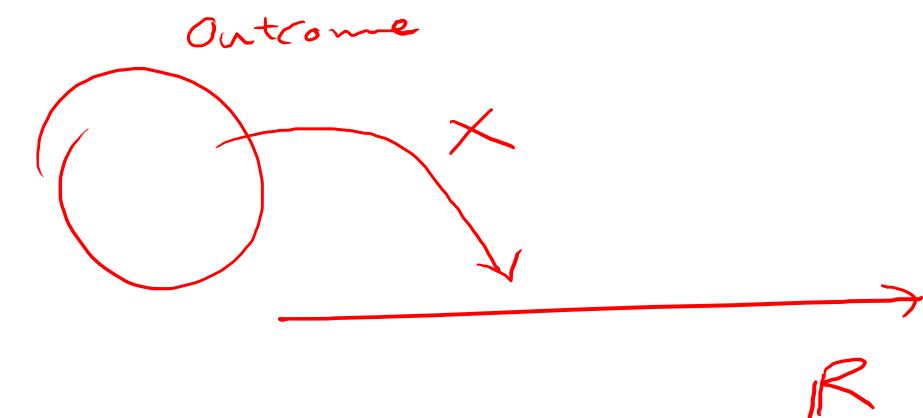


Stochastic Process

Markov Process

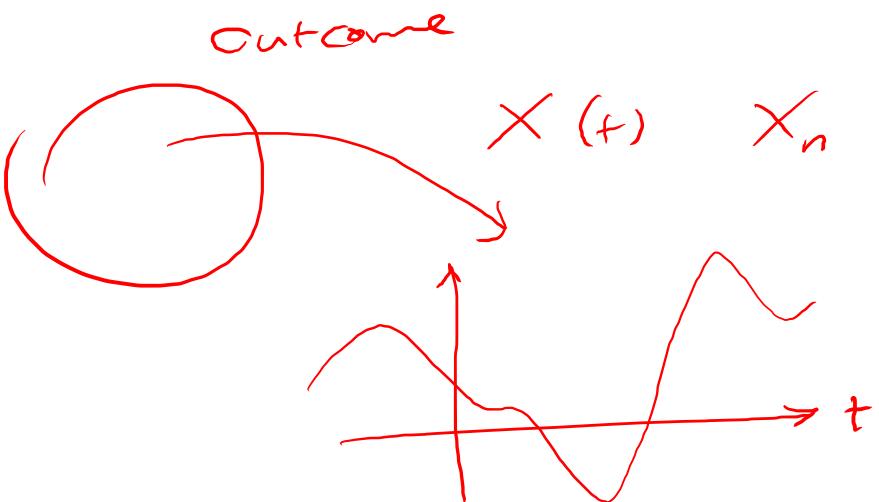


متغير عشوائي في الزمن



متغير عشوائي في الزمن

$$\{S_1, S_2, \dots, S_n, \dots\}$$



$$P(S_n | S_{n-1}, \dots, S_1) = P(S_n | S_{n-1})$$

$$P(S_n) =: \pi_n$$

$$\pi_0$$

trans. prob.

matrix

$$P = \begin{bmatrix} & & j \\ & & | \\ i & - & P_{ij} \\ & & | \\ & & \end{bmatrix}$$

$$P_{ij} = P(S_n=j \mid S_{n-1}=i)$$

$$\pi_n = \pi_0 P^n$$

Does $\lim_{n \rightarrow \infty} \pi_n$ exists?

Stationary prob. π^* : $\pi^* = \underline{\pi^* P}$

Under certain conditions

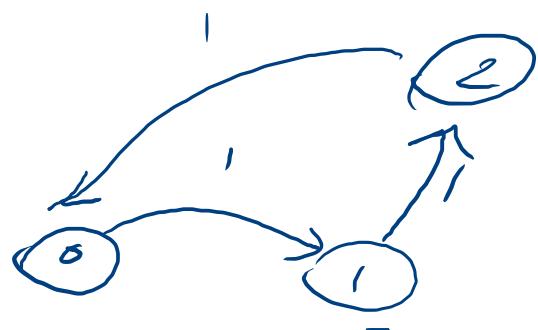
$\lim_{n \rightarrow \infty} \pi_n$ exists & it's equal

π^* , which is unique.

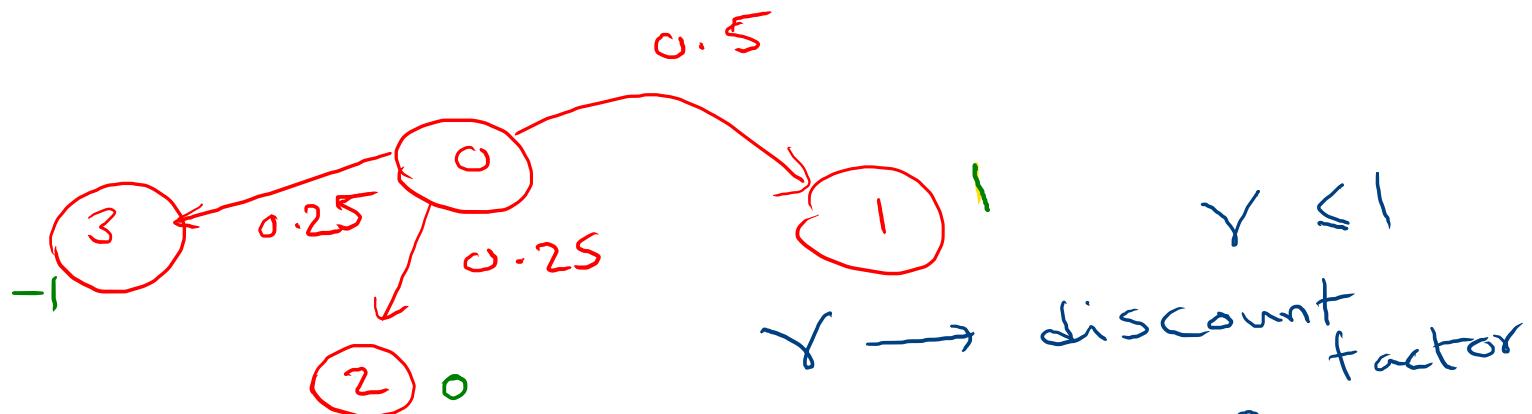
Conditions: - Finite State

جنبش، پرسنلیتی،
و انتقالات
جنبشی، پرسنلیتی، انتقالات

{ - Irreducible
- Aperiodic



Markov Reward Process



$$\underbrace{V(S)}_{\text{Value function}} := E \left(\sum_{n=0}^{\infty} \gamma^n R_n \mid S_0 = s \right)$$

$$= E \left(R_0 + \gamma \sum_{n=1}^{\infty} \gamma^n R_{n+1} \mid S_0 = s \right)$$

$$= \mathbb{E}(R_0 | S_0 = s) + \mathbb{E} \left(\gamma \sum_{n=0}^{\infty} R_{n+1} | S_0 = s \right)$$

Recall :

$$\mathbb{E} \left(\underbrace{\mathbb{E}(X|Y)}_{g(Y)} \right) = \mathbb{E}(X)$$

$$V(S_1)$$

$$\mathbb{E} \left(\mathbb{E} \left(\underbrace{\gamma \sum_{n=0}^{\infty} R_{n+1}}_{S_1 = S_1} \right) | S_0 = s \right)$$

$$S_1 = S_1$$

$$| S_0 = s \Big)$$

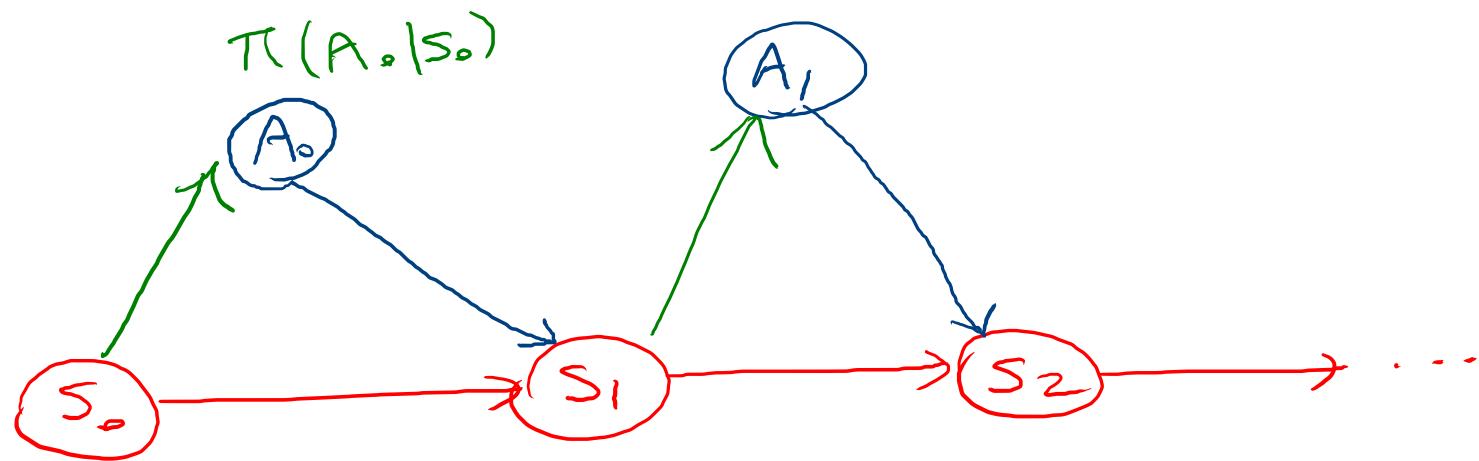
trans. prob.

$$= \mathbb{E} \left(R_0 + \gamma V(S_1) \mid S = S_0 \right)$$

$$V(S_0)$$

Bellman's Eq.

Markov Decision Process

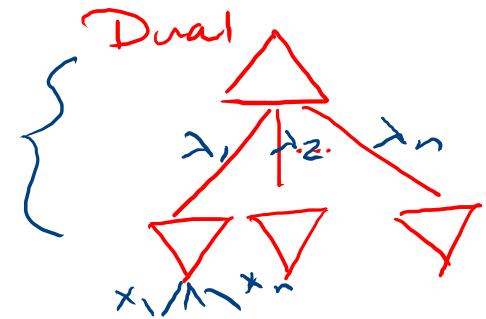


$$P(S_0, A_0, S_1, A_1, \dots) = \overbrace{P(S_0)}^{} \pi(A_0|S_0) \overbrace{P(S_1|S_0, A_0)}^{} \\ \pi(A_1|S_1) \cdot \overbrace{P(S_2|S_1, A_1)}^{} \dots$$

$$\max_{\pi} E \left\{ \sum \gamma^n R(s_n, A_n, s_{n+1}) \right\}$$

$$\min_{\alpha} f(\alpha)$$

$$\text{s.t. } \forall i \ g_i(\alpha) \leq 0$$



Primal

$$\min_{\alpha} \max_{\lambda_i \geq 0} f(\alpha) + \sum_{i=1}^n \lambda_i g_i(\alpha)$$

$$g_i(x) < 0 \\ \lambda_i^* = 0$$

$$0 = g_i(x) \\ \lambda_i^* \\ \text{could be pos.}$$

↙ ↘ Duality Gap

Dual

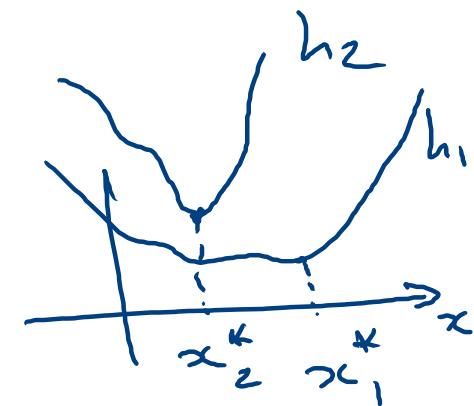
$$\max_{\lambda_i \geq 0} \min_{\alpha} f(\alpha) + \sum_{i=1}^n \lambda_i g_i(\alpha)$$

$$f(\alpha) + \sum_{i=1}^n \lambda_i g_i(\alpha)$$

$$0 = \lambda_i^* g_i(x^*)$$

λ_i^* 's are Lagrange Multipliers
of the optimal
The Dual problem.

$$\begin{aligned} & \max_{\lambda_i \geq 0} h_1(x) \\ & \sqrt{x} \min_x f(x) + \sum_{i=1}^n \lambda_i g_i(x) \leq h_1(x) \\ \rightarrow & \min_x \max_{\lambda_i \geq 0} f(x) + \sum_{i=1}^n \lambda_i g_i(x) \end{aligned}$$



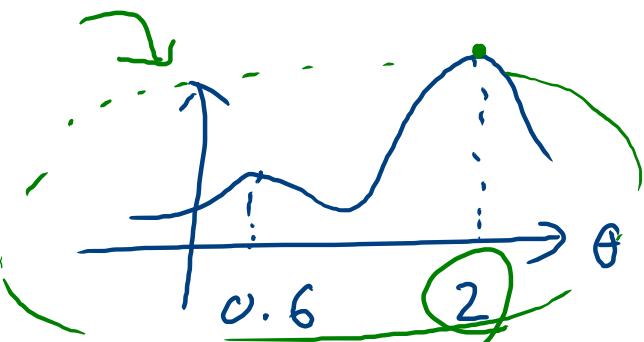
$$\forall x \quad h_1(x) \leq h_2(x) \Rightarrow h_1(x_1^*) \leq h_2(x_2^*)$$

How to estimate unknown params?

$$x_1, \dots, x_n \stackrel{iid}{\sim} f$$

$$\theta$$

↑
point estimation



$$P(\theta | x_1, \dots, x_n) \text{ full posterior}$$
$$P(\theta)$$

MLE :

$$\max_{\theta} P(x_1, \dots, x_n | \theta)$$

MAP

Bayesian

$$\theta \sim P_\theta$$

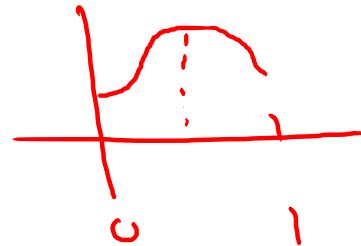
$$\max_{\theta} P(\theta | x_1, \dots, x_n)$$

Likelihood

$$\max_{\theta} \underbrace{P(\theta | x_1, \dots, x_n)}_{\cdot P(\theta)} = \frac{P(x_1, \dots, x_n | \theta)}{\underbrace{P(x_1, \dots, x_n)}_{\cdot}}$$

$$x_1, \dots, x_n \stackrel{iid}{\sim} \text{Bernoulli}(\theta)$$

$$\theta \sim \underbrace{C \theta^\alpha (1-\theta)^{1-\alpha}}_{0 \leq \theta \leq 1}$$

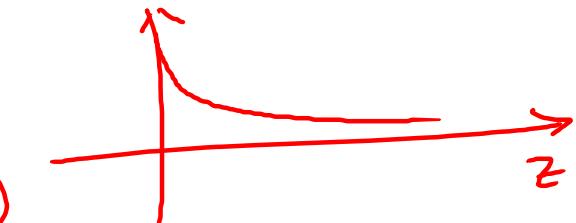


$$\begin{aligned}
 P(\theta | x_1, \dots, x_n) &\propto \underbrace{P(x_1, \dots, x_n | \theta)}_{\prod_{i=1}^n P(x_i | \theta)} \cdot \underbrace{P(\theta)}_{C \theta^\alpha (1-\theta)^{1-\alpha}} \\
 &= C \theta^{\sum x_i + \alpha} (1-\theta)^{n - \sum x_i + 1 - \alpha}
 \end{aligned}$$

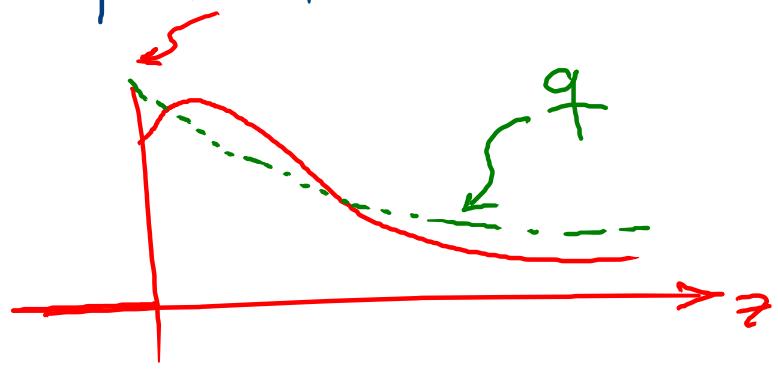
$$x_i \stackrel{iid}{\sim} \mathcal{N}(z; 1)$$

$$z \sim e^{-z} \mathbb{I}(z \geq 0)$$

$$\lambda e^{-\lambda z} \mathbb{I}(z \geq 0)$$

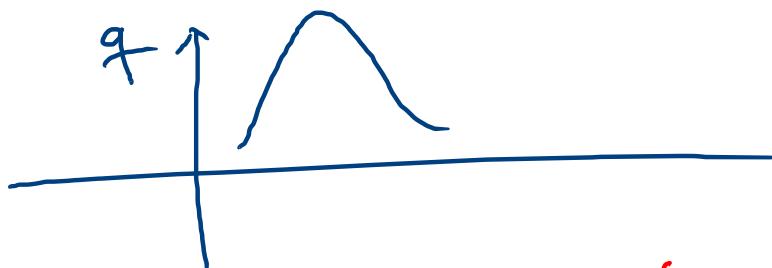
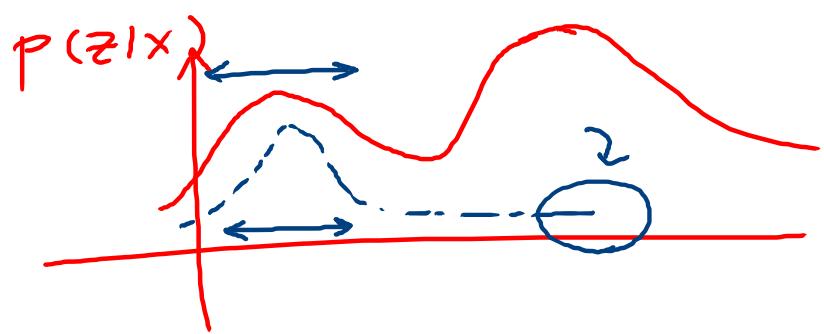


$$p(z|x_1) = p(x_1|z) \cdot p(z)$$

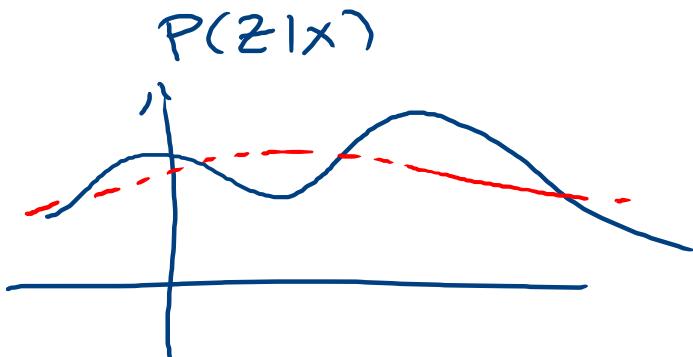


$$\frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-z)^2}{2}} \cdot e^{-z} \mathbb{I}(z \geq 0)}{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-z)^2}{2}} e^{-z} \mathbb{I}(z \geq 0)}$$

$$\min_q D(q \| p(z|x_1))$$



$$D(q \parallel p(z|x))$$



$$= \mathbb{E}_{\underbrace{z \sim q}_{}} \left[\log \frac{q}{p(z|x)} \right]$$

$$\log \left\{ C \theta^{\sum x_i + \alpha} (1-\theta)^{n - \sum x_i + 1 - \alpha} \right\}$$

~~$\log C$~~

$$+ (\sum x_i + \alpha) \log \theta + (n - \sum x_i + 1 - \alpha) \log(1-\theta)$$

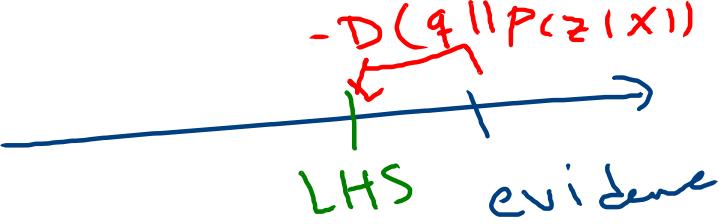
$$\frac{\sum x_i + \alpha}{\theta} + \frac{-(n - \sum x_i + 1 - \alpha)}{1-\theta} = 0$$

$$(1-\theta)(\sum x_i + \alpha) = \theta(n - \sum x_i + 1 - \alpha)$$

$$\sum x_i + \alpha = (n+1)\theta \Rightarrow$$

$$\theta = \frac{1}{n+1} \left\{ \sum x_i + \alpha \right\}$$

$$D(q \parallel P(z|x))$$



$$= \mathbb{E}_{\substack{z \sim q}} \left[\log \frac{q(z)}{\cancel{P(z|x)}} \right] + \cancel{P(z,x)/P(x)}$$

$$= \mathbb{E}_{\substack{z \sim q}} \left[\log \frac{q(z)}{P(x, z)} + \log P(x) \right]$$

a lower bound on $\log P(x)$ --- ELBO

$$= \mathbb{E}_{\substack{z \sim q}} \left[\log \frac{P(x, z)}{q(z)} \right] + \underbrace{\log P(x)}_{\text{evidence}}$$

$$- D(q \parallel P(z|x))$$

$$E(\text{BO}) = E_{z \sim q} \left[\log \frac{P(x, z)}{q(z)} \right]$$

proposal

$$q(z) = \theta e^{-\theta z} \mathbb{I}(z \geq 0)$$

$$\frac{-(x-z)^2}{2} - z$$

$$F_{\text{LBO}} = E_{z \sim q} \left[\log \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-z)^2}{2}} e^{-z} \mathbb{I}(z \geq 0) \right]$$

$$- \underbrace{\log \theta e^{-\theta z}}_{-\log \theta + \theta z} \mathbb{I}(z \geq 0)$$

$$\mathbb{E}_{z \sim q}(z) = \frac{1}{\theta}$$

$$\text{Var}(z) = \frac{1}{\theta^2}$$

$$\begin{aligned} \mathbb{E}_{z \sim q} & \left\{ -\frac{x^2 - z^2}{2} + zx - z - \log \theta + \theta z \right\} \\ \frac{\partial}{\partial \theta} & \left\{ -\frac{1}{\theta^2} + \frac{1}{\theta} - \frac{1}{\theta} - \log \theta + 1 \right\} = 0 \end{aligned}$$

$$\frac{+2}{\theta^3} - \frac{x}{\theta^2} + \frac{1}{\theta^2} - \frac{1}{\theta} = 0$$

$$2 - \theta x + \theta - \theta^2 = 0 \rightarrow \theta^2 + (x-1)\theta - 2 = 0$$

$$\theta = (-x) + \frac{\sqrt{(x-1)^2 + 8}}{2}$$