Chapter 8
Random-Variate Generation

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Discrete-Event System Simulation

Purpose & Overview

- Develop understanding of generating samples from a specified distribution as input to a simulation model.

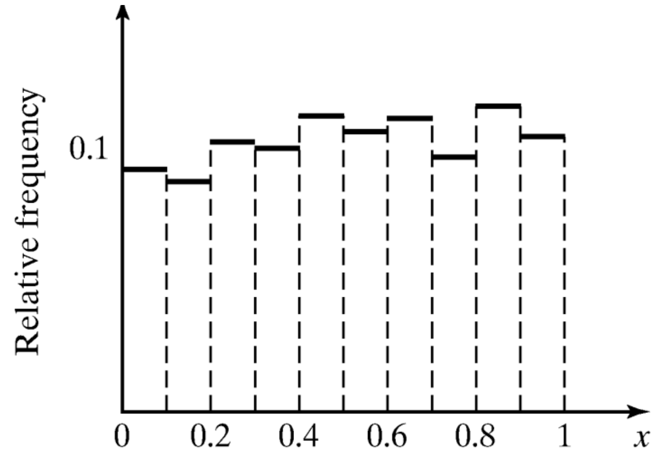
- Illustrate some widely-used techniques for generating random variates.
  - Inverse-transform technique
  - Acceptance-rejection technique

Uniform Distribution

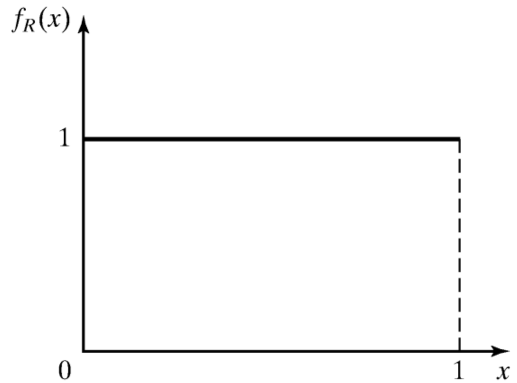
[Inverse-transform]

- Histogram for 200 R_i:

(empirical)



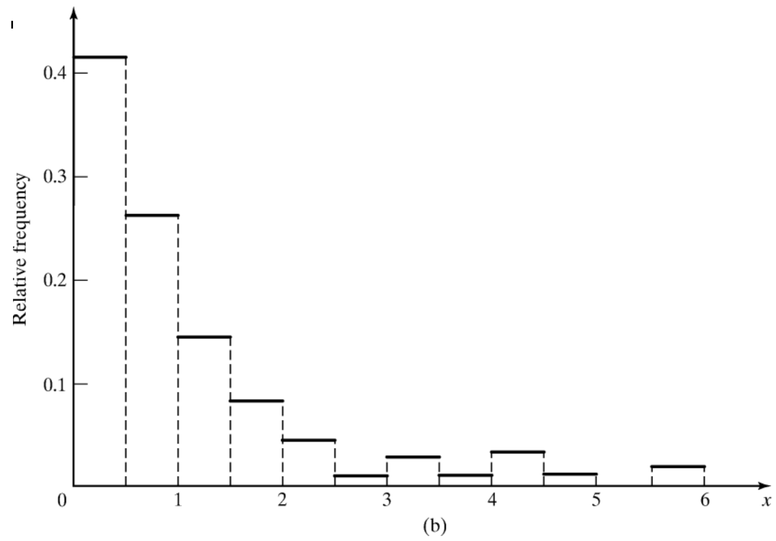
(theoretical)



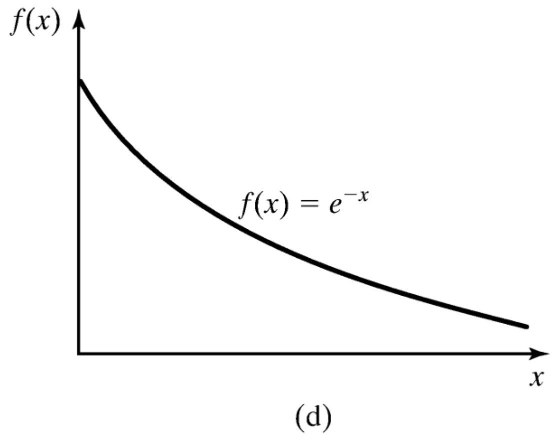
(c)

Exponential Distribution [Inverse-transform]

- Histogram for 200 X_i :
  (empirical)



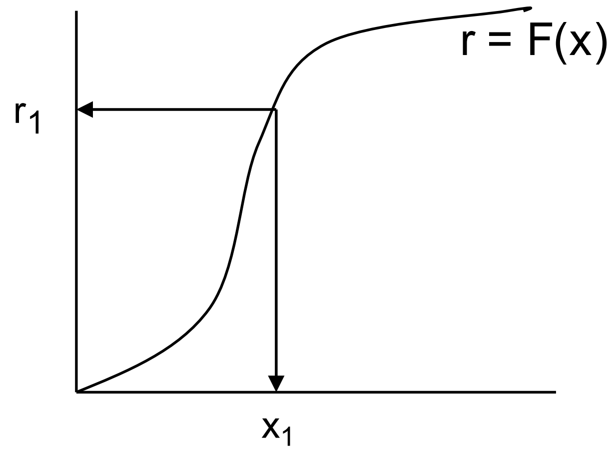
  (theoretical)



Inverse-transform Technique

- The concept:
  - For cdf function: $r = F(x)$
  - Generate r from uniform (0,1)
  - Find x:

$$\boxed{x = F^{-1}(r)}$$



Exponential Distribution [Inverse-transform]

- Exponential Distribution:
  - Exponential cdf:

$$r = F(x)$$
$$= 1 - e^{-\lambda x} \qquad \text{for } x \geq 0$$

  - To generate $X_1, X_2, X_3 \ldots$

$$X_i = F^{-1}(R_i)$$
$$= -(1/\lambda)\, ln(1-R_i) \qquad \text{[Eq'n 8.3]}$$

$$X_i = -\frac{1}{\lambda} \ln R_i$$

Exponential Distribution [Inverse-transform]

- Generate using the graphical view:

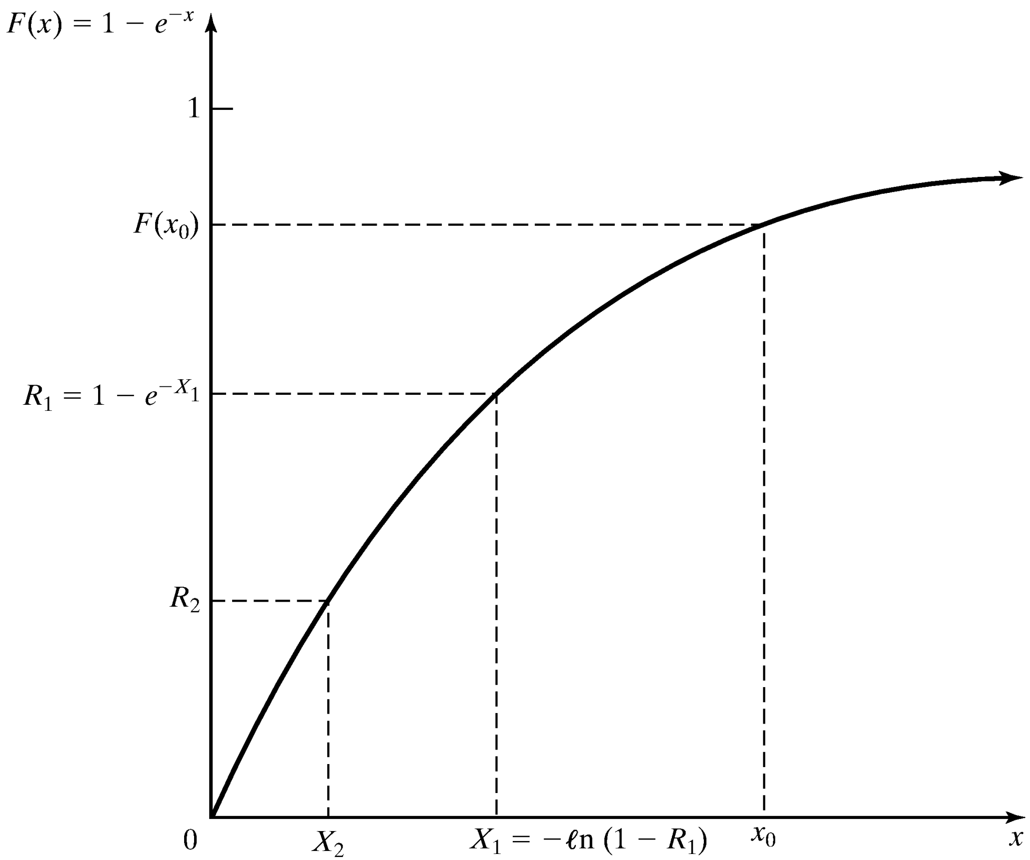


Figure: Inverse-transform technique for e*xp(λ = 1)*

Exponential Distribution [Inverse-transform]

- Draw a horizontal line from R_1 (randomly generated), find to corresponding *x* to obtain X_1.

- We can see that:
  - $X_1 \le x_0$ when and only when $R_1 \le F(x_0)$

    so : $P(X_1 \le x_0) = P(R_1 \le F(x_0))$

  - $R_1 \sim U(0,1)$

    so : $P(R_1 \le F(x_0)) = F(x_0)$

Other Distributions [Inverse-transform]

- Examples of other distributions for which inverse cdf works are:
  - Uniform distribution

$$X \sim U(a,b) \Rightarrow X = a + (b-a)R$$

  - Weibull distribution

    (v = 0)

$$f(x) = \begin{cases} \dfrac{\beta}{\alpha^{\beta}} x^{\beta-1} e^{-(x/a)^{\beta}}, & x \ge 0 \\ 0, & otherwise \end{cases}$$

$$F(X) = 1 - e^{-(x/\alpha)^{\beta}}, x \ge 0$$

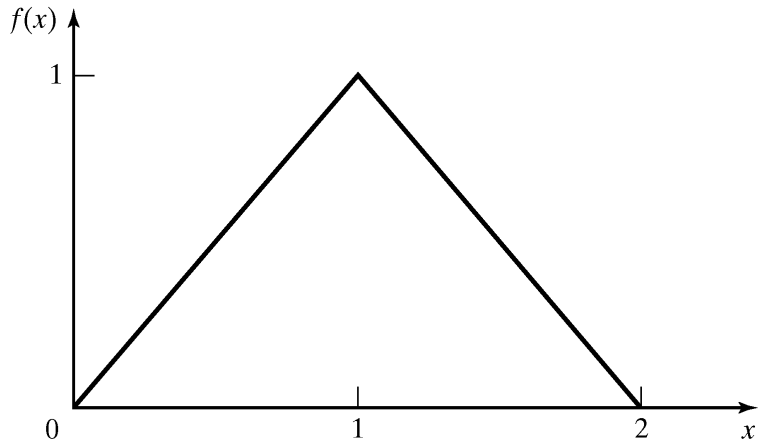
$$\Rightarrow \quad X = \alpha[-\ln(1-R)]^{1/\beta}$$

Other Distributions

[Inverse-transform]

- Triangular distribution

$$f(x)=\begin{cases} x. & 0\le x\le 1 \\ 2-x, & 1\le x\le 2 \\ 0 & otherwise \end{cases}$$



$$\Rightarrow \quad X=\begin{cases} \sqrt{2R}, & 0\le R\le \frac{1}{2} \\ 2-\sqrt{2(1-R)}, & \frac{1}{2}\prec R\le 1 \end{cases}$$

Empirical Continuous Dist'n [Inverse-transform]

- When theoretical distribution is not applicable
- To collect empirical data:
  - Resample the observed data
  - Interpolate between observed data points to fill in the gaps
- For a small sample set (size *n*):
  - Arrange the data from smallest to largest

$$x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$$

  - Assign the probability 1/n to each interval $x_{(i-1)} \le x \le x_{(i)}$

$$\boxed{X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i\left(R - \frac{(i-1)}{n}\right)}$$

where

$$a_i = \frac{x_{(i)} - x_{(i-1)}}{i/n - (i-1)/n} = \frac{x_{(i)} - x_{(i-1)}}{1/n}$$

Empirical Continuous Dist'n [Inverse-transform]

- Five observations of fire-crew response times (in mins.):
  - 2.76 1.83 0.80 1.45 1.24

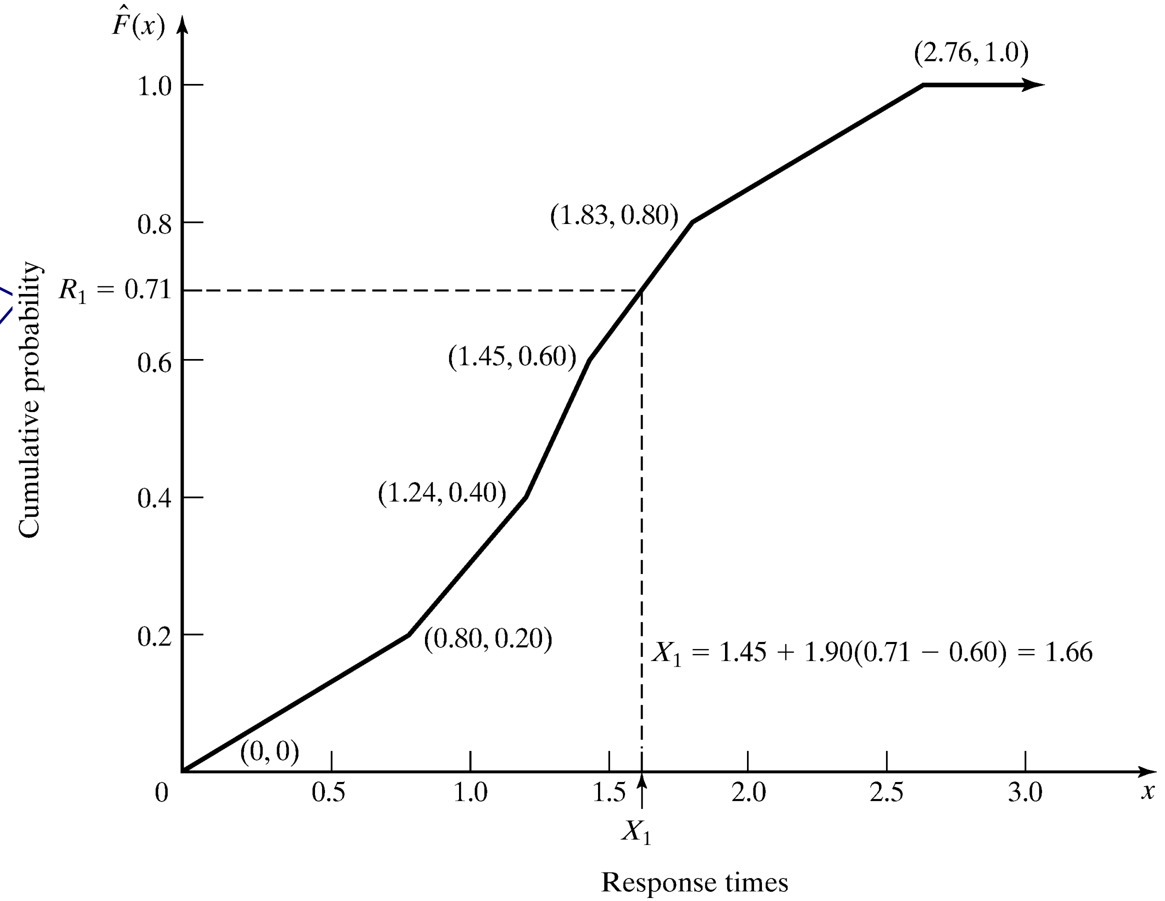
i	*Interval (Hours)*	*Probability 1/n*	*Cumalative Probability, i/n*	*Slope, a_i*
1	$0.0 \le x \le 0.80$	0.2	0.20	4.00
2	$0.8 \le x \le 1.24$	0.2	0.40	2.20
3	$1.24 \le x \le 1.45$	0.2	0.60	1.05
4	$1.45 \le x \le 1.83$	0.2	0.80	1.90
5	$1.83 \le x \le 2.76$	0.2	1.00	4.65

Empirical Continuous Dist'n [Inverse-transform]

Consider $R_1 = 0.71$:

$(i-1)/n = 0.6 < R_1 < i/n = 0.8$

$X_1 = x_{(4-1)} + a_4(R_1 - (4-1)/n)$
$= 1.45 + 1.90(0.71-0.60)$
$= 1.66$



Empirical Continuous Dist'n [Inverse-transform]

- For a large sample set :
  - Summarize into frequency distribution with small number of intervals.
    (equal intervals)
    $x_{(i-1)} < x \le x_{(i)}$: i th interval
  - Fit a continuous empirical cdf to the frequency distribution.
    if $c_{i-1} < R \le c_i$ (ci : cumulative frequency)

$$X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i (R - c_{i-1})$$

Where

$$a_i = \frac{x_{(i)} - x_{(i-1)}}{c_i - c_{i-1}}$$

Empirical Continuous Dist'n [Inverse-transform]

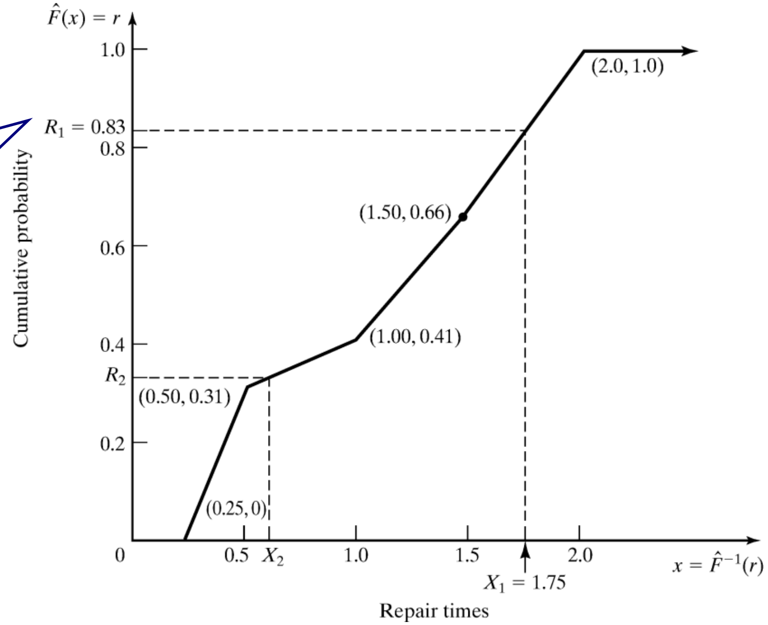
- Example: Suppose the data collected for100 broken-widget repair times are:

i	*Interval (Hours)*	*Frequency*	*Relative Frequency*	*Cumulative Frequency, c_i*	*Slope, a_i*
1	$0.25 \le x \le 0.5$	31	0.31	0.31	0.81
2	$0.5 \le x \le 1.0$	10	0.10	0.41	5.0
3	$1.0 \le x \le 1.5$	25	0.25	0.66	2.0
4	$1.5 \le x \le 2.0$	34	0.34	1.00	1.47

Consider $R_1 = 0.83$:

$c_3 = 0.66 < R_1 < c_4 = 1.00$

$X_1 = x_{(4-1)} + a_4(R_1 - c_{(4-1)})$
$= 1.5 + 1.47(0.83-0.66)$
$= 1.75$



Discrete Distribution [Inverse-transform]

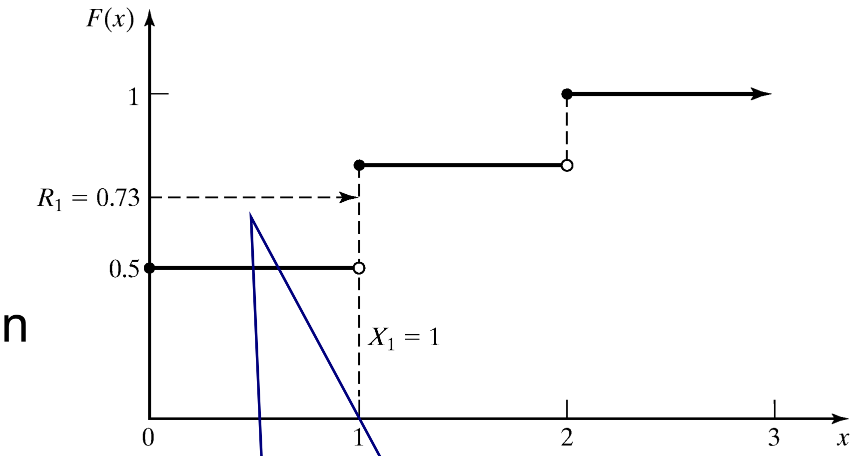
- All discrete distributions can be generated via inverse-transform technique
- Method: numerically, table-lookup procedure, algebraically, or a formula
- Examples of application:
  - Empirical
  - Discrete uniform
  - Gamma

Discrete Distribution

[Inverse-transform]

- Example: Suppose the number of shipments, x, on the loading dock of IHW company is either *0*, *1*, or *2*
  - Data - Probability distribution:

x	*p(x)*	*F(x)*
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00



  - Method - Given R, the generation scheme becomes:

$$x = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 \prec R \le 0.8 \\ 2, & 0.8 \prec R \le 1.0 \end{cases}$$

Consider $R_1 = 0.73$:

$F(x_{i-1}) < R <= F(x_i)$

$F(x_0) < 0.73 <= F(x_1)$

Hence, $x_1 = 1$

Acceptance-Rejection technique

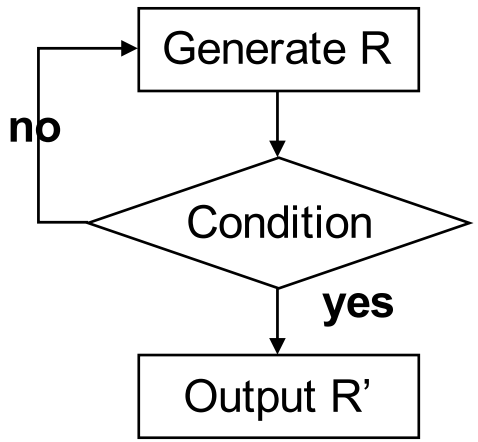
- Useful particularly when inverse cdf does not exist in closed form, a.k.a. thinning
- Illustration: To generate random variates, $X \sim U(1/4, 1)$

Procedures:

Step 1. Generate $R \sim U[0,1]$

Step 2a. If $R >= 1/4$, accept $X=R$.

Step 2b. If $R < 1/4$, reject R, return to Step 1



- *R* does not have the desired distribution, but *R* conditioned (R') on the event $\{R \geq 1/4\}$ does.
- Efficiency: Depends heavily on the ability to minimize the number of rejections.

Poisson Distribution

$$p(n) = P(N = n) = \frac{e^{-\alpha}\alpha^n}{n!}, \mathrm{n} = 0,1,2, \ldots$$

$$N = n$$

$$A_1 + A_2 + \cdots + A_n \leq 1 < A_1 + \cdots + A_n + A_{n+1}$$

$$\sum_{i=1}^{n} -\frac{1}{\alpha} \ln R_i \leq 1 < \sum_{i=1}^{n+1} -\frac{1}{\alpha} \ln R_i$$

$$\ln \prod_{i=1}^{n} R_i = \sum_{i=1}^{n} \ln R_i \geq -\alpha > \sum_{i=1}^{n+1} \ln R_i = \ln \prod_{i=1}^{n+1} R_i$$

Poisson Distribution

$$\prod_{i=1}^{n} R_i \geq e^{-\alpha} > \prod_{i=1}^{n+1} R_i$$

- **Step 1:** Set $n = 0$, $P = 1$.
- **Step 2:** Generate a random number R_{n+1}, and replace P by $P.\, R_{n+1}$.
- **Step 3:** If $P < e^{-\alpha}$, then accept $N = n$, otherwise, reject the current n, increase n by one, and return to step 2.

Poisson Distribution

$$\alpha = 0.2 \Rightarrow e^{-\alpha} = 0.8187$$

n	R_{n+1}	P	Accept/Reject	Result
0	0.4357	0.4357	$P < e^{-\alpha}$ (accept)	$N = 0$
0	0.4146	0.4146	$P < e^{-\alpha}$ (accept)	$N = 0$
0	0.8353	0.8353	$P \geq e^{-\alpha}$ (reject)	
1	0.9952	0.8313	$P \geq e^{-\alpha}$ (reject)	
2	0.8004	0.6654	$P < e^{-\alpha}$ (accept)	$N = 2$

Summary

- Principles of random-variate generate via
  - Inverse-transform technique
  - Acceptance-rejection technique
- Important for generating continuous and discrete distributions