Admissibility

Def An estimator $\hat{\theta}$ is called inadmissible if there exists another one $\hat{\theta}'$ such that

$$R(\hat{\theta}',\theta) \le R(\hat{\theta},\theta) \quad \forall \theta$$
$$R(\hat{\theta}',\theta) < R(\hat{\theta},\theta) \quad \text{for some } \theta.$$

Otherwise $\hat{\theta}$ is called admissible.

Ex If $X \sim N(\theta,1)$ and $\hat{\theta}(X)=3$. Then $\hat{\theta}$ is admissible because otherwise, for $\hat{\theta}'$ we would have

$$R(\hat{\theta},\theta) = (3-\theta)^2$$

and if there exists $\hat{\theta}'$ with

$$R(\hat{\theta}',\theta) = \int (\hat{\theta}'(x)-\theta)^2 f(x|\theta)\,dx \le (3-\theta)^2$$

which for $\theta=3$ yields that

$$\int (\hat{\theta}'(x)-\theta)^2 f(x;\theta)\,dx = 0$$

from which $\hat{\theta}'(x)=3$ or $\hat{\theta}'=\theta$ a.s. This is okay with the first assumption, but not so with the second one.

We have a number of theorems which give some criteria for admissibility.

Theorem Assume $R(\hat{\theta},\theta)$ is continuous in θ for any $\hat{\theta}$. If f is a prior which is supported on the whole parameter space, then $\hat{\theta}^f$, the Bayes rule associated to f is admissible.

Pf If not so, we have $\hat{\theta}$ such that

$$R(\hat{\theta},\theta) \le R(\hat{\theta}^f,\theta) \text{ for all } \theta \text{ and}$$
$$R(\hat{\theta},\theta_0) < R(\hat{\theta}^f,\theta_0)$$

In particular we have that for some $\delta, \varepsilon > 0$ (from cont of $R(\hat{\theta},\theta)$)

$$R(\hat{\theta}^f,\theta) > R(\hat{\theta},\theta)+\varepsilon, \quad |\theta-\theta_0|<\delta$$

Now

$$r(f;\hat{\theta}^f) - r(f:\hat{\theta}) = \int (R(\hat{\theta}^f,\theta)-R(\hat{\theta},\theta))f(\theta)d\theta$$
$$\geq \int_{\theta_0-\delta}^{\theta_0+\delta} \varepsilon f(\theta)\,d\theta > 0$$

which contradicts the fact that $\hat{\theta}^f$ is a Bayes rule.

Theorem If $X_1,\dots,X_n \sim N(\theta,\sigma^2)$, then $\bar{X}$ is admissible.

Idea: From the previous theorem, if we take $\theta \sim N(a,b^2)$, then the Bayesian estimator is given by

$$\hat{\theta} = \frac{b^2}{b^2+\sigma^2/n}\bar{X} + \frac{\sigma^2}{nb^2+\sigma^2}a$$

For $b\to\infty$ $\hat{\theta}$ and $\bar{X}$ are close to each other and there is a way of showing that $\bar{X}$ if $\bar{X}$ is assumed inadmissible, then $\hat{\theta}$ is also inadmissible.

Theorem If $\hat{\theta}$ has constant risk and is admissible, then $\hat{\theta}$ is minimax.

Pf If $\hat{\theta}$ were not minimax, then $\exists\ \hat{\theta}'$ such that

$$\bar{R}(\hat{\theta}') = \sup_{\theta} R(\hat{\theta}',\theta) < \bar{R}(\hat{\theta}) = R(\hat{\theta},\theta) \quad \forall \theta.$$

Thus $R(\hat{\theta}',\theta) \leq R(\hat{\theta},\theta)\ \forall\ \theta$ and this is strict for some θ which implies that $\hat{\theta}$ is not admissible.

Theorem If $X_1, \ldots, X_n \sim N(\theta,1)$ then $\bar{X}$ is minimax

Pf This follows from the fact that $\bar{X}$ is admissible and because it has constant risk it follows that it must be minimax.

There is a notion of strongly inadmissible which states that $\hat{\theta}$ is so if $\exists\ \hat{\theta}'$ and $\varepsilon > 0$ such that

$$R(\hat{\theta}',\theta) < R(\hat{\theta},\theta) - \varepsilon \quad \forall \theta.$$

If this is the case then any minimax is NOT stronglyadmissible.

Indeed if it where, then

$$R(\hat{\theta}',\theta) < R(\hat{\theta},\theta) - \varepsilon \text{ and thus}$$

$$\sup_{\theta} R(\hat{\theta}',\theta) < R(\hat{\theta},\theta) - \varepsilon/2 <$$

$$< \sup_{\theta} R(\hat{\theta},\theta) - \varepsilon/2$$

which is a contradiction.