pmf } p_{X,Y}\\ \int x\frac{f_{X,Y}(x,y)}{f_Y(y)}dx & (X,Y) \text{ have joint pdf } f_{X,Y}.\end{cases}$$

Problem 3. *Compute the following conditional expectations:*
(1) $\mathbb{E}[X|X^2]$ *if $X\sim Exp(\lambda)$.*
(2) $\mathbb{E}[X|X^3]$ *if $X\sim N(0,2)$.*
(3) $\mathbb{E}[X-2X^4|X^2]$ *for $X\sim N(0,2)$.*
(4) *Give an example of two different functions ϕ_1 and ϕ_2 such that $\mathbb{E}[X|X^2]=\phi_1(X^2)$ and also $\mathbb{E}[X|X^2]=\phi_2(X^2)$. Are $\phi_1(X^2)$ and $\phi_2(X^2)$ equal?*
(5) $\mathbb{E}[\cos(X)|X^2]$ *if $X\sim N(0,1)$.*
(6) $\mathbb{E}[X|Y]$ *if X,Y are iid $N(0,1)$.*
(7) $\mathbb{E}[X+X^2|X^4]$ *for $X\sim N(0,1)$.*
(8) $\mathbb{E}[X|X+2Y]$ *if X,Y are iid $N(0,1)$.*

Problem 4. (1) *If (X,Y) are iid uniform on $(0,1)$, find $\mathbb{E}[X|X+Y]$. Can you explain?*
(2) *If (X,Y) are uniform on $0<x<y<1$, find $\mathbb{E}[X|Y]$.*
(3) *Assume (X,Y) take the values $(0,1),(2,3),(3,4),(2,4),(4,1)$ with equal probability. Compute $\mathbb{E}[X|Y]$.*

Problem 5. *If (X,Y) have joint pdf given by $f_{X,Y}(x,y)=12(2x+y^2)/7$ on the set $0<x<y<1$ and 0 otherwise, find $\mathbb{E}[X|Y]$.*

Problem 6. *Flip a coin until the head comes up and let X be the number of flips. Compute $\mathbb{E}[X|\cos(\pi X/2)]$.*

Problem 7. *Let $\mathbb{E}[X|Y] = \phi(Y)$ and using (0.1) show that $\mathbb{E}[X] = \mathbb{E}[\phi(Y)]$ and*
$$Var(X) = Var(\phi(Y)) + \mathbb{E}[(X - \phi(Y))^2].$$
In particular argue that $Var(X) \geq Var(\phi(Y))$ with equality if and only if $X = \phi(Y)$ or that X is a function of Y.

Problem 8. *Show that if $X_1, X_2, \ldots, X_n$ is a sample and T is a sufficient statistic, then for any one-to-one map $g : \mathbb{R} \to \mathbb{R}$, $\bar{T} = g(T)$ is also a sufficient statistic.*

Problem 9. *Assume $X_1, X_2, \ldots, X_n$ is a sample with density*
$$f(x:\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$
Show that $T = \prod_{k=1}^n X_k$ is a sufficient statistic. Also show that $T = \sum_{k=1}^n \ln(X_k)$ is also a sufficient statistic.

Show that $U = \sum_{k=1}^n X_i$ IS NOT a sufficient statistic.

Problem 10. *Let $X_1, X_2, \ldots, X_n$ be a sample from a uniform distribution on $[0, \theta]$. Show that $T = \max\{X_1, X_2, \ldots, X_n\}$ is a sufficient statistic. Do this using the definition and also using the factorization theorem.*

Is this a complete statistic? Why or why not?

Problem 11. *Find a sufficient statistic for a sample from $Beta(\theta, 2)$. Recall that the density of $Beta(\alpha, \beta)$ has density $\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ where $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.*

Problem 12. *Find a sufficient statistics for a sample from a Bernoulli distribution with parameter θ. Is this statistic sufficient? Can you find the MVUE for the sample? Hint: write the pmf as $f_X(x;\theta) = \theta^x(1-\theta)^{1-x}$.*

Problem 13. *If $X_1, X_2, \ldots, X_n$ is a sample from the $Poisson(\lambda)$, then $\sum_{k=1}^n X_k$ is a sufficient statistic. Find a MVUE? Justify your answer.*

Problem 14. *Assume that $X_1, X_2, \ldots, X_n$ is a sample from the density*
$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & x > \theta \\ 0, & \text{otherwise.} \end{cases}.$$

1. *Is $f(x;\theta)$ a regular exponential family?*
2. *Find a sufficient statistic for θ.*

Problem 15. *If X is a single sample from $N(0, \theta)$, $\theta > 0$ then X is a sufficient but not complete statistic for θ. Can you give an example of a sufficient and complete statistic?*

Problem 16. *Show that the family $N(\theta, \theta)$ for $\theta > 0$ is a regular exponential family, but $N(\theta, \theta^2)$ is not. Can you find a MVUE for θ.*

Problem 17. *Let $f(x;\theta)$ for θ positive integer be the uniform distribution on $\{1, 2, 3, \ldots, \theta\}$. Take a sample $X_1, X_2, \ldots, X_n$.*

1. *Set $T = \max\{X_1, X_2, \ldots, X_n\}$. Show that T is a sufficient statistic.*
2. *Show that T is also a complete statistic.*
3. *Prove that $\frac{T^{n+1}-(T-1)^{n+1}}{T^n-(T-1)^n}$ is the unique MVUE of θ.*