Assignment 1 (ML for TS) - MVA 2021/2022

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1 Introduction

Objective. The goal is to learn to apply the convolutional dictionary learning procedure and the dynamic time warping distance on the real medical application.

Warning and advice.

- Use code from the tutorials as well as from other sources. Do not code yourself well-known procedures (e.g. cross validation or k-means), use an existing implementation.
- The associated notebook contains some hints and several helper functions.
- Be concise. Answers are not expected to be longer than a few sentences (omitting calculations).

Instructions.

- Fill in your names and emails at the top of the document.
- Hand in your report (one per pair of students) by Friday 28$^{\text{th}}$ January 11:59 PM.
- Rename your report and notebook as follows:
  `FirstnameLastname1_FirstnameLastname2.pdf` and
  `FirstnameLastname1_FirstnameLastname2.ipynb`.
  For instance, `LaurentOudre_CharlesTruong.pdf`.
- Upload your report (PDF file) and notebook (IPYNB file) using this link: dropbox.com/request/IgDvMyznSh4fco8J8slk.

2 General questions

Question 1

Consider the following Lasso regression:

$$\min_{\beta \in \mathbb{R}^p} \frac{1}{2} \|y - X\beta\|_2^2 \quad + \quad \lambda \|\beta\|_1 \tag{1}$$

where $y \in \mathbb{R}^n$ is the response vector, $X \in \mathbb{R}^{n \times p}$ the design matrix, $\beta \in \mathbb{R}^p$ the vector of regressors and $\lambda > 0$ the smoothing parameter.

Show that there exists $\lambda_{\max}$ such that the minimizer of (**??**) is $\mathbf{0}_p$ (a p-dimensional vector of zeros) for any $\lambda > \lambda_{\max}$.

Answer 1

We want to minimize $\frac{1}{2}\|y - X\beta\|_2^2 \quad + \quad \lambda\|\beta\|_1$. Then with first order condition $-X^T(y - X\beta) + \lambda z_\beta = 0(2)$ with $z_\beta = sign(\beta)$ if $\beta \neq 0_p$ and $z_\beta \in [-1, 1]$ if $\beta = 0_p$.

Plugging $\beta = 0_p$ in the first order condition, we get $X^T y = \lambda z_\beta$ and $||X^T y||_\infty = \lambda||z_\beta||_\infty$. If $||z_\beta||_\infty < 1$, we can decrease λ and $||z_\beta||_\infty$ will increase up to 1. If $||z_\beta||_\infty = 1$, we have $\lambda_{max} = ||X^T y||_\infty$.

$$\lambda_{\max} = ||X^T y||_\infty \tag{2}$$

Question 2

For a univariate signal $\mathbf{x} \in \mathbb{R}^n$ with n samples, the convolutional dictionary learning task amounts to solving the following optimization problem:

$$\min_{(\mathbf{d}_k)_k, (\mathbf{z}_k)_k \|\mathbf{d}_k\|_2^2 \leq 1} \quad \left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2 \quad + \quad \lambda \sum_{k=1}^{K} \|\mathbf{z}_k\|_1 \tag{3}$$

where $\mathbf{d}_k \in \mathbb{R}^L$ are the K dictionary atoms (patterns), $\mathbf{z}_k \in \mathbb{R}^{N-L+1}$ are activations signals, and $\lambda > 0$ is the smoothing parameter.

Show that

- for a fixed dictionary, the sparse coding problem is a lasso regression (explicit the response vector and the design matrix);
- for a fixed dictionary, there exists $\lambda_{\max}$ (which depends on the dictionary) such that the sparse codes are only 0 for any $\lambda > \lambda_{\max}$.

Answer 2

We suppose the dictionnary fixed. $\forall k$, we can express the convolution $\mathbf{z}_k * d_k$ as the scalar product $\mathbf{z}_k * D_k$ with $D_k \in \mathcal{M}_{N,N-L+1}$.

In order to wite the sum as a single scalar product, let's define the concatenated versions:

$$z = (\mathbf{z}_1 ... \mathbf{z}_K)$$

and:

$$D = (D_1 ... D_K)$$

Thus: $D \in \mathcal{M}_{N,K\times(L-N+1)}$ an $\mathbf{z} \in \mathbb{R}^{(L-N+1)\times K}$

And because we have:

$$\sum_{k=1}^{K} \|\mathbf{z}_k\|_1 = \|\mathbf{z}\|_1$$

We obtain the following problem form:

$$\min_{\mathbf{z}} \quad \|\mathbf{x} - D\mathbf{z}\|_2^2 \quad + \quad \lambda \, \|\mathbf{z}\|_1 \tag{4}$$

Then, according to question 1 :

$$\lambda_{\max} = 2||D^T x||_\infty \tag{5}$$

3 Data study

3.1 General information

Context. The study of human gait is a central problem in medical research with far-reaching consequences in the public health domain. This complex mechanism can be altered by a wide range of pathologies (such as Parkinson's disease, arthritis, stroke,...), often resulting in a significant loss of autonomy and an increased risk of fall. Understanding the influence of such medical disorders on a subject's gait would greatly facilitate early detection and prevention of those possibly harmful situations. To address these issues, clinical and bio-mechanical researchers have worked to objectively quantify gait characteristics.

Among the gait features that have proved their relevance in a medical context, several are linked to the notion of step (step duration, variation in step length, etc.), which can be seen as the core atom of the locomotion process. Many algorithms have therefore been developed to automatically (or semi-automatically) detect gait events (such as heel-strikes, heel-off, etc.) from accelerometer and gyrometer signals.

Data. Data are described in the associated notebook.

3.2 Step detection with convolutional dictionary learning

Task. The objective is to perform **step detection**, that is to estimate the start and end times of footsteps contained in accelerometer and gyrometer signals recorded with Inertial Measurement Units (IMUs).

Performance metric. Step detection methods will be evaluated with the **F-score**, based on the following precision/recall definitions. The F-score is first computed per signal then averaged over all instances. Precision and recall rely on the "intersection over union" metric (IoU) that measures the overlap of two intervals $[s_1, e_1]$ and $[s_2, e_2]$:

$$\text{IoU} = \frac{\big|[s_1, e_1] \cap [s_2, e_2]\big|}{\big|[s_1, e_1] \cup [s_2, e_2]\big|}$$

- Precision (or positive predictive value). A detected (or predicted) step is counted as correct if it overlaps (measured by IoU) an annotated step by more than 75%. The precision is the number of correctly predicted steps divided by the total number of predicted steps.
- Recall (or sensitivity). An annotated step is counted as detected if it overlaps (measured by IoU) a predicted step by more than 75%. The recall is the number of detected annotated steps divided by the total number of annotated steps.

The F-score is the geometric mean of the precision and recall:

$$2 \times \frac{\text{precision} \times \text{recall}}{\text{precision} + \text{recall}}.$$

Note that an annotated step can only be detected once, and a predicted step can only be used to detect one annotated step. If several predicted steps correspond to the same annotated step, all but one are considered as false. Conversely, if several annotated steps are detected with the same predicted step, all but one are considered undetected.

Example 1.

- Annotation ("ground truth label"): $\big[[80, 100], [150, 250], [260, 290]\big]$ (three steps)
- Prediction: $\big[[80, 98], [105, 120], [256, 295], [298, 310]\big]$ (four steps)

Here, precision is $0.5 = (1 + 0 + 1 + 0)/4$, recall is $0.67 = (1 + 0 + 1)/3$ and the F-score is 0.57.

Example 2.

- Annotation ("ground truth label"): $\big[[80, 120]\big]$ (one step)
- Prediction: $\big[[80, 95]\big]$ (one step)

Here, precision is $0 = 0/1$, recall is $0 = 0/1$ and the F-score is 0.

Question 3

For a single signal, learn a dictionary with manually chosen penalty, number of atoms and length.

Modify Figure 1 to display the original signal and its reconstruction. Modify Figure 2 to display the individual atoms.

Answer 3

For this fitting, the chosen parameters are $\lambda = 0.2$,K= 3, $L = 20$.

The reconstruction error (MSE) is equal to 0.00265. The F-score is equal to 0.

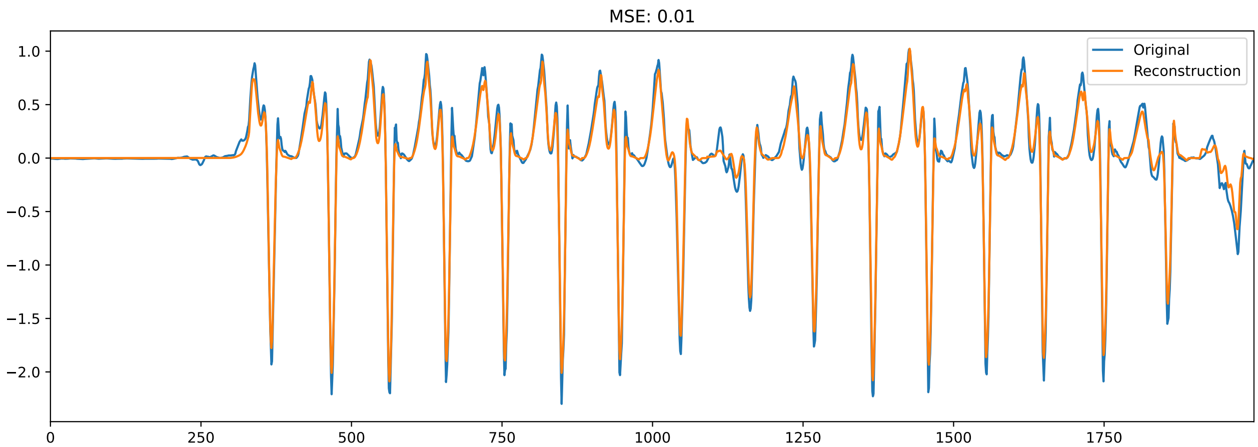


Figure 1: Original signal and its reconstruction (see Question **??**).

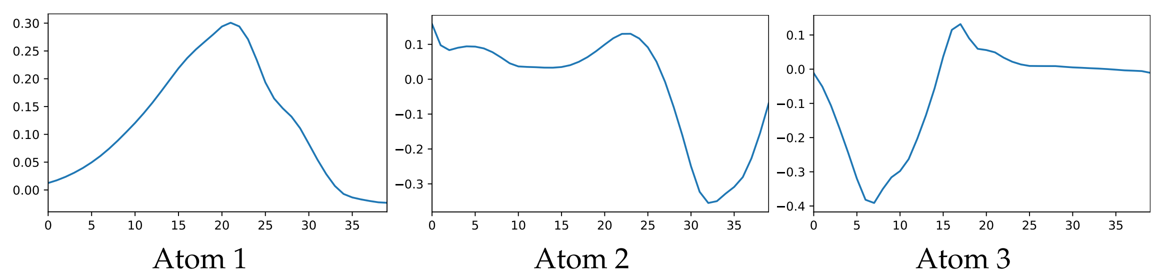


Figure 2: Individual atoms (see Question **??**).

Question 4

Using only the training set, find with a 5-fold cross-validation among the candidates values (see notebook) the best combination of (λ, K, L) for the step detection task.

Provide the optimal values of (λ, K, L) and the associated average F-score and MSE.

Answer 4

The cross-validation gave the following optimal parameters:

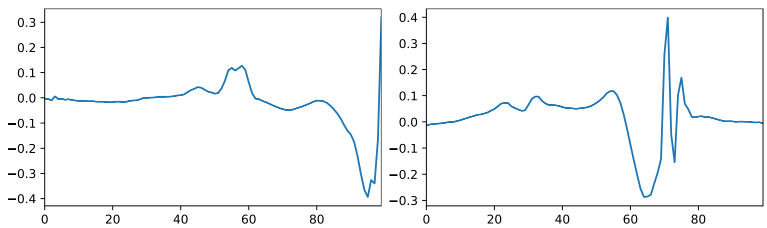
- $\lambda = 0.1$
- $K = 2$
- $L = 100$

which results in: MSE: 0.014, F-score: 0.088

Question 5

Display on Figure 3 the atoms learned for Question 4.

Answer 5



Atom 1 Atom 2

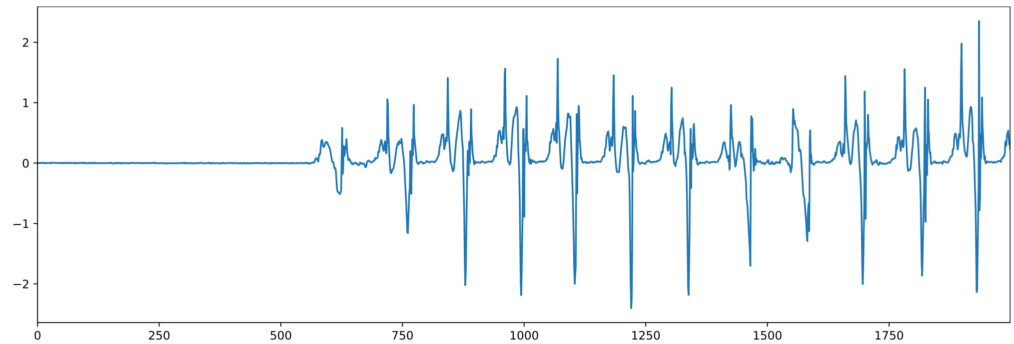
Figure 3: Individual atoms (see Question **??**).

Question 6

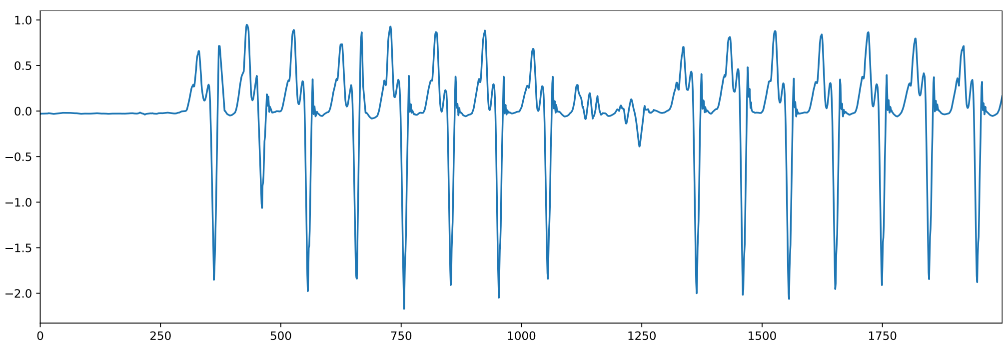
Display on Figure 4 the signals from the test set with highest and lowest F-score. Comment briefly.

Answer 6

The bad score of the second signal can be explained by the fact the signal is heterogeneous in a given part (between samples 1200 to 1300 approximately). We might have predicted steps that were not existing in this zone. On the opposite, the first signal has clearer step sections, and appears not to have different movements.



Test signal with the best F-score



Test signal with the worst F-score

Figure 4: Best and worst scores (see Question **??**).

3.3 Step classification with the dynamic time warping (DTW) distance

Task. The objective is to classify footsteps then walk signals between healthy and non-healthy.

Performance metric. The performance of this binary classification task is measured by the F-score.

Question 7

Combine the DTW and a k-neighbors classifier to classify each step. Find the optimal number of neighbors with 5-fold cross-validation and report the optimal number of neighbors and the associated F-score. Comment briefly.

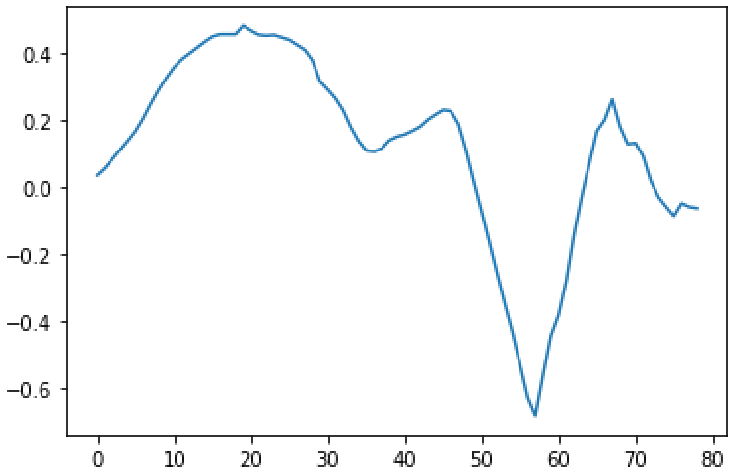
Answer 7

The optimal number of neighbors is 1, and the associated mean F1-score on 5-fold cross-validation is 0.8804. It means that the object is assigned to the class of it's closest neighbor, this is the nearest neighbor algorithm. Using nearest neighbor algorithm makes k-neighbors tractable even for wide datasets.

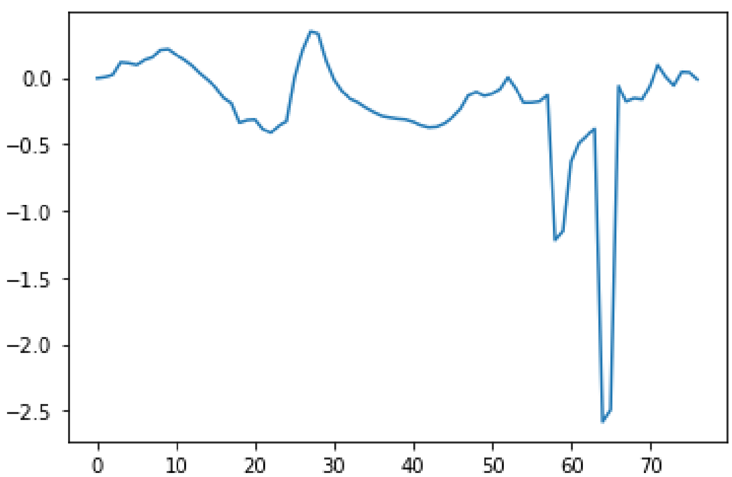
Question 8

Display on Figure **??** a badly classified step from each class (healthy/non-healthy).

Answer 8



Badly classified healthy step



Badly classified non-healthy step

Figure 5: Examples of badly classified steps (see Question **??**).