VID55 OPTIMERING PÅ KOMPAKT OMRÅDE, EXEMPEL 1

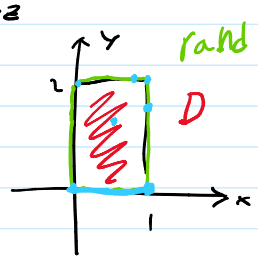
den 2 augusti 2024 13:38

<u>Optimering</u>. Hitta ett största/minsta värde.

$f(x,y) = ye^x - xy^2$, $D: 0 \le x \le 1,\ 0 \le y \le 2$

f kontinuerlig, D sluten & begränsad dvs kompakt

$\Rightarrow f_{max}$ o f_{min} existerar.



1) stationära punkter:

$$\begin{cases} f'_x = ye^x - y^2 = y(e^x - y) = 0 & (1) \\ f'_y = e^x - 2xy = 0 & (2) \end{cases}$$

(1) $\underline{y=0}$: sätter in i (2) $\Rightarrow e^x = 0$ saknar lösning ty $e^x > 0$

$\underline{y = e^x}$: – \|\| – : $e^x - 2xe^x = 0 = e^x(1-2x) = 0$

$\Leftrightarrow x = \frac{1}{2}$ stat punkt $(\frac{1}{2}, e^{1/2})$ $f(\frac{1}{2}, e^{1/2}) = e^{1/2} e^{1/2} - \frac{1}{2} e = \frac{1}{2} e$

<u>Intressanta:</u> $f(\frac{1}{2}, e^{1/2}) = \boxed{\cancel{\tfrac{1}{2}e}}$ $\cancel{<}$ 2

2) <u>Randen</u>: $\bullet f(x,0) = \boxed{0}$

$\bullet g_1(x) = f(x,2) = \underline{2e^x - 4x} \Rightarrow g_1'(x) = 2e^x - 4 = 0 \Leftrightarrow e^x = 2$

$\Leftrightarrow x = \ln 2$

intressant punkt $f(\ln 2, 2) = 2e^{\ln 2} - 4\ln 2 = \boxed{4(1-\ln 2)}$

$(1 - \ln 2) < \frac{1}{2}$?

$\Leftrightarrow \frac{1}{2} < \ln 2 \Leftrightarrow 1 < 2\ln 2$

$\Leftrightarrow 1 < \ln 4$ sant ty $\ln e = 1 < \ln 4$

$\bullet g_2(y) = f(0,y) = y \Rightarrow g_2'(y) = 1 \neq 0$

$\bullet g_3(y) = f(1,y) = ye - y^2 \Rightarrow g_3'(y) = e - 2y = 0 \Leftrightarrow y = \frac{e}{2}$

$f(1, \frac{e}{2}) = \frac{e}{2} e - \frac{e^2}{4} = \boxed{\frac{e^2}{4}}$

3) <u>Hörn</u>: $f(0,0) = f(1,0) = \boxed{0}$

$f(1,2) = 2e^1 - 1 \cdot 2^2 = \boxed{\cancel{2e-4}} < 2 \cdot 3 - 4 = 2$

$f(0,2) = 2e^0 = \boxed{2}$

4) <u>Jämför</u>: $\boxed{f_{min} = 0 \quad f_{max} = 2}$

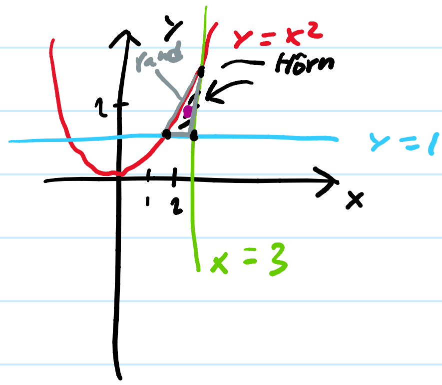
$$\frac{e^2}{4} < \frac{(2,8)^2}{4} = \frac{2.8(3-0,2)}{4} = \frac{8.4 - 0.56}{4} = \frac{7.84}{4} < 2$$

$\Rightarrow f_{max} = 2$

VID56 OPTIMERING PÅ KOMPAKT OMRÅDE, EX 2

den 3 augusti 2024 11:54

$$f(x,y) = \frac{x^2+2y-4}{x^2 y}, \quad y = x^2,\ y=1,\ x=3$$



1) $$f'_x = \frac{2x\cdot x^2 \cancel{y} - 2x\cancel{y}(x^2+2y-4)}{\cancel{x^4}\cancel{y^2}} =$$

$$= \frac{2x^2-2x^2-4y+8}{x^3 y} = \frac{4(2-y)}{x^3 y} = 0$$

$$\Leftrightarrow \boxed{y=2}$$

$$f'_y = \frac{2\cancel{x^2}y - \cancel{x^2}(x^2+2y-4)}{x^4y^2} = \frac{\cancel{2y}-x^2-\cancel{2y}+4}{x^2y^2} = \frac{4-x^2}{x^2y^2} = 0$$

$$\Leftrightarrow \boxed{x=\pm 2}$$

$(2,2)$:

$\cancel{(-2,2)}$: inte i området

$$f(2,2) = \frac{4+4-4}{8} = \frac{1}{2}$$

2) Randen: $f(x,1) = \frac{x^2-2}{x^2} = 1 - \frac{2}{x^2}$ växer då x växer så derivatan inte 0

$$f(3,y) = \frac{9+2y-4}{9y} = \frac{2y+5}{9y} = \frac{2}{9} + \frac{5}{9y}$$ avtar inga intressanta punkter

$$g(x) = f(x,x^2) = \frac{x^2+2x^2-4}{x^4} = \frac{3x^2-4}{x^4} = \frac{3x^2}{x^4} - \frac{4}{x^4}$$

$(3x^{-2} \quad 4x^{-4})$

$$\Rightarrow g'(x) = \frac{-6}{x^3} + \frac{16}{x^5} = \frac{16-6x^2}{x^5} = 0$$

$$\Leftrightarrow x^2 = \frac{16}{6} = \frac{8}{3}$$

$$\Leftrightarrow x = \pm\frac{4}{\sqrt{6}}$$

$$f\left(\frac{4}{\sqrt{6}}, \frac{8}{3}\right) = \frac{3\cdot\frac{16}{6}-4}{\left(\frac{16}{6}\right)^2} = \frac{4\,(36)}{16\cdot 16(4)} = \frac{36}{64} = \boxed{\frac{9}{16}}$$

$$f\left(\frac{4}{\sqrt{6}}, \frac{8}{3}\right) = \boxed{\frac{9}{16}}$$

3) <u>Hörnen:</u>

$$f(1,1) = \frac{1+2-4}{1} = \boxed{-1}$$

$$f(3,1) = \frac{9+2-4}{9} = \boxed{\frac{7}{9}}, \quad f(3,9) = \frac{9+18-4}{81} = \boxed{\frac{23}{81}}$$

<u>Svar:</u> $\boxed{\begin{array}{l} f_{min} = -1 \\ f_{max} = \frac{7}{9} \end{array}}$

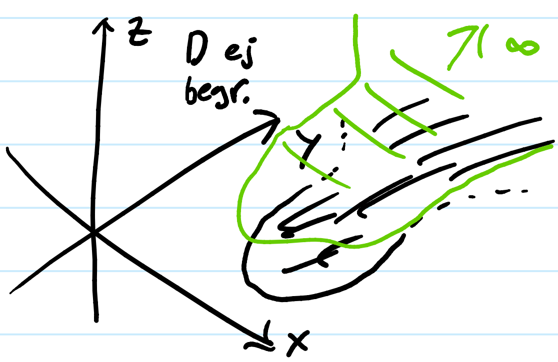
VID57 OPTIMERING PÅ ICKE KOMPAKT OMRÅDE, INTRO

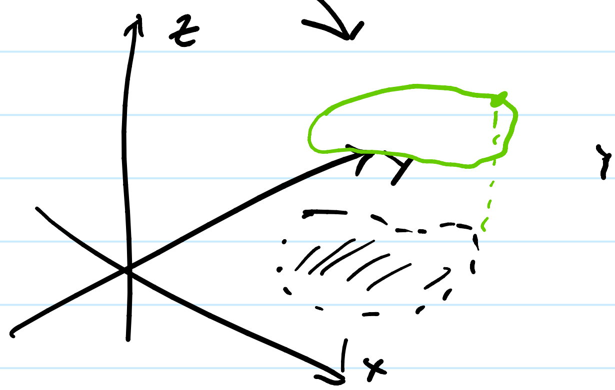
den 4 augusti 2024 13:54

Optimering på icke-kompakta områden

$f(x,y)$ kont. $\quad$ f_{max} o f_{min}

D icke-kompakt $\quad$ behöver ej existera





VID58 OPTIMERIG PÅ ICKE-KOMPAKT OMRÅDE, EX 1

den 4 augusti 2024 14:00

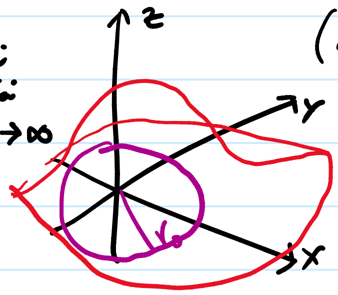
Ex: Bestäm ev. största & minsta värde till

$f(x,y) = (x+2y)e^{-(x^2+y^2)}$, $D: \mathbb{R}^2$ obegr. ej kompakt

kont.

Lösn: $f(x,y) = \dfrac{x+2y}{e^{x^2+y^2}}$ borde gå mot 0 då $\sqrt{x^2+y^2} \to \infty$

(anta r_0 tillr. stort)



Optimera först på D_{r_0} (kompakt) för något r_0.

D_{r_0}

$D_{r_0}: x^2+y^2 \le r_0^2$ kompakt

I) Stat. Punkter:

$$\begin{cases} f'_x = 1 \cdot \cancel{e^{-(x^2+y^2)}} + (x+2y)\cancel{e^{-(x^2+y^2)}}(-2x) = 0 \\ f'_y = 2\cancel{e^{-(x^2+y^2)}} + (x+2y)\cancel{e^{-(x^2+y^2)}}(-2y) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x+2y)2x = 1 & (1) \\ (x+2y)2y = 2 & (2) \end{cases} \qquad (1): x+2y = \frac{1}{2x} \quad (x \neq 0)$$

Sätt in i (2): $\frac{1}{2x} \cdot 2y = 2 \Leftrightarrow \boxed{y = 2x}$

Sätt in i (1): $(x+4x)2x = 1$

$\Leftrightarrow 10x^2 = 1$

$\Leftrightarrow x = \pm\frac{1}{\sqrt{10}}$

ger $y = \pm\frac{2}{\sqrt{10}}$ (samma tecken)

Stat. punkter: $\left(\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{10}}, \frac{-2}{\sqrt{10}}\right)$

$$f\left(\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) = \left(\frac{1}{\sqrt{10}} + \frac{4}{\sqrt{10}}\right)e^{-\left(\frac{1}{10}+\frac{4}{10}\right)} =$$

$$= \boxed{\frac{5}{\sqrt{10}} e^{-1/2}}$$

$$f\left(-\frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = \boxed{-\frac{5}{\sqrt{10}} e^{-1/2}}$$

<u>Gränsvärdet då $\sqrt{x^2+y^2} \to \infty$:</u> $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

$$0 \leq |f(x,y)| = |(r\cos\varphi + 2r\sin\varphi)e^{-r^2}|$$

$$= \underbrace{|\cos\varphi + 2\sin\varphi|}_{\leq 3} r e^{-r^2} \leq 3\frac{r}{e^{r^2}} \to 0$$

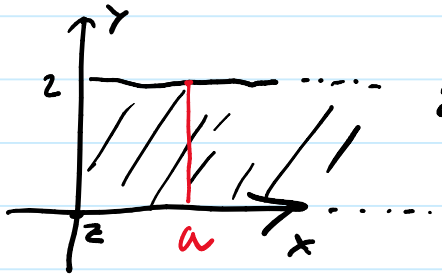
$0 \to 0$, $|f(x,y)| \to 0$

II) $f(x,y) \to 0$ då $\sqrt{x^2+y^2} \to \infty$

Alltså finns ett tal r_0 sådant att $|f(x,y)| < \frac{5}{\sqrt{10}} e^{-1/2}$ då $\sqrt{x^2+y^2} \geq r_0$. På den kompakta mängden D_{r_0} så har f ett största & minsta värde och dessa måste vara i de stationära punkterna $\frac{5}{\sqrt{10}} e^{-1/2}$ respektive $-\frac{5}{\sqrt{10}} e^{-1/2}$, vilket också måste gälla på hela $\mathbb{R}^2$.

Ex: Bestäm ev. största/minsta värde till

$$f(x,y) = x^2 e^{-x-y} \; ; \; D: x \geq 0, \; 0 \leq y \leq 2$$



D obeg
ej kompakt

Lsg. Studera först linjer $x = a$, $0 \leq y \leq 2$

Vi får

$$g(y) = f(a,y) = a^2 e^{-a} \cdot e^{-y}$$
$$= a^2 e^{-a} \cdot \frac{1}{e^y}$$

strängt avtagande
(icke neg konst framför viktigt)

Störst då $y=0$ och minst då $y=2$

OBS! $f(x,y) \geq 0$, och $f(0,0)=0$

så minsta värde är $\boxed{0}$

Återstår bara bestämma f_{max}:

$$h(x) = f(x,0) = x^2 e^{-x}, \quad x \geq 0$$

$$h'(x) = 2x e^{-x} - x^2 e^{-x} = x(2-x)e^{-x}$$

$\Leftrightarrow$ $x=0$ eller $x=2$

	0		2	
$h'(x)$		+	0	−
$h(x)$		↗	$4e^{-2}$	↘

$h(2) = 4e^{-2}$

Svar: $f_{min} = 0$

$f_{max} = 4e^{-2}$

VID60 OPTIMERING MED BIVILLKOR, INTRO

den 4 augusti 2024 15:03

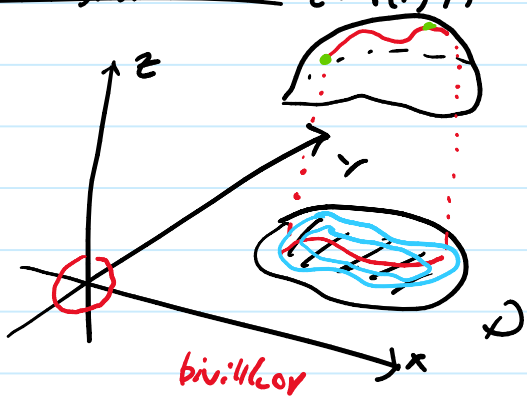
Optimering med bivillkor

- Parametrisera

$h(t) = f(x,y) = f(x(t), y(t))$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

$z = f(x,y)$



bivillkor

$g(x,y) = c$

$g(x,y) = x^2 + y^2 = 1$

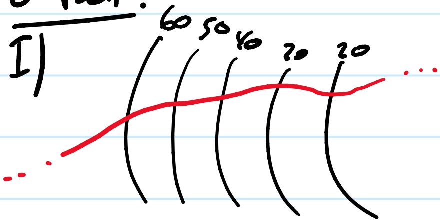
funktion $f(x,y)$

bivillkor $g(x,y) = c$

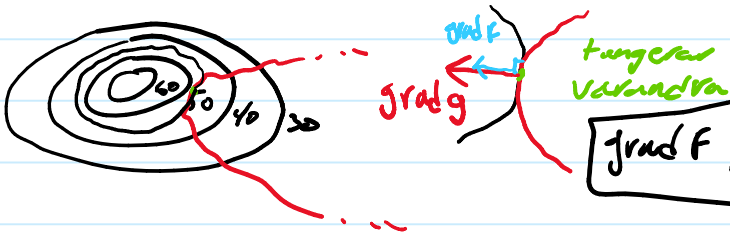
nivåkurvor $f(x,y) = D$

3 fall:

I)



II)



(//)

$\boxed{\text{grad } f \text{ parallell med grad } g}$

III)

Ändpunkter till bivillkoret

Så optimering med bivillkor:

$f(x,y)$

$f(x,y)$

bivillkor $g(x,y)=c$

Intressanta punkter:

(parallell)

- $\operatorname{grad} f \parallel \operatorname{grad} g$
- ändpunkter till bivillkoret

alt. $\begin{vmatrix} f'_x & f'_y \\ g'_x & g'_y \end{vmatrix} = 0$

\+ bivillkoret $g(x,y)=c$

\- $\operatorname{grad} f = (f'_x, f'_y)$ $\operatorname{grad} f \parallel \operatorname{grad} g$ innebär att

$\operatorname{grad} g = (g'_x, g'_y)$

$$\begin{cases} f'_x = \lambda g'_x \\ f'_y = \lambda g'_y \end{cases} \quad ((g'_x, g'_y) \neq (0,0))$$

VID61 OPTIMERING MED BIVILLKOR, EXEMPEL

den 4 augusti 2024 15:50

Ex: Bestäm största & minsta avstånd från kurvan

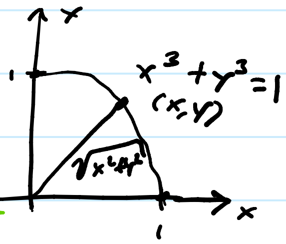
$x^3+y^3=1$, $x\geq 0$, $y\geq 0$, till origo.

Problemet: Vi vill optimera

$f(x,y)=x^2+y^2$ då

$g(x,y)=x^3+y^3=1$, $x\geq 0$, $y\geq 0$

(kompakt mängd)



Ändp.: $(1,0)$, $(0,1)$ $\quad f(1,0), f(0,1)=\boxed{1}$

grad f // grad g

$\text{grad} f=(2x,2y)$

$\text{grad} g=(3x^2,3y^2)$

$$\begin{vmatrix} 2x & 2y \\ 3x^2 & 3y^2 \end{vmatrix} = 6xy^2-6x^2y=0 \quad (1)$$

$$+ \quad x^3+y^3=1 \quad (2)$$

(1): $6xy(y-x)=0 \iff x=0$ eller $y=0$ eller $\boxed{y=x}$

sätter $y=x$ i (2)

(2): $x^3+y^3=1 \iff x^3=\frac{1}{2} \iff x=\frac{1}{2^{1/3}}$

infr. p. $\left(\frac{1}{2^{1/3}},\frac{1}{2^{1/3}}\right)$ $\quad f\left(\frac{1}{2^{1/3}},\frac{1}{2^{1/3}}\right)=\frac{1}{2^{2/3}}\cdot 2=\boxed{2^{1/3}}$

så

$$\boxed{\begin{array}{l} f_{max}=2^{1/3} \\ f_{min}=1 \end{array}}$$

Svar: Längsta avstånd $\sqrt{2^{1/3}}=2^{1/6}$

kortaste —"— $\sqrt{1}=1$.

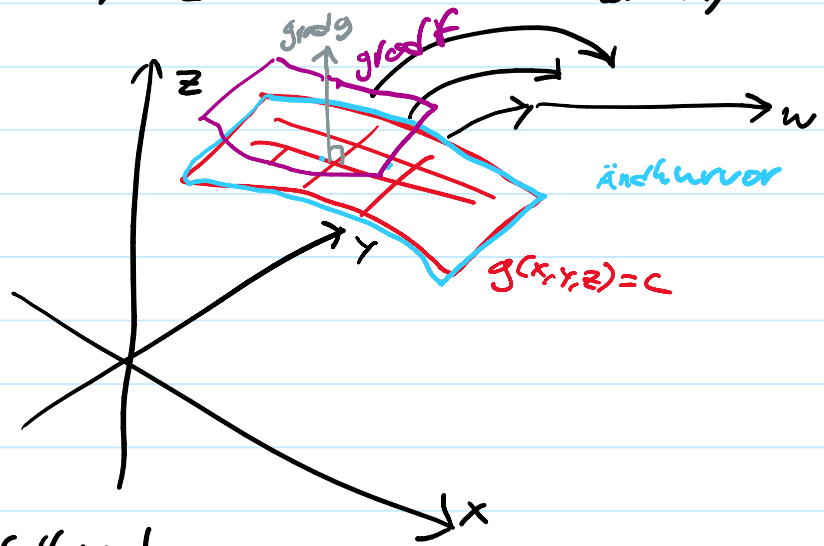
VID62 OPTIMERING MED BIVILLKOR I RUMMET, EX

den 4 augusti 2024 20:26

$f(x,y,z)$

$g(x,y,z)=C$ ——— yta i rummet (bivillkor)

($g(x,y,z)=x^2+y^2+z^2=1$ skulle vara enhetssfären)



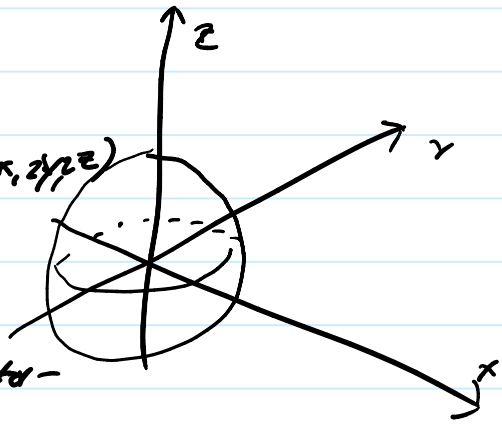
i) grad f // grad g

ii) ändkurvor till bivillkoret

Ex: Bestäm största & minsta värde till

$f(x,y,z) = x+y^2+z$

på enhetssfären $g(x,y,z)=x^2+y^2+z^2=1$ (Bivillkoret).



Lsg. grad f // grad g.

grad $f=(1,2y,1)$, grad $g=(2x,2y,2z)$

$$\begin{cases} 1 = \lambda 2x & (1) \\ 2y = \lambda 2y & (2) \\ 1 = \lambda 2z & (3) \end{cases}$$

Lagranges multiplikator-metod

+ bivillkoret $x^2+y^2+z^2=1$ (4)

(2) : $2y-\lambda 2y = 2y(1-\lambda)=0$

$\Leftrightarrow y=0$ eller $\lambda=1$

(2) : $2y-\lambda 2y = 2y(1-\lambda)=0$

$\Leftrightarrow y=0$ eller $\lambda=1$

$\underline{\lambda=1}$ i (1): $x=\frac{1}{2}$, (3): $z=\frac{1}{2}$, (4): $\frac{1}{4}+y^2+\frac{1}{4}=1 \Leftrightarrow y=\pm\frac{1}{\sqrt{2}}$

Intr. punkter $\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

$\underline{y=0}$: $\begin{cases} 1=\lambda\cdot 2x & (1) \\ 1=\lambda\cdot 2z & (3) \\ x^2+z^2=1 & (4) \end{cases}$ (1),(3): $x=\frac{1}{2\lambda}=z$ $(\lambda\neq 0)$

(4): $x^2+x^2=1 \Leftrightarrow x=\pm\frac{1}{\sqrt{2}}$, $z=\pm\frac{1}{\sqrt{2}}$

(samma tecken)

Intr. punkter $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

$\underline{\text{Jämf.}}$ $f\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)=\boxed{\frac{3}{2}}$

$f\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)=\frac{2}{\sqrt{2}}=\frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}}=\boxed{\sqrt{2}} \approx 1.4$

$f\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)=\boxed{-\sqrt{2}}$

$\underline{\text{Svar:}}$ $f_{max}=1.5$, $f_{min}=-\sqrt{2}$

<u>"undantagspunkter"</u>: grad $g=(0,0,0)$

dvs då $(x,y,z)=(0,0,0)$.

Men $(0,0,0)$ uppfyller ej bivillkoret,

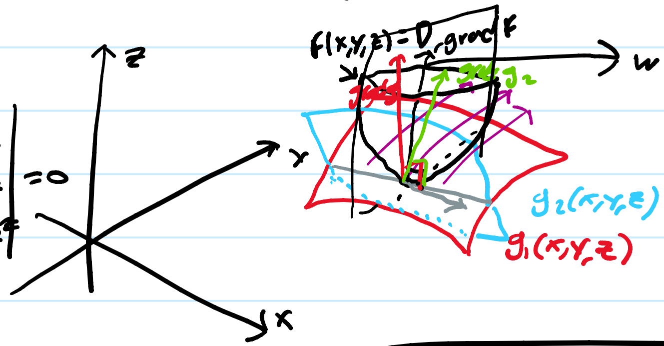
dvs "undantagspunkter" saknas

VID63 OPTIMERING MED TVÅ BIVILLKOR I RUMMET, EX

den 4 augusti 2024 20:44

$f(x,y,z)$ bivillkor: $\begin{cases} g_1(x,y,z)=c_1 \\ g_2(x,y,z)=c_2 \end{cases}$

$$\begin{vmatrix} f'_x & f'_y & f'_z \\ g'_{1x} & g'_{1y} & g'_{1z} \\ g'_{2x} & g'_{2y} & g'_{2z} \end{vmatrix} = 0$$



grad f, grad g_1, grad g_2 ligger i samma plan

alt — " — linjärt beroende

Ex: För vilka punkter på kurvan

$\begin{cases} x^2+y^2+z^2=1 \\ x+2y+2z=0 \end{cases}$ är z-koordinaten störst & minst?

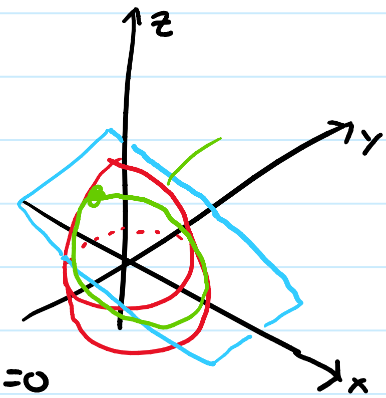
Lsg. Optimera

$f(x,y,z)=z$

Med avseende på bivillkoren

$g_1(x,y,z)=x^2+y^2+z^2=1$

$g_2(x,y,z)=x+2y+2z=0$



Kurvan är kompakt (ändpunkter saknas)

$\Rightarrow f_{max}$ & f_{min} existerar.

- grad f, grad g_1, grad g_2 när linjärt beroende?

grad $f=(0,0,1)$

grad $g_1=(2x,2y,2z)$

grad $g_2=(1,2,2)$

$$\begin{vmatrix} 0 & 0 & 1 \\ 2x & 2y & 2z \\ 1 & 2 & 2 \end{vmatrix} = 0+0+4x-2y-0-0 = 4x-2y=0 \Leftrightarrow \boxed{y=2x}$$

$\text{grad}\, g_2 = (1,2,2)$

$= 4x - 2y = 0 \iff \boxed{y = 2x}$

$y = 2x$

$+ 2$ bivillkor

$$\begin{cases} y = 2x \\ x^2 + y^2 + z^2 = 1 \\ x + 2y + 2z = 0 \end{cases} \iff \begin{cases} \boxed{y = 2x} \\ x^2 + 4x^2 + z^2 = 1 \text{ ger } \\ x + 4x + 2z = 0 \iff \boxed{z = -\frac{5}{2}x} \end{cases}$$

$5x^2 + \frac{25}{4}x^2 = 1 \iff x^2 = \frac{4}{45}$

$x = \pm \frac{2}{\sqrt{45}}$

Två intressanta punkter

$\Rightarrow \left(\frac{2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, -\frac{5}{\sqrt{45}}\right)$

och $\left(-\frac{2}{\sqrt{45}}, -\frac{4}{\sqrt{45}}, \frac{5}{\sqrt{45}}\right)$

<u>Svar:</u> $\min z = -\frac{5}{\sqrt{45}}$

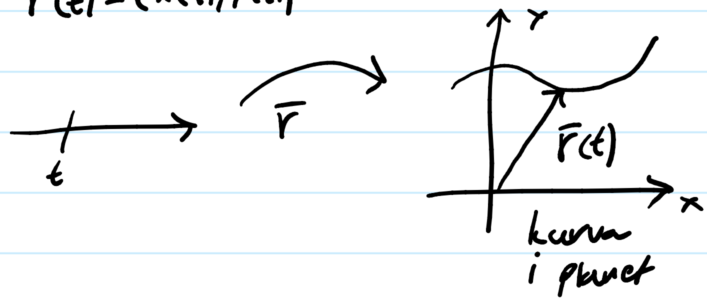
$\max z = \frac{5}{\sqrt{45}}$

VID64 DIFFERENTIALKALKYL VEKTORVÄRD - DERIVATA FÖR KURVOR

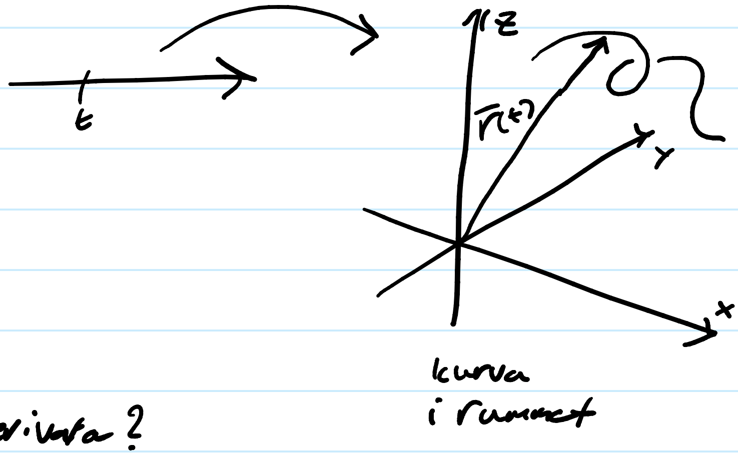
den 5 augusti 2024 19:46

Derivator för vektorvärda funktioner

$\mathbb{R} \to \mathbb{R}^2 \qquad \bar{r}(t) = (x(t), y(t))$



$\mathbb{R} \to \mathbb{R}^3 \qquad \bar{r}(t) = (x(t), y(t), z(t))$

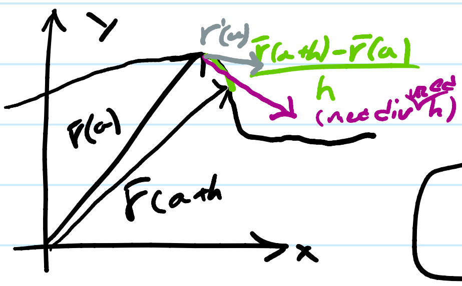


Derivata?

Derivata för vektorvärda funktioner

$\mathbb{R} \to \mathbb{R}^2 \qquad \bar{r}(t) = (x(t), y(t))$

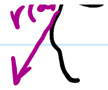
$$\frac{\bar{r}(a+h) - \bar{r}(a)}{h}$$



$$\boxed{\bar{r}'(a) = \lim_{h \to 0} \frac{\bar{r}(a+h) - \bar{r}(a)}{h}}$$

$r'(a)$ är en tangentvektor till kurvan i punkten $\bar{r}(a)$

$\mathbb{R} \to \mathbb{R}^3 \quad \bar{r}(t) = (x(t), y(t), z(t))$



$\mathbb{R} \to \mathbb{R}^2$: $\bar{r}(t) = (x(t), y(t))$

$$\bar{r}'(a) = \lim_{h\to 0} \frac{(x(a+h), y(a+h)) - (x(a), y(a))}{h} =$$

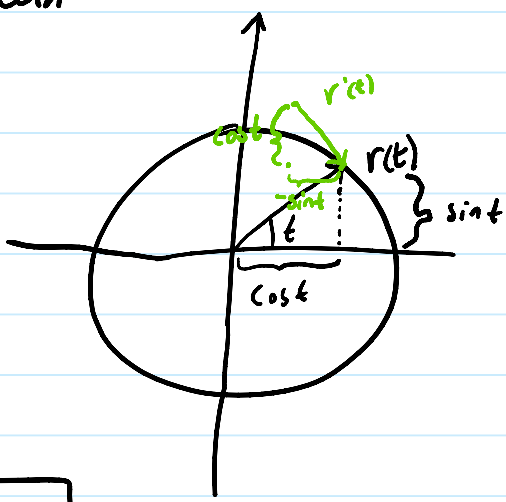
$$= \lim_{h\to 0} \frac{(x(a+h)-x(a),\ y(a+h)-y(a))}{h} =$$

$$= \lim_{h\to 0} \left(\frac{x(a+h)-x(a)}{h}, \frac{y(a+h)-y(a)}{h} \right) = \underline{\underline{(x'(a), y'(a))}}$$

Samma princip för $\mathbb{R} \to \mathbb{R}^3$

Ex: $\bar{r}(t) = (\cos t, \sin t)$ $\quad 0 \le t \le 2\pi$

enhetscirkeln

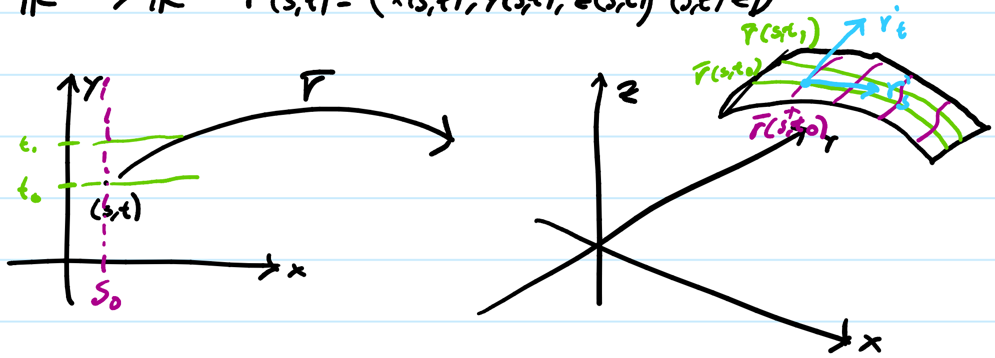


$$\boxed{r'(t) = (-\sin t, \cos t)}$$

VID65 DIFFERENTIALKALKYL VEKTORVÄRD - DERIVATOR FÖR YTOR

den 5 augusti 2024 20:05

$\mathbb{R}^2 \to \mathbb{R}^3 \quad \bar{r}(s,t) = (x(s,t), y(s,t), z(s,t)) \quad (s,t) \in D$

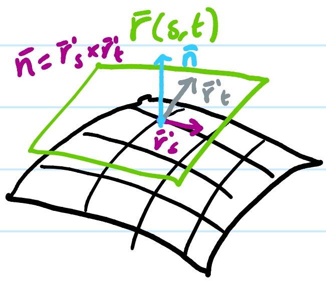


Partiell derivata

$$\boxed{\bar{r}'_s(a,b) = \lim_{h \to 0} \frac{\bar{r}(a+h) - \bar{r}(a,b)}{h}}$$

Samma princip för $\bar{r}_t'(a,b)$.

$$\boxed{\bar{r}'_s = (x'_s, y'_s, z'_s)}$$



$\bar{r}'_s \times r'_t$

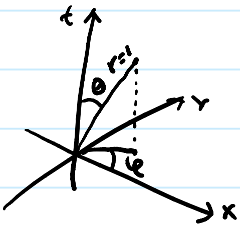
VID66 DIFFERENTIALKALKYK VEKTORVÄRD - TANGENTPLAN, EXEMPEL

den 5 augusti 2024 20:17

<u>Ex:</u> ($\mathbb{R}^2 \to \mathbb{R}^3$)

$$r(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$
$$0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$$

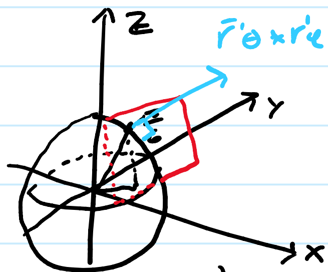
$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$$



Enhetsfären $x^2+y^2+z^2=1$

Söker: ekv. för tangentplanet

då $\theta = \frac{\pi}{6}$, $\varphi = \frac{\pi}{4}$ dvs, i punkten $\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2}\right)$



$$r_\theta' \times r_\varphi' = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \times (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) \times \left(-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}, \frac{1}{2}\cdot\frac{1}{\sqrt{2}}, 0\right) =$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}\left(1, 1, -\frac{\sqrt{2}}{\sqrt{3}}\right) \times \frac{1}{2\sqrt{2}}(-1, 1, 0) =$$

$$= \left(\frac{\sqrt{3}}{2\sqrt{2}} \cdot \frac{1}{2\sqrt{2}}\right)\left(1, 1, -\frac{\sqrt{2}}{\sqrt{3}}\right) \times (-1, 1, 0)$$

Kryssprod:
$$\begin{array}{cccccc} 1 & 1 & -\frac{\sqrt{2}}{\sqrt{3}} & 1 & 1 & -\frac{\sqrt{2}}{\sqrt{3}} \\ -1 & 1 & 0 & -1 & 1 & 0 \end{array}$$

så
$$r_\theta' \times r_\varphi' = \frac{\sqrt{3}}{8}\left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, 2\right) = \frac{1}{8}(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$$

normal-vektor

<u>Planets ekv.</u> $\sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z + D = 0$

Sätt in P: $\cancel{\sqrt{2}}\cdot\frac{1}{2\cancel{\sqrt{2}}} + \cancel{\sqrt{2}}\frac{1}{2\cancel{\sqrt{2}}} + \cancel{2}\sqrt{3}\frac{\sqrt{3}}{\cancel{2}} + D = 0$

$\Leftrightarrow D = -4$

Svar: $\sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z - 4 = 0$

– – – – – – – –

$F(x,y,z) = x^2 + y^2 + z^2 = 1$

Alt. 1: $\operatorname{grad} F = (F'_x, F'_y, F'_z) = \underset{\text{norm.}}{(2x, 2y, 2z)}$

Alt. 2: $z = (\pm)\sqrt{1-(x^2+y^2)} = f(x,y)$

$$\boxed{z - f(a,b) = f'_x(a,b)(x-a) + f'_y(a,b)(y-b)}$$

VID67 DIFFERENTIALKALKYL VEKTORVÄRD - FUNKTIONALMATRIS, INTRO

den 5 augusti 2024 20:34

Funktionalmatris $\quad \mathbb{R}^n \to \mathbb{R}^p$

$(\mathbb{R}^2 \to \mathbb{R}) \quad f(x_1, x_2) \quad f'_{x_1}, f'_{x_2}$

skala $\quad \text{grad} f = (f'_{x_1} \; f'_{x_2})$

radmatris (inget kommatecken)

$(\mathbb{R}^2 \to \mathbb{R}^2) \quad \bar{f}(\bar{x}) = (f_1(x_1, x_2), f_2(x_1, x_2))$

$\bar{x} = (x_1, x_2)$

$f'_{1x_1}, f'_{1x_2}, f'_{2x_1}, f'_{2x_2}$

$$\bar{f}'(\bar{x}) = \begin{pmatrix} f'_{1x_1} & f'_{1x_2} \\ f'_{2x_1} & f'_{2x_2} \end{pmatrix} \leftarrow \text{funktionalmatris}$$

$(\mathbb{R}^n \to \mathbb{R}^p)$ Allmänt

$$\bar{f}(\bar{x}) = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n))$$

$$\bar{f}'(\bar{x}) = \begin{pmatrix} f'_{1x_1} & f'_{1x_2} & f'_{1x_3} & \ldots & f'_{1x_4} \\ f'_{2x_1} & f'_{2x_2} & & \ldots & f'_{2x_n} \\ & & & \vdots & \\ f'_{px_1} & f'_{px_2} & & \ldots & f'_{px_n} \end{pmatrix}$$

Ex: $(\mathbb{R}^2 \to \mathbb{R}^2) \quad \bar{f}(\bar{x}) = (f_1(x_1, x_2), f_2(x_1, x_2)) =$

$= (x_1^2 + 2x_2, \; x_1 x_2)$

$$\bar{F}'(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 2x_1 & 2 \\ x_2 & x_1 \end{pmatrix} \begin{array}{l} = \text{grad } f_1 \\ = \text{grad } f_2 \end{array} \\ \begin{array}{cc} \| & \| \\ \bar{F}'_{x_1}(\bar{x}) & \bar{F}'_{x_2}(\bar{x}) \end{array} \end{array}$$

$$\bar{F}'(1,3) = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$$

VID68 DIFFERENTIALKALKYL VEKTORVÄRD - FUNKTIONALMATRIS, TOLKNING

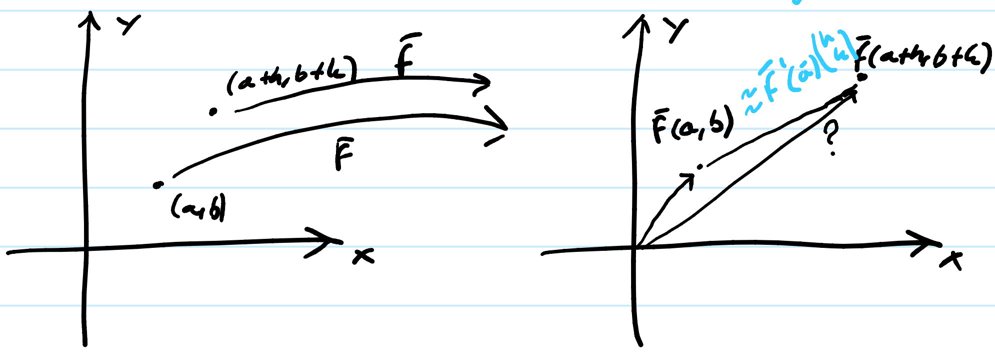
den 5 augusti 2024 20:53

$(\mathbb{R}^2 \to \mathbb{R}^2)$ $\quad \bar{F}(x,y) = (f_1(x,y), f_2(x,y))$

$\bar{a} = (a,b)$

Vill approximera f väl nära punkten (a,b)

linjär avb. med $\bar{F}'(\bar{a})$ som avb. matris



Taylorutveckling av $\bar{F} = (f_1, f_2)$ nära (a,b):

liten om (h,k) är små

$$f_1(a+h, b+k) \approx f_1(a,b) + f'_{1x}(a,b)h + f'_{1y}(a,b)k + \cancel{Rest}$$

$$f_2(a+h, b+k) \approx f_2(a,b) + f'_{2x}(a,b)h + f'_{2y}(a,b)k + \cancel{Rest}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} f_1(a+h, b+k) \\ f_2(a+h, b+k) \end{pmatrix}}_{= \bar{F}(a+h,\, b+k)} \approx \underbrace{\begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix}}_{= \bar{F}(a,b)} + \underbrace{\begin{pmatrix} f'_{1x}(a,b) & f'_{1y}(a,b) \\ f'_{2x}(a,b) & f'_{2y}(a,b) \end{pmatrix}}_{= \bar{F}'(\bar{a})} \begin{pmatrix} h \\ k \end{pmatrix}$$

VID69 FUNKTIONSDETERMINANT, INTRO

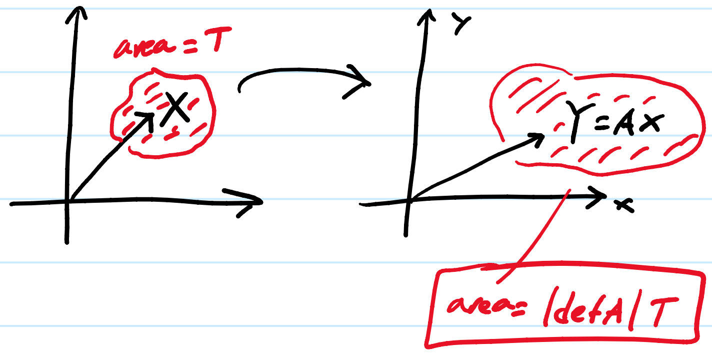
den 5 augusti 2024 21:03

Funktionaldeterminant

Avb.matris

Lin. alg. $(\mathbb{R}^2 \to \mathbb{R}^2)$ $Y = AX$ linj.avb

$$\begin{pmatrix}\cdot\\\cdot\end{pmatrix}=\begin{pmatrix}\cdot&\cdot\\\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot\\\cdot\end{pmatrix}$$



$(\mathbb{R}^3 \to \mathbb{R}^3$, volym$)$

$(\mathbb{R}^2 \to \mathbb{R}^2)$ $\bar{F}(x,y)$

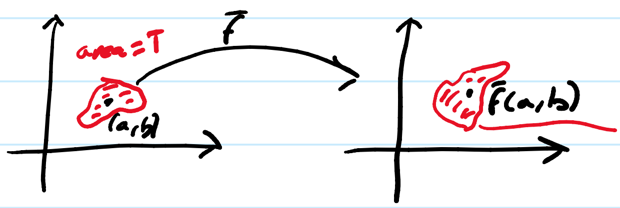
linj.avb

(h,k) små

$$\underline{\bar{F}(a+h, b+k) - \bar{F}(a,b)} \approx \bar{F}'(a,b)\begin{pmatrix}h\\k\end{pmatrix}$$

funk. matr.

avb.matris



area $\approx \underbrace{|\det \bar{F}'(a,b)|}_{\text{Funktionaldeterminant}} T$

$$\det \bar{F}'(a,b) = \frac{d(f_1, f_2)}{d(x,y)}(a,b)$$

$$\bar{F} = (f_1, f_2)$$

VID70 DIFFERENTIALKALKYL VEKTORVÄRD - FUNKTIONALDETERMINANT, EXEMPEL

den 5 augusti 2024 21:13

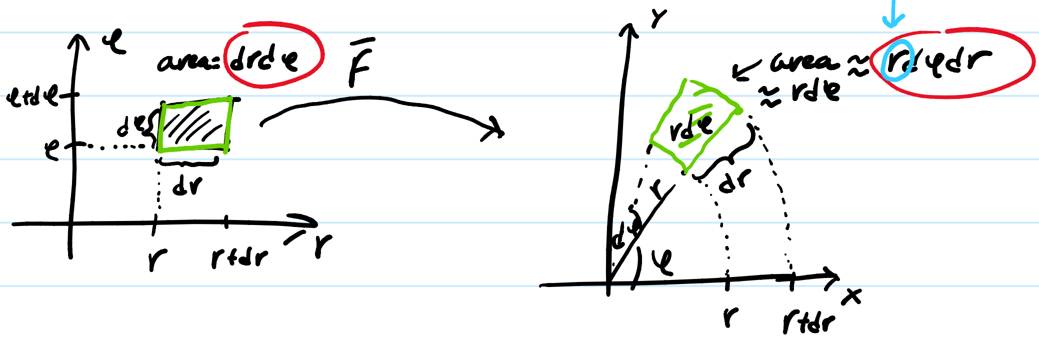
Ex: Byte till polära koord:

$$\begin{cases} x = r\cos\varphi = f_1 \\ y = r\sin\varphi = f_2 \end{cases}$$

$(\mathbb{R}^2 \to \mathbb{R}^2)$ $\qquad \bar{F}(r,\varphi) = (r\cos\varphi, r\sin\varphi)$

Areaförändring (lokalt)?

Skalfaktor $\boxed{r}$



Funktionaldeterminant:

$$\frac{d(x,y)}{d(r,\varphi)} = \begin{vmatrix} f_1'r & f_1'\varphi \\ f_2'r & f_2'\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} =$$

$$= r\cos^2\varphi - (-r\sin^2\varphi) =$$

$$= r(\underbrace{\cos^2\varphi + \sin^2\varphi}_{=1}) = r$$

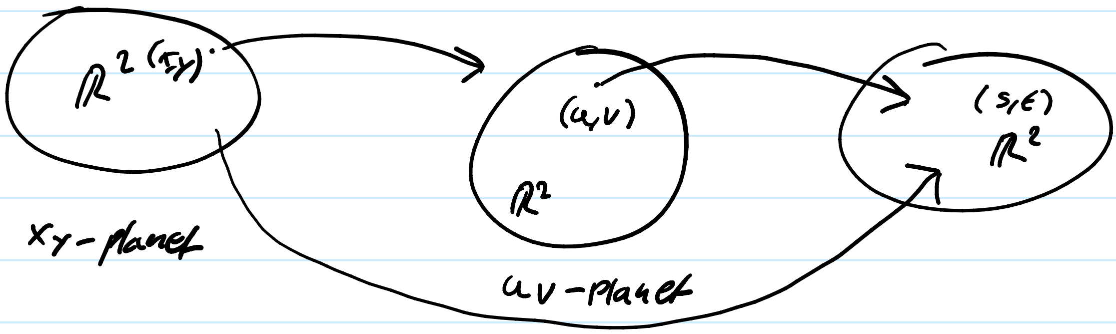
VID71 DIFFERENTIALKALKYL - FUNKTIONALMATRIS, KEDJEREGELN

den 5 augusti 2024 21:34

Kedjeregeln – funk. matriser ($\mathbb{R}^n \to \mathbb{R}^p$)

Funktionen
$$\begin{cases} s = f(u(x,y), v(x,y)) \\ t = g(u(x,y), v(x,y)) \end{cases}$$

är en sammansättning



Kedjeregeln:

$$f'_x = f'_u \cdot u'_x + f'_v \cdot v'_x \quad , \quad f'_y = f'_u \cdot u'_y + f'_v \cdot v'_y$$

$$g'_x = g'_u \cdot u'_x + g'_v \cdot v'_x \quad , \quad g'_y = g'_u \cdot u'_y + g'_v \cdot v'_y$$

$$\begin{tikzcd} \underset{(x,y)}{\mathbb{R}^2} \arrow[r] \arrow[rr, bend left] & \underset{(u,v)}{\mathbb{R}^2} \arrow[r] & \underset{(s,t)}{\mathbb{R}^2} \end{tikzcd}$$

I matrisform:
$$\begin{pmatrix} f'_x & f'_y \\ g'_x & g'_y \end{pmatrix} = \begin{pmatrix} f'_u & f'_v \\ g'_u & g'_v \end{pmatrix} \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix}$$

Funktionalmatriser

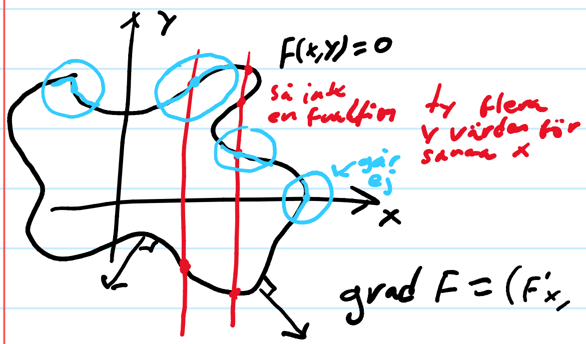
VID72 DIFFERENTIALKALKYL VEKTORVÄRD - IMPLICITA FUNKTIONSSATSEN

den 5 augusti 2024 21:43

Implicita funktionssatsen:

$F(x,y) = y^5 + xy - y = 0$

$\boxed{y = f(x)\ ?}$ F — funktion



$\text{grad } F = (F'_x, F'_y) \neq 0$

Implicita funktionssatsen:

> Låt $F(x,y)$ vara en funk. med kont. part. derivator, och låt (a,b) vara punkt på kurvan $F(x,y)=C$.
> Om $F'_y(a,b) \neq 0$ så kan y uttryckas som en kont. deriverbar funktion av x (dvs $y = y(x)$) i en omgivning av (a,b).

$\boxed{y = y(x)}$

Ex: Visa att ekvationen

$$F(x,y) = y^5 + xy - y = 0$$

definierar y som en funktion ✓ av x nära punkten $P: (3,1)$.

på kurvan Ok!

Bestäm $y'(3)$.

Bestäm $y'(3)$.

Lösning: $F'_y = 5y^4 + x \Rightarrow F'_y(3,1) = 5 \cdot 1 + 3 = 8 \neq 0$ Ok! (impl. funk. sats)

Implicit derivering: $y(5)^5 + xy(x) - 4 = 0$

Deriverar ledvis $5y(x)^4 \cdot y'(x) + 1 \cdot y(x) + xy'(x) - 0 = 0$

$\Leftrightarrow y'(x)(5y(x)^4 + x) = -y(x)$

$\Leftrightarrow y'(x) = \dfrac{-y(x)}{5y(x)^4 + x}$ $\quad$ $x=3$: $y'(3) = \dfrac{-y(3)}{5y(3)^4 + 3}$ (P(3,1) ger y värdet 1.)

$y(3) = 1$

$\Leftrightarrow y'(3) = \dfrac{-1}{8}$