Chapter 9
Input Modeling

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Discrete-Event System Simulation

Purpose & Overview

- Input models provide the driving force for a simulation model.
- The quality of the output is no better than the quality of inputs.
- In this chapter, we will discuss the 4 steps of input model development:
  - Collect data from the real system
  - Identify a probability distribution to represent the input process
  - Choose parameters for the distribution
  - Evaluate the chosen distribution and parameters for goodness of fit.

Data Collection

- One of the biggest tasks in solving a real problem. GIGO – garbage-in-garbage-out
- Suggestions that may enhance and facilitate data collection:
  - Plan ahead: begin by a practice or pre-observing session, watch for unusual circumstances
  - Analyze the data as it is being collected: check adequacy
  - Combine homogeneous data sets, e.g. successive time periods, during the same time period on successive days
  - Be aware of data censoring: the quantity is not observed in its entirety, danger of leaving out long process times
  - Check for relationship between variables, e.g. build scatter diagram
  - Check for autocorrelation
  - Collect input data, not performance data

Identifying the Distribution

- Histograms
- Selecting families of distribution
- Parameter estimation
- Goodness-of-fit tests
- Fitting a non-stationary process

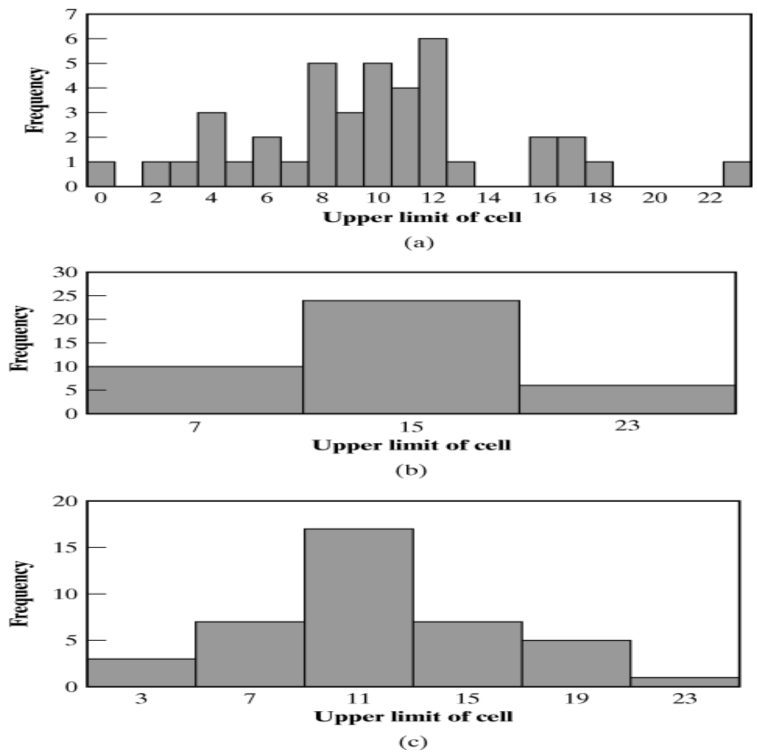
Histograms

[Identifying the distribution]

- A frequency distribution or histogram is useful in determining the shape of a distribution
- The number of class intervals depends on:
  - The number of observations
  - The dispersion of the data
  - Suggested: the square root of the sample size
- For continuous data:
  - Corresponds to the probability density function of a theoretical distribution
- For discrete data:
  - Corresponds to the probability mass function
- If few data points are available: combine adjacent cells to eliminate the ragged appearance of the histogram

Histograms

[Identifying the distribution]



Same data with different interval sizes

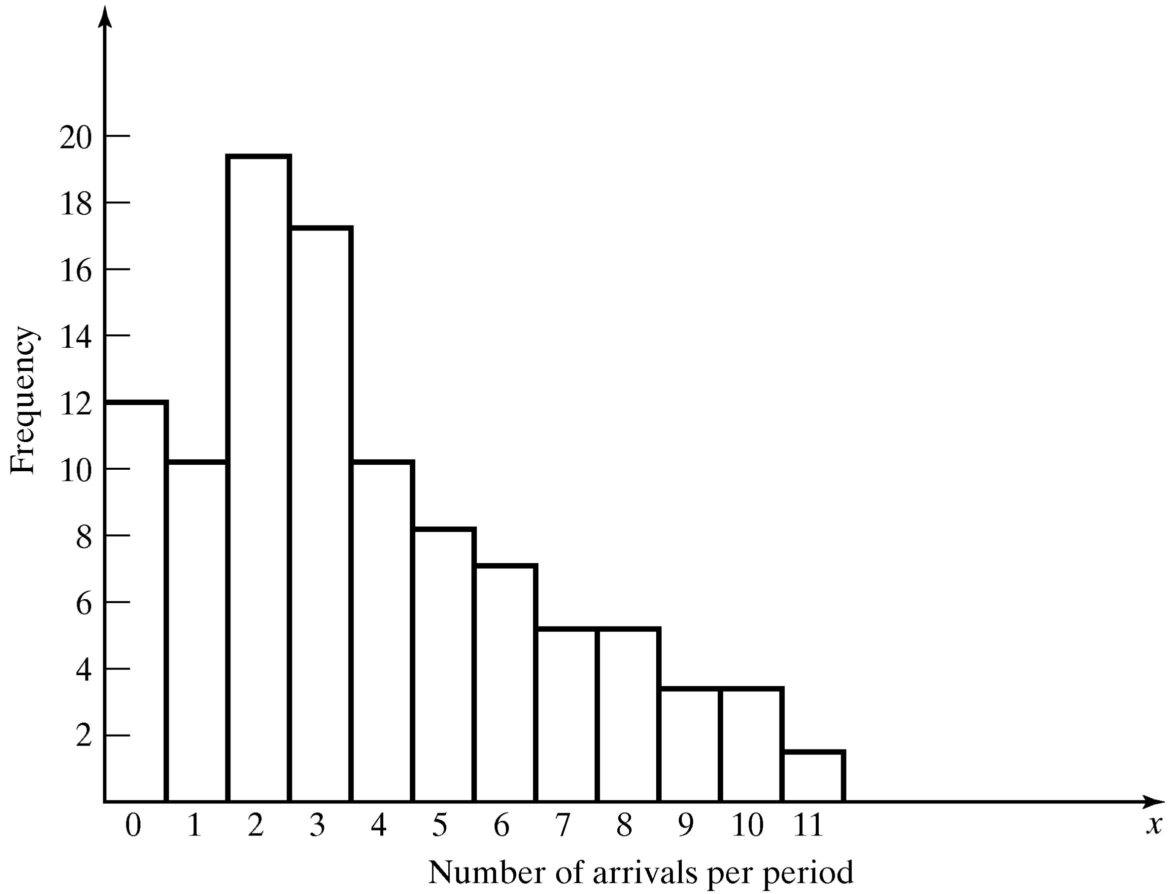
Histograms

- Vehicle Arrival Example: # of vehicles arriving at an intersection between 7 am and 7:05 am was monitored for *100* random workdays.

Arrivals per Period	Frequency
0	12
1	10
2	19
3	17
4	10
5	8
6	7
7	5
8	5
9	3
10	3
11	1

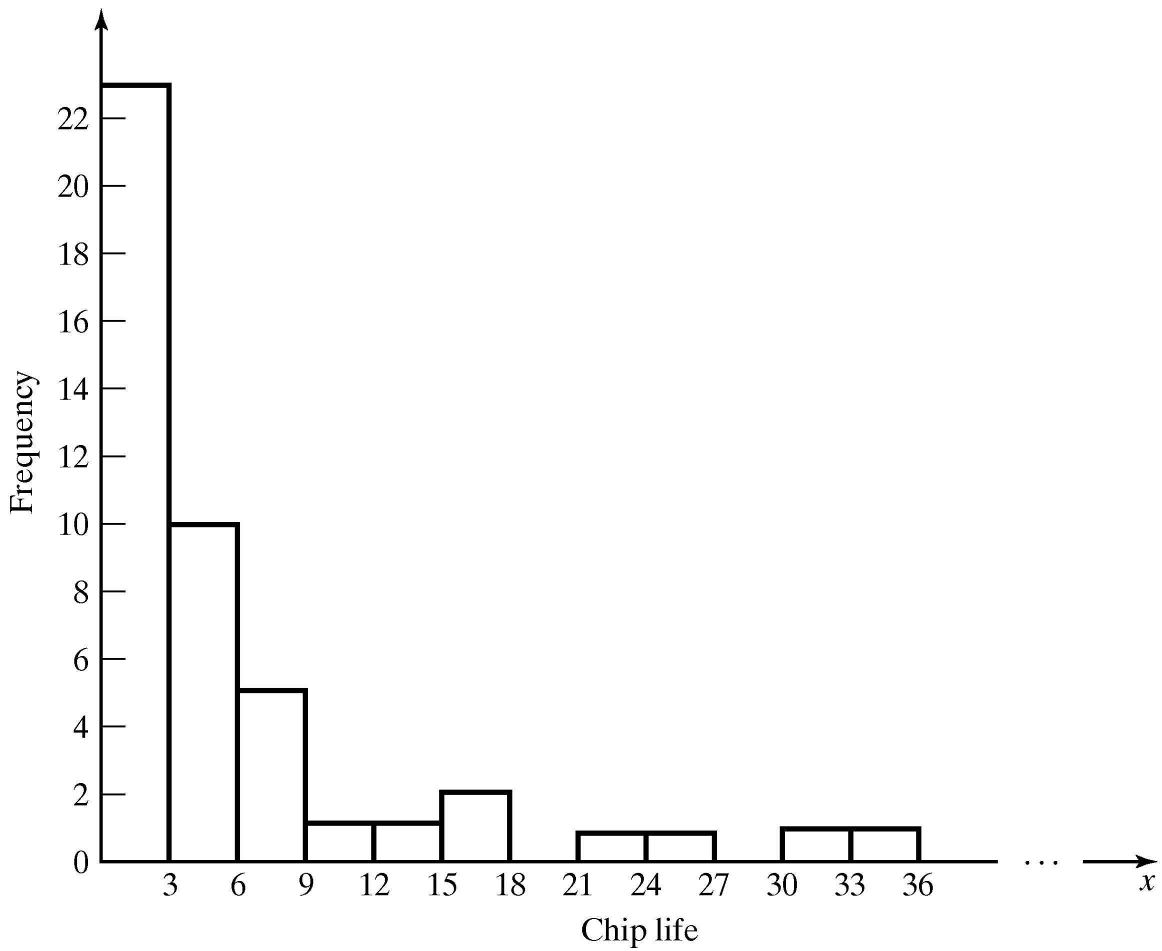
- There are sample data, so the histogram may have a cell for each possible value in the data range

Histograms



Frequency
20
18
16
14
12
10
8
6
4
2
0 1 2 3 4 5 6 7 8 9 10 11
x
Number of arrivals per period

Histograms



Frequency
0
2
4
6
8
10
12
14
16
18
20
22
3
6
9
12
15
18
21
24
27
30
33
36
. . .
x
Chip life

Selecting the Family of Distributions

- A family of distributions is selected based on:
  - The context of the input variable
  - Shape of the histogram
- Frequently encountered distributions:
  - Easier to analyze: exponential, normal and Poisson
  - Harder to analyze: beta, gamma and Weibull

Selecting the Family of Distributions

[Identifying the distribution]

- Use the physical basis of the distribution as a guide, for example:
  - Binomial: # of successes in *n* trials
  - Poisson: # of independent events that occur in a fixed amount of time or space
  - Normal: dist'n of a process that is the sum of a number of component processes
  - Exponential: time between independent events, or a process time that is memoryless
  - Weibull: time to failure for components
  - Discrete or continuous uniform: models complete uncertainty
  - Triangular: a process for which only the minimum, most likely, and maximum values are known
  - Empirical: resamples from the actual data collected

Selecting the Family of Distributions

[Identifying the distribution]

- Remember the physical characteristics of the process
  - Is the process naturally discrete or continuous valued?
  - Is it bounded?
- No “true” distribution for any stochastic input process
- Goal: obtain a good approximation

Quantile-Quantile Plots [Identifying the distribution]

- *Q-Q* plot is a useful tool for evaluating distribution fit
- If X is a random variable with cdf F, then the q-quantile of X is the γ such that

$$F(\gamma) = P(X \leq \gamma) = q, \qquad \text{for } 0 < q < 1$$

  - When F has an inverse, $\gamma = F^{-1}(q)$

- Let $\{x_i,\ i = 1,2, \ldots, n\}$ be a sample of data from X and $\{y_j,\ j = 1,2, \ldots, n\}$ be the observations in ascending order:

$$y_j \text{ is approximately } F^{-1}\left(\frac{j - 0.5}{n}\right)$$

where j is the ranking or order number

Quantile-Quantile Plots [Identifying the distribution]

- The plot of y_j versus $F^{-1}((j-0.5)/n)$ is
  - Approximately a straight line if F is a member of an appropriate family of distributions
  - The line has slope 1 if F is a member of an appropriate family of distributions with appropriate parameter values

Quantile-Quantile Plots [Identifying the distribution]

- Example: Check whether the door installation times follows a normal distribution.
  - The observations are now ordered from smallest to largest:

j	Value	j	Value	j	Value
1	99.55	6	99.98	11	100.26
2	99.56	7	100.02	12	100.27
3	99.62	8	100.06	13	100.33
4	99.65	9	100.17	14	100.41
5	99.79	10	100.23	15	100.47

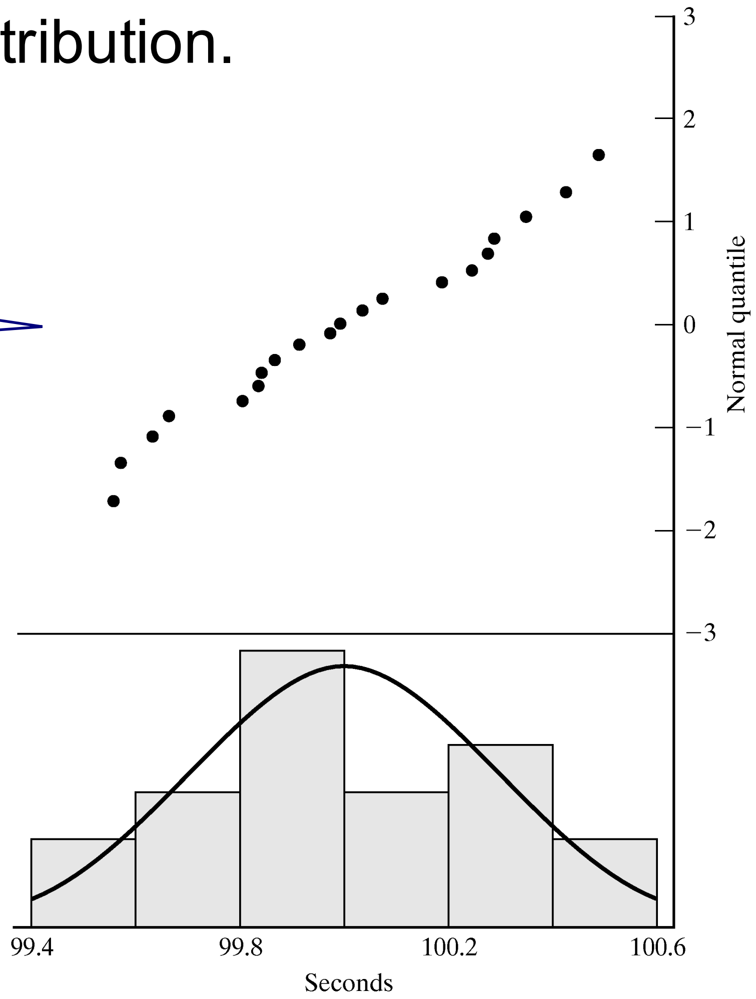
  - y_j are plotted versus $F^{-1}((j-0.5)/n)$ where F has a normal distribution with the sample mean *(99.99 sec)* and sample variance ($0.2832^2\ sec^2$)

Quantile-Quantile Plots [Identifying the distribution]

- Example (continued): Check whether the door installation times follow a normal distribution.

Straight line, supporting the hypothesis of a normal distribution

Superimposed density function of the normal distribution



Quantile-Quantile Plots [Identifying the distribution]

- Consider the following while evaluating the linearity of a *q-q* plot:
  - The observed values never fall exactly on a straight line
  - The ordered values are ranked and hence not independent, unlikely for the points to be scattered about the line
  - Variance of the extremes is higher than the middle. Linearity of the points in the middle of the plot is more important.
- *Q-Q* plot can also be used to check homogeneity
  - Check whether a single distribution can represent both sample sets
  - Plotting the order values of the two data samples against each other

Parameter Estimation

- Next step after selecting a family of distributions
- If observations in a sample of size n are $X_1, X_2, \ldots, X_n$ (discrete or continuous), the sample mean and variance are:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}$$

- If the data are discrete and have been grouped in a frequency distribution:

$$\bar{X} = \frac{\sum_{j=1}^{k} f_j X_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{k} f_j X_j^2 - n\bar{X}^2}{n-1}$$

where f_j is the observed frequency of value X_j

Parameter Estimation [Identifying the distribution]

- When raw data are unavailable (data are grouped into class intervals), the approximate sample mean and variance are:

$$\bar{X} = \frac{\sum_{j=1}^{c} f_j m_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{c} f_j m_j^2 - n\bar{X}^2}{n-1}$$

where f_j is the observed frequency of in the jth class interval

m_j is the midpoint of the jth interval, and c is the number of class intervals

- A parameter is an unknown constant, but an estimator is a statistic.
- Estimator depends on the sample values.

Parameter Estimation [Identifying the distribution]

- Vehicle Arrival Example (continued): Table in the histogram example on slide 6 (Table 9.1 in book) can be analyzed to obtain:

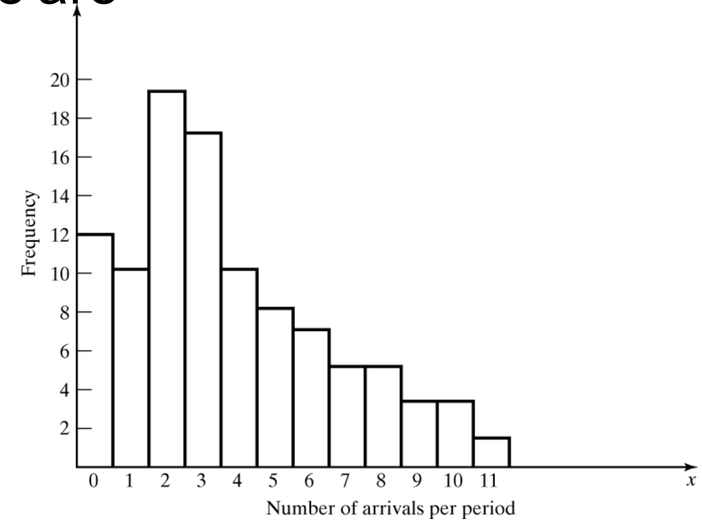
$$n = 100, f_1 = 12, X_1 = 0, f_2 = 10, X_2 = 1, \ldots,$$

$$\text{and } \sum_{j=1}^{k} f_j X_j = 364, \text{ and } \sum_{j=1}^{k} f_j X_j^2 = 2080$$

  - The sample mean and variance are

$$\bar{X} = \frac{364}{100} = 3.64$$

$$S^2 = \frac{2080 - 100*(3.64)^2}{99} = 7.63$$



  - The histogram suggests *X* to have a Possion distribution
    - However, note that sample mean is not equal to sample variance.
    - Reason: each estimator is a random variable, is not perfect.

Exponential Distribution

- $$f(x) = \lambda e^{-\lambda x}$$

$$f(x_1, x_2, \ldots, x_n) = \lambda e^{-\lambda x_1} . \lambda e^{-\lambda x_2} \ldots \lambda e^{-\lambda x_n} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ln L(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 => \lambda = \frac{n}{\sum_{i=1}^{n} x_i} = 1/\bar{X}$$

Goodness-of-Fit Tests [Identifying the distribution]

- Conduct hypothesis testing on input data distribution using:
  - Kolmogorov-Smirnov test
  - Chi-square test
- No single correct distribution in a real application exists.
  - If very little data are available, it is unlikely to reject any candidate distributions
  - If a lot of data are available, it is likely to reject all candidate distributions
- So,
  - Failing to reject a candidate == one piece of evidence in favor of that choice
  - rejecting a input model == one piece of evidence against that choice

Chi-Square test [Goodness-of-Fit Tests]

- Intuition: comparing the histogram of the data to the shape of the candidate density or mass function
- Valid for **large** sample sizes when parameters are estimated by maximum likelihood
- By arranging the *n* observations into a set of *k* class intervals or cells, the test statistics is:

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Observed Frequency

Expected Frequency

$E_i = n*p_i$

where p_i is the theoretical prob. of the *i*th interval.

Suggested Minimum = 5

which **approximately** follows the chi-square distribution with *k-s-1* degrees of freedom, where s = # of parameters of the hypothesized distribution estimated by the sample statistics.

Chi-Square test [Goodness-of-Fit Tests]

- The hypothesis of a chi-square test is:

  H_0: The random variable, X, conforms to the distributional assumption with the parameter(s) given by the estimate(s).

  H_1: The random variable X does not conform.

- If the distribution tested is discrete and combining adjacent cell is not required (so that E_i > minimum requirement):
  - Each value of the random variable should be a class interval, unless combining is necessary, and

$$p_i = p(x_i) = P(X = x_i)$$

Chi-Square test [Goodness-of-Fit Tests]

- If the distribution tested is continuous:

$$p_i = \int_{a_{i-1}}^{a_i} f(x)\,dx = F(a_i) - F(a_{i-1})$$

where a_{i-1} and a_i are the endpoints of the i^{th} class interval and $f(x)$ is the assumed pdf, $F(x)$ is the assumed cdf.

- Recommended number of class intervals (k):

Sample Size, n	Number of Class Intervals, k
20	Do not use the chi-square test
50	5 to 10
100	10 to 20
> 100	$n^{1/2}$ to n/5

- Caution: Different grouping of data (i.e., k) can affect the hypothesis testing result.

Chi-Square test [Goodness-of-Fit Tests]

- Vehicle Arrival Example (continued):

  H_0: the random variable is Poisson distributed.

  H_1: the random variable is not Poisson distributed.

x_i	Observed Frequency, O_i	Expected Frequency, E_i	$(O_i - E_i)^2/E_i$
0	12	2.6	7.87
1	10	9.6	
2	19	17.4	0.15
3	17	21.1	0.8
4	19	19.2	4.41
5	6	14.0	2.57
6	7	8.5	0.26
7	5	4.4	
8	5	2.0	
9	3	0.8	11.62
10	3	0.3	
> 11	1	0.1	
	100	100.0	27.68

$$E_i = np(x) = n\frac{e^{-\alpha}\alpha^x}{x!}$$

Combined because of min E_i

- Degree of freedom is $k-s-1 = 7-1-1 = 5$, hence, the hypothesis is rejected at the 0.05 level of significance.

$$\chi_0^2 = 27.68 > \chi_{0.05,5}^2 = 11.1$$

Chi-Square test, equal probabilities

- Use class intervals that are equal in probability
- Only when raw data is available
- Using equal probabilities:

$$E_i = np_i \geq 5, p_i = \frac{1}{k}$$

$$\frac{n}{k} \geq 5 => k \leq \frac{n}{5}$$

Chi-Square test, equal probabilities

- Failure data Example (continued):

  $\hat{\lambda} = \frac{1}{\bar{X}} = 0.084, \quad \mathrm{n} = 50 \Rightarrow \mathrm{k} \le 10$

  H_0: the random variable is exponentially distributed.

  H_1: the random variable is not exponentially distributed

- $\mathrm{k} = 8 \Rightarrow \mathrm{p} = 0.125$
- $F(a_i) = 1 - e^{-\lambda a_i}, \mathrm{i} = 1,2, \ldots, \mathrm{k}$
- $ip = 1 - e^{-\lambda a_i} \Rightarrow a_i = -\frac{1}{\lambda}\ln(1 - ip), i = 0,1, \ldots, k$
- $\alpha = 0.05, \quad k - s - 1 = 8 - 1 - 1 = 6$
- $\chi^2_{\alpha,k-s-1} = 12.6$

Chi-Square test, equal probabilities

Class Interval	Observed Frequency O_i	Expected Frequency E_i	$\frac{(O_i - E_i)^2}{E_i}$
[0, 1.590)	19	6.25	26.01
[1.590, 3.425)	10	6.25	2.25
[3.425, 5.595)	3	6.25	0.81
[5.595, 8.252)	6	6.25	0.01
[8.252, 11.677)	1	6.25	4.41
[11.677, 16.503)	1	6.25	4.41
[16.503, 24.755)	4	6.25	0.81
[24.755, ∞)	6	6.25	0.01
	50	50	39.6

Kolmogorov-Smirnov Test

- Intuition: formalize the idea behind examining a *q-q* plot
- Recall from Chapter 7.4.1:
  - The test compares the **continuous** cdf, $F(x)$, of the hypothesized distribution with the empirical cdf, $S_N(x)$, of the N sample observations.
  - Based on the maximum difference statistics (Tabulated in A.8):

$$D = max| F(x) - S_N(x)|$$

- A more powerful test, particularly useful when:
  - Sample sizes are small,
  - No parameters have been estimated from the data.

Kolmogorov-Smirnov Test

- Compares the continuous cdf, *F(x)*, of the uniform distribution with the empirical cdf, $S_N(x)$, of the *N* sample observations.
  - We know: $F(x) = x, \quad 0 \le x \le 1$
  - If the sample from the RN generator is $R_1, R_2, \ldots, R_N$, then the empirical cdf, $S_N(x)$ is:

$$S_N(x) = \frac{\text{number of } R_1, R_2, \ldots, R_n \text{ which are} \le x}{N}$$

- Based on the statistic: $D = max| F(x) - S_N(x)|$
  - Sampling distribution of *D* is known (a function of *N*, tabulated in Table A.8.)

Kolmogorov-Smirnov Test

[Goodness-of-Fit Tests]

- Example: Suppose *5* generated numbers are *0.44, 0.81, 0.14, 0.05, 0.93*.

Step 1:

$R_{(i)}$	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
$i/N - R_{(i)}$	0.15	0.26	0.16	0.01	0.07
$R_{(i)} - (i-1)/N$	0.05	0.06	0.04	0.21	0.13

Step 2:

Arrange $R_{(i)}$ from smallest to largest

$D^{+} = max\ \{i/N - R_{(i)}\}$

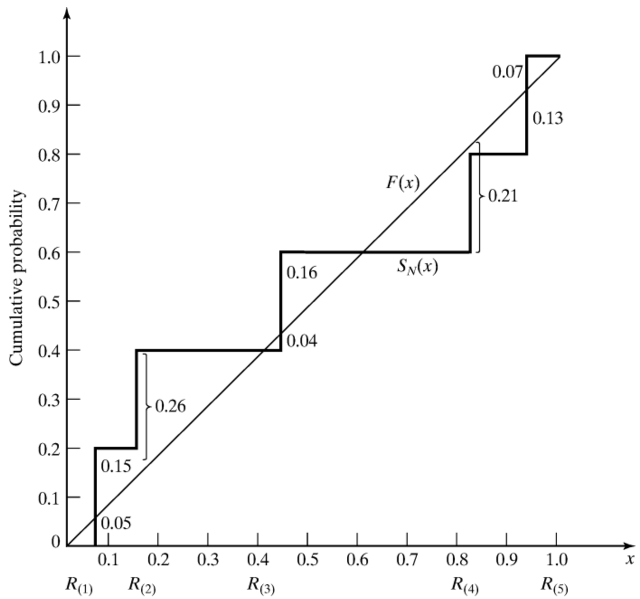
$D^{-} = max\ \{R_{(i)} - (i-1)/N\}$

Step 3: $D = max(D^{+}, D^{-}) = 0.26$

Step 4: For $\alpha = 0.05$,

$D_{\alpha} = 0.565 > D$

Hence, H_0 is not rejected.



p-Values and "Best Fits"

- *p-value* for the test statistics
  - The significance level at which one would just reject H_0 for the given test statistic value.
  - A measure of fit, the larger the better
  - Large *p-value*: good fit
  - Small *p-value*: poor fit

- Vehicle Arrival Example (cont.):
  - H_0: data is Possion
  - Test statistics: $\chi_0^2 = 27.68$, with *5* degrees of freedom
  - *p-value = 0.00004*, meaning we would reject H_0 with *0.00004* significance level, hence Poisson is a poor fit.

p-Values and "Best Fits"

- Many software use *p-value* as the ranking measure to automatically determine the "best fit". Things to be cautious about:
  - Software may not know about the physical basis of the data, distribution families it suggests may be inappropriate.
  - Close conformance to the data does not always lead to the most appropriate input model.
  - *p-value* does not say much about where the lack of fit occurs
- Recommended: always inspect the automatic selection using graphical methods.

Fitting a Non-stationary Poisson Process

- Fitting a NSPP to arrival data is difficult, possible approaches:
  - Fit a very flexible model with lots of parameters or
  - Approximate constant arrival rate over some basic interval of time, but vary it from time interval to time interval.

Our focus

- Suppose we need to model arrivals over time [0,T], our approach is the most appropriate when we can:
  - Observe the time period repeatedly and
  - Count arrivals / record arrival times.

Fitting a Non-stationary Poisson Process

- The estimated arrival rate during the *i*th time period is:

$$\hat{\lambda}(t) = \frac{1}{n\Delta t}\sum_{j=1}^{n} C_{ij}$$

where n = # of observation periods, Δt = *time interval length*

C_{ij} = # of arrivals during the i^{th} time interval on the j^{th} observation period

- Example: Divide a *10*-hour business day [*8am,6pm*] into equal intervals *k* = *20* whose length Δt = ½, and observe over n =3 days

Time Period	Number of Arrivals			Estimated Arrival Rate (arrivals/hr)
	Day 1	Day 2	Day 3	
8:00 - 8:00	12	14	10	24
8:30 - 9:00	23	26	32	54
9:00 - 9:30	27	18	32	52
9:30 - 10:00	20	13	12	30

For instance,
1/3(0.5)*(23+26+32)
= 54 arrivals/hour

Selecting Model without Data

- If data is not available, some possible sources to obtain information about the process are:
  - Engineering data: often product or process has performance ratings provided by the manufacturer or company rules specify time or production standards.
  - Expert option: people who are experienced with the process or similar processes, often, they can provide optimistic, pessimistic and most-likely times, and they may know the variability as well.
  - Physical or conventional limitations: physical limits on performance, limits or bounds that narrow the range of the input process.
  - The nature of the process.
- The uniform, triangular, and beta distributions are often used as input models.

Multivariate and Time-Series Input Models

- Multivariate:
  - For example, lead time and annual demand for an inventory model, increase in demand results in lead time increase, hence variables are dependent.

- Time-series:
  - For example, time between arrivals of orders to buy and sell stocks, buy and sell orders tend to arrive in bursts, hence, times between arrivals are dependent.

Covariance and Correlation

[Multivariate/Time Series]

- Consider the model that describes relationship between X_1 and X_2:

$$(X_1 - \mu_1) = \beta(X_2 - \mu_2) + \varepsilon$$

ε is a random variable with mean *0* and is independent of X_2

  - $\beta = 0$, X_1 and X_2 are statistically independent
  - $\beta > 0$, X_1 and X_2 tend to be above or below their means together
  - $\beta < 0$, X_1 and X_2 tend to be on opposite sides of their means

- Covariance between X_1 and X_2 :

$$\operatorname{cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] = E(X_1 X_2) - \mu_1 \mu_2$$

  - where $\quad cov(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases} \quad$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$

Covariance and Correlation

[Multivariate/Time Series]

- Correlation between X_1 and X_2 (values between -1 and 1):

$$\rho = \operatorname{corr}(X_1, X_2) = \frac{\operatorname{cov}(X_1, X_2)}{\sigma_1 \sigma_2}$$

  - where $corr(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases}$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$
  - The closer ρ is to -1 or 1, the stronger the linear relationship is between X_1 and X_2.

Covariance and Correlation

- Atimeseriesisasequenceofrandomvariables $X_1, X_2, X_3, \ldots,$ are identically distributed (same mean and variance) but dependent.
  - cov(X_t, X_{t+h}) is the *lag-h autocovariance*
  - corr(X_t, X_{t+h}) is the *lag-h autocorrelation*
  - If the autocovariance value depends only on *h* and not on *t*, the time series is covariance stationary

Multivariate Input Models

- If X_1 and X_2 are normally distributed, dependence between them can be modeled by the bivariate normal distribution with $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ and correlation ρ
  - To Estimate $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, see "Parameter Estimation" (slide 15-17, Section 9.3.2 in book)
  - To Estimate ρ, suppose we have n independent and identically distributed pairs $(X_{11}, X_{21}), (X_{12}, X_{22}), \ldots (X_{1n}, X_{2n})$, then:

$$\hat{\text{cov}}(X_1, X_2) = \frac{1}{n-1}\sum_{j=1}^{n}(X_{1j} - \hat{X}_1)(X_{2j} - \hat{X}_2)$$

$$= \frac{1}{n-1}\left(\sum_{j=1}^{n} X_{1j}X_{2j} - n\hat{X}_1\hat{X}_2\right)$$

$$\hat{\rho} = \frac{\hat{\text{cov}}(X_1, X_2)}{\hat{\sigma}_1\hat{\sigma}_2}$$

Sample deviation

Summary

- In this chapter, we described the 4 steps in developing input data models:
  - Collecting the raw data
  - Identifying the underlying statistical distribution
  - Estimating the parameters
  - Testing for goodness of fit