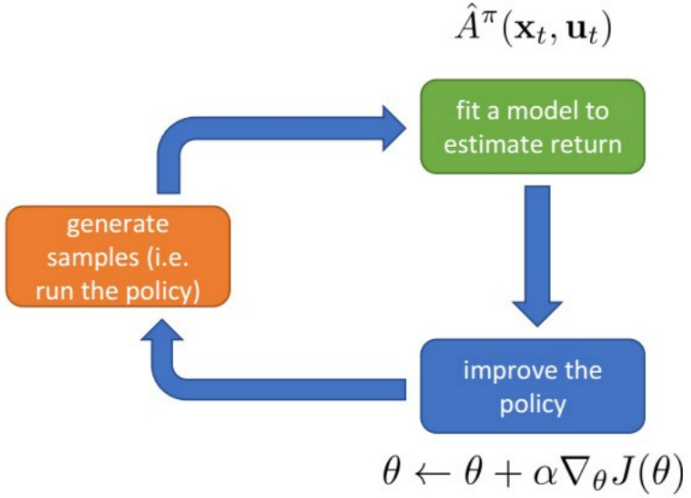
به سمت پالیسی بهتر رفتن در On-policy ← جمع‌آوری اینبار بهتر نسبت قبلی
اینا RL ه.

Policy Gradient as Policy Iteration

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t}) \hat{A}^\pi_{i,t}$$

main steps of policy gradient algorithm:

1. Estimate $\hat{A}^\pi(\mathbf{s}_t, \mathbf{a}_t)$ for current policy π
2. Use $\hat{A}^\pi(\mathbf{s}_t, \mathbf{a}_t)$ to get *improved* policy π'



Familiar to policy iteration algorithm:

1. evaluate $A^\pi(\mathbf{s}, \mathbf{a})$
2. set $\pi \leftarrow \pi'$

Policy Gradient as Policy Iteration

$\theta \longrightarrow \theta'$

$$J(\theta) = E_{\tau \sim p_\theta(\tau)}\left[\sum_t \gamma^t r(\mathbf{s}_t, \mathbf{a}_t)\right]$$

$$\theta' = \arg\max_{\theta'} J(\theta') - J(\theta)$$

$$\text{s.t. } d(\theta', \theta) \leq \epsilon$$

claim: $J(\theta') - J(\theta) = E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_t \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t)\right]$

could be interpreted as policy improvement!

$$\max_{\theta'} J(\theta') - J(\theta) \quad E_{\tau \sim p_{\theta'}}\left[\sum_t \gamma^t A^{\pi_\theta}(s_t, a_t)\right]$$

$$A^{\pi_\theta}(s_t, a_t) = r(s_t, a_t) + \gamma V^{\pi_\theta}(s_{t+1}) - V^{\pi_\theta}(s_t)$$

هدف کلی ⇐ چون تو RL آپدیت کردن به حالت چیز آوری ایتای بد و دوباره

Policy Gradient as Policy Iteration

claim: $J(\theta') - J(\theta) = E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_t \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t)\right]$

proof: $J(\theta') - J(\theta) = J(\theta') - E_{\mathbf{s}_0 \sim p(\mathbf{s}_0)}\left[V^{\pi_\theta}(\mathbf{s}_0)\right]$

$$= J(\theta') - E_{\tau \sim p_{\theta'}(\tau)}\left[V^{\pi_\theta}(\mathbf{s}_0)\right]$$

$$= J(\theta') - E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t V^{\pi_\theta}(\mathbf{s}_t) - \sum_{t=1}^{\infty} \gamma^t V^{\pi_\theta}(\mathbf{s}_t)\right]$$

$$= J(\theta') + E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t (\gamma V^{\pi_\theta}(\mathbf{s}_{t+1}) - V^{\pi_\theta}(\mathbf{s}_t))\right]$$

$$= E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t r(\mathbf{s}_t, \mathbf{a}_t)\right] + E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t (\gamma V^{\pi_\theta}(\mathbf{s}_{t+1}) - V^{\pi_\theta}(\mathbf{s}_t))\right]$$

$$= E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t (r(\mathbf{s}_t, \mathbf{a}_t) + \gamma V^{\pi_\theta}(\mathbf{s}_{t+1}) - V^{\pi_\theta}(\mathbf{s}_t))\right]$$

$$= E_{\tau \sim p_{\theta'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t)\right]$$

Policy Gradient as Policy Iteration

$$J(\theta') - J(\theta) = E_{\tau \sim p_{\theta'}(\tau)} \left[\sum_t \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right]$$

expectation under $\pi_{\theta'}$

advantage under π_θ

$J(\theta') - J(\theta)$

$\pi_\theta = \pi_{\theta'} \Rightarrow p_\theta = p_{\theta'}$

θ خیلی آپدیت نشده باشه یا به عبارت بهتر توزیع اون‌ها خیلی فرق نکنه

$$E_{\tau \sim p_{\theta'}(\tau)} \left[\sum_t \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] = \sum_t E_{\mathbf{s}_t \sim p_{\theta'}(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)} \left[\gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$$= \sum_t E_{\mathbf{s}_t \sim p_{\theta'}(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

p_θ

importance weight

is it OK to use $p_\theta(\mathbf{s}_t)$ instead?

Policy Gradient as Policy Iteration

Can we ignore distribution mismatch?

$$\sum_t E_{\mathbf{s}_t \sim p_{\theta'}(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right] \overset{?}{\approx} \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$\bar{A}(\theta')$

why do we want this to be true?

$$J(\theta') - J(\theta) \approx \bar{A}(\theta') \quad \Rightarrow \quad \theta' \leftarrow \arg\max_{\theta'} \bar{A}(\theta)$$

2. Use $\hat{A}^\pi(\mathbf{s}_t, \mathbf{a}_t)$ to get *improved* policy π'

is it true? and when?

$p_\theta(\mathbf{s}_t)$ is *close* to $p_{\theta'}(\mathbf{s}_t)$ when π_θ is *close* to $\pi_{\theta'}$

Bounding the distribution change

Claim: $p_\theta(\mathbf{s}_t)$ is *close* to $p_{\theta'}(\mathbf{s}_t)$ when π_θ is *close* to $\pi_{\theta'}$

Simple case: assume π_θ is a *deterministic* policy $\mathbf{a}_t = \pi_\theta(\mathbf{s}_t)$

$\pi_{\theta'}$ is *close* to π_θ if $\pi_{\theta'}(\mathbf{a}_t \neq \pi_\theta(\mathbf{s}_t)|\mathbf{s}_t) \leq \epsilon$

$$p_{\theta'}(\mathbf{s}_t) \geq (1-\epsilon)^t p_\theta(\mathbf{s}_t) + (1-(1-\epsilon)^t) p_{\text{mistake}}(\mathbf{s}_t)$$

probability we made no mistakes — some *other* distribution

$$|p_{\theta'}(\mathbf{s}_t) - p_\theta(\mathbf{s}_t)| = (1-(1-\epsilon)^t)|p_{\text{mistake}}(\mathbf{s}_t) - p_\theta(\mathbf{s}_t)| \leq 2(1-(1-\epsilon)^t)$$

useful identity: $(1-\epsilon)^t \geq 1 - \epsilon t$ for $\epsilon \in [0,1]$

$$\leq 2\epsilon t$$

Total Variation distance $(p_\theta, p_{\theta'}) = \sum_\tau |p_\theta(\tau) - p_{\theta'}(\tau)|$

seem familiar?

$p_\theta(s_t) = (1-\epsilon)^t p_\theta(s_t) + (1-(1-\epsilon)^t) p_\theta(s_t)$

بزرگ‌ترین حداکثر ۲ است

not a great bound, but a bound!

Bounding the distribution change

Claim: $p_\theta(\mathbf{s}_t)$ is *close* to $p_{\theta'}(\mathbf{s}_t)$ when π_θ is *close* to $\pi_{\theta'}$

General case: assume π_θ is an arbitrary distribution

$\pi_{\theta'}$ is *close* to π_θ if $|\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) - \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)| \leq \epsilon$ for all $\mathbf{s}_t$

Useful lemma: if $|p_X(x) - p_Y(x)| = \epsilon$, exists $p(x, y)$ such that $p(x) = p_X(x)$ and $p(y) = p_Y(y)$ and $p(x = y) = 1 - \epsilon$

$\Rightarrow$ $p_X(x)$ "agrees" with $p_Y(y)$ with probability ϵ

$\Rightarrow$ $\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)$ takes a different action than $\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$ with probability at most ϵ

$$
\begin{aligned}
|p_{\theta'}(\mathbf{s}_t) - p_\theta(\mathbf{s}_t)| = (1 - (1 - \epsilon)^t)|p_{\text{mistake}}(\mathbf{s}_t) - p_\theta(\mathbf{s}_t)| &\leq 2(1 - (1 - \epsilon)^t) \\
&\leq 2\epsilon t
\end{aligned}
$$

$\sum_{t=0}^{\infty} t\rho^t < \infty \quad \rho < 1$

Bounding the objective value

$x \geq -|x|$

$\pi_{\theta'}$ is *close* to π_θ if $|\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) - \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)| \leq \epsilon$ for all $\mathbf{s}_t$

$|p_{\theta'}(\mathbf{s}_t) - p_\theta(\mathbf{s}_t)| \leq 2\epsilon t$

$= \sum (P_{\theta'} - P_\theta + P_\theta) f = \sum P_\theta f +$

$$E_{p_{\theta'}(\mathbf{s}_t)}[f(\mathbf{s}_t)] = \sum_{\mathbf{s}_t} p_{\theta'}(\mathbf{s}_t) f(\mathbf{s}_t) \geq \sum_{\mathbf{s}_t} p_\theta(\mathbf{s}_t) f(\mathbf{s}_t) - |p_\theta(\mathbf{s}_t) - p_{\theta'}(\mathbf{s}_t)| \max_{\mathbf{s}_t} |f(\mathbf{s}_t)|$$

$\sum (P_{\theta'} - P_\theta) f$

x

$$\geq E_{p_\theta(\mathbf{s}_t)}[f(\mathbf{s}_t)] - 2\epsilon t \max_{\mathbf{s}_t} |f(\mathbf{s}_t)|$$

$J(\theta') - J(\theta)$

$$\sum_t E_{\mathbf{s}_t \sim p_{\theta'}(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right] \geq$$

$$\sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right] - \sum_t 2\epsilon t C$$

$O(Tr_{\max})$ or $O\left(\frac{r_{\max}}{1-\gamma}\right)$

maximizing this maximizes a bound on the thing we want!

$x \geq -|x| = -\sum |P_{\theta'} - P_\theta|$

$\geq -\left(\sum |P_{\theta'} - P_\theta|\right) \max |f|$

Some Useful Preliminaries: Taylor Series

Approximation of a differentiable function around a given point with sum of terms of the function's derivatives:

$$f(x) \approx f(x_0) + (x - x_0)^T \nabla f(x_0) + \frac{1}{2}(x - x_0)^T H (x - x_0) + \cdots$$

Some Useful Preliminaries: Constrained Optimization

- equality constraints: method of Lagrange multipliers

$$\begin{array}{l} optimize\ f(x) \\ subject\ to: g(x) = 0 \end{array} \longrightarrow \mathcal{L}(x, \lambda) = f(x) + \lambda g(x)$$

- Inequality constraints: KKT

$$\begin{array}{l} optimize\ f(x) \\ subject\ to: \\ \quad g_i(x) \leq 0, \\ \quad h_j(x) = 0 \end{array} \longrightarrow \begin{array}{l} \mathcal{L}(x, \mu, \lambda) = f(x) + \mu^T g(x) + \lambda^T h(x) \\ subject\ to: \\ \qquad \mu_i \geq 0, \\ \qquad \mu^T g(x) = 0 \end{array}$$

Some Useful Preliminaries: KL-Divergence

A Common distance measure for distributions:

$$D_{KL}(p||q) = \int_x p(x) log \frac{p(x)}{q(x)} dx$$

Other useful distance measures:

- Total variation distance
- Wasserstein distance
- Jensen–Shannon divergence
- …

Some Useful Preliminaries: Fisher Information

likelihood function: $p_\theta(x)$

score function: $\nabla_\theta log p_\theta(x)$

Fisher information: measuring the amount of information that a random variable (x) carries about likelihood parameters (θ):

$$[I(\theta)]_{i,j} = E_{x\sim p_\theta(x)}[\,(\frac{\partial}{\partial\theta_i} log p_\theta(x))(\frac{\partial}{\partial\theta_j} log p_\theta(x))]$$

$$= -E_{x\sim p_\theta(x)}[\,\frac{\partial^2}{\partial\theta_i\partial\theta_j} log p_\theta(x)]$$

variance (covariance) of score function

curvature of score function

Where are we at so far?

$\approx J(\theta') - J(\theta)$

$$\theta' \leftarrow \arg\max_{\theta'} \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

such that $|\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) - \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)| \leq \epsilon$

Tot Var

for small enough ϵ, this is guaranteed to improve $J(\theta') - J(\theta)$

A more convenient bound

Claim: $p_\theta(\mathbf{s}_t)$ is *close* to $p_{\theta'}(\mathbf{s}_t)$ when π_θ is *close* to $\pi_{\theta'}$

$\pi_{\theta'}$ is *close* to π_θ if $|\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) - \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)| \leq \epsilon$ for all $\mathbf{s}_t$

$$|p_{\theta'}(\mathbf{s}_t) - p_\theta(\mathbf{s}_t)| \leq 2\epsilon t$$

a more convenient bound: $|\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) - \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)| \leq \sqrt{\frac{1}{2} D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)\|\pi_\theta(\mathbf{a}_t|\mathbf{s}_t))}$

$\Rightarrow$ $D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)\|\pi_\theta(\mathbf{a}_t|\mathbf{s}_t))$ bounds state marginal difference

$$D_{\mathrm{KL}}(p_1(x)\|p_2(x)) = E_{x\sim p_1(x)}\left[\log \frac{p_1(x)}{p_2(x)}\right]$$

KL divergence has some very convenient properties that make it much easier to approximate!

How do we optimize the objective?

$$\theta' \leftarrow \arg\max_{\theta'} \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$$\text{such that } D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$$

for small enough ϵ, this is guaranteed to improve $J(\theta') - J(\theta)$

How do we enforce the constraint?

$$\theta' \leftarrow \arg\max_{\theta'} \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

such that $D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$

$$\mathcal{L}(\theta', \lambda) = \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right] - \lambda (D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) - \epsilon)$$

1. Maximize $\mathcal{L}(\theta', \lambda)$ with respect to θ' ← **can do this incompletely (for a few grad steps)**
2. $\lambda \leftarrow \lambda + \alpha (D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) - \epsilon)$

Intuition: raise λ if constraint violated too much, else lower it

an instance of *dual gradient descent* (more on this later!)

How (else) do we optimize the objective?

$$\theta' \leftarrow \arg\max_{\theta'} \sum_t \underbrace{E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]}_{\bar{A}(\theta')}$$

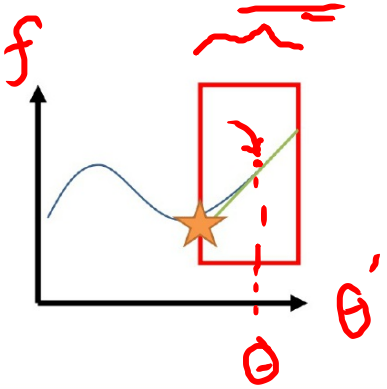
such that $D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$

for small enough ϵ, this is guaranteed to improve $J(\theta') - J(\theta)$

max $f(\theta)$ s.t. $g(\theta) \geq 0$

linearize

Trust Region



$$\theta' \leftarrow \arg\max_{\theta'} \nabla_\theta \bar{A}(\theta)^T (\theta' - \theta)$$

such that $D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$

$f(\theta') = f(\theta) + \nabla f^T \Big|_{\theta'=\theta} (\theta' - \theta) + \dots$

Use first order Taylor approximation for objective (a.k.a., linearization)

How (else) do we optimize the objective?

$$\theta' \leftarrow \arg\max_{\theta'} \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$$\text{such that } D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$$

$$\theta' \leftarrow \arg\max_{\theta'} \nabla_\theta \bar{A}(\theta)^T (\theta' - \theta)$$

$$\text{such that } D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$$

$$\nabla_{\theta'} \bar{A}(\theta') = \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t \nabla_{\theta'} \log \pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$\theta' = \theta$

(see policy gradient lecture for derivation)

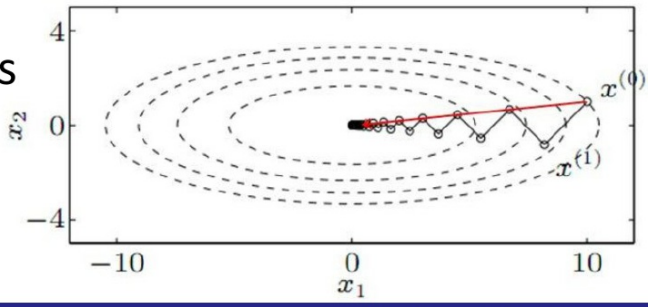
$$\nabla_\theta \bar{A}(\theta) = \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t \nabla_\theta \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t) A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$$\nabla_\theta \bar{A}(\theta) = \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\gamma^t \nabla_\theta \log \pi_\theta(\mathbf{a}_t|\mathbf{s}_t) A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right] = \nabla_\theta J(\theta)$$

exactly the normal policy gradient!

Importance of step size in RL

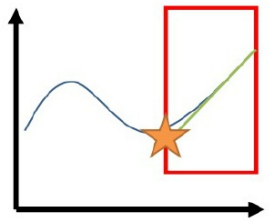
- Supervised learning: Step too far → next updates will fix it
- Reinforcement learning: Policy is determining data collection!
  - Step too far → bad policy
  - Next batch: collected under bad policy
  - May not be able to recover from a bad choice, collapse in performance!
- Learning rate tuning is hard
  - Poor conditioning could be more dangerous in RL settings
  - More sophisticated optimizers can reduce numerical issues
  - Need for advanced learning rate adjustment methods!



Natural Policy Gradient

$$\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$$

$\nabla_\theta D_{KL}(\pi_{\theta'} \| \pi_\theta) = 0$



Could be shown as a constraint problem:

$$\theta' \leftarrow \arg\max_{\theta'} (\theta' - \theta)^T \nabla_\theta J(\theta) \text{ s.t. } \|\theta' - \theta\|^2 \leq \epsilon$$

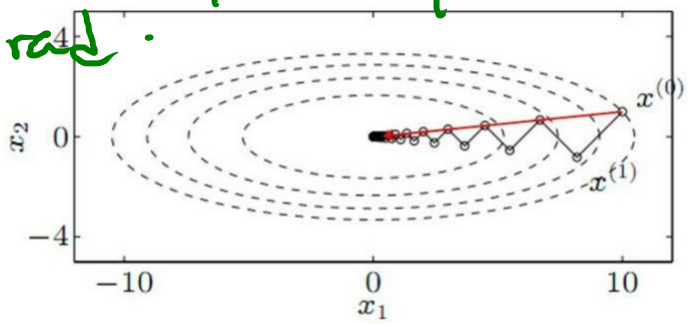
controls how far we go

Can we rescale the gradients to reduce the problem with poor conditioning?

Gradient That Comes From policy grad.

$$\theta' \leftarrow \arg\max_{\theta'} (\theta' - \theta)^T \nabla_\theta J(\theta) \text{ s.t. } D(\pi_{\theta'}, \pi_\theta) \leq \epsilon$$

parameterization-independent divergence measure



این بهینه‌ساز در policy Grad با این تفاوت که این قید در

بنابراین ← روش‌های پیشرفته در این حیطه کار می‌کنن

آپدیت θ به یه چیز دور ⇐ یعنی پالیسی ⇒ حتمی بد میشه

اگه P.G رو آپدیت کنیم

خلاصه اینکه PG قابلیت اینو داره که خیلی بد fail شه چیزی به عنیه
هرگاه زیاد آپدیت نشه

$$\nabla_{\theta'} \underbrace{D_{KL}(\pi_{\theta'} \| \pi_\theta)}_{f(\theta')} = \nabla_{\theta'} \sum_a \pi_{\theta'}(a) \log \frac{\pi_{\theta'}(a)}{\pi_\theta(a)}$$

$$= \sum_a \nabla \pi_{\theta'}(a) \underbrace{\log \frac{\pi_{\theta'}}{\pi_\theta}}_{0} + \underbrace{\frac{\pi_{\theta'}}{\pi_{\theta'}} \frac{\nabla \pi_{\theta'}}{\pi_{\theta'}}}_{\nabla \pi_{\theta'}} \Big|_{\theta'=\theta}$$

$$= \sum_a \nabla \pi_{\theta'} \Big|_{\theta'=\theta} = \nabla_{\theta'} \underbrace{\sum_a \pi_{\theta'}(a)}_{1} = 0$$

grad w.r.t θ'

$$= \sum_a H_{\pi_{\theta'}} \log \frac{\pi_{\theta'}}{\pi_\theta} + \nabla \pi_{\theta'} \cdot \frac{\nabla \pi_{\theta'}}{\pi_{\theta'}} + H_{\pi_{\theta'}} \Big|_{\theta'=\theta}$$

$$= \sum_a \frac{\nabla \pi_{\theta'} \nabla \pi_{\theta'}}{\pi_{\theta'}} = F$$

$$= H \sum \pi_{\theta'}(a)$$

$$a a^T = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & a_i a_j & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$\begin{cases} \max\limits_{\theta'} & \nabla J^T (\theta' - \theta) \\ \text{s.t.} & (\theta' - \theta)^T F (\theta' - \theta) \le \epsilon \end{cases}$$

$$\nabla J^T F^{-1} \nabla J$$

$$\alpha^2 (F^{-1} \nabla J)^T F (F^{-1} \nabla J) \le \epsilon$$

$$\alpha \le \sqrt{\frac{\epsilon}{\nabla J^T F^{-1} \nabla J}}$$

$$\hookrightarrow \nabla_{\theta'} J^T \Big|_{\theta'=\theta} (\theta' - \theta) - \lambda \left[(\theta' - \theta)^T F (\theta' - \theta) - \epsilon \right]$$

$$\nabla_{\theta'} J \Big|_{\theta'=\theta} - 2\lambda F (\theta' - \theta) = 0$$

$$\theta' - \theta = \alpha F^{-1} \nabla J$$

$$\theta' = \theta + \alpha F^{-1} \nabla J$$

$$f(\theta') = f(\theta) + \nabla_{\theta'} f^T \Big|_{\theta'=\theta} (\theta' - \theta) \quad (\nabla_{\theta'} f^T \Big|_{\theta'=\theta} \to 0)$$

$$+ \frac{1}{2} (\theta' - \theta)^T H_f (\theta' - \theta) + \cdots$$

$$H_f(\eta)$$

Natural Policy Gradient

$$\theta' \leftarrow \arg\max_{\theta'} (\theta' - \theta)^T \nabla_\theta J(\theta) \text{ s.t. } D(\pi_{\theta'}, \pi_\theta) \leq \epsilon$$

parameterization-independent divergence measure

usually KL-divergence: $D_{\mathrm{KL}}(\pi_{\theta'} \| \pi_\theta) = E_{\pi_{\theta'}}[\log \pi_\theta - \log \pi_{\theta'}]$

Taylor expansion:

$$D_{\mathrm{KL}}(\pi_{\theta'} \| \pi_\theta) \approx (\theta' - \theta)^T \mathbf{F} (\theta' - \theta)$$

Fisher-information matrix

$$\mathbf{F} = E_{\pi_\theta}[\nabla_\theta \log \pi_\theta(\mathbf{a}|\mathbf{s}) \nabla_\theta \log \pi_\theta(\mathbf{a}|\mathbf{s})^T]$$

can estimate with samples

$$F = \sum_a \frac{\nabla_\theta \pi_\theta \nabla_\theta \pi_\theta^T}{\pi_\theta} \nabla \log \pi_\theta$$

$$= \sum_a \frac{\pi_\theta \nabla \pi_\theta \nabla^T \pi_\theta}{\pi_\theta \pi_\theta} \nabla \log \pi_\theta$$

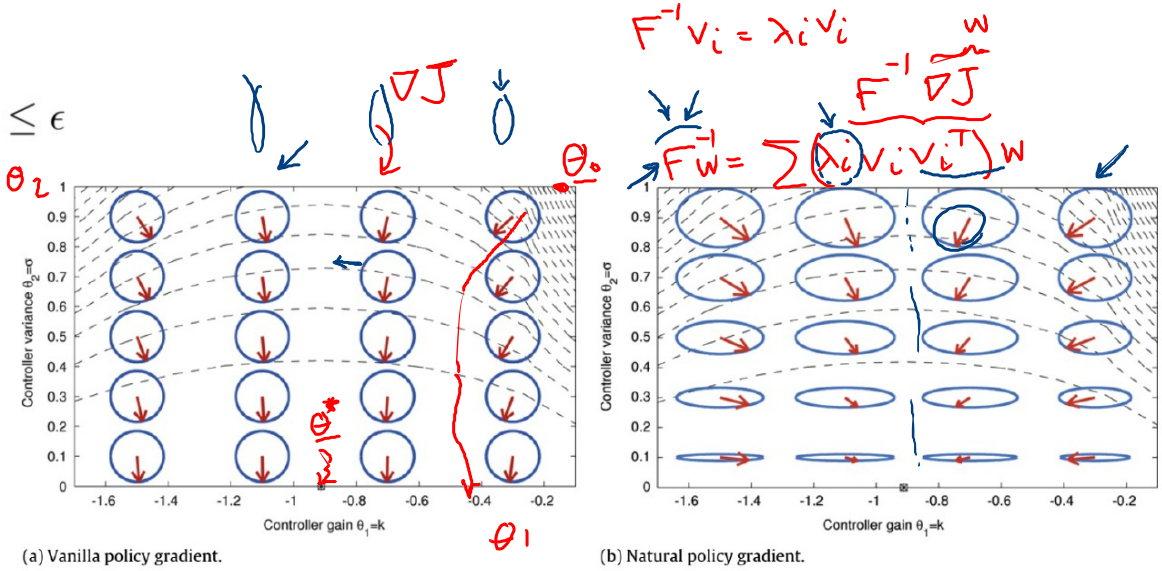
Natural Policy Gradient

$$D_{\mathrm{KL}}(\pi_{\theta'} \| \theta_\pi) \approx (\theta' - \theta)^T \mathbf{F} (\theta' - \theta)$$

$$\theta' \leftarrow \arg\max_{\theta'} (\theta' - \theta)^T \nabla_\theta J(\theta) \text{ s.t. } D(\pi_{\theta'}, \pi_\theta) \leq \epsilon$$

$$\theta \leftarrow \theta + \alpha \mathbf{F}^{-1} \nabla_\theta J(\theta)$$

natural gradient: pick α



(a) Vanilla policy gradient. (b) Natural policy gradient.

(figure from Peters & Schaal 2008)

trust region policy optimization: pick ϵ

can solve for optimal α while solving $\mathbf{F}^{-1} \nabla_\theta J(\theta)$

$x^T A x$

$A \succeq 0$

$\Delta\theta \propto \nabla_\theta J$

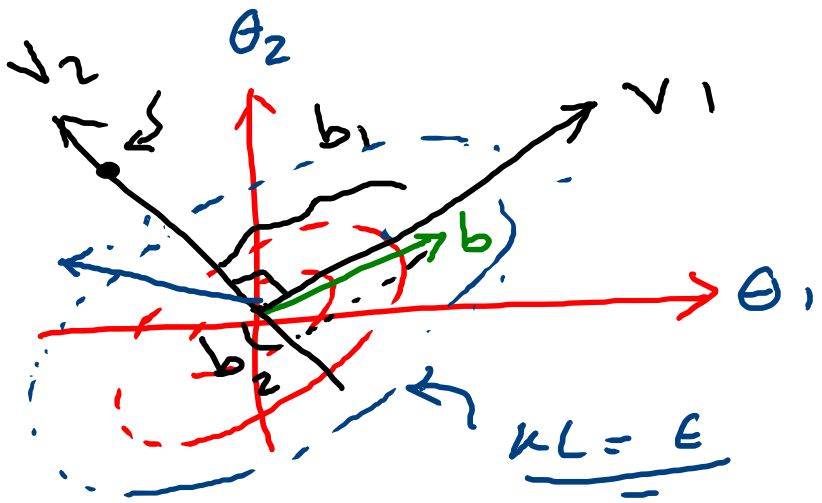
$\Delta\theta^T F^{-1} \Delta\theta = \sum \frac{1}{\lambda_i} \Delta\theta^T v_i v_i^T \Delta\theta$

$D_{KL}(\pi_\theta \| \pi_\theta)$

$$\begin{cases} \max_{\theta'} & (\theta' - \theta)^T \nabla_\theta J \\ \text{s.t.} & (\theta' - \theta)^T F (\theta' - \theta) \le \epsilon \end{cases}$$

assume that $F = I \Rightarrow \|\theta' - \theta\|^2 \le \epsilon$

$$\theta' \leftarrow \theta + \alpha \nabla J \qquad \alpha = \sqrt{\frac{2\epsilon}{\|\nabla J\|^2}}$$



$$\max_{b} \langle \nabla J, \Delta\theta \rangle \text{ s.t. } D_{KL} \leq \epsilon$$

$$D_{KL} = \frac{b_1}{\lambda_1} + \frac{b_2}{\lambda_2}$$

$$\Delta\theta = \alpha F^{-1} \nabla J$$

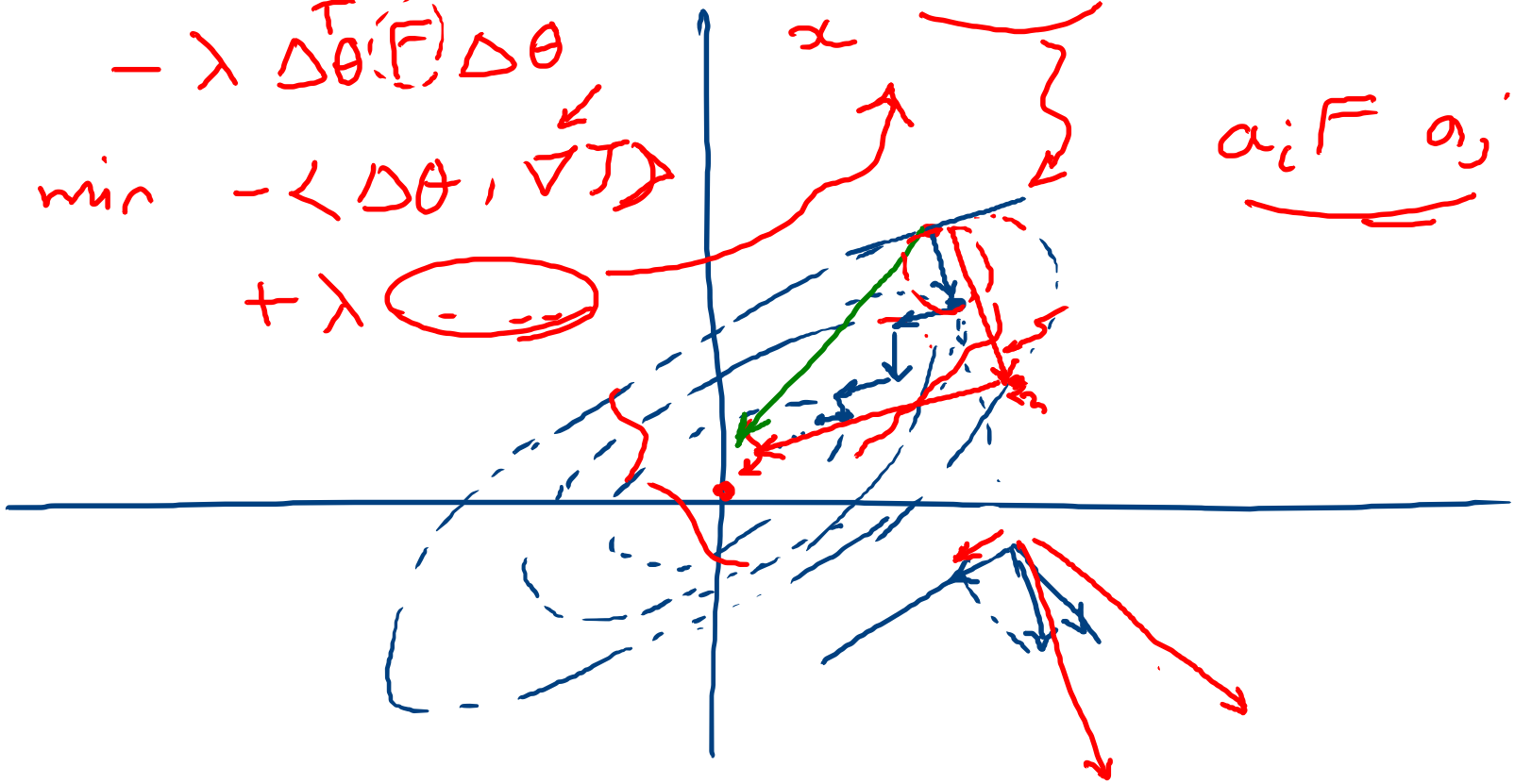
$$\nabla J$$

$$\max \; \langle \Delta\theta, \nabla J \rangle - \lambda \Delta\theta^T F \Delta\theta$$

$$\equiv \min \; -\langle \Delta\theta, \nabla J \rangle + \lambda \left(\quad \right)$$

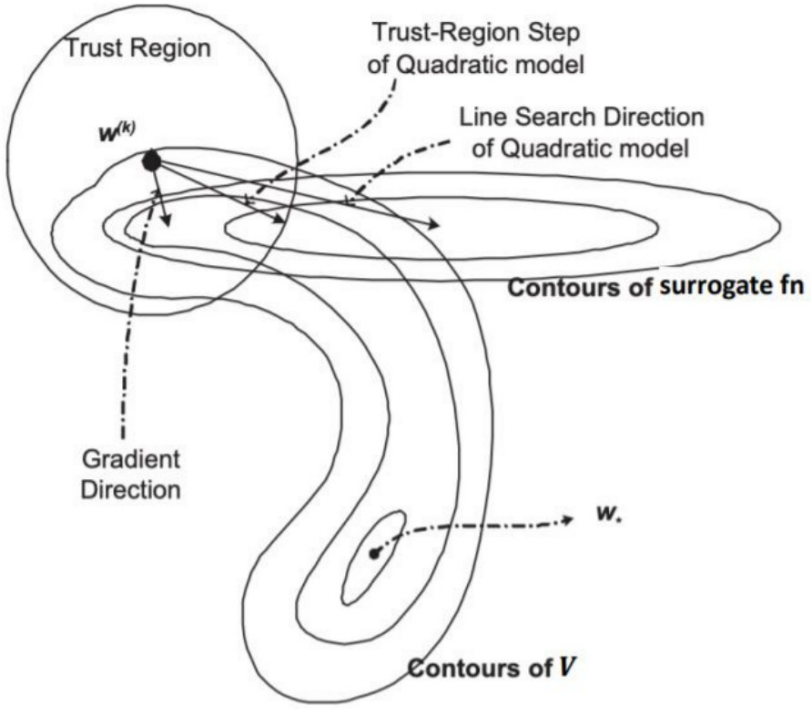
$$\min_x \; x^T A x + b^T x$$

$$a_i^T F a_j = 0$$



Trust Region Method

- We often optimize a surrogate objective
- Surrogate objective may be trustable only in a small region
- Limit search to small trust region



Cs885, waterloo 2022

Trust Region Policy Optimization

Recall from "Policy Gradient as Policy Iteration":

$$\bar{A}(\theta')$$

$$\theta' \leftarrow \arg\max_{\theta'} \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$$\text{such that } D_{\text{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$$

surrogate loss

trust region

for small enough ϵ, this is guaranteed to improve $J(\theta') - J(\theta)$

Policy Gradient with Constraints

How do we enforce the constraint?

$$\theta' \leftarrow \arg\max_{\theta'} \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right]$$

$$\text{such that } D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$$

$$\mathcal{L}(\theta', \lambda) = \sum_t E_{\mathbf{s}_t \sim p_\theta(\mathbf{s}_t)} \left[E_{\mathbf{a}_t \sim \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \left[\frac{\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)} \gamma^t A^{\pi_\theta}(\mathbf{s}_t, \mathbf{a}_t) \right] \right] - \lambda (D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) - \epsilon)$$

1. Maximize $\mathcal{L}(\theta', \lambda)$ with respect to θ'
2. $\lambda \leftarrow \lambda + \alpha(D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) - \epsilon)$

Intuition: raise λ if constraint violated too much, else lower it

an instance of *dual gradient descent*

Natural Gradient Based on Trust Region

Recall:

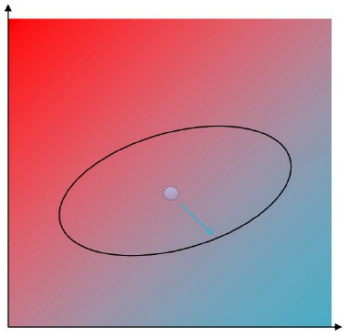
$$\theta' \leftarrow \arg\max_{\theta'} \nabla_\theta J(\theta)^T (\theta' - \theta)$$

$$\text{such that } D_{\mathrm{KL}}(\pi_{\theta'}(\mathbf{a}_t|\mathbf{s}_t) \| \pi_\theta(\mathbf{a}_t|\mathbf{s}_t)) \leq \epsilon$$

$$D_{\mathrm{KL}}(\pi_{\theta'} \| \pi_\theta) \approx \frac{1}{2}(\theta' - \theta)^T \mathbf{F} (\theta' - \theta)$$

$$\theta' = \theta + \alpha \mathbf{F}^{-1} \nabla_\theta J(\theta)$$

natural gradient



$$\alpha = \sqrt{\frac{2\epsilon}{\nabla_\theta J(\theta)^T \mathbf{F} \nabla_\theta J(\theta)}}$$

Proximal Policy Optimization

- TRPO is conceptually and computationally challenging in large part because of the constraint in the optimization.

$$D_{KL}(\pi_{\theta'}(.|s)\,||\pi_{\theta}(.|s)) \leq \epsilon$$

- What is the effect of the constraint?
- Recall KL-Divergence:

$$D_{KL}(\pi_{\theta'}(.|s)\,||\pi_{\theta}(.|s)) = \Sigma_a \pi_{\theta'}(a|s) log \frac{\pi_{\theta'}(a|s)}{\pi_{\theta}(a|s)}$$

We are effectively constraining the ratio $\frac{\pi_{\theta'}(a|s)}{\pi_{\theta}(a|s)}$

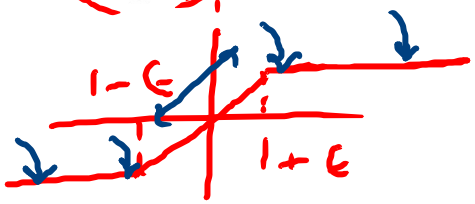
Proximal Policy Optimization

$$J(\theta') - J(\theta) = \sum_t \frac{\pi_{\theta'}}{\pi_\theta} A^{\pi_\theta}$$

- Let's design a simpler objective that directly constrains $\frac{\pi_{\theta'}(a|s)}{\pi_\theta(a|s)}$

$$\underset{\theta'}{\text{argmax}}\, E_{\{s\sim\mu_\theta, a\sim\pi_\theta\}}\, min \begin{cases} \frac{\pi_{\theta'}(a|s)}{\pi_\theta(a|s)} A^{\pi_\theta}(s,a), \\ clip\left(\frac{\pi_{\theta'}(a|s)}{\pi_\theta(a|s)}, 1-\epsilon, 1+\epsilon\right) A^{\pi_\theta}(s,a) \end{cases}$$

could be easily implemented with auto-diff packages



where $clip(x, 1-\epsilon, 1+\epsilon) = \begin{cases} 1-\epsilon & if\ x < 1-\epsilon \\ x & if\ 1-\epsilon \le x \le 1+\epsilon \\ 1+\epsilon & if\ x > 1+\epsilon \end{cases}$