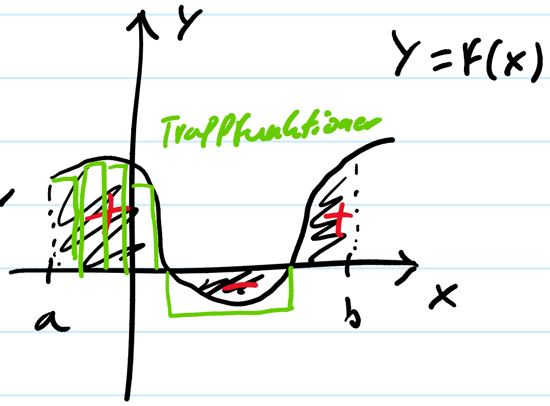
VID73 INTEGRALKALKYL - DUBBELINTEGRAL, INTRO

den 6 augusti 2024 19:36

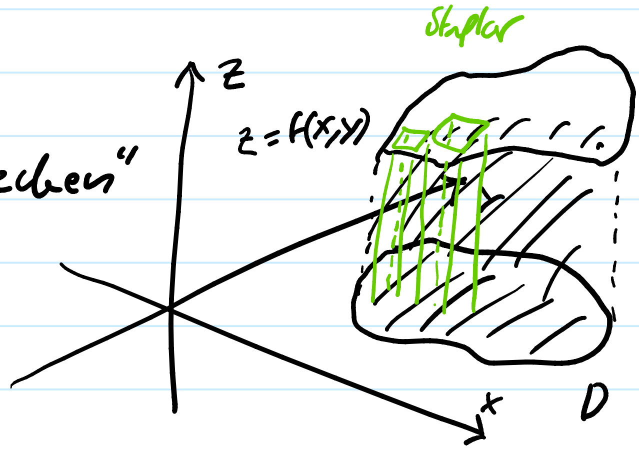
<u>Integralkalkyl</u> <u>Endim</u>:

$$\int_a^b f(x)\,dx = \text{"area med tecken"}$$



$$\iint_D f(x,y)\,dx\,dy =$$

$$= \text{"volym med tecken"}$$



VID74 INTEGRALKALKYL - DUBBELINTEGRAL ÖVER REKTANGEL, FORMEL

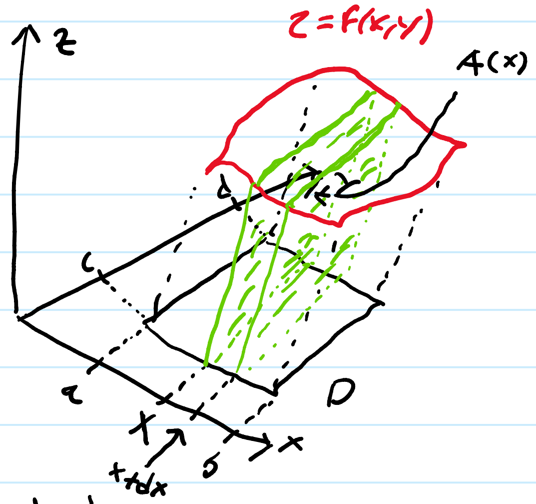
den 6 augusti 2024 19:50

$$\iint_D f(x,y)\,dx\,dy$$

Metod, D rektangel

Approximativ likhet

$$dV = A(x)\,dx = \left(\int_c^d f(x,y)\,dy\right)dx$$



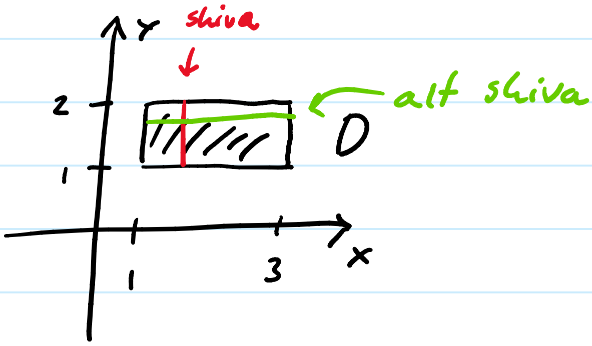
$$\iint_D f(x,y)\,dx\,dy = \int_a^b dV = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$$

$$\text{alt. } \iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$$

VID75 INTEGRALKALKYL - DUBBELINTEGRAL ÖVER REKTANGEL, EXEMPEL

den 6 augusti 2024 20:00

Ex: $\iint\limits_D x^2 y \, dx\, dy$ rektangel $D: 1 \leq x \leq 3, \ 1 \leq y \leq 2$



$$I = \int_1^3 \left(\int_1^2 x^2 y \, dy \right) dx =$$

$$= \int_1^3 \left[\frac{x^2 y^2}{2} \right]_1^2 dx =$$

$$= \int_1^3 \left(\frac{4}{2} x^2 - \frac{1}{2} x^2 \right) dx = \int_1^3 \frac{3}{2} x^2 dx = \left[\frac{1}{2} x^3 \right]_1^3 = \underline{\underline{13}}$$

alt. $I = \int_1^2 \left(\int_1^3 x^2 y \, dx \right) dy = \int_1^2 \left[\frac{1}{3} x^3 y \right]_1^3 dy =$

$$= \int_1^2 \left(\frac{27}{3} y - \frac{1}{3} y \right) dy = \int_1^2 \frac{26}{3} y \, dy = \left[\frac{13}{3} y^2 \right]_1^2 = \frac{52}{3} - \frac{13}{3} = \underline{\underline{13}}$$

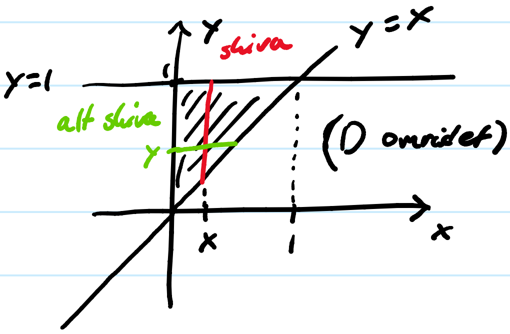
VID76 INTEGRALKALKYL - DUBBELINTEGRAL ALLMÄNT OMRÅDE, EXEMPEL 1

den 6 augusti 2024 20:09

Ex:

$$I = \iint_D xy^2\,dx\,dy$$

$D: y \geq x,\ y \leq 1,\ x \geq 0$



$$I = \int_0^1 \left(\int_x^1 xy^2\,dy \right) dx =$$

$$= \int_0^1 \left[\frac{xy^3}{3} \right]_x^1 dx =$$

$$= \int_0^1 \left(\frac{x}{3} - \frac{x^4}{3} \right) dx = \left[\frac{x^2}{6} - \frac{x^5}{15} \right]_0^1 = \frac{1}{6} - \frac{1}{15} = \frac{3}{30} = \frac{1}{10}$$

alt.

$$I = \int_0^1 \left(\int_0^y xy^2\,dx \right) dy = \int_0^1 \left[\frac{x^2}{2} y^2 \right]_0^y dy =$$

$$= \int_0^1 \frac{y^4}{4}\,dy = \left[\frac{y^5}{10} \right]_0^1 = \frac{1}{10}$$

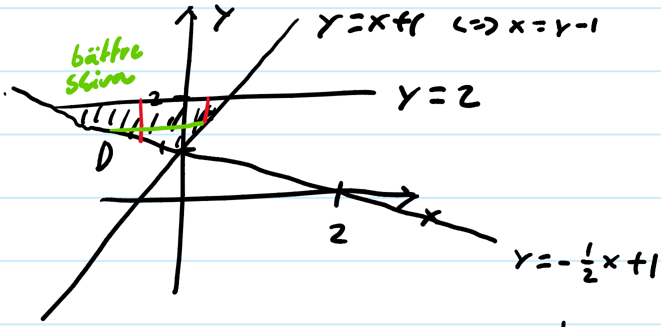
VID77 INTEGRALKALKYL - DUBBELINTEGRAL ALLMÄNT OMRÅDE, EXEMPEL 2

den 6 augusti 2024 20:19

Ex:

$$I = \iint_D y \, dx\,dy$$

$D: \ y \geq x+1, \ y \geq -\frac{1}{2}x+1, \ y \leq 2$



$\Leftrightarrow \frac{1}{2}x = -y+1$

$\Leftrightarrow x = -2y+2$

$$I = \int_1^2 \left(\int_{-2y+2}^{y-1} y \, dx \right) dy =$$

$$= \int_1^2 \left[yx \right]_{-2y+2}^{y-1} dy =$$

$$= \int_1^2 \left(y(y-1) - y(-2y+2) \right) dy =$$

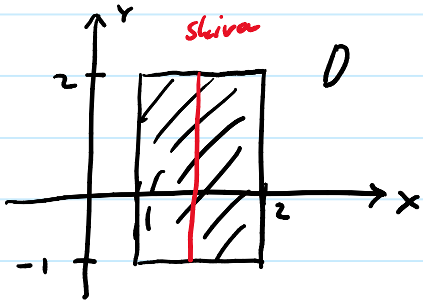
$$= \int_1^2 \left(3y^2 - 3y \right) dy = \left[y^3 - \frac{3}{2}y^2 \right]_1^2 = 8 - \frac{3}{2}\cdot 4 - \left(1 - \frac{3}{2}\right) =$$

$$= \frac{5}{2}$$

VID78 INTEGRALKALKYL - DUBBELINTEGRAL SOM PRODUKT AV ENKELINTEGRALER

den 6 augusti 2024 20:35

Ex: $\iint_D (x+xy^2)\,dx\,dy$ D: $1\le x\le 2,\ -1\le y\le 2$



Ty x konstant

$$I = \int_1^2 \left(\int_{-1}^2 x(1+y^2)\,dy\right)dx =$$

konstanta gränser

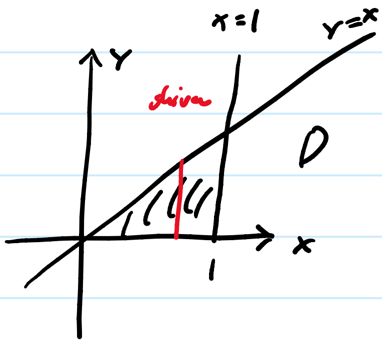
$$= \int_1^2 x\left(\int_{-1}^2 (1+y^2)\,dy\right)dx =$$

$$= \left(\int_{-1}^2 (1+y^2)\,dy\right)\cdot\left(\int_1^2 x\,dx\right) = \left[y+\frac{y^3}{3}\right]_{-1}^2 \cdot \left[\frac{x^2}{2}\right]_1^2 =$$

$$= \left(2+\frac{8}{3}-\underbrace{\left(-1-\frac{1}{3}\right)}_{-4/3}\right)\cdot\left(2-\frac{1}{2}\right) = 6\cdot\frac{3}{2} = 9$$

Hade vi startat med
sin(xy) kan vi inte göra
såhär då vi inte
kan separera x från y.

Ex: $I = \iint_D (x+xy^2)\,dx\,dy$ D: $x\le 1,\ 0\le y\le x$



$$I = \int_0^1 \left(\int_0^x x(1+y^2)\,dy\right)dx =$$

← ej konstant

$$= \int_0^1 x\left(\int_0^x (1+y^2)\,dy\right)dx$$

VID79 INTEGRALKALKYL - DUBBELINTEGRAL VARIABELBYTE, INTRO

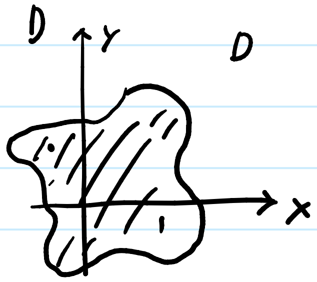
den 6 augusti 2024 20:45

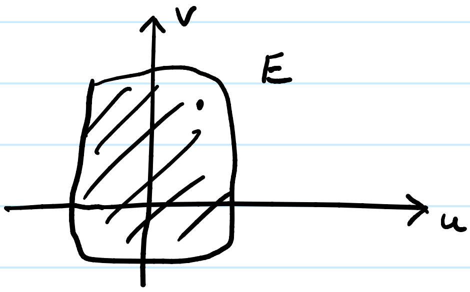
<u>Variabelbyte</u>

$$\iint_D f(x,y)\,dx\,dy$$

Förändringsfaktor

$$\iint_D f(x,y)\,dx\,dy = \left[\begin{cases} x = x(u,v) \\ y = y(u,v)\end{cases}\right] = \iint_E f(x(u,v), y(u,v))\; ?\; du\,dv$$

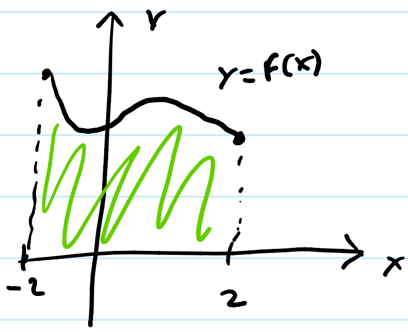


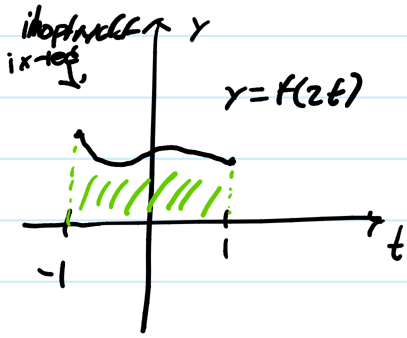


<u>Endim:</u>

Skalfaktor

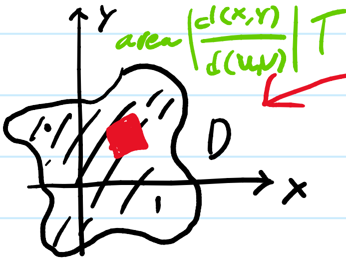
$$\int_{-2}^{2} f(x)\,dx = \left[\begin{matrix} x = 2t & x=2 \Rightarrow t=1 \\ & x=-2 \Rightarrow t=-1 \\ \frac{dx}{dt} = 2 \Rightarrow dx = 2dt & \end{matrix}\right] \int_{-1}^{1} f(2t)\,\underset{\overset{\shortparallel}{\frac{dx}{dt}}}{2}\,dt$$

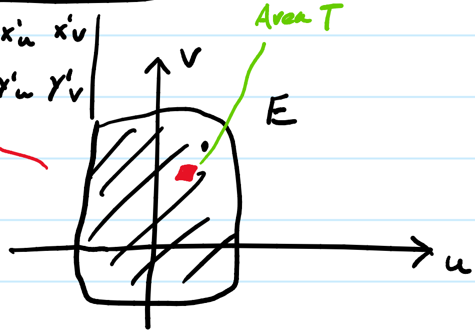




$$\iint_D f(x,y)\,dx\,dy = \left[\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}\right] = \iint_E f(x(u,v), y(u,v)) \overbrace{\left|\frac{d(x,y)}{d(u,v)}\right|}^{\text{Förändringsfaktor}} du\,dv$$

$$\frac{d(x,y)}{d(u,v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}$$

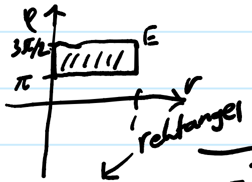


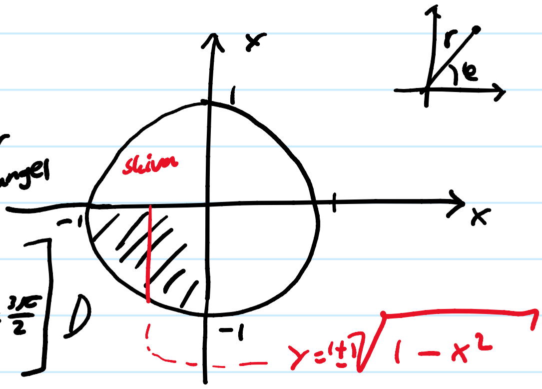


Ex: Beräkna

$$\iint_D xy^2\,dx\,dy \qquad D: x^2+y^2\le 1,\quad x\le 0,\ y\le 0$$

Vid byte till polära koord gäller att $\dfrac{d(x,y)}{d(r,\varphi)} = r$





$$= \left[\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \quad E: \begin{cases} 0\le r\le 1 \\ \pi\le\varphi\le \frac{3\pi}{2} \end{cases} \right] D$$

$$= \iint_E r\cos\varphi\, r^2\sin^2\varphi \underbrace{\left|\frac{d(x,y)}{d(r,\varphi)}\right|}_{=|r|=r} dr\,d\varphi =$$

$$= \iint_{E_1} r^4 \sin^2\varphi\cos\varphi\, dr\,d\varphi = \qquad \frac{d(x,y)}{d(r,\varphi)} = \begin{vmatrix} x'_r & x'_\varphi \\ y'_r & y'_\varphi \end{vmatrix} =$$

$$= \left(\int_0^1 r^4\,dr\right)\cdot\left(\int_\pi^{\frac{3\pi}{2}} \sin^2\varphi\cos\varphi\,d\varphi\right) \qquad = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = (*)$$

$$(*) = \left|(\cos\varphi\cdot r\cos\varphi) - (-r\sin\varphi\cdot\sin\varphi)\right| =$$

$$= \left|r\cos^2\varphi + r\sin^2\varphi\right| = \left|r(\underbrace{\cos^2\varphi+\sin^2\varphi}_{=1})\right| = |r| = r$$

$$\left(\int_0^1 r^4\,dr\right)\cdot\left(\int_\pi^{3\pi/2} \sin^2\varphi\cos\varphi\,d\varphi\right) = \left[\frac{r^5}{5}\right]_0^1 \cdot \left[\frac{1}{3}\sin^3\varphi\right]_\pi^{\frac{3\pi}{2}} =$$

-1, $\sin\pi = 0$

$$= \frac{1}{5}\cdot\left(-\frac{1}{3}\right) = -\frac{1}{15}$$

$$= \frac{1}{5} \cdot \left(-\frac{1}{3}\right) = -\frac{1}{15}$$

VID81 INTEGRALKALKYL - DUBBELINTEGRAL VARIABELBYTE, EXEMPEL 2

den 6 augusti 2024 21:18

Ex:

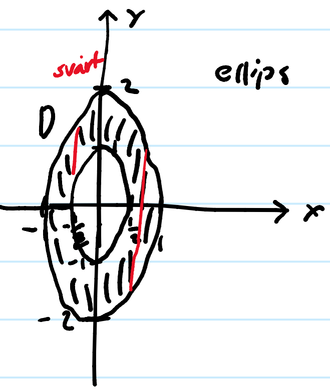
$$I=\iint_D x^2\,dx\,dy, \quad D: 1\le 4x^2+y^2\le 4$$

Rita D!

rand:

$4x^2+y^2=4 \iff x^2+\frac{y^2}{4}=1 \iff x^2+\frac{y^2}{2^2}=1$

$4x^2+y^2=1 \iff \frac{x^2}{1/4}+y^2=1 \iff \frac{x^2}{(\frac{1}{2})^2}+y^2=1$



$$1\le (2x)^2+y^2\le 4 \qquad 1\le \underbrace{r^2\cos^2\varphi+r^2\sin^2\varphi}_{\Rightarrow r^2}\le 1$$

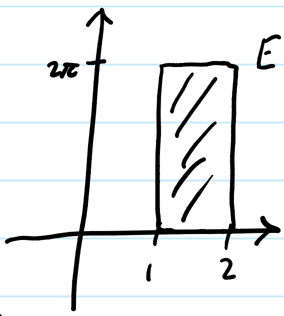
$$\begin{cases} 2x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad E:\begin{cases} 1\le r\le 2 \\ 0\le \varphi\le 2\pi \end{cases}$$

$$\iff \begin{cases} x=\frac{r}{2}\cos\varphi \\ y=r\sin\varphi \end{cases}$$

$$\left|\frac{d(x,y)}{d(r,\varphi)}\right| = \begin{vmatrix} x'_r & x'_\varphi \\ y'_r & x'_\varphi \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\cos\varphi & -\frac{r}{2}\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$$= \frac{r}{2}\cos^2\varphi+\frac{r}{2}\sin^2\varphi = \frac{r}{2}(\underbrace{\cos^2\varphi+\sin^2\varphi}_{=1}) = \frac{r}{2}$$

(Formel ⟹ nedan)



$$I=\iint_E \frac{r^2}{4}\cos^2\varphi\cdot\frac{r}{2}\,dr\,d\varphi$$

$$\cos^2\varphi=\frac{1+\cos 2\varphi}{2}$$

$$=\frac{1}{8}\left(\int_1^2 r^3\,dr\right)\cdot\left(\int_0^{2\pi}\cos^2\varphi\,d\varphi\right)=\frac{1}{8}\left[\frac{r^4}{4}\right]_1^2\cdot\left[\frac{\varphi}{2}+\frac{\sin 2\varphi}{4}\right]_0^{2\pi}=$$

$$=\frac{1}{8}\left(\frac{16}{4}-\frac{1}{4}\right)\pi=\frac{1}{8}\cdot\frac{15}{4}\cdot\pi=\frac{15}{32}\pi$$

VID82 INTEGRALKALKYL - DUBBELINTEGRAL VARIABELBYTE, EXEMPEL 3

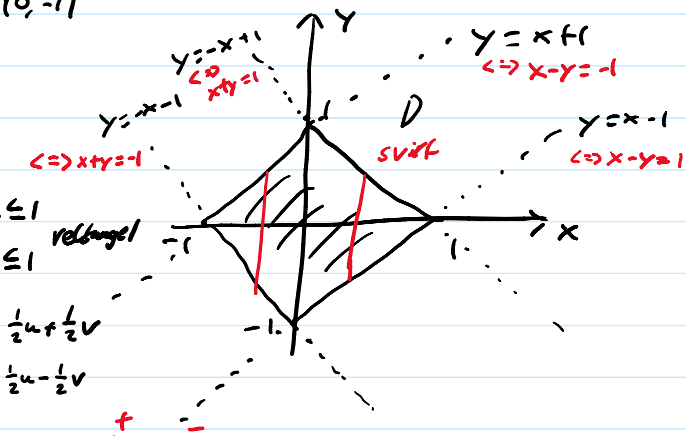
den 6 augusti 2024 21:48

Ex:

$$I = \iint_D \frac{(x+y)^2}{1+(x-y)^2}\,dx\,dy$$

D är den kvadratiska skivan med hörn $(1,0), (0,1), (-1,0), (0,-1)$

Alt. D $\begin{cases} -1 \le x+y \le 1 \\ -1 \le x-y \le 1 \end{cases}$ ger E: $\begin{cases} -1 \le u \le 1 \\ -1 \le v \le 1 \end{cases}$ rektangel



Sätt

$$\begin{cases} x+y=u \\ x-y=v \end{cases} \iff \begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u - \frac{1}{2}v \end{cases}$$

$$\left|\frac{d(x,y)}{d(u,v)}\right| = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$I = \iint_E \underbrace{\frac{u^2}{1+v^2}}_{= u^2 \cdot \frac{1}{1+v^2}} \underbrace{\left|-\frac{1}{2}\right|}_{= +\frac{1}{2}} du\,dv = \frac{1}{2}\left(\int_{-1}^{1} u^2\,du\right)\left(\int_{-1}^{1} \frac{1}{1+v^2}\,dv\right) =$$

$$= \frac{1}{2}\left[\frac{u^3}{3}\right]_{-1}^{1}\left[\arctan v\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right)\left(\overset{(\frac{\pi}{4})}{\arctan(1)} - \overset{(-\frac{\pi}{4})}{\arctan(-1)}\right) =$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6}$$

VID83 INTEGRALKALKYL - DUBBELINTEGRAL, AREABERÄKNING

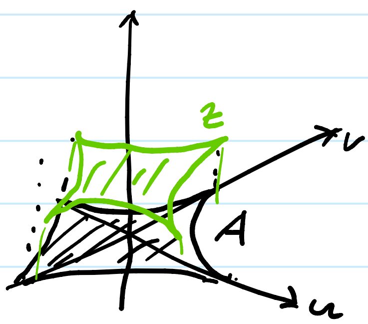
den 6 augusti 2024 22:00

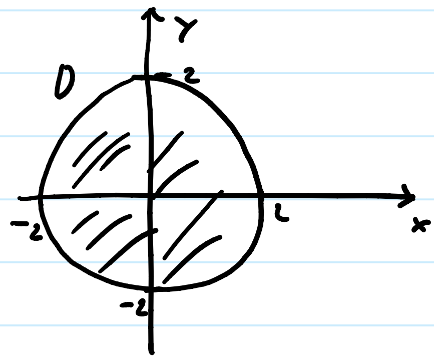
Ex: Genom avbildningen

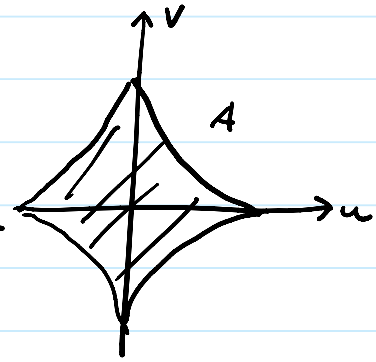
$$\begin{cases} u = x^3 \\ v = y^3 \end{cases} \qquad (\mathbb{R}^2 \to \mathbb{R}^2)$$

överförs cirkelskivan $x^2+y^2 \le 4$ i ett område A. Beräkna arean av A!

Trick!







arean av A = volym av kropp med höjd 1 $= \iint_A 1\,du\,dv =$

$$= \left[\begin{cases} u = x^3 \\ v = y^3 \end{cases} \right] = \iint_D 1 \left| \frac{d(u,v)}{d(x,y)} \right| dx\,dy$$

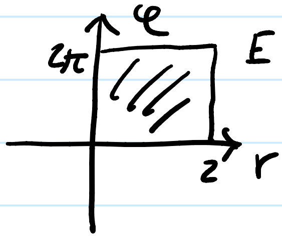
Så

area av $A = \iint_D 1 \cdot \left| \frac{d(u,v)}{d(x,y)} \right| dx\,dy = (*)$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} 3x^2 & 0 \\ 0 & 3y^2 \end{vmatrix} =$$

$= 9x^2y^2$ så $\left| \frac{d(u,v)}{d(x,y)} \right| = 9x^2y^2$

$(*) = \iint_D 9x^2y^2\,dx\,dy =$



$\overset{\cup}{D}{}^{\cup}$ | $z' \; r$

$$= \left[\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad E: \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le 2\pi \end{cases} \right] = \iint_E 9r^2\cos^2\varphi\, r^2\sin^2\varphi \cdot r\, dr\, d\varphi =$$

$$= \left(\int_0^2 9r^5\, dr \right) \cdot \left(\int_0^{2\pi} \cos^2\varphi \sin^2\varphi\, d\varphi \right) =$$

$320 - 32 = 288$

$\frac{9 \cdot 32}{3} = \frac{288}{3} = 96$

$$= \left[\frac{9}{6} r^6 \right]_0^2 \cdot \left[\frac{\varphi}{8} - \frac{\sin 4\varphi}{32} \right]_0^{2\pi} = \frac{\overset{3}{\cancel{9}} \cdot \overset{32}{\cancel{64}}}{\underset{3}{\cancel{6}}} \overset{96}{//} \cdot \frac{\pi}{4} = 24\pi$$

$\frac{96}{4} = 24$

$$\cos^2\varphi \sin^2\varphi = (\underline{\cos\varphi \sin\varphi})^2 = \left(\tfrac{1}{2}\sin 2\varphi\right)^2 = \tfrac{1}{4}\sin^2 2\varphi = \frac{1 - \cos 4\varphi}{8}$$

$$\sin^2\varphi = \frac{1 - \cos^2\varphi}{2}$$

$$\Rightarrow \sin^2 2\varphi = \frac{1 - \cos 4\varphi}{2}$$

$$\boxed{\text{area } A = \iint_A 1\, dx\, dy}$$

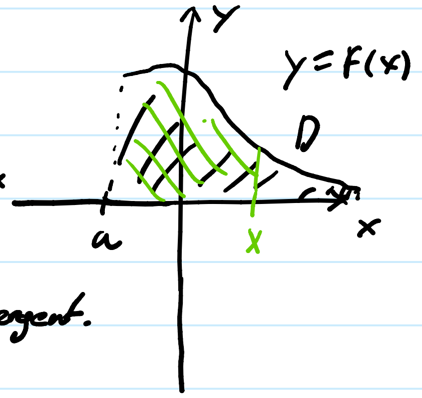
VID84 INTEGRALKALKYL - GENERALISERAD INTEGRAL, INTRO

den 7 augusti 2024 19:24

Generaliserade integraler

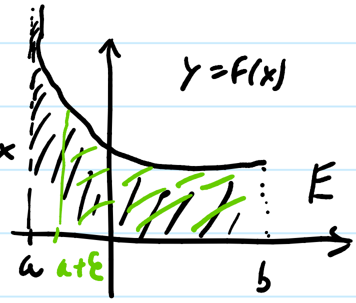
Endim:

$$\int_a^\infty f(x)\,dx = \lim_{x\to\infty}\int_a^x f(x)\,dx$$

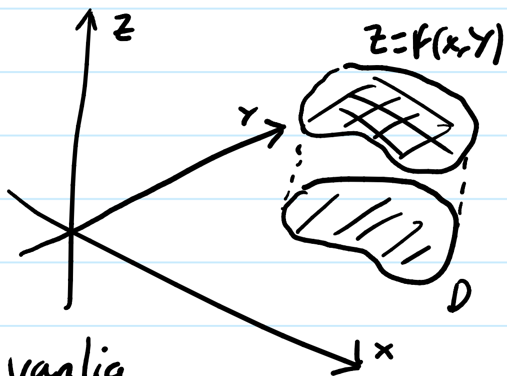


konvergent/divergent.

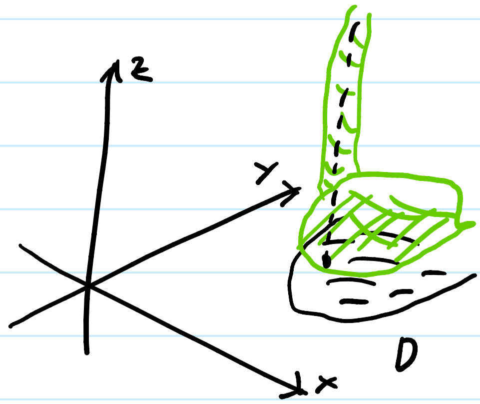
$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f(x)\,dx$$



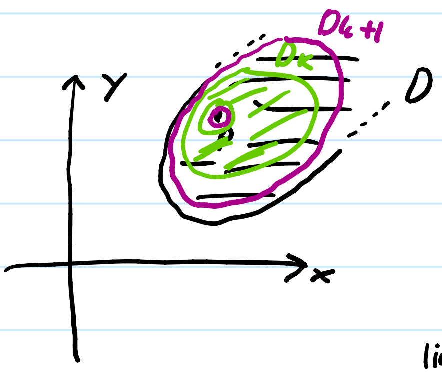
Flerdim (dubbelint.)



vanlig



generellt



delmängd

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$$

$$D_k \nearrow D$$

$$\lim_{k\to\infty}\iint f(x,y)\,dx\,dy$$

$$\lim_{k\to\infty} \iint_{D} f(x,y)\,dx\,dy$$

Def: Om $I = \lim_{k\to\infty} \iint_{D_k} f(x,y)\,dx\,dy$ existerar ändligt och är oberoende av svit D_k så säger vi att $\iint_D f(x,y)$ är konvergent med värde I.

Annars divergent.

Sats: Om $f(x,y) \geq 0$ (≤ 0) i hela D så är

$\lim_{k\to\infty} \iint_{D_k} f(x,y)\,dx\,dy$ är oberoende av svit D_k

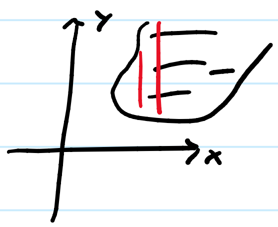
Om:

$$\boxed{f(x,y) \geq 0 \quad (\leq 0)}$$

så:

$$\iint_D f(x,y)\,dx\,dy = \int\Big(\int \quad dy\Big)dx$$

gen gen.



VID85 INTEGRALKALKYL - GENERALISERAD INTEGRAL, EXEMPEL 1

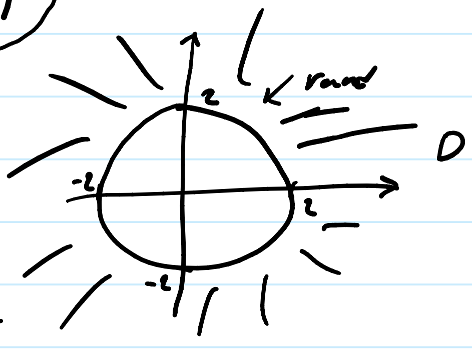
den 7 augusti 2024 20:19

Ex: Beräkna

byter ej tecken

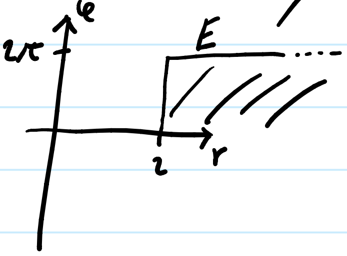
Obegr.

$$\iint_D \overbrace{\frac{1}{(x^2+y^2)^{3/2}}}^{>0}\,dx\,dy,\quad D: x^2+y^2 \ge 4$$



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$E: \begin{cases} r \ge 2 \\ 0 \le \varphi \le 2\pi \end{cases}$$



$$I = \iint_E \frac{1}{\underbrace{(r^2)^{3/2}}_{r^3}} \cdot r\,dr\,d\varphi = \iint_E \frac{1}{r^2}\,dr\,d\varphi =$$

$$= \left(\int_2^{\infty} \frac{1}{r^2}\,dr\right) \cdot \underbrace{\left(\int_0^{2\pi} 1\,d\varphi\right)}_{\left(2\pi \text{ ty } [\varphi]_0^{2\pi} = 2\pi - 0 = 2\pi\right)} =$$

$$= 2\pi \lim_{X\to\infty} \int_2^X \frac{1}{r^2}\,dr = 2\pi \lim_{X\to\infty} \left[-\frac{1}{r}\right]_2^X =$$

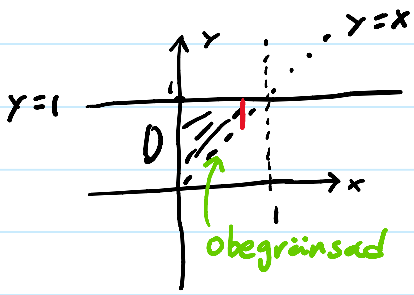
$$= 2\pi \lim_{X\to\infty} \left(\overbrace{-\frac{1}{X}}^{\to 0} + \frac{1}{2}\right) = \underline{\underline{\pi}}$$

konvergent med värdet π.

Ex:

Beräkna

$$\iint_D \overset{>0}{\frac{1}{\sqrt{y-x}}}\,dx\,dy, \quad D:\ x<y\le 1,\ x\ge 0$$



$$I=\int_0^1\left(\overset{\text{gen.}}{\int_x^1}\frac{1}{\sqrt{y-x}}\,dy\right)dx=$$

$$=\int_0^1\left(\lim_{\varepsilon\to 0}\int_{x+\varepsilon}^1\frac{1}{\sqrt{y-x}}\,dy\right)dx=$$

$$=\int_0^1\left(\lim_{\varepsilon\to 0^+}\Big[2\sqrt{y-x}\Big]_{x+\varepsilon}^1\right)dx=\int_0^1\left(\lim_{\varepsilon\to 0^+}\left(2\sqrt{1-x}-2\sqrt{\varepsilon}\right)\right)dx=$$

$$=\int_0^1 2\sqrt{1-x}\,dx=\left[-\frac{4}{3}(1-x)^{3/2}\right]_0^1=-\left(-\frac{4}{3}\right)\cdot 1=\underline{\underline{\frac{4}{3}}}$$

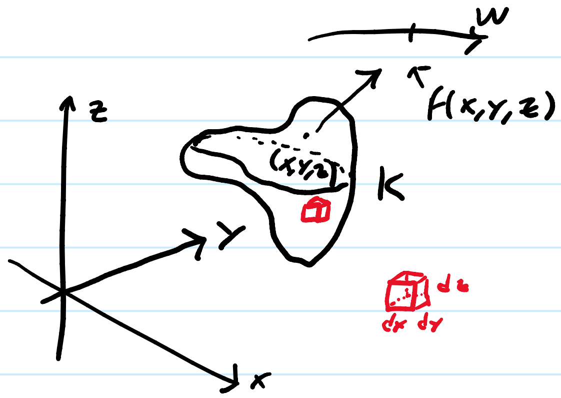
Konvergent med värde $\frac{4}{3}$.

VID87 INTEGRALKALKYL - TRIPPELINTEGRAL, INTRODUKTION

den 7 augusti 2024 20:39

Trippelintegraler

$$\iiint_K f(x,y,z)\,dxdydz$$



Fysikalisk tolkning

Densitet $f(x,y,z)$ (kg/m^3)

$dm = f(x,y,z)\,dxdydz$

$$\text{massa} = \iiint_K dm = \iiint_K f(x,y,z)\,dxdydz$$

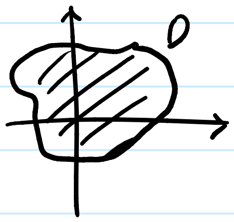
Volym

$f(x,y,z) = 1$ (kg/m^3)

$$\boxed{\text{Volym} = \iiint 1\,dxdydz}$$

Jämför

$$\text{Area} = \iint_D 1\,dxdy$$



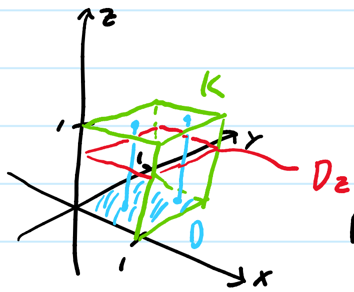
VID88 INTEGRALKALKYL - TRIPPELINTEGRAL ÖVER RÄTBLOCK, EXEMPEL

den 7 augusti 2024 20:47

Ex: Beräkna

$$\iiint_K x^3y^2z\,dxdydz$$

där $K: 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$



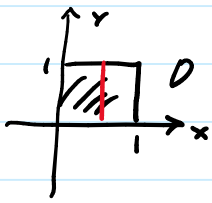
D_z: $0\le x\le 1$
$0\le y\le 1$
"$z=z$"

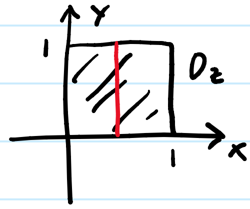
Alt. $\iint\left(\int\right)$ eller $\int\left(\iint\right)$

alt 1 $\iint\left(\int\right)$:

$$I=\iint_D\left(\int_0^1 x^3y^2z\,dz\right)dxdy=$$

D: $0\le x\le 1$
$0\le y\le 1$





$$\int_0^1\left(\int_0^1\left(\int_0^1 x^3y^2z\,dz\right)dy\right)dx$$

$$=\int_0^1\left(\int_0^1\left[\tfrac{1}{2}x^3y^2z\right]_0^1 dy\right)dx=$$

$$=\int_0^1\left(\int_0^1\tfrac{1}{2}x^3y^2\,dy\right)dx=\int_0^1\left[\tfrac{1}{2}\cdot\tfrac{1}{3}x^3y^3\right]_0^1 dx=$$

$$=\int_0^1\tfrac{1}{2}\cdot\tfrac{1}{3}x^3\,dx=\left[\tfrac{1}{2}\cdot\tfrac{1}{3}\cdot\tfrac{1}{4}x^4\right]_0^1=\tfrac{1}{2}\cdot\tfrac{1}{3}\cdot\tfrac{1}{4}=\frac{1}{24}$$

alt 2 $\int\left(\iint\right)$:

$$I=\int_0^1\left(\iint_{D_z}x^3y^2z\,dxdy\right)dz=$$

$$=\int_0^1\left(\int_0^1\left(\int_0^1 x^3y^2z\,dy\right)dx\right)dz=$$

$$=\ldots=\frac{1}{24}$$

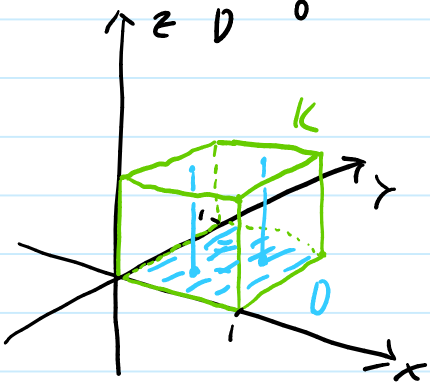
VID89 INTEGRALKALKYL - TRIPPELINTEGRAL, PRODUKT AV ENKELINTEGRALER

den 7 augusti 2024 21:09

$$\iiint_K x^3y^2z\,dx\,dy\,dz = \int_0^1\left(\int_0^1\left(\int_0^1 x^3y^2z\,dz\right)dy\right)dx$$

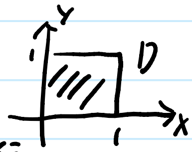
konstant i dz

$$\iint_D\left(\int_0^1 x^3y^2z\,dz\right)dx\,dy$$



kont i dy

$$= \int_0^1\left(\int_0^1 x^3y^2\left(\int_0^1 z\,dz\right)dy\right)dx =$$



$D: 0 \le x \le 1$

$\quad 0 \le y \le 1$

$K: 0 \le x \le 1$

$\quad 0 \le y \le 1$

$\quad 0 \le z \le 1$

konst i dx

$$= \int_0^1 x^3\left(\int_0^1 z\,dz\right)\left(\int_0^1 y^2\,dy\right)dx =$$

$$= \left(\int_0^1 x^3\,dx\right)\left(\int_0^1 y^2\,dy\right)\left(\int_0^1 z\,dz\right) = \left[\frac{x^4}{4}\right]_0^1\cdot\left[\frac{y^3}{3}\right]_0^1\cdot\left[\frac{z^2}{2}\right]_0^1 = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{24}$$

VID90 INTEGRALKALKYL - TRIPPELINTEGRAL ALLMÄNT OMRÅDE, EXEMPEL

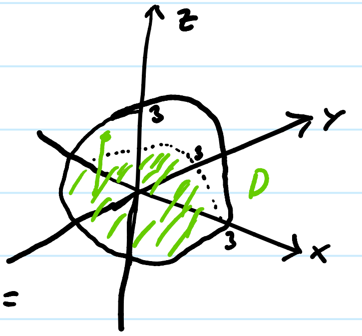
den 7 augusti 2024 21:19

__Ex:__ Beräkna

$$I = \iiint_K z \, dx dy dz$$

där K är halvklotet $x^2+y^2+z^2 \le 9$, $z \ge 0$

__alt.__ $\iint(\int)$, $\int(\iint)$



$\underline{\iint(\int)}$: $I = \iint_D \left(\int_0^{\sqrt{9-(x^2+y^2)}} z \, dz \right) dx dy$

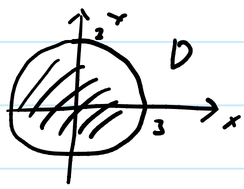
$$= \iint_D \left[\frac{1}{2} z^2 \right]_0^{\sqrt{9-(x^2+y^2)}} dx dy =$$

$x^2+y^2+z^2 = 9$

$\Leftrightarrow z = \overset{+}{(-)}\sqrt{9-(x^2+y^2)}$

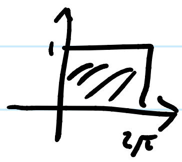
$$= \frac{1}{2} \iint_D (9-(x^2+y^2)) \, dx dy =$$

$D: \; x^2+y^2 \le 9$



$$= \left[\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad E: \begin{cases} 0 \le r \le 2\pi \\ 0 \le \varphi \le 2\pi \end{cases} \right] =$$

$$= \frac{1}{2} \iint_E (9-r^2) \cdot r \cdot 1 \, dr d\varphi =$$



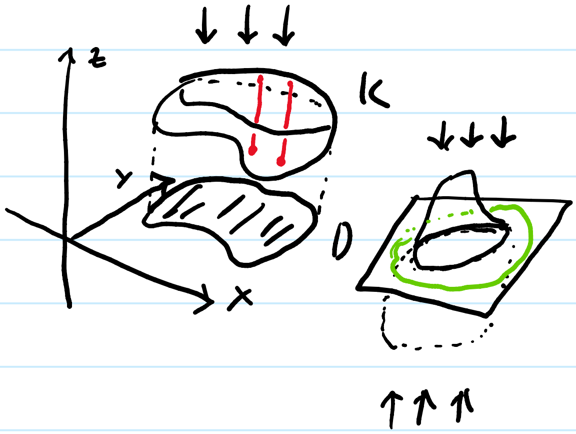
$$= \frac{1}{2} \left(\int_0^3 (9r - r^3) dr \right) \underbrace{\left(\int_0^{2\pi} 1 \, d\varphi \right)}_{=2\pi} =$$

$$= \frac{1}{2} \left[\frac{9}{2} r^2 - \frac{r^4}{4} \right]_0^3 \cdot 2\pi =$$

$$= \frac{1}{\cancel{2}} \left(\frac{81}{2} - \frac{81}{4} \right) \cancel{2}\pi =$$

$= \frac{1}{2}\left(\frac{81}{2} - \frac{81}{4}\right) \times \pi =$

$= \underline{\underline{\frac{81}{4}\pi}}$



VID91 INTEGRALKALKYL - TRIPPELINTEGRAL ALLMÄNT OMRÅDE, EXEMPEL (FORTS.)

den 7 augusti 2024 21:32

Ex: Beräkna

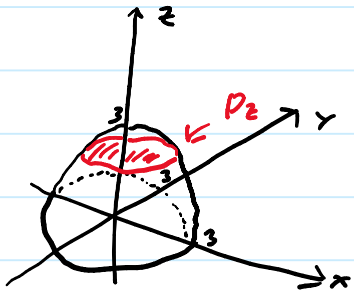
$$I = \iiint_K z \, dx dy dz$$

där k är halvklotet $x^2+y^2+z^2 \le 9$, $z \ge 0$

S(SS):

konst i dxdy

$$I = \int_0^3 \left(\iint_{D_z} z \, dx dy \right) dz =$$



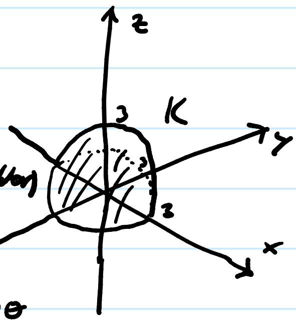
$$= \int_0^3 z \left(\underbrace{\iint 1 \, dx dy}_{\text{area } D_z = \pi(\sqrt{9-z^2})^2} \right) dz =$$

D_z: $x^2+y^2 \le 9-z^2$ cirkelskiva med radie $\sqrt{9-z^2}$

$$= \pi \int_0^3 z(9-z^2) dz = \pi \left[\frac{9}{2}z^2 - \frac{1}{4}z^4 \right]_0^3 = \pi \left(\frac{81}{2} - \frac{81}{4} \right) = \frac{81}{4}\pi$$

<u>Ex:</u> Beräkna

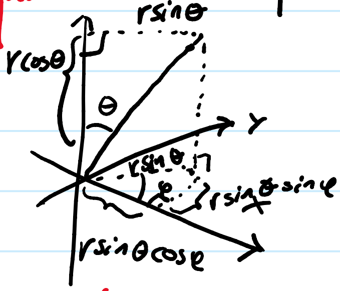
$$\iiint_K z\,dxdydz \qquad K: x^2+y^2+z^2 \le 9,\quad z \ge 0$$



<u>Rymdpolära koordinater</u>

$$\boxed{\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}}$$

Måste ta fram funk.determ av denna (studfallet)



$$E: \begin{cases} 0 \le r \le 3 \\ 0 \le r \le \frac{\pi}{2} \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\begin{vmatrix} x'_r & x'_\theta & x'_\varphi \\ y'_r & y'_\theta & y'_\varphi \\ z'_r & z'_\theta & z'_\varphi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$= r^2\cos^2\theta\sin\theta\cos^2\varphi + r^2\sin^3\theta\sin^2\varphi$$

$$+ r^2\cos^2\theta\sin\theta\sin^2\varphi + r^2\sin^3\theta\cos^2\varphi =$$

$$= r^2\cos^2\theta\sin\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1} + r^2\sin^3\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1} =$$

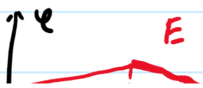
kom ihåg!

$$= r^2\sin\theta\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1} = |r^2\sin\theta|$$

men $r^2\sin\theta$ alltid ≥ 0 och $0 \le \sin\theta \le 2\pi$

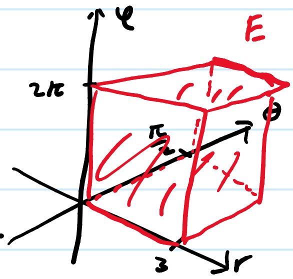
så $|r^2\sin\theta| = r^2\sin\theta$

$$I = \iiint_K z\,dxdydz = \iiint_E r\cos\theta \cdot r^2\sin\theta\, drd\theta d\varphi =$$



$$= \iiint_E r^3 \cos\theta \sin\theta \, dr \, d\theta \, d\varphi =$$

$$= \left(\int_0^3 r^3 \, dr\right)\left(\int_0^{2\pi} \cos\theta\sin\theta \, d\theta\right)\underbrace{\left(\int 1 \, d\varphi\right)}_{2\pi} =$$



$$= 2\pi \left[\frac{r^4}{4}\right]_0^3 \left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} = 2\pi \cdot \frac{81}{4} \cdot \frac{1}{2} = \frac{81}{4}\pi$$

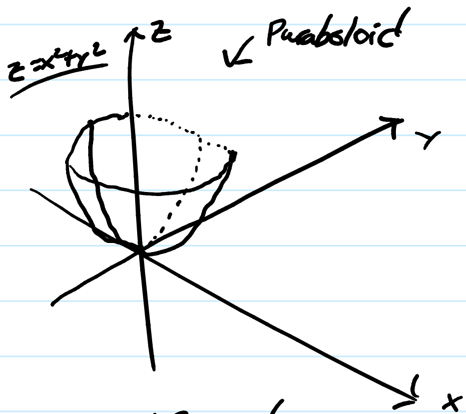
VID93 INTEGRALER TILLÄMPNINGAR - VOLYMBERÄKNING

den 7 augusti 2024 22:01

Volym K: $z \geq x^2+y^2$, $z \leq 2-\sqrt{x^2+y^2}$.

$z = x^2+y^2$: $y=0$: $z=x^2$

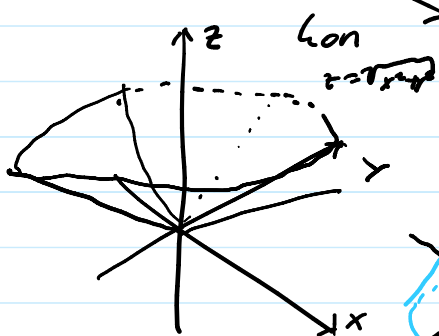
$x=0$: $z=y^2$

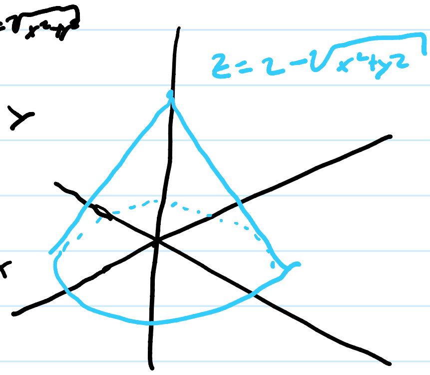


$z = 2 - x^2+y^2$:

$z = \sqrt{x^2+y^2}$: $y=0$: $z=\sqrt{x^2} = |x|$

$$V = \iiint_K 1 \, dxdydz = \iint_D \left(\int_{x^2+y^2}^{2-\sqrt{x^2+y^2}} 1 \, dz \right) dxdy$$





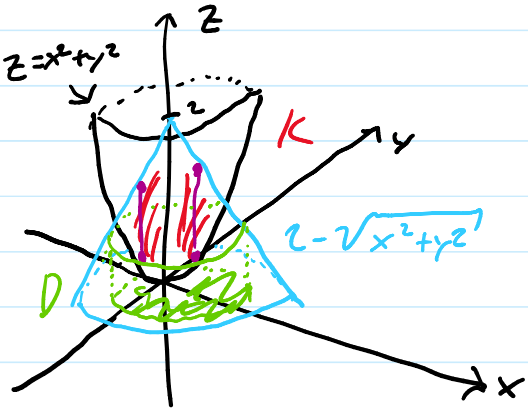
Skärning: $\sqrt{x^2+y^2} = 2-z \Rightarrow$

$\Rightarrow x^2+y^2 = (2-z)^2 \Rightarrow$ $z=4>2$ går ej

$\Rightarrow z = (2-z)^2 \Leftrightarrow \ldots \Leftrightarrow \begin{cases} z=4 \\ z=1 \end{cases}$

insätt $z=1$: $D: x^2+y^2 \leq 1$

cirkelskiva!



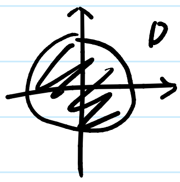
så

$$V = \iiint_K 1 \, dxdydz =$$

$$= \iint_D \left(\int_{x^2+y^2}^{2-\sqrt{x^2+y^2}} 1 \, dz \right) dxdy =$$

$D: x^2+y^2 \leq 1$

$$= \iint_D \left[z \right]_{x^2+y^2}^{2-\sqrt{x^2+y^2}} =$$



$$= \iint_D \left(2-\sqrt{x^2+y^2} - (x^2+y^2)\right) dxdy =$$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

$$= \iint_E (2-r-r^2) \cdot r \, dr d\varphi =$$

(inget φ)

E: $\{ 0 \leq r \leq 1$

$$= \iint_E (2-r-r^2)\cdot r\,dr\,d\varphi =$$

(inget φ)

$$\begin{cases} y = r\sin\varphi \end{cases}$$

$$E: \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$= \left(\int_0^{2\pi} 1\,d\varphi\right)\left(\int_0^1 (2r - r^2 - r^3)\,dr\right) =$$

$$= 2\pi \cdot \left[r^2 - \frac{1}{3}r^3 - \frac{1}{4}r^4\right]_0^1 = 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi\frac{5}{12} = \frac{10}{12}\pi$$

$$\frac{12}{12} - \frac{7}{12} = \frac{5}{12}$$

VID94 INTEGRALER TILLÄMPNINGAR - MASSBERÄKNING, INTRO

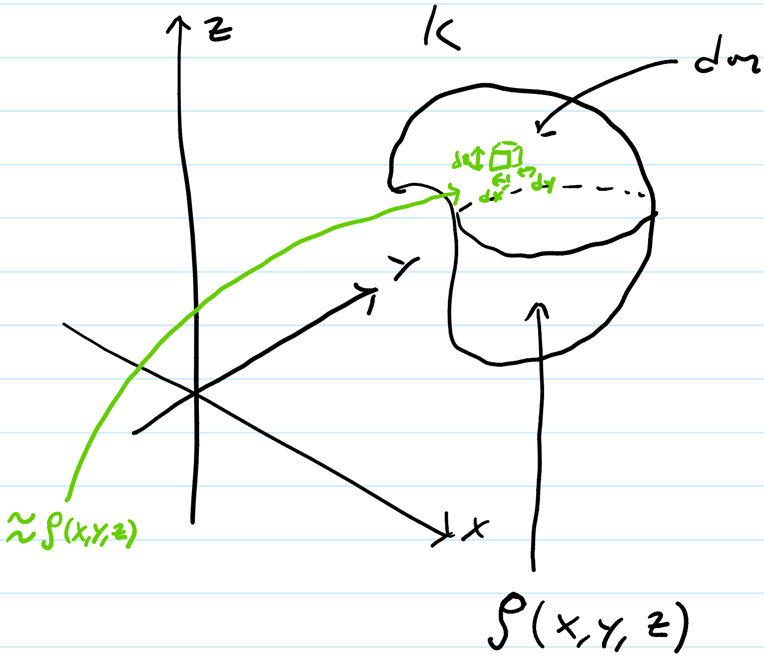
den 9 augusti 2024 10:24

$m = \rho V$

Om ρ konstant

$dm \approx \rho(x,y,z) \cdot dxdydz =$

$$\boxed{m = \int_K dm = \iiint_K \rho(x,y,z)\, dxdydz}$$



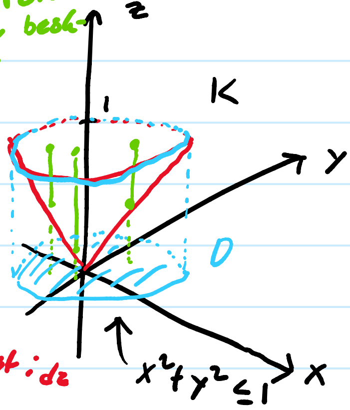
VID95 INTEGRALER TILLÄMPNINGAR - MASSBERÄKNING, EXEMPEL

den 9 augusti 2024 10:32

$K: \sqrt{x^2+y^2} \le z \le 1$

Vi integrerar i z-led först då gränserna beskrivna

$\rho(x,y,z) = x^2+y^2 \quad (kg/m^3)$

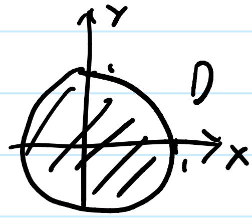


$z=\sqrt{x^2+y^2}$

$$m = \iiint_K \rho(x,y,z)\,dxdydz =$$

konst: dz

$$= \iiint_K (x^2+y^2)\,dxdydz = \iint_D \left(\int_{\sqrt{x^2+y^2}}^{1} (x^2+y^2)\,dz\right) dxdy$$



$\sqrt{x^2+y^2}=1 \iff x^2+y^2=1$

$$= \iint_D (x^2+y^2)\left(\int_{\sqrt{x^2+y^2}}^{1} 1\,dz\right) dxdy = \iint_D (x^2+y^2)\left(1-\sqrt{x^2+y^2}\right) dxdy =$$

$$= \left[\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \; E: \begin{cases} 0\le r\le 1 \\ 0\le\varphi\le 2\pi \end{cases}\right] =$$

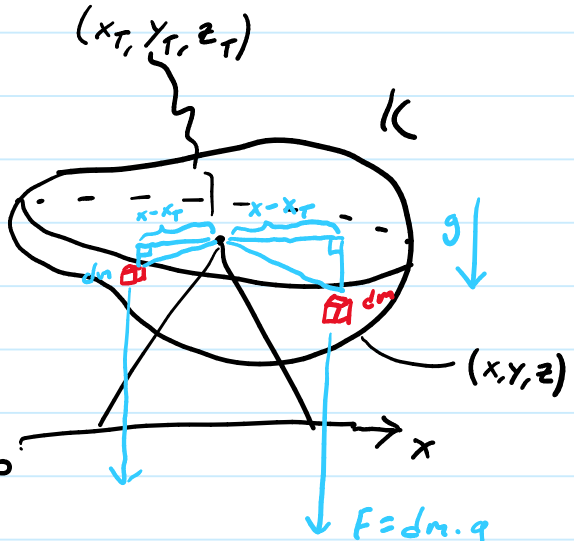
$$= \iint_E r^2(1-r)\cdot r\,drd\varphi = \left(\int_0^{2\pi} 1\,d\varphi\right)\left(\int_0^1 (r^3-r^4)\,dr\right) =$$

$$= 2\pi\cdot\left[\frac{r^4}{4}-\frac{r^5}{5}\right]_0^1 = 2\pi\underbrace{\left(\frac{1}{4}-\frac{1}{5}\right)}_{\frac{1}{20}} = \frac{\pi}{10}$$

VID96 INTEGRALER TILLÄMPNINGAR - TYNGDPUNKT, INTRO

den 9 augusti 2024 10:44

Tyngdpunkt (eller "masscentrum")



$$\int_K dm\, g(x-x_T) = 0$$

$$\Leftrightarrow \int (dm\, g\, x - dm\, g\, x_T) = 0$$

$$\Leftrightarrow \cancel{g}\int_K x\,dm - \cancel{g}\,x_T \int_K dm = 0$$

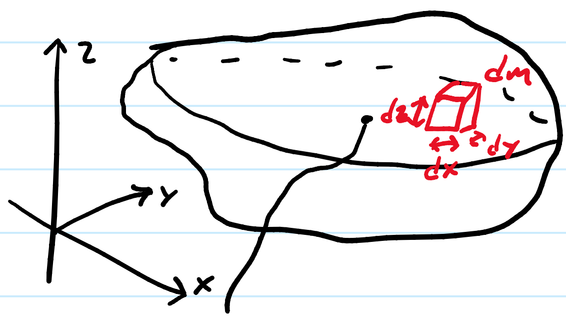
Vridningsmoment $\tau = F(x-x_T) = $
$= dm\, g\,(x-x_T)$

$$\Leftrightarrow x_T \int_K dm = \int_K x\,dm \Leftrightarrow \boxed{x_T = \frac{1}{\int_K dm}\int_K x\,dm}$$

$$y_T = \frac{1}{m}\int_K y\,dm$$

m Totalmassan

$$x_T = \frac{1}{\int_K dm}\int_K x\,dm =$$



$\rho(x,y,z)$

$$dm \approx \rho(x,y,z)\,dx\,dy\,dz$$

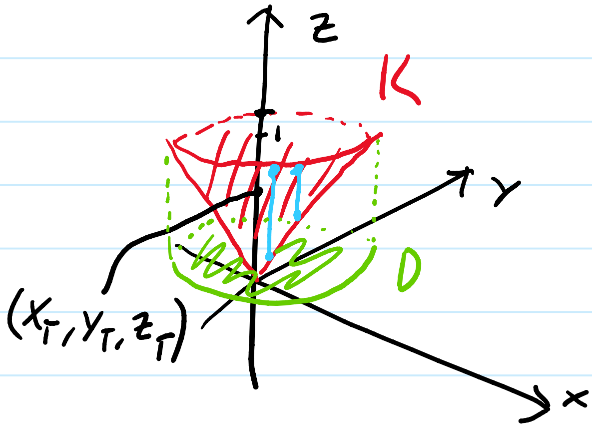
$$= \frac{1}{\iiint_K \rho(x,y,z)\,dx\,dy\,dz}\iiint_K x\,\rho(x,y,z)\,dx\,dy\,dz$$

6 integraler

VID97 INTEGRALER TILLÄMPNINGAR - TYNGDPUNKT, EXEMPEL

den 9 augusti 2024 10:58

$K: \sqrt{x^2+y^2} \le z \le 1$



$\rho(x,y,z) = x^2+y^2$

Symmetriskäl $x_T = y_T = 0$

z_T ?

$$z_T = \frac{1}{\iiint_K \rho(x,y,z)\,dx\,dy\,dz} \iiint_K z\rho(x,y,z)\,dx\,dy\,dz$$

$\frac{\pi}{10}$ från tidigare ex.

konst i dz

$$\iiint_K z(x^2+y^2)\,dx\,dy\,dz = \iint_D \left(\int_{\sqrt{x^2+y^2}}^{1} z(x^2+y^2)\,dz \right) dx\,dy$$

$$= \iint_D (x^2+y^2) \underbrace{\left(\int_{\sqrt{x^2+y^2}}^{1} z\,dz \right)}_{=\left[\frac{1}{2}z^2\right]_{\sqrt{x^2+y^2}}^{1}} dx\,dy = \frac{1}{2} \iint_D (x^2+y^2)(1-(x^2+y^2))\,dx\,dy$$

$D: x^2+y^2 \le 1$

$$= \left[\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad E: \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases} \right] =$$

$$= \frac{1}{2} \iint_E r^2(1-r^2)\cdot r\,dr\,d\varphi = \frac{1}{2} \left(\int_0^{2\pi} 1\,d\varphi \right) \left(\int_0^1 (r^3-r^5)\,dr \right) =$$

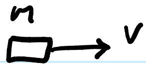
$$= \frac{1}{2} \cdot 2\pi \left[\frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 = \pi \left(\frac{1}{4} - \frac{1}{6} \right) = \frac{\pi}{12}$$

$$z_T = \frac{1}{\pi/10} \cdot \frac{\pi}{12} = \frac{10}{\pi} \cdot \frac{\pi}{12} = \frac{5}{6}$$

VID98 INTEGRALER TILLÄMPNINGAR - TRÖGHETSMOMENT, INTRO

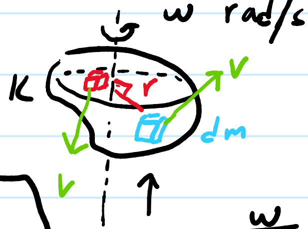
den 9 augusti 2024 11:10

Tröghetsmoment



Rörelse-energi $\boxed{E = \frac{1}{2} v^2 m}$

Massa



$$\boxed{dE = \frac{1}{2} v^2 \, dm = \frac{1}{2} \omega^2 r^2 \, dm}$$

$$v = \frac{\omega}{\cancel{2\pi}} \cdot \cancel{2\pi} r = \omega r$$

$$\boxed{E = \int_K dE = \int_K \frac{1}{2} \omega^2 r^2 \, dm = \frac{1}{2} \omega^2 \int_K r^2 \, dm}$$

$$dm = \rho(x,y,z) \, dx\,dy\,dz$$

tröghetsmomentet $J = \int_K r^2 \, dm$

$$J = \iiint_K r(x,y,z)^2 \rho(x,y,z) \, dx\,dy\,dz$$

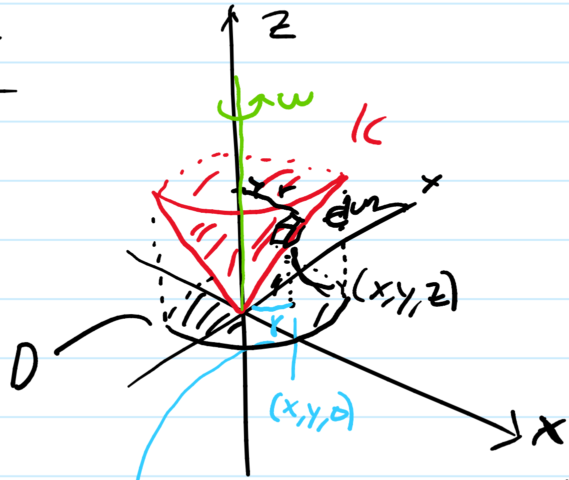
Om tröghetsmomentet minskar kommer rotationshastigheten att öka. Likt en konståkare som roterar och drar in armarna.

VID99 INTEGRALER TILLÄMPNINGAR - TRÖGHETSMOMENT, EXEMPEL

den 9 augusti 2024 13:24

$K: \sqrt{x^2+y^2} \leq z \leq 1$

$\rho(x,y,z) = x^2+y^2$



$dm = \rho(x,y,z)\,dx\,dy\,dz$

$\sqrt{x^2+y^2} = r$

$m^2 \cdot kg$

$$J = \iiint_K r^2\,dm = \iiint_K r(x,y,z)^2 \rho(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_K \underbrace{(x^2+y^2)(x^2+y^2)}_{(x^2+y^2)^2}\,dx\,dy\,dz$$

konst i dz

$$= \iint_D \left(\int_{\sqrt{x^2+y^2}}^{1} (x^2+y^2)^2\,dz \right) dx\,dy =$$

$D: x^2+y^2 \leq 1$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

$$= \iint_D (x^2+y^2)^2 \left(1 - \sqrt{x^2+y^2}\right) dx\,dy =$$

$$= \iint_E r^4(1-r)\cdot r\,dr\,d\varphi = \ldots = 2\pi\left(\frac{1}{6} - \frac{1}{7}\right) = 2\pi\frac{1}{42} = \frac{\pi}{21}$$