



Computer Engineering Department

Reinforcement Learning: Model Based RL

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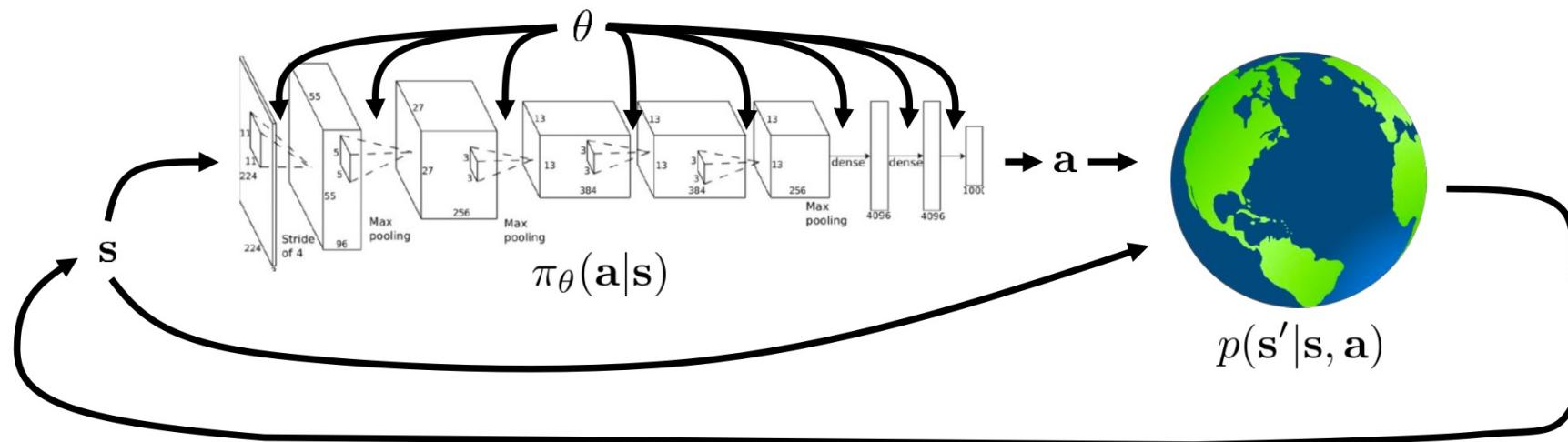
Spring 2023

Courtesy: Most of slides are adopted from CS 285 Berkeley.

Overview

- Introduction to model-based reinforcement learning
- What if we know the dynamics? How can we make decisions?
- Stochastic optimization methods
- Monte Carlo tree search (MCTS)
- Trajectory optimization
- Goal: Understand how we can perform planning with known dynamics models in discrete and continuous spaces

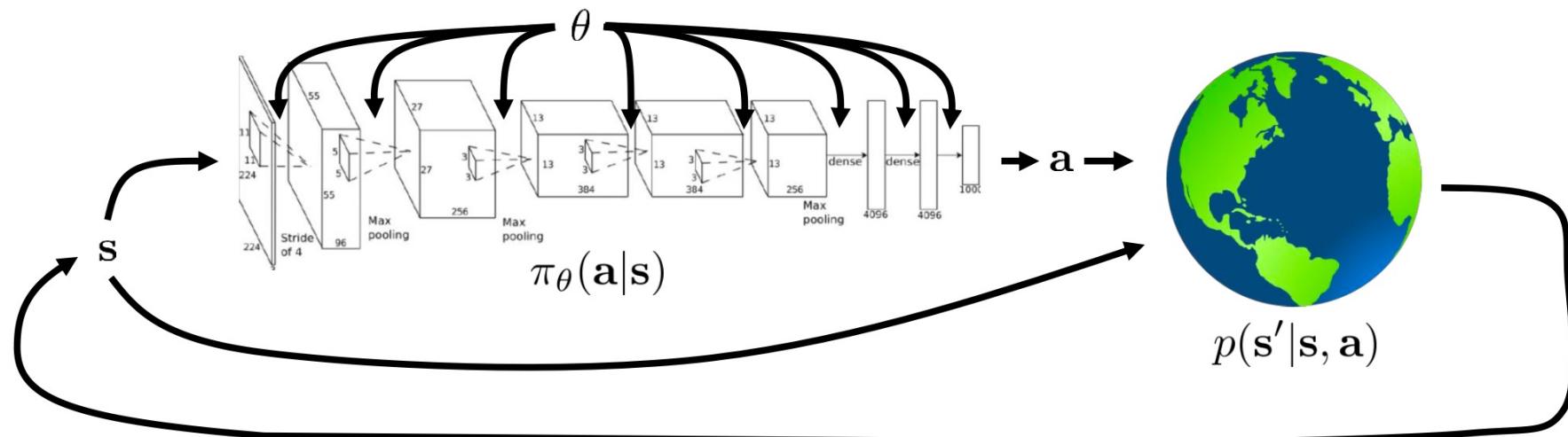
Recap: Model-Free RL



$$p_\theta(s_1, a_1, \dots, s_T, a_T) = \underbrace{p(s_1)}_{\pi_\theta(\tau)} \prod_{t=1}^T \pi_\theta(a_t | s_t) p(s_{t+1} | s_t, a_t)$$

$$\theta^\star = \arg \max_\theta E_{\tau \sim p_\theta(\tau)} \left[\sum_t r(s_t, a_t) \right]$$

Recap: Model-Free RL



$$p_\theta(s_1, a_1, \dots, s_T, a_T) = \underbrace{p(s_1)}_{\pi_\theta(\tau)} \prod_{t=1}^T \pi_\theta(a_t | s_t) \cancel{p(s_{t+1} | s_t, a_t)}$$

assume this is unknown
don't even attempt to learn it

$$\theta^\star = \arg \max_\theta E_{\tau \sim p_\theta(\tau)} \left[\sum_t r(s_t, a_t) \right]$$

What if we knew the transition dynamics?

- Often we do know the dynamics
 - Games (e.g., Atari games, chess, Go)
 - Easily modeled systems (e.g., navigating a car)
 - Simulated environments (e.g., simulated robots, video games)
- Often we can learn the dynamics
 - System identification – fit unknown parameters of a known model
 - Learning – fit a general-purpose model to observed transition data

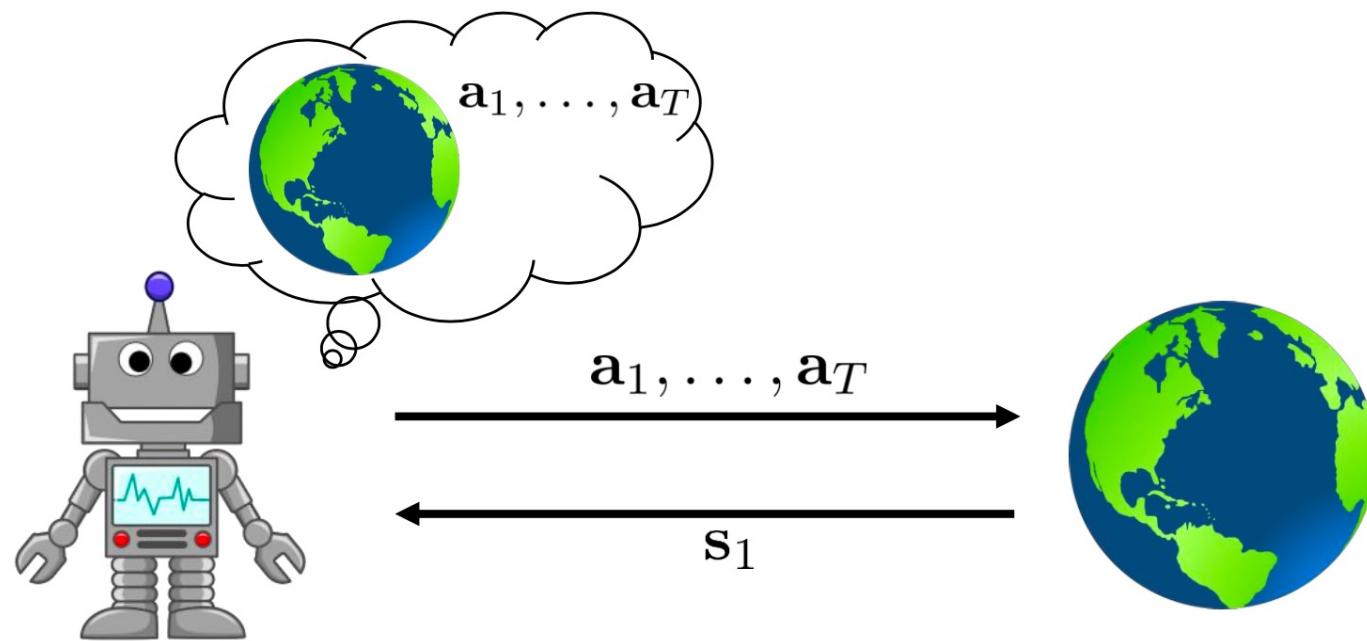
Does knowing the dynamics make things easier?

Often, yes!

Model-based RL

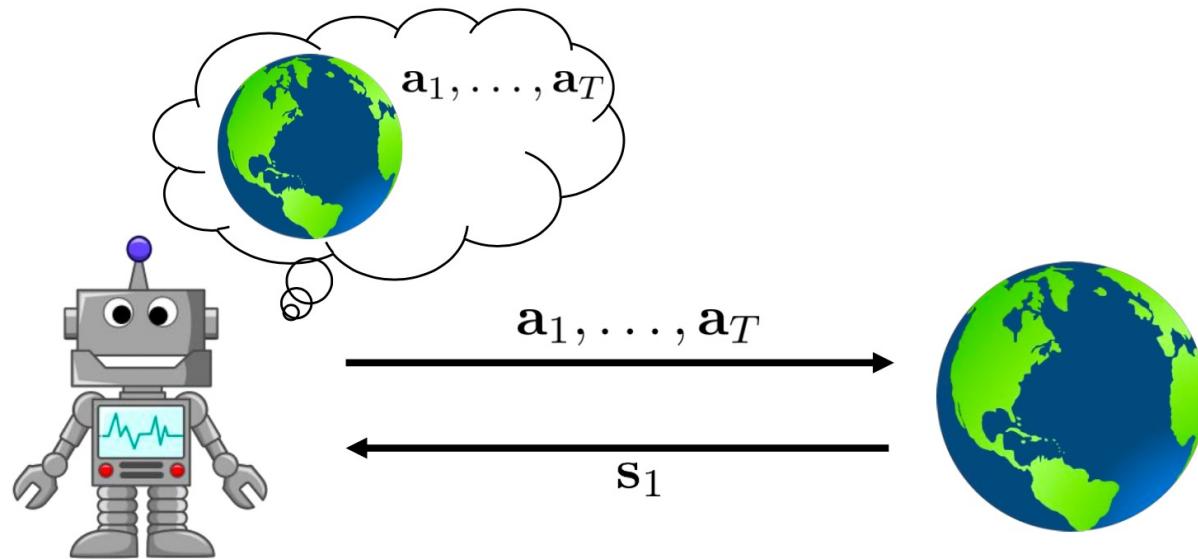
- Model-based reinforcement learning: learn the transition dynamics, then figure out how to choose actions.
- Today: how can we make decisions if we know the dynamics?
 - a. How can we choose actions under **perfect knowledge** of the system dynamics?
 - b. Optimal control, trajectory optimization, planning

The deterministic case



$$\mathbf{a}_1, \dots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \dots, \mathbf{a}_T} \sum_{t=1}^T r(\mathbf{s}_t, \mathbf{a}_t) \text{ s.t. } \mathbf{a}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t)$$

The stochastic open-loop case



$$p_{\theta}(\mathbf{s}_1, \dots, \mathbf{s}_T | \mathbf{a}_1, \dots, \mathbf{a}_T) = p(\mathbf{s}_1) \prod_{t=1}^T p(\mathbf{s}_{t+1} | \mathbf{s}_t, \mathbf{a}_t)$$

$$\mathbf{a}_1, \dots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \dots, \mathbf{a}_T} E \left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t) | \mathbf{a}_1, \dots, \mathbf{a}_T \right]$$

why is this suboptimal?

The stochastic open-loop case

کری می خواست به عیادت بیماری برود. اندیشید که هنگام احوال پرسی ممکن است صدای اورانشنوم و پاسخی ناشایسته بدهم. ازین رودپی چاره برآمد و بالاخره با خود گفت: بهتر است پرسشهارا پیش از رفتن بسنجم و پاسخ رانیزبرآورد کنم تا دچار اشتباہ نشوم.
بنابراین پرسشهای خود را چنین پیش بینی کرد:

- ابتدا زومی پرسم حالت بهتر است؟ او خواهد گفت "آری" من در جواب می گویم: خدا را شکر

- بعد از این پرسم چه خورده ای؟ لابد نام غذایی را خواهد آورد. من می گویم گوارا باد.

- در پایان می پرسم پزشکت کیست؟ نام پزشکی رامی گوید و من پاسخ می دهم: مقدمش مبارک باد.

.....

چون به خانه‌ی بیمار رسید همان گونه که از پیش آماده شده بود به احوال پرسی پرداخت:

- کر گفت: "چگونه ای؟"

بیمار گفت: مُردم

کر گفت: خدارا شکر

بیمار از این سخن بیجا برآشافت.

- بعد از آن پرسید: "چه خورده ای؟"

بیمار گفت: زهر

کر گفت: گوارابا داروی خوبی است.

بیمار از این پاسخ نیز بیشتر به خود پیچید.

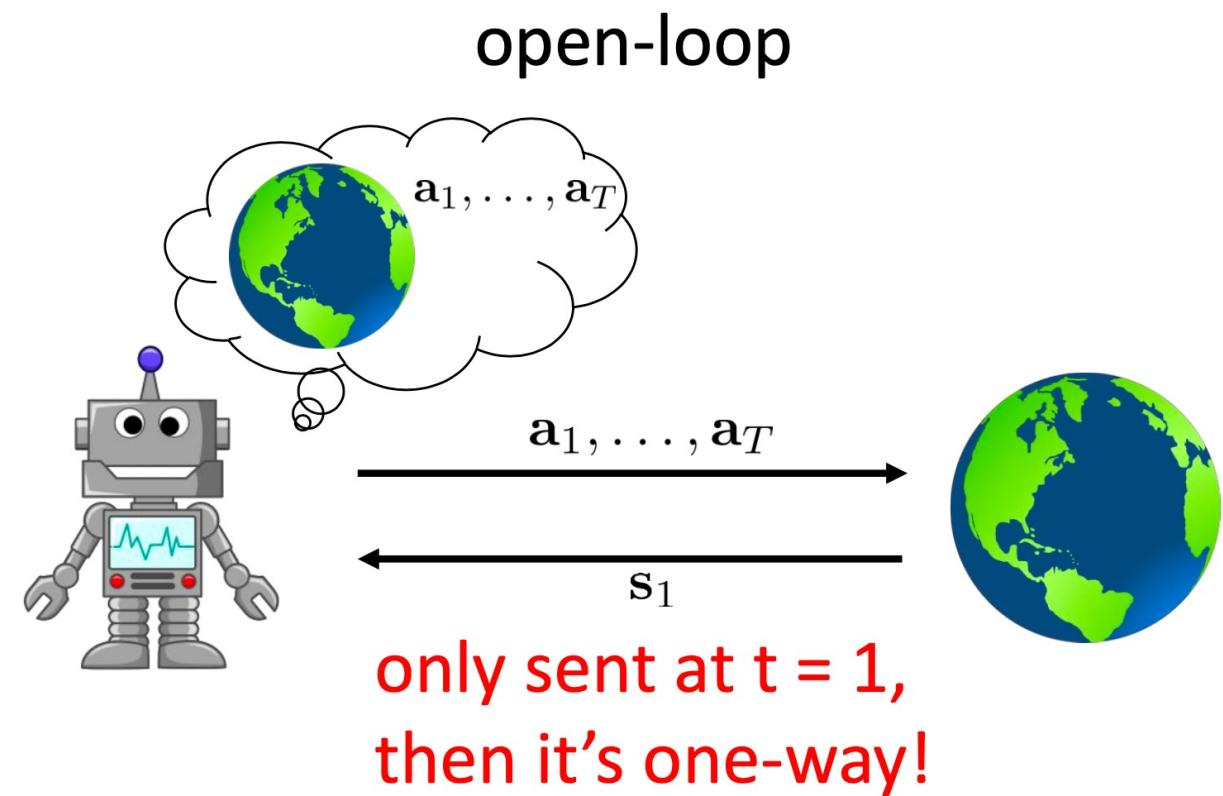
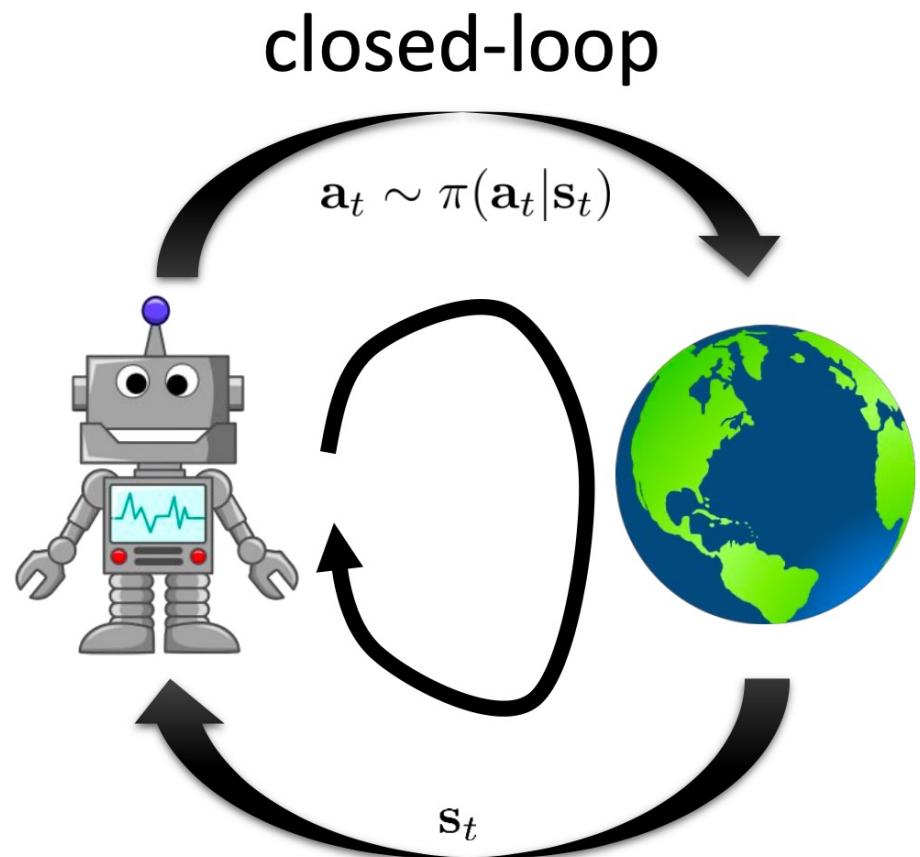
- بعد از آن کر گفت: "از طبیعت کیست او" کاوه‌می آید به چاره پیش تو؟"

بیمار که آشتفتگی و ناراحتی اش به نهایت رسیده بود در پاسخ گفت:

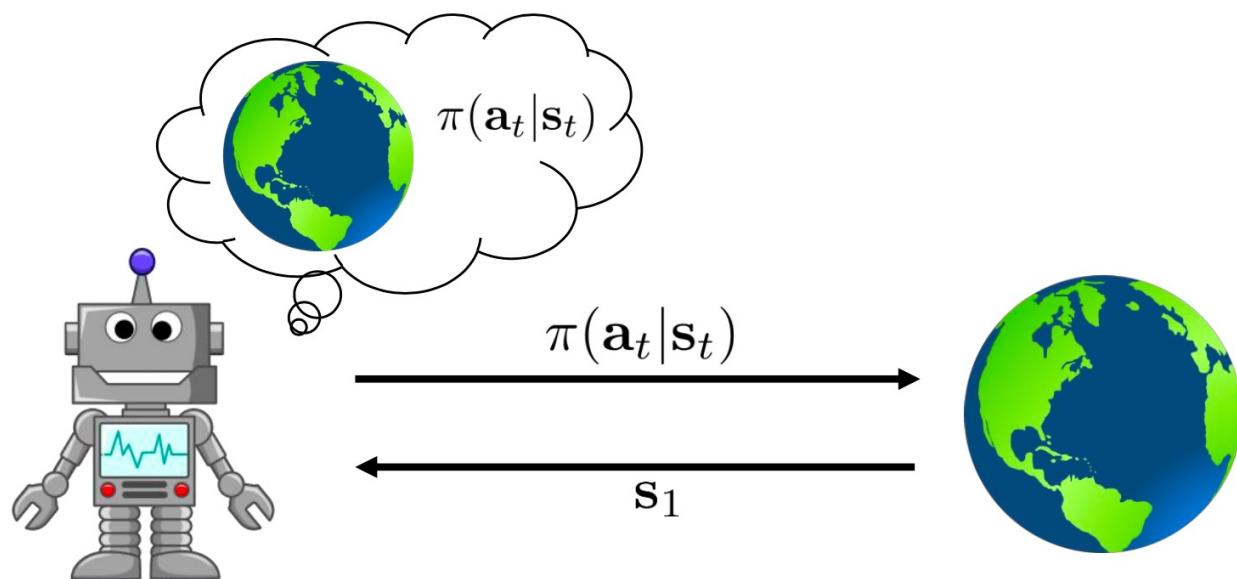
عذرایل می آید، برو.

کر گفت: پایش بس مبارک. شاد شو!

open-loop vs. closed-loop case



The stochastic open-loop case



$$p(s_1, a_1, \dots, s_T, a_T) = p(s_1) \prod_{t=1}^T \pi(a_t | s_t) p(s_{t+1} | s_t, a_t)$$

$$\pi = \arg \max_{\pi} E_{\tau \sim p(\tau)} \left[\sum_t r(s_t, a_t) \right]$$

form of π ?

neural net

time-varying linear

$$\mathbf{K}_t \mathbf{s}_t + \mathbf{k}_t$$

global

local

Stochastic optimization

abstract away optimal control/planning:

$$\mathbf{a}_1, \dots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \dots, \mathbf{a}_T} J(\underbrace{\mathbf{a}_1, \dots, \mathbf{a}_T}_{})$$

$$\mathbf{A} = \arg \max_{\mathbf{A}} J(\mathbf{A})$$

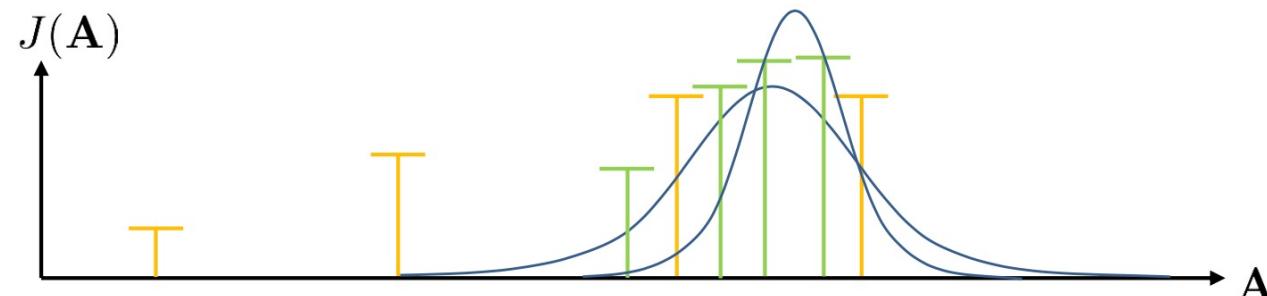
don't care what this is

simplest method: guess & check “random shooting method”

1. pick $\mathbf{A}_1, \dots, \mathbf{A}_N$ from some distribution (e.g., uniform)
2. choose \mathbf{A}_i based on $\arg \max_i J(\mathbf{A}_i)$

Cross-entropy Method (CEM)

1. pick $\mathbf{A}_1, \dots, \mathbf{A}_N$ from some distribution (e.g., uniform)
2. choose \mathbf{A}_i based on $\arg \max_i J(\mathbf{A}_i)$ **can we do better?**



cross-entropy method with continuous-valued inputs:

- 1. sample $\mathbf{A}_1, \dots, \mathbf{A}_N$ from $p(\mathbf{A})$
- 2. evaluate $J(\mathbf{A}_1), \dots, J(\mathbf{A}_N)$
- 3. pick the *elites* $\mathbf{A}_{i_1}, \dots, \mathbf{A}_{i_M}$ with the highest value, where $M < N$
- 4. refit $p(\mathbf{A})$ to the elites $\mathbf{A}_{i_1}, \dots, \mathbf{A}_{i_M}$

Pros and Cons

- Pros
 - Could be very fast (Parallelizable)
 - Extremely simple
- Cons
 - Very harsh dimensionality limit
 - Only open-loop planning

Discrete Case: Monte Carlo Tree Search

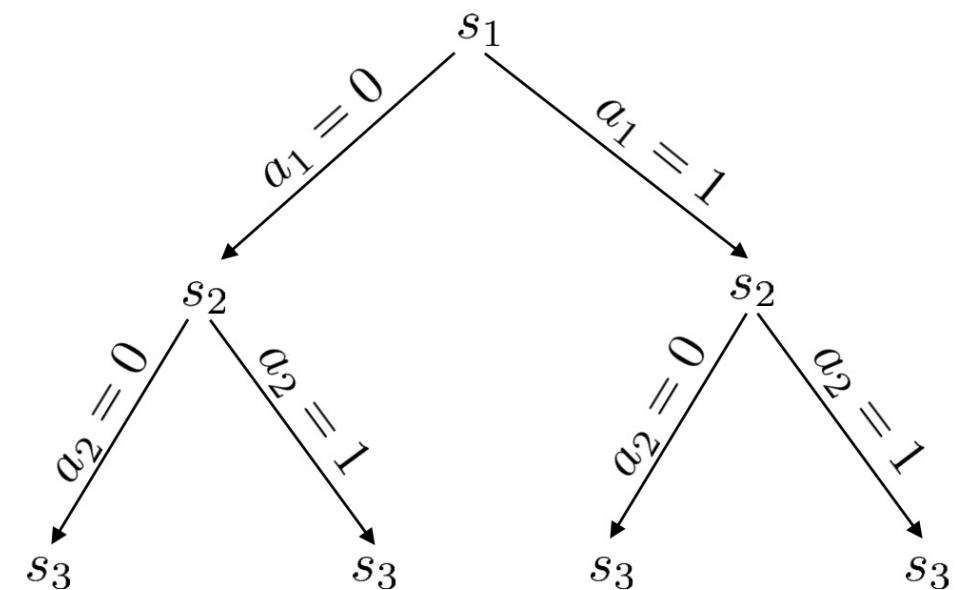


s_t



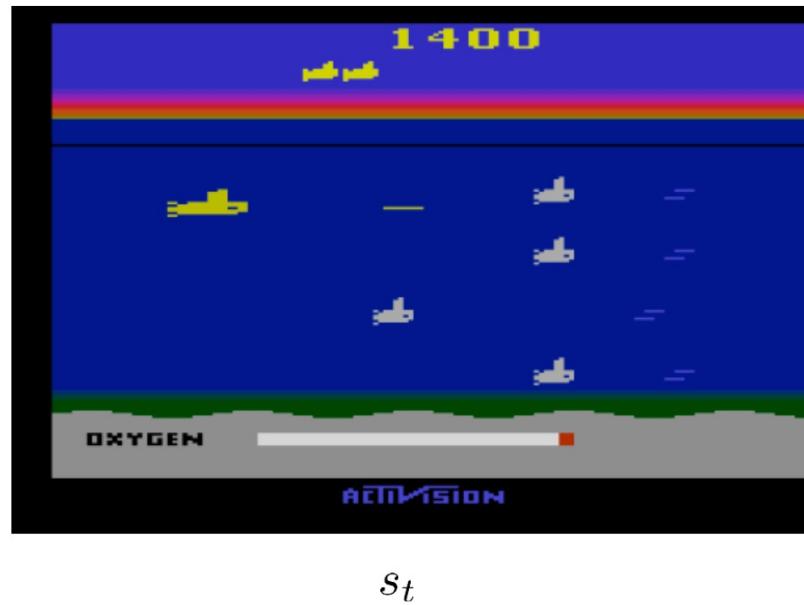
a_t

discrete planning as a search problem

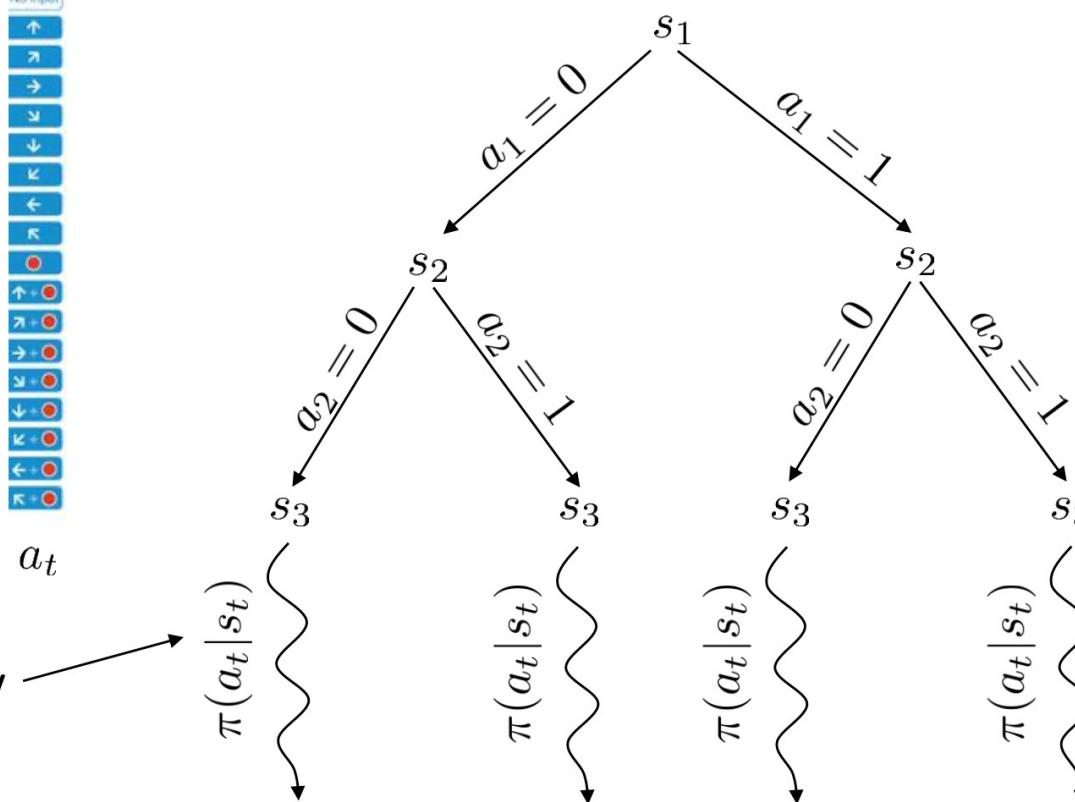


Discrete Case: Monte Carlo Tree Search

how to approximate value without full tree?



e.g., random policy



Discrete Case: Monte Carlo Tree Search

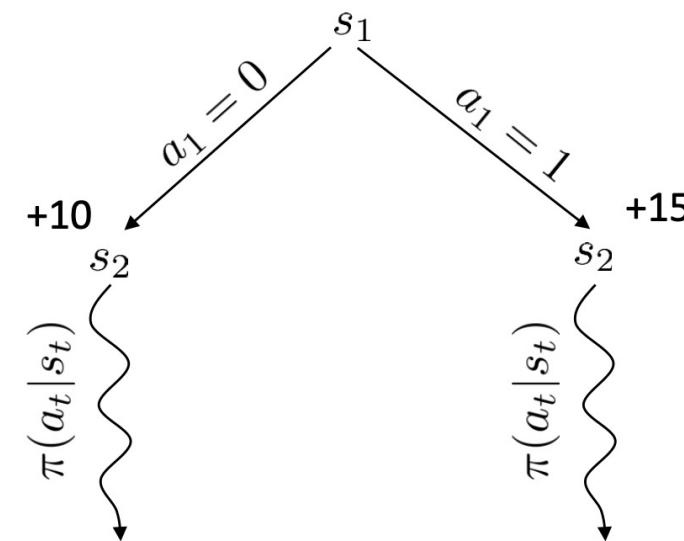
can't search all paths – where to search first?



s_t



a_t



intuition: choose nodes with best reward, but also prefer rarely visited nodes

Discrete Case: Monte Carlo Tree Search

generic MCTS sketch

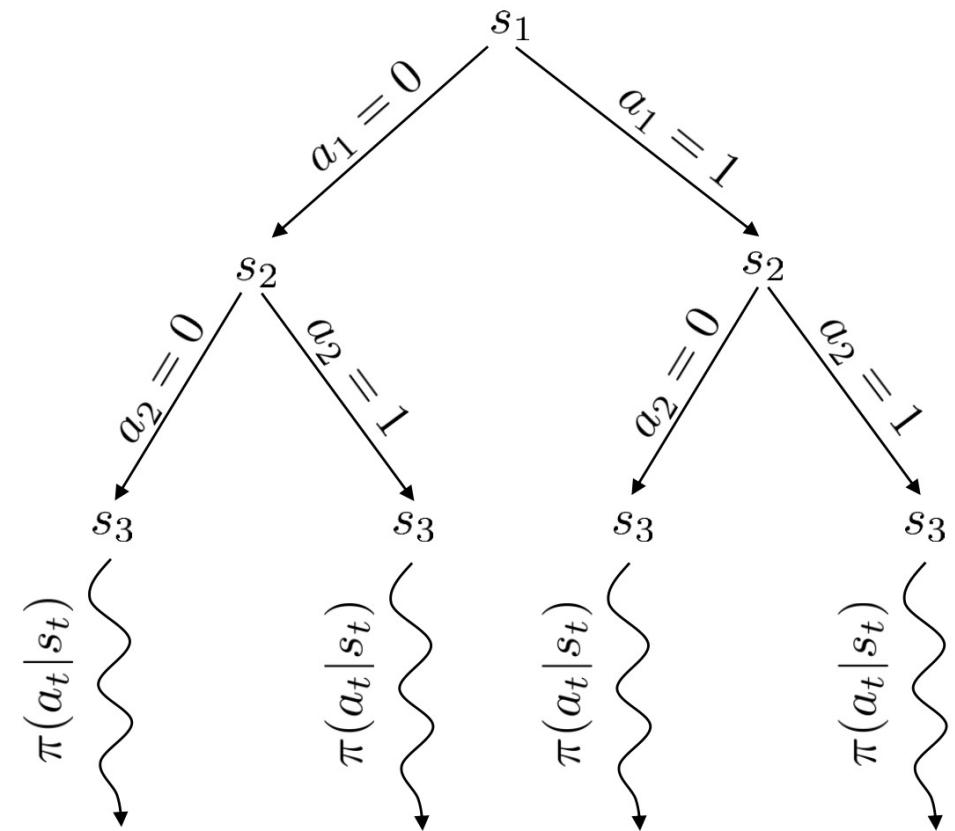
- 1. find a leaf s_l using $\text{TreePolicy}(s_1)$
- 2. evaluate the leaf using $\text{DefaultPolicy}(s_l)$
- 3. update all values in tree between s_1 and s_l

take best action from s_1

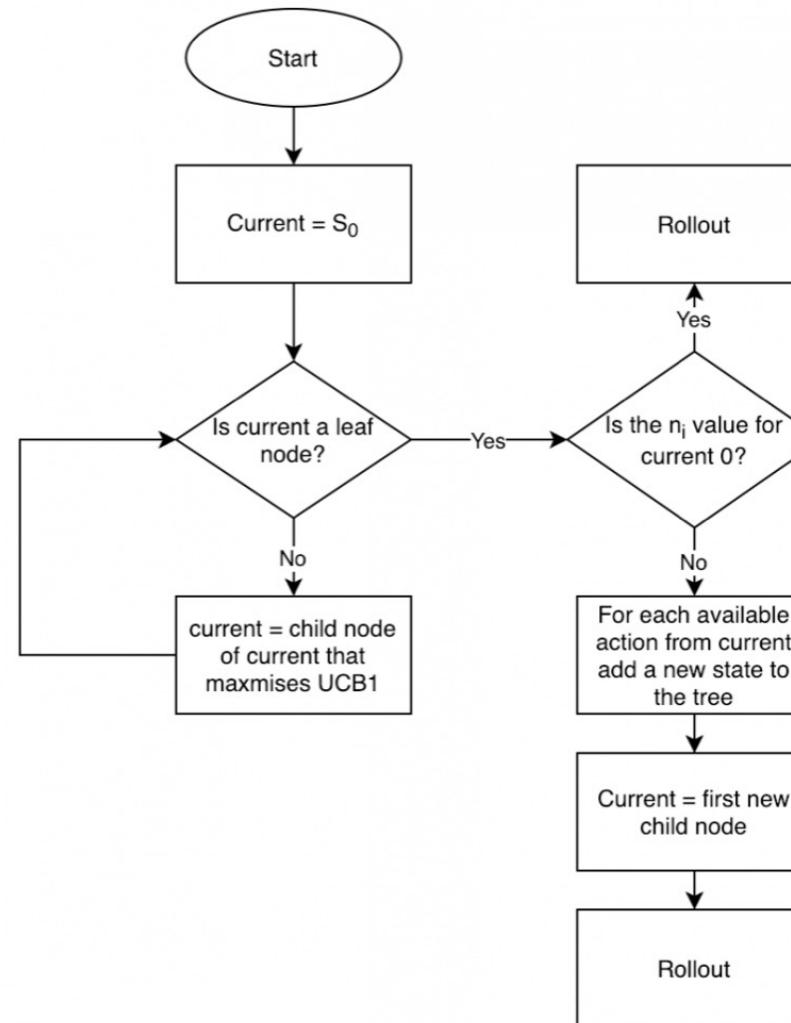
UCT TreePolicy(s_t)

if s_t not fully expanded, choose new a_t
else choose child with best Score(s_{t+1})

$$\text{Score}(s_t) = \frac{Q(s_t)}{N(s_t)} + 2C \sqrt{\frac{2 \ln N(s_{t-1})}{N(s_t)}}$$



Discrete Case: Monte Carlo Tree Search



Additional reading

- Browne, Powley, Whitehouse, Lucas, Cowling, Rohlfshagen, Tavener, Perez, Samothrakis, Colton. (2012). A Survey of Monte Carlo Tree Search Methods.
 - Survey of MCTS methods and basic summary.

Trajectory Optimization: Can we use derivatives?

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} \sum_{t=1}^T c(\mathbf{x}_t, \mathbf{u}_t) \text{ s.t. } \mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})$$

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots)), \mathbf{u}_T)$$

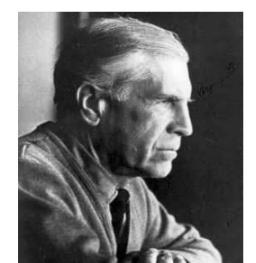
usual story: differentiate via backpropagation and optimize!

need $\frac{df}{d\mathbf{x}_t}, \frac{df}{d\mathbf{u}_t}, \frac{dc}{d\mathbf{x}_t}, \frac{dc}{d\mathbf{u}_t}$

\mathbf{s}_t – state
 \mathbf{a}_t – action

\mathbf{x}_t – state
 \mathbf{u}_t – action

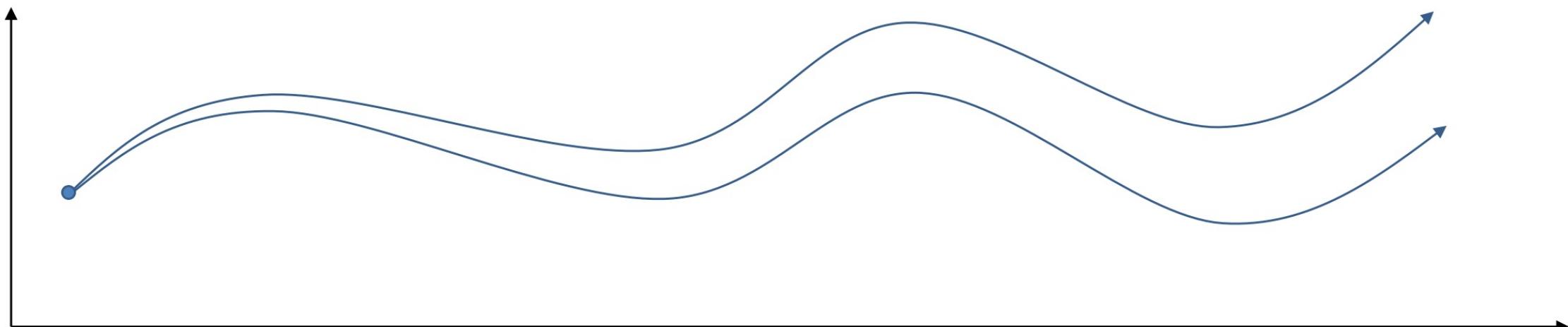
in practice, it really helps to use a 2nd order method!



Shooting methods vs collocation

shooting method: optimize over actions only

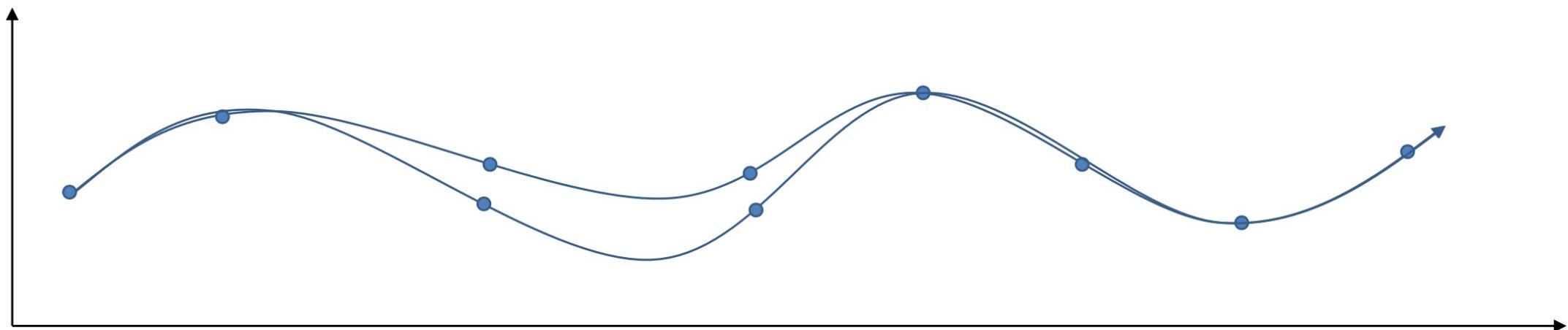
$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots) \dots), \mathbf{u}_T)$$



Shooting methods vs collocation

collocation method: optimize over actions and states, with constraints

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T, \mathbf{x}_1, \dots, \mathbf{x}_T} \sum_{t=1}^T c(\mathbf{x}_t, \mathbf{u}_t) \text{ s.t. } \mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})$$



Linear case: LQR

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots) \dots), \mathbf{u}_T)$$

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

linear

$$c(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

quadratic

Linear case: LQR

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + \underbrace{c(f(f(\dots) \dots), \mathbf{u}_T)}_{\mathbf{x}_T \text{ (unknown)}}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

only term that depends on \mathbf{u}_T

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

Base case: solve for \mathbf{u}_T only

$$\mathbf{C}_T = \begin{bmatrix} \mathbf{C}_{\mathbf{x}_T, \mathbf{x}_T} & \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \\ \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} & \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \end{bmatrix}$$

$$Q(\mathbf{x}_T, \mathbf{u}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{c}_T$$

$$\mathbf{c}_T = \begin{bmatrix} \mathbf{c}_{\mathbf{x}_T} \\ \mathbf{c}_{\mathbf{u}_T} \end{bmatrix}$$

$$\nabla_{\mathbf{u}_T} Q(\mathbf{x}_T, \mathbf{u}_T) = \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} \mathbf{x}_T + \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{u}_T + \mathbf{c}_{\mathbf{u}_T}^T = 0$$

$$\mathbf{K}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T}$$

$$\mathbf{u}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} (\mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} \mathbf{x}_T + \mathbf{c}_{\mathbf{u}_T})$$

$$\mathbf{u}_T = \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T$$

$$\mathbf{k}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{c}_{\mathbf{u}_T}$$

Linear Case: LQR

$$\mathbf{u}_T = \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T$$

$$\mathbf{K}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T}$$

$$\mathbf{k}_T = -\mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T}^{-1} \mathbf{c}_{\mathbf{u}_T}$$

$$Q(\mathbf{x}_T, \mathbf{u}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{c}_T$$

Since \mathbf{u}_T is fully determined by \mathbf{x}_T , we can eliminate it via substitution!

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix}^T \mathbf{c}_T$$

$$\begin{aligned} V(\mathbf{x}_T) &= \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T, \mathbf{x}_T} \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{K}_T \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{K}_T \mathbf{x}_T + \\ &\quad \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{k}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{k}_T + \mathbf{x}_T^T \mathbf{c}_{\mathbf{x}_T} + \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{c}_{\mathbf{u}_T} + \text{const} \end{aligned}$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \mathbf{x}_T^T \mathbf{V}_T \mathbf{x}_T + \mathbf{x}_T^T \mathbf{v}_T$$

$$\mathbf{V}_T = \mathbf{C}_{\mathbf{x}_T, \mathbf{x}_T} + \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{K}_T + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{x}_T} + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{K}_T$$

$$\mathbf{v}_T = \mathbf{c}_{\mathbf{x}_T} + \mathbf{C}_{\mathbf{x}_T, \mathbf{u}_T} \mathbf{k}_T + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T} + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T, \mathbf{u}_T} \mathbf{k}_T$$

Linear Case: LQR

Solve for \mathbf{u}_{T-1} in terms of \mathbf{x}_{T-1}

\mathbf{u}_{T-1} affects \mathbf{x}_T !

$$f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \mathbf{x}_T = \mathbf{F}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \mathbf{f}_{T-1}$$

$$Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{C}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{c}_{T-1} + V(f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}))$$

$$\overbrace{\quad\quad\quad}^{V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \mathbf{x}_T^T \mathbf{V}_T \mathbf{x}_T + \mathbf{x}_T^T \mathbf{v}_T}$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{F}_{T-1}}_{\text{quadratic}} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{f}_{T-1}}_{\text{linear}} + \underbrace{\begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{F}_{T-1}^T \mathbf{v}_T}_{\text{linear}}$$

Linear Case: LQR

$$Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{C}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{c}_{T-1} + V(f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}))$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{F}_{T-1}}_{\text{quadratic}} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{f}_{T-1}}_{\text{linear}} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{v}_T}_{\text{linear}}$$

$$Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{Q}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{q}_{T-1}$$

$$\mathbf{Q}_{T-1} = \mathbf{C}_{T-1} + \mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{F}_{T-1}$$

$$\mathbf{q}_{T-1} = \mathbf{c}_{T-1} + \mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{f}_{T-1} + \mathbf{F}_{T-1}^T \mathbf{v}_T$$

$$\nabla_{\mathbf{u}_{T-1}} Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{x}_{T-1}} \mathbf{x}_{T-1} + \mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{u}_{T-1}} \mathbf{u}_{T-1} + \mathbf{q}_{\mathbf{u}_{T-1}}^T = 0$$

$$\mathbf{u}_{T-1} = \mathbf{K}_{T-1} \mathbf{x}_{T-1} + \mathbf{k}_{T-1} \quad \mathbf{K}_{T-1} = -\mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{u}_{T-1}}^{-1} \mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{x}_{T-1}}$$

$$\mathbf{k}_{T-1} = -\mathbf{Q}_{\mathbf{u}_{T-1}, \mathbf{u}_{T-1}}^{-1} \mathbf{q}_{\mathbf{u}_{T-1}}$$

Linear Case: LQR

Backward recursion

for $t = T$ to 1:

$$\mathbf{Q}_t = \mathbf{C}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{F}_t$$

$$\mathbf{q}_t = \mathbf{c}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{f}_t + \mathbf{F}_t^T \mathbf{v}_{t+1}$$

$$Q(\mathbf{x}_t, \mathbf{u}_t) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{Q}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{q}_t$$

$$\mathbf{u}_t \leftarrow \arg \min_{\mathbf{u}_t} Q(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

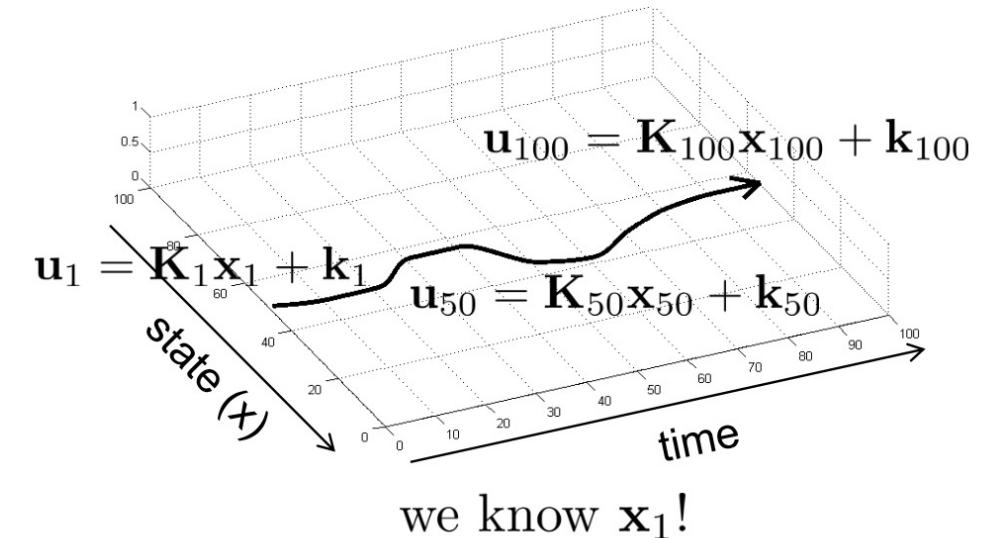
$$\mathbf{K}_t = -\mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t}^{-1} \mathbf{Q}_{\mathbf{u}_t, \mathbf{x}_t}$$

$$\mathbf{k}_t = -\mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t}^{-1} \mathbf{q}_{\mathbf{u}_t}$$

$$\mathbf{V}_t = \mathbf{Q}_{\mathbf{x}_t, \mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t, \mathbf{u}_t} \mathbf{K}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{x}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t} \mathbf{K}_t$$

$$\mathbf{v}_t = \mathbf{q}_{\mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t, \mathbf{u}_t} \mathbf{k}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t, \mathbf{u}_t} \mathbf{k}_t$$

$$V(\mathbf{x}_t) = \text{const} + \frac{1}{2} \mathbf{x}_t^T \mathbf{V}_t \mathbf{x}_t + \mathbf{x}_t^T \mathbf{v}_t$$



Forward recursion

for $t = 1$ to T :

$$\mathbf{u}_t = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$$

LQR for Stochastic Dynamics

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

$$\mathbf{x}_{t+1} \sim p(\mathbf{x}_{t+1} | \mathbf{x}_t, \mathbf{u}_t)$$

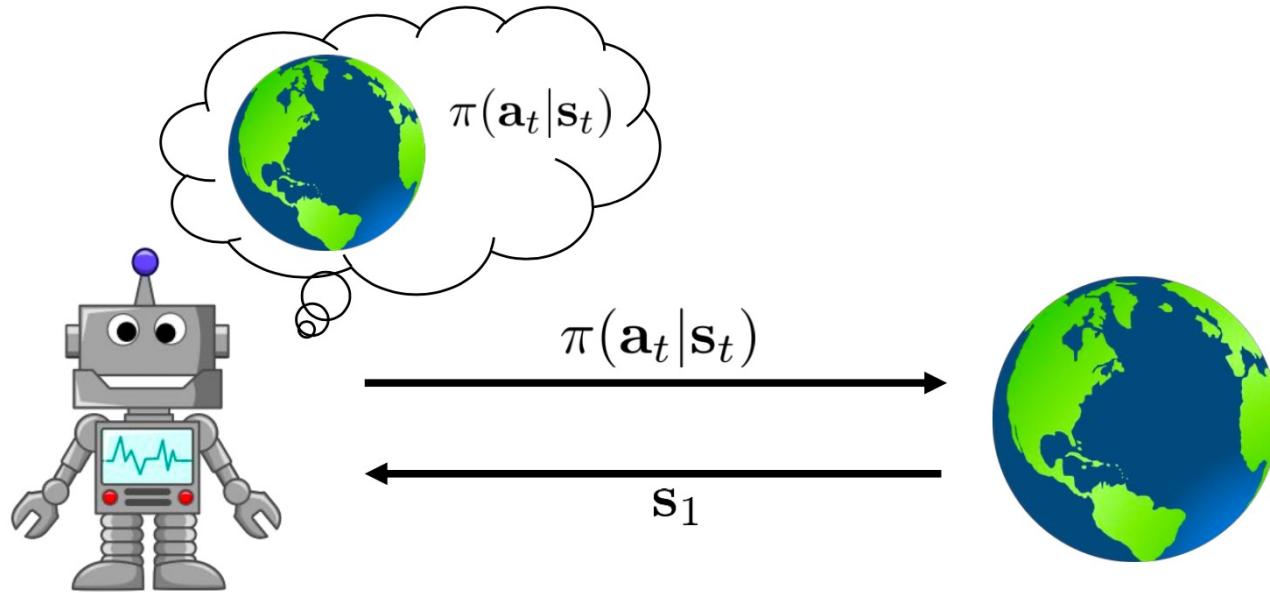
$$p(\mathbf{x}_{t+1} | \mathbf{x}_t, \mathbf{u}_t) = \mathcal{N} \left(\mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t, \Sigma_t \right)$$

Solution: choose actions according to $\mathbf{u}_t = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$

$\mathbf{x}_t \sim p(\mathbf{x}_t)$, no longer deterministic, but $p(\mathbf{x}_t)$ is Gaussian

no change to algorithm! can ignore Σ_t due to symmetry of Gaussians
(checking this is left as an exercise; hint: the expectation of a quadratic under a Gaussian has an analytic solution)

The stochastic closed-loop case



form of π ?

$$p(\mathbf{s}_1, \mathbf{a}_1, \dots, \mathbf{s}_T, \mathbf{a}_T) = p(\mathbf{s}_1) \prod_{t=1}^T \pi(\mathbf{a}_t | \mathbf{s}_t) p(\mathbf{s}_{t+1} | \mathbf{s}_t, \mathbf{a}_t)$$

time-varying linear
 $\mathbf{K}_t \mathbf{s}_t + \mathbf{k}_t$

$$\pi = \arg \max_{\pi} E_{\tau \sim p(\tau)} \left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t) \right]$$

Nonlinear case: DDP/iterative LQR

Linear-quadratic assumptions:

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t \quad c(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

Can we *approximate* a nonlinear system as a linear-quadratic system?

$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) \approx c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}^T \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

Nonlinear case: DDP/iterative LQR

$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) \approx c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}^T \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$\bar{f}(\delta \mathbf{x}_t, \delta \mathbf{u}_t) = \underbrace{\mathbf{F}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)} \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} \quad \bar{c}(\delta \mathbf{x}_t, \delta \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}^T \underbrace{\mathbf{C}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)} \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}^T \underbrace{\mathbf{c}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)}$$

$$\delta \mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$$

$$\delta \mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$$

Now we can run LQR with dynamics \bar{f} , cost \bar{c} , state $\delta \mathbf{x}_t$, and action $\delta \mathbf{u}_t$

Nonlinear case: DDP/iterative LQR

Iterative LQR (simplified pseudocode)

until convergence:

$$\mathbf{F}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{c}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{C}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

Run LQR backward pass on state $\delta \mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$ and action $\delta \mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$

Run forward pass with real nonlinear dynamics and $\mathbf{u}_t = \mathbf{K}_t(\mathbf{x}_t - \hat{\mathbf{x}}_t) + \mathbf{k}_t + \hat{\mathbf{u}}_t$

Update $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{u}}_t$ based on states and actions in forward pass

Nonlinear case: DDP/iterative LQR

Why does this work?

Compare to Newton's method for computing $\min_{\mathbf{x}} g(\mathbf{x})$:

until convergence:

$$\mathbf{g} = \nabla_{\mathbf{x}} g(\hat{\mathbf{x}})$$

$$\mathbf{H} = \nabla_{\mathbf{x}}^2 g(\hat{\mathbf{x}})$$

$$\hat{\mathbf{x}} \leftarrow \arg \min_{\mathbf{x}} \frac{1}{2} (\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{H} (\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{g}^T (\mathbf{x} - \hat{\mathbf{x}})$$

Iterative LQR (iLQR) is the same idea: locally approximate a complex nonlinear function via Taylor expansion

In fact, iLQR is an approximation of Newton's method for solving

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots)), \mathbf{u}_T)$$

Nonlinear case: DDP/iterative LQR

In fact, iLQR is an approximation of Newton's method for solving

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots) \dots), \mathbf{u}_T)$$

To get Newton's method, need to use *second order* dynamics approximation:

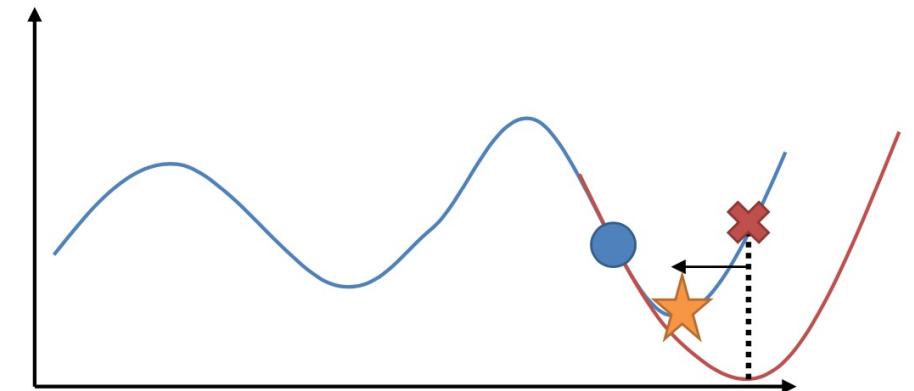
$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} + \frac{1}{2} \left(\nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \cdot \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} \right) \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}$$

differential dynamic programming (DDP)

Nonlinear case: DDP/iterative LQR

$$\hat{\mathbf{x}} \leftarrow \arg \min_{\mathbf{x}} \frac{1}{2} (\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{H} (\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{g}^T (\mathbf{x} - \hat{\mathbf{x}})$$

why is this a bad idea?



until convergence:

$$\mathbf{F}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{c}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{C}_t = \nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

Run LQR backward pass on state $\delta \mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$ and action $\delta \mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$

Run forward pass with $\mathbf{u}_t = \mathbf{K}_t(\mathbf{x}_t - \hat{\mathbf{x}}_t) + \alpha \mathbf{k}_t + \hat{\mathbf{u}}_t$

Update $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{u}}_t$ based on states and actions in forward pass

Nonlinear case: DDP/iterative LQR

In fact, iLQR is an approximation of Newton's method for solving

$$\min_{\mathbf{u}_1, \dots, \mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \dots + c(f(f(\dots) \dots), \mathbf{u}_T)$$

To get Newton's method, need to use *second order* dynamics approximation:

$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} + \frac{1}{2} \left(\nabla_{\mathbf{x}_t, \mathbf{u}_t}^2 f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \cdot \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix} \right) \begin{bmatrix} \delta \mathbf{x}_t \\ \delta \mathbf{u}_t \end{bmatrix}$$

differential dynamic programming (DDP)

Additional Reading

- Mayne, Jacobson. (1970). Differential dynamic programming.
 - Original differential dynamic programming algorithm.
- Tassa, Erez, Todorov. (2012). Synthesis and Stabilization of Complex Behaviors through Online Trajectory Optimization.
 - Practical guide for implementing non-linear iterative LQR.
- Levine, Abbeel. (2014). Learning Neural Network Policies with Guided Policy Search under Unknown Dynamics.
 - Probabilistic formulation and trust region alternative to deterministic line search.