VID100 VEKTORANALYS - KURVINTEGRAL, INTRO

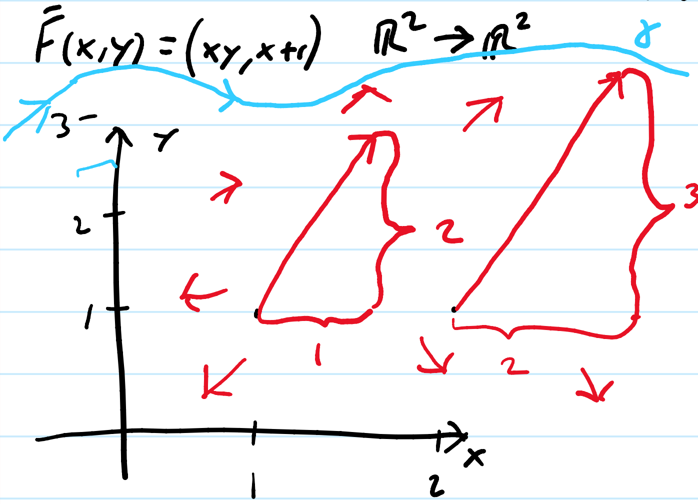
den 9 augusti 2024 13:38

Vektoranalys i $\mathbb{R}^2$

Kurvintegral

Vektorfält

$\bar{F}(x,y) = (xy, x+y) \qquad \mathbb{R}^2 \to \mathbb{R}^2$



Hastighetsfält

Kraftfält

Elektriskt fält

Magnetfält

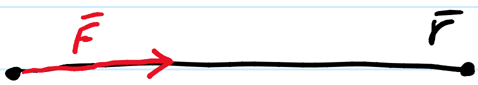
Kraftfält

$\bar{F}(1,1) = (1,2)$

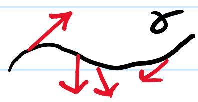
$\bar{F}(2,1) = (2,3)$

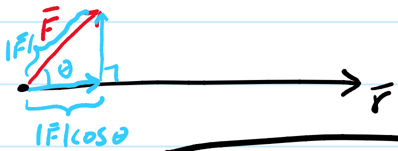
$\boxed{\text{Arbete som } F(x,y) \text{ uträttar}}$

$\boxed{\text{Arbete} = \text{kraft} \cdot \text{sträcka}}$



$W = |\bar{F}| \cdot |\bar{r}|$

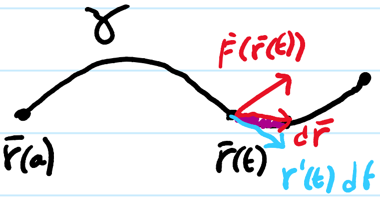




skalär produkt

$$\boxed{W = |\bar{F}|\cos\theta\,|\bar{r}| = \bar{F} \cdot \bar{r}}$$

(gamma)



$\bar{F}(x,y)$

$\bar{r}(t) = (x(t), y(t)) \qquad t : A \to b$

$\bar{r}'(t) = (x'(t), y'(t))$

tid $\underline{\underline{dt}}$ litet

$$\underline{dW} \approx \bar{F}(\bar{r}(t)) \cdot d\bar{r} = \bar{F} \cdot \frac{d\bar{r}}{dt} dt \approx \bar{F} \cdot \bar{r}'(t)\,dt$$

$$\underline{\underline{dW}} \approx \bar{F}(\bar{r}(t))\cdot d\bar{r} = \bar{F}\cdot \underbrace{\frac{d\bar{r}}{dt}}_{\approx \bar{r}'(t)}dt \approx \bar{F}\cdot \bar{r}'(t)\,dt$$

tangentriktn.

för små dt

$$\boxed{W = \int_\gamma dW = \int_a^b \bar{F}\cdot\bar{r}'(t)\,dt} = \int_\gamma \bar{F}\cdot d\bar{r}$$

Kurvintegral

$$\bar{F}(x,y) = (xy, x+1)$$

$$\int_a^b \bar{F}\cdot\bar{r}'(t)\,dt = \int_a^b \big(P(x,y), Q(x,y)\big)\cdot\big(x'(t), y'(t)\big)dt = \quad P \quad Q$$

$$= \int_a^b \Big(P\underbrace{x'(t)}_{=\frac{dx}{dt}} + Q\underbrace{y'(t)}_{=\frac{dy}{dt}}\Big)\cancel{dt} = \boxed{\int_\gamma P\,dx + Q\,dy}$$

VID101 VEKTORANALYS - KURVINTEGRAL, EXEMPEL

den 9 augusti 2024 14:01

$\bar{F}(x,y) = (\underbrace{x^2y}_{P}, \underbrace{xy+1}_{Q})$

$\gamma: y = x^2$ från $(0,0)$ till $(2,4)$

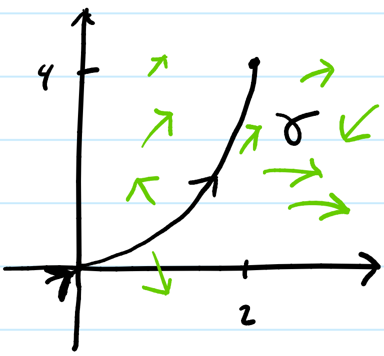
$$\int_\gamma P\,dx + Q\,dy =$$

$$= \int_0^2 t^4 \cdot dt + (t^2+1)2t\,dt =$$

$$= \int \big(\underbrace{t^4 + 2t^4}_{=3t^4} + 2t\big)\,dt =$$

$$= \left[\frac{3}{5}t^5 + t^2\right]_0^2 = \frac{96}{5} + 4 = \frac{96+20}{5} =$$

$$= \underline{\underline{\frac{116}{5}}} \leftarrow$$



Parametriserar:

$$\begin{cases} y = t^2 \\ x = t \end{cases} \qquad t: 0 \to 2$$

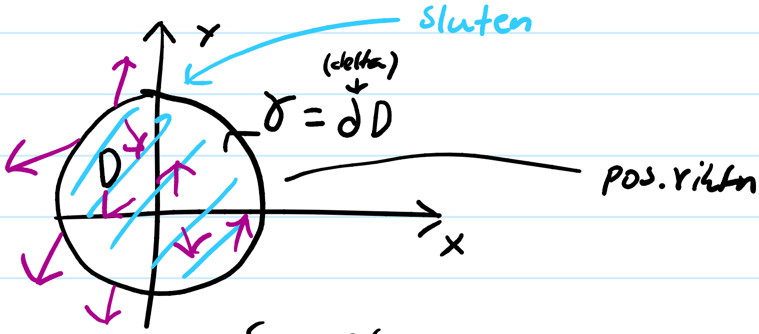
$$\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 2t \end{cases} \Rightarrow \begin{cases} dx = dt \\ dy = 2t\,dt \end{cases}$$

VID102 VEKTORANALYS - GREENS FORMEL, INTRO + EXEMPEL

den 9 augusti 2024 14:16

$\bar{F}(x,y) = (xy, x+y)$

$\gamma = x^2+y^2=1$ pos. riktn. (moturs)



$\gamma : \begin{cases} x = \cos t \\ y = \sin t \end{cases}$

$t: 0 \to 2\pi$

$\begin{cases} \frac{dx}{dt} = -\sin t \\ \frac{dy}{dt} = \cos t \end{cases}$

$$\int_\gamma P\,dx + Q\,dy = \int_0^{2\pi} \Big(\cos t \sin t(-\sin t) + (\cos t + \sin t)\cos t\Big)\,dt =$$

$= \ldots =$

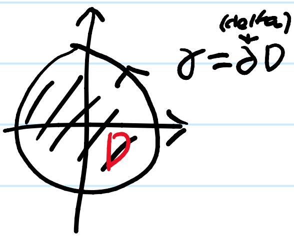
Greens formel

Om

$\bar{F}$ kont, P deriverbar i D, ∂D pos. or.

så

$$\int_\gamma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$\bar{F} = (P, Q) = (xy, x+y)$

$$\int_{\gamma} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$$

$$= \iint_{D} (1-x)\,dx\,dy = \left[\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad \overset{E}{\begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}}\right] =$$

$$= \iint_{E} (1 - r\cos\varphi)\,r\,dr\,d\varphi = \int_{0}^{2\pi}\left(\int_{0}^{1} (r - r^2\cos\varphi)\,dr\right) d\varphi =$$

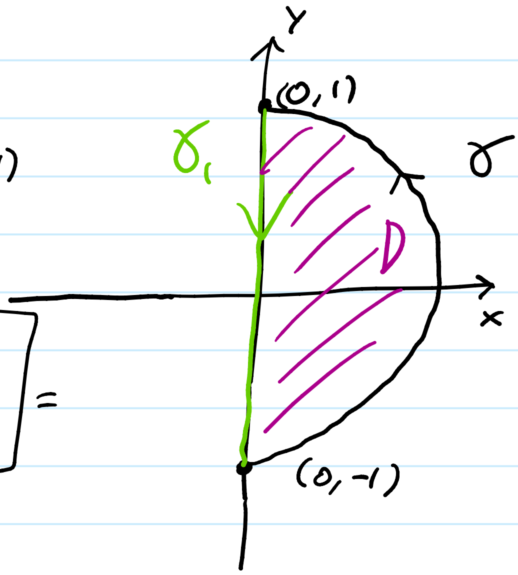
$$= \int_{0}^{2\pi} \left[\frac{r^2}{2} - \frac{r^3}{3}\cos\varphi\right]_{0}^{1} d\varphi = \int_{0}^{2\pi} \left(\frac{1}{2} - \frac{1}{3}\cos\varphi\right) d\varphi = \left[\frac{\varphi}{2} - \frac{1}{3}\sin\varphi\right]_{0}^{2\pi} = \boxed{\pi}$$

VID103 VEKTORANALYS - GREENS FORMEL, EXEMPEL 2

den 9 augusti 2024 14:28

$(P, Q) = (e^x, 1 + xy^2)$

$\gamma = x^2 + y^2 = 1$, $x \geq 0$, från $(0,-1)$ till $(0,1)$



$$\int_\gamma P\,dx + Q\,dy = \left[\begin{cases} x = \cos t \\ y = \sin t \end{cases}, t: -\frac{\pi}{2} \to \frac{\pi}{2} \right] =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\cos t}(-\sin t) + (1 + \cos t \sin^2 t)\cos t)\,dt = \ldots \text{ (jobbig att beräkna)}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - 0 = y^2 \quad \underline{\text{lättare}}.$$

$$\int_{\gamma + \gamma_1} P\,dx + Q\,dy = \iint_D y^2\,dx\,dy =$$

$$= \left[\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, E: \begin{cases} 0 \leq r \leq 1 \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \end{cases} \right] = \iint_E r^2 \sin^2\varphi \cdot \underline{\underline{r}} \cdot dr\,d\varphi =$$

$$= \iint_E r^3 \sin^2\varphi\,dr\,d\varphi = \left(\int_0^1 r^3\,dr \right) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sin^2\varphi}_{=\frac{1-\cos 2\varphi}{2}} \right) =$$

$$= \left[\frac{r^4}{4} \right]_0^1 \cdot \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$$

$$\int_{\gamma_1} P\,dx + Q\,dy = \left[\begin{cases} x=0 \\ y=t \end{cases}, \; t: 1 \to -1\right] =$$

$$= \int_{1}^{-1} (e^0 \cdot 0) + (1+0)\,dt = -2$$

$$\int_{\gamma} P\,dx + Q\,dy = \frac{\pi}{8} - (-2) = \frac{\pi}{8} + 2$$

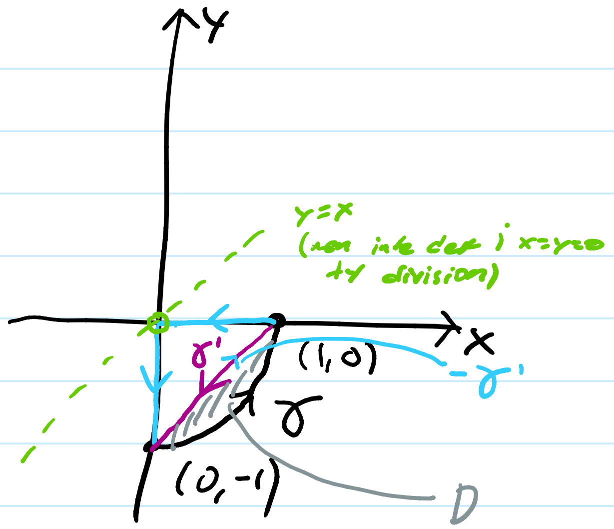
VID104 VEKTORANALYS - GREENS FORMEL, EXEMPEL 3

den 9 augusti 2024 14:39

$$(P,Q) = \left(-\frac{2y}{(x-y)^3}, \frac{x+y}{(x-y)^3}\right)$$

$$\gamma: x^2+y^2=1,\ x\geq 0,\ y\leq 0$$

(0,-1) till (1,0)



(kvotregeln)

$$\frac{\partial Q}{\partial x} = \frac{1\cdot(x-y)^{\cancel{3}1} - 3\cancel{(x-y)^2}\cdot(x+y)}{(x-y)^{\cancel{6}4}} =$$

$$= \frac{x-y-3(x+y)}{(x-y)^3} = \frac{-2x-4y}{(x-y)^3}$$

(samma)

(kvotregeln)

$$\frac{\partial P}{\partial y} = \frac{-2(x-y)^{\cancel{3}} - 3\cancel{(x-y)^2}\cdot(-1)(-2y)}{(x-y)^{\cancel{6}}} = \frac{-2(x-y)-6y}{(x-y)^3} = \frac{-2x-4y}{(x-y)^3}$$

Greens formel lämplig. Måste sluta området först. Följer axlarna.
Men $x=y\neq 0$ ty division går inte. Använd γ'

$$\int_{\gamma+\gamma'} P\,dx+Q\,dy = \iint_D 0\,dx\,dy = 0 \Rightarrow \int_\gamma \dots = -\int_{\gamma'} \dots = \int_{-\gamma'} \dots$$

$$\int_\gamma P\,dx+Q\,dy = \int_{-\gamma'} P\,dx+Q\,dy =$$

$$= \left[-\gamma': \begin{cases} x=t \\ y=t-1 \end{cases},\ t: 0\to 1\right] = \int_0^1 \left(\overbrace{\frac{-2(t-1)}{1}}^{\cancel{-2t}+2}\cdot 1 + \overbrace{\frac{t+t-1}{1}}^{\cancel{2t}-1}\cdot 1\right)dt =$$

$$= \int_0^1 1\,dt = 1$$

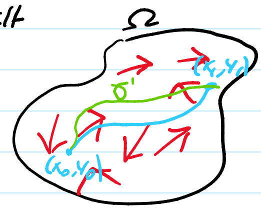
VID105 VEKTORANALYS - POTENTIALFÄLT INTRO

den 9 augusti 2024 14:53

Potentialfält, konservativt fält

$\vec{F} = (P, Q)$ i Ω (område)

$\boxed{U(x(t), y(t))}$



$U(x,y)$ i Ω sådan att $\text{grad}\, U = (P, Q)$ dvs

$$\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right) = (P, Q)$$

Potential (funktion)

Sats: Potentialfält

$$\int_\gamma P\,dx + Q\,dy = U(x_1, y_1) - U(x_0, y_0)$$

Bevis:

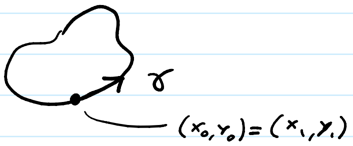
$$\int_\gamma P\,dx + Q\,dy = \int_a^b \left(\frac{\partial U}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial U}{\partial y} \cdot \frac{\partial y}{\partial t}\right) dt \overset{\text{kedjeregeln}}{=} \int_a^b \frac{d}{dt} U(x(t), y(t))\, dt = \Big[U(x(t), y(t))\Big]_a^b =$$

$\gamma: (x(t), y(t))$, $t: a \to b$

$$= U(x(b), y(b)) - U(x(a), y(a)) =$$

$$= U(x_1, y_1) - U(x_0, y_0). \quad \square$$

Kurvintegraler i potentialfält är oberoende av vägen. Vägoberoende



$$\int_\gamma P\,dx + Q\,dy = \underline{0}$$

VID106 VEKTORANALYS - POTENTIALFÄLT, EXEMPEL

den 9 augusti 2024 15:40

$$(P,Q) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$ Potentialfält?

$U(x,y)$

$$\begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{x}{x^2+y^2} & ① \\ \dfrac{\partial U}{\partial y} = \dfrac{y}{x^2+y^2} & ② \end{cases}$$

$$②: U(x,y) = \int \frac{y}{x^2+y^2}\,dy = \left[\frac{1}{2}\ln(x^2+y^2) + \varphi(x)\right]$$

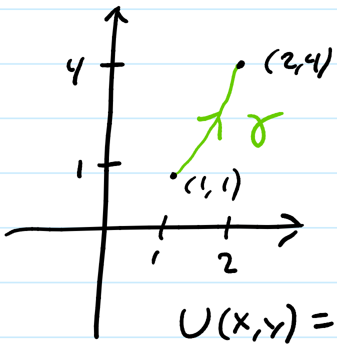
Deriverar m.a.på x

$$\frac{\partial U}{\partial x} = \frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+y^2} + \varphi'(x) = \frac{x}{x^2+y^2} \Leftrightarrow \varphi'(x) = 0 \Leftrightarrow \varphi(x) = C$$

pot. funk. $U(x,y) = \frac{1}{2}\ln(x^2+y^2) + C$, (P,Q) Potentialfält

$$(P,Q) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$ Potentialfält

γ: $y = x^2$ från $(1,1)$ till $(2,4)$



$U(x,y) = \frac{1}{2}\ln(x^2+y^2) + C$

$$\int_\gamma P\,dx + Q\,dy = U(2,4) - U(1,1) =$$

$$= \frac{1}{2}\ln(2^2+4^2) - \frac{1}{2}\ln(1^2+1^2) =$$

$$= \frac{1}{2}\ln 20 - \frac{1}{2}\ln 2 =$$

$$= \frac{1}{2}(\ln 20 - \ln 2) = \frac{1}{2}\ln\frac{20}{2} = \frac{1}{2}\ln 10$$