VID30 Partiell derivata, intro

den 24 juli 2024 21:36

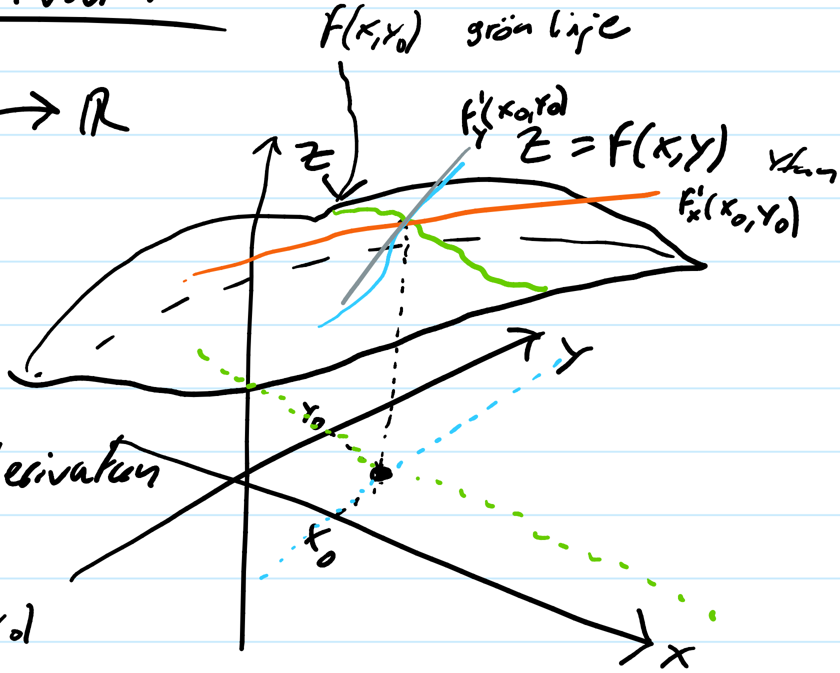
Partiella derivator

$f(x,y)$ $\mathbb{R}^2 \to \mathbb{R}$

$f(x, y_0)$

$\Rightarrow g'(x_0)$

↑ Partiella derivatan av f i punkten (x_0,y_0) med avseende på variabeln x.



$$f'_x(x_0,y_0),\ \frac{\partial f}{\partial x}(x_0,y_0)$$

Def: $f'_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$

$$f'_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

VID31 Partiell derivata, exempel

den 24 juli 2024 21:46

<u>Ex:</u> $f(x,y) = x^4 y^2$

$f'_x(x,y) = 4x^3 \cdot y^2$

$f'_x(2,3) = 4 \cdot 2^3 \cdot 3^2 =$
$= 4 \cdot 8 \cdot 9 = \underline{\underline{288}}$

$f'_y(x,y) = x^4 2y = 2x^4 y$

<u>Ex:</u> $f(x,y) = xe^{xy^2}$

produktregeln

$f'_x = 1 \cdot e^{xy^2} + xy^2 e^{xy}$

kedjeregeln

$f'_y = x \cdot 2xye^{xy^2} = 2x^2ye^{xy^2}$

Vid32 Partiell derivata, primitiv funktion

den 24 juli 2024 21:52

Ex: Finns $f(x,y)$ sådan att

$$\begin{cases} f'_x = 3x^2y + y^2 & (1) \\ f'_y = x^3 + 2xy + 3y^2 & (2) \end{cases}$$

(1) $f(x,y) = \int (3x^2y + y^2)\,dx = x^3y + y^2x + \varphi(y)$

godtycklig (deriverbar) funktion i en variabel

Deriverar med avseende på y

$$f'_y = x^3 + 2xy + \varphi'(y)$$

Jämför med (2) $\cancel{x^3} + \cancel{2xy} + \varphi'(y) = \cancel{x^3} + \cancel{2xy} + 3y^2$

$\Leftrightarrow \varphi'(y) = 3y^2 \Leftrightarrow \varphi(y) = \int 3y^2\,dy + C$

$\Leftrightarrow \varphi(y) = y^3 + C$

Svar: Ja, $f(x,y) = x^3y + xy^2 + y^3 + C$

Ex: Finns det en funktion $f(x,y)$ sådan att

$$\begin{cases} f'_x = x + x^2y & (1) \\ f'_y = \frac{1}{3}x^3 + xy & (2) \end{cases}$$?

(1): $f(x,y) = \int (x + x^2y)\,dx = \frac{1}{2}x^2 + \frac{1}{3}x^3y + \varphi(y)$

Deriverar: $f'_y = \cancel{\frac{1}{3}x^3} + \varphi'(y) \overset{(2)}{=} \cancel{\frac{1}{3}x^3} + xy$

Deriverar: $F'_y = \cancel{3x} + \varphi'(y) \overset{}{\underset{(2)}{=}} \cancel{3x} + xy$

$\Leftrightarrow \varphi'(y) = xy$ Omöjligt ty HL beror av både x och y.

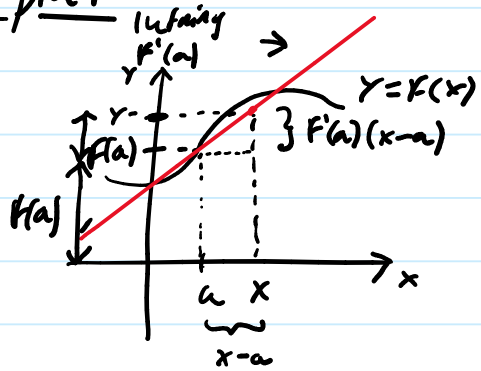
<u>Svar:</u> Nej.

VID33 Tangentplan

den 27 juli 2024 14:27

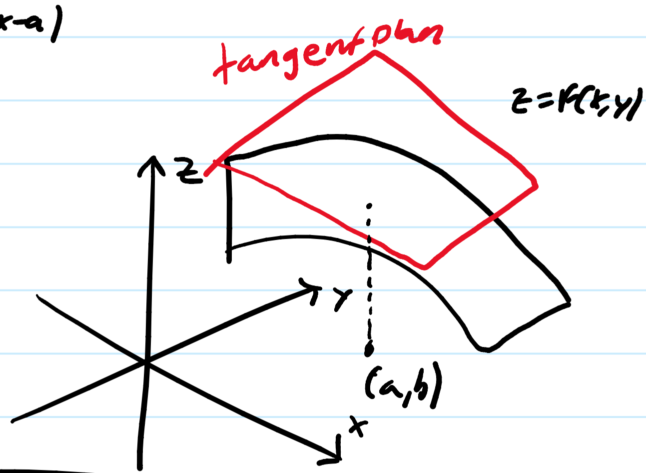
Tangentplan

Endim:

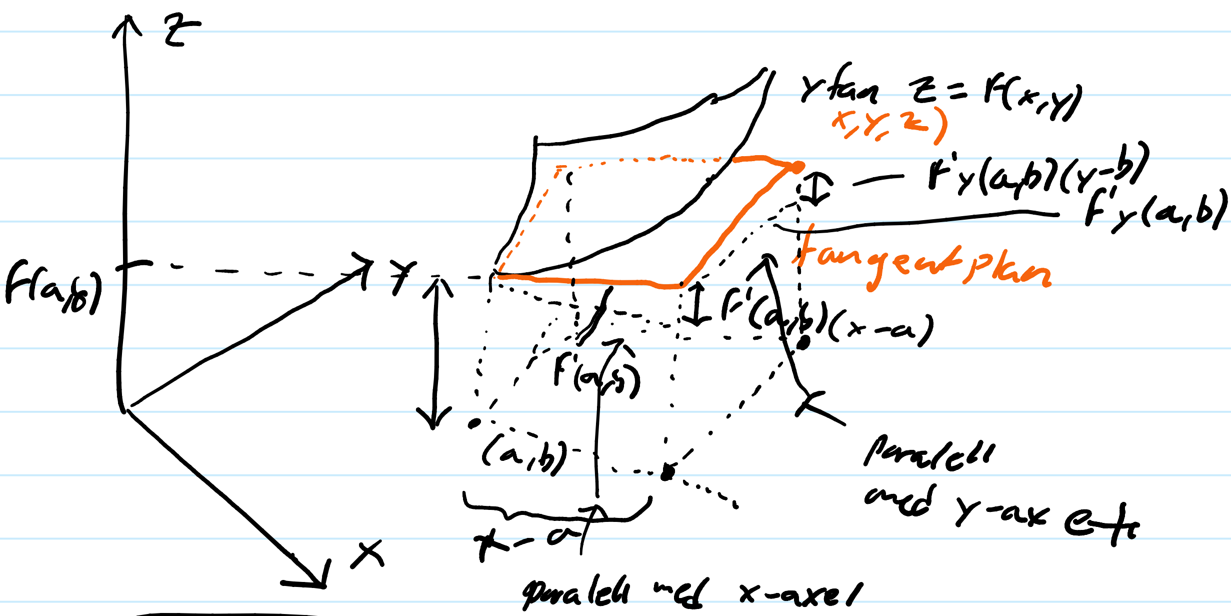


$y = f(a) + f'(a)(x-a)$

Flerdim: $f(x,y)$



$$\boxed{z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)}$$



$$z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)$$

$$\boxed{z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)} \quad (*)$$

Ex: Bestäm en ekvation för tangentplanet till

$z = f(x,y) = x^2y + y^2$ i punkten

$P:(\underset{a}{1}, \underset{b}{-2}, \underset{f(a,b)}{2})$.

$$\begin{cases} f'_x = 2xy \\ f'_y = x^2 + 2y \end{cases} \qquad \begin{cases} f'_x(1,-2) = -4 \\ f'_y(1,-2) = -3 \end{cases}$$

$(*)$ $\quad z = 2 + (-4)(x-1) + (-3)(y-(-2)) =$

$\Leftrightarrow \quad z = 2 - 4x + 4 - 3y - 6$

$\Leftrightarrow \quad \underline{\underline{4x + 3y + z = 0}}$

VID34 Differentierbarhet

den 27 juli 2024 14:46

<u>Deriverbar</u> (Differentierbar kallas det i flervar)

<u>Endim</u>: f är deriverbar i $x=a$ med derivatan A

$$\iff \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = A$$

$$\iff \frac{f(a+h)-f(a)}{h} = A + \rho(h)$$

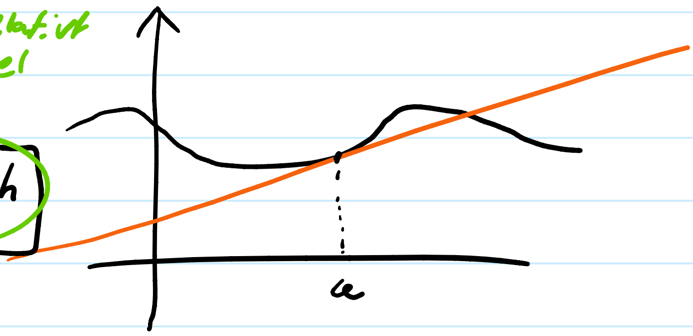
där $\lim_{h\to 0} \rho(h) = 0$

relativt fel

Tangenten

$$\iff \boxed{\underbrace{f(a+h)-f(a) = Ah}_{\text{Tangenten}} + \underbrace{\rho(h)h}_{\text{Fel}}}$$

$A = f'(a)$



<u>Flerdim</u>: $f(x,y)$ är <u>differentierbar</u> i $(x,y)=(a,b)$ om det finns tal A o B sådana att

$$\underbrace{f(a+h, b+k) - f(a,b) = Ah + Bk}_{\text{Tangentplanet}} + \underbrace{\rho(h,k)\sqrt{h^2+k^2}}_{\text{Felet}}$$

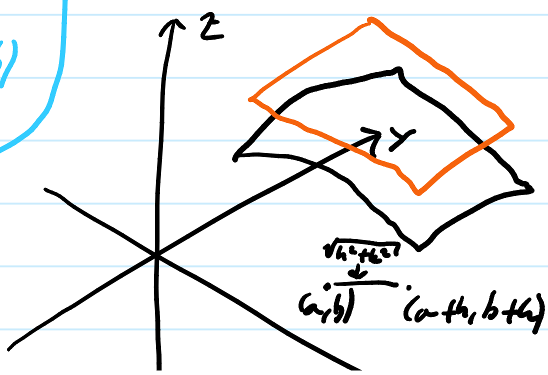
där $\rho(h,k) \to 0$ då $(h,k) \to (0,0)$

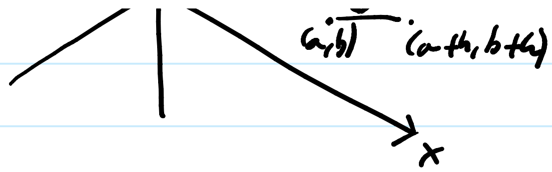
Felet

Tangentplanet

$A = f'_x(a,b)$

$B = f'_y(a,b)$





Ex: Visa att $f(x,y) = xy$ är differentierbar i $(1,2)$.
$a=1,\ b=2$

$2 = f'_x(1,2),\ 1 = f'_y(1,2)$

$$f(1+h,\ 2+k) - f(1,2) = (1+h)(2+k) - (1\cdot 2) =$$
$$= \cancel{2} + 2h + k + hk - \cancel{2} = \underbrace{2}_{A}h + \underbrace{1}_{B}k + \underbrace{\frac{hk}{\sqrt{h^2+k^2}}}_{\rho(h,k)}\sqrt{h^2+k^2}$$

$$\frac{hk}{\sqrt{h^2+k^2}} = \left[\begin{array}{l} h = r\cos\theta \\ k = r\sin\theta \end{array},\ r \to 0\right] \underset{\text{trig. ettan}}{=} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2} = r} = r\cos\theta\sin\theta$$

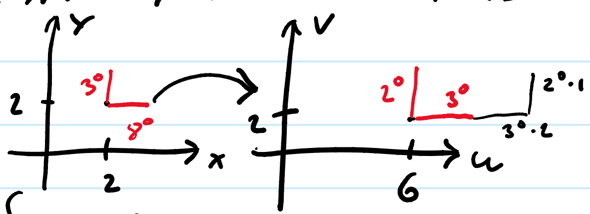
$$\underset{\swarrow 0}{0} \le |\underbrace{r\cos\theta\sin\theta}_{\to 0}| = r\underbrace{|\cos\theta|}_{\le 1}\underbrace{|\sin\theta|}_{\le 1} \le r\cdot 1\cdot 1 = r \to 0$$

VID35 KEDJEREGELN, INTRO

den 27 juli 2024 15:11

tempökning (°)

Ex: $f(x,y) = x^2y - y = v^2(u-2v) - (u-2v) = uv^2 - 2v^3 - u + 2v = \tilde{f}(u,v)$



$$\begin{cases} u = 2x + y \\ v = x \end{cases} \Leftrightarrow \begin{cases} x = v \\ y = u - 2x = u - 2v \end{cases}$$

$f'_x = 2xy,\ f'_x(2,2) = \boxed{8}$, $\tilde{f}'_u = v^2 - 1,\ \tilde{f}'_u(6,2) = \boxed{3}$

$f'_y = x^2 - 1,\ f'_y(2,2) = \boxed{3}$, $\tilde{f}'_v = 2uv - 6v^2 + 2,\ \tilde{f}'_v(6,2) = \boxed{2}$

$8 = 3 \cdot 2 + 2 \cdot 1$

$$\boxed{f'_x = \underbrace{\tilde{f}'_u \cdot u'_x + \tilde{f}'_v \cdot v'_x}_{\text{kedjeregeln}}}$$

Samma för y

VID36 KEDJEREGELN, INLEDANDE EXEMPEL

den 27 juli 2024 15:22

$$f(x,y) = f(u,v) = f(u(x,y), v(x,y)) \quad \begin{cases} u = 2x+y \\ v = x \end{cases}$$

$$f'_x = f'_u \cdot \underbrace{u'_x}_{=2} + f'_v \cdot \underbrace{v'_x}_{=1} = \underline{2f'_u + f'_v}$$

$$f'_y = f'_u \cdot \underbrace{u'_y}_{=1} + f'_v \cdot \underbrace{v'_y}_{=0} = \underline{f'_u}$$

$$\underline{\underline{f'_x - 2f'_y}} = \cancel{2f'_u} + f'_v - \cancel{2f'_u} = f'_v$$

VID37 KEDJEREGELN, PARTIELL DIFFERENTIALEKVATION

den 27 juli 2024 15:28

Ex: Lös $f'_x - 2f'_y = x$ villkor $f(0,y) = e^y$.

$$\begin{cases} u = 2x + y \\ v = x \end{cases}$$

$$f'_x = f'_u \cdot \underbrace{u'_x}_{=2} + f'_v \cdot \underbrace{v'_x}_{=1} = 2f'_u + f'_v$$

$$f'_y = f'_u \cdot \underbrace{u'_y}_{=1} + f'_v \cdot \underbrace{v'_y}_{=0} = f'_u$$

$$(*)\quad \cancel{2f'_u} + f'_v - \cancel{2f'_u} = v \iff \boxed{f'_v = v}$$

$$f(u,v) = \int v\,dv = \frac{1}{2}v^2 + \varphi(u)$$

$$\boxed{f(x,y) = \frac{1}{2}x^2 + \varphi(2x+y)}$$

med villkoret: $f(0,y) = \varphi(y) = e^y$

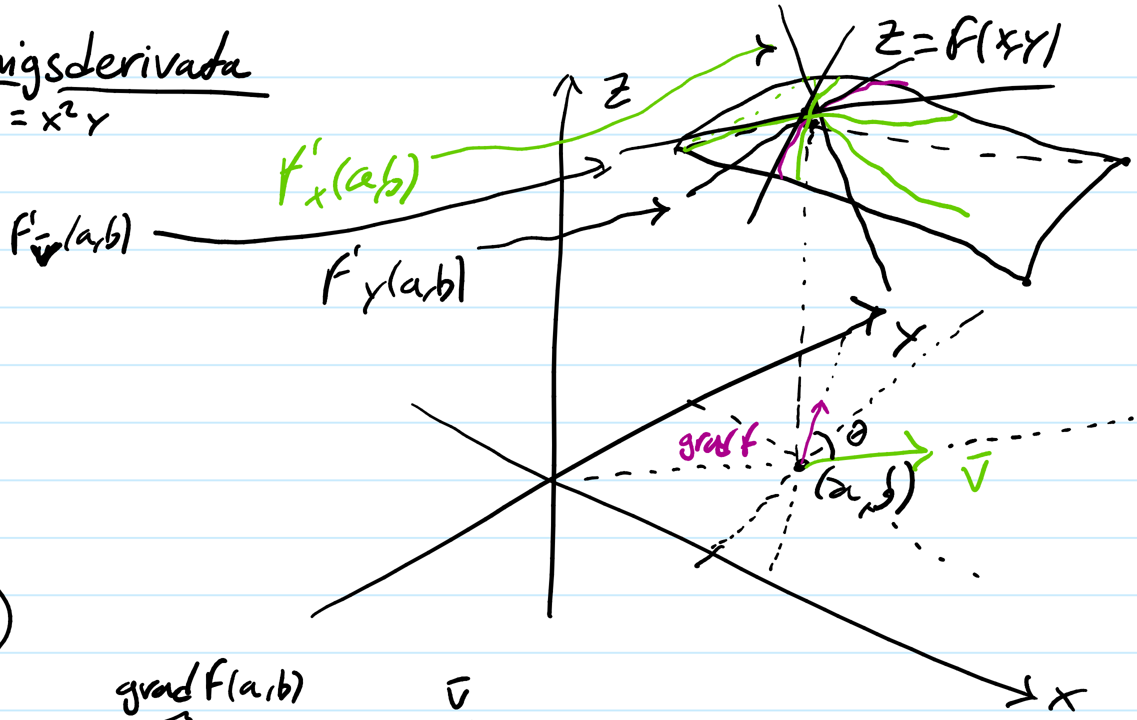
Svar: $f(x,y) = \frac{1}{2}x^2 + e^{2x+y}$

VID38 GRADIENT OCH RIKTNINGSDERIVATA, INTRODUKTION

den 27 juli 2024 15:46

Riktnigsderivata

$f(x,y) = x^2 y$



$|\bar{v}| = 1$

$$f'_{\bar{v}}(a,b) = \underbrace{\left(f'_x(a,b),\ f'_y(a,b)\right)}_{\text{grad } f(a,b)} \cdot \underbrace{(v_1, v_2)}_{\bar{v}} =$$

$$= \left|\text{grad } f(a,b)\right| \underbrace{|\bar{v}|}_{=1} \cos\theta = |\text{grad } f(a,b)| \cos\theta$$

$-1 \le \cos\theta \le 1$

Som störst när $\cos\theta = 1$

dvs $\theta = 0$

VID39 RIKTNINGSDERIVATA, EXEMPEL

den 27 juli 2024 15:58

$f(x,y) = x^2y + xy$ Punkt $(1,2)$

riktning $(3,1)$

$$\bar{v} = \frac{1}{|(3,1)|}(3,1) = \frac{1}{\sqrt{3^2+1^2}}(3,1) = \frac{1}{\sqrt{10}}(3,1)$$

normerat $\bar{v}$ så $|\bar{v}| = 1$.

$$\boxed{f'_{\bar{v}}(a,b) = \text{grad } f(a,b) \cdot \bar{v}}$$

$$\text{grad } f = (f'_x, f'_y) = (2xy + y, x^2 + x)$$

$$f'_{\bar{v}}(1,2) = \text{grad } f(1,2) \cdot \frac{1}{\sqrt{10}}(3,1) =$$

$$= (6,2) \cdot \frac{1}{\sqrt{10}}(3,1) =$$

$$\bar{u} \cdot \bar{v} = |\bar{u}| \cdot |\bar{v}| \cos\theta$$

$$= \frac{1}{\sqrt{10}}(6 \cdot 3 + 2 \cdot 1) = \underline{\underline{\frac{1}{\sqrt{10}} 20}}$$

VID40 GRADIENT OCH NIVÅKURVA

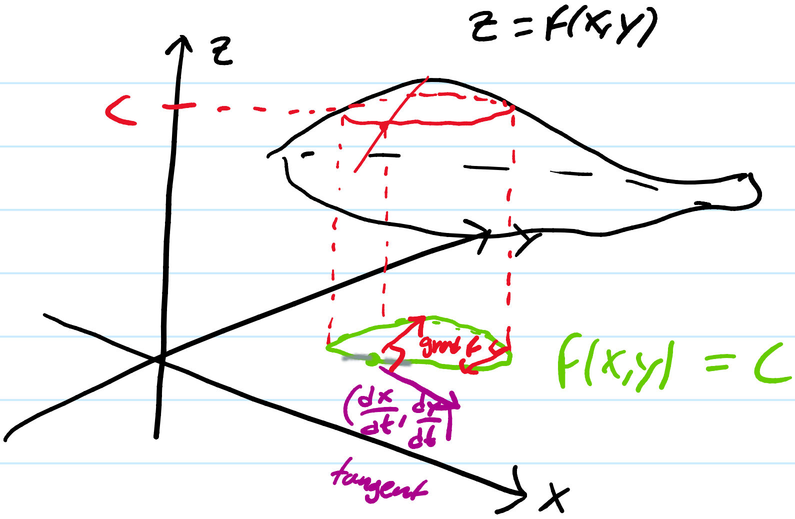
den 27 juli 2024 16:03

$f(x,y) = C$

nivåkurva

$(x,y) = (x(t), y(t))$

$$\boxed{f(x(t), y(t)) = C}$$



$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\Leftrightarrow \underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}_{\text{grad } f} \cdot \underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}_{\text{tangent}} = 0$$

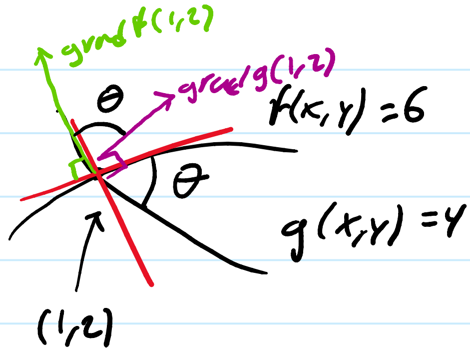
vinkelrät mot nivåkurvan

gradienter är vinkelräta mot nivåkurvor

VID41 GRADIENT OCH VINKEL MELLAN KURVOR

den 27 juli 2024 16:12

$$\begin{cases} f(x,y) = x^3y + xy + y = 6 \\ g(x,y) = xy^3 - x^2y^2 = 4 \end{cases}$$



$$\text{grad} f = (3x^2y + y,\ x^3 + x + 1)$$
$$\text{grad} g = (y^3 - 2xy^2,\ 3xy^2 - 2x^2y)$$

$$\bar{u} = \text{grad } f(1,2) = (8, 3)$$
$$\bar{v} = \text{grad } g(1,2) = (0, 8)$$

$$\begin{cases} \bar{u} \cdot \bar{v} = (8,3) \cdot (0,8) = 8 \cdot 0 + 3 \cdot 8 = 24 \\ \bar{u} \cdot \bar{v} = |\bar{u}||\bar{v}| \cos\theta \end{cases}$$

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{|\bar{u}||\bar{v}|} = \frac{24}{\sqrt{8^2 + 3^2} \cdot 8} = \frac{24}{8\sqrt{73}} = \frac{3}{\sqrt{73}}$$

Så $\theta = \arccos\left(\frac{3}{\sqrt{73}}\right)$

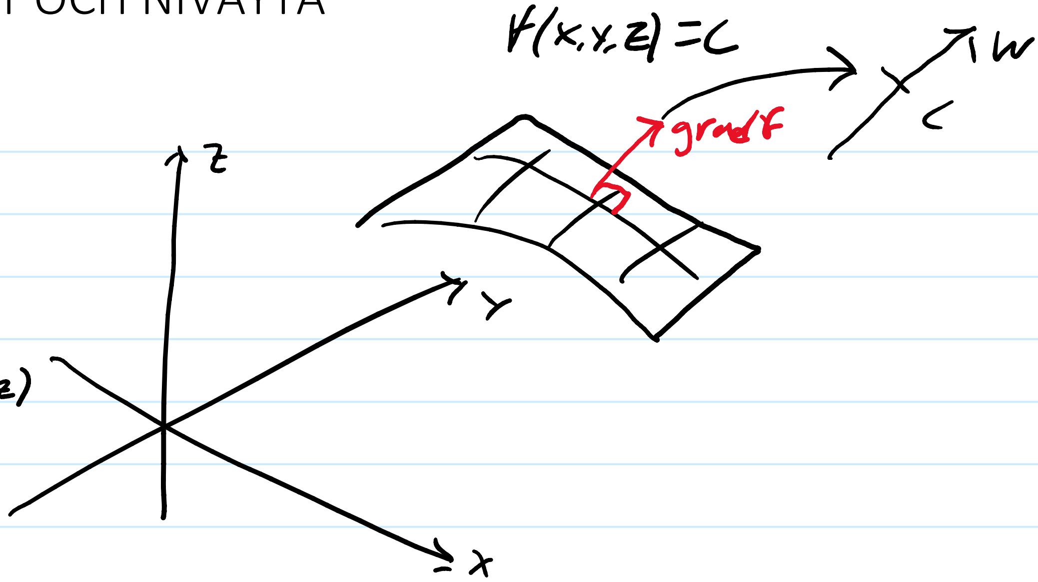
VID42 GRADUEBT OCH NIVÅYTA

den 27 juli 2024 16:20

$f(x,y,z) = C$

nivåyta

$\operatorname{grad} f = (f'_x, f'_y, f'_z)$
vinkelrät mot
nivåytan



Ex: Bestäm en ekvation för tangentplanet π till ellipsoiden

$$E: x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 3 \quad \text{i } P:(1,3,2).$$

E kan ses som en nivåyta till funktionen $f(x,y,z) = x^2 + \frac{y^2}{9} + \frac{z^2}{4}$

$$\operatorname{grad} f = \left(2x, \frac{2}{9}y, \frac{1}{2}z\right) \Rightarrow \operatorname{grad} f(1,3,2) = \left(2, \frac{2}{3}, 1\right) =$$
$$= \frac{1}{3}(6,2,3)$$

är normal till E i P

$\pi: 6x + 2y + 3z + D = 0$

Även P ligger i π:

$6 \cdot 1 + 2 \cdot 3 + 3 \cdot 2 + D = 0 \iff D = -18$

Svar: $6x + 2y + 3z - 18 = 0$

VID43 PARTIELLA ANDRADERIVATOR, INTRO

den 27 juli 2024 16:30

$f(x,y) = x^3y^2 + xy^2 + x$

$f'_x = 3x^2y^2 + y^2 + 1 \quad = \frac{\partial f}{\partial x}$

$f'_y = 2x^3y + 2xy \quad = \frac{\partial f}{\partial y}$

$f''_{xx} \; (f'_x)'_x = 6xy^2 \quad = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

osv

$f''_{xy} \; (f'_x)'_y = 6x^2y + 2y$

$f''_{yy} \; (f'_y)'_y = 2x^3 + 2x$

$f''_{yx} \; (f'_y)'_x = 6x^2y + 2y$

om kontinuerliga så blir det såhär

VID44 TRANSFORMATION AV ANDRADERIVATOR VID KOORDINATBYTE

den 28 juli 2024 19:58

$$\boxed{f(x,y) = f(u(x,y), v(x,y))}$$

$$\begin{cases} u = x^2 + y \\ v = x \end{cases} \qquad \left(\Leftrightarrow \begin{cases} x = v \\ y = u - v^2 \end{cases} \right)$$

$$f'_x = f'_u \cdot u'_x + f'_v \cdot v'_x =$$

$$= f'_u \cdot 2x + f'_v \cdot 1 \left(= f'_u \cdot 2v + f'_v\right)$$

$$f''_{xx} = (f'_x)'_x = (\underbrace{f'_u \cdot 2x}_{\text{produktregeln}} + f'_v)'_x =$$

$$= (f'_u)'_x \cdot 2x + f'_u \cdot 2 + (f'_v)'_x =$$

$$\boxed{f'_u(u(x,v), v(x,y))}$$

kedjeregeln

$$= \left((f'_u)'_u \cdot \overset{=2x}{u'_x} + (f'_u)'_v \cdot \overset{=1}{v'_x}\right) \cdot 2x + 2f'_u + (f'_v)'_u \cdot \overset{=2x}{u'_x} + (f'_v)'_v \cdot \overset{=1}{v'_x} =$$

$$= f''_{uu} \cdot 4x^2 + f''_{uv} \cdot 2x + 2f'_u +$$

$$+ f''_{vu} \cdot 2x + f''_{vv} = 4v^2 f''_{uu} + 4v f''_{uv} + f''_{vv} + 2f'_u$$

VID45 PARTIELL DIFFERENTIALKALKYL AV ORDNING TVÅ

den 28 juli 2024 20:14

Ex: Lös $f''_{xx} + 4x^2 f''_{yy} - 4x f''_{xy} - 2f'_y = 0$ (*)

$$\begin{cases} u = x^2 + y \\ v = x \end{cases}$$

Måste använda kedjeregeln ty: $f'(u(x,y), v(x,y))$

kan välja xy eller yx derivatan ty vi söker f som är kontinuerligt deriverbar två gånger

$$f''_{xy} = (f'_y)'_x = (f'_u)'_x = (f'_u)'_u \cdot \overset{=2x}{u'_x} + (f'_u)'_v \cdot \overset{=1}{v'_x} =$$

$$= f''_{uu} 2x + f''_{uv} = \underline{f''_{uu} 2v + f''_{uv}}$$

$$f''_{yy} = (f'_y)'_y = (f'_u)'_y = (f'_u)'_u \cdot \overset{=1}{u'_y} + (f'_u)'_v \cdot \overset{=0}{v'_y} = f''_{uu}$$

$$(*)\ \cancel{4v^2 f''_{uu}} + \cancel{4v f''_{uv}} + f''_{vv} + \cancel{2f'_u} + \cancel{4v^2 f''_{uu}}$$

$$- \cancel{8v^2 f''_{uu}} - \cancel{4v f''_{uv}} - \cancel{2f'_u} = 0 \iff \boxed{f''_{vv} = 0}$$

$$(f'_v)'_v = 0 \iff f'_v = g(u)$$

$$\iff f(u,v) = g(u)v + h(u)$$

$$\iff \underline{\underline{f(x,y) = g(x^2+y)x + h(x^2+y)}}$$

Så t.ex $f(x,y) = (x^2+y)^2 x + e^{x^2+y}$