Chapter 5
Statistical Models in Simulation

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Discrete-Event System Simulation

Outlines

- Purpose & Overview
- Discrete random variables
- Continuous random variables
- Cumulative distribution function
- Expectation
- Empirical distribution
- Discrete distributions
- Continuous distributions
- Useful Statistical Models
- Poisson Process

Purpose & Overview

- The world the model-builder sees is probabilistic rather than deterministic.
  - Some statistical model might well describe the variations.

- An appropriate model can be developed by sampling the phenomenon of interest:
  - Select a known distribution through educated guesses
  - Make estimate of the parameter(s)
  - Test for goodness of fit

- In this chapter:
  - Review several important probability distributions
  - Present some typical application of these models

Discrete Random Variables

- X is a discrete random variable if the number of possible values of X is finite, or countably infinite.
- Example: Consider jobs arriving at a job shop.
  - Let X be the number of jobs arriving each week at a job shop.
  - R_x = possible values of X (range space of X) = $\{0,1,2,\ldots\}$
  - $p(x_i)$ = probability the random variable is $x_i = P(X = x_i)$
- $p(x_i)$, $i = 1,2, \ldots$ must satisfy:

$$1.\ \ p(x_i) \geq 0,\ \text{ for all } i$$

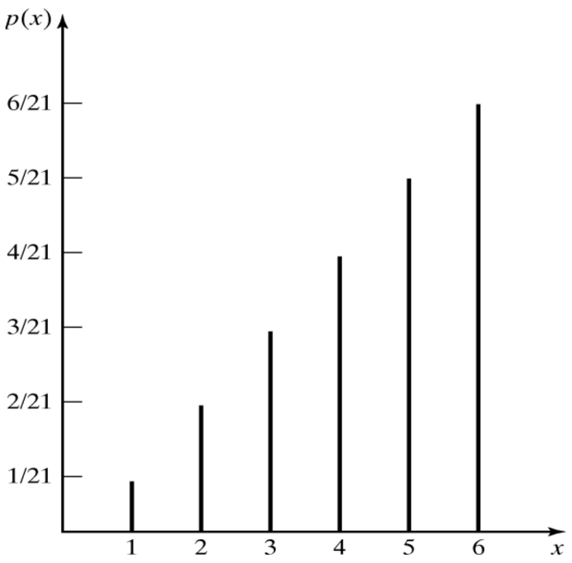
$$2.\ \ \sum_{i=1}^{\infty} p(x_i) = 1$$

- The collection of pairs $[x_i, p(x_i)]$, $i = 1,2,\ldots$, is called the probability distribution of X, and $p(x_i)$ is called the probability mass function (pmf) of X.

Discrete Random Variables

- Example: Assume the die is loaded so that the probability that a given face lands up is proportional to the number of spots showing.

x_i	1	2	3	4	5	6
$P(x_i)$	1/21	2/21	3/21	4/21	5/21	6/21



- *$p(x_i)$, $i = 1,2, \ldots$* must satisfy:

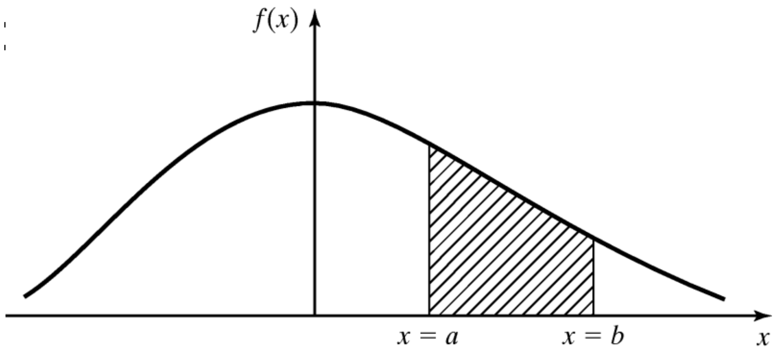
1. $p(x_i) \geq 0$, for all i
2. $\sum_{i=1}^{\infty} p(x_i) = 1$

Continuous Random Variables

- X is a continuous random variable if its range space R_x is an interval or a collection of intervals.
- The probability that X lies in the interval *[a,b]* is given by:

$$P(a \le X \le b) = \int_a^b f(x)dx$$

- *f(x)*, denoted as the pdf of *X*, satisfies:



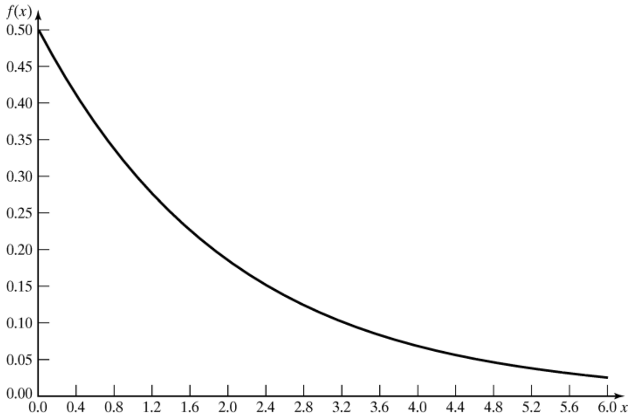
1. $f(x) \ge 0$, for all x in R_X
2. $\int_{R_X} f(x)dx = 1$
3. $f(x) = 0$, if x is not in R_X

- Properties
  1. $P(X = x_0) = 0$, because $\int_{x_0}^{x_0} f(x)dx = 0$
  2. $P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$

Continuous Random Variables [Probability Review]

- Example: Life of an inspection device is given by *X*, a continuous random variable with pdf:

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



  - *X* has an exponential distribution with mean 2 years
  - Probability that the device's life is between 2 and 3 years is:

$$P(2 \leq x \leq 3) = \frac{1}{2} \int_{2}^{3} e^{-x/2} dx = 0.14$$

Cumulative Distribution Function [Probability Review]

- Cumulative Distribution Function (cdf) is denoted by *F(x)*, where *F(x) = P(X <= x)*
  - If *X* is discrete, then $$F(x) = \sum_{\substack{\text{all} \\ x_i \le x}} p(x_i)$$
  - If *X* is continuous, then $$F(x) = \int_{-\infty}^{x} f(t)\,dt$$
- Properties

  1. F is nondecreasing function. If $a < b$, then $F(a) \le F(b)$
  2. $\lim_{x \to \infty} F(x) = 1$
  3. $\lim_{x \to -\infty} F(x) = 0$

- All probability question about *X* can be answered in terms of the cdf, e.g.:

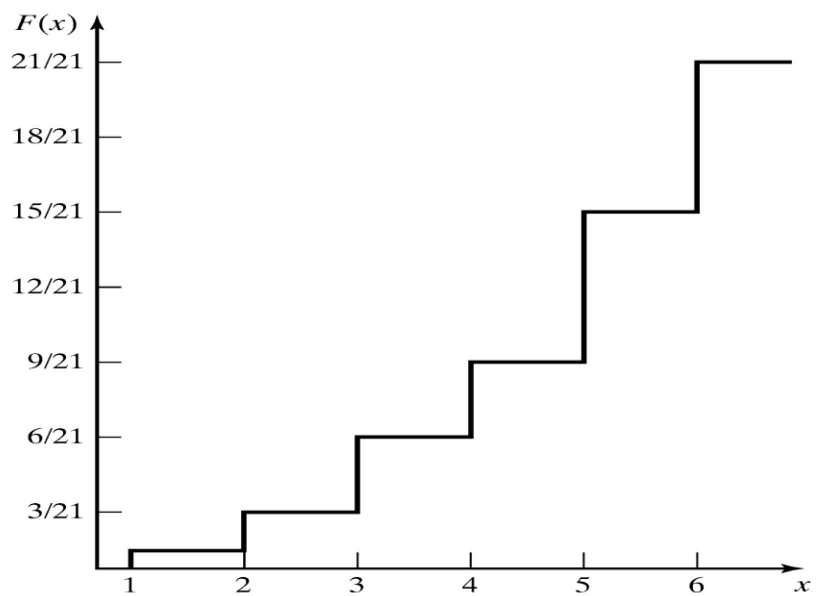
$$P(a < X \le b) = F(b) - F(a), \text{ for all } a < b$$

Cumulative Distribution Function [Probability Review]

- Example: The die-tossing experiment described in last example has a cdf given as follows:

x	(-∞,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,∞)
F(x)	0	1/21	3/21	6/21	10/21	15/21	21/21

  - [a,b) = {a ≤ x < b}



Cumulative Distribution Function [Probability Review]

- Example: An inspection device has cdf:

$$F(x) = \frac{1}{2}\int_0^x e^{-t/2}dt = 1 - e^{-x/2}$$

  - The probability that the device lasts for less than 2 years:

$$P(0 \le X \le 2) = F(2) - F(0) = F(2) = 1 - e^{-1} = 0.632$$

  - The probability that it lasts between 2 and 3 years:

$$P(2 \le X \le 3) = F(3) - F(2) = (1 - e^{-(3/2)}) - (1 - e^{-1}) = 0.145$$

Expectation

- The expected value of *X* is denoted by *E(X)*
  - If *X* is discrete $$E(x) = \sum_{\text{all } i} x_i p(x_i)$$
  - If *X* is continuous $$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$
  - The mean, m, or the 1st moment of *X*
  - A measure of the central tendency
- The variance of *X* is denoted by *V(X)* or *var(X)* or σ^2
  - Definition: $V(X) = E[(X - E[X]^2]$
  - Also, $V(X) = E(X^2) - [E(x)]^2$
  - A measure of the spread or variation of the possible values of X around the mean
- The standard deviation of *X* is denoted by σ
  - Definition: square root of *V(X)*
  - Expressed in the same units as the mean

Expectations

[Probability Review]

- Example: The mean of life of the previous inspection device is:

$$E(X) = \frac{1}{2}\int_0^\infty x e^{-x/2} dx = -xe^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2} dx = 2$$

- To compute variance of *X*, we first compute *E(X²)*:

$$E(X^2) = \frac{1}{2}\int_0^\infty x^2 e^{-x/2} dx = -x^2 e^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2} dx = 8$$

- Hence, the variance and standard deviation of the device's life are:

$$V(X) = 8 - 2^2 = 4$$

$$\sigma = \sqrt{V(X)} = 2$$

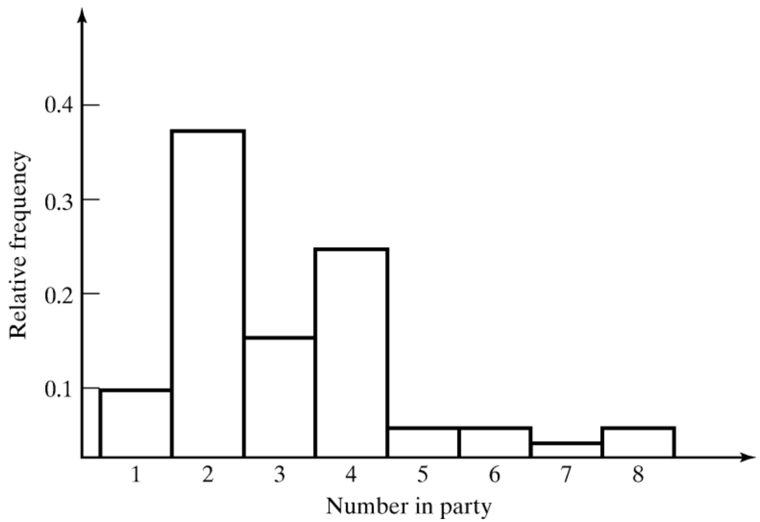
Empirical Distributions

- Example:
- Customers arrive at lunchtime in groups of from one to eight persons.
- The number of persons per party in the last 300 groups has been observed.
- The results are summarized in a table.
- The histogram of the data is also included.

Empirical Distributions (cont.)

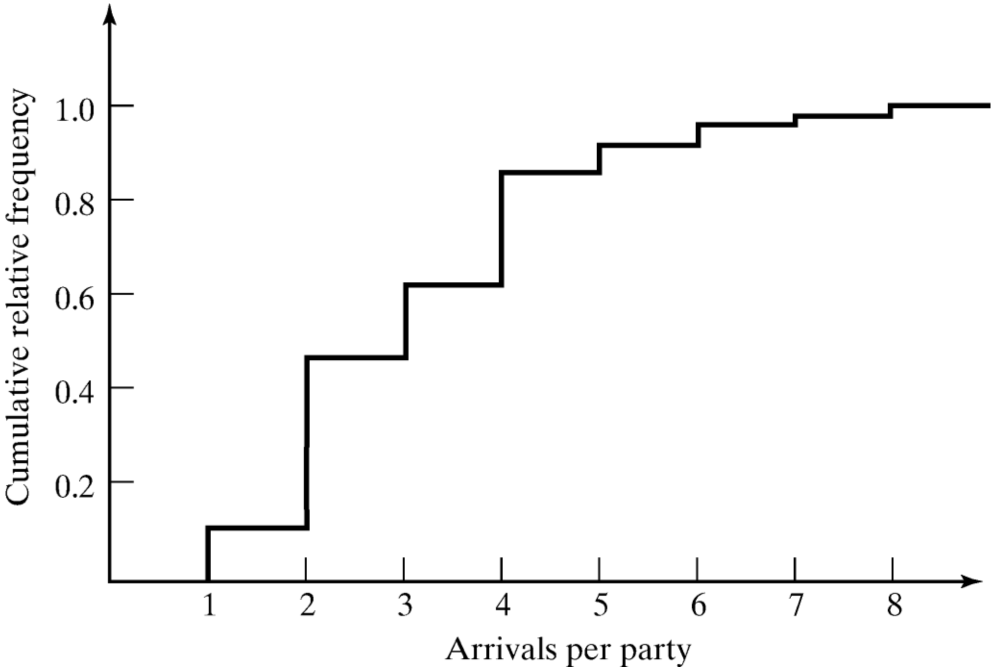
[Probability Review]

Arrivals per Party	*Frequency*	*Relative Frequency*	*Cumulative Relative Frequency*
1	30	0.10	0.10
2	110	0.37	0.47
3	45	0.15	0.62
4	71	0.24	0.86
5	12	0.04	0.90
6	13	0.04	0.94
7	7	0.02	0.96
8	12	0.04	1.00



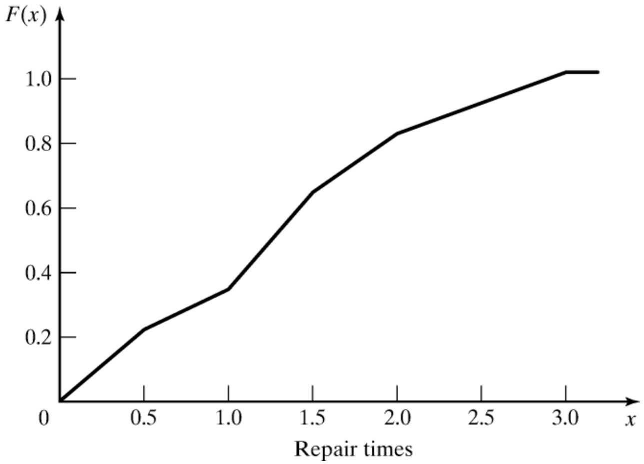
Empirical Distributions (cont.)

- The CDF in the figure is called the empirical distribution of the given data.



Empirical Distributions

- Example:
- The time required to repair a system that has suffered a failure has been collected for the last 100 instances.
- The empirical CDF is shown in the figure



Empirical Distributions (cont.)

Intervals (Hours)	Frequency	Relative Frequency	Cumulative Frequency
0<x<0.5	21	0.21	0.21
0.5<x<1.0	12	0.12	0.33
1.0<x<1.5	29	0.29	0.62
1.5<x<2.0	19	0.19	0.81
2.0<x<2.5	8	0.08	0.89
2.5<x<3.0	11	0.11	1.00

Discrete Distributions

- Discrete random variables are used to describe random phenomena in which only integer values can occur.
- In this section, we will learn about:
  - Bernoulli trials and Bernoulli distribution
  - Binomial distribution
  - Geometric and negative binomial distribution
  - Poisson distribution

Bernoulli Trials and Bernoulli Distribution [Discrete Dist'n]

- Bernoulli Trials:
  - Consider an experiment consisting of n trials, each can be a success or a failure.
    - Let $X_j = 1$ if the jth experiment is a success
    - and $X_j = 0$ if the jth experiment is a failure
  - The Bernoulli distribution (one trial):

$$p_j(x_j) = p(x_j) = \begin{cases} p, & x_j = 1,\ j = 1,2,\ldots,n \\ 1 - p = q, & x_j = 0, j = 1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

  - where $E(X_j) = p$ *and* $V(X_j) = p(1-p) = pq$
- Bernoulli process:
  - The n Bernoulli trials where trails are independent:

$$p(x_1, x_2, \ldots, x_n) = p_1(x_1) p_2(x_2) \ldots p_n(x_n)$$

Binomial Distribution

[Discrete Dist’n]

- The number of successes in n Bernoulli trials, X, has a binomial distribution.

$$p(x) = \begin{cases} \binom{n}{x} \underbrace{p^x q^{n-x}}, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

The number of outcomes having the required number of successes and failures

Probability that there are x successes and (n-x) failures

  - The mean, $E(x) = p + p + \ldots + p = n^*p$
  - The variance, $V(X) = pq + pq + \ldots + pq = n^*pq$

Binomial Distribution [Discrete Dist’n]

- Example

A production process manufactures computer chips on the average at 2% nonconforming. Every day, a random sample of size 50 is taken from the process. If the sample contains more than two nonconforming chips, the process will be stopped. Compute the probability that the process is stopped by the sampling scheme.

Binomial Distribution [Discrete Dist'n]

Solution

$$p(x) = \begin{cases} \binom{50}{x} \ (0.02)^x (0.98)^{50-x}, & x = 0,1,2,\dots,50 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X > 2) = 1 - P(X \le 2)$$

$$P(X \le 2) = \sum_{x=0}^{2} \binom{50}{x} \ (0.02)^x (0.98)^{50-x}$$

$$= (0.98)^{50} + 50(0.02)(0.98)^{49} + 1225(0.02)^2 (0.98)^{48}$$

$$= 0.92$$

$$P(X > 2) = 1 - 0.92 = 0.08$$

Geometric & Negative Binomial Distribution

[Discrete Dist'n]

- Geometric distribution
  - The number of Bernoulli trials, X, to achieve the 1st success:

$$p(x) = \begin{cases} q^{x-1}p, & x = 1, 2, \dots, n \\ 0, & \text{otherwise} \end{cases}$$

  - $E(x) = 1/p$, and $V(X) = q/p^2$

- Negative binomial distribution
  - The number of Bernoulli trials, Y, until the k^{th} success
  - If Y is a negative binomial distribution with parameters p and k, then:

$$p(y) = \begin{cases} \binom{y-1}{k-1} q^{y-k} p^k, & y = k, k+1, k+2, \dots \\ 0, & \text{otherwise} \end{cases}$$

  - $E(Y) = k/p$, and $V(Y) = kq/p^2$

Geometric & Negative Binomial Distribution

- Example

Forty percent of the assembled ink-jet printers are rejected at the inspection station. Find the probability that the first acceptable ink-jet printer is the third one inspected.

Geometric & Negative Binomial Distribution

- Solution

- Considering each inspection as a Bernoulli trial with q=0.4 and p=0.6 yields

$$p(3) = 0.4^2(0.6) = 0.096$$

- Thus, in only about 10% of the cases is the first acceptable printer the third one from any arbitrary starting point.

- To determine the probability that the third printer inspected the second acceptable printer, we use the negative binomial distribution.

$$p(3) = \binom{3-1}{2-1} \; 0.4^{(3-2)}(0.6)^2 = \binom{2}{1} \; 0.4(0.6)^2 = .0288$$

Poisson Distribution

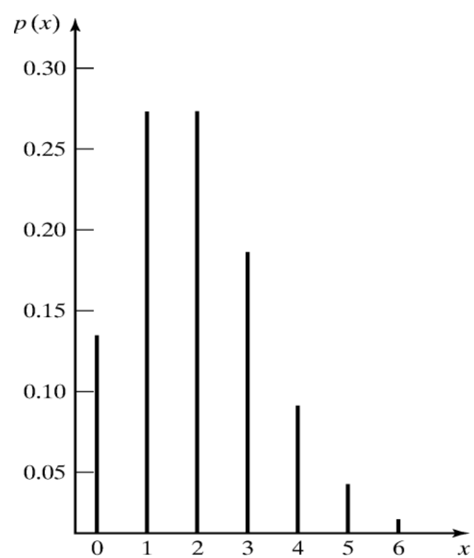
[Discrete Dist'n]

- Poisson distribution describes many random processes quite well and is mathematically quite simple.
  - where $\alpha > 0$, pdf and cdf are:

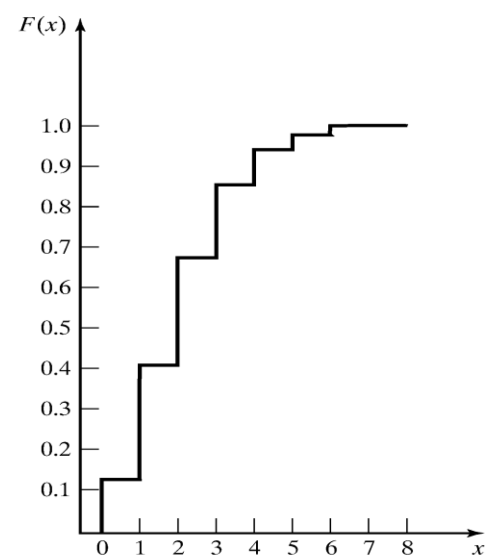
$$p(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^{x}}{x!}, & x = 0,1,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^{i}}{i!}$$

  - $E(X) = \alpha = V(X)$



(a)



(b)

Poisson Distribution

- Example: A computer repair person is "beeped" each time there is a call for service. The number of beeps per hour ~ Poisson(α = 2 per hour).
  - The probability of three beeps in the next hour:

    $p(3) = e^{-2}2^3/3! = 0.18$

    also, $p(3) = F(3) – F(2) = 0.857-0.677=0.18$

  - The probability of two or more beeps in a 1-hour period:

    $$\begin{aligned} p(2 \text{ or more}) &= 1 – p(0) – p(1) \\ &= 1 – F(1) \\ &= 0.594 \end{aligned}$$

Continuous Distributions

- Continuous random variables can be used to describe random phenomena in which the variable can take on any value in some interval.
- In this section, the distributions studied are:
  - Uniform
  - Exponential
  - Gamma
  - Normal
  - Weibull
  - Lognormal

Uniform Distribution [Probability Review]

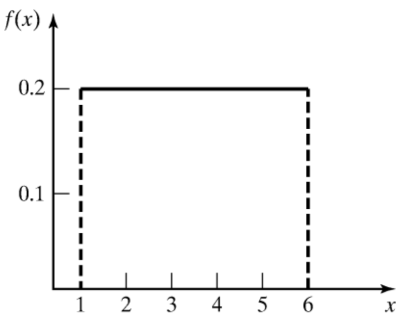
- A random variable *X is* uniformly distributed on the interval *(a, b)* if its PDF is given by

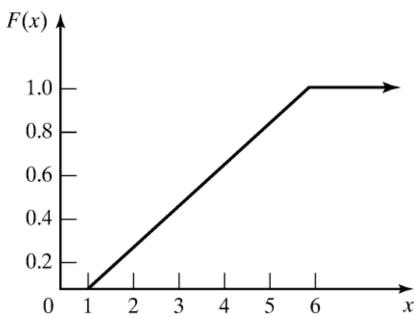
$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \mathrm{a} \le x \le \mathrm{b} \\ 0, & \text{otherwise} \end{cases}$$

- The CDF is given by

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

- The PDF and CDF when *a=1* and *b=6:*





Uniform Distribution

[Continuous Dist'n]

- A random variable *X* is uniformly distributed on the interval (*a,b*), *U*(*a,b*), if its pdf and cdf are:

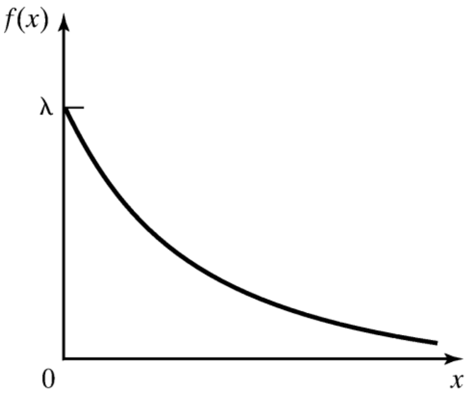
$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

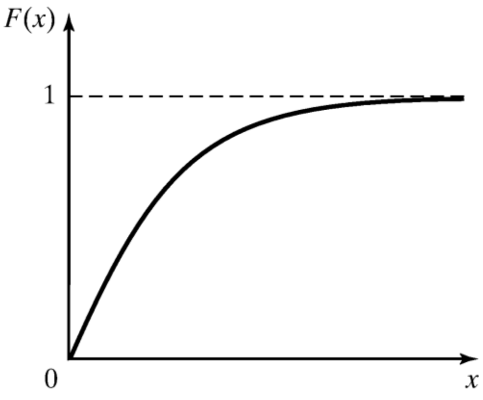
- Properties
  - $P(x_1 < X < x_2)$ is proportional to the length of the interval $[F(x_2) - F(x_1) = (x_2 - x_1)/(b-a)]$
  - $E(X) = (a+b)/2$ $\qquad V(X) = (b-a)^2/12$
- U(0,1) provides the means to generate random numbers, from which random variates can be generated.

Exponential Distribution [Probability Review]

- A random variable *X* is said to be exponentially distributed with parameter $\lambda > 0$ if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



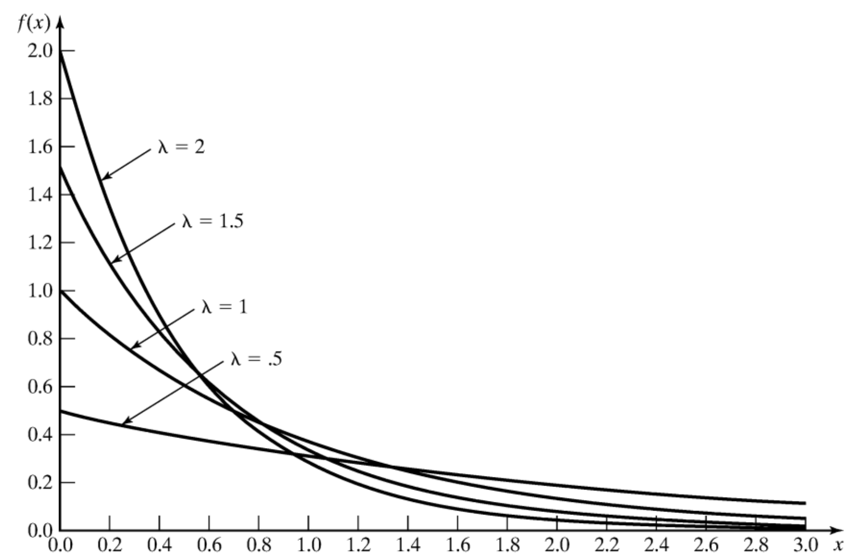


Exponential Distribution [Continuous Dist'n]

- A random variable *X* is exponentially distributed with parameter λ > 0 if its pdf and cdf are:

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\ge 0\\ 0, & \text{elsewhere}\end{cases}\qquad F(x)=\begin{cases}0, & x<0\\ \int_0^x \lambda e^{-\lambda t}dt=1-e^{-\lambda x}, & x\ge 0\end{cases}$$

  - $E(X) = 1/\lambda$ $V(X) = 1/\lambda^2$
  - Used to model interarrival times when arrivals are completely random, and to model service times that are highly variable
  - For several different exponential pdf's (see figure), the value of intercept on the vertical axis is λ, and all pdf's eventually intersect.



Exponential Distribution

- **Memoryless property**
  - For all *s* and *t* greater or equal to 0:

$$P(X > s+t \mid X > s) = P(X > t)$$

  - Example: A lamp ~ exp(λ = 1/3 per hour), hence, on average, 1 failure per 3 hours.
    - The probability that the lamp lasts longer than its mean life is:

$$P(X > 3) = 1-(1-e^{-3/3}) = e^{-1} = 0.368$$

    - The probability that the lamp lasts between 2 to 3 hours is:

$$P(2 <= X <= 3) = F(3) - F(2) = 0.145$$

    - The probability that it lasts for another hour given it is operating for 2.5 hours:

$$P(X > 3.5 \mid X > 2.5) = P(X > 1) = e^{-1/3} = 0.717$$

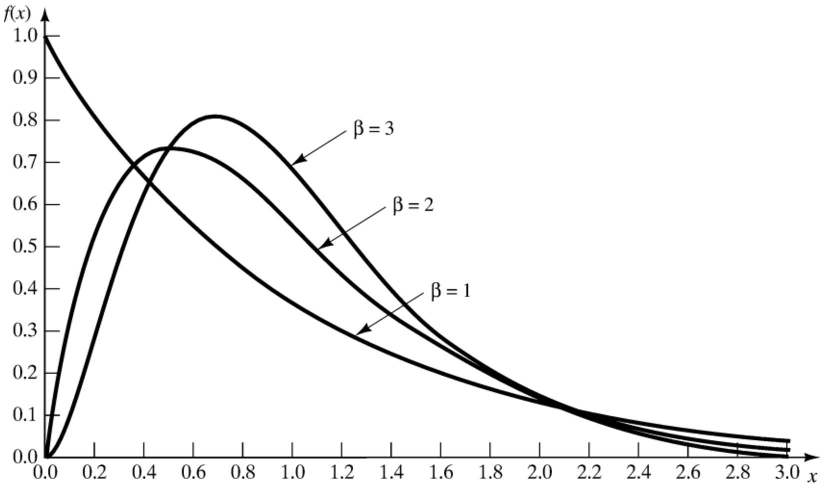
Gamma Distribution [Probability Review]

- A function used in defining the gamma distribution is the gamma function, which is defined for all $\beta > 0$ as

$$\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx$$

- A random variable X is gamma distributed with parameters β and θ if its PDF is given by

$$f(x) = \begin{cases} \dfrac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta-1} e^{-\beta\theta x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

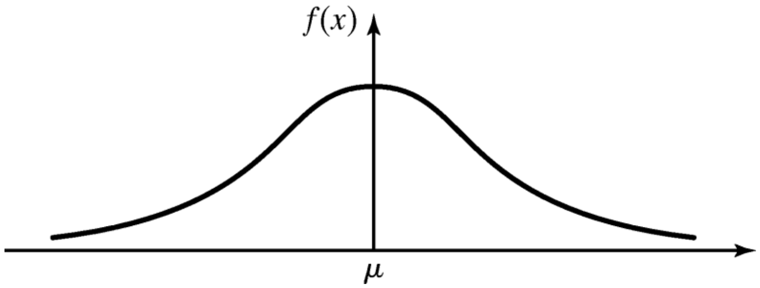


Normal Distribution [Continuous Dist'n]

- A random variable X is normally distributed has the pdf:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right], \quad -\infty < x < \infty$$

  - Mean: $-\infty < \mu < \infty$
  - Variance: $\sigma^2 > 0$
  - Denoted as $X \sim N(\mu, \sigma^2)$



- Special properties:
  - $\lim_{x \to -\infty} f(x) = 0$, and $\lim_{x \to \infty} f(x) = 0$.
  - $f(\mu - x) = f(\mu + x)$; the pdf is symmetric about μ.
  - The maximum value of the pdf occurs at $x = \mu$; the mean and mode are equal.

Normal Distribution [Continuous Dist'n]

- Evaluating the distribution:
  - Use numerical methods (no closed form)
  - Independent of μ and σ, using the standard normal distribution:

    Z ~ N(0,1)
  - Transformation of variables: let Z = (X - μ) / σ,

$$F(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \phi(z) dz = \Phi(\tfrac{x-\mu}{\sigma})$$

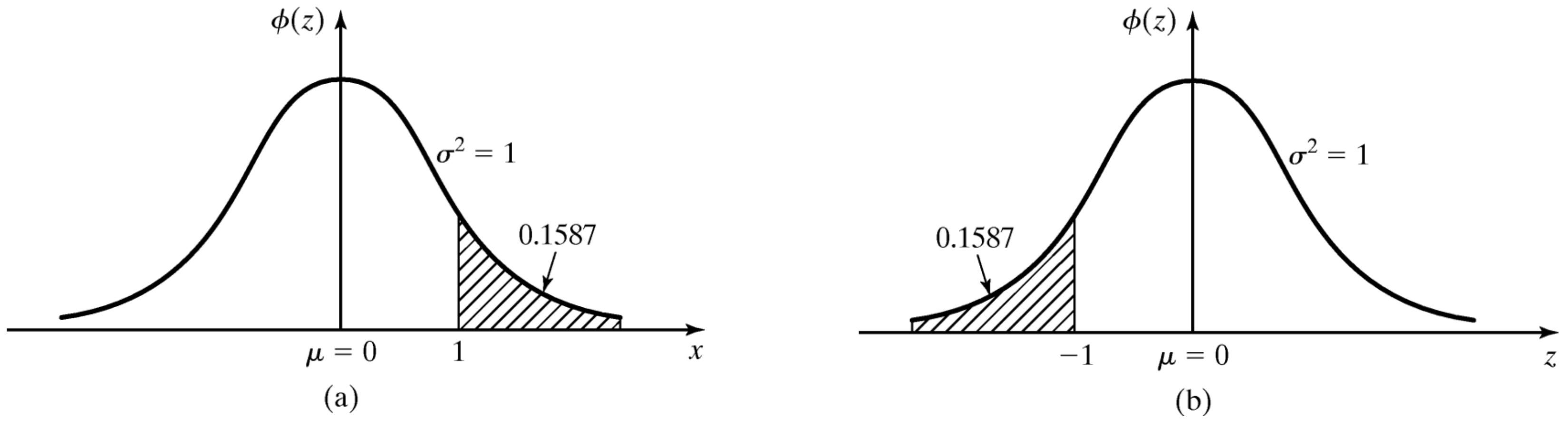
, where $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$

Normal Distribution

- Example: The time required to load an oceangoing vessel, *X*, is distributed as *N(12,4)*
  - The probability that the vessel is loaded in less than 10 hours:

$$F(10) = \Phi\left(\frac{10-12}{2}\right) = \Phi(-1) = 0.1587$$

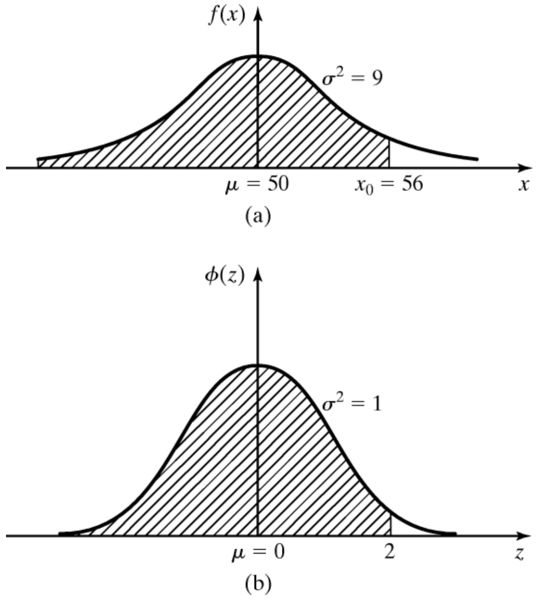
    - Using the symmetry property, $\Phi(1)$ is the complement of Φ (-1)



Normal Distribution [Probability Review]

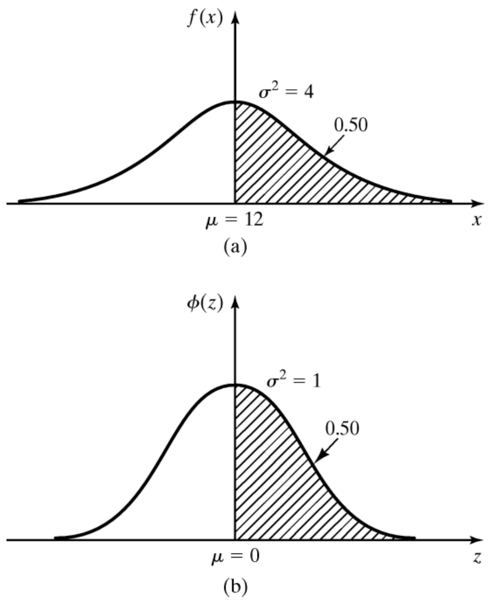
- Example: Suppose that *X ~ N (50, 9).*

$$F(56) = \Phi\left(\frac{56-50}{3}\right) = \Phi(2) = 0.9772$$



Normal Distribution [Probability Review]

- Example: The time in hours required to load a ship, X, is distributed as $N(12, 4)$. The probability that 12 or more hours will be required to load the ship is:



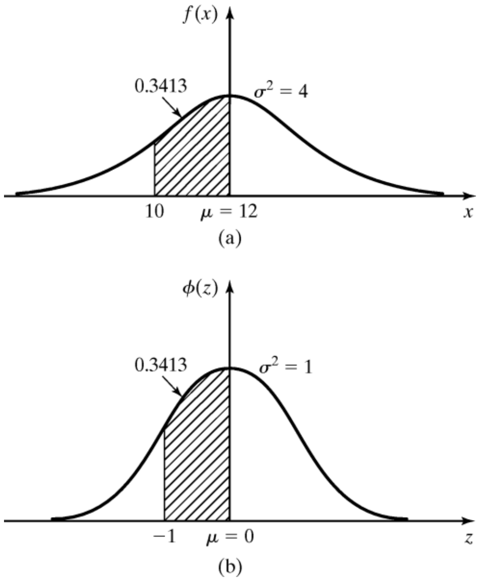
$$P(X > 12) = 1 - F(12) = 1 - 0.50 = 0.50$$

(The shaded portions in both figures)

Normal Distribution [Probability Review]

- Example (cont.):

  The probability that between 10 and 12 hours will be required to load a ship is given by

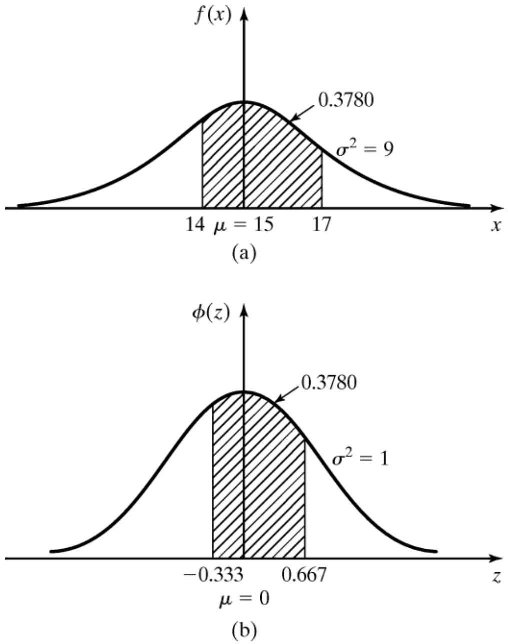


$$P(10 \leq X \leq 12) = F(12) - F(10) = 0.5000 - 0.1587 = 0.3413$$

The area is shown in shaded portions of the figure

Normal Distribution [Probability Review]

- Example: The time to pass through a queue is *N(15, 9)*. The probability that an arriving customer waits between 14 and 17 minutes is:



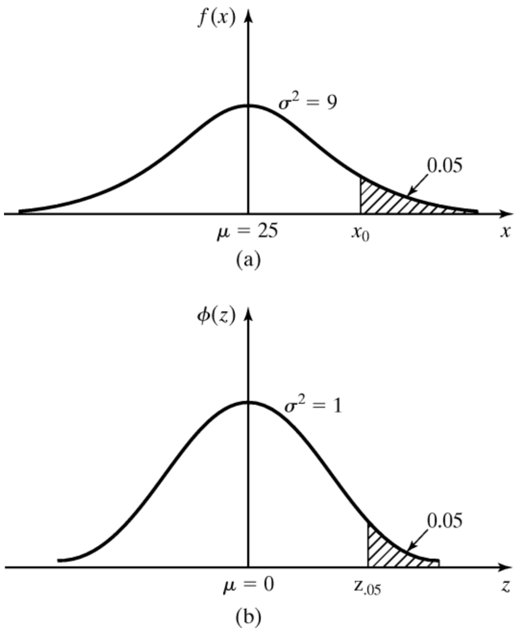
$$P(14 \le X \le 17) = F(17) - F(14) =$$

$$\Phi\left(\frac{17-15}{3}\right) - \Phi\left(\frac{14-15}{3}\right) = \Phi(0.667) - \Phi(-0.333) = 0.7476 - 0.3696 = 0.3780$$

Normal Distribution [Probability Review]

- Example: Lead-time demand, *X*, for an item is *N(25, 9)*.

  Compute the value for lead-time that will be exceeded only *5%* of time.



$$P(X > x_0) = P(Z > \frac{x_0 - 25}{3}) = 1 - \Phi(\frac{x_0 - 25}{3}) = 0.05$$

$$\frac{x_0 - 25}{3} = 1.645$$

$$x_0 = 29.935$$

Weibull Distribution

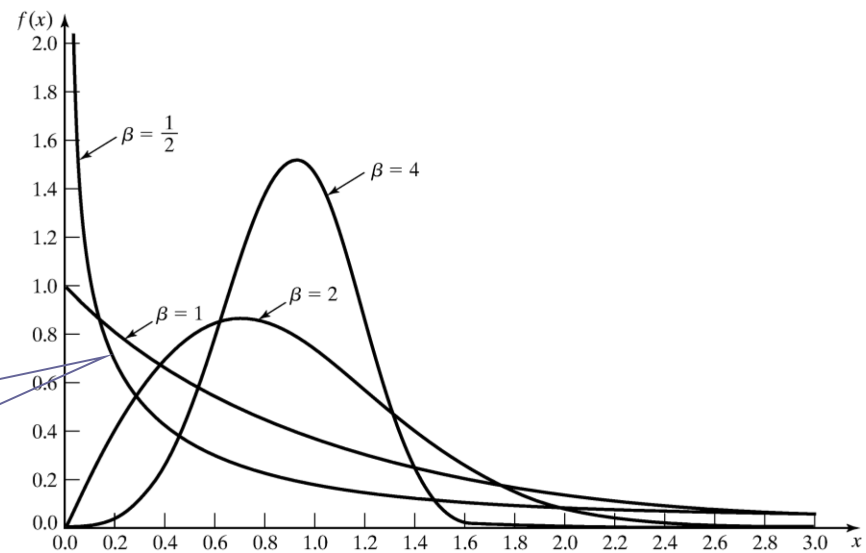
[Continuous Dist'n]

- A random variable X has a Weibull distribution if its pdf has the form:

$$f(x) = \begin{cases} \dfrac{\beta}{\alpha}\left(\dfrac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left[-\left(\dfrac{x-\nu}{\alpha}\right)^{\beta}\right], & x \ge \nu \\ 0, & \text{otherwise} \end{cases}$$

- 3 parameters:
  - Location parameter: υ, $(-\infty < \nu < \infty)$
  - Scale parameter: β, $(\beta > 0)$
  - Shape parameter. α, (> 0)

When $\upsilon = 0$ $\beta = 1$
$X \sim exp(\lambda = 1/\alpha)$



Weibull Distribution [Continuous Dist'n]

- Exponential

when β=1 and ν=0

$$f(x) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

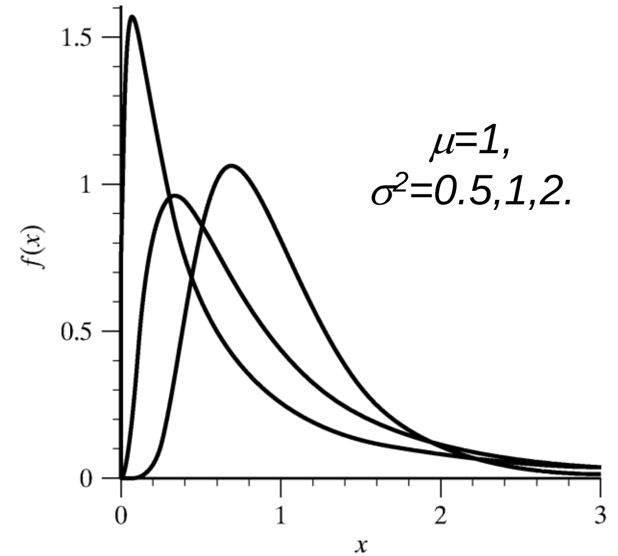
- Rayleigh Distribution when α=2
- used to model
  - multipath fading,
  - radiation,
  - wind speeds

$$f(x) = \begin{cases} \frac{\beta}{2}\left(\frac{x-\nu}{2}\right)^{\beta-1} \exp\left[-\left(\frac{x-\nu}{2}\right)^{\beta}\right], & x \geq \nu \\ 0, & \text{otherwise} \end{cases}$$

Lognormal Distribution [Continuous Dist'n]

- A random variable X has a lognormal distribution if its pdf has the form:

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right], & x > 0 \\ 0, & \text{otherwise} \end{cases}$$



  - Mean $E(X) = e^{\mu+\sigma^2/2}$
  - Variance $V(X) = e^{2\mu+\sigma^2/2}\,(e^{\sigma^2} - 1)$

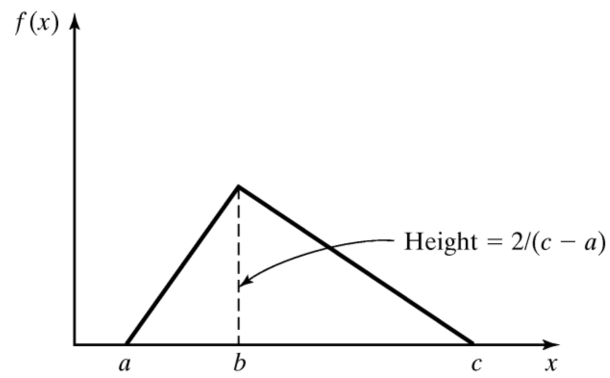
- Relationship with normal distribution
  - When $Y \sim N(\mu, \sigma^2)$, then $X = e^Y \sim \text{lognormal}(\mu, \sigma^2)$
  - Parameters μ and σ^2 are not the mean and variance of the lognormal

Triangular Distribution [Probability Review]

- A random variable *X* has a triangular distribution if its PDF is given by

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x \le b \\ \\ \dfrac{2(c-x)}{(c-b)(c-a)}, & b < x \le c \\ \\ 0, & \text{elsewhere} \end{cases}$$

where $a \le b \le c$.



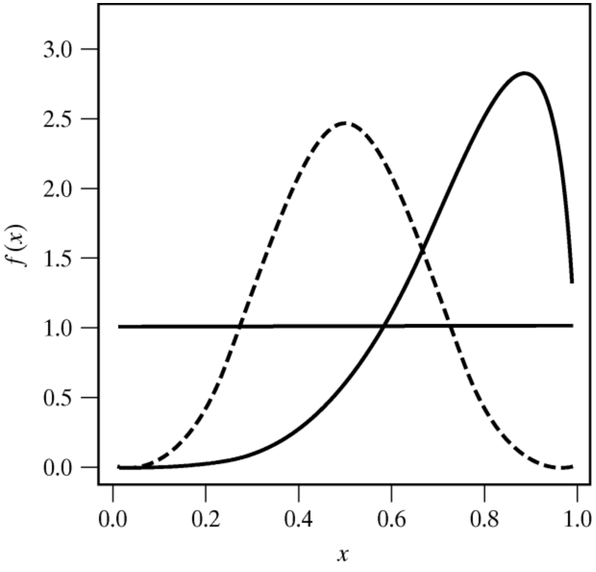
Beta Distribution

- A random variable *X* is beta-distributed with parameters $\beta_1 > 0$ and $\beta_2 > 0$ if its PDF is given by

$$f(x) = \begin{cases} \dfrac{x^{\beta_1 - 1}(1-x)^{\beta_2 - 1}}{B(\beta_1, \beta_2)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where

$$B(\beta_1, \beta_2) = \Gamma(\beta_1)\Gamma(\beta_2) \big/ \Gamma(\beta_1 + \beta_2)$$



Useful Statistical Models

- In this section, statistical models appropriate to some application areas are presented. The areas include:
  - Queueing systems
  - Inventory and supply-chain systems
  - Reliability and maintainability
  - Limited data

Queueing Systems [Useful Models]

- In a queueing system, interarrival and service-time patterns can be probablistic (for more queueing examples, see Chapter 2).
- Sample statistical models for interarrival or service time distribution:
  - Exponential distribution: if service times are completely random
  - Normal distribution: fairly constant but with some random variability (either positive or negative)
  - Truncated normal distribution: similar to normal distribution but with restricted value.
  - Gamma and Weibull distribution: more general than exponential (involving location of the modes of pdf's and the shapes of tails.)

Inventory and supply chain [Useful Models]

- In realistic inventory and supply-chain systems, there are at least three random variables:
  - The number of units demanded per order or per time period
  - The time between demands
  - The lead time
- Sample statistical models for lead time distribution:
  - Gamma
- Sample statistical models for demand distribution:
  - Poisson: simple and extensively tabulated.
  - Negative binomial distribution: longer tail than Poisson (more large demands).
  - Geometric: special case of negative binomial given at least one demand has occurred.

Reliability and maintainability [Useful Models]

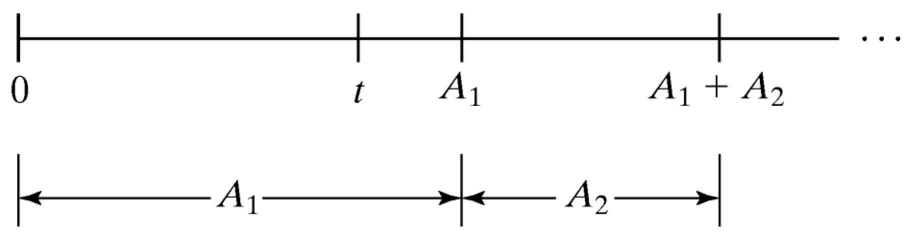
- Time to failure (TTF)
  - Exponential: failures are random
  - Gamma: for standby redundancy where each component has an exponential TTF
  - Weibull: failure is due to the most serious of a large number of defects in a system of components
  - Normal: failures are due to wear

Other areas

- For cases with limited data, some useful distributions are:
  - Uniform, triangular and beta
- Other distribution: Bernoulli, binomial and hyperexponential.

Poisson Process [Probability Review]

- Consider the time at which arrivals occur.
- Let the first arrival occur at time A_1, the second occur at time A_1+A_2, and so on.



- The probability that the first arrival will occur in *[0, t]* is given by

$$P(A_1 \leq t) = 1 - e^{-\lambda t}$$

Poisson Process

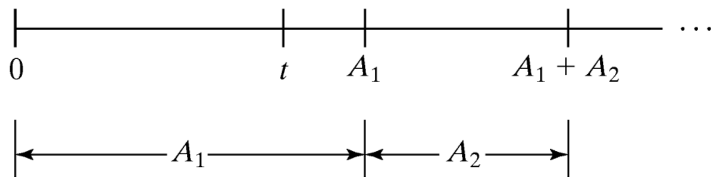
- Definition: *N(t)* is a counting function that represents the number of events occurred in *[0,t]*.
- A counting process {*N(t), t>=0*} is a Poisson process with mean rate λ if:
  - Arrivals occur one at a time
  - {*N(t), t>=0*} has stationary increments
  - {*N(t), t>=0*} has independent increments
- Properties

$$P[N(t)=n]=\frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad \text{for } t\geq 0 \text{ and } n=0,1,2,\ldots$$

  - Equal mean and variance: $E[N(t)] = V[N(t)] = \lambda t$
  - Stationary increment: The number of arrivals in time *s* to *t* is also Poisson-distributed with mean $\lambda(t-s)$

Interarrival Times

- Consider the interarrival times of a Possion process ($A_1, A_2, \ldots$), where A_i is the elapsed time between arrival *i* and arrival *i+1*



  - The 1st arrival occurs after time t iff there are no arrivals in the interval [0,t], hence:

$$P\{A_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

$$P\{A_1 <= t\} = 1 - e^{-\lambda t} \qquad \text{[cdf of exp}(\lambda)\text{]}$$

  - Interarrival times, $A_1, A_2, \ldots$, are exponentially distributed and independent with mean $1/\lambda$

Arrival counts $\sim Poi(\lambda)$ ⟺ Interarrival time $\sim Exp(1/\lambda)$

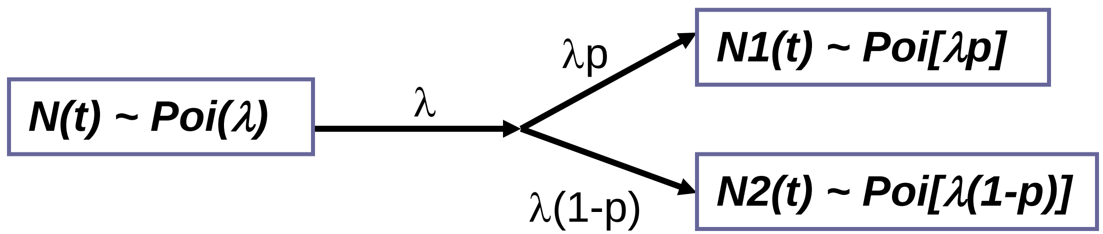
Stationary & Independent

Memoryless

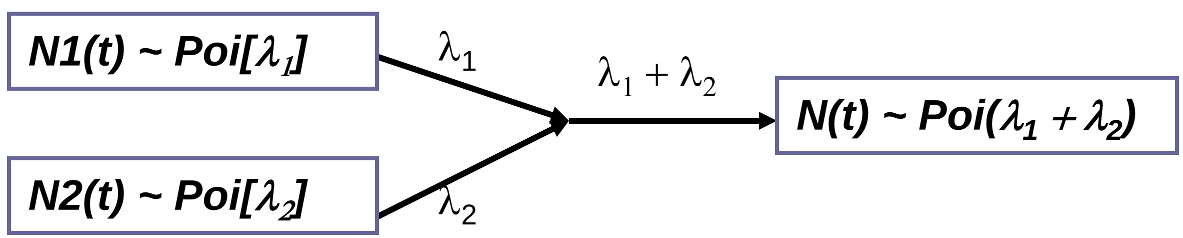
Splitting and Pooling

[Poisson Dist'n]

- Splitting:
  - Suppose each event of a Poisson process can be classified as Type I, with probability *p* and Type II, with probability *1-p.*
  - *N(t)* = *N1(t)* + *N2(t)*, where *N1(t)* and *N2(t)* are both Poisson processes with rates λp and $\lambda(1-p)$



- Pooling:
  - Suppose two Poisson processes are pooled together
  - *N1(t)* + *N2(t)* = *N(t)*, where *N(t)* is a Poisson processes with rates $\lambda_1 + \lambda_2$



Summary

- The world that the simulation analyst sees is probabilistic, not deterministic.
- In this chapter:
  - Reviewed several important probability distributions.
  - Showed applications of the probability distributions in a simulation context.
- Important task in simulation modeling is the collection and analysis of input data, e.g., hypothesize a distributional form for the input data. Reader should know:
  - Difference between discrete, continuous, and empirical distributions.
  - Poisson process and its properties.