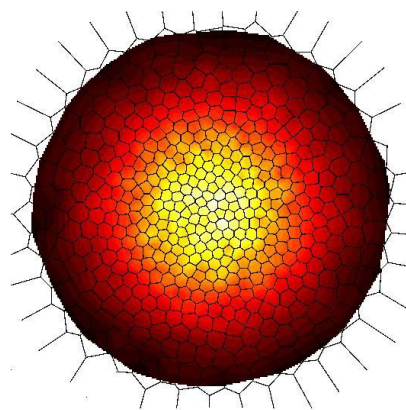# Introduction
to
Numerical Probability for Finance

——

2015-16

——

GILLES PAGÈS

LPMA-Université Pierre et Marie Curie

—

E-mail: gilles.pages@upmc.fr

—

http: //www.proba.jussieu.fr/pageperso/pages

—

Contents

Notations

▷ GENERAL NOTATIONS

- $\lfloor x \rfloor$ denotes the integral part of the real number x i.e. the highest integer not greater than x. $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of the real number x.

- The notation $u \in \mathbb{K}^d$ will denote the *column* vector u of the vector space $\mathbb{K}^d$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *row* vector will be denoted u^* or $^t u$.

- $(u|v) = \sum_{1 \le i \le d} u^i v^i$ denotes the canonical inner product of vectors $u = (u^1, \ldots, u^d)$ and $v = (v^1, \ldots, v^d)$ of $\mathbb{R}^d$.

- $\mathcal{M}(d, q, \mathbb{K})$ will denote the vector space of matrices with d rows and q columns with $\mathbb{K}$-valued entries.

- The transpose of a matrix A will be denoted A^* or A^t.

- $\mathcal{C}_b(S, \mathbb{R}^d) := \{ f : (S, \delta) \to \mathbb{R}^d, \text{continuous and bounded} \}$ where (S, δ) denotes a metric space.

- For a function $f : (\mathbb{R}^d, |\,.\,|_q) \to (\mathbb{R}^p, |\,.\,|_q)$

$$[f]_{\text{Lip}} = \sup_{x \ne y} \frac{|f(x) - f(y)|_q}{|x - y|_d}.$$

  and f is Lipschitz continuous continuous with coefficient $[f]_{\text{Lip}}$ if $[f]_{\text{Lip}} < +\infty$.

- An assertion $\mathcal{P}(x)$ depending on a generic element x of a measured space $(E, \mathcal{E}, \mu)$ is true μ-*almost everywhere* (denoted μ-a.e.) if it is true outside a μ-negligible set of E.

▷ PROBABILISTIC NOTATIONS

- The distribution on $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$ of a random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ is denoted $\mathbb{P}_X$.

- $X \overset{d}{=} Y$ stands for equality in distribution of the random vectors X and Y.

- $\overset{\mathcal{L}}{\longrightarrow}$ denotes the convergence in distribution of random vectors (*i.e.* the weak convergence of their distributions).

- $\mathcal{L}^p_{\mathbb{R}^d}(\Omega, \mathcal{A}, \mu) = \left\{ f : (\Omega, \mathcal{A}) \to \mathbb{R}^d \text{ s.t. } \int_X |f|^p d\mu < +\infty \right\}$ which does not depend upon the selected norm on $\mathbb{R}^d$.

Chapter 1

Simulation of random variables

1.1 Pseudo-random numbers

From a mathematical point of view, the definition of a sequence of (uniformly distributed) random numbers (over the unit interval $[0, 1]$) should be:

"**Definition.**" *A sequence $x_n, n \geq 1$, of $[0, 1]$-valued real numbers is a sequence of random numbers if there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a sequence $U_n, n \geq 1$, of i.i.d. random variables with uniform distribution $\mathcal{U}([0, 1])$ and $\omega \in \Omega$ such that $x_n = U_n(\omega)$ for every $n \geq 1$.*

But this naive and abstract definition is not satisfactory because the "scenario" $\omega \in \Omega$ may be not a "good" one *i.e.* not a "generic"...? Many probabilistic properties (like the law of large numbers to quote the most basic one) are only satisfied $\mathbb{P}$-*a.s.*. Thus, if ω precisely lies in the negligible set that does not satisfy one of them, the induced sequence will not be "admissible".

Whatever, one usually cannot have access to an i.i.d. sequence of random variables (U_n) with distribution $\mathcal{U}([0, 1])$! Any physical device would too slow and not reliable. Some works by logicians like Martin-Löf lead to consider that a sequence (x_n) that can be generated by an algorithm cannot be considered as "random" one. Thus the digits of π are not random in that sense. This is quite embarrassing since an essential requested feature for such sequences is to be generated almost instantly on a computer!

The approach coming from computer and algorithmic sciences is not really more tractable since their definition of a sequence of random numbers is that the complexity of the algorithm to generate the first n terms behaves like $O(n)$. The rapidly growing need of good (pseudo-)random sequences with the explosion of Monte Carlo simulation in many fields of Science and Technology (I mean not only neutronics) after World War II lead to adopt a more pragmatic approach – say – heuristic – based on a statistical tests. The idea is to submit some sequences to statistical tests (uniform distribution, block non correlation, rank tests, etc)

For practical implementation, such sequences are finite, as the accuracy of computers is. One considers some sequences (x_n) of so-called *pseudo-random* numbers defined by

$$x_n = \frac{y_n}{N}, \quad y_n \in \{0, \ldots, N - 1\}.$$

One classical process is to generate the integers y_n by a congruential induction:

$$y_{n+1} \equiv a y_n + b \mod. N$$

where $\gcd(a, N) = 1$, so that $\bar{a}$ (class of a modulo N) is invertible for the multiplication (modulo N). Let $(\mathbb{Z}/N\mathbb{Z})^*$ denote the set of such invertible classes (modulo N). The multiplication of classes (module N) is an internal law on $(\mathrm{Z}/\mathrm{NZ})^*$ and $((\mathbb{Z}/N\mathbb{Z})^*, \times)$ is a commutative group for this operation. By the very definition of the Euler function $\varphi(N)$ as the number of integers a in $\{0, \ldots, N-1\}$ such that $\gcd(a, N) = 1$, the cardinality of $(\mathbb{Z}/N\mathbb{Z})^*$ is equal to $\varphi(N)$. Let us recall that the Euler function is multiplicative and given by the following closed formula

$$\varphi(N) = N \prod_{p|N, \ p \text{ prime}} \left(1 - \frac{1}{p}\right).$$

Thus $\varphi(p) = p - 1$ for very prime integer of $\mathbb{N}$ and $\varphi(p^k) = p^k - p$ (primary numbers), etc. In particular, if p is prime it shows that $(\mathbb{Z}/N\mathbb{Z})^* = (\mathbb{Z}/N\mathbb{Z}) \setminus \{0\}$.

If $b = 0$ (the most common case), one speaks of *homogeneous generator*. We will focus on this type of generators in what follows.

$\triangleright$ *Homogeneous congruent generators.* When $b = 0$, the period of the sequence (y_n) is given by the multiplicative order of a in $((\mathbb{Z}/N\mathbb{Z})^*, \times)$ *i.e.*

$$\tau_a := \min\{k \geq 1, \ | \ a^k \equiv 1 \text{ mod. } N\} = \min\{k \geq 1, \ | \ \bar{a}^k\}.$$

Moreover, we know by Lagrange's Theorem that τ_a divides the cardinality $\varphi(N)$ of $(\mathbb{Z}/N\mathbb{Z})^*$.

For pseudo-random number simulation purpose, we search for couples (N, a) such that the period τ_a of a in $((\mathbb{Z}/N\mathbb{Z})^*, \times)$ is very large. The needs an in-depth study of the multiplicative groups $((\mathbb{Z}/N\mathbb{Z})^*, \times)$, with in mind that N should be large itself to allow a having a large period. This suggests to focus on prime numbers or primary numbers since, as seen above, their Euler function is itself large.

In fact the structure of these groups have been elucidated for a long time and we sum up below these results.

Theorem 1.1 *Let $N = p^\alpha$, p prime, $\alpha \in \mathbb{N}^*$ be a primary number.*

(a) *If $\alpha = 1$ (i.e. $N = p$ prime), then $((\mathbb{Z}/N\mathbb{Z})^*, \times)$ (whose cardinality is $p-1$) is a cyclic group. This means that there exists $a \in \{1, \ldots, p-1\}$ s.t. $(\mathbb{Z}/p\mathbb{Z})^* = \langle \bar{a} \rangle$. Hence the maximal period is $\tau = p - 1$.*

(b) *If $p = 2$, $\alpha \geq 3$, $(\mathbb{Z}/N\mathbb{Z})^*$ (whose cardinality is $2^{\alpha-1} = N/2$) is not cyclic. The maximal period is then $\tau = 2^{\alpha-2}$ with $a \equiv \pm 3$ mod. 8. (If $\alpha = 2$ ($N = 4$), the group of size 2 is trivially cyclic!)*

(c) *If $p \neq 2$, then $(\mathbb{Z}/N\mathbb{Z})^*$ (whose cardinality is $p^{\alpha-1}(p-1)$) is cyclic, hence $\tau = p^{\alpha-1}(p-1)$. It is generated by any element a whose class $\tilde{a}$ in $(\mathbb{Z}/p\mathbb{Z})$ spans the cyclic group $((\mathbb{Z}/p\mathbb{Z})^*, \times)$.*

What does this theorem say in connection with our pseudo-random number generation problem?

First a very good news: *when N is a prime number* the group $((\mathbb{Z}/N\mathbb{Z})^*, \times)$ is *cyclic i.e.* there exists $a \in \{1, \ldots, N\}$ such that $(\mathbb{Z}/N\mathbb{Z})^* = \{\bar{a}^n, 1 \leq n \leq N - 1 \in\}$. The bad news is that *we do not know which a satisfy this property*, not all do and, even worse, we do not know how to find any. Thus, if $p = 7$, $\varphi(7) = 7 - 1 = 6$ and $o(3) = o(5) = 6$ but $o(2) = o(4) = 3$ (which divides 6) and $o(6) = 2$ (which again divides 6).

The send bad news is that the length of a sequence, though a necessary property of a sequence $(y_n)_n$, provides no guarantee or even clue that i$(x_n)_n$ is a *good* sequence of pseudo-random numbers! Thus, the (homogeneous) generator of the FORTRAN IMSL library does not fit in the formerly described setting: one sets $N := 2^{31} - 1 = 2\,147\,483\,647$ (which is a Mersenne prime number (see below)) discovered by Leonhard Euler in 1772), $a := 7^5$, and $b := 0$ ($a \not\equiv 0$ mod.8), the point being that the period of 7^5 is not maximal.

Another approach to random number generation is based on shift register and relies upon the theory of finite fields.

At this stage, a sequence must pass successfully through various statistical tests, having in mind that such a sequence is finite by contraction and consequently cannot satisfy asymptotically as common properties as the Law of Large Numbers, the Central Limit Theorem or the Law of the Iterated Logarithm (see next chapter). Dedicated statistical toolboxes like *DIEHARD* (Marsaglia, 1998) have been devised to test and "certify" sequences of pseudo-random numbers.

The aim of this introductory section is just to give the reader the flavour of pseudo-random numbers generation, but in no case we recommend the specific use of any of the above generators or discuss the virtues of such or such generator.

For more recent developments on random numbers generators (like shift register, etc, we refer *e.g.* to [42], [117], etc). Nevertheless, let us mention the Mersenne twister generators. This family of generator has been introduced in 1997 by Makoto Matsumata and Takuji Nishimura in [111]. The first level of Mersenne Twister Generators (denoted MT-p) are congruential generators whose period N_p is a prime Mersenne number *i.e.* an integer of the form $N_p = 2^p - 1$ where p is itself prime. The most popular and now worldwide implemented is the MT-19937, owing to its unrivaled period $2^{19937} - 1 \approx 10^{6000}$ since $2^{10} \approx 10^3$). It can simulate a uniform distribution in 623 dimension (*i.e.* on $[0, 1]^{623}$. A second "shuffling" device still improves it. For recent improvements and their implementations in $C++$, use the link

www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html

Recently, new developments in massively parallel computing drew the attention back to pseudo-random number simulation. In particular the GPU based intensive computations which use the graphic device of a computer as a computing unit which may run in parallel hundreds of parallel computations. One can imagine that each pixel is a (virtual) small computing unit achieving a small chain of elementary computations (thread). What is really new is the access to such an intensive parallel computing becomes cheap, although it requires some specific programming language (like CUDA on Nvidia GPU). As concerns its use for intensively parallel Monte Carlo simulation, some new questions arise: in particular the ability to generate independently (in parallel!) many "independent" sequences of pseudo-random numbers since the computing units of a GPU never "speak" to each other or to anybody else while running: each pixel is a separate (virtual) thread. The Wichmann-Hill pseudo-random numbers generator (which is in fact a family of 273 different methods) seems to be a good candidate for Monte Carlo simulation on GPU. For more insight on that topic we refer to [150] and the references therein.

1.2 The fundamental principle of simulation

Theorem 1.2 (Fundamental Theorem of simulation) *Let (E, d_E) be a Polish space (complete and separable) and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{B}or_{d_E}(E))$ be a random variable with distribution $\mathbb{P}_X$. Then there exists a Borel function $\varphi : ([0,1], \mathcal{B}([0,1]), \lambda_{[0,1]}) \to (E, \mathcal{B}or_{d_E}(E), \mathbb{P}_X)$ such that*

$$\mathbb{P}_X = \lambda_{[0,1]} \circ \varphi^{-1}$$

where $\lambda_{[0,1]} \circ \varphi^{-1}$ denotes the image of the Lebesgue measure $\lambda_{[0,1]}$ on the unit interval by φ.

We will admit this theorem. For a proof we refer to [27] (Theorem A.3.1p.38). It also appears as a "brick" in the proof of the Skorokhod representation theorem for random variables having values in a Polish space.

As a consequence this means that, if U denotes any random variable with uniform distribution on $(0,1)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then

$$X \stackrel{d}{=} \varphi(U).$$

The interpretation is that any E-valued random variable can be simulated from a uniform distribution... provided the function φ is computable. If so is the case, the *yield* of the simulation is 1 since every (pseudo-)random number $u \in [0,1]$ produces a $\mathbb{P}_X$-distributed random number. Except in very special situations (see below), this result turns out to be of purely theoretical nature and is of little help for practical simulation. However the fundamental theorem of simulation is important from a theoretical point view in Probability Theory since it is the fundamental step of Skorokhod representation Theorem.

In the three sections below we provide a short background on the most classical simulation methods (inversion of the distribution function, acceptance-rejection method, Box-Müller for Gaussian vectors). This is of course far from being exhaustive. For an overview of the different aspects of simulation of non uniform random variables or vectors, we refer to [42]. But in fact, a large part of the results from Probability Theory can give rise to simulation methods.

1.3 The (inverse) distribution function method

Let μ be a probability distribution on $(\mathbb{R}, \mathcal{B}or(\mathbb{R}))$ with distribution function F defined for every $x \in \mathbb{R}$ by

$$F(x) := \mu((-\infty, x]).$$

The function F is always non-decreasing, "càdlàg" (French acronym for "right continuous with left limits") and $\lim_{x \to +\infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$.

One can always associate to F its canonical left inverse F_l^{-1} defined on the open unit interval $(0,1)$ by

$$\forall u \in (0,1), \quad F_l^{-1}(u) = \inf\{x \mid F(x) \geq u\}.$$

One easily checks that F_l^{-1} is non-decreasing, left-continuous and satisfies

$$\forall u \in (0,1), \qquad F_l^{-1}(u) \leq x \Longleftrightarrow u \leq F(x).$$

Proposition 1.1 *If* $U \overset{d}{=} U((0,1))$*, then* $X := F_l^{-1}(U) \overset{d}{=} \mu$.

Proof. Let $x \in \mathbb{R}$. It follows from what precedes that

$$\{X \leq x\} = \{F_l^{-1}(U) \leq x\} = \{U \leq F(x)\}$$

so that $\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(X)) = F(x)$. $\quad \diamond$

Remarks. • When F is increasing and continuous on the real line, then F has an inverse function denoted F^{-1} defined $(0,1)$ (increasing and continuous as well) such that $F \circ F^{-1} = Id_{|(0,1)}$ and $F^{-1} \circ F = Id_{\mathbb{R}}$. Clearly $F^{-1} = F_l^{-1}$ by the very definition of F_l^{-1}. But the above proof can be made even more straightforward since $\{F^{-1}(U) \leq x\} = \{U \leq F(x)\}$ by simple left composition of F^{-1} by the increasing function F.

• If μ has a probability density f such that $\{f = 0\}$ has an empty interior, then $F(x) = \displaystyle\int_{-\infty}^{x} f(u)du$ is continuous and increasing.

• One can replace $\mathbb{R}$ by any interval $[a,b] \subset \mathbb{R}$ or $\overline{\mathbb{R}}$ (with obvious conventions).

• One could also have considered the *right continuous canonical inverse* F_r^{-1} by:

$$\forall u \in (0,1), \quad F_r^{-1}(u) = \inf\{x \,|\, F(x) > u\}.$$

One shows that F_r^{-1} is non-decreasing, right continuous and that

$$F_r^{-1}(u) \leq x \Longrightarrow u \leq F(x) \quad \text{and} \quad u < F(x) \Longrightarrow F_r^{-1}(u) \leq x.$$

Hence $F_r^{-1}(U) \overset{d}{=} X$ since

$$\mathbb{P}(F_r^{-1}(U) \leq x) \left\{ \begin{array}{l} \leq \mathbb{P}(F(x) \geq U) = F(x) \\ \geq \mathbb{P}(F(x) > U) = F(x) \end{array} \right\} = \mathbb{P}(X \leq x) = F(x).$$

When X takes finitely many values in $\mathbb{R}$, we will see in Example 4 below that this simulation method corresponds to the standard simulation method of such random variables.

▷ **Exercise.** (*a*) Show that, for every $u \in (0,1)$

$$F\big(F_l^{-1}(u) - \big) \leq u \leq F \circ F_l^{-1}(u)$$

so that if F is continuous (or equivalently μ has no atom: $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$), then $F \circ F_l^{-1} = Id_{(0,1)}$

(*b*) Show that if F is continuous, then $F(X) \overset{d}{=} U([0,1])$.

(*c*) Show that if F is (strictly) increasing, then F_l^{-1} is continuous and $F_l^{-1} \circ F = Id_{\mathbb{R}}$.

(*d*) One defines the survival function of μ by $\bar{F}(x) = 1 - F(x) = \mu\big((x, +\infty)\big), x \in \mathbb{R}$. One defines the canonical *right* inverse of $\bar{F}$ by

$$\forall u \in (0,1), \quad \bar{F}_r^{-1}(u) = \inf\{x \,|\, \bar{F}(x) \leq u\}.$$

Show that $\bar{F}_r^{-1}(u) = F_l^{-1}(1-u)$. Deduce that $\bar{F}_r^{-1}$ is right continuous on $(0,1)$ and that $\bar{F}_r^{-1}(U)$ has distribution μ. Define $\bar{F}_l^{-1}$ and show that $\bar{F}_r^{-1}(U)$ has distribution μ. Finally establish for $\bar{F}_r^{-1}$ similar properties to (a)-(b)-(c).

(Unformal) Definition *The* yield *(often denoted r) of a simulation procedure is defined as the inverse of the number of pseudo-random numbers used to generate one $\mathbb{P}_X$-distributed random number.*

One must keep in mind that the yield is attached to a simulation method, not to a probability distribution (the fundamental theorem of simulation always provides a simulation method with yield 1, except that it is usually not tractable).

Example. Typically, if $X = \varphi(U_1, \ldots, U_m)$ where φ is a Borel function defined on $[0,1]^m$ and $U_1, \ldots, U_m$ are independent and uniformly distributed over $[0,1]$, the yield of this φ-based procedure to simulate the distribution of X is $r = 1/m$.

Thus, the yield r of the (inverse) distribution function is consequently always equal to $r = 1$.

Examples: **1.***Simulation of an exponential distribution.* Let $X \overset{d}{=} \mathcal{E}(\lambda)$, $\lambda > 0$. Then

$$\forall\, x \in (0,\infty), \qquad F_X(x) = \lambda \int_0^x e^{-\lambda \xi} d\xi = 1 - e^{-\lambda x}.$$

Consequently, for every $y \in (0,1)$, $F_X^{-1}(u) = -\log(1-u)/\lambda$. Now, using that $U \overset{d}{=} 1 - U$ if $U \overset{d}{=} U((0,1))$ yields

$$X = -\frac{\log(U)}{\lambda} \overset{d}{=} \mathcal{E}(\lambda).$$

2. *Simulation of a Cauchy(c), $c > 0$, distribution.* We know that $\mathbb{P}_X(dx) = \dfrac{c}{\pi(x^2 + c^2)} dx$.

$$\forall x \in \mathbb{R}, \qquad F_X(x) = \int_{-\infty}^x \frac{c}{u^2 + c^2} \frac{du}{\pi} = \frac{1}{\pi}\left(\operatorname{Arctan}\left(\frac{x}{c}\right) + \frac{\pi}{2} \right).$$

Hence $F_X^{-1}(x) = c \tan\big(\pi(u - 1/2)\big)$. It follows that

$$X = c \tan\big(\pi(U - 1/2)\big) \overset{d}{=} \text{Cauchy}(c).$$

3. *Simulation of a Pareto(θ), $\theta > 0$, distribution.* We know that $\mathbb{P}_X(dx) = \dfrac{\theta}{x^{1+\theta}} \mathbf{1}_{\{x \geq 1\}} dx$. The distribution function $F_X(x) = 1 - x^{-\theta}$ so that, still using $U \overset{d}{=} 1 - U$ if $U \overset{d}{=} U((0,1))$,

$$X = U^{-\frac{1}{\theta}} \overset{d}{=} \text{Pareto}(\theta).$$

4. *Simulation of a distribution supported by a finite set E.* Let $E := \{x_1, \ldots, x_N\}$ be a subset of $\mathbb{R}$ indexed so that $i \mapsto x_i$ is increasing. Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to E$ be an E-valued random variable with

distribution $\mathbb{P}(X = x_k) = p_k$, $1 \le k \le N$, where $p_k \in [0, 1]$, $p_1 + \cdots + p_N = 1$. Then, one checks that its distribution function F_X reads

$$\forall\, x \in \mathbb{R}, \quad F_X(x) = p_1 + \cdots + p_i \ \text{ if } x_i \le x < x_{i+1}$$

with the convention $x_0 = -\infty$ and $x_{N+1} = +\infty$. As a consequence, its left continuous canonical inverse is given by

$$\forall u \in (0, 1), \qquad F_{X,l}^{-1}(u) = \sum_{k=1}^{N} x_k \mathbf{1}_{\{p_1 + \cdots + p_{k-1} < u \le p_1 + \cdots + p_k\}}$$

so that

$$X \stackrel{d}{=} \sum_{k=1}^{N} x_k \mathbf{1}_{\{p_1 + \cdots + p_{k-1} < U \le p_1 + \cdots + p_k\}}.$$

The yield of the procedure is still $r = 1$. However, when implemented naively, its complexity – which corresponds to (at most) N comparisons for every simulation – may be quite high. See [42] for some considerations (in the spirit of quick sort algorithms) which lead to a $O(\log N)$ complexity. Furthermore, this procedure underlines that one has access to the probability weights p_k with an arbitrary accuracy. This is not always the case even in *a priori simple* situations as emphasized in Example 6 below.

It should be noticed of course that the above simulation formula is still appropriate for a random variable taking values in *any* set E, not only for subsets of $\mathbb{R}$!

5. *Simulation of a Bernoulli random variable $B(p)$, $p \in (0, 1)$.* This is the simplest application of the previous method since

$$X = \mathbf{1}_{\{U \le p\}} \stackrel{d}{=} B(p).$$

6. *Simulation of a Binomial random variable $B(n, p)$, $p \in (0, 1)$, $n \ge 1$.* One relies on the very definition of the binomial distribution as the law of the sum of n independent $B(p)$-distributed random variables *i.e.*

$$X = \sum_{k=1}^{n} \mathbf{1}_{\{U_k \le p\}} \stackrel{d}{=} B(n, p).$$

where $U_1, \ldots, U_n$ are i.i.d. random variables, uniformly distributed over $[0, 1]$.

Note that this procedure has a very bad yield, namely $r = \frac{1}{n}$. Moreover, it needs n comparisons like the standard method (without any shortcut).

Why not using the above standard method for random variable taking finitely many values? Because the cost of the computation of the probability weights p_k's is much too high as n grows.

7. *Simulation of geometric random variables $G(p)$ and $G^*(p)$, $p \in (0, 1)$.* It is the distribution of the first success instant when repeating independently the same Bernoulli experiment with parameter p. Conventionaly, $G(p)$ starts at time 0 whereas $G^*(p)$ starts at time 1.

To be precise, if $(X_k)_{k \ge 0}$ denotes an i.i.d. sequence of random variables with distribution $B(p)$, $p \in (0, 1)$ then

$$\tau^* := \min\{k \ge 1 \,|\, X_k = 1\} \stackrel{d}{=} G^*(p)$$

and
$$\tau := \min\{k \geq 0 \,|\, X_k = 1\} \overset{d}{=} G(p).$$

Hence
$$\mathbb{P}(\tau^* = k) = p(1-p)^{k-1}, \ k \in \mathbb{N}^* := \{1, 2, \ldots, n, \ldots\}$$

and
$$\mathbb{P}(\tau = k) = p(1-p)^k, \ k \in \mathbb{N} := \{0, 1, 2, \ldots, n, \ldots\}$$

(so that both random variables are $\mathbb{P}$-*a.s.* finite since $\sum_{k\geq 0} \mathbb{P}(\tau = k) = \sum_{k\geq 1} \mathbb{P}(\tau^* = k) = 1$). Note that $\tau + 1$ has the same $G^*(p)$-distribution as τ^*.

The (random) yields of the above two procedures are $r^* = \frac{1}{\tau^*}$ and $r = \frac{1}{\tau+1}$ respectively. Their common mean (*average yield*) $\bar{r} = \bar{r}^*$ is given by

$$
\begin{aligned}
\mathbb{E}\left(\frac{1}{\tau+1}\right) = \mathbb{E}\left(\frac{1}{\tau^*}\right) \ &= \ \sum_{k\geq 1} \frac{1}{k} p(1-p)^{k-1} \\
&= \ \frac{p}{1-p} \sum_{k\geq 1} \frac{1}{k}(1-p)^k \\
&= \ -\frac{p}{1-p} \log(1-x)_{|x=1-p} \\
&= \ -\frac{p}{1-p} \log(p).
\end{aligned}
$$

▷ **Exercises. 1.** Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}or(\mathbb{R}))$ be a real-valued random variable with distribution function F and left continuous canonical inverse F_l^{-1}. Let $I = [a, b]$, $-\infty \leq a < b \leq +\infty$, be a nontrivial interval of $\mathbb{R}$. Show that, if $U \overset{d}{=} U([0,1])$, then

$$F_l^{-1}\big(F(a) + (F(b) - F(a))U\big) \overset{d}{=} \mathcal{L}(X \,|\, X \in I).$$

2. *Negative binomial distributions* The negative binomial distribution with parameter (n, p) is the law $\mu_{n,p}$ of the nth success in an infinite sequence of independent Bernoulli trials, namely, with above notations used for the geometric distributions, the distribution of

$$\tau^{(n)} = \min\big\{k \geq 1 \,|\, \mathrm{card}\{1 \leq \ell \leq k \,|\, X_\ell = 1\} = n\big\}.$$

Show that
$$\mu_{n,p}(k) = 0, \ k \leq n-1, \quad \mu_{n,p}(k) = C_{k-1}^{n-1} p^n (1-p)^{k-n}, \ k \geq n.$$

Compute the mean yield of its (natural and straightforward) simulation method.

1.4   The acceptance-rejection method

This method is due to Von Neumann (1951). It is contemporary of the development of computers and of the Monte Carlo method. The original motivation was is to devise a simulation method for a probability distribution ν on a measurable space $(E, \mathcal{E})$, absolutely continuous with respect a σ-finite non-negative measure μ with a density given, *up to a constant*, by $f : (E, \mathcal{E}) \to \mathbb{R}_+$ when

we know that f is *dominated* by a probability distribution $g.\mu$ which can be simulated at "low cost". (Note that $\nu = \frac{f}{\int_E f d\mu}.\mu$.)

In most elementary applications (see below), E is either a Borel set of $\mathbb{R}^d$ equipped with its Borel σ-field and μ is the trace of the Lebesgue measure on E or a subset $E \subset \mathbb{Z}^d$ equipped with the counting measure.

Let us be more precise. So, let μ be a non-negative σ-finite measure on $(E, \mathcal{E})$ and let $f, g : (E, \mathcal{E})) \longrightarrow \mathbb{R}_+$ be two Borel functions. Assume that $f \in L^1_{\mathbb{R}_+}(\mu)$ with $\int_E f \, d\mu > 0$ and that g is a *probability density* with respect to μ satisfying furthermore $g > 0$ μ-a.s. and there exists a positive real constant $c > 0$ such that

$$f(x) \leq c \, g(x) \quad \mu(dx)\text{-a.e.}$$

Note that this implies $c \geq \int_E f d\mu$. As mentioned above, the aim of this section is to show how to simulate some random numbers distributed according to the probability distribution

$$\nu = \frac{f}{\int_{\mathbb{R}^d} f \, d\mu} \cdot \mu$$

using some $g.\mu$-distributed (pseudo-)random numbers. In particular, to make the problem consistent, we will assume that $\nu \neq \mu$ which in turn implies that

$$c > \int_{\mathbb{R}^d} f d\mu.$$

The underlying requirements on f, g and μ to undertake a practical implementation of the method described below are the following:

- the numerical value of the real constant c is known,

- we know how to simulate (at a reasonable cost) on a computer a sequence of i.i.d. random vectors $(Y_k)_{k \geq 1}$ with the distribution $g \cdot \mu$

- we can compute on a computer the ratio $\frac{f}{g}(x)$ at every $x \in \mathbb{R}^d$ (again at a reasonable cost).

As a first (not so) preliminary step, we will explore a natural connection (in distribution) between an E-valued random varaible X with distribution ν and Y an E-valued random variable with distribution $g.\mu$. W swill see that he key idea is completely elementary. let $h : (E, \mathcal{E}) \to \mathbb{R}$ a test function (measurable and bounded or non-negative). On the one hand

$$
\begin{aligned}
\mathbb{E}\,h(X) &= \frac{1}{\int_E f d\mu} \int_E h(x)f(x)\mu(dx) \\
&= \frac{1}{\int_E f d\mu} \int_E h(y)\frac{f}{g}(y)g(y)\mu(dy) \quad \text{since } g > 0 \ \mu\text{-p.p.} \\
&= \mathbb{E}\left(h(Y)\frac{f}{g}(Y)\right).
\end{aligned}
$$

We can also stay on the state space E and note in a somewhat artificial way that

$$
\begin{aligned}
\mathbb{E}\,h(X) &= \frac{c}{\int_E f d\mu} \int_E h(y) \left(\int_0^1 \mathbf{1}_{\{u \le \frac{1}{c}\frac{f}{g}(y)\}} du \right) g(y)\mu(dy) \\
&= \frac{c}{\int_E f d\mu} \int_E \int_0^1 h(y) \mathbf{1}_{\{u \le \frac{1}{c}\frac{f}{g}(y)\}} g(y)\mu(dy) du \\
&= \frac{c}{\int_E f d\mu} \mathbb{E}\left(h(Y) \mathbf{1}_{\{cU \le \frac{f}{g}(Y)\}} \right)
\end{aligned}
$$

where U is uniformly distributed over $[0,1]$ and *independent* of Y.

By considering $h \equiv 1$, we derive from the above identity that $\dfrac{c}{\int_E f d\mu} = \dfrac{1}{\mathbb{P}\left(cU \le \frac{f}{g}(Y) \right)}$ so that finally

$$
\mathbb{E}\,h(X) = \mathbb{E}\left(h(Y) \,|\, \{cU \le \frac{f}{g}(Y))\} \right).
$$

The proposition below takes advantage of this identity in distribution to propose a simulation procedure. In fact it is simply a reverse way to make (and interpret) the above computations.

Proposition 1.2 (Acceptance-rejection simulation method) *Let $(U_n, Y_n)_{n \ge 1}$ be a sequence of i.i.d. random variable with distribution $U([0,1]) \otimes \mathbb{P}_Y$ (independent marginals) defined on $(\Omega, \mathcal{A}, \mathbb{P})$ where $\mathbb{P}_Y(dy) = g(y)\mu(dy)$ is the distribution of Y on $(E, \mathcal{E})$. Set*

$$
\tau := \min\{k \ge 1 \,|\, c\,U_k g(Y_k) \le f(Y_k)\}.
$$

Then, τ has a geometric distribution $G^*(p)$ with parameter $p := \mathbb{P}(c\,U_1 g(Y_1) \le f(Y_1)) = \dfrac{\int_E f\,d\mu}{c}$ and

$$
X := Y_\tau \overset{d}{=} \nu.
$$

Remark. The (random) *yield* of the method is $\frac{1}{\tau}$. Hnece we know that its mean yield is given by

$$
\mathbb{E}\left(\frac{1}{\tau}\right) = -\frac{p \log p}{1-p} = \frac{\int_E f\,d\mu}{c - \int_E f\,d\mu} \log\left(\frac{c}{\int_E f\,d\mu} \right).
$$

Since $\lim\limits_{p \to 1} -\dfrac{p \log p}{1-p} = 1$, the closer to $\displaystyle\int_{\mathbb{R}^d} f\,d\mu$ the constant c is, the higher the yield of the simulation is.

Proof. STEP 1: Let (U, Y) be a couple of random variables with distribution $U([0,1]) \otimes \mathbb{P}_Y$. Let

$h : \mathbb{R}^d \to \mathbb{R}$ be a bounded Borel test function. We have

$$
\begin{aligned}
\mathbb{E}\big(h(Y)\mathbf{1}_{\{cUg(Y)\leq f(Y)\}}\big) &= \int_{E\times[0,1]} h(y)\mathbf{1}_{\{cug(y)\leq f(y)\}}g(y)\mu(dy)\otimes du \\
&= \int_E \left[\int_{[0,1]} \mathbf{1}_{\{cug(y)\leq f(y)\}\cap\{g(y)>0\}}du\right] h(y)g(y)\mu(dy) \\
&= \int_E h(y)\left[\int_0^1 \mathbf{1}_{\{u\leq \frac{f(y)}{cg(y)}\}\cap\{g(y)>0\}}du\right] g(y)\mu(dy) \\
&= \int_{\{g(y)>0\}} h(y)\frac{f(y)}{cg(y)}g(y)\mu(dy) \\
&= \frac{1}{c}\int_E h(y)f(y)\mu(dy)
\end{aligned}
$$

where we used successively Fubini's Theorem, $g(y) > 0$ $\mu(dy)$-a.e., Fubini's Theorem again and $\frac{f}{cg}(y) \leq 1$ $\mu(dy)$-a.e. Note that we can apply Fubini's Theorem since the reference measure μ is σ-finite.

Putting $h \equiv 1$ yields

$$
c = \frac{\int f d\mu}{\mathbb{P}(cUg(Y) \leq f(Y))}.
$$

Then, elementary conditioning yields

$$
\mathbb{E}\big(h(Y)|\{cUg(Y) \leq f(Y)\}\big) = \int_E h(y)\frac{f(y)}{\int_E f\,d\mu}\mu(dy) = \int h(y)\nu(dy)
$$

i.e.

$$
\mathcal{L}\left(Y|\{cUg(Y) \leq f(Y)\}\right) = \nu.
$$

STEP 2: Then (using that τ is $\mathbb{P}$-*a.s.* finite)

$$
\begin{aligned}
\mathbb{E}(h(Y_\tau)) &= \sum_{n\geq 1}\mathbb{E}\left(\mathbf{1}_{\{\tau=n\}}h(Y_n)\right) \\
&= \sum_{n\geq 1}\mathbb{P}\big(\{cU_1g(Y_1) > f(Y_1)\}\big)^{n-1}\mathbb{E}\big(h(Y_1)\mathbf{1}_{\{cU_1g(Y_1)\leq f(Y_1)\}}\big) \\
&= \sum_{n\geq 1}(1-p)^{n-1}\mathbb{E}\big(h(Y_1)\mathbf{1}_{\{cU_1g(Y_1)\leq f(Y_1)\}}\big) \\
&= \sum_{n\geq 1}p(1-p)^{n-1}\mathbb{E}\big(h(Y_1)\,|\,\{cU_1g(Y_1) \leq f(Y_1)\}\big) \\
&= 1 \times \int h(y)\nu(dy) \\
&= \int h(y)\nu(dy). \quad \diamond
\end{aligned}
$$

Remark. An important point to be noticed is that we do not need to know the numerical value either of $\int_E f\,d\mu$ to implement the above acceptance-rejection procedure.

Corollary 1.1 *Set by induction for every $n \geq 1$*

$$\tau_1 := \min\{k \geq 1 \,|\, c\, U_k g(Y_k) \leq f(Y_k)\} \quad and \quad \tau_{n+1} := \min\{k \geq \tau_n + 1 \,|\, c\, U_k g(Y_k) \leq f(Y_k)\}.$$

(a) The sequence $(\tau_n - \tau_{n-1})_{n \geq 1}$ (with the convention $\tau_0 = 0$) is i.i.d. with distribution $G^*(p)$ and the sequence

$$X_n := Y_{\tau_n}$$

is an i.i.d. $\mathbb{P}_X$-distributed sequence of random variables.

(b) Furthermore the random yield of the simulation of the first n $\mathbb{P}_X$-distributed random variables Y_{τ_k}, $k = 1, \ldots, n$ is

$$\rho_n = \frac{n}{\tau_n} \overset{a.s.}{\longrightarrow} p \quad as \quad n \to +\infty.$$

Proof. (a) Left as an exercise (see below).

(b) The fact that $\rho_n = \frac{n}{\tau_n}$ is obvious. The announced $a.s.$ convergence follows form the Strong Law of Large Numbers since $(\tau_n - \tau_{n-1})_{n \geq 1}$ is i.i.d. and $\mathbb{E}\,\tau_1 = \frac{1}{p}$ which implies that $\frac{\tau_n}{n} \overset{a.s.}{\longrightarrow} \frac{1}{p}$. $\diamond$

▷ **Exercise.** Prove item (a) of the above corollary.

Before proposing first applications, let us briefly present a more applied point of view which is closer to what is really implemented in practice when performing a Monte Carlo simulation based on the acceptance-rejection method.

The user's viewpoint (practitioner's corner).   The practical implementation of the acceptance-rejection method is rather simple. Let $h : E \to \mathbb{R}$ be a $\mathbb{P}_X$-integrable Borel function. How to compute $\mathbb{E}\,h(X)$ using Von Neumann's acceptance-rejection method? It amounts to the simulation an n-sample $(U_k, Y_k)_{1 \leq k \leq n}$ on a computer and to the computation of

$$\frac{\displaystyle\sum_{k=1}^{n} \mathbf{1}_{\{cU_k g(Y_k) \leq f(Y_k)\}} h(Y_k)}{\displaystyle\sum_{k=1}^{n} \mathbf{1}_{\{cU_k g(Y_k) \leq f(Y_k)\}}}.$$

Note that

$$\frac{\displaystyle\sum_{k=1}^{n} \mathbf{1}_{\{cU_k g(Y_k) \leq f(Y_k)\}} h(Y_k)}{\displaystyle\sum_{k=1}^{n} \mathbf{1}_{\{cU_k g(Y_k) \leq f(Y_k)\}}} = \frac{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \mathbf{1}_{\{cU_k g(Y_k) \leq f(Y_k)\}} h(Y_k)}{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \mathbf{1}_{\{cU_k g(Y_k) \leq f(Y_k)\}}}, \quad n \geq 1.$$

Hence, owing to the strong Law of Large Number (see next chapter if necessary) this quantity *a.s.* converges as n goes to infinity toward

$$\frac{\int_0^1 du \int \mu(dy) \mathbf{1}_{\{cug(y) \leq f(y)\}} h(y) g(y)}{\int_0^1 du \int \mu(dy) \mathbf{1}_{\{cug(y) \leq f(y)\}} g(y)} = \frac{\int_E \frac{f(y)}{cg(y)} h(y) g(y) \mu(dy)}{\int_{\mathbb{R}^d} \frac{f(y)}{cg(y)} g(y) \mu(dy)}$$

$$= \frac{\int_E \frac{f(y)}{c} h(y) \mu(dy)}{\int_{\mathbb{R}^d} \frac{f(y)}{c} \mu(dy)}$$

$$= \int_{\mathbb{R}^d} h(y) \frac{f(y)}{\int_{\mathbb{R}^d} f \, d\mu} \mu(dy)$$

$$= \int_E h(y) \nu(dy).$$

This third way to present the same computations show that in term of practical implementation, this method is in fact very elementary.

CLASSICAL APPLICATIONS. ▷ *Uniform distributions on a bounded domain D.*

Let $D \subset a + [-M, M]^d$, $\lambda_d(D) > 0$ (where $a \in \mathbb{R}^d$, $M > 0$) and let $Y \overset{d}{=} U(a + [-M, M]^d)$, let $\tau_D := \min\{n \mid Y_n \in D\}$ where $(Y_n)_{n \geq 1}$ is an i.i.d. sequence defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the same distribution as Y. Then,

$$Y_{\tau_D} \overset{d}{=} U(D)$$

where $U(D)$ denotes the uniform distribution over D.

This follows from the above Proposition 1.2 with $E = a + [-M, M]^d$, $\mu := \lambda_{d|a+[-M,M]^d}$ (Lebesgue measure on $a + [-M, M]^d$),

$$g(u) := (2M)^{-d} \mathbf{1}_{a+[-M,M]^d}(u)$$

and

$$f(x) = \mathbf{1}_D(x) \leq \underbrace{(2M)^d}_{=:c} g(x)$$

so that $\dfrac{f}{\int_{\mathbb{R}^d} f \, d\mu} . \mu = U(D)$.

As a matter of fact, with the notations of the above proposition,

$$\tau = \min\left\{k \geq 1 \mid c\, U_k \leq \frac{f}{g}(Y_k)\right\}.$$

However, $\frac{f}{g}(y) > 0$ if and only if $y \in D$ and if so, $\frac{f}{g}(y) = 1$. Consequently $\tau = \tau_D$.

A standard application is to consider the unit ball of $\mathbb{R}^d$, $D := B_d(0; 1)$. When $d = 2$, this is involved in the so-called *polar method*, see below, for the simulation of $\mathcal{N}(0; I_2)$ random vectors.

▷ *The $\gamma(\alpha)$-distribution.*

Let $\alpha > 0$ and $\mathbb{P}_X(dx) = f_\alpha(x)\frac{dx}{\Gamma(\alpha)}$ where

$$f_\alpha(x) = x^{\alpha-1}e^{-x}\mathbf{1}_{\{x>0\}}(x).$$

(Keep in mind that $\Gamma(a) = \int_0^{+\infty} u^{a-1}e^{-u}du$, $a > 0$). Note that when $\alpha = 1$ the gamma distribution is but the exponential distribution. We will consider $E = (0, +\infty)$ and the reference σ-finite measure $\mu = \lambda_{1|(0,+\infty)}$.

– *Case $0 < \alpha < 1$*. We use the rejection method, based on the probability density

$$g_\alpha(x) = \frac{\alpha e}{\alpha + e}\left(x^{\alpha-1}\mathbf{1}_{\{0<x<1\}} + e^{-x}\mathbf{1}_{\{x\geq1\}}\right).$$

The fact that g_α is a probability density function follows from an elementary computation. First, one checks that $f_\alpha(x) \leq c_\alpha g_\alpha(x)$ for every $x \in \mathbb{R}_+$, where

$$c_\alpha = \frac{\alpha + e}{\alpha e}.$$

Note that this choice of c_α is optimal since $f_\alpha(1) = c_\alpha g_\alpha(1)$. Then, one uses the inverse distribution function to simulate the random variable with distribution $\mathbb{P}_Y(dy) = g_\alpha(y)\lambda(dy)$. Namely, if G_α denotes the distribution function of Y, one checks that, for every $x \in \mathbb{R}$,

$$G_\alpha(x) = \frac{e}{\alpha + e}x^\alpha\mathbf{1}_{\{0<x<1\}} + \frac{\alpha e}{\alpha + e}\left(\frac{1}{e} + \frac{1}{\alpha} - e^{-x}\right)\mathbf{1}_{\{x>1\}}$$

so that for every $u \in (0, 1)$,

$$G_\alpha^{-1}(u) = \left(\frac{\alpha + e}{e}u\right)^{\frac{1}{\alpha}}\mathbf{1}_{\{u<\frac{e}{\alpha+e}\}} - \log\left((1-u)\frac{\alpha + e}{\alpha e}\right)\mathbf{1}_{\{u\geq\frac{e}{\alpha+e}\}}.$$

Note that the computation of the Γ function is never required to implement this method.

– *Case $\alpha \geq 1$*. We rely on the following classical lemma about the γ distribution.

Lemma 1.1 *Let X' and X'' two independent random variables with distributions $\gamma(\alpha')$ and $\gamma(\alpha'')$ respectively. Then $X = X' + X''$ has a distribution $\gamma(\alpha' + \alpha'')$.*

Consequently, if $\alpha = n \in \mathbb{N}$, an induction based on the lemma shows that

$$X = \xi_1 + \cdots + \xi_n$$

where ξ_k are i.i.d. with exponential distribution since $\gamma(1) = \mathcal{E}(1)$. Consequently, if $U_1, \ldots, U_n$ are i.i.d. uniformly distributed random variables

$$X \overset{d}{=} -\log\left(\prod_{k=1}^n U_k\right).$$

To simulate a random variable with general distribution $\gamma(\alpha)$, one writes $\alpha = \lfloor\alpha\rfloor + \{\alpha\}$ where $\lfloor\alpha\rfloor := \max\{k \leq \alpha, \ k \in \mathbb{N}\}$ denotes the integral value of α and $\{\alpha\} \in [0, 1)$ its fractional part.

▷ **Exercises. 1.** Prove the above lemma.

2. Show that considering the normalized probability density function of the $\gamma(\alpha)$-distribution (which involves the computation of $\Gamma(\alpha)$) as function f_α will not improve the yield of the simulation.

3. Let $\alpha = \alpha' + n$, $\alpha' = \lfloor \alpha \rfloor \in [0, 1)$. Show that the yield of the simulation id given by $r = \frac{1}{n+\tau_\alpha}$ where τ_α has a $G^*(p_\alpha)$ distribution with p_α is related to the simulation of the $\gamma(\lfloor \alpha \rfloor)$-distribution. Show that

$$\bar{r} := \mathbb{E}r = -\frac{p}{(1-p)^{n+1}}\Big(\log p + \sum_{k=1}^{n} \frac{(1-p)^k}{k} \Big).$$

▷ *Acceptance-rejection method for a bounded density with compact support.*
Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be a bounded Borel function with compact support (hence integral with respect to the Lebesgue measure). If f can be computed at a reasonable cost, one may simulate the distribution $\nu = \frac{f}{\int_{\mathbb{R}^d} f.d\lambda_d}\lambda_d$ by simply considering a uniform distribution on a hypercube dominating f. To be more precise let $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)$, $c \in \mathbb{R}^d$ such that

$$\operatorname{supp}(f) = \overline{\{f \neq 0\}} \subset K = c + \prod_{i=1}^{d}[a_i, b_i].$$

Let $E = K$, let $\mu = \lambda_{d|E}$ be the reference measure and $g = \frac{1}{\lambda_d(K)}\mathbf{1}_K$ the density of the uniform distribution over K (this distribution is very easy to simulate as emphasized in a former example). Then

$$f(x) \le c\,g(x)\ x \in K \quad \text{with} \quad c = \|f\|_{\sup}\lambda_d(K) = \|f\|_{\sup}\prod_{i=1}^{d}(b_i - a_i).$$

Then, if $(Y_n)_{n\geq 1}$ denotes an i.i.d. sequence defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the uniform distribution over K, the stopping strategy τ of the Von Neumann's acceptance-rejection method reads

$$\tau = \min\big\{k \mid \|f\|_{\sup}U_k \le f(Y_k)\big\}.$$

Equivalently this van be rewritten in more intuitive way as follows: let $V_n = (V_n^1, V_n^2)_{n\geq 1}$ be an i.i.d. sequence of random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ having *a uniform distribution over* $K \times [0, \|f\|_\infty]$. Then

$$V_\tau^1 \overset{d}{=} \nu \quad \text{where} \quad \tau = \min\big\{k \ge 1 \mid V_k^2 \le f(V_k^1)\big\}.$$

▷ *Simulation of a discrete distribution supported by the non-negative integers.*
Let $Y \overset{d}{=} G^*(p)$, $p \in (0, 1)$. *i.e.* such its distribution satisfies $\mathbb{P}_Y = g.m$ where m is the counting measure on $\mathbb{N}^*$ ($m(k) = 1$, $k \in \mathbb{N}^*$) and $g_k = p(1-p)^{k-1}$, $k \ge 1$. Let $f = (f_k)_{k\geq 1}$ be a function from $\mathbb{N} \to \mathbb{R}_+$ defined by $f_k = a_k(1-p)^{k-1}$ and satisfying $\kappa := \sup_n a_n < +\infty$ (so that $\sum_n f_n < +\infty$). Then

$$f_k \le c\,g_k \quad \text{with} \quad c = \frac{\kappa}{p}.$$

Consequently, if $\tau := \min\big\{k \ge 1 \mid U_k \le \frac{pa_k}{\kappa}\big\}$, then $Y_\tau \overset{d}{=} \nu$ where $\nu_k := \dfrac{a_k(1-p)^{k-1}}{\sum_n a_n(1-p)^{n-1}}$, $k \ge 1$.

There are many other applications of Von Neumann's acceptance-rejection method, *e.g.* in Physics, to take advantage of the fact the density to be simulated is only known up to constant. Several methods have been devised to speed it up *i.e.* to increase its yield. Among them let us cite the *Zigurat method* for which we refer to [112]. It has been developed by Marsaglia and Tsang in the early 2000's.

1.5   Simulation of Poisson distributions (and Poisson processes)

The Poisson distribution with parameter $\lambda > 0$, denoted $\mathcal{P}(\lambda)$, is an integral valued probability measure analytically defined by

$$\forall\, k \in \mathbb{N}, \quad \mathcal{P}(\lambda)(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

To simulate this distribution in an exact way, one relies on its close connection with the Poisson counting process. The (normalized) Poisson counting process is the counting process induced by the Exponential random walk (with parameter 1). It is defined by

$$\forall\, t \geq 0, \quad N_t = \sum_{n\geq 1} \mathbf{1}_{\{S_n \leq t\}} = \min\left\{n \mid S_{n+1} > t\right\}$$

where $S_n = X_1 + \cdots + X_n$, $n \geq 1$, $S_0 = 0$, $(X_n)_{n\geq 1}$ is an i.i.d. sequence of random variables with distribution $\mathcal{E}(1)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Proposition 1.3 *The process $(N_t)_{t\geq 0}$ has càdlàg ([1]) paths, independent stationary increments. In particular for every s, $t \geq 0$, $s \leq t$, $N_t - N_s$ is independent of N_s and has the same distribution as N_{t-s}. Furthermore, for every $t \geq 0$, N_t has a Poisson distribution with parameter t.*

Proof. Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. random variables with an exponential distribution $\mathcal{E}(1)$. Set, for every $n \geq 1$,
$$S_n = X_1 + \cdots + X_n,$$
Let t_1, $t_2 \in \mathbb{R}_+$, $t_1 < t_2$ and let k_1, $k_2 \in \mathbb{N}$. Assume temporarily $k_2 \geq 1$.

$$\begin{aligned}
\mathbb{P}(N_{t_2} - N_{t_1} = k_2,\, N_{t_1} = k_1) &= \mathbb{P}(N_{t_1} = k_1,\, N_{t_2} = k_1 + k_2)\\
&= \mathbb{P}(S_{k_1} \leq t_1 < S_{k_1+1} \leq S_{k_1+k_2} \leq t_2 < S_{k_1+k_2+1}).
\end{aligned}$$

Now, if set $A = \mathbb{P}(S_{k_1} \leq t_1 < S_{k_1+1} \leq S_{k_1+k_2} \leq t_2 < S_{k_1+k_2+1})$ for convenience, we get

$$A = \int_{\mathbb{R}_+^{k_1+k_2+1}} \mathbf{1}_{\{x_1+\cdots+x_{k_1} \leq t_1 \leq x_1+\cdots+x_{k_1+1},\, x_1+\cdots+x_{k_1+k_2} \leq t_2 < x_1+\cdots+x_{k_1+k_2+1}\}} e^{-(x_1+\cdots+x_{k_1+k_2+1})}dx_1\cdots dx_{k_1+k_2+1}$$

The usual change of variable $x_1 = u_1$ and $x_i = u_i - u_{i-1}$, $i = 2, \ldots, k_1 + k_2 + 1$, yields

$$A = \int_{\{0\leq u_1 \leq \cdots \leq u_{k_1} \leq t_1 \leq u_{k_1+1} \leq \cdots \leq u_{k_1+k_2} \leq t_2 < u_{k_1+k_2+1}\}} e^{-u_{k_1+k_2+1}}du_1\cdots du_{k_1+k_2+1}$$

---

[1]French acronym for right continuous with left limits (continu à droite limitée à gauche).

Integrating downward from $u_{k_1+k_2+1}$ down to u_1 we get owing to Fubini's Theorem,

$$
\begin{aligned}
A &= \int_{\{0 \leq u_1 \leq \cdots \leq u_{k_1} \leq t_1 \leq u_{k_1+1} \leq \cdots \leq u_{k_1+k_2} \leq t_2 < u_{k_1+k_2+1}\}} du_1 \cdots du_{k_1+k_2}\, e^{-t_2} \\
&= \int_{\{0 \leq u_1 \leq \cdots \leq u_{k_1} \leq t_1\}} du_1 \cdots du_{k_1} \frac{(t_2 - t_1)^{k_2}}{k_2!} e^{-t_2} \\
&= \frac{t_1^{k_1}}{k_1!} \frac{(t_2 - t_1)^{k_2}}{k_2!} e^{-t_2} \\
&= e^{-t_1} \frac{t_1^{k_1}}{k_1!} \times e^{-(t_2 - t_1)} \frac{(t_2 - t_1)^{k_2}}{k_2!}.
\end{aligned}
$$

When $k_2 = 0$, one computes likewise

$$
\mathbb{P}(S_{k_1} \leq t_1 < t_2 < S_{k_1+1}) = \frac{t_1^{k_1}}{k_1!} \times e^{-t_2} = e^{-t_1} \frac{t_1^{k_1}}{k_1!} \times e^{-(t_2-t_1)}.
$$

Summing over $k_2 \in \mathbb{N}$ shows that, for every $k_1 \in \mathbb{N}$,

$$
\mathbb{P}(N_{t_1} = k_1) = e^{-t_1} \frac{t_1^{k_1}}{k_1!}
$$

i.e. $N_{t_1} \overset{d}{=} \mathcal{P}(t_1)$. Summing over $k_1 \in \mathbb{N}$ shows that, for every $k_2 \in \mathbb{N}$,

$$
N_{t_2} - N_{t_1} \overset{d}{=} N_{t_2 - t_1} \overset{d}{=} \mathcal{P}(t_2 - t_1).
$$

Finally, this yields for every $k_1, k_2 \in \mathbb{N}$,

$$
\mathbb{P}(N_{t_2} - N_{t-1} = k_2,\, N_{t_1} = k_1) = \mathbb{P}(N_{t_2} - N_{t_1} = k_2) \times \mathbb{P}(N_{t_1} = k_1)
$$

i.e. the increments $N_{t_2} - N_{t_1}$ and N_{t_1} are independent.

One shows likewise, with a bit more technicalities, that in fact, if $0 < t_1 < t_2 < \cdots < t_n$, $n \geq 1$, then the increments $(N_{t_i} - N_{t_{i-1}})_{i=1,\dots,n}$ are independent and stationary in the sense that $N_{t_i} - N_{t_{i-1}} \overset{d}{=} \mathcal{P}(t_i - t_{i-1})$. $\diamond$

Corollary 1.2 (Simulation of a Poisson distribution) *Let $(U_n)_{n \geq 1}$ be an i.i.d. sequence of uniformly distributed random variables on the unit interval. The process null at zero and defined for every $t > 0$ by*

$$
N_t = \min \{k \geq 0 \,|\, U_1 \cdots U_{k+1} < e^{-t}\} \overset{d}{=} \mathcal{P}(t)
$$

is a normalized Poisson process.

Proof. It follows from Example 1 in the former Section 1.3 that the exponentially distributed i.i.d. sequence $(X_k)_{k \geq 1}$ can be written in the following form

$$
X_k = -\log U_k, \quad k \geq 1.
$$

Using that the random walk $(S_n)_{n \geq 1}$ is non-decreasing it follows from the definition of a Poisson process that for every $t \geq 0$,

$$
\begin{aligned}
N_t &= \min\{k \geq 0 \text{ such that } S_{k+1} > t\}, \\
&= \min \{k \geq 0 \text{ such that } -\log\left(U_1 \cdots U_{k+1}\right) > t\}, \\
&= \min \{k \geq 0 \text{ such that } U_1 \cdots U_{k+1} < e^{-t}\}. \qquad\qquad \diamond
\end{aligned}
$$

1.6   The Box-Müller method for normally distributed random vectors

1.6.1 d-dimensional normally distributed random vectors

One relies on the Box-Müller method, which is probably the most efficient method to simulate the bi-variate normal distribution $\mathcal{N}(0; I_2)$.

Proposition 1.4 *Let R^2 and $\Theta : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be two independent r.v. with distributions $\mathcal{L}(R^2) = \mathcal{E}(\frac{1}{2})$ and $\mathcal{L}(\Theta) = U([0, 2\pi])$ respectively. Then*

$$X := (R \cos \Theta, R \sin \Theta) \overset{d}{=} \mathcal{N}(0, I_2)$$

where $R := \sqrt{R^2}$.

Proof. Let f be a bounded Borel function.

$$\iint_{\mathbb{R}^2} f(x_1, x_2) \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) \frac{dx_1 dx_2}{2\pi} = \iint f(\rho \cos \theta, \rho \sin \theta) e^{-\frac{\rho^2}{2}} \mathbf{1}_{\mathbb{R}_+^*}(\rho) \mathbf{1}_{(0, 2\pi)}(\theta) \rho \frac{d\rho d\theta}{2\pi}$$

using the standard change of variable: $x_1 = \rho \cos \theta, x_2 = \rho \sin \theta$. We use the facts that $(\rho, \theta) \mapsto (\rho \cos \theta, \rho \sin \theta)$ is a $\mathcal{C}^1$-diffeomorphism from $(0, 2\pi) \times (0, \infty) \to \mathbb{R}^2 \setminus (\mathbb{R}_+ \times \{0\})$ and $\lambda_2(\mathbb{R}_+ \times \{0\}) = 0$. Setting now $\rho = \sqrt{r}$, one has:

$$
\begin{aligned}
\iint_{\mathbb{R}^2} f(x_1, x_2) \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) \frac{dx_1 dx_2}{2\pi} &= \iint f(\sqrt{r} \cos \theta, \sqrt{r} \sin \theta) \frac{e^{-\frac{r}{2}}}{2} \mathbf{1}_{\mathbb{R}_+^*}(\rho) \mathbf{1}_{(0, 2\pi)}(\theta) \frac{dr d\theta}{2\pi} \\
&= \mathbb{E}\left(f\left(\sqrt{R^2} \cos \Theta, \sqrt{R^2} \sin \Theta\right)\right) \\
&= \mathbb{E}(f(X)). \qquad \diamond
\end{aligned}
$$

Corollary 1.3 (Box-Müller method) *One can simulate a distribution $\mathcal{N}(0; I_2)$ from a couple (U_1, U_2) of independent random variable with distribution $U([0, 1])$ by setting*

$$X := \left(\sqrt{-2 \log(U_1)} \cos(2\pi U_2), \sqrt{-2 \log(U_1)} \sin(2\pi U_2)\right).$$

The yield of the simulation is $r = 1/2$ with respect to the $\mathcal{N}(0; 1)\ldots$ and $r = 1$ when the aim is to simulate a $\mathcal{N}(0; I_2)$-dsitributed (psuedo-)random vector or, equivalently, two $\mathcal{N}(0; 1)$-distributed (pseudo-)random numbers.

Proof. Simulate the exponential distribution using the inverse distribution function using $U_1 \overset{d}{=} U([0, 1])$ and note that if $U_2 \overset{d}{=} U([0, 1])$, then $2\pi U_2 \overset{d}{=} U([0, 2\pi])$ (where U_2 is taken independent of U_1. $\quad \diamond$

APPLICATION TO THE SIMULATION OF THE MULTIVARIATE NORMAL DISTRIBUTION To simulate a d-dimensional vector $\mathcal{N}(0; I_d)$, one may assume that d is even and "concatenate" the above process.

We consider a d-tuple $(U_1, \ldots, U_d) \overset{d}{=} U([0,1]^d)$ random vector (so that $U_1, \ldots, U_d$ are i.i.d. with distribution $U([0,1])$) and we set

$$(X_{2i-1}, X_{2i}) = \left(\sqrt{-2\log(U_{2i-1})} \cos(2\pi U_{2i}), \sqrt{-2\log(U_{2i-1})} \sin(2\pi U_{2i}) \right), \quad i = 1, \ldots, d/2.$$

▷ **Exercise (Marsaglia's Polar method).** See [112] Let $(V_1, V_2) \overset{d}{=} U(B(0;1))$ where $B(0,1)$ denotes the canonical Euclidean unit ball in $\mathbb{R}^2$. Set $R^2 = V_1^2 + V_2^2$ and

$$X := \left(V_1 \sqrt{-2\log(R^2)/R^2}, V_2 \sqrt{-2\log(R^2)/R^2} \right).$$

(a) Show that $R^2 \overset{d}{=} U([0,1])$, $\left(\frac{V_1}{R}, \frac{V_2}{R} \right) \sim \left(cos(\theta), \sin(\theta) \right)$, R^2 and $\left(\frac{V_1}{R}, \frac{V_2}{R} \right)$ are independent. Deduce that $X \overset{d}{=} \mathcal{N}(0; I_2)$.

(b) Let $(U_1, U_2) \overset{d}{=} U([-1,1]^2)$. Show that $\mathcal{L}\left((U_1, U_2) \,|\, U_1^2 + U_2^2 \le 1 \right) = U\left(B(0;1) \right)$. Derive a simulation method for $\mathcal{N}(0; I_2)$ combining the above identity and an appropriate acceptance-rejection algorithm to simulate the $\mathcal{N}(0; I_2)$ distribution. What us the yield of the resulting procedure.

(c) Compare the performances of this so-called Marsaglia's polar method with those of the Box-Müller algorithm (*i.e.* the acceptance-rejection rule *versus* the computation of trigonometric functions). Conclude.

1.6.2 Correlated d-dimensional Gaussian vectors, Gaussian processes

Let $X = (X^1, \ldots, X^d)$ be a centered $\mathbb{R}^d$-valued Gaussian vector with covariance matrix $\Sigma = \left[\mathrm{cov}(X^i, X^j) \right]_{1 \le i,j \le d}$. The matrix Σ is symmetric non-negative (but possibly non-invertible). It follows from the very definition of Gaussian vectors that any linear combination $(u|X) = \sum_i u^i X^i$ has a (centered) Gaussian distribution with variance $u^* \Sigma u$ so that the characteristic function of X is given by

$$\Phi_X(u) := \mathbb{E}\left(e^{i(u|X)} \right) = e^{-\frac{1}{2} u^* \Sigma u}, \quad u \in \mathbb{R}^d.$$

As a well-known consequence the covariance matrix Σ characterizes the distribution of X which allows us to denote $\mathcal{N}(0; \Sigma)$ the distribution of X.

The key to simulate such a random vector is the following general lemma (which has nothing to do with Gaussian vectors). It describes how the covariance is modified by a linear transform.

Lemma 1.2 *Let Y be an $\mathbb{R}^d$-valued square integrable random vector and let $A \in \mathcal{M}(q,d)$ be a $q \times d$ matrix. Then the covariance matrix C_{AY} of the random vector AY is given by*

$$C_{AY} = A C_Y A^*$$

where A^* stands for the transpose of A.

This result can be used in two different ways.

– *Square root of Σ.* It is classical background that there exists a unique non-negative symmetric matrix commuting with Σ, denoted $\sqrt{\Sigma}$, such that $\Sigma = \sqrt{\Sigma}^2 = \sqrt{\Sigma}\sqrt{\Sigma}^*$. Consequently, owing to the above lemma,

$$\text{If} \qquad Z \overset{d}{=} \mathcal{N}(0; I_d) \qquad \text{then} \qquad \sqrt{\Sigma} Z \overset{d}{=} \mathcal{N}(0; \Sigma).$$

One can compute $\sqrt{\Sigma}$ by diagonalizing Σ in the orthogonal group: if $\Sigma = P\mathrm{Diag}(\lambda_1, \ldots, \lambda_d)P^*$ with $PP^* = I_d$ and $\lambda_1, \ldots, \lambda_d \geq 0$. Then, by uniqueness of the square root of Σ as defined above, it is clear that $\sqrt{\Sigma} = P\mathrm{Diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d})P^*$.

– *Cholesky decomposition of* Σ. When the covariance matrix Σ is invertible ((*i.e.* definite), it is much more efficient to rely on the Cholesky decomposition (see *e.g.* Numerical Recipes [118]) by decomposing Σ as

$$\Sigma = TT^*$$

where T is a lower triangular matrix (*i.e.* such that $T_{ij} = 0$ if $i < j$). Then, owing to Lemma 1.2,

$$TZ \stackrel{d}{=} \mathcal{N}(0; \Sigma).$$

In fact, the Cholevsky decomposition is the matrix formulation of the Hilbert-Schmidt orthonormalization procedure. In particular, there exists *a unique such lower triangular matrix T with positive diagonal terms* ($T_{ii} > 0$, $i = 1, \ldots, d$) called the Cholesky matrix of Σ.

The Cholesky based approach is more performing since it approximately divides the complexity of this phase of the simulation almost by a factor 2.

▷ **Exercises. 1.** Let $Z = (Z_1, Z_2)$ be a Gaussian vector such that $Z_1 \stackrel{d}{=} Z_2 \stackrel{d}{=} \mathcal{N}(0; 1)$ and $\mathrm{Cov}(Z_1, Z_2) = \rho \in [-1, 1]$.

(*a*) Compute for every $u \in \mathbb{R}^2$ the Laplace transform $L(u) = \mathbb{E}\, e^{(u|Z)}$ of Z.

(*b*) Compute for every σ_1, $\sigma_2 > 0$ the correlation ([2]) ρ_{X_1, X_2} between the random variables $X_1 = e^{\sigma_1 Z_1}$ and $X_2 = e^{\sigma_2 Z_2}$.

(*c*) Show that $\inf_{\rho \in [-1,1]} \rho_{X_1, X_2} \in (-1, 0)$ and that, when $\sigma_i = \sigma > 0$, $\inf_{\rho \in [-1,1]} \rho_{X_1, X_2} = -e^{-\sigma^2}$.

2. Let Σ be a positive definite matrix. Show the existence of a unique lower triangular matrix T and a diagonal matrix D such that both T and D have positive diagonal entries and $\sum_{i,j=1} T_{ij}^2 = 1$ for every $i = 1, \ldots, d$. [Hint: change the reference Euclidean norm to perform the Hilbert-Schmidt decomposition] ([3]).

APPLICATION TO THE SIMULATION OF THE STANDARD BROWNIAN MOTION AT FIXED TIMES. Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $(t_1, \ldots, t_n)$ be a non-decreasing $n$tuples $(0 \leq t_1 < t_2 < \ldots, t_{n-1} < t_n)$ of instants. One elementary definition of the standard Brownian motion is that it is a centered Gaussian process with covariance given by $C^W(s, t) = \mathbb{E}(W_s W_t) = s \wedge t$ ([4]). The resulting simulation method relying on the Cholesky decomposition of the covariance structure of the Gaussian vector $(W_{t_1}, \ldots, W_{t_n})$ given by

$$\Sigma^W_{(t_1, \ldots, t_n)} = [t_i \wedge t_j]_{1 \leq i, j \leq n}$$

---

[2]The correlation ρ_{X_1, X_2} between two square integrable non *a.s.* constant random variables defined on the same probability space is defined by

$$\rho_{X_1, X_2} = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma(X_1)\sigma(X_2)} = \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}}.$$

[3]This modified Cholesky decomposition is faster than the standard one (see *e.g.* [154]) since it avoids square root computations even if, in practice, the cost of the decomposition itself remains negligible compared to that of a large-scale Monte Carlo simulation.

[4]This definition does not include the fact that W has continuous paths, however, it can be derived using the celebrated Kolmogorov criterion, that it has a modification with continuous paths (see *e.g.* [140]).

is a first possibility.

However, it seems more natural to use the independence and the stationarity of the increments *i.e.* that

$$\left(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}\right) \overset{d}{=} \mathcal{N}\left(0; \mathrm{Diag}(t_1, t_2 - t_1, \ldots, t_n - t_{n-1})\right)$$

so that

$$\begin{bmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ \vdots \\ W_{t_n} - W_{t_{n-1}} \end{bmatrix} \overset{d}{=} \mathrm{Diag}\left(\sqrt{t_1}, \sqrt{t_2 - t_1}, \ldots, \sqrt{t_n - t_{n-1}}\right) \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$$

where $(Z_1, \ldots, Z_n) \overset{d}{=} \mathcal{N}\left(0; I_n\right)$. The simulation of $(W_{t_1}, \ldots, W_{t_n})$ follows by summing up the increments.

Remark. To be more precise, one derives from the above result that

$$\begin{bmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{bmatrix} = L \begin{bmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ \vdots \\ W_{t_n} - W_{t_{n-1}} \end{bmatrix} \qquad \text{where} \qquad L = \left[\mathbf{1}_{\{i \geq j\}}\right]_{1 \leq i,j \leq n}.$$

Hence, if we set $T = L\,\mathrm{Diag}\left(\sqrt{t_1}, \sqrt{t_2 - t_1}, \ldots, \sqrt{t_n - t_{n-1}}\right)$ one checks on the one hand that $\left[\sqrt{t_j - t_{j-1}}\mathbf{1}_{\{i \geq j\}}\right]_{1 \leq i,j \leq n}$ and on the other hand that

$$\begin{bmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{bmatrix} = T \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}.$$

We derive, owing to Lemma 1.2, that $TT^* = TI_nT^* = \Sigma^W_{t_1,\ldots,t_n}$. The matrix T being lower triangular, it provides the Cholesky decomposition of the covariance matrix $\Sigma^W_{t_1,\ldots,t_n}$.

PRACTITIONER'S CORNER: WARNING! In quantitative finance, especially when modeling the dynamics of several risky assets, say d, the correlation between the Brownian sources of randomness $B = (B^1, \ldots, B^d)$ attached to the log-return is often misleading in terms of notations: since it is usually written as

$$\forall\, i \in \{1, \ldots, d\}, \ B_t^i = \sum_{j=1}^q \sigma_{ij} W_t^j.$$

where $W = (W^1, \ldots, W^q)$ is a standard q-dimensional Brownian motion (*i.e.* each component W^j, $j = 1, \ldots, q$, is a standard Brownian motion and all these components are independent). The normalized covariance matrix of B (at time 1) is given by

$$\mathrm{Cov}\left(B_1^i, B_1^j,\right) = \sum_{\ell=1}^q \sigma_{i\ell}\sigma_{j\ell} = \left(\sigma_{i.}|\sigma_{j.}\right) = (\sigma\sigma^*)_{ij}$$

where $\sigma_{i\cdot}$ is the column vector $[\sigma_{ij}]_{1\leq j\leq q}$, $\sigma = [\sigma_{ij}]_{1\leq i\leq d,1\leq j\leq q}$ and $(.|.)$ denotes here the canonical inner product on $\mathbb{R}^q$. So one should proceed the Cholesky decomposition on the symmetric non-negative matrix $\Sigma_{B_1} = \sigma\sigma^*$. Also have in mind that, if $q < d$, then σ_{B_1} has rank at most q and cannot be invertible.

APPLICATION TO THE SIMULATION OF THE FRACTIONAL BROWNIAN MOTION AT FIXED TIMES.

The fractional Brownian motion (fBm) $W^H = (W_t^H)_{t\geq 0}$ with Hurst coefficient $H \in (0,1)$ is defined as a centered Gaussian process with a correlation function given for every $s,t \in \mathbb{R}_+$ by

$$C^{W^H}(s,t) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t - s|^{2H}\Big).$$

When $H = \frac{1}{2}$, W^H is simply a standard Brownian motion. When $H \neq \frac{1}{2}$, W^H has none of the usual properties of the Brownian motion (except the stationarity of its increments and some self-similarity properties) : it has dependent increments, it is not a martingale (or even a semi-martingale). It is not either a Markov process.

A natural approach to simulate the fBm W^H at times $t_1, \ldots, t_n$ is to rely on the Cholesky decomposition of its auto-covariance matrix $[C^{W^H}(t_i, t_j)]_{1\leq i,j\leq n}$. On the one hand, this matrix is ill-conditioned which induces instability in the computation of the Cholesky decomposition. This should be out in balance with the fact that such a computation can be made only once and offline to be stored (at least for usual values of the t_is like $T_i = \frac{iT}{n}$, $i = 1, \ldots, n$).

Other methods have been introduced based in which the auto-covariance matrix is embed in a circuit matrix. It relies on a (fast) Fourier transform procedure. This method, originally introduced in [41], has been recently improved in [156] where a precise algorithm is described.

Chapter 2

Monte Carlo method and applications to option pricing

2.1 The Monte Carlo method

The basic principle of the so-called *Monte Carlo* method is to implement on a computer the Strong Law of Large Number (*SLLN*): if $(X_k)_{k\geq 1}$ denotes a sequence (defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$) of independent copies of an integrable random variable X then

$$\mathbb{P}(d\omega)\text{-}a.s. \qquad \overline{X}_M(\omega) := \frac{X_1(\omega) + \cdots + X_M(\omega)}{M} \overset{M\to+\infty}{\longrightarrow} m_X := \mathbb{E}\, X.$$

The sequence $(X_k)_{k\geq 1}$ is also called an i.i.d. sequence of random variables with distribution $\mu = \mathbb{P}_X$ (that of X) or an (infinite) *sample* of the distribution μ. Two conditions are requested to *implement the above SLLN on a computer* (which can be seen as the true definition of a *Monte Carlo simulation* of the distribution μ or of the random vector X).

$\triangleright$ One can generate on the computer some as perfect as possible (pseudo-)random numbers which can be seen as a "generic" sequence $(U_k(\omega))_{k\geq 1}$ of a sample $(U_k)_{k\geq 1}$ of the uniform distribution on $[0, 1]$. Note that $((U_{(k-1)d+\ell}(\omega))_{1\leq\ell\leq d})_{k\geq 1}$ is then generic for the sample $((U_{(k-1)d+\ell})_{1\leq\ell\leq d})_{k\geq 1}$ of the uniform distribution on $[0, 1]^d$ for any $d \geq 1$ (theoretically speaking) since $U([0,1]^d) = \big(U([0,1])\big)^{\otimes d}$. This problem has already been briefly discussed in the introductory chapter.

$\triangleright$ One can write "represent" X either as

  – $X = \varphi(U)$ where U is a uniformly distributed random vector on $[0, 1]^d$ where the Borel function $\varphi : u \mapsto \varphi(u)$ can be computed at any $u \in [0, 1]^d$

 or

  – $X = \varphi_\tau(U_1, \ldots, U_\tau)$ where $(U_n)_{n\geq 1}$ is a sequence of i.i.d. uniformly distributed over $[0, 1]$, τ is a simulatable *finite stopping rule* (or stopping time) for the sequence $(U_k)_{k\geq 1}$, taking values in $\mathbb{N}^*$. By "stopping rule" we mean that the event $\{\tau = k\} \in \sigma(U_1, \ldots, U_k)$ for every $k \geq 1$ and by simulatable we mean that $\mathbf{1}_{\{\tau=k\}} = \psi_k(U_1, \ldots, U_k)$ where ψ_k has an explicit form for every $k \geq 1$. We also assume that for every $k \geq 1$, the function φ_k is a computable functions (defined on the set of finite $[0, 1]^k$-valued sequences) as well.

This procedure is the core of the Monte Carlo simulation. We provided several examples of such representations in the previous chapter. For further developments on that wide topic, we refer to [42] and the references therein, but in some way an important part of the scientific activity in Probability Theory is motivated by or can be applied to simulation.

Once these conditions are fulfilled, one can perform a Monte Carlo simulation. But this leads to two important issues:
– what about the rate of convergence of the method?
and
– how can the resulting error be controlled?
The answer to these questions call upon fundamental results in Probability Theory and Statistics.

▷ *Rate(s) of convergence.* The (weak) rate of convergence in the $SLLN$ is ruled by the Central Limit Theorem (CLT) which says that if X is square integrable $(X \in L^2(\mathbb{P}))$, then

$$\sqrt{M}\left(\overline{X}_M - m_X\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0; \sigma_X^2) \quad \text{as} \quad M \to +\infty$$

where $\sigma_X^2 = \mathrm{Var}(X) := \mathbb{E}((X - \mathbb{E}\,X)^2) = \mathbb{E}(X^2) - (\mathbb{E}\,X)^2$ is the variance (its square root σ_X is called *standard deviation*) ([1]). Also note that the mean quadratic rate of convergence (*i.e.* the rate in $L^2(\mathbb{P})$) is exactly

$$\|\overline{X}_M - m_X\|_2 = \frac{\sigma_X}{\sqrt{M}}.$$

If $\sigma_X = 0$, then $\overline{X}_M = X = m_X$ $\mathbb{P}$-*a.s.*. So, from now on, we may assume without loss of generality that $\sigma_X > 0$.

The Law of the Iterated Logarithm (LIL) provides an *a.s.* rate of convergence, namely

$$\varlimsup_{M}\sqrt{\frac{M}{2\log(\log M)}}\left(\overline{X}_M - m_X\right) = \sigma_X \quad \text{and} \quad \varliminf_{M}\sqrt{\frac{M}{2\log(\log M)}}\left(\overline{X}_M - m_X\right) = -\sigma_X.$$

A proof of this (difficult) result can be found *e.g.* in [148]. All these rates stress the main drawback of the Monte Carlo method: it is a slow method since dividing the error by 2 needs to increase the size of the simulation by 4.

---

[1] The symbol $\xrightarrow{\mathcal{L}}$ stands for the convergence in distribution (or "in law" whence the notation "$\mathcal{L}$") : a sequence of random variables $(Y_n)_{n \geq 1}$ converges in distribution toward Y_∞ if

$$\forall f \in \mathcal{C}_b(\mathbb{R}, \mathbb{R}), \quad \mathbb{E}f(Y_n) \longrightarrow \mathbb{E}f(Y_\infty) \quad \text{as} \quad n \to +\infty.$$

It can be defined equivalently as the weak convergence of the distributions $\mathbb{P}_{Y_n}$ toward the distribution $\mathbb{P}_{Y_\infty}$. Convergence in distribution is also characterized by the following property

$$\forall A \in \mathcal{B}(\mathbb{R}), \ \mathbb{P}(Y_\infty \in \partial A) = 0 \Longrightarrow \mathbb{P}(Y_n \in A) \longrightarrow \mathbb{P}(Y_\infty \in A) \quad \text{as} \quad n \to +\infty.$$

The extension to $\mathbb{R}^d$-valued random vectors is straightforward. See [24] for a presentation of weak convergence of probability measures in a general framework.

▷ *Data driven control of the error: confidence level and confidence interval.* Assume that $\sigma_X > 0$ (otherwise the problem is empty). As concerns the control of the error, one relies on the CLT. It is obvious that the CLT also reads

$$\sqrt{M}\frac{\overline{X}_M - m_X}{\sigma_X} \xrightarrow{\mathcal{L}} \mathcal{N}(0;1) \quad \text{as} \quad M \to +\infty.$$

Furthermore, the normal distribution having a density, it has no atom. Consequently, this convergence implies (in fact it is equivalent, see [24]) that for every real numbers $a > b$,

$$\lim_M \mathbb{P}\left(\sqrt{M}\frac{\overline{X}_M - m_X}{\sigma_X} \in [b, a]\right) = \mathbb{P}(\mathcal{N}(0;1) \in [b, a]) \quad \text{as} \quad M \to +\infty.$$
$$= \Phi_0(a) - \Phi_0(b)$$

where Φ_0 denotes the distribution function of the standard normal distribution *i.e.*

$$\Phi_0(x) = \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\frac{d\xi}{\sqrt{2\pi}}.$$

▷ **Exercise.** Show that, as well as any distribution function of a symmetric random variable without atom, the distribution function of the centered normal distribution on the real line satisfies

$$\forall\, x \in \mathbb{R}, \qquad \Phi_0(x) + \Phi_0(-x) = 1.$$

Deduce that $\mathbb{P}(|\mathcal{N}(0;1)| \leq a) = 2\Phi_0(a) - 1$ for every $a > 0$.

In turn, if $X_1 \in L^3(\mathbb{P})$, the convergence rate in the Central Limit Theorem is ruled by the Berry-Essen Theorem (see [148]). It turns out to be again $1/\sqrt{M}$ ([2]) which is rather slow from a statistical viewpoint but is not a real problem within the usual range of Monte Carlo simulations (at least many thousands, usually one hundred thousands or one million paths). Consequently, one can assume that $\sqrt{M}\dfrac{\overline{X}_M - m_X}{\sigma_X}$ does have a standard normal distribution. Which means in particular that one can design a *probabilistic control of the error* directly derived from statistical concepts: let $\alpha \in (0, 1)$ denote a *confidence level* (close to 1) and let q_α be the *two-sided α-quantile* defined as the unique solution to the equation

$$\mathbb{P}(|\mathcal{N}(0;1)| \leq q_\alpha) = \alpha \qquad i.e. \qquad 2\Phi_0(q_\alpha) - 1 = \alpha.$$

Then, setting $a = q_\alpha$ and $b = -q_\alpha$, one defines the theoretical random confidence interval at level α

$$J_M^\alpha := \left[\overline{X}_M - q_\alpha\,\frac{\sigma_X}{\sqrt{M}}, \overline{X}_M + q_\alpha\,\frac{\sigma_X}{\sqrt{M}}\right],$$

---

[2]Namely, if $\sigma_X > 0$,

$$\forall\, n \geq 1, \quad \forall\, x \in \mathbb{R}, \quad \left|\mathbb{P}(\sqrt{M}(\bar{X}_M - m_X) \leq x\sigma_X) - \Phi_0(x)\right| \leq \frac{C\,\mathbb{E}|X_1 - \mathbb{E}X_1|^3}{\sigma_X^3\,\sqrt{M}}\,\frac{1}{1 + |x|^3}.$$

See [148].

which satisfies

$$\mathbb{P}(m_X \in J_M^\alpha) \quad = \quad \mathbb{P}\left(\sqrt{M}\frac{|\overline{X}_M - m_X|}{\sigma_X} \leq q_\alpha\right)$$

$$\overset{M \to +\infty}{\longrightarrow} \quad \mathbb{P}(|\mathcal{N}(0;1)| \leq q_\alpha) = \alpha.$$

However, at this stage this procedure remains purely theoretical since the confidence interval J_M involves the standard deviation σ_X of X which is usually unknown.

Here comes out the "trick" which made the tremendous success of the Monte Carlo method: the variance of X can in turn be evaluated on-line by simply adding a *companion Monte Carlo simulation* to estimate the variance σ_X^2, namely

$$\overline{V}_M \quad = \quad \frac{1}{M-1}\sum_{k=1}^{M}(X_k - \overline{X}_M)^2$$

$$= \quad \frac{1}{M-1}\sum_{k=1}^{M}X_k^2 - \frac{M}{M-1}\overline{X}_M^2 \longrightarrow \mathrm{Var}(X) = \sigma_X^2 \text{ as } M \to +\infty.$$

The above convergence (^{3}) follows from the *SLLN* applied to the i.i.d. sequence of integrable random variables $(X_k^2)_{k\geq 1}$ (and the convergence of $\overline{X}_M$ which follows from the *SLLN* as well). It is an easy exercise to show that moreover $\mathbb{E}(\overline{V}_M) = \sigma_X^2$ *i.e.*, with the terminology of Statistics, $\overline{V}_M$ is an *unbiased* estimator of σ_X^2. This remark is of little importance in practice due to the usual – large – range of Monte Carlo simulations. Note that the above *a.s.* convergence is again ruled by a CLT if $X \in L^4(\mathbb{P})$ (do that $X^2 \in L^2(\mathbb{P})$).

▷ **Exercises. 1.** Show that $\overline{V}_M$ is unbiased, that is $\mathbb{E}\,\overline{V}_M = \sigma_X^2$.

2. Show that the sequence $(\overline{X}_M, \overline{V}_M)_{M\geq 1}$ satisfies the following recursive equation

$$\overline{X}_M \quad = \quad \overline{X}_{M-1} + \frac{1}{M}\left(X_M - \overline{X}_{M-1}\right)$$

$$= \quad \frac{M-1}{M}\overline{X}_{M-1} + \frac{X_M}{M}, \ M \geq 1$$

(with the convention $\overline{X}_0 = 0$) and

$$\overline{V}_M \quad = \quad \overline{V}_{M-1} - \frac{1}{M-1}\left(\overline{V}_{M-1} - (X_M - \overline{X}_{M-1})(X_M - \overline{X}_M)\right)$$

$$= \quad \frac{M-2}{M-1}\overline{V}_{M-1} + \frac{(X_M - \overline{X}_{M-1})^2}{M}, \ M \geq 2.$$

---

^{3}When X "already" has an $\mathcal{N}(0;1)$ distribution, then $\overline{V}_M$ has a $\chi^2(M-1)$-distribution. The $\chi^2(\nu)$-distribution, known as the χ^2-distribution with $\nu \in \mathbb{N}^*$ degrees of freedom is the distribution of the sum $Z_1^2 + \cdots + Z_\nu^2$ where $Z_1, \ldots, Z_\nu$ are i.i.d. with $\mathcal{N}(0;1)$ distribution. The loss of one degree of freedom for $\overline{V}_M$ comes form the fact that $X_1, \ldots, X_M$ and $\overline{X}_M$ satisfies a linear equation which induces a linear constraint. This result is known as Cochran's Theorem.

As a consequence, one derives from Slutsky's Theorem that

$$\sqrt{M}\,\frac{\overline{X}_M - m_X}{\sqrt{\overline{V}_M}} = \sqrt{M}\,\frac{\overline{X}_M - m_X}{\sigma_X} \times \frac{\sigma_X}{\sqrt{\overline{V}_M}} \overset{M \to +\infty}{\longrightarrow} \mathcal{N}(0;1).$$

Of course, within the usual range of Monte Carlo simulations, one can always consider that, for large M,

$$\sqrt{M}\,\frac{\overline{X}_M - m_X}{\sqrt{\overline{V}_M}} \approx \mathcal{N}(0;1).$$

Note that when X is itself normally distributed, one shows that the empirical mean $\overline{X}_M$ and the empirical variance $\overline{V}_M$ are independent so the that the true distribution of the left hand side of the above (approximate) equation is a *Student distribution with $M-1$ degree of freedom* ([4]), denoted $T(M-1)$.

Finally, one defines the *confidence interval* at level α of the Monte Carlo simulation by

$$I_M = \left[\overline{X}_M - q_\alpha \sqrt{\frac{\overline{V}_M}{M}}, \overline{X}_M + q_\alpha \sqrt{\frac{\overline{V}_M}{M}}\right]$$

which will still satisfy (for large M)

$$\mathbb{P}(m_X \in I_M) \approx \mathbb{P}\big(|\mathcal{N}(0;1)| \leq q_\alpha\big) = \alpha$$

For numerical implementation one often considers $q_\alpha = 2$ which corresponds to the confidence level $\alpha = 95,44\,\% \approx 95\,\%$.

The "academic" birth of the Monte Carlo method started in 1949 with the publication of a seemingly seminal paper by Metropolis and Ulam "The Monte Carlo method" in *J. of American Statistics Association* (**44**, 335-341). In fact, the method had already been extensively used for several years as a secret project of the U.S. Defense Department.

One can also consider that, in fact, the Monte Carlo method goes back to the celebrated Buffon's needle experiment and should subsequently be credited to the French naturalist Georges Louis Leclerc, Comte de Buffon (1707-1788).

As concerns Finance and more precisely the pricing of derivatives, it seems difficult to trace the origin the implementation of Monte Carlo method for the pricing and hedging of options. In the academic literature, the first paper dealing in a systematic manner with a Monte Carlo approach seems to go back to Boyle in [28].

---

[4]The Student distribution with ν degrees of freedom, denoted $T(\nu)$ is the law of the random variable $\dfrac{\sqrt{\nu}Z_0}{\sqrt{Z_1^2 + \cdots + Z_\nu^2}}$ where $Z_0, Z_1, \ldots, Z_\nu$ are i.i.d. with $\mathcal{N}(0;1)$ distribution. As expected for the coherence of what precedes, one shows that $T(\nu)$ converges in distribution to the normal distribution $\mathcal{N}(0;1)$ as $\nu \to +\infty$.

2.2 Vanilla option pricing in a Black-Scholes model: the premium

For the sake of simplicity, one considers a 2-dimensional correlated Black-Scholes model (under its unique risk neutral probability) but a general d-dimensional can be defined likewise.

$$
\begin{aligned}
dX_t^0 &= rX_t^0 dt, \quad X_0^0 = 1, \\
dX_t^1 &= X_t^1(rdt + \sigma_1 dW_t^1), \quad X_0^1 = x_0^1, \\
dX_t^2 &= X_t^2(rdt + \sigma_2 dW_t^2), \quad X_0^2 = x_0^2,
\end{aligned}
\tag{2.1}
$$

with the usual notations (r interest rate, σ_i volatility of X^i). In particular, $W = (W^1, W^2)$ denotes a correlated bi-dimensional Brownian motion such that $d < W^1, W^2 >_t = \rho\, dt$ (this means that $W_t^2 = \rho W_t^1 + \sqrt{1 - \rho^2}\, \widetilde{W}_t^2$ where $(W^1, \widetilde{W}^2)$ is a standard 2-dimensional Brownian motion). The filtration $\mathcal{F}$ of this market is the augmented filtration of W *i.e.* $\mathcal{F}_t = \mathcal{F}_t^W := \sigma(W_s, 0 \le s \le t, \mathcal{N}_{\mathbb{P}})$ where $\mathcal{N}_{\mathbb{P}}$ denote the family of $\mathbb{P}$-negligible sets of $\mathcal{A}$ ([5]). By "filtration of the market" we mean that $\mathcal{F}$ is the smallest filtration satisfying the usual conditions to which the process (X_t) is adapted.

Then, for every $t \in [0, T]$,

$$
X_t^0 = e^{rt}, \quad X_t^i = x_0^i e^{(r - \frac{\sigma_i^2}{2})t + \sigma_i W_t^i}, \; i = 1, 2.
$$

A European *vanilla* option with maturity $T > 0$ is an option related to a European payoff

$$
h_T := h(X_T)
$$

which only depends on X at time T. In such a complete market the option premium at time 0 is given by

$$
V_0 = e^{-rT} \mathbb{E}(h(X_T))
$$

and more generally, at any time $t \in [0, T]$,

$$
V_t = e^{-r(T-t)} \mathbb{E}(h(X_T) \,|\, \mathcal{F}_t).
$$

The fact that W has independent stationary increments implies that X^1 and X^2 have independent stationary ratios *i.e.*

$$
\frac{X_T^i}{X_t^i} \stackrel{d}{=} \frac{X_{T-t}^i}{x_0^i} \text{ is independent of } \mathcal{F}_t^W.
$$

As a consequence, if

$$
V_0 := v(x_0, T),
$$

---

[5]One shows that, owing to the 0-1 Kolmogorov law, that this filtration is right continuous *i.e.* $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$. A right continuous filtration which contains the $\mathbb{P}$-negligible sets satisfies the so-called "usual conditions".

then

$$
\begin{aligned}
V_t &= e^{-r(T-t)}\mathbb{E}\big(h(X_T)\,|\,\mathcal{F}_t\big) \\
&= e^{-r(T-t)}\mathbb{E}\left(h\left(X_t^i \times \left(\frac{X_T^i}{X_t^i}\right)_{i=1,2}\right)\,\Big|\,\mathcal{F}_t\right) \\
&= e^{-r(T-t)}\mathbb{E}\left(h\left(\left(x^i\frac{X_{T-t}^i}{x_0^i}\right)_{i=1,2}\right)\right)_{|x^i=X_t^i} \\
&= v(X_t, T-t).
\end{aligned}
$$

EXAMPLES. ● *Vanilla call* with strike price K:

$$
h(x^1, x^2) = (x^1 - K)_+.
$$

There is a closed form for this option which is but the celebrated *Black-Scholes formula* for option on stock (without dividend)

$$
\text{Call}_0^{BS} = C(x_0, K, r, \sigma_1, T) = x_0\Phi_0(d_1) - e^{-rT}K\Phi_0(d_2) \tag{2.2}
$$

with
$$
d_1 = \frac{\log(x_0/K) + (r + \frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}, \qquad d_2 = d_1 - \sigma_1\sqrt{T}
$$

where Φ_0 denotes the distribution function of the $\mathcal{N}(0;1)$-distribution.

● *Best-of call* with strike price K:

$$
h_T = \big(\max(X_T^1, X_T^2) - K\big)_+.
$$

A quasi-closed form is available involving the distribution function of the bi-variate (correlated) normal distribution. Laziness may lead to price it by MC (*PDE* is also appropriate but needs more care) as detailed below.

● *Exchange Call Spread* with strike price K:

$$
h_T = \big((X_T^1 - X_T^2) - K\big)_+.
$$

For this payoff no closed form is available. One has the choice between a PDE approach (quite appropriate in this 2-dimensional setting but requiring some specific developments) and a Monte Carlo simulation.

We will illustrate below the regular Monte Carlo procedure on the example of a *Best-of* call.

▷ PRICING A *Best-of Call* BY A MONTE CARLO SIMULATION. To implement a (crude) Monte Carlo simulation one needs to write the payoff as a function of independent uniformly distributed random variables, or equivalently as a function of independent random variables that are simple

functions of independent uniformly distributed random variables, namely a centered and normalized Gaussian. In our case, it amounts to writing

$$e^{-rT}h_T \overset{d}{=} \varphi(Z^1, Z^2)$$

$$:= \left(\max \left(x_0^1 \exp\left(-\frac{\sigma_1^2}{2}T + \sigma_1\sqrt{T}Z^1 \right), \; x_0^2 \exp\left(-\frac{\sigma_2^2}{2}T + \sigma_2\sqrt{T}\left(\rho Z^1 + \sqrt{1-\rho^2}Z^2 \right) \right) \right) - Ke^{-rT} \right)_+$$

where $Z = (Z^1, Z^2) \overset{d}{=} \mathcal{N}(0; I_2)$ (the dependence of φ in x_0^i, etc, is dropped). Then, simulating a M-sample $(Z_m)_{1 \le m \le M}$ of the $\mathcal{N}(0; I_2)$ distribution using *e.g.* the Box-Müller yields the estimate

$$\textit{Best-of Call}_0 \;=\; e^{-rT}\mathbb{E}\big(\max(X_T^1, X_T^2) - K \big)_+ \big)$$

$$=\; \mathbb{E}(\varphi(Z^1, Z^2)) \approx \overline{\varphi}_M := \frac{1}{M}\sum_{m=1}^{M} \varphi(Z_m).$$

One computes an estimate for the variance using the same sample

$$\overline{V}_M(\varphi) = \frac{1}{M-1}\sum_{m=1}^{M} \varphi(Z_m)^2 - \frac{M}{M-1}\overline{\varphi}_M^2 \approx \mathrm{Var}(\varphi(Z))$$

since M is large enough. Then one designs a confidence interval for $\mathbb{E}\,\varphi(Z)$ at level $\alpha \in (0,1)$ by setting

$$I_M^\alpha = \left[\overline{\varphi}_M - q_\alpha\sqrt{\frac{\overline{V}_M(\varphi)}{M}}, \overline{\varphi}_M + q_\alpha\sqrt{\frac{\overline{V}_M(\varphi)}{M}} \right]$$

where q_α is defined by $\mathbb{P}(|\mathcal{N}(0;1)| \le q_\alpha) = \alpha$ (or equivalently $2\Phi_0(q_\alpha) - 1 = \alpha$).

Numerical Application: We consider a European "Best-of-Call" option with the following parameters

$$r = 0.1, \; \sigma_i = 0.2 = 20\,\%, \; \rho = 0, \; X_0^i = 100, \; T = 1, \; K = 100$$

(being aware that the assumption $\rho = 0$ is not realistic). The confidence level is set (as usual) at $\alpha = 0.95$.

The Monte Carlo simulation parameters are $M = 2^m$, $m = 10, \ldots, 20$ (have in mind that $2^{10} = 1024$). The (typical) results of a numerical simulation are reported in the Table below.

M	$\overline{\varphi}_M$	I_M
$2^{10} = 1024$	18.8523	[17.7570 ; 19.9477]
$2^{11} = 2048$	18.9115	[18.1251 ; 19.6979]
$2^{12} = 4096$	18.9383	[18.3802 ; 19.4963]
$2^{13} = 8192$	19.0137	[18.6169 ; 19.4105]
$2^{14} = 16384$	19.0659	[18.7854 ; 19.3463]
$2^{15} = 32768$	18.9980	[18.8002 ; 19.1959]
$2^{16} = 65536$	19.0560	[18.9158 ; 19.1962]
$2^{17} = 131072$	19.0840	[18.9849 ; 19.1831]
$2^{18} = 262144$	19.0359	[18.9660 ; 19.1058]
$2^{19} = 524288$	19.0765	[19.0270 ; 19.1261]
$2^{20} = 1048576$	19.0793	[19.0442 ; 19.1143]
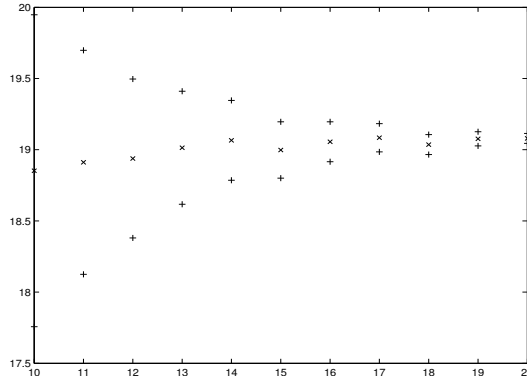
Figure 2.1: BLACK-SCHOLES BEST OF CALL. *The Monte Carlo estimator ($\times$) and its confidence interval (+) at level $\alpha = 95\%$.*

▷ **Exercise.** Proceed likewise with an Exchange Call spread option.

Remark. Once the script is written for one option, *i.e.* one *payoff* function, it is almost instantaneous to modify it to price another option based on a new payoff function: the Monte Carlo method is very flexible, much more than a *PDE* approach.

Conclusion for practitioners. Performing a **Monte Carlo simulation** to compute $m_X = \mathbb{E}X$, consists of **three steps**:

▷ Specification of a confidence level $\alpha \in (0, 1)$ ($\alpha \approx 1$).

▷ Simulation of an M-sample $X_1, X_2, \ldots, X_M$ of i.i.d. random vectors having the same distribution as X and (possibly recursive) computation of both its empirical mean $\bar{X}_M$ and its empirical variance $\bar{V}_M$.

▷ Computation of the resulting *confidence interval at confidence level* α which will be the result of the Monte Carlo simulation.

We will see further on in Chapter 3 how to specify the size M of the simulation to comply with an *a priori* accuracy level.

2.3 Greeks (sensitivity to the option parameters): a first approach

2.3.1 Background on differentiation of function defined by an integral

The *greeks* or sensitivities denote the set of parameters obtained as derivatives of the premium of an option with respect to some of its parameters: the starting value, the volatility, etc. In elementary situations, one simply needs to apply some more or less standard theorems like the one reproduced below (see [30], Chapter 8 for a proof). A typical example of such a "elementary situation" is the case of a (possibly multi-dimensional) Black-Scholes model.

Theorem 2.1 (Interchanging differentiation and expectation) *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let I be a nontrivial interval of $\mathbb{R}$. Let $\varphi : I \times \Omega \to \mathbb{R}$ be a $\mathcal{B}or(I) \otimes \mathcal{A}$-measurable function.*

(a) LOCAL VERSION. *Let $x_0 \in I$. If the function φ satisfies:*

   (i) for every $x \in I$, the random variable $\varphi(x, \,.\,) \in L^1_{\mathbb{R}}(\Omega, \mathcal{A}, \mathbb{P})$,

   (ii) $\mathbb{P}(d\omega)$-a.s. $\dfrac{\partial \varphi}{\partial x}(x_0, \omega)$ exists,

   (iii) There exists $Y \in L^1_{\mathbb{R}_+}(\mathbb{P})$ such that, for every $x \in I$,

$$\mathbb{P}(d\omega)\text{-}a.s. \ \ |\varphi(x, \omega) - \varphi(x_0, \omega)| \leq Y(\omega)|x - x_0|,$$

then, the function $\Phi(x) := \mathbb{E}\big(\varphi(x, \,.\,)\big) = \displaystyle\int_\Omega \varphi(x, \omega)\mathbb{P}(d\omega)$ is defined at every $x \in I$, differentiable at x_0 with derivative

$$\Phi'(x_0) = \mathbb{E}\left(\frac{\partial \varphi}{\partial x}(x_0, \,.\,)\right).$$

(b) GLOBAL VERSION. *If φ satisfies (i) and*

   $(ii)_{glob}$ $\mathbb{P}(d\omega)$-a.s., $\dfrac{\partial \varphi}{\partial x}(x, \omega)$ exists at every $x \in I$,

   $(iii)_{glob}$ There exists $Y \in L^1_{\mathbb{R}_+}(\Omega, \mathcal{A}, \mathbb{P})$ such that, for every $x \in I$,

$$\mathbb{P}(d\omega)\text{-}a.s. \ \ \left|\frac{\partial \varphi(x, \omega)}{\partial x}\right| \leq Y(\omega),$$

then, the function $\Phi(x) := \mathbb{E}\big(\varphi(x, \,.\,)\big)$ is defined and differentiable at every $x \in I$, with derivative

$$\Phi'(x) = \mathbb{E}\left(\frac{\partial \varphi}{\partial x}(x, \,.\,)\right).$$

Remarks. • The local version of the above theorem can be necessary to prove the differentiability of a function defined by an expectation over the whole real line (see exercise below).

• All what precedes remains true if one replaces the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ by any measurable space $(E, \mathcal{E}, \mu)$ where μ is a non-negative measure (see again Chapter 8 in [30]). However, this extension is no longer true as set when dealing under the uniform integrability assumption mentioned in the exercises below.

• Some variants of the result can be established to get a theorem for right or left differentiability of functions $\Phi = \mathbb{E}_\omega \varphi(x, \omega)$ defined on the real line, (partially) differentiable functions defined on $\mathbb{R}^d$, for holomorphic functions on $\mathbb{C}$, etc.

• There exists a local continuity result for such functions φ defined as an expectation which are quite similar to Claim (a). The domination property by an integrable non-negative random variable Y is requested on $\varphi(x, \omega)$ itself. A precise statement is provided in Chapter 11 (with the same notations). For a proof we still refer to [30], Chapter 8. This result is in particular useful to establish the differentiability of a multi-variable function by combining the existence and the continuity of its partial derivatives.

▷ **Exercise.** Let $Z \overset{d}{=} \mathcal{N}(0;1)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\varphi(x, \omega) = (x - Z(\omega))_+$ and $\Phi(x) = \mathbb{E}\,\varphi(x - Z)_+$, $x \in \mathbb{R}$.

(a) Show that Φ is differentiable on the real line by applying the local version of Theorem 2.1 and compute its derivative.

(b) Show that if I denotes a non-trivial interval of $\mathbb{R}$. Show that if $\omega \in \{Z \in I\}$ (*i.e.* $Z(\omega) \in I$), the function $x \mapsto (x - Z(\omega))_+$ is never differentiable on the whole interval I.

▷ **Exercises (extension to uniform integrability).** One can replace the domination property (iii) in Claim (a) (local version) of the above Theorem 2.1 by the less stringent *uniform integrability* assumption

$$(iii)_{ui} \qquad \left(\frac{\varphi(x,.) - \varphi(x_0,.)}{x - x_0} \right)_{x \in I \setminus \{x_0\}} \quad \text{is } \mathbb{P}\text{-uniformly integrable on } (\Omega, \mathcal{A}, \mathbb{P}).$$

For the definition and some background on uniform integrability, see Chapter 11, Section 11.5.

1. Show that $(iii)_{ui}$ implies (iii).

2. Show that (i)-(ii)-$(iii)_{ui}$ implies the conclusion of Claim (a) (local version) in the above Theorem 2.1.

3. State a "uniform integrable" counterpart of $(iii)_{glob}$ to extend Claim (b) (global version) of Theorem 2.1.

4. Show that uniform integrability of the above family of random variables follows from its L^p-boundedness for (any) $p > 1$.

2.3.2 Working on the scenarii space (Black-Scholes model)

To illustrate the different methods to compute the sensitivities, we will consider the 1-dimensional the Black-Scholes model
$$dX_t^x = X_t^x(r\,dt + \sigma\,dW_t), \quad X_0^x = x > 0,$$
so that $X_t^x = x \exp\left((r - \frac{\sigma^2}{2})t + \sigma W_t \right)$. Then we consider, for every $x \in (0, \infty)$,

$$\Phi(x) := \mathbb{E}\,\varphi(X_T^x), \tag{2.3}$$

where $\varphi : (0, +\infty) \to \mathbb{R}$ lies in $L^1(\mathbb{P}_{X_T^x})$ for every $x \in (0, \infty)$ and $T > 0$. This corresponds (when φ is non-negative), to *vanilla payoffs* with maturity T. However, we skip on purpose the discounting factor in what follows to alleviate notations: one can always imagine it is included as a constant in the function φ below since we work at a fixed time T. The updating of formulas is obvious.

First, we are interested in computing the first two derivatives $\Phi'(x)$ and $\Phi''(x)$ of the function Φ which correspond (up to the discounting factor) to the δ-hedge of the option and its γ parameter respectively. The second parameter γ is involved in the so-called "tracking error". But other sensitivities are of interest to the practitioners like the *vega i.e.* the derivative of the (discounted) function Φ with respect to the volatility parameter, the ρ (derivative with respect to the interest rate r), etc. The aim is to derive some representation of these sensitivities as expectations in order to compute them using a Monte Carlo simulation, in parallel with the premium computation. Of

course the Black-Scholes model is simply a toy model to illustrate such an approach since, for this model, many alternative and more efficient methods can be implemented to carry out these computations.

We will first work on the scenarii space $(\Omega, \mathcal{A}, \mathbb{P})$, because this approach contains the "seed" of methods that can be developed in much more general settings in which the SDE has no explicit solution like it has in the Black-Scholes model. On the other hand, as soon as an explicit expression is available for the density $p_T(x, y)$ of X_T^x, it is more efficient to use the next section 2.3.3.

Proposition 2.1 (*a*) *If* $\varphi : (0, +\infty) \to \mathbb{R}$ *is differentiable and* φ' *has polynomial growth, then the function* Φ *defined by (2.3) is differentiable and*

$$\forall\, x > 0, \qquad \Phi'(x) = \mathbb{E}\left(\varphi'(X_T^x)\frac{X_T^x}{x}\right). \tag{2.4}$$

(*b*) *If* $\varphi : (0, +\infty) \to \mathbb{R}$ *is differentiable outside a countable set and is* locally Lipschitz continuous with polynomial growth *in the following sense*

$$\exists\, m \geq 0, \ \forall\, u, v \in \mathbb{R}_+, \qquad |\varphi(u) - \varphi(v)| \leq C|u - v|(1 + |u|^m + |v|^m),$$

then Φ *is differentiable everywhere on* $(0, \infty)$ *and* Φ' *is given by (2.4).*
(*c*) *If* $\varphi : (0, +\infty) \to \mathbb{R}$ *is simply a Borel function with polynomial growth, then* Φ *is still differentiable and*

$$\forall\, x > 0, \qquad \Phi'(x) = \mathbb{E}\left(\varphi(X_T^x)\frac{W_T}{x\,\sigma\,T}\right). \tag{2.5}$$

Proof. (*a*) This straightforwardly follows from the explicit expression for X_T^x and the above differentiation Theorem 2.1 (global version) since, for every $x \in (0, \infty)$,

$$\frac{\partial}{\partial x}\varphi(X_T^x) = \varphi'(X_T^x)\frac{\partial X_T^x}{\partial x} = \varphi'(X_T^x)\frac{X_T^x}{x}.$$

Now $|\varphi'(u)| \leq C(1 + |u|^m)$ ($m \in \mathbb{N}$, $C \in (0, \infty)$) so that, if $0 < x \leq L < +\infty$,

$$\left|\frac{\partial}{\partial x}\varphi(X_T^x)\right| \leq C'_{r,\sigma,T}(1 + L^m \exp\left((m+1)\sigma W_T\right)) \in L^1(\mathbb{P})$$

where $C'_{r,\sigma,T}$ is another positive real constant. This yields the domination condition of the derivative.

(*b*) This claim follows from Theorem 2.1(*a*) (local version) and the fact that, for every $T > 0$, $\mathbb{P}(X_T^x = \xi) = 0$ for every $\xi \geq 0$.

(*c*) Now, still under the assumption (*a*) (with $\mu := r - \frac{\sigma^2}{2}$),

$$
\begin{aligned}
\Phi'(x) &= \int_{\mathbb{R}} \varphi'\big(x \exp\left(\mu T + \sigma\sqrt{T}u\right)\big) \exp\left(\mu T + \sigma\sqrt{T}u\right) e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \\
&= \frac{1}{x\,\sigma\sqrt{T}} \int_{\mathbb{R}} \frac{\partial\varphi\big(x \exp\left(\mu T + \sigma\sqrt{T}u\right)\big)}{\partial u} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \\
&= -\frac{1}{x\,\sigma\sqrt{T}} \int_{\mathbb{R}} \varphi\big(x \exp\left(\mu T + \sigma\sqrt{T}u\right)\big)\frac{\partial e^{-u^2/2}}{\partial u}\frac{du}{\sqrt{2\pi}} \\
&= \frac{1}{x\,\sigma\sqrt{T}} \int_{\mathbb{R}} \varphi\big(x \exp\left(\mu T + \sigma\sqrt{T}u\right)\big)u\, e^{-u^2/2}\frac{du}{\sqrt{2\pi}} \\
&= \frac{1}{x\,\sigma T} \int_{\mathbb{R}} \varphi\big(x \exp\left(\mu T + \sigma\sqrt{T}u\right)\big)\sqrt{T}\, u\, e^{-u^2/2}\frac{du}{\sqrt{2\pi}}
\end{aligned}
$$

where we used an integration by parts to obtain the third equality taking advantage of the fact that, owing to the polynomial growth assumptions on φ,

$$\lim_{|u| \to +\infty} \varphi\big(x \exp\big(\mu T + \sigma\sqrt{T}u\big)\big)e^{-u^2/2} = 0.$$

Finally, coming back to Ω,

$$\Phi'(x) = \frac{1}{x\,\sigma T}\mathbb{E}\big(\varphi(X_T^x)W_T\big). \tag{2.6}$$

When φ is not differentiable, let us first sketch the extension of the formula by a density argument. When φ is continuous and has compact support in $\mathbb{R}_+$, one may assume without loss of generality that φ is defined on the whole real line as a continuous function with compact support. Then φ can be uniformly approximated by differentiable functions φ_n with compact support (use a convolution by *mollifiers*, see [30], Chapter 8). Then, with obvious notations, $\Phi'_n(x) := \frac{1}{x\,\sigma T}\mathbb{E}\big(\varphi_n(X_T^x)W_T\big)$ converges uniformly on compact sets of $(0,\infty)$ to $\Phi'(x)$ defined by (2.6) since

$$|\Phi'_n(x) - \Phi'(x)| \leq \|\varphi_n - \varphi\|_{\sup}\frac{\mathbb{E}\,|W_T|}{x\,\sigma\,T}.$$

Furthermore $\Phi_n(x)$ converges (uniformly) toward $f(x)$ on $(0,\infty)$. Consequently Φ is differentiable with derivative f'. $\quad\diamond$

Remark. We will see in the next section a much quicker way to establish claim (c). The above method of proof, based on an integration by parts, can be seen as a toy-introduction to a systematic way to produce random weights like $\frac{W_T}{x\sigma T}$ as a substitute of the differentiation of the function φ, especially when the differential does not exist. The most general extension of this approach, developed on the Wiener space[6] for functionals of the Brownian motion is known as Malliavin-Monte Carlo method.

$\triangleright$ **Exercise** *(Extension to Borel functions with polynomial growth).* (a) Show that as soon as φ is a Borel function with polynomial growth, the function f defined by (2.3) is continuous. [Hint: use that the distribution X_T^x has a probability density $p_T(x,y)$ on the positive real line which continuously depends on x and apply the continuity theorem for functions defined by an integral, see Theorem 11.3(a) in the "Miscellany" Chapter 11.]

(b) Show that (2.6) holds true as soon as φ is a bounded Borel function. [Hint: Apply the Functional Monotone Class Theorem (see Theorem 11.5 in the "Miscellany" Chapter 11) to an appropriate vector subspace of functions φ and use the Baire σ-field Theorem.]

(c) Extend the result to Borel functions φ with polynomial growth [Hint: use that $\varphi(X_T^x) \in L^1(\mathbb{P})$ and $\varphi = \lim_n \varphi_n$ with $\varphi_n = (n \wedge \varphi \vee (-n))$].

(d) Derive from what precedes a simple expression for $\Phi(x)$ when $\varphi = \mathbf{1}_I$ is the indicator function of the interval I.

COMMENTS: The extension to Borel functions φ always needs at some place an argument based on the regularizing effect of the diffusion induced by the Brownian motion. As a matter of fact if

---

[6] The Wiener space $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$ and its Borel σ-field for the topology of uniform convergence on compact sets, namely $\sigma\big(\omega \mapsto \omega(t),\ t\in\mathbb{R}_+\big)$, endowed with the Wiener measure *i.e.* the distribution of a standard d-dimensional Brownian motion $(W^1,\ldots,W^d)$.

X_t^x were the solution to a regular ODE this extension to non-continuous functions φ would fail. We propose in the next section an approach (the log-likelihood method) directly based on this regularizing effect through the direct differentiation of the probability density $p_{_T}(x,y)$ of X_T^x.

$\triangleright$ **Exercise.** Prove claim (b) in details.

Note that the assumptions of claim (b) are satisfied by usual payoff functions like $\varphi_{Call}(x) = (x-K)_+$ or $\varphi_{Put} := (K-x)_+$ (when X_T^x has a continuous distribution). In particular, this shows that

$$\frac{\partial \mathbb{E}(\varphi_{Call}(X_T^x))}{\partial x} = \mathbb{E}\left(\mathbf{1}_{\{X_T^x \geq K\}} \frac{X_T^x}{x} \right)$$

The computation of this quantity – which is part of that of the Black-Scholes formula – finally yields as expected:

$$\frac{\partial \mathbb{E}(\varphi_{Call}(X_T^x))}{\partial x} = e^{rT}\Phi_0(d_1)$$

(keep in mind that the discounting factor is missing).

$\triangleright$ **Exercises. 0.** *A comparison.* Try a direct differentiation of the Black-Scholes formula (2.2) and compare with a (formal) differentiation based on Theorem 2.1. You should find by both methods

$$\frac{\partial Call_0^{BS}}{\partial x}(x) = \Phi_0(d_1).$$

But the true question is: "how long did it take you to proceed?"

1. *Application to the computation of the γ (i.e. $\Phi''(x)$).* Show that, if φ is differentiable with a derivative having polynomial growth,

$$\Phi''(x) := \frac{1}{x^2 \sigma T}\mathbb{E}\left((\varphi'(X_T^x)X_T^x - \varphi(X_T^x))W_{_T} \right)$$

and that, if φ is continuous with compact support,

$$\Phi''(x) := \frac{1}{x^2 \sigma T}\mathbb{E}\left(\varphi(X_T^x)\left(\frac{W_T^2}{\sigma T} - W_{_T} - \frac{1}{\sigma} \right) \right)$$

Extend this identity to the case where φ is simply Borel with polynomial growth. Note that a (somewhat simpler) formula also exists when the function φ is itself twice differentiable but such an assumption is not realistic for financial applications.

2. *Variance reduction for the δ ([7]).* The above formulae are clearly not the unique representations of the δ as an expectation: using that $\mathbb{E}\,W_{_T} = 0$ and $\mathbb{E}\,X_T^x = xe^{rT}$, one derives immediately that

$$\Phi'(x) = \varphi'(xe^{rT})e^{rT} + \mathbb{E}\left((\varphi'(X_T^x) - \varphi'(xe^{rT}))\frac{X_T^x}{x} \right)$$

---

[7]In this exercise we slightly anticipate on the next chapter entirely devoted to Variance reduction.

as soon as φ is differentiable at xe^{rT}. When φ is simply Borel

$$\Phi'(x) = \frac{1}{x\,\sigma\,T}\mathbb{E}\left((\varphi(X_T^x) - \varphi(xe^{rT}))W_T\right).$$

3. *Variance reduction for the* γ. Show that

$$\Phi''(x) = \frac{1}{x^2\sigma T}\mathbb{E}\left((\varphi'(X_T^x)X_T^x - \varphi(X_T^x) - xe^{rT}\varphi'(xe^{rT}) + \varphi(xe^{rT}))W_T\right).$$

4. *Testing the variance reduction if any.* Although the former two exercises are entitled "variance reduction" the above formulae do not guarantee a variance reduction at a fixed time T. It seems intuitive that they do only when the maturity T is small. Do some numerical experiments to test whether or not the above formulae induce some variance reduction.

When the maturity increases, test whether the regression method introduced in Chapter 3, Section 3.2 works or not with these "control variates".

5. *Computation of the vega* ([8]). Show likewise that $\mathbb{E}(\varphi(X_T^x))$ is differentiable with respect to the volatility parameter σ under the same assumptions on φ, namely

$$\frac{\partial}{\partial\sigma}\mathbb{E}(\varphi(X_T^x)) = \mathbb{E}\left(\varphi'(X_T^x)X_T^x(W_T - \sigma T)\right)$$

if φ is differentiable with a derivative with polynomial growth. Derive without any further computations – but with the help of the previous exercises – that

$$\frac{\partial}{\partial\sigma}\mathbb{E}(\varphi(X_T^x)) = \mathbb{E}\left(\varphi(X_T^x)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right).$$

if φ is simply Borel with polynomial growth. [Hint: use the former exercises.]

This derivative is known (up to an appropriate discounting) as the *vega* of the option related to the payoff $\varphi(X_T^x)$. Note that the γ and the *vega* of a Call satisfy

$$vega(x, K, r, \sigma, T) = x^2\sigma T\gamma(x, K, r, \sigma, T)$$

which is the key of the tracking error formula.

In fact the beginning of this section can be seen as an introduction to the so-called tangent process method (see the section that ends this chapter and section 9.2.2 in Chapter 9).

2.3.3 Direct differentiation on the state space: the log-likelihood method

In fact, one can also achieve similar computations directly on the state space of a family of random variables (or vectors) X_T^x (indexed by its stating value x), provided this random variable (or vector) has an explicit probability density $p_T(x, y)$ with respect to a reference measure $\mu(dy)$ on the real line (or $\mathbb{R}^d$). In general $\mu = \lambda_d$, Lebesgue measure.

---

[8]Which is not a greek letter...

In fact, one may imagine in full generality that X_T^x depends on a parameter θ a real parameter of interest : thus, $X_T^x = X_T^x(\theta)$ may be the solution at time T to a stochastic differential equation which coefficients depend on θ. This parameter can be the starting x itself, or more specifically in a Black-Scholes model, the volatility σ, the interest rate r, the maturity T or even the strike K of a *Call* or *Put* option written on the risky asset X^x.

Formally, one has

$$\Phi(\theta) = \mathbb{E}(\varphi(X_T^x(\theta))) = \int_{\mathbb{R}} \varphi(y) p_T(\theta, x, y) \mu(dy)$$

so that, formally,

$$
\begin{aligned}
\Phi'(\theta) &= \int_{\mathbb{R}} \varphi(y) \frac{\partial p_T}{\partial \theta}(\theta, x, y) \mu(dy) \\
&= \int_{\mathbb{R}} \varphi(y) \frac{\frac{\partial p_T}{\partial \theta}(\theta, x, y)}{p_T(\theta, x, y)} p_T(\theta, x, y) \mu(dy) \\
&= \mathbb{E}\left(\varphi(X_T^x) \frac{\partial \log p_T}{\partial \theta}(\theta, x, X_T^x)\right).
\end{aligned}
\tag{2.7}
$$

Of course, the above computations need to be supported by appropriate assumptions (domination, etc) to justify interchange of integration and differentiation (see Exercises below). Using this approach, one retrieves the formulas obtained in Section 2.3.2 for the Black-Scholes model much more easily, especially when the function φ is only Borel. This is the aim of the exercises at the end of this section.

A multi-dimensional version of this result can be established the same way round. However, this straightforward and simple approach to "greek" computation remains limited beyond the Black-Scholes world by the fact that it needs to have access not only to the regularity of the probability density $p_T(x, y, \theta)$ of the asset at time T but also to its explicit expression in order to include it in a simulation process.

A solution in practice is to replace the true process X by an approximation, typically its Euler scheme, with step $\frac{T}{n}$, $(\bar{X}_{t_k^n}^x)_{0 \le k \le n}$ where $\bar{X}_0^x = x$ and $t_k^n := \frac{kT}{n}$ (see Chapter 7 for details). Then, under some natural ellipticity assumption (non degeneracy of the volatility) this Euler scheme starting at x (with step $= T/n$) does have a density $\bar{p}_{\frac{kT}{n}}(x, y)$, and more important its increments have conditional densities which makes possible to proceed using a Monte Carlo simulation.

An alternative is to "go back" to the "scenarii" space Ω. Then, some extensions of the first two approaches are possible: if the function and the diffusion coefficients (when the risky asset prices follow a Brownian diffusion) are smooth enough and the payoff function is smooth too, one relies on the tangent process (derivative of the process with respect to its starting value, or more generally with respect to one of its structure parameters, see below).

When the payoff function has no regularity, a more sophisticated method is to introduce some Malliavin calculus methods which correspond to a differentiation theory with respect to the generic Brownian paths. An integration by parts formula plays the role of the elementary integration by parts used in Section 2.3.2. This second topic is beyond the scope of the present course, although some aspects will be mentioned through Bismut and Clark-Occone formulas.

▷ **Exercises: 0.** Provide simple assumptions to justify the above formal computations in (2.7), at some point θ_0 or for all θ running over a non empty pen interval Θ of $\mathbb{R}$ (or domain of $\mathbb{R}^d$ if θ is vector valued). [Hint: use the remark right below Theorem 2.1.]

1. Compute the probability density $p_{_T}(\sigma, x, y)$ of $X_T^{x,\sigma}$ in a Black-Scholes model ($\sigma > 0$ stands for the volatility parameter).

2. Re-establish all the sensitivity formulae established in the former section 2.3.2 (including the exercises at the end of the section) using this approach.

3. Apply these formulae to the case $\varphi(x) := e^{-rT}(x - K)_+$ and retrieve the classical expressions for the greeks in a Black-Scholes model: the δ, the γ and the *vega*.

2.3.4 The tangent process method

In fact when both the payoff function and the coefficients of the SDE are regular enough, one can differentiate directly the function/functional of the process with respect to a given parameter. The former section 2.3.2 was a special case of this method for vanilla payoffs in a Black-Schole model. We refer to section 9.2.2 for a more detailed developments.

Chapter 3

Variance reduction

3.1 The Monte Carlo method revisited: static control variate

Let $X \in L^2_{\mathbb{R}}(\Omega, \mathcal{A}, \mathbb{P})$ be a random variable, assumed to be easy to simulate. One wishes to compute

$$m_X = \mathbb{E}\, X \in \mathbb{R}$$

as the result of a Monte Carlo simulation.

▷ CONFIDENCE INTERVAL REVISITED FROM THE SIMULATION VIEWPOINT. The parameter m is to be computed by a Monte Carlo simulation. Let X_k, $k \geq 1$ be a sequence of i.i.d. copies of X. Then $(SLLN)$

$$m_X = \lim_{M \to +\infty} \overline{X}_M \quad \mathbb{P}\text{-}a.s. \quad \text{with} \quad \overline{X}_M := \frac{1}{M} \sum_{k=1}^{M} X_k.$$

This convergence is ruled by the Central Limit Theorem (CLT)

$$\sqrt{M} \left(\overline{X}_M - m_X \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0; \mathrm{Var}(X)) \quad \text{as} \quad M \to +\infty.$$

Hence for large enough M, and $q \in \mathbb{R}$

$$\mathbb{P}\left(m \in \left[\overline{X}_M - q\, \frac{\sigma(X)}{\sqrt{M}}, \overline{X}_M + q\, \frac{\sigma(X)}{\sqrt{M}} \right] \right) \approx 2\Phi_0(q) - 1$$

where $\sigma(X) := \sqrt{\mathrm{Var}(X)}$ and $\Phi_0(x) = \displaystyle\int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}}$.

In numerical probability, we adopt the following *reverse point of view* based on the *target* or *prescribed accuracy* $\varepsilon > 0$: to make $\overline{X}_M$ enter a confidence interval $[m - \varepsilon, m + \varepsilon]$ with a confidence level $\alpha := 2\Phi_0(q_\alpha) - 1$, one needs to process a Monte Carlo simulation of size

$$M \geq M^X(\varepsilon, \alpha) = \frac{q_\alpha^2 \mathrm{Var}(X)}{\varepsilon^2}. \tag{3.1}$$

In practice, of course, the estimator $\overline{V}_M$ is computed on-line to estimate the variance $\mathrm{Var}(X)$ as presented in the previous chapter. This estimate needs not to be as sharp as the estimation of m, so it can be processed at he beginning of the simulation on a smaller sample size.

As a first conclusion, this shows that, a confidence level being fixed, the size of a Monte Carlo simulation grows linearly with the variance of X for a given accuracy and quadratically as the inverse of the prescribed accuracy for a given variance.

▷ VARIANCE REDUCTION: (NOT SO) NAIVE APPROACH. Assume now that we know *two* random variables X, $X' \in L^2_{\mathbb{R}}(\Omega, \mathcal{A}, \mathbb{P})$ satisfying

$$m = \mathbb{E}\, X = \mathbb{E}\, X' \in \mathbb{R}, \quad \mathrm{Var}(X),\ \mathrm{Var}(X'),\ \mathrm{Var}(X - X') > 0.$$

(the last condition only says that X and X' are not *a.s.* equal).

QUESTION: Which random vector (distribution...) is more appropriate?

Several examples of such a situation have already been pointed out in the previous chapter: usually many formulae are available to compute a greek parameter, even more if one takes into account the (potential) control variates introduced in the exercises.

A natural answer is: if both X and X' *can be simulated with an equivalent cost* (complexity), then the one with the lowest variance is the best choice *i.e.*

$$X' \quad \text{if} \quad \mathrm{Var}(X') < \mathrm{Var}(X), \qquad X \quad \text{otherwise,}$$

provided this fact is known *a priori*.

▷ PRACTITIONER'S CORNER. Usually, the problem appears as follows: there exists a random variable $\Xi \in L^2_{\mathbb{R}}(\Omega, \mathcal{A}, \mathbb{P})$ such that

(i) $\mathbb{E}\,\Xi$ can be computed at a very low cost by a deterministic method (closed form, numerical analysis method),

(ii) the random variable $X - \Xi$ can be simulated with the same cost (complexity) than X,

(iii) the variance $\mathrm{Var}(X - \Xi) < \mathrm{Var}(X)$.

Then, the random variable
$$X' = X - \Xi + \mathbb{E}\,\Xi$$

can be simulated at the same cost as X,

$$\mathbb{E}\, X' = \mathbb{E}\, X = m \quad \text{and} \quad \mathrm{Var}(X') = \mathrm{Var}(X - \Xi) < \mathrm{Var}(X).$$

Definition 3.1 *A random variable Ξ satisfying* (i)-(ii)-(iii) *is called a* control variate *for X.*

▷ **Exercise.** Show that if the simulation process of X and $X - \Xi$ have complexity κ and κ' respectively, then (iii) becomes

(iii)$'$ $\kappa'\mathrm{Var}(X - \Xi) < \kappa\mathrm{Var}(X).$

Example (Toy-): In the previous chapter, we showed that, in *B-S* model, if the payoff function is differentiable outside a countable set and locally Lipschitz continuous with polynomial growth at infinity then the function $\Phi(x) = \mathbb{E}\,\varphi(X^x_T)$ is differentiable on $(0, +\infty)$ and

$$\Phi'(x) = \mathbb{E}\Big(\varphi'(X^x_T)\frac{X^x_T}{x}\Big) = \mathbb{E}\Big(\varphi(X^x_T)\frac{W_T}{x\sigma T}\Big).$$

So we have at hand two formulas for $\Phi'(x)$ that can be implemented in a Monte Carlo simulation. Which one should we choose to compute $\Phi'(x)$ (*i.e.* the δ-hedge up to an e^{-rT}-factor)? Since they have the same expectations, the two random variables should be discriminated through (the square of) their L^2- norm, namely

$$\mathbb{E}\left[\left(\varphi'(X_T^x)\frac{X_T^x}{x}\right)^2\right] \quad \text{and} \quad \mathbb{E}\left[\left(\varphi(X_T^x)\frac{W_T}{x\sigma T}\right)^2\right].$$

It is clear, owing to the *LIL* for the Brownian motion that at 0 that if $\varphi(x) \neq 0$,

$$\liminf_{T\to 0}\mathbb{E}\left(\varphi(X_T^x)\frac{W_T}{x\sigma T}\right)^2 = +\infty$$

and that, if φ' is bounded

$$\mathbb{E}\left(\varphi(X_T^x)\frac{W_T}{x\sigma T}\right)^2 = O\left(\frac{1}{T}\right) \to 0 \text{ as } T \to +\infty.$$

On the other hand, the first formula

$$\liminf_{T\to 0}\mathbb{E}\left(\varphi'(X_T^x)\frac{X_T^x}{x}\right)^2 = (\varphi'(x))^2$$

whereas

$$\liminf_{T\to +\infty}\mathbb{E}\left(\varphi'(X_T^x)\frac{X_T^x}{x}\right)^2 = +\infty.$$

As a consequence the first formula is more appropriate for short maturities whereas the second formula has less variance for long maturities. This appears somewhat as an exception: usually Malliavin-Monte Carlo weights tend to introduce variance, not only for short maturities. One way to partially overcome this drawback is to introduce some localization methods (see *e.g.* [26] for an in-depth analysis or Section 9.4.3 for an elementary introduction).

▷ A VARIANT (PSEUDO-CONTROL VARIATE). In option pricing, when the random variable X is a payoff it is usually non-negative. In that case, any random variable Ξ satisfying (i)-(ii) and

(iii)" $0 \leq \Xi \leq X$

can be considered as a good *candidate* to reduce the variance, especially if Ξ is close to X so that $X - \Xi$ is "small".

However, note that it does not imply (iii). Here is a trivial counter-example: let $X \equiv 1$, then $\text{Var}(X) = 0$ whereas a uniformly distributed random variable Ξ on $[1 - \eta, 1]$, $0 < \eta < 1$, will (almost) satisfy (i)-(ii) but $\text{Var}(X - \Xi) > 0$... Consequently, this variant is only a heuristic method to reduce the variance which often works, but with no *a priori* guarantee.

3.1.1   Jensen inequality and variance reduction

This section is in fact an illustration of the notion of pseudo-control variate described above.

Proposition 3.1 (Jensen Inequality) *Let* $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ *be a random variable and let* $g : \mathbb{R} \to \mathbb{R}$ *be a* convex *function. Suppose* X *and* $g(X)$ *are integrable. Then, for any sub-σ-field* $\mathcal{B}$ *of* $\mathcal{A}$,

$$g\left(\mathbb{E}(X \mid \mathcal{B})\right) \leq \mathbb{E}\left(g(X) \mid \mathcal{B}\right) \qquad \mathbb{P}\text{-}a.s.$$

In particular considering $\mathcal{B} = \{\emptyset, \Omega\}$ *yields the above inequality for regular expectation i.e.*

$$g\left(\mathbb{E}(X)\right) \leq \mathbb{E}\, g(X).$$

Proof. The inequality is a straightforward consequence of the following classical characterization of a convex function

$$g \text{ is convex} \quad \text{if and only if} \quad \forall\, x \in \mathbb{R},\ g(x) = \sup\left\{ax + b,\ a,\ b \in \mathbb{Q},\ ay + b \leq g(y),\ \forall\, y \in \mathbb{R}\right\}. \quad \diamond$$

Jensen inequality is an efficient tool to design control variate when dealing with path-dependent or multi-asset options as emphasized by the following examples:

EXAMPLES: **1.** *Basket or index option.* We consider a payoff on a basket of d (positive) risky assets (this basket can be an index). For the sake of simplicity we suppose it is a call with strike K i.e.

$$h_T = \left(\sum_{i=1}^{d} \alpha_i X_T^{i,x_i} - K\right)_+$$

where $(X^{1,x_1}, \ldots, X^{d,x_d})$ models the price of d traded risky assets on a market and the α_k are some positive $(\alpha_i > 0)$ weights satisfying $\sum_{1 \leq i \leq d} \alpha_i = 1$. Then the convexity of the exponential implies that

$$0 \leq e^{\sum_{1 \leq i \leq d} \alpha_i \log(X_T^{i,x_i})} \leq \sum_{i=1}^{d} \alpha_i X_T^{i,x_i}$$

so that

$$h_T \geq k_T := \left(e^{\sum_{1 \leq i \leq d} \alpha_i \log(X_T^{i,x_i})} - K\right)_+ \geq 0.$$

The motivation for this example is that in a (possibly correlated) d-dimensional Black-Scholes model (see below), $\sum_{1 \leq i \leq d} \alpha_i \log(X_T^{i,x_i})$ still has a normal distribution so that the *Call* like European option written on the payoff

$$k_T := \left(e^{\sum_{1 \leq i \leq d} \alpha_i \log(X_T^{i,x_i})} - K\right)_+$$

has a closed form.

Let us be more specific on the model and the variance reduction procedure.

The *correlated d-dimensional Black-Scholes model* (under the risk-neutral probability measure with $r > 0$ denoting the interest rate) can be defined by the following system of *SDE*'s which governs the price of d risky assets denoted $i = 1, \ldots, d$:

$$dX_t^{i,x_i} = X_t^{i,x_i}\left(r\, dt + \sum_{j=1}^{q} \sigma_{ij} dW_t^j\right), \quad t \in [0, T], \quad x_i > 0,\ i = 1, \ldots, d$$

where $W = (W^1, \ldots, W^q)$ is a standard q-dimensional Brownian motion and $\sigma = [\sigma_{ij}]_{1 \leq i \leq d, 1 \leq j \leq q}$ is a is given by $d \times q$ matrix with real entries. Its solution

$$X_t^{i,x_i} = x_i \exp\left(\left(r - \frac{\sigma_{i.}^2}{2}\right)t + \sum_{j=1}^{q} \sigma_{ij} W_t^j\right), \quad t \in [0, T], \quad x_i > 0, \; i = 1, \ldots, d.$$

where

$$\sigma_{i.}^2 = \sum_{j=1}^{q} \sigma_{ij}^2, \quad i = 1, \ldots, d.$$

▷ **Exercise.** Show that if the matrix $\sigma\sigma^*$ is positive definite (then $q \geq d$) and one may assume without modifying the model that X^{i,x_i} only depends on the first i components of a d-dimensional standard Brownian motion. [Hint: think to Cholesky decomposition in Section 1.6.2.]

Now, let us describe the two phases of the variance reduction procedure:

– PHASE I: $\Xi = e^{-rT} k_T$ *as a pseudo-control variate and computation of its expectation* $\mathbb{E}\,\Xi$.

The vanilla call option has a closed form in a Black-Scholes model and elementary computations show that

$$\sum_{1 \leq i \leq d} \alpha_i \log(X_T^{i,x_i}/x_i) \overset{d}{=} \mathcal{N}\left(\left(r - \frac{1}{2}\sum_{1 \leq i \leq d} \alpha_i \sigma_{i.}^2\right)T; \alpha^* \sigma \sigma^* \alpha\, T\right)$$

where α is the column vector with components α_i, $i = 1, \ldots, d$.

Consequently, the premium at the origin $e^{-rT}\mathbb{E}\,k_T$ admits a closed form given by

$$e^{-rT}\mathbb{E}\,k_T = Call_{BS}\left(\prod_{i=1}^{d} x_i^{\alpha_i} e^{-\frac{1}{2}\left(\sum_{1 \leq i \leq d} \alpha_i \sigma_{i.}^2 - \alpha^* \sigma \sigma^* \alpha\right)T}, K, r, \sqrt{\alpha^* \sigma \sigma^* \alpha}, T\right).$$

– PHASE II: *Joint simulation of the couple* (h_T, k_T).

We need to simulate M independent copies of the couple (h_T, k_T) or, to be more precise of the quantity

$$e^{-rT}(h_T - k_T) = e^{-rT}\left(\left(\sum_{i=1}^{d} \alpha_i X_T^{i,x_i} - K\right)_+ - \left(e^{\sum_{1 \leq i \leq d} \alpha_i \log(X_T^{i,x_i})} - K\right)_+\right).$$

This task clearly amounts to simulating M independent copies $W_T^{(m)} = (W_T^{1,(m)}, \ldots, W_T^{q,(m)})$, $m = 1, \ldots, M$ of the d-dimensional standard Brownian motion W at time T *i.e.* M independent copies $Z^{(m)} = (Z_1^{(m)}, \ldots, Z_q^{(m)})$ of the $\mathcal{N}(0; I_q)$ distribution in order to set $W_T^{(m)} \overset{d}{=} \sqrt{T}Z^{(m)}$, $m = 1, \ldots, M$.

The resulting pointwise estimator of the premium is given, with obvious notations, by

$$\frac{e^{-rT}}{M}\sum_{m=1}^{m}\left(h_T^{(m)} - k_T^{(m)}\right) + Call_{BS}\left(\prod_{i=1}^{d} x_i^{\alpha_i} e^{-\frac{1}{2}\left(\sum_{1 \leq i \leq d} \alpha_i \sigma_{i.}^2 - \alpha^* \sigma \sigma^* \alpha\right)T}, K, r, \sqrt{\alpha^* \sigma \sigma^* \alpha}, T\right).$$

Remark. The extension to more general payoffs of the form $\varphi\left(\sum_{1\leq i\leq d}\alpha_i X_T^{i,x_i}\right)$ is straightforward provided φ is non-decreasing and a closed form exists for the vanilla option with payoff $\varphi\left(e^{\sum_{1\leq i\leq d}\alpha_k \log(X_T^{i,x_i})}\right)$.

▷ **Exercise.** Other ways to take advantage of the convexity of the exponential function can be explored: thus one can start from

$$\sum_{1\leq i\leq d}\alpha_i X_T^{i,x_i} = \left(\sum_{1\leq i\leq d}\alpha_i x_i\right)\sum_{1\leq i\leq d}\widetilde{\alpha}_i\frac{X_T^{i,x_i}}{x_i}$$

where $\widetilde{\alpha}_i = \dfrac{\alpha_i x_i}{\sum_k \alpha_k x_k}$, $i = 1,\ldots,d$. Compare on simulations the respective performances of these different approaches.

2. *Asian options and Kemna-Vorst control variate in a Black-Scholes model (see [80]).*

Let

$$h_T = \varphi\left(\frac{1}{T}\int_0^T X_t^x dt\right)$$

be a generic *Asian* payoff where φ is non-negative, non-decreasing function defined on $\mathbb{R}_+$ and let

$$X_t^x = x\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad x > 0,\ t \in [0,T],$$

be a regular Black-Scholes dynamics with volatility $\sigma > 0$ and interest rate r. Then, (standard) Jensen inequality applied to the probability measure $\frac{1}{T}\mathbf{1}_{[0,T]}(t)dt$ implies

$$\frac{1}{T}\int_0^T X_t^x\,dt \ \geq\ x\exp\left(\frac{1}{T}\int_0^T\left((r-\sigma^2/2)t + \sigma W_t\right)dt\right)$$

$$= \ x\exp\left((r-\sigma^2/2)\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_t dt\right).$$

Now

$$\int_0^T W_t dt = T\,W_T - \int_0^T s\,dW_s = \int_0^T (T-s)dW_s$$

so that

$$\frac{1}{T}\int_0^T W_t dt \overset{d}{=} \mathcal{N}\left(0;\frac{1}{T^2}\int_0^T s^2 ds\right) = \mathcal{N}\left(0;\frac{T}{3}\right).$$

This suggests to rewrite the right hand side of the above inequality in a "Black-Scholes asset" style, namely:

$$\frac{1}{T}\int_0^T X_t^x\,dt \geq x\,e^{\alpha T}\exp\left((r-(\sigma^2/3)/2)T + \sigma\frac{1}{T}\int_0^T W_t dt\right)$$

where a straightforward computation shows that

$$\alpha = -\left(\frac{r}{2} + \frac{\sigma^2}{12}\right).$$

This naturally leads to introduce the so-called *Kemna-Vorst* (pseudo-)control variate

$$k_T^{KV} := \varphi\left(xe^{-(\frac{r}{2}+\frac{\sigma^2}{12})T}\,\exp\left(\left(r-\frac{1}{2}\frac{\sigma^2}{3}\right)T+\sigma\frac{1}{T}\int_0^T W_t\,dt\right)\right)$$

which is clearly of Black-Scholes type and satisfies moreover

$$h_T \geq k_T^{KV}.$$

– PHASE I: The random variable k_T^{KV} is an admissible control variate *as soon as the vanilla option related to the payoff $\varphi(X_T^x)$ has a closed form.* Indeed if a vanilla option related to the payoff $\varphi(X_T^x)$ has a closed form

$$e^{-rT}\mathbb{E}\,\varphi(X_T^x) = \mathrm{Premium}_{BS}^\varphi(x,r,\sigma,T),$$

then, one has

$$e^{-rT}\mathbb{E}\,k_T^{KV} = \mathrm{Premium}_{BS}^\varphi\left(x\,e^{-(\frac{r}{2}+\frac{\sigma^2}{12})T},r,\frac{\sigma}{\sqrt{3}},T\right).$$

– PHASE II: One has to simulate independent copies of $h_T - k_T^{KV}$ *i.e.* in practice, independent copies of the couple (h_T, k_T^{KV}). This requires theoretically speaking to know how to simulate exactly paths of the standard Brownian motion $(W_t)_{t\in[0,T]}$ and, moreover to impute with an infinite accuracy integral of the form $\frac{1}{T}\int_0^T f(t)dt$.

In practice these two tasks are clearly impossible (one cannot even compute a real-valued function $f(t)$ *at every* $t \in [0,T]$ with a computer). In fact one relies on quadrature formulas to approximate the time integrals in both payoffs which makes this simulation possible since only finitely many random marginals of the Brownian motion, say $W_{t_1},\ldots,W_{t_n}$ are necessary which is then quite realistic. Typically, one uses a mid-point quadrature formula

$$\frac{1}{T}\int_0^T f(t)dt \approx \frac{1}{n}\sum_{k=1}^n f\left(\frac{2k-1}{2n}T\right)$$

or any other numerical integration method, having in mind nevertheless that the (continuous) functions f of interest are here given for the first and the second payoff function by

$$f(t) = \varphi\left(x\exp\left(\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t(\omega)\right)\right) \quad \text{and} \quad f(t) = W_t(\omega)$$

respectively. Hence, their regularity is (almost) $\frac{1}{2}$-Hölder (in fact α-Hölder, for every $\alpha < \frac{1}{2}$, locally as for the payoff h_T). Finally, in practice it will amount to simulating independent copies of the n-tuple

$$\left(W_{\frac{T}{2n}},\ldots,W_{\frac{(2k-1)T}{2n}},\ldots,W_{\frac{(2n-1)T}{2n}}\right).$$

from which one can reconstruct both a mid -point approximation of both integrals appearing in h_T and k_T.

In fact one can improve this first approach by taking advantage of the fact that W is a Gaussian process as detailed in the exercise below

Further developments to reduce the time discretization error are proposed in Section 8.2.6 (see [97] where an in-depth study of the Asian option pricing in a Black-Scholes model is carried out).

▷ **Exercises. 1.** (a) Show that if $f : [0, T] \to \mathbb{R}$ is continuous then

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{kT}{n}\right) = \int_0^T f(t)\, dt$$

Show that $t \mapsto x \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t(\omega)\right)$ is α-Hölder for very $\alpha \in (0, \frac{1}{2})$ (with a random Hölder ratio of course).

(b) Show that

$$\mathcal{L}\left(\frac{1}{T} \int_0^T W_t dt \mid W_{\frac{kT}{n}} - W_{\frac{(k-1)T}{n}} = \delta w_k,\ 1 \le k \le n\right) = \mathcal{N}\left(\sum_{k=1}^{n} a_k\, \delta w_k;\ \frac{T^2}{12n^2}\right)$$

with $a_k = \frac{2(k-n)+1}{2n} T$, $k = 1, \dots, n$. [Hint: we know that for gaussian vector conditional expectation and affine regression coincide.]

(c) Propose a variance reduction method in which the pseudo-control variate $e^{-rT} k_T$ will be simulated exactly.

2. Check that what precedes can be applied to payoffs of the form

$$\varphi\left(\frac{1}{T} \int_0^T X_t^x dt - X_T^x\right)$$

where φ is still a non-negative, non-decreasing function defined on $\mathbb{R}$.

2. *Best-of-call option.* We consider the *Best-of-call* payoff given by

$$h_T = \left(\max(X_T^1, X_T^2) - K\right)_+.$$

(a) Using the convexity inequality (that can still be seen as an application of Jensen inequality)

$$\sqrt{ab} \le \max(a, b), \quad a,\, b > 0,$$

show that

$$k_T := \left(\sqrt{X_T^1 X_T^2} - K\right)_+.$$

is a natural (pseudo-)control variate for h_T.

(b) Show that, in a 2-dimensional Black-Scholes (possibly correlated) model (see expel in Section 2.2), the premium of the option with payoff k_T (known as *geometric mean option*) has a closed form. Show that this closed form can be written as a Black-Scholes formula with appropriate parameters (and maturity T).

(c) Check on (at least one) simulation(s) that this procedure does reduce the variance (use the parameters of the model specified in Section 2.2).

(d) When σ_1 and σ_2 are not equal, improve the above variance reduction protocol by considering a parametrized family of (pseudo-)control variate, obtained from the more general inequality $a^\theta b^{1-\theta} \le \max(a, b)$ when $\theta \in (0, 1)$.

▷ **Exercise.** Compute the premium of the European option with payoff k_T.

3.1.2 Negatively correlated variables, antithetic method

In this section we assume that X and X' have not only the same expectation m_X but also the same variance, *i.e.* $\mathrm{Var}(X) = \mathrm{Var}(X')$, and can be simulated with the same complexity $\kappa = \kappa_X = \kappa_{X'}$. In such a situation, choosing between X or X' may seem *a priori* a question of little interest. However, it is possible to take advantage of this situation to reduce the variance of a simulation when X and X' are *negatively correlated..*

Set

$$\chi = \frac{X + X'}{2}$$

(This corresponds to $\Xi = \frac{X-X'}{2}$ with our formalism). It is reasonable (when no further information on (X, X') is available) to assume that the simulation complexity of χ is twice that of X and X', *i.e.* $\kappa_\chi = 2\kappa$. On the other hand

$$
\begin{aligned}
\mathrm{Var}\,(\chi) &= \frac{1}{4}\,\mathrm{Var}\,(X + X') \\
&= \frac{1}{4}\,\left(\mathrm{Var}(X) + \mathrm{Var}(X') + 2\mathrm{Cov}(X, X')\right) \\
&= \frac{\mathrm{Var}(X) + \mathrm{Cov}(X, X')}{2}.
\end{aligned}
$$

The size $M^X(\varepsilon, \alpha)$ and $M^\chi(\varepsilon, \alpha)$ of the simulation using X and χ respectively to enter a given interval $[m - \varepsilon, m + \varepsilon]$ with the same confidence level α is given, following (3.1), by

$$M^X = \left(\frac{q_\alpha}{\varepsilon}\right)^2 \mathrm{Var}(X) \quad \text{with } X \quad \text{and} \quad M^\chi = X = \left(\frac{q_\alpha}{\varepsilon}\right)^2 \mathrm{Var}(\chi) \quad \text{with } \chi.$$

Taking into account the complexity like in the exercise that follows Definition 3.1, that means in terms of *CPU* computation time that one should better use χ if and only if

$$\kappa_\chi M^\chi < \kappa_X M^X \iff 2\kappa\, M^\chi < \kappa\, M^X$$

i.e.

$$2\mathrm{Var}(\chi) < \mathrm{Var}(X).$$

Given the above inequality, this reads

$$\mathrm{Cov}(X, X') < 0.$$

To use this remark in practice, one usually relies on the following result.

Proposition 3.2 (co-monotony) (*a*) *Let* $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ *be a random variable and. Let* $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ *be two monotone (hence Borel) functions with the same monotony. Assume that* $\varphi(Z), \psi(Z) \in L^2_{\mathbb{R}}(\Omega, \mathcal{A}, \mathbb{P})$. *Then*

$$\mathrm{Cov}(\varphi(Z), \psi(Z)) \geq 0.$$

If, mutatis mutandis, φ *and* ψ *have opposite monotony, then*

$$\mathrm{Cov}(\varphi(Z), \psi(Z)) \leq 0.$$

Furthermore, the inequity is strict holds as an equality if and only if $\varphi(Z) = \mathbb{E}\,\varphi(Z)$ $\mathbb{P}$-a.s. or $\psi(Z) = \mathbb{E}\,\psi(Z)$ $\mathbb{P}$-a.s..

(b) Assume there exists a non-increasing (hence Borel) function $T : \mathbb{R} \to \mathbb{R}$ such that $Z \overset{d}{=} T(Z)$. Then $X = \varphi(Z)$ and $X' = \varphi(T(Z))$ are identically distributed and satisfy

$$\mathrm{Cov}(X, X') \leq 0.$$

In that case, the random variables X and X' are called antithetic.

Proof. (*a*) *Inequality.* The functions Let Z, Z' be two independent random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $\mathbb{P}_Z$ ([1]). Then, using that φ and ψ are monotone with the same monotony, we have

$$\big(\varphi(Z) - \varphi(Z')\big)\big(\psi(Z) - \psi(Z')\big) \geq 0$$

so that the expectation of this product is non-positive (and finite since all random variables are square integrable). Consequently

$$\mathbb{E}\big(\varphi(Z)\psi(Z)\big) + \mathbb{E}\big(\varphi(Z')\psi(Z')\big) - \mathbb{E}\big(\varphi(Z)\psi(Z')\big) - \mathbb{E}\big(\varphi(Z')\psi(Z)\big) \quad \geq \quad 0.$$

Using that $Z' \overset{d}{=} Z$ and that Z, Z' are independent, we get

$$2\mathbb{E}\big(\varphi(Z)\psi(Z)\big) \geq \mathbb{E}\big(\varphi(Z)\big)\mathbb{E}\big(\psi(Z')\big) + \mathbb{E}\big(\varphi(Z')\big)\mathbb{E}\big(\psi(Z)\big) = 2\mathbb{E}\big(\varphi(Z)\big)\mathbb{E}\big(\psi(Z)\big),$$

that is

$$\mathrm{Cov}\big(\varphi(Z), \psi(Z)\big) = \mathbb{E}\big(\varphi(Z)\psi(Z)\big) - \mathbb{E}\big(\varphi(Z)\big)\mathbb{E}\big(\psi(Z)\big) \geq 0.$$

If the functions φ and ψ have opposite monotony, then

$$\big(\varphi(Z) - \varphi(Z')\big)\big(\psi(Z) - \psi(Z')\big) \leq 0$$

and one concludes as above up to sign changes.

Equality case. As for the equality case under the co-monotony assumption, we may assume without loss of unreality that φ and ψ are non-decreasing. Moreover,we make the following convention: if a is not an atom of the distribution $\mathbb{P}_Z$ of Z, then set $\varphi(a) = \varphi(a_+)$, $\psi(a) = \psi(a_+)$, idem for b.

Now, if $\varphi(Z)$ or $\psi(Z)$ are $\mathbb{P}$-a.s. constant (hence to their expectation) then equality clearly holds.

Conversely, it follows by reading backward the above proof that if equality holds, then $\mathbb{E}\big(\varphi(Z) - \varphi(Z')\big)\big(\psi(Z) - \psi(Z')\big) = 0$ so that $\big(\varphi(Z) - \varphi(Z')\big)\big(\psi(Z) - \psi(Z')\big)$ $\mathbb{P}$-a.s.. Now, let I be the (closed) convex hull of the support of the distribution $\mu = \mathbb{P}_Z$ of Z on the real line. Assume *e.g.* that $I = [a, b] \subset \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$ (other cases can be adapted easily from that one).

By construction a and b are in the support of μ so that for every ε, $\varepsilon \in (0, b - a)$ both $\mathbb{P}(a \leq Z < a + \varepsilon)$ and $\mathbb{P}(b - \varepsilon < Z \leq b)$ are (strictly) positive. If a is an atom of $\mathbb{P}_Z$, one may choose $\varepsilon_a = 0$, idem for b. Hence the event $C_\varepsilon = \{a \leq Z < Z + \varepsilon\} \cap \{b - \varepsilon < Z' \leq b\}$ has a positive probability since Z and Z' are independent.

---

[1]This is always possible owing to Fubini's Theorem for product measures by considering the probability space $(\Omega^2, \mathcal{A}^{\otimes 2}, \mathbb{P}^{\otimes 2})$: extend Z by $Z(\omega, \omega') = Z(\omega)$ and define Z' by $Z'(\omega, \omega') = Z(\omega')$.

Now assume that $\varphi(Z)$ is not $\mathbb{P}$-*a.s.* constant. Then, φ cannot be constant on I and $\varphi(a) < \varphi(b)$ (with the above convention on atoms). Consequently, on C_ε, $\varphi(Z) - \varphi(Z') > 0$ *a.s.* so that $\psi(Z) - \Psi(Z') = 0$ $\mathbb{P}$-*a.s.*; which in turn implies that $\psi(a + \varepsilon) = \psi(b - \varepsilon)$. Then, letting ε and ε to 0, one derives that $\psi(a) = \psi(b)$ (still having in mind the convention on atoms)). Finally this shows that $\psi(Z)$ is $\mathbb{P}$-*a.s.* constant. ◇

(*b*) Set $\psi = \varphi \circ T$ so that φ and ψ have opposite monotony. Noting that X and X' do have the same distribution and applying claim (*a*) completes the proof. ◇

This leads to the well-known "*antithetic random variables method.*

ANTITHETIC RANDOM VARIABLES METHOD. This terminology is shared by two classical situations in which the above approach applies:

– the *symmetric random variable* Z: $Z \overset{d}{=} -Z$ (*i.e.* $T(z) = -z$).

– the $[0, L]$-valued random variable Z such that $Z \overset{d}{=} L - Z$ (*i.e.* $T(z) = L - z$). This is satisfied by $U \overset{d}{=} U([0, 1])$ with $L = 1$.

EXAMPLES. **1.** *European option pricing in a BS model.* Let $h_{_T} = \varphi(X_T^x)$ with φ monotone (like for Calls, Puts, spreads, etc). Then $h_{_T} = \varphi\left(xe^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}\right)$, $Z = \frac{W_T}{\sqrt{T}} \overset{d}{=} \mathcal{N}(0; 1)$. The function $z \mapsto \varphi(xe^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z})$ is monotone as the composition of two monotone functions and Z is symmetric.

2. *Uniform distribution on the unit interval.* If φ is monotone on $[0, 1]$ and $U \overset{d}{=} U([0, 1])$ then

$$\mathrm{Var}\left(\frac{\varphi(U) + \varphi(1 - U)}{2}\right) \leq \frac{1}{2}\mathrm{Var}(\varphi(U)).$$

The above one dimensional Proposition 3.2 admits a multi-dimensional extension that reads as follows:

Theorem 3.1 *Let $d \in \mathbb{N}^*$ and let $\Phi, \varphi : \mathbb{R}^d \to \mathbb{R}$ two functions satisfying the following joint marginal monotony assumption: for every $i \in \{1, \ldots, d\}$, for every $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_d) \in \mathbb{R}^{d-1}$,*

$$z_i \mapsto \Phi(z_1, \ldots, z_i, \ldots, z_d) \quad and \quad z_i \mapsto \varphi(z_1, \ldots, z_i, \ldots, z_d) \quad have \ the \ same \ monotony$$

which may depend on i and on $(z_{i+1}, \ldots, z_d)$. For every $i \in \{1, \ldots, d\}$, let $Z_1, \ldots, Z_d$ be independent real valued random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $T_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, d$, be non-increasing functions such that $T_i(Z_i) \overset{d}{=} Z_i$. Then, if $\Phi(Z_1, \ldots, Z_d), \varphi(Z_1, \ldots, Z_d) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$,

$$\mathrm{Cov}\big(\Phi(Z_1, \ldots, Z_d), \varphi(T_1(Z_1), \ldots, T_d(Z_d))\big) \leq 0.$$

Remarks. • This result may be successfully applied to functions $f(\bar{X}_{\frac{T}{n}}, \ldots, \bar{X}_{k\frac{T}{n}}, \ldots, \bar{X}_{n\frac{T}{n}})$ of the Euler scheme with step $\frac{T}{n}$ of a one-dimensional Brownian diffusion with a non-decreasing drift

and a deterministic strictly positive diffusion coefficient, provided f is "marginally monotonic" *i.e.* monotonic in each of its variable with the same monotony. (We refer to Chapter 7 for an introduction to the Euler scheme of a diffusion.) The idea is to rewrite the functional as a "marginally monotonic" function of the n (independent) Brownian increments which play the role of the r.v. Z_i. Furthermore, passing to the limit as the step size goes to zero, yields some correlation results for a class of monotone continuous functionals defined on the canonical space $\mathbb{C}([0,T],\mathbb{R})$ of the diffusion itself (the monotony should be understood with respect to the naive partial order.

• For more insight about this kind of *co-monotony* properties and their consequences for the pricing of derivatives, we refer to [124].

▷ **Exercises. 1.** Prove the above theorem by induction on d. [Hint: use Fubini's Theorem.]

2. Show that if φ and ψ are *non-negative* Borel functions defined on $\mathbb{R}$, monotone with *opposite monotony*,

$$\mathbb{E}\big(\varphi(Z)\psi(Z)\big) \leq \mathbb{E}\,\varphi(Z)\,\mathbb{E}\,\psi(Z)$$

so that, if $\varphi(Z),\ \psi(Z) \in L^1(\mathbb{P})$, then $\varphi(Z)\psi(Z) \in L^1(\mathbb{P})$.

3. Use Proposition 3.2(a) and Proposition 2.1(b) to derive directly (in the Black-Scholes model) from its representation as an expectation that the δ-hedge of a European option whose payoff function is convex is non-negative.

3.2   Regression based control variate

3.2.1   Optimal mean square control variate

We come back to the original situation of two *square integrable* random variables X and X', having the same expectation

$$\mathbb{E}\,X = \mathbb{E}\,X' = m$$

with nonzero variances *i.e.*

$$\mathrm{Var}(X),\ \mathrm{Var}(X') > 0.$$

We assume again that X and X' are not identical in the sense that $\mathbb{P}(X \neq X') > 0$ which turns out to be equivalent to

$$\mathrm{Var}(X - X') > 0.$$

We saw that if $\mathrm{Var}(X') \ll \mathrm{Var}(X)$, one will naturally choose X' to implement the Monte Carlo simulation and we provided several classical examples in that direction. However we will see that with a little more effort it is possible to improve this naive strategy.

This time we simply (and temporarily) set

$$\Xi := X - X' \qquad \text{with} \quad \mathbb{E}\,\Xi = 0 \qquad \text{and} \qquad \mathrm{Var}(\Xi) > 0.$$

The idea is simply to parametrize the impact of the control variate Ξ by a factor λ *i.e.* we set for every $\lambda \in \mathbb{R}$,

$$X^\lambda = X - \lambda\,\Xi.$$

Then the strictly convex parabolic function Φ defined by

$$\Phi(\lambda) := \mathrm{Var}(X^\lambda) = \lambda^2 \mathrm{Var}(\Xi) - 2\,\lambda\,\mathrm{Cov}(X,\Xi) + \mathrm{Var}(X)$$

reaches its minimum value at $\lambda_{\min}$ with

$$
\begin{aligned}
\lambda_{\min} \quad &:= \quad \frac{\mathrm{Cov}(X,\Xi)}{\mathrm{Var}(\Xi)} = \frac{\mathbb{E}(X\,\Xi)}{\mathbb{E}\,\Xi^2} \\
&= \quad 1 + \frac{\mathrm{Cov}(X',\Xi)}{\mathrm{Var}(\Xi)} = 1 + \frac{\mathbb{E}(X'\,\Xi)}{\mathbb{E}\,\Xi^2}.
\end{aligned}
$$

Consequently

$$\sigma^2_{\min} := \mathrm{Var}(X^{\lambda_{\min}}) = \mathrm{Var}(X) - \frac{(\mathrm{Cov}(X,\Xi))^2}{\mathrm{Var}(\Xi)} = \mathrm{Var}(X') - \frac{(\mathrm{Cov}(X',\Xi))^2}{\mathrm{Var}(\Xi)}.$$

so that

$$\sigma^2_{\min} \le \min\big(\mathrm{Var}(X), \mathrm{Var}(X')\big)$$

and $\sigma^2_{\min} = \mathrm{Var}(X)$ if and only if $\mathrm{Cov}(X,\Xi) = 0$.

Remark. Note that $\mathrm{Cov}(X,\Xi) = 0$ if and only if $\lambda_{\min} = 0$ *i.e.* $\mathrm{Var}(X) = \min_{\lambda \in \mathbb{R}} \Phi(\lambda)$.

If we denote by $\rho_{X,\Xi}$ the correlation coefficient between X and Ξ, one gets

$$\sigma^2_{\min} = \mathrm{Var}(X)(1 - \rho^2_{X,\Xi}) = \mathrm{Var}(X')(1 - \rho^2_{X',\Xi}).$$

A more symmetric expression for $\mathrm{Var}(X^{\lambda_{\min}})$ is

$$
\sigma^2_{\min} \quad = \quad \frac{\mathrm{Var}(X)\mathrm{Var}(X')(1 - \rho^2_{X,X'})}{(\sqrt{\mathrm{Var}(X)} - \sqrt{\mathrm{Var}(X')})^2 + 2\sqrt{\mathrm{Var}(X)\mathrm{Var}(X')}(1 - \rho_{X,X'})}
$$

$$
\le \quad \sigma_X \sigma_{X'} \frac{1 + \rho_{X,X'}}{2}.
$$

where σ_X and $\sigma_{X'}$ denote the standard deviations of X and X' respectively.

3.2.2 Implementation of the variance reduction: batch *vs* adaptive

Let $(X_k, X'_k)_{k \ge 1}$ be an i.i.d. sequence of random vectors with the same distribution as (X, X') and let $\lambda \in \mathbb{R}$. Set for every $k \ge 1$

$$\Xi_k = X_k - X'_k, \quad X^\lambda_k = X_k - \lambda\,\Xi_k.$$

Now, set for every size $M \geq 1$ of the simulation:

$$V_M := \frac{1}{M}\sum_{k=1}^{M}\Xi_k^2, \qquad C_M := \frac{1}{M}\sum_{k=1}^{M}X_k\Xi_k$$

and

$$\lambda_M := \frac{C_M}{V_M} \tag{3.2}$$

with the convention $\lambda_0 = 0$.

The "batch" approach. $\triangleright$ *Definition of the batch estimator.* The Strong Law of Large Numbers implies that both

$$V_M \longrightarrow \mathrm{Var}(X - X') \quad \text{and} \quad C_M \longrightarrow \mathrm{Cov}(X, X - X') \quad \mathbb{P}\text{-}a.s. \quad as \quad M \to +\infty.$$

so that

$$\lambda_M \to \lambda_{\min} \quad \mathbb{P}\text{-}a.s. \quad as \quad M \to +\infty.$$

This suggests to introduce the *batch* estimator of m, defined for every size $M \geq 1$ of the simulation by

$$\overline{X}_M^{\lambda_M} = \frac{1}{M}\sum_{k=1}^{M}X_k^{\lambda_M}.$$

One checks that, for every $M \geq 1$,

$$
\begin{aligned}
\overline{X}_M^{\lambda_M} &= \frac{1}{M}\sum_{k=1}^{M}X_k - \lambda_M\frac{1}{M}\sum_{k=1}^{M}\bar{\Xi}_k \\
&= \bar{X}_M - \lambda_M\bar{\Xi}_M
\end{aligned} \tag{3.3}
$$

with standard notations for empirical means.

$\triangleright$ *Convergence of the batch estimator.* The asymptotic behaviour of the batch estimator is summed up in the proposition below.

Proposition 3.3 *The* batch *estimator* $\mathbb{P}$-a.s. *converges to m (consistency) i.e.*

$$\overline{X}_M^{\lambda_M} = \frac{1}{M}\sum_{k=1}^{M}X_k^{\lambda_M} \xrightarrow{a.s.} \mathbb{E}X = m$$

and satisfies a CLT (asymptotic normality) with an optimal asymptotic variance $\sigma_{\min}^2$ *i.e.*

$$\sqrt{M}\left(\overline{X}_M^{\lambda_M} - m\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0; \sigma_{\min}^2\right).$$

Remark. However, note that the batch estimator is a *biased* estimator of m since $\mathbb{E}\,\lambda_M \bar{\Xi}_M \neq 0$.

Proof. First, one checks from (3.3) that

$$\frac{1}{M}\sum_{k=1}^{M} X_k^{\lambda_M} \xrightarrow{a.s.} m - \lambda_{\min} \times 0 = m.$$

Now, it follows from the regular CLT that

$$\sqrt{M}\left(\frac{1}{M}\sum_{k=1}^{M} X_k^{\lambda_{\min}} - m\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0; \sigma_{\min}^2\right)$$

since $\mathrm{Var}(X - \lambda_{\min}\Xi) = \sigma_{\min}^2$. On the other hand

$$\sqrt{M}\left(\frac{1}{M}\sum_{k=1}^{M} X_k^{\lambda_M} - X_k^{\lambda_{\min}}\right) = \left(\lambda_M - \lambda_{\min}\right) \times \frac{1}{\sqrt{M}}\sum_{k=1}^{M} \Xi_k \xrightarrow{\mathbb{P}} 0$$

owing to Slutsky Lemma $(^2)$ since $\lambda_M - \lambda_{\min} \to 0$ *a.s.* as $M \to +\infty$ and

$$\frac{1}{\sqrt{M}}\sum_{k=1}^{M} \Xi_k \xrightarrow{\mathcal{L}} \mathcal{N}\left(0; \mathbb{E}\,\Xi^2\right)$$

by the regular CLT applied to the centered square integrable i.i.d. random variables Ξ_k, $k \geq 1$. Combining these two convergence results yields the announced CLT. $\quad\diamond$

▷ **Exercise.** Let $\bar{X}_m$ and $\bar{\Xi}_m$ denote the empirical mean processes of the sequences $(X_k)_{k\geq}$ and $\Xi_k)_{k\geq 1}$ respectively. Show that the quadruplet $(\bar{X}_M, \bar{\Xi}_M, C_M, V_M)$ can be computed in a recursive way from the sequence $(X_k, X_k')_{k\geq 1}$. Derive a recursive way to compute the batch estimator

   ▷ *Practitioner's corner.* One may proceed as follows:

    – *Recursive implementation:* Use the recursion satisfied by the sequence $(\bar{X}_k, \bar{\Xi}_k, C_k, V_k)_{k\geq 1}$ to compute λ_M and the resulting batch estimator for each size M.

    – *True batch implementation:* A first phase of the simulation of size M', $M' \ll M$ (say $M' \approx 5\%$ or 10% of the total budget M of the simulation) devoted to a rough estimate $\lambda_{M'}$ of $\lambda_{\min}$, based on the Monte Carlo estimator (3.2).

A second phase of the simulation to compute the estimator of m defined by

$$\frac{1}{M - M'}\sum_{k=M'+1}^{M} X_k^{\lambda_{M'}}$$

whose asymptotic variance – given the first phase of the simulation – is $\dfrac{\mathrm{Var}(X^\lambda)_{|\lambda=\hat{\lambda}_{M'}}}{M - M'}$. This approach is not recursive at all. On the other hand, the above resulting estimator satisfies a CLT with asymptotic variance $\Phi(\hat{\lambda}_{M'}) = \mathrm{Var}(X^\lambda)_{|\lambda=\hat{\lambda}_{M'}}$. In particular we most likely observe a significant – although not optimal – variance reduction? So, from this point of view, you can stop reading this section at this point....

---

^{2}If $Y_n \to c$ in probability and $Z_n \to Z_\infty$ in distribution then, $Y_n Z_n \to c\,Z_\infty$ in distribution. In particular if $c = 0$ the last convergence holds in probability.

The adaptive unbiased approach.   Another approach is to design an adaptive estimator of m by considering at each step k the (predictable) estimator λ_{k-1} of $\lambda_{\min}$. This adaptive estimator is defined and analyzed below.

Theorem 3.2 *Assume $X, X' \in L^{2+\delta}(\mathbb{P})$ for some $\delta > 0$. Let $(X_k, X'_k)_{k \geq 1}$ be an i.i.d. sequence with the same distribution as (X, X'). We set for every $k \geq 1$*

$$\widetilde{X}_k = X_k - \tilde{\lambda}_{k-1}\Xi_k = (1 - \tilde{\lambda}_{k-1})X_k + \tilde{\lambda}_{k-1}X'_k \quad where \quad \tilde{\lambda}_k = (-k) \vee (\lambda_k \wedge k)$$

and λ_k is defined by (3.2). Then the adaptive estimator of m defined by

$$\overline{X}^{\widetilde{\lambda}}_M = \frac{1}{M} \sum_{k=1}^{M} \widetilde{X}_k$$

is unbiased ($\mathbb{E}\,\overline{X}^{\widetilde{\lambda}}_M = m$), convergent i.e.

$$\overline{X}^{\widetilde{\lambda}}_M \xrightarrow{a.s.} m \quad as \quad M \to +\infty,$$

and asymptotically normal with minimal variance i.e.

$$\sqrt{M}\left(\overline{X}^{\widetilde{\lambda}}_M - m\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0; \sigma^2_{\min}) \quad as \quad M \to +\infty.$$

The rest of this section can be omitted at the occasion of a first reading although the method of proof exposed below is quite standard when dealing with the efficiency of an estimator by martingale methods.

Proof. STEP 1 (*a.s.* convergence): Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let $\mathcal{F}_k := \sigma(X_1, X'_1, \ldots, X_k, X'_k)$, $k \geq 1$, be the filtration of the simulation.

First we show that $(\widetilde{X}_k - m)_{k \geq 1}$ is a sequence of square integrable $(\mathcal{F}_k, \mathbb{P})$-martingale increments. It is clear by construction that $\widetilde{X}_k$ is $\mathcal{F}_k$-measurable. Moreover,

$$\mathbb{E}\,(\widetilde{X}_k)^2 \leq 2\big(\mathbb{E}\,X_k^2 + \mathbb{E}(\tilde{\lambda}_{k-1}\Xi_k)^2\big)$$

$$= 2\big(\mathbb{E}X_k^2 + \mathbb{E}\,\tilde{\lambda}_{k-1}^2\,\mathbb{E}\,\Xi_k^2\big) < +\infty$$

where we used that Ξ_k and $\tilde{\lambda}_{k-1}$ are independent. Finally, for every $k \geq 1$,

$$\mathbb{E}(\widetilde{X}_k \,|\, \mathcal{F}_{k-1}) = \mathbb{E}(X_k \,|\, \mathcal{F}_{k-1}) - \tilde{\lambda}_{k-1}\mathbb{E}(\Xi_k \,|\, \mathcal{F}_{k-1}) = m.$$

This shows that the adaptive estimator is unbiased since $\mathbb{E}(\widetilde{X}_k) = m$ for every $k \geq 1$. In fact, we can also compute the conditional variance increment process:

$$\mathbb{E}((\widetilde{X}_k - m)^2 \,|\, \mathcal{F}_{k-1}) = \mathrm{Var}(X^\lambda)_{|\lambda = \tilde{\lambda}_{k-1}} = \Phi(\tilde{\lambda}_{k-1}).$$

Now, we set for every $k \geq 1$,

$$N_k := \sum_{\ell=1}^{k} \frac{\widetilde{X}_\ell - m}{\ell}.$$

It follows from what precedes that the sequence $(N_k)_{k \geq 1}$ is a square integrable $((\mathcal{F}_k)_k, \mathbb{P})$-martingale since $(\widetilde{X}_k - m)_{k \geq 1}$, is a sequence of square integrable $(\mathcal{F}_k, \mathbb{P})$-martingale increments. Its conditional variance increment process (also known as "bracket process") $\langle N \rangle_k$, $k \geq 1$ given by

$$
\begin{aligned}
\langle N \rangle_k &= \sum_{\ell=1}^{k} \frac{\mathbb{E}((\widetilde{X}_\ell - m)^2 \,|\, \mathcal{F}_{\ell-1})}{\ell^2} \\
&= \sum_{\ell=1}^{k} \frac{\Phi(\tilde{\lambda}_{\ell-1})}{\ell^2}.
\end{aligned}
$$

Now, the above series is *a.s.* converging since we know that $\Phi(\tilde{\lambda}_k)$ *a.s.* converges towards $\Phi(\lambda_{\min})$ as $kn \to +\infty$ since $\tilde{\lambda}_k$ *a.s.* converges toward $\lambda_{\min}$ and Φ is continuous. Consequently,

$$
\langle N \rangle_\infty = a.s. \lim_{M \to +\infty} \langle N \rangle_M < +\infty \quad a.s.
$$

Hence, it follows from Proposition 11.4 in Chapter 11 that $N_M \to N_\infty$ $\mathbb{P}$-*a.s.* as $M \to +\infty$ where N_∞ is an *a.s.* finite random variable. In turn, the *Kronecker Lemma* (see below) implies,

$$
\frac{1}{M} \sum_{k=1}^{M} \widetilde{X}_k - m \xrightarrow{a.s.} 0 \quad \text{as} \quad M \to \infty
$$

i.e.

$$
\overline{\widetilde{X}}_M := \frac{1}{M} \sum_{k=1}^{M} \widetilde{X}_k \xrightarrow{a.s.} m \quad \text{as} \quad M \to \infty.
$$

We will need this classical lemma (see Chapter 11 (Miscellany), Section 11.9 for a proof).

Lemma 3.1 (Kronecker Lemma) *Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let $(b_n)_{n \geq 1}$ be a non-decreasing sequence of positive real numbers with $\lim_n b_n = +\infty$. Then*

$$
\left(\sum_{n \geq 1} \frac{a_n}{b_n} \text{ converges in } \mathbb{R} \text{ as a series} \right) \implies \left(\frac{1}{b_n} \sum_{k=1}^{n} a_k \longrightarrow 0 \text{ as } n \to +\infty \right).
$$

STEP 2 (*CLT*, weak rate of convergence): It is a consequence of the Lindeberg Central Limit Theorem for (square integrable) martingale increments (see Theorem 11.7 the Miscellany chapter or Theorem 3.2 and its Corollary 3.1, p.58 in [70] refried to as "Lindeberg's *CLT*" in what follows). In our case, the array of martingale increments is defined by

$$
X_{M,k} := \frac{\widetilde{X}_k - m}{\sqrt{M}}, \ 1 \leq k \leq M.
$$

There are basically two assumptions to be checked. First the convergence of the conditional variance increment process toward $\sigma_{\min}^2$:

$$
\begin{aligned}
\sum_{k=1}^{M} \mathbb{E}(X_{M,k}^2 \,|\, \mathcal{F}_{k-1}) &= \frac{1}{M} \sum_{k=1}^{M} \mathbb{E}((\widetilde{X}_k - m)^2 \,|\, \mathcal{F}_{k-1}) \\
&= \frac{1}{M} \sum_{k=1}^{M} \Phi(\tilde{\lambda}_{k-1}) \\
&\longrightarrow \sigma_{\min}^2 := \min_\lambda \Phi(\lambda).
\end{aligned}
$$

The second one is the so-called Lindeberg condition (see again Miscellany or [70], p.58) which reads in our framework:

$$\forall \varepsilon > 0, \quad \sum_{\ell=1}^{M} \mathbb{E}(X_{M,\ell}^2 \mathbf{1}_{\{|X_{M,\ell}|>\varepsilon\}} \,|\, \mathcal{F}_{\ell-1}) \xrightarrow{\mathbb{P}} 0.$$

In turn, owing to conditional Markov inequality and the definition of $X_{M,\ell}$, this condition classically follows from the slightly stronger

$$\sup_{\ell \geq 1} \mathbb{E}(|\widetilde{X}_\ell - m|^{2+\delta} \,|\, \mathcal{F}_{\ell-1}) < +\infty \quad \mathbb{P}\text{-}a.s.$$

since

$$\sum_{\ell=1}^{M} \mathbb{E}(X_{M,\ell}^2 \mathbf{1}_{\{|X_{M,\ell}|>\varepsilon\}} \,|\, \mathcal{F}_{\ell-1}) \leq \frac{1}{\varepsilon^\delta M^{1+\frac{\delta}{2}}} \sum_{\ell=1}^{M} \mathbb{E}(|\widetilde{X}_\ell - m|^{2+\delta} \,|\, \mathcal{F}_{\ell-1}).$$

Now, using that $(u+v)^{2+\delta} \leq 2^{1+\delta}(u^{2+\delta} + v^{2+\delta})$, $u,v \geq 0$, and the fact that $X, X' \in L^{2+\delta}(\mathbb{P})$, one gets

$$\mathbb{E}(|\widetilde{X}_\ell - m|^{2+\delta} \,|\, \mathcal{F}_{\ell-1}) \leq 2^{1+\delta}(\mathbb{E}\,|X - m|^{2+\delta} + |\tilde{\lambda}_{\ell-1}|^{2+\delta}\mathbb{E}\,|\Xi|^{2+\delta}).$$

Finally, the Lindeberg Central Limit Theorem implies

$$\sqrt{M}\left(\frac{1}{M}\sum_{k=1}^{M}\widetilde{X}_k - m\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0;\sigma_{\min}^2).$$

This means that the expected variance reduction does occur if one implements the recursive approach described above. $\diamond$

3.3   Application to option pricing: using parity equations to produce control variates

The variance reduction by regression introduced in the former section still relies on the fact that $\kappa_X \approx \kappa_{X-\lambda\Xi}$ or, equivalently that the additional complexity induced by the simulation of Ξ given that of X is negligible. This condition may look demanding but we will see that in the framework of derivative pricing this requirement is always fulfilled as soon as the payoff of interest satisfies a so-called *parity equation i.e.* that the original payoff can by *duplicated* by a "synthetic" version.

Furthermore these *parity equations* are *model free* so they can be applied for various specifications of the dynamics of the underlying asset.

In this paragraph, we denote by $(S_t)_{t\geq 0}$ the risky asset (with $S_0 = s_0 > 0$) and set $S_t^0 = e^{rt}$ the riskless asset. We work under the risk-neutral probability which means that

$$\left(e^{-rt}S_t\right)_{t\in[0,T]} \quad \text{is a martingale on the scenarii space } (\Omega, \mathcal{A}, \mathbb{P})$$

(with respect to the augmented filtration of $(S_t)_{t\in[0,T]}$). This means that $\mathbb{P}$ is a *risk-neutral* probability (supposed to exist). furthermore, to comply with usual assumptions of AOA theory, we will assume that this risk neutral probability is unique (complete market) to justify that we may price any derivative under this probability. However this has no real impact on what follows.

$\triangleright$ Vanilla Call-Put parity $(d = 1)$: We consider a *Call* and a *Put* with common maturity T and strike K. We denote by

$$\mathrm{Call}_0 = e^{-rT}\mathbb{E}\big((S_T - K)_+\big) \qquad \text{and} \qquad \mathrm{Put}_0 = e^{-rT}\mathbb{E}\big(K - S_T)_+\big)$$

the premia of this *Call* and this *Put* option respectively. Since

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

and $\left(e^{-rt} S_t\right)_{t \in [0,T]}$ is a martingale, one derives the classical *Call-Put* parity equation:

$$\text{Call}_0 - \text{Put}_0 = s_0 - e^{-rT} K$$

so that $\text{Call}_0 = \mathbb{E}(X) = \mathbb{E}(X')$ with

$$X := e^{-rT}(S_T - K)_+ \quad \text{and} \quad X' := e^{-rT}(K - S_T)_+ + s_0 - e^{-rT} K.$$

As a result one sets

$$\Xi = X - X' = e^{-rT} S_T - s_0$$

which turns out to be the terminal value of a martingale null at time 0 (this is in fact the generic situation of application of this parity method).

Note that the simulation of X involves that of S_T so that the additional cost of the simulation of Ξ is definitely negligible.

▷ ASIAN CALL-PUT PARITY: We consider an :em asian Call and an *Asian Put* with common maturity T, strike K and averaging period $[T_0, T]$, $0 \le T_0 > T]$.

$$\text{Call}_0^{As} = e^{-rT} \mathbb{E}\left(\left(\frac{1}{T - T_0} \int_{T_0}^{T} S_t \, dt - K\right)_+\right) \quad \text{and} \quad \text{Put}_0^{As} = e^{-rT} \mathbb{E}\left(\left(K - \frac{1}{T - T_0} \int_{T_0}^{T} S_t \, dt\right)_+\right).$$

Still using that $\widetilde{S}_t = e^{-rt} S_t$ is a $\mathbb{P}$-martingale and, this time, Fubini-Tonelli's Theorem yield

$$\text{Call}_0^{As} - \text{Put}_0^{As} = s_0 \frac{1 - e^{-r(T - T_0)}}{r(T - T_0)} - e^{-rT} K$$

so that

$$\text{Call}_0^{As} = \mathbb{E}(X) = \mathbb{E}(X')$$

with

$$X := e^{-rT} \left(\frac{1}{T - T_0} \int_{T_0}^{T} S_t \, dt - K\right)_+$$

$$X' := s_0 \frac{1 - e^{-r(T - T_0)}}{r(T - T_0)} - e^{-rT} K + e^{-rT} \left(K - \frac{1}{T - T_0} \int_{T_0}^{T} S_t \, dt\right)_+.$$

This leads to

$$\Xi = e^{-rT} \frac{1}{T - T_0} \int_{T_0}^{T} S_t dt - s_0 \frac{1 - e^{-r(T - T_0)}}{r(T - T_0)}.$$

Remark. In both cases, the parity equation directly follows from the $\mathbb{P}$-*martingale property* of $\widetilde{S}_t = e^{-rt} S_t$.

3.3.1   Complexity aspects in the general case

In practical implementations, one often neglects the cost of the computation of $\lambda_{\min}$ since only a rough estimate is computed: this leads to stop its computation after the first 5% or 10% of the simulation.

 – However, one must be aware that the case of the existence of parity equations is quite specific since the random variable Ξ is involved in the simulation of X, so the complexity of the simulation process *is not* increased: thus in the recursive approach the updating of λ_M and of (the empirical mean) $\overline{\widetilde{X}}_M$ is (almost) costless. Similar observations can be made to some extent on batch approaches. As a consequence, in that specific setting, the complexity of the adaptive linear regression procedure and the original one are (almost) the same!

 – WARNING ! This is no longer true in general... and in a general setting the complexity of the simulation of X and X' is double of that of X itself. Then the regression method is efficient iff

$$\sigma_{\min}^2 < \frac{1}{2} \min(\mathrm{Var}(X), \mathrm{Var}(X'))$$

(provided one neglects the cost of the estimation of the coefficient $\lambda_{\min}$).

 The exercise below shows the connection with antithetic variables which then appears as a special case of regression methods.

▷ **Exercise (*Connection with the antithetic variable methdod*).** Let X, $X' \in L^2(\mathbb{P})$ such that $\mathbb{E}\,X = \mathbb{E}\,X' = m$ and $\mathrm{Var}(X) = \mathrm{Var}(X')$.

(*a*) Show that

$$\lambda_{\min} = \frac{1}{2}.$$

(*b*) Show that

$$X^{\lambda_{\min}} = \frac{X + X'}{2}$$

and

$$\mathrm{Var}\left(\frac{X + X'}{2}\right) = \frac{1}{2}\left(\mathrm{Var}(X) + \mathrm{Cov}(X, X')\right).$$

Characterize the couple (X, X') with which the regression method does reduce the variance. Make the connection with the antithetic method.

3.3.2   Examples of numerical simulations

▷ VANILLA B-S CALLS. The model parameters are specified as follows

$$T = 1,\ x_0 = 100,\ r = 5,\ \sigma = 20,\ K = 90, \ldots, 120.$$

 The simulation size is set at $M = 10^6$.

▷ ASIAN CALLS IN A HESTON MODEL.
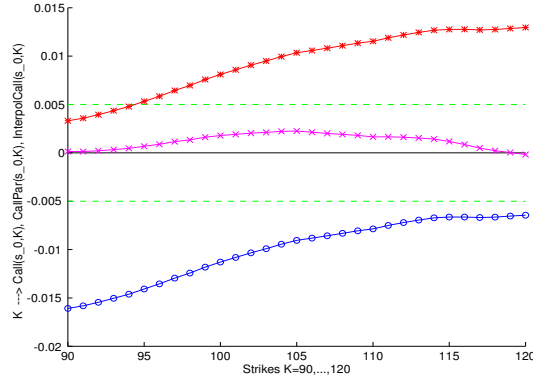
Figure 3.1: BLACK-SCHOLES CALLS: *Error = Reference BS−(MC Premium).* $K = 90, \dots, 120$. *–o–o–o– Crude Call. –∗–∗–∗– Synthetic Parity Call. ×××– Interpolated synthetic Call.*
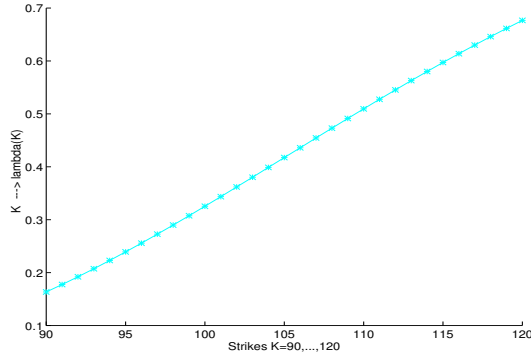
Figure 3.2: BLACK-SCHOLES CALLS: $K \mapsto 1 - \lambda_{\min}(K)$, $K = 90, \dots, 120$, *for the Interpolated synthetic Call.*

The dynamics of the risky asset is this time a stochastic volatility model, namely the Heston model, defined as follows. Let ϑ, k, a such that $\vartheta^2/(4ak) < 1$ (so that v_t remains *a.s.* positive, see [91]).

$$dS_t = S_t\big(r\,dt + \sqrt{v_t}dW_t^1\big), \qquad s_0 = x_0 > 0, \quad \text{(risky asset)}$$

$$dv_t = k(a - v_t)dt + \vartheta\sqrt{v_t}dW_t^2, \; v_0 > 0 \quad \text{with} \; <W^1, W^2>_t = \rho\,t, \; \rho \in [-1, 1].$$

The payoff is a an Asian call with strike price K

$$\mathrm{AsCall}^{Hest} = e^{-rT}\mathbb{E}\left(\left(\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+\right)\right).$$

Usually, no closed form are available for Asian payoffs even in the Black-Scholes model and so is the case in the Heston model. Note however that (quasi-)closed forms do exist for vanilla European
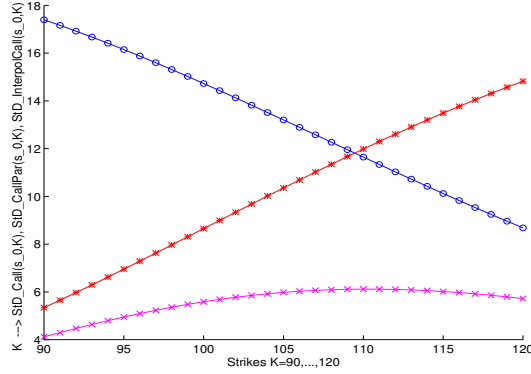
Figure 3.3: BLACK-SCHOLES CALLS.   *Standard Deviation(MC Premium).* $K =$
$90, \ldots, 120$. –o–o–o– *Crude Call.* –∗–∗–∗– *Parity Synthetic Call.* ××× *Interpolated Synthetic Call.*

options in this model (see [72]) which is the origin of its success. The simulation has been carried
out by replacing the above diffusion by an Euler scheme (see Chapter 7 for an introduction to
Euler scheme). In fact the dynamics of the stochastic volatility process does not fulfill the standard
Lipschitz continuous continuous assumptions required to make the Euler scheme converge at least
at its usual rate. In the present case it is even difficult to define this scheme because of the term
$\sqrt{v_t}$. Since our purpose here is to illustrate the efficiency of parity relations to reduce variance we
adopted a rather "basic" scheme, namely

$$\bar{S}_{\frac{kT}{n}} = \bar{S}_{\frac{(k-1)T}{n}}\left(1 + \frac{rT}{n} + \sqrt{|\bar{v}_{\frac{(k-1)T}{n}}|}\sqrt{\frac{T}{n}}\left(\rho U_k^2 + \sqrt{1-\rho^2}U_k^1\right)\right), \qquad \bar{S}_0 = s_0 > 0$$

$$\bar{v}_{\frac{kT}{n}} = k\left(a - \bar{v}_{\frac{(k-1)T}{n}}\right)\frac{T}{n} + \vartheta\sqrt{|v_{\frac{(k-1)T}{n}}|}\,U_k^2, \ \bar{v}_0 = v_0 > 0$$

where $U_k = (U_k^1, U_k^2)_{k\geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0, I_2)$-distributed random vectors.

   This scheme is consistent but its rate of convergence if not optimal. The simulation of the Heston
model has given rise to an extensive literature. Se *e.g.* among others Diop, Alfonsi, Andersen, etc.....

– Parameters of the model:

$$s_0 = 100,\ k = 2,\ a = 0.01,\ \rho = 0.5,\ v_0 = 10\%,\ \vartheta = 20\%.$$

– Parameters of the option portfolio:

$$T = 1,\ \ K = 90, \cdots, 120 \qquad (31 \text{ strikes}).$$

▷ **Exercises.** One considers a 1-dimensional Black-Scholes model with market parameters

$$r = 0, \quad \sigma = 0.3, \quad x_0 = 100, \quad T = 1.$$

1. One considers a vanilla Call with strike $K = 80$. The random variable Ξ is defined as above.
Estimate the $\lambda_{\min}$ (one should be not too far from 0.825). Then, compute a confidence interval for
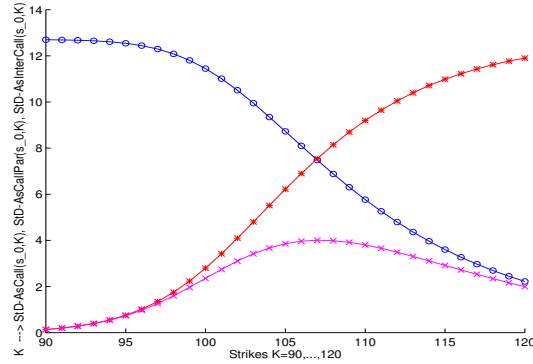
Figure 3.4: HESTON ASIAN CALLS. *Standard Deviation (MC Premium).* $K = 90, \ldots, 120.$
–o–o–o– *Crude Call.* –∗–∗–∗– *Synthetic Parity Call.* ×××– *Interpolated synthetic Call.*
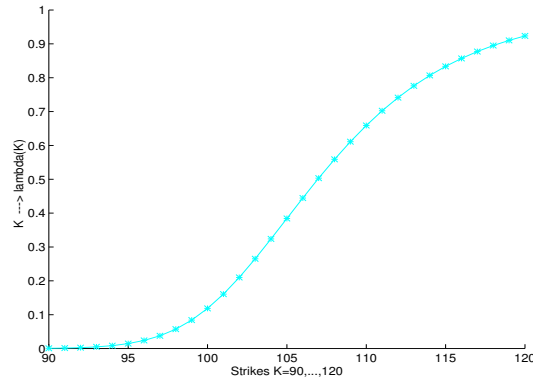
Figure 3.5: HESTON ASIAN CALLS. $K \mapsto 1 - \lambda_{\min}(K)$, $K = 90, \ldots, 120$, *for the Interpolated Synthetic Asian Call.*

the Monte Carlo pricing of the Call with and without the linear variance reduction for the following sizes of the simulation: $M = 5\,000,\ 10\,000, 100\,000,\ 500\,000$.

2. Proceed as above but with $K = 150$ (true price 1.49). What do you observe ? Provide an interpretation.

3.3.3 Multidimensional case

▷ Let $X := (X^1, \ldots, X^d)$, $\Xi := (\Xi^1, \ldots, \Xi^q) : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \mathbb{R}^q$, be *square integrable* random vectors.

$$\mathbb{E}\,X = m \in \mathbb{R}^d, \qquad \mathbb{E}\,\Xi = 0 \in \mathbb{R}^q.$$

Let $D(X) := [\mathrm{Cov}(X^i, X^j)]_{1 \le i,j \le d}$ and let $D(\Xi)$ denote the covariance (dispersion) matrices of X and Ξ respectively. Assume

$$D(X) \text{ and } D(\Xi) > 0$$
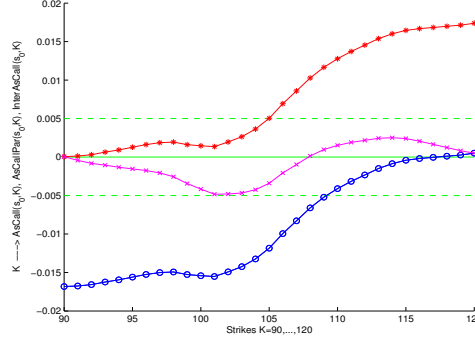
Figure 3.6: Heston Asian Calls. $M = 10^6$ (Reference: MC with $M = 10^8$). $K = 90, \ldots, 120$. –o-o-o– Crude Call. –∗–∗–∗– Parity Synthetic Call. ⨯⨯⨯– Interpolated Synthetic Call.

as positive definite symmetric matrices.

▷ Problem: Find a matrix $\Lambda \in \mathcal{M}(d, q)$ solution to the optimization problem

$$\mathrm{Var}(X - \Lambda\, \Xi) = \min \left\{ \mathrm{Var}(X - L\, \Xi),\ L \in \mathcal{M}(d, q) \right\}$$

where $\mathrm{Var}(Y)$ is defined by $\mathrm{Var}(Y) := \mathbb{E}|Y - \mathbb{E}\, Y|^2 = \mathbb{E}\, |Y|^2 - |\mathbb{E}\, Y|^2$ for any $\mathbb{R}^d$-valued random vector.

▷ Solution

$$\Lambda = D(\Xi)^{-1} C(X, \Xi)$$

where

$$C(X, \Xi) = [\mathrm{Cov}(X^i, \Xi^j)]_{1 \leq i \leq d, 1 \leq j \leq q}.$$

▷ **Examples-Exercises:** Let $X_t = (X_t^1, \ldots, X_t^d)$, $t \in [0, T]$, be the price process of d risky traded assets (be careful about the notations that collide at this point: X is here for the traded assets and the aim is to reduce the variance of the discounted payoff usually denoted with the letter h).

1. Options on various baskets:

$$h_T^i = \left(\sum_{j=1}^{d} \theta_j^i X_T^j - K \right)_+ , \ i = 1, \ldots, d.$$

Remark. This approach also produces an *optimal asset selection* (since it is essentially a *PCA*) which helps for hedging.

2. Portfolio of *forward start options*

$$h^{i,j} = \left(X_{T_{i+1}}^j - X_{T_i}^j \right)_+ , \ i = 1, \ldots, d - 1$$

where T_i, $i = 0, \ldots, d$ is an increasing sequence of maturities.

3.4 Pre-conditioning

The principle of the pre-conditioning method – also known as the Blackwell-Rao method – is based on the very definition of conditional expectation: let $(\Omega, \mathcal{A}, \mathbb{P})$ be probability space and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be a square integrable random variable. The practical constraint for implementation is the ability to simulate at a competitive cost $\mathbb{E}(X \,|\, \mathcal{B})$. The examples below show some typical situations where so is the case.

For every sub-σ-field $\mathcal{B} \subset \mathcal{A}$

$$\mathbb{E}\, X = \mathbb{E}\,(\mathbb{E}(X \,|\, \mathcal{B}))$$

and

$$
\begin{aligned}
\mathrm{Var}\,(\mathbb{E}(X \,|\, \mathcal{B})) &= \mathbb{E}(\mathbb{E}(X \,|\, \mathcal{B})^2) - (\mathbb{E}\, X)^2 \\
&\leq \mathbb{E}(X^2) - (\mathbb{E}\, X)^2 = \mathrm{Var}(X)
\end{aligned}
$$

since conditional expectation is a contraction in $L^2(\mathbb{P})$ as an orthogonal projection. Furthermore the above inequality is strict except if X is $\mathcal{B}$-measurable *i.e.* in any non-trivial case.

The archetypal situation is the following: assume

$$X = g(Z_1, Z_2), \qquad g \in L^2(\mathbb{P}_{(Z_1, Z_2)}),$$

where Z_1, Z_2 are independent random vectors. Set $\mathcal{B} := \sigma(Z_2)$. Then standard results on conditional expectations show that

$$\mathbb{E}\, X = \mathbb{E}\, G(Z_2) \quad \text{where} \quad G(z_2) = \mathbb{E}\,(g(Z_1, Z_2) \,|\, Z_2 = z_2) = \mathbb{E}\, g(Z_1, z_2)$$

is a version of the conditional expectation of $g(Z_1, Z_2)$ given $\sigma(Z_2)$. At this stage, the pre-conditionning method can be implemented as soon as the following conditions are satisfied:

– a closed form is available for the function G and

– the (distribution of) Z_2 can be simulated with the same complexity as (the distribution of) X.

EXAMPLES. **1.** *Exchange spread options.* Let $X_T^i = x_i e^{(r - \frac{\sigma_i^2}{2})T + \sigma_i W_T^i}$, x_i, $\sigma_i > 0$, $i = 1, 2$, be two "Black-Scholes" assets at time T related to two Brownian motions W_T^i, $i = 1, 2$, with correlation $\rho \in [-1, 1]$. One considers an exchange spread options with strike K *i.e.* related to the payoff

$$h_T = (X_T^1 - X_T^2 - K)_+.$$

Then one can write

$$(W_T^1, W_T^2) = \sqrt{T}\left(\sqrt{1 - \rho^2}\, Z_1 + \rho\, Z_2, Z_2\right)$$

where $Z = (Z_1, Z_2)$ is an $\mathcal{N}(0; I_2)$-distributed random vector. Then

$$
\begin{aligned}
e^{-rT}\mathbb{E}\,(h_T \,|\, Z_2) &= e^{-rT}\mathbb{E}\left(\left(x_1 e^{(r - \frac{\sigma_1^2}{2})T + \sigma_1\sqrt{T}(\sqrt{1-\rho^2}\, Z_1 + \rho z_2)} - x_2 e^{(r - \frac{\sigma_2^2}{2})T + \sigma_2\sqrt{T}z_2} - K\right)_+\right)_{|z_2 = Z_2} \\
&= \mathrm{Call}_{BS}\left(x_1 e^{-\frac{\rho^2\sigma_1^2 T}{2} + \sigma_1\rho\sqrt{T}Z_2}, x_2 e^{(r - \frac{\sigma_2^2}{2})T + \sigma_2\sqrt{T}Z_2} + K, r, \sigma_1\sqrt{1 - \rho^2}, T\right).
\end{aligned}
$$

Then one takes advantage of the closed form available for vanilla call options in a Black-scholes model to compute

$$\mathrm{Premium}_{BS}(X_0^1, x_0^2, K, \sigma_1, \sigma_2, r, T) = \mathbb{E}\Big(\mathbb{E}(e^{-rT}h_T \,|\, Z_2)\Big)$$

with a smaller variance than with the original payoff.

2. *Barrier options.* This example will be detailed in Section 8.2.3 devoted to the pricing of (some classes) of barrier options in a general model using the simulation of a continuous Euler scheme (using the so-called Brownian bridge method).

3.5   Stratified sampling

The starting idea of stratification is to localize the Monte Carlo method on the elements of a measurable partition of the state space E of a random variable $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{E})$.

Let $(A_i)_{i \in I}$ be a finite $\mathcal{E}$-measurable partition of the state space E. The A_i's are called *strata*. Assume that the weights

$$p_i = \mathbb{P}(X \in A_i), \ i \in I,$$

are *known*, (strictly) *positive* and that, still for every $i \in I$,

$$\mathcal{L}(X \,|\, X \in A_i) \overset{d}{=} \varphi_i(U)$$

where U is uniformly distributed on $[0, 1]^{r_i}$ (with $r_i \in \mathbb{N} \cup \{\infty\}$, the case $r_i = \infty$ corresponding to the acceptance-rejection method) and $\varphi_i : [0, 1]^{r_i} \to E$ is an (easily) computable function. This second condition simply means that $\mathcal{L}(X \,|\, X \in A_i)$ is easy to simulate on a computer. To be more precise we implicitly assume in what follows that the the simulation of X and of the conditional distribution $\mathcal{L}(X \,|\, X \in A_i)$, $i \in I$, or equivalently the random vectors $\varphi_i(U)$ have the same complexity. One must always keep that in mind since *it is a major constraint for practical implementations of stratification methods*.

This simulability condition usually has a strong impact on the possible design of the strata.

For convenience, we will assume in what follows that $r_i = r$.

Let $F : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}or(\mathbb{R}))$ such that $\mathbb{E}\, F^2(X) < +\infty$,

$$
\begin{aligned}
\mathbb{E}\, F(X) &= \sum_{i \in I} \mathbb{E}\Big(\mathbf{1}_{\{X \in A_i\}} F(X)\Big) \\
&= \sum_{i \in I} p_i \,\mathbb{E}\big(F(X) \,|\, X \in A_i\big) \\
&= \sum_{i \in I} p_i \,\mathbb{E}\big(F(\varphi_i(U))\big).
\end{aligned}
$$

The stratification idea takes place now. Let M be the global "budget" allocated to the simulation of $\mathbb{E}\, F(X)$. We split this budget into $|I|$ groups by setting

$$M_i = q_i M, \quad i \in I,$$

be the allocated budget to compute $\mathbb{E}\, F(\varphi_i(U))$ in each stratum "A_i". This leads to define the following (unbiased) estimator

$$\widehat{F(X)}_M^I := \sum_{i \in I} p_i \frac{1}{M_i} \sum_{k=1}^{M_i} F(\varphi_i(U_i^k))$$

where $(U_i^k)_{1 \le k \le M_i, i \in I}$ are uniformly distributed on $[0,1]^r$, i.i.d. random variables. Then, elementary computations show that

$$\mathrm{Var}\left(\widehat{F(X)}_M^I \right) = \frac{1}{M} \sum_{i \in I} \frac{p_i^2}{q_i} \sigma_{F,i}^2$$

where, for every $i \in I$,

$$
\begin{aligned}
\sigma_{F,i}^2 &= \mathrm{Var}\Big(F(\varphi_i(U)) \Big) = \mathrm{Var}\big(F(X) \mid X \in A_i \big) \\
&= \mathbb{E}\Big(\big(F(X) - \mathbb{E}(F(X)|X \in A_i) \big)^2 \mid X \in A_i \Big) \\
&= \frac{\mathbb{E}\Big(\big(F(X) - \mathbb{E}(F(X)|X \in A_i) \big)^2 \mathbf{1}_{\{X \in A_i\}} \Big)}{p_i}.
\end{aligned}
$$

Optimizing the simulation allocation to each stratum amounts to solving the following minimization problem

$$\min_{(q_i) \in \mathcal{P}_I} \sum_{i \in I} \frac{p_i^2}{q_i} \sigma_{F,i}^2 \qquad \text{where} \qquad \mathcal{P}_I := \left\{ (q_i)_{i \in I} \in \mathbb{R}_+^I \mid \sum_{i \in I} q_i = 1 \right\}.$$

▷ A *sub-optimal choice*, but natural and simple, is to set

$$q_i = p_i, \quad i \in I.$$

Such a choice is first motivated by the fact that the weights p_i are known and of course because it does reduce the variance since

$$
\begin{aligned}
\sum_{i \in I} \frac{p_i^2}{q_i} \sigma_{F,i}^2 &= \sum_{i \in I} p_i\, \sigma_{F,i}^2 \\
&= \sum_{i \in I} \mathbb{E}\left(\mathbf{1}_{\{X \in A_i\}} (F(X) - \mathbb{E}(F(X)|X \in A_i))^2 \right) \\
&= \| F(X) - \mathbb{E}(F(X)|\sigma(\{X \in A_i\}, i \in I)) \|_2^2.
\end{aligned}
\tag{3.4}
$$

where $\sigma(\{X \in A_i\}, i \in I))$ denotes the σ-field spanned by the partition $\{X \in A_i\}, i \in I$. Consequently

$$\sum_{i \in I} \frac{p_i^2}{q_i} \sigma_{F,i}^2 \le \| F(X) - \mathbb{E}(F(X)) \|_2^2 = \mathrm{Var}(F(X)) \tag{3.5}$$

with equality if and only if $\mathbb{E}\,(F(X)|\sigma(\{X \in A_i\}, i \in I)) = \mathbb{E}\,F(X) \; \mathbb{P}\text{-}a.s.$ Or, equivalently, equality holds in this inequality if and only if

$$\mathbb{E}(F(X) \mid X \in A_i) = \mathbb{E}\,F(X), \; i \in I.$$

So this choice always reduces the variance of the estimator since we assumed that the stratification is not trivial. It corresponds in the *opinion poll* world to the so-called *quota method*.

▷ The *optimal choice* is the solution to the above constrained minimization problem. It follows from a simple application of Schwarz Inequality (and its equality case) that

$$
\begin{aligned}
\sum_{i \in I} p_i \sigma_{F,i} &= \sum_{i \in I} \frac{p_i \sigma_{F,i}}{\sqrt{q_i}} \sqrt{q_i} \\
&\leq \left(\sum_{i \in I} \frac{p_i^2 \sigma_{F,i}^2}{q_i} \right)^{\frac{1}{2}} \left(\sum_{i \in I} q_i \right)^{\frac{1}{2}} \\
&= \left(\sum_{i \in I} \frac{p_i^2 \sigma_{F,i}^2}{q_i} \right)^{\frac{1}{2}} \times 1 \\
&= \left(\sum_{i \in I} \frac{p_i^2 \sigma_{F,i}^2}{q_i} \right)^{\frac{1}{2}}.
\end{aligned}
$$

Consequently, the optimal choice for the allocation parameters q_i's *i.e.* the solution to the above constrained minimization problem is given by

$$
q_{F,i}^* = \frac{p_i \sigma_{F,i}}{\sum_j p_j \sigma_{F,j}}, \quad i \in I,
$$

with a resulting minimal variance given by

$$
\left(\sum_{i \in I} p_i \sigma_{F,i} \right)^2.
$$

At this stage the problem is that the local inertia $\sigma_{F,i}^2$ are not known which makes the implementation less straightforward. Some attempts have been made to circumvent this problem, see *e.g.* [45] for a recent reference based on an adaptive procedure for the computation of the local F-inertia $\sigma_{F,i}^2$.

However, using that the L^p-norms with respect to a probability measure are non-decreasing in p, one derives that $\sigma_{F,i} \geq \mathbb{E}\left(|X - \mathbb{E}(X|\{X \in A_i\})| \mid \{X \in A_i\} \right)$ so that

$$
\left(\sum_{i \in I} p_i \sigma_{F,i} \right)^2 \geq \| X - \mathbb{E}(X | \sigma(\{X \in A_i\}, i \in I)) \|_1^2.
$$

EXAMPLES. Stratifications for the computation of $\mathbb{E}\, F(X)$, $X \overset{d}{=} \mathcal{N}(0; I_d)$, $d \geq 1$.

(*a*) *Stripes.* Let v be a fixed unitary vector (a simple and natural choice for v is $v = e_1 = (1,0,0,\cdots,0)$): it is natural to define the strata as hyper-stripes perpendicular to the main axis $\mathbb{R}e_1$ of X). So, we set, for a given size N of the stratification ($I = \{1, \ldots, N\}$),

$$
A_i := \{ x \in \mathbb{R}^d \ \text{ s.t. } \ (v|x) \in [y_{i-1}, y_i] \}, \quad i = 1, \ldots, N
$$

where $y_i \in \overline{\mathbb{R}}$ is defined by $\Phi_0(y_i) = \frac{i}{N}$, $i = 0, \ldots, N$ (the N-quantiles of the $\mathcal{N}(0;1)$ distribution). Then, if Z denotes a $\mathcal{N}(0;1)$-distributed random variable,

$$p_i = \mathbb{P}(X \in A_i) = \mathbb{P}(Z \in [y_{i-1}, y_i]) = \Phi_0(y_i) - \Phi_0(y_{i-1}) = \frac{1}{N}$$

where Φ_0 denotes the distribution function of the $\mathcal{N}(0;1)$-distribution. Other choices are possible for the y_i, leading to a non uniform distribution of the p_i's. The simulation of the conditional distributions follows from the fact that

$$\mathcal{L}(X \,|\, (v|X) \in [a, b]) = \xi_1 v + \pi_{v^\perp}^\perp(\xi_2)$$

where $\xi_1 \stackrel{d}{=} \mathcal{L}(Z \,|\, Z \in [a, b])$ is independent of $\xi_2 \stackrel{d}{=} \mathcal{N}(0, I_{d-1})$,

$$\mathcal{L}(Z \,|\, Z \in [a, b]) \stackrel{d}{=} \Phi_0^{-1}\Big((\Phi_0(b) - \Phi_0(a))U + \Phi_0(a)\Big), \quad U \stackrel{d}{=} U([0, 1])$$

and $\pi_{v^\perp}^\perp$ denotes the orthogonal projection on $v^\perp$. When $v = e_1$, this reads simply

$$\mathcal{L}(X \,|\, (v|X) \in [a, b]) = \mathcal{L}(Z \,|\, Z \in [a, b]) \otimes \mathcal{N}(0, I_{d-1}).$$

(b) *Hyper-rectangles.* We still consider $X = (X^1, \ldots, X^d) \stackrel{d}{=} \mathcal{N}(0; I_d)$, $d \geq 2$. Let $(e_1, \ldots, e_d)$ denote the canonical basis of $\mathbb{R}^d$. We define the strata as hyper-rectangles. Let $N_1, \ldots, N_d \geq 1$.

$$A_{\underline{i}} := \prod_{\ell=1}^{d} \Big\{ x \in \mathbb{R}^d \ \text{s.t.} \ (e_\ell|x) \in [y_{i_\ell-1}^\ell, y_{i_\ell}^\ell] \Big\}, \quad \underline{i} \in \prod_{\ell=1}^{d} \{1, \ldots, N_\ell\}$$

where the $y_i^\ell \in \overline{\mathbb{R}}$ are defined by $\Phi_0(y_i^\ell) = \frac{i}{N_\ell}$, $i = 0, \ldots, N_\ell$. Then, for every multi-index $\underline{i} = (i_1, \ldots, i_d) \in \prod_{\ell=1}^{d} \{1, \ldots, N_\ell\}$,

$$\mathcal{L}(X \,|\, X \in A_{\underline{i}}) = \bigotimes_{\ell=1}^{d} \mathcal{L}(Z \,|\, Z \in [y_{i_\ell-1}^\ell, y_{i_\ell}^\ell]). \tag{3.6}$$

Optimizing the allocation to each stratum in the simulation for a given function F in order to to reduce the variance is of course interesting and can be highly efficient but with the drawback to be strongly F-dependent, especially when this allocation needs an extra procedure like in [45]. An alternative and somewhat dual approach is to try optimizing the strata themselves uniformly with respect to a class of functions F (namely Lipschitz continuous continuous functions) prior to the allocation of the allocation.

This approach emphasizes the connections between stratification and optimal quantization and provides bounds on the best possible variance reduction factor that can be expected from a stratification. Some elements are provided in Chapter 5, see also [38] for further developments in infinite dimension.

3.6   Importance sampling

3.6.1   The abstract paradigm of important sampling

The basic principle of importance sampling is the following: let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{E})$ be an E-valued random variable. Let μ be a σ-finite *reference* measure on $(E, \mathcal{E})$ so that $\mathbb{P}_X \ll \mu$ *i.e.* there exists a probability density $f : (E, \mathcal{E}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$\mathbb{P}_X = f.\mu.$$

In practice, we will have to simulate several r.v., whose distributions are all absolutely continuous with respect to this *reference* measure μ. For a first reading one may assume that $E = \mathbb{R}$ and μ is the Lebesgue measure but what follows can also be applied to more general measured spaces like the Wiener space (equipped with the Wiener measure), etc. Let $h \in L^1(\mathbb{P}_X)$. Then,

$$\mathbb{E}\, h(X) = \int_E h(x)\mathbb{P}_X(dx) = \int_E h(x)f(x)\mu(dx).$$

Now for any μ-*a.s.* positive probability density function g defined on $(E, \mathcal{E})$ (with respect to μ), one has

$$\mathbb{E}\, h(X) = \int_E h(x)f(x)\mu(dx) = \int_E \frac{h(x)f(x)}{g(x)}g(x)\mu(dx).$$

One can always enlarge (if necessary) the original probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to design a random variable $Y : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{E})$ having g as a probability density with respect to μ. Then, going back to the probability space yields for every non-negative or $\mathbb{P}_X$-integrable function $h : E \to \mathbb{R}$,

$$\mathbb{E}\, h(X) = \mathbb{E}\left(\frac{h(Y)f(Y)}{g(Y)}\right). \tag{3.7}$$

So, in order to compute $\mathbb{E}\, h(X)$ one can also implement a Monte Carlo simulation based on the simulation of independent copies of the random variable Y *i.e.*

$$\mathbb{E}\, h(X) = \mathbb{E}\left(\frac{h(Y)f(Y)}{g(Y)}\right) = \lim_{M \to +\infty} \frac{1}{M}\sum_{k=1}^{M} h(Y_k)\frac{f(Y_k)}{g(Y_k)} \quad a.s.$$

PRACTITIONER'S CORNER

   ▷ *Practical requirements (to undertake the simulation).* To proceed it is necessary to simulate independent copies of Y and to compute the ratio of density functions f/g at a reasonable cost (note that only the ratio is needed which makes useless the computation of some "structural" constants like $(2\pi)^{d/2}$ when both f and g are Gaussian densities, *e.g.* with different means, see below). By "reasonable cost" for the simulation of Y, we mean *at the same cost as* that of X (in terms of complexity). As concerns the ratio f/g this means that its computation remains negligible with respect to that of h.

   ▷ *Sufficient conditions (to undertake the simulation).* Once the above conditions are fulfilled, the question is: is it profitable to proceed like that? So is the case if the complexity of the simulation for a given accuracy (in terms of confidence interval) is lower with the second method.

If one assumes as above that,simulating X and Y on the one hand, and computing $h(x)$ and $(hf/g)(x)$ on the other hand are both comparable in terms of complexity, *the question amounts to comparing the variances* or equivalently the squared quadratic norm of the estimators since they have the same expectation $\mathbb{E}\,h(X)$.

Now

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{h(Y)f(Y)}{g(Y)}\right)^2\right] &= \mathbb{E}\left[\left(\frac{hf}{g}(Y)\right)^2\right] \\
&= \int_E \left(\frac{h(x)f(x)}{g(x)}\right)^2 g(x)\mu(dx))^2 \\
&= \int_E h(x)^2 \frac{f(x)}{g(x)} f(x)\mu(dx) \\
&= \mathbb{E}\left(h(X)^2 \frac{f}{g}(X)\right).
\end{aligned}
$$

As a consequence simulating $\dfrac{hf}{g}(Y)$ rather than $h(X)$ will reduce the variance if and only if

$$
\mathbb{E}\left(h(X)^2 \frac{f}{g}(X)\right) < \mathbb{E}\left(h(X)^2\right). \tag{3.8}
$$

Remark. In fact, theoretically, as soon as h is non-negative and $\mathbb{E}\,h(X) \neq 0$, one may reduce the variance of the new simulation to... 0. As a matter of fact, using Schwarz Inequality one gets, as if trying to "reprove" that $\mathrm{Var}\left(\frac{h(Y)f(Y)}{g(Y)}\right) \geq 0$,

$$
\begin{aligned}
(\mathbb{E}\,h(X))^2 &= \left(\int_E h(x)f(x)\mu(dx)\right)^2 \\
&= \left(\int_E \frac{h(x)f(x)}{\sqrt{g(x)}} \sqrt{g(x)}\mu(dx)\right)^2 \\
&\leq \int_E \frac{(h(x)f(x))^2}{g(x)}\mu(dx) \times \int_E g\,d\mu \\
&= \int_E \frac{(h(x)f(x))^2}{g(x)}\mu(dx)
\end{aligned}
$$

since g is a probability density function. Now the equality case in Schwarz inequality says that the variance is 0 if and only if $\sqrt{g(x)}$ and $\frac{h(x)f(x)}{\sqrt{g(x)}}$ are proportional $\mu(dx)$-a.s. *i.e.* $h(x)f(x) = c\,g(x)$ $\mu(dx)$-a.s. for a (non-negative) real constant c. Finally this leads, *when h has a constant sign and $\mathbb{E}\,h(X) \neq 0$* to

$$
g(x) = f(x)\frac{h(x)}{\mathbb{E}\,h(X)}\,\mu(dx) \qquad \text{a.s.}
$$

This choice is clearly impossible to make since this would mean that $\mathbb{E}\,h(X)$ is known since it is involved in the formula... and would then be of no use. *A contrario* this may suggest a direction to design the (distribution) of Y.

3.6.2   How to design and implement importance sampling

The intuition that must guide practitioners when designing an importance sampling method is to replace a random variable X by a random variable Y so that $\frac{h}{fg}(Y)$ is in some way often "closer" than $h(X)$ to their common mean. Let us be more specific.

We consider a *Call* on the risky asset $(X_t)_{t\in[0,T]}$ with strike price K and maturity $T > 0$ (with interest rate $r \equiv 0$ for simplicity). If $X_0 = x \ll K$ *i.e.* the option is deep out-of-the money at the origin of time so that most of the scenarii payoff $X_T(\omega)$ will satisfy $X_T(\omega) \leq K$ or equivalently $(X_T(\omega) - K)_+ = 0$. In such a setting, the event $\{(X_T - K)_+ > 0\}$ – the payoff is positive – is a *rare event* so that the number of scenarios that will produce a nonzero value for $(X_T - K)_+$ will be small, inducing a too rough estimate of the quantity of interest $\mathbb{E}\,(X_T - K)_+$.

By contrast, if we switch from $(X_t)_{t\in[0,T]}$ to $(Y_t)_{t\in[0,T]}$ so that

– we can compute the ratio $\frac{f_{X_T}}{g_{Y_T}}(y)$ where f_{X_T} and g_{Y_T} are the probability densities of X_T and Y_T respectively,

– Y_T takes most, or at least i significant part, of its values in $[K, +\infty)$.

Then

$$\mathbb{E}\,(X_T - K)_+ = \mathbb{E}\left((Y_T - K)_+ \frac{f_{X_T}}{g_{Y_T}}(Y_T)\right)$$

and we can reasonably hope that we will simulate more significant scenarios for $(Y_T - K)_+ \frac{f_{X_T}}{g_{Y_T}}(Y_T)$ than for $(X_T - K)_+$. This effect will be measured by the variance reduction.

This interpretation in terms of "rare events" is in fact the core of importance sampling, more than the plain "variance reduction" feature. In particular, this is what a practitioner must have in in mind when searching for a "good" probability distribution g: importance sampling is more a matter of "focusing light where it is needed" than reducing variance.

When dealing with vanilla options in simple models (typically local volatility), one usually works on the state space $E = \mathbb{R}_+$ and importance sampling amounts to a change of variable in one dimensional integrals as emphasized above. However, in a more involved frameworks, one considers the scenarii space as a state space, typically $E = \Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and uses Girsanov's Theorem instead of the usual change of variable with respect to the Lebesgue measure.

3.6.3   Parametric importance sampling

In practice, the starting idea is to introduce a parametric family of random variables $(Y_\theta)_{\theta\in\Theta}$ (often defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ as X) such that,

– for every $\theta \in \Theta$, Y_θ has a probability density $g_\theta > 0$ μ-*a.e.* with respect to a reference measure μ and Y_θ is as easy to simulate as X in terms of complexity.

– The ratio $\frac{f}{g_\theta}$ has a small computational cost where f is the probability density of the distribution of X with respect to μ.

Furthermore we can always assume, by adding a value to Θ if necessary, that for a value $\theta_0 \in \Theta$ of the parameter, $Y_{\theta_0} = X$ (at least in distribution).

The problem becomes a parametric optimization problem, typically solving the minimization problem

$$\min_{\theta \in \Theta} \left\{ \mathbb{E}\left[\left(h(Y_\theta)\frac{f}{g_\theta}(Y_\theta)\right)^2\right] = \mathbb{E}\left(h(X)^2\frac{f}{g_\theta}(X)\right)\right\}.$$

Of course there is no reason why the solution to the above problem should be θ_0 (if so, such a parametric model is inappropriate). At this stage one can follow two strategies:

– Try to solve by numerical means the above minimization problem,

– Use one's intuition to select *a priori* a good (although sub-optimal) $\theta \in \Theta$ by applying the heuristic principle: "focus light where it is needed".

EXAMPLE (Cameron-Martin formula and Importance sampling by mean translation).

This example takes place in a Gaussian framework. We consider (as a starting motivation) a 1-dimensional Black-Scholes model defined by

$$X_T^x = xe^{\mu T + \sigma W_T} = xe^{\mu T + \sigma\sqrt{T}Z}, \qquad Z \overset{d}{=} \mathcal{N}(0;1)$$

with $x > 0$, $\sigma > 0$ and $\mu = r - \frac{\sigma^2}{2}$. Then, the premium of an option with payoff $\varphi : (0, +\infty) \to (0, +\infty)$,

$$\begin{aligned}
e^{-rT}\mathbb{E}\,\varphi(X_T^x) &= \mathbb{E}\,h(Z) \\
&= \int_{\mathbb{R}} h(z)e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}
\end{aligned}$$

where $h(z) = e^{-rT}\varphi\left(xe^{\mu T + \sigma\sqrt{T}z}\right)$, $z \in \mathbb{R}$.

From now on we forget about the financial framework and deal with

$$\mathbb{E}\,h(Z) = \int_{\mathbb{R}} h(z)f(z)dz \quad \text{where} \quad f(z) = \frac{e^{\frac{z^2}{2}}}{\sqrt{2\pi}}$$

and the random variable Z will play the role of X in the above theoretical part. The idea is to introduce the parametric family

$$Y_\theta = Z + \theta, \quad \theta \in \Theta := \mathbb{R}.$$

We consider the Lebesgue measure on the real line λ_1 as a reference measure, so that

$$g_\theta(y) = \frac{e^{-\frac{(y-\theta)^2}{2}}}{\sqrt{2\pi}}, \; y \in \mathbb{R}, \; \theta \in \Theta := \mathbb{R}.$$

Elementary computations show that

$$\frac{f}{g_\theta}(y) = e^{-\theta y + \frac{\theta^2}{2}}, \; y \in \mathbb{R}, \; \theta \in \Theta := \mathbb{R}.$$

Hence, we derive the *Cameron-Marin formula*

$$
\begin{aligned}
\mathbb{E}\, h(Z) &= e^{\frac{\theta^2}{2}} \mathbb{E}\left(h(Y_\theta) e^{-\theta Y_\theta} \right) \\
&= e^{\frac{\theta^2}{2}} \mathbb{E}\left(h(Z+\theta) e^{-\theta(Z+\theta)} \right) \\
&= e^{-\frac{\theta^2}{2}} \mathbb{E}\left(h(Z+\theta) e^{-\theta Z} \right).
\end{aligned}
$$

In fact, a standard change of variable based on the invariance of the Lebesgue measure by translation yields the same result in a much more straightforward way: setting $z = u + \theta$ shows that

$$
\begin{aligned}
\mathbb{E}\, h(Z) &= \int_{\mathbb{R}} h(u+\theta) e^{-\frac{\theta^2}{2} - \theta u - \frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} \\
&= e^{-\frac{\theta^2}{2}} \mathbb{E}\left(e^{-\theta Z} h(Z+\theta) \right) \\
&= e^{\frac{\theta^2}{2}} \mathbb{E}\left(h(Z+\theta) e^{-\theta(Z+\theta)} \right).
\end{aligned}
$$

It is to be noticed again that there is no need to account for the normalization constants to compute the ratio $\dfrac{f}{g} = \dfrac{g_0}{g_\theta}$.

The next step is to choose a "good" θ which significantly reduces the variance *i.e.*, following Condition (3.8) (using the formulation involving with "$Y_\theta = Z + \theta$") such that

$$
\mathbb{E}\left(e^{\frac{\theta^2}{2}} h(Z+\theta) e^{-\theta(Z+\theta)} \right)^2 < \mathbb{E}\, h^2(Z)
$$

i.e.

$$
e^{-\theta^2} \mathbb{E}\left(h^2(Z+\theta) e^{-2\theta Z} \right) < \mathbb{E}\, h^2(Z)
$$

or, equivalently if one uses the formulation of (3.8) based on the original random variable (here Z)

$$
\mathbb{E}\left(h^2(Z) e^{\frac{\theta^2}{2} - \theta Z} \right) < \mathbb{E}\, h^2(Z).
$$

Consequently the variance minimization amounts to the following problem

$$
\min_{\theta \in \mathbb{R}} \left[e^{\frac{\theta^2}{2}} \mathbb{E}\left(h^2(Z) e^{-\theta Z} \right) = e^{-\theta^2} \mathbb{E}\left(h^2(Z+\theta) e^{-2\theta Z} \right) \right].
$$

It is clear that the solution of this optimization problem and the resulting choice of θ choice highly depends on the function h.

– OPTIMIZATION APPROACH: When h is smooth enough, an approach based on large deviation estimates has been proposed by Glasserman and al. (see [57]). We propose a simple recursive/adaptive

approach in Chapter 6 based on Stochastic Approximation methods which does not depend upon the regularity of the unction h (see also [4] for a pioneering work in that direction).

– HEURISTIC SUBOPTIMAL APPROACH: Let us come back temporarily to our pricing problem involving the specified function $h(z) = \left(x \exp\left(\mu T + \sigma\sqrt{T}z\right) - K\right)_+$, $z \in \mathbb{R}$. When $x \ll K$ (deep-out-of-the-money option), most simulations of $h(Z)$ will produce 0 as a result. A first simple idea – if one does not wish to carry out the above optimization... – can be to "re-center the simulation" of X_T^x around K by replacing Z by $Z + \theta$ where θ satisfies

$$\mathbb{E}\left(x \exp\left(\mu T + \sigma\sqrt{T}(Z + \theta)\right)\right) = K$$

which yields, since $\mathbb{E}X_T^x = xe^{rT}$,

$$\theta := -\frac{\log(x/K) + rT}{\sigma\sqrt{T}}. \tag{3.9}$$

Solving the similar, although slightly different equation,

$$\mathbb{E}\left(x \exp\left(\mu T + \sigma\sqrt{T}(Z + \theta)\right)\right) = e^{rT}K$$

would lead to

$$\theta := -\frac{\log(x/K)}{\sigma\sqrt{T}}. \tag{3.10}$$

A third simple, intuitive idea is to search for θ such that

$$\mathbb{P}\left(x \exp\left(\mu T + \sigma\sqrt{T}(Z + \theta)\right) < K\right) = \frac{1}{2}$$

which yields

$$\theta := -\frac{\log(x/K) + \mu T}{\sigma\sqrt{T}}. \tag{3.11}$$

This choice is also the solution to the equation $xe^{\mu T + \sigma\sqrt{T}\theta} = K$, etc.

All these choices are suboptimal but reasonable when $x \ll K$. However, if we need to price a whole portfolio including many options with various strikes, maturities (and underlyings...), the above approach is no longer possible and a data driven optimization method like the one developed in Chapter 6 becomes mandatory.

Other parametric methods can be introduced, especially in non-Gaussian frameworks, like for example the so-called "exponential tilting" (or Esscher transform) for distributions having a Laplace transform on the whole real line (see *e.g.* [102]). Thus, when dealing with the NIG distribution (for Normal Inverse Gaussian) this transform has an impact on the thickness oft the tail of the distribution. Of course there is no *a priori* limit to what can be designed on as specific problem. When dealing with path-dependent options, one usually relies on the Girsanov theorem to modify likewise the drift of the risky asset dynamics (see [102]). Of course all this can be adapted to multi-dimensional models...

▷ **Exercises. 1.** (*a*) Show that under appropriate integrability assumptions on h to be specified, the function

$$\theta \mapsto \mathbb{E}\left(h^2(Z)e^{\frac{\theta^2}{2} - \theta Z}\right)$$

is strictly convex and differentiable on the whole real line with a derivative given by $\theta \mapsto \mathbb{E}\Big(h^2(Z)(\theta-Z)e^{\frac{\theta^2}{2}-\theta Z}\Big)$.

(b) Show that if h is an even function, then this parametric importance sampling procedure by mean translation is useless. Give a necessary and sufficient condition (involving h and Z) that makes it always useful.

2. Set $r = 0$, $\sigma = 0.2$, $X_0 = x = 70$, $T = 1$. One wishes to price a *Call* option with strike price $K = 100$ (*i.e.* deep out-of-the-money). The true Black-Scholes price is 0.248.

Compare the performances of

(i) a "crude" Monte Carlo simulation,

(ii) the above "intuitively guided" heuristic choices for θ.

Assume now that $x = K = 100$. What do you think of the heuristic choice suboptimal choice?

3. Write all what precedes when $Z \stackrel{d}{=} \mathcal{N}(0; I_d)$.

4. *Randomization of an integral.* Let $h \in L^1(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d), \lambda)$.

(a) Show that for any $\mathbb{R}^d$-valued random vector Y having an absolutely continuous distribution $\mathbb{P}_Y = g.\lambda_d$ with $g > 0$ λ_d *a.s.* on $\{h > 0\}$, one has

$$\int_{\mathbb{R}^d} h d\lambda = \mathbb{E}\left(\frac{h}{g}(X)\right)$$

Derive a probabilistic method to compute $\displaystyle\int_{\mathbb{R}^d} h\, d\lambda$.

(b) Propose an importance sampling approach to that problem inspired by the above examples.

3.6.4   Computing the Value-at-Risk by Monte Carlo simulation: first approach.

Let X be a real-valued random variable defined on a probability space, representative of a *loss*. For the sake of simplicity we suppose here that X has a continuous distribution *i.e.* that its distribution function defined for every $x \in \mathbb{R}$ by $F(x) := \mathbb{P}(X \leq x)$ is continuous. For a given confidence level $\alpha \in (0,1)$ (usually closed to 1), the Value-at-Risk at level α (denoted $V@R_\alpha$ or $V@R_{\alpha,X}$ following [49]) is any real number satisfying the equation

$$\mathbb{P}(X \leq V@R_{\alpha,X}) = \alpha \in (0,1). \tag{3.12}$$

Equation (3.12) has at least one solution since F is continuous which may not be unique in general. For convenience one often assumes that the lowest solution of Equation (3.12) is the $V@R_{\alpha,X}$. In fact Value-at-Risk (of X) is not consistent as a measure of risk (as emphasized in [49]), but nowadays it is still widely used to measure financial risk.

One naive way to compute $V@R_{\alpha,X}$ is to estimate the empirical distribution function of a (large enough) Monte Carlo simulation at some points ξ lying in a grid $\Gamma := \{\xi_i, \ i \in I\}$, namely

$$\widehat{F(\xi)}_M := \frac{1}{M}\sum_{k=1}^{M}\mathbf{1}_{\{X_k \leq \xi\}}, \ \xi \in \Gamma,$$

where $(X_k)_{k\geq 1}$ is an i.i.d. sequence of X-distributed random variables. Then one solves the equation $\widehat{F}(\xi)_M = \alpha$ (using an interpolation step of course).

Such an approach based on empirical distribution of X needs to simulate extreme values of X since α is usually close to 1. So implementing a Monte Carlo simulation directly on the above equation is usually a slightly meaningless exercise. Importance sampling becomes the natural way to "re-center" the equation.

Assume *e.g.* that

$$X := \varphi(Z), \qquad Z \stackrel{d}{=} \mathcal{N}(0,1).$$

Then, for every $\xi \in \mathbb{R}$,

$$\mathbb{P}(X \leq \xi) = e^{-\frac{\theta^2}{2}} \mathbb{E}\left(\mathbf{1}_{\{\varphi(Z+\theta)\leq\xi\}}e^{-\theta Z}\right)$$

so that the above Equation (3.12) now reads (θ being fixed)

$$\mathbb{E}\left(\mathbf{1}_{\{\varphi(Z+\theta)\leq V@R_{\alpha,X}\}}e^{-\theta Z}\right) = e^{\frac{\theta^2}{2}}\alpha.$$

It remains to find good θ's. Of course this choice of depends on ξ but in practice it should be fitted to reduce the variance in the neighbourhood of $V@R_{\alpha,X}$. We will see in Chapter 6 devoted that more efficient methods based on Stochastic Approximation can be devised. But they also need variance reduction to be implemented. Furthermore, similar ideas can be used to compute a consistent measure of risk called the Conditional Value-at-Risk (or Averaged Value-at-Risk). All these topics will be investigated in Chapter 6.

Chapter 4

The Quasi-Monte Carlo method

In this chapter we present the so-called *quasi-Monte Carlo* *(QMC)* method which can be seen as a deterministic alternative to the standard Monte Carlo method: the pseudo-random numbers are replaced by deterministic computable sequences of $[0,1]^d$-valued vectors which, once substituted *mutatis mutandis* to pseudo-random numbers in the Monte Carlo may speed up significantly its rate of convergence making it *almost* independent of the structural dimension d of the simulation.

4.1 Motivation and definitions

Computing an expectation $\mathbb{E}\,\varphi(X)$ using a Monte Carlo simulation ultimately amounts to computing either a finite-dimensional integral

$$\int_{[0,1]^d} f(u^1,\dots,u^d)du^1\cdots du^d$$

or an infinite dimensional integral

$$\int_{[0,1]^{(\mathbb{N}^*)}} f(u)\lambda_\infty(du)$$

where $[0,1]^{(\mathbb{N}^*)}$ denotes the set of finite $[0,1]$-valued sequences (or, equivalently, sequences vanishing at a finite range) and $\lambda_\infty = \lambda^{\otimes\mathbb{N}}$ is the Lebesgue measure on $([0,1]^{\mathbb{N}}, \mathcal{B}or([0,1]^{\mathbb{N}}))$. Integrals of the first type show up when X can be simulated by standard methods like inverse distribution function, Box-Müller, etc, so that $X = g(U)$, $U = (U^1,\cdots,U^d) \overset{d}{=} U([0,1]^d)$ whereas the second type is typical of a simulation using an acceptance-rejection method.

As concerns finite dimensional integrals, we saw that if $(U_n)_{n\geq 1}$ denotes an i.i.d. sequence of uniformly distributed random vectors on $[0,1]^d$, then for every function $f \in L^1_{\mathbb{R}}([0,1]^d, \lambda_d)$,

$$\mathbb{P}(d\omega)\text{-}a.s. \qquad \frac{1}{n}\sum_{k=1}^n f(U_k(\omega)) \longrightarrow \mathbb{E}(f(U_1)) = \int_{[0,1]^d} f(u^1,\dots,u^d)du^1\cdots du^d \qquad (4.1)$$

where the subset Ω_f of $\mathbb{P}$-probability 1 one which this convergence does hold depends on the function f. In particular the above *a.s.*-convergence holds for any continuous function on $[0,1]^d$. But in fact, taking advantage of the separability of the space of continuous functions, one shows

that, we will show below that this convergence *simultaneously* holds for all continuous functions on $[0,1]^d$ and even on the larger class of functions of *Riemann integrable functions* on $[0,1]^d$.

First we briefly recall the basic definition of weak convergence of probability measures on metric spaces (see [24] Chapter 1, for a general introduction to weak convergence of measures on metric spaces).

Definition 4.1 (Weak convergence) *Let (S, δ) be a metric space and $\mathcal{S} := \mathcal{B}or_\delta(S)$ its Borel σ-field. Let $(\mu_n)_{n \geq 1}$ be a sequence of probability measures on $(S, \mathcal{S})$ and μ a probability measure on the same space. The sequence $(\mu_n)_{n \geq 1}$ weakly converges to μ (denoted $\mu_n \overset{(S)}{\Longrightarrow} \mu$ throughout this chapter) if, for every function $f \in \mathcal{C}_b(S, \mathbb{R})$,*

$$\int_S f d\mu_n \longrightarrow \int_S f d\mu \quad as \quad n \to +\infty. \tag{4.2}$$

One important result on weak convergence of probability measures that we will use in this chapter is the following. The result below is important for applications.

Proposition 4.1 *(See Theorem 5.1 in [24]) If $\mu_n \overset{(S)}{\Longrightarrow} \mu$, then the above convergence (4.2) holds for every bounded Borel function $f : (S, \mathcal{S}) \to \mathbb{R}$ such that*

$$\mu(Disc(f)) = 0 \quad where \quad Disc(f) = \big\{ x \in S, \ f \ is \ discontinuous \ at \ x \big\} \in \mathcal{S}.$$

Functions f such that $\mu(Disc(f)) = 0$ are called *μ-a.s. continuous functions.*

Theorem 4.1 (Glivenko-Cantelli) *If $(U_n)_{n \geq 1}$ is an i.i.d. sequence of uniformly distributed random variables on the unit hypercube $[0,1]^d$, then*

$$\mathbb{P}(d\omega)\text{-}a.s. \qquad \frac{1}{n} \sum_{k=1}^n \delta_{U_k(\omega)} \overset{([0,1]^d)}{\Longrightarrow} \lambda_{d|[0,1]^d} = U([0,1]^d)$$

i.e.

$$\mathbb{P}(d\omega)a.s. \qquad \forall f \in \mathcal{C}([0,1]^d, \mathbb{R}), \quad \frac{1}{n} \sum_{k=1}^n f(U_k(\omega)) \longrightarrow \int_{[0,1]^d} f(x)\lambda_d(dx). \tag{4.3}$$

Proof. The vector space $\mathcal{C}([0,1]^d, \mathbb{R})$ endowed with the sup-norm $\|f\|_\infty := \sup_{x \in [0,1]^d} |f(x)|$ is separable in the sense that there exists a sequence $(f_m)_{m \geq 1}$ of continuous functions on $[0,1]^d$ which is everywhere dense in $\mathcal{C}([0,1]^d, \mathbb{R})$ for the sup-norm ([1]). Then, a countable union of $\mathbb{P}$-negligible sets remaining $\mathbb{P}$-negligible, one derives from (4.1) the existence of $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and

$$\forall \omega \in \Omega_0, \ \forall m \geq 1, \qquad \frac{1}{n} \sum_{k=1}^n f_m(U_k(\omega)) \longrightarrow \mathbb{E}(f_m(U_1)) = \int_{[0,1]^d} f_m(u)\lambda_d(du) \quad as \ n \to +\infty.$$

---

[1]When $d = 1$, an easy way to construct this sequence is to consider the countable family of piecewise affine functions with monotony breaks at rational points of the unit interval and taking rational values at this break points (and at 0 and 1). The density follows form that of the set $\mathbb{Q}$ of rational numbers. When $d \geq 2$, one proceeds likewise by considering functions which are affine on hyper-rectangles with rational vertices which tile the unit hypercube.

On the other hand, it is straightforward that, for any $f \in \mathcal{C}([0,1]^d, \mathbb{R})$, for every $n \geq 1$ and every $\omega \in \Omega_0$,

$$\left| \frac{1}{n} \sum_{k=1}^{n} f_m(U_k(\omega)) - \frac{1}{n} \sum_{k=1}^{n} f(U_k(\omega)) \right| \leq \|f - f_m\|_\infty \quad \text{and} \quad |\mathbb{E}(f_m(U_1)) - \mathbb{E}(f(U_1))| \leq \|f - f_m\|_\infty$$

As a consequence, for every $m \geq 1$.

$$\limsup_n \left| \frac{1}{n} \sum_{k=1}^{n} f(U_k(\omega)) - \mathbb{E} \, f(U_1) \right| \leq 2\|f - f_m\|_\infty.$$

Now, the fact that the sequence $(f_m)_{m \geq 1}$ is everywhere dense in $\mathcal{C}([0,1]^d, \mathbb{R})$ for the sup-norm exactly means that

$$\liminf_m \|f - f_m\|_\infty = 0.$$

This completes the proof since it shows that, for every $\omega \in \Omega_0$, the expected convergence holds for every continuous function on $[0,1]^d$. $\diamond$

Corollary 4.1 *Owing to Proposition 4.1, one may replace in (4.3) the set of continuous functions on $[0,1]^d$ by that of all bounded Borel λ_d-a.s. continuous functions on $[0,1]^d$*

Remark. In fact, one may even replace the Borel measurability by the "Lebesgue"-measurable namely, for a function $f : [0,1]^d \to \mathbb{R}$, replace the $(\mathcal{B}([0,1]^d), \mathcal{B}or(\mathbb{R}))$-measurability by the $(\mathcal{L}([0,1]^d), \mathcal{B}or(\mathbb{R}))$-measurability where $\mathcal{L}([0,1]^d)$ denotes the completion of the Borel σ-field on $[0,1]^d$ by the λ_d-negligible sets (see [30], Chapter 13). Such functions are known as *Riemann integrable functions* on $[0,1]^d$ (see again [30], Chapter 13).

What precedes suggests that, as long as one wishes to compute some quantities $\mathbb{E} \, f(U)$ for (reasonably) smooth functions f, we only need to have access to a sequence that satisfies the above convergence property for its empirical distribution. Furthermore, we know form the first chapter devoted to simulation that this situation is generic since all distributions can be simulated from a uniformly distributed random variable. This leads to formulate the following definition of a *uniformly distributed* sequence .

Definition 4.2 *A $[0,1]^d$-valued sequence $(\xi_n)_{n \geq 1}$ is uniformly distributed (u.d.) on $[0,1]^d$ if*

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{\xi_k} \overset{([0,1]^d)}{\Longrightarrow} U([0,1]^d) \qquad \text{as } n \to +\infty.$$

We need some characterizations of uniform distribution which can be established more easily on examples. These are provided by the proposition below. To this end we need to introduce further definitions and notations.

Definition 4.3 *(a) We define a partial order, denoted "$\leq$", on $[0,1]^d$ is defined by: for every $x = (x^1, \dots, x^d)$, $y = (y^1, \dots, y^d) \in [0,1]^d$*

$$x \leq y \quad \text{if} \quad x^i \leq y^i, \; 1 \leq i \leq d.$$

(b) *The "box" $[\![x, y]\!]$ is defined for every $x = (x^1, \ldots, x^d)$, $y = (y^1, \ldots, y^d) \in [0, 1]^d$, $x \leq y$, by*

$$[\![x, y]\!] := \{\xi \in [0, 1]^d, \ x \leq \xi \leq y\}.$$

Note that $[\![x, y]\!] \neq \emptyset$ if and only if $x \leq y$ and, if so is the case, $[\![x, y]\!] = \Pi_{i=1}^d [x^i, y^i]$.

Notation. In particular the unit hypercube $[0, 1]^d$ can be denoted $[\![0, \mathbf{1}]\!]$.

Proposition 4.2 (Portemanteau Theorem) *(See among others [86, 27]) Let $(\xi_n)_{n \geq 1}$ be a $[0, 1]^d$-valued sequence. The following assertions are equivalent.*

(i) *$(\xi_n)_{n \geq 1}$ is uniformly distributed on $[0, 1]^d$.*

(ii) *For every $x \in [\![0, 1]\!] = [0, 1]^d$,*

$$\frac{1}{n} \sum_{k=1}^n \mathbf{1}_{[\![0, x]\!]}(\xi_k) \longrightarrow \lambda_d([\![0, x]\!]) = \prod_{i=1}^d x^i \qquad as \quad n \to +\infty.$$

(iii) *("Discrepancy at the origin" or "star discrepancy")*

$$D_n^*(\xi) := \sup_{x \in [0,1]^d} \left| \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{[\![0, x]\!]}(\xi_k) - \prod_{i=1}^d x^i \right| \longrightarrow 0 \qquad as \quad n \to \infty.$$

(iv) *("Extreme discrepancy")*

$$D_n^\infty(\xi) := \sup_{x, y \in [0,1]^d} \left| \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{[\![x, y]\!]}(\xi_k) - \prod_{i=1}^d (y^i - x^i) \right| \longrightarrow 0 \qquad as \quad n \to +\infty.$$

(v) *(Weyl's criterion) For every integer $p \in \mathbb{N}^d \setminus \{0\}$*

$$\frac{1}{n} \sum_{k=1}^n e^{2\tilde{\imath}\pi(p|\xi_k)} \longrightarrow 0 \qquad as \quad n \to +\infty \quad (where \ \tilde{\imath}^2 = -1).$$

(vi) *(Bounded Riemann integrable function) For every bounded λ_d-a.s. continuous Lebesgue-measurable function $f : [0, 1]^d \to \mathbb{R}$*

$$\frac{1}{n} \sum_{k=1}^n f(\xi_k) \longrightarrow \int_{[0,1]^d} f(x)\lambda_d(dx) \quad as \quad n \to +\infty.$$

Remark. The two moduli introduced in items *(iii)* and *(iv)* define the discrepancy at the origin and the extreme discrepancy respectively.

Sketch of proof. The ingredients of the proof come from the theory of weak convergence of probability measures. For more details in the multi-dimensional setting we refer to [24] (Chapter 1 devoted to the general theory of weak convergence of probability measures on a Polish space)

or [86] (old but pleasant book devoted to *u.d.* sequences). We provide some elements of proof in 1-dimension.

The equivalence $(i) \Longleftrightarrow (ii)$ is simply the characterization of weak convergence of probability measures by the convergence of their distributions functions ([2]) since the distribution function F_{μ_n} of $\mu_n = \frac{1}{n} \sum_{1 \le k \le n} \delta_{\xi_k}$ is given by $F_{\mu_n}(x) = \frac{1}{n} \sum_{1 \le k \le n} \mathbf{1}_{\{0 \le \xi_k \le x\}}$.

Owing to second Dini's Lemma, this convergence of non-decreasing (distribution) functions is uniform as soon as it holds pointwise since its pointwise limiting function is $F_{U([0,1])}(x) = x$ is continuous. This remark yields the equivalence $(ii) \Longleftrightarrow (iii)$. Although more technical, the d-dimensional extension is remains elementary and relies on a similar principle.

The equivalence $(iii) \Longleftrightarrow (iv)$ is trivial since $D_n^*(\xi) \le D^\infty(\xi) \le 2D_n^*(\xi)$ in 1-dimension. In d-dimension the equality on the right reads with 2^d instead of 2.

Item (v) is based on the fact that weak convergence of finite measures on $[0,1]^d$ is characterized by that of the sequences of their *Fourier coefficients* (the Fourier coefficients of a finite measure μ on $([0,1]^d, \mathcal{B}or([0,1]^d)$ are defined by $c_p(\mu) := \int_{[0,1]^d} e^{2\tilde{\imath}\pi(p|u)} \mu(du)$, $p \in \mathbb{N}^d$, $\tilde{\imath}2 = -1$). One checks that the Fourier coefficients $\left(c_p(\lambda_{d,[0,1]^d})\right)_{p \in \mathbb{N}^d}$ are simply $c_p(\lambda_{d,0,1]^d}) = 0$ if $p \ne 0$ and 1 if $p = 0$.

Item (vi) follows from (i) and Proposition 4.1 since for every $x \in [\![0,1]\!]$, $f_c(\xi) := \mathbf{1}_{\{\xi \le x\}}$ is continuous outside $\{x\}$ which is clearly Lebesque negligible. Conversely (vi) implies the pointwise convergence of the distribution function F_{μ_n} as defined above toward $F_{U([0,1])}$. $\diamond$

The discrepancy at the origin $D_n^*(\xi)$ plays a central rôle in the theory of uniformly distributed sequences: it is not only provide a criterion for uniform distribution, it also appears in as an upper error modulus for numerical integration when the function f has the appropriate regularity (see Koksma-Hlawka Inequality below).

Remark. One can define the discrepancy $D_n^*(\xi_1, \dots, \xi_n)$ at the origin of a given n-tuple $(\xi_1, \dots, \xi_n) \in ([0,1]^d)^n$ by the expression in (iii) of the above Proposition.

The geometric interpretation of the discrepancy is the following: if $x := (x^1, \dots, x^d) \in [\![0, \mathbf{1}]\!]$, the hyper-volume of $[\![0, x]\!]$ is equal to the product $x^1 \cdots x^d$ and

$$\frac{1}{n} \sum_{k=1}^n \mathbf{1}_{[\![0,x]\!]}(\xi_k) = \frac{\text{card}\{k \in \{1, \dots, n\}, \, \xi_k \in [\![0,x]\!]\}}{n}$$

is but the frequency with which the first n points ξ_k's of the sequence fall into $[\![0, x]\!]$. The discrepancy measures the maximum induced error when x runs over $[\![0,1]\!]$.

The first exercise below provides a first example of uniformly distributed sequence.

$\triangleright$ **Exercises. 1.** *Rotations of the torus* $([0,1]^d, +)$. Let $(\alpha^1, \dots, \alpha^d) \in (\mathbb{R} \backslash \mathbb{Q})^d$ (irrational numbers) such that $(1, \alpha^1, \dots, \alpha^d)$ are linearly independent over $\mathbb{Q}$ ([3]). Let $x = (x^1, \dots, x^d) \in \mathbb{R}^d$. Set for every $n \ge 1$

$$\xi_n := (\{x^i + n\,\alpha^i\})_{1 \le i \le d}.$$

---

[2]The distribution function F_μ of a probability measure μ on $[0,1]$ is defined by $F_\mu(x) = \mu([0,x])$. One shows that a sequence of probability measures μ_n converges toward a probability measure μ if and only if their distribution functions F_{μ_n} and F_μ satisfy $F_{\mu_n}(x)$ converges to $F_\mu(x)$ at every $x \in \mathbb{R}$ such that F_μ is continuous at x (see [24], Chapter 1)

[3]This means that if $\lambda^i \in \mathbb{Q}$, $i = 0, \dots, d$ satisfy $\lambda^0 + \lambda^1 \alpha^1 + \cdots + \lambda^d \alpha^d = 0$ then $\lambda^0 = \lambda^1 = \cdots = \lambda^d = 0$.

where $\{x\}$ denotes the fractional part of a real number x. Show that the sequence $(\xi_n)_{n\geq 1}$ is uniformly distributed on $[0,1]^d$ (and can be recursively generated). [Hint: use the Weyl criterion.]

2.(*a*) Assume $d = 1$. Show that for every n-tuple $(\xi_1, \ldots, \xi_n) \in [0,1]^n$

$$D_n^*(\xi_1, \ldots, \xi_n) = \max_{1 \leq k \leq n} \left(\left| \frac{k-1}{n} - \xi_k^{(n)} \right|, \left| \frac{k}{n} - \xi_k^{(n)} \right| \right)$$

where $(\xi_k^{(n)})_{1 \leq k \leq n}$ is the reordering of the n-tuple $(\xi_1, \ldots, \xi_n)$. [Hint: How does the "càdlàg" function $x \mapsto \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{\xi_k \leq x\}} - x$ reach its supremum?].

(*b*) Deduce that

$$D_n^*(\xi_1, \ldots, \xi_n) = \frac{1}{2n} + \max_{1 \leq k \leq n} \left| \xi_k^{(n)} - \frac{2k-1}{2n} \right|.$$

(*c*) *Minimal discrepancy at the origin.* Show that the n-tuple with the lowest discrepancy (at the origin) is $\left(\frac{2k-1}{2n} \right)_{1 \leq k \leq n}$ (the "mid-point" uniform n-tuple) with discrepancy $\frac{1}{2n}$.

4.2 Application to numerical integration: functions with finite variations

Definition 4.4 *(see [132, 27]) A function $f : [0,1]^d \to \mathbb{R}$ has finite variation in the measure sense if there exists a signed measure ([4]) ν on $([0,1]^d, \mathcal{B}or([0,1]^d))$ such that $\nu(\{0\}) = 0$ and*

$$\forall\, x \in [0,1]^d, \qquad f(x) = f(\mathbf{1}) + \nu(\llbracket 0, \mathbf{1} - x \rrbracket)$$

(or equivalently $f(x) = f(0) - \nu(^c\llbracket 0, \mathbf{1} - x \rrbracket)$). The variation $V(f)$ is defined by

$$V(f) := |\nu|([0,1]^d)$$

where $|\nu|$ is the variation measure of ν.

▷ **Exercises. 1.** Show that f has finite variation in the measure sense if and only if there exists a signed ν measure with $\nu(\{\mathbf{1}\}) = 0$ such that

$$\forall\, x \in [0,1]^d, \qquad f(x) = f(\mathbf{1}) + \nu(\llbracket x, \mathbf{1} \rrbracket) = f(0) - \nu(^c\llbracket x, \mathbf{1} \rrbracket)$$

and that its variation is given by $|\nu|([0,1]^d)$. This could of course be taken as the definition equivalently to the above one.

2. Show that the function f defined on $[0,1]^2$ by

$$f(x^1, x^2) := (x^1 + x^2) \wedge 1, \quad (x^1, x^2) \in [0,1]^2$$

---

[4] A signed measure ν on a space $(X, \mathcal{X})$ is a mapping from $\mathcal{X}$ to $\mathbb{R}$ which satisfies the two axioms of a measure, namely $\nu(\emptyset) = 0$ and if A_n, $n \geq 1$, are pairwise disjoint, then $\nu(\cup_n A_n) = \sum_{n \geq 1} \nu(A_n)$ (the series is commutatively convergent hence absolutely convergent). Such a measure is finite and can be decomposed as $\nu = \nu_1 - \nu_2$ where ν_1, ν_2 are non-negative finite measures supported by disjoint sets *i.e.* there exists $A \in \mathcal{X}$ such that $\nu_1(A^c) = \nu_2(A) = 0$ (see [144]).

has finite variation in the measure sense [Hint: consider the distribution of $(U, 1-U)$, $U \stackrel{d}{=} U([0, 1])$].

For the class of functions with finite variations, the Koksma-Hlawka Inequality provides an error bound for $\frac{1}{n} \sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^d} f(x) dx$ based on the star discrepancy.

Proposition 4.3 *(Koksma-Hlawka Inequality (1943 when $d = 1$)) Let $\xi = (\xi_1, \ldots, \xi_n)$ be an n-tuple of $[0, 1]^d$-valued vectors and let $f : [0, 1]^d \to \mathbb{R}$ be a function with finite variation (in the measure sense). Then*

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^d} f(x) \lambda_d(dx) \right| \leq V(f) D_n^*(\xi).$$

Proof. Set $\widetilde{\mu}_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\xi_k} - \lambda_{d|[0,1]^d}$. It is a signed measure with 0-mass. Then, if f has finite variation with respect to a signed measure ν,

$$
\begin{aligned}
\frac{1}{n} \sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^d} f(x) \lambda_d(dx) &= \int f(x) d\widetilde{\mu}_n(dx) \\
&= f(\mathbf{1}) \widetilde{\mu}_n([0, 1]^d) + \int_{[0,1]^d} \nu([\![0, \mathbf{1} - x]\!]) \widetilde{\mu}_n(dx) \\
&= 0 + \int_{[0,1]^d} \left(\int \mathbf{1}_{\{v \leq \mathbf{1}-x\}} \nu(dv) \right) \widetilde{\mu}_n(dx) \\
&= \int_{[0,1]^d} \widetilde{\mu}_n([\![0, \mathbf{1} - v]\!]) \nu(dv)
\end{aligned}
$$

where we used Fubini's Theorem to interchange the integration order (which is possible since $|\nu| \otimes |\widetilde{\mu}_n|$ is a finite measure). Finally, using the extended triangular inequality for integrals with respect to signed measures,

$$
\begin{aligned}
\left| \frac{1}{n} \sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^d} f(x) \lambda_d(dx) \right| &= \left| \int_{[0,1]^d} \widetilde{\mu}_n([\![0, \mathbf{1} - v]\!]) \nu(dv) \right| \\
&\leq \int_{[0,1]^d} |\widetilde{\mu}_n([\![0, \mathbf{1} - v]\!])| \, |\nu|(dv) \\
&\leq \sup_{v \in [0,1]^d} |\widetilde{\mu}_n([\![0, v]\!])| |\nu|([0, 1]^d) \\
&= D_n^*(\xi) V(f). \qquad \diamond
\end{aligned}
$$

Remarks. • The notion of finite variation in the measure sense has been introduced in [27] and [132]. When $d = 1$, it coincides with the notion of *left continuous functions* with finite variation. When $d \geq 2$, it is a slightly more restrictive notion than "finite variation" in the Hardy and Krause sense (see *e.g.* [86, 117] for a definition of this slightly more general notion). However, it is much easier to handle, in particular to establish the above Koksma-Hlawka Inequality! Furthermore $V(f) \leq V_{H\&K}(f)$. Conversely, one shows that a function with finite variation in the Hardy and

Krause sense is λ_d-a.s. equal to a function $\widetilde{f}$ having finite variations in the measure sense satisfying $V(\tilde{f}) \leq V_{H\&K}(f)$. In one dimension, finite variation in the Hardy & Krause sense exactly coincides with the standard definition of finite variation.

• A classical criterion (see [27, 132]) for finite variation in the measure sense is the following: if $f : [0,1]^d \to \mathbb{R}$ having a cross derivative $\dfrac{\partial^d f}{\partial x^1 \cdots \partial x^d}$ in the distribution sense which is an integrable function *i.e.*

$$\int_{[0,1]^d} \left| \frac{\partial^d f}{\partial x^1 \cdots \partial x^d}(x^1, \ldots, x^d) \right| dx^1 \cdots dx^d < +\infty,$$

then f has finite variation in the measure sense. This class includes the functions f defined by

$$f(x) = f(\mathbf{1}) + \int_{[\![0,\mathbf{1}-x]\!]} \varphi(u^1, \ldots, u^d) du^1 \ldots du^d, \quad \varphi \in L^1([0,1]^d, \lambda_d). \tag{4.4}$$

▷ **Exercises. 1.** Show that the function f on $[0,1]^3$ defined by

$$f(x^1, x^2, x^3) := (x^1 + x^2 + x^3) \wedge 1$$

has not finite variations in the measure sense [Hint: its third derivative in the distribution sense is not a measure] ([5]).

2. (*a*) Show directly that if f satisfies (4.4), then for any n-tuple $(\xi_1, \ldots, \xi_n)$,

$$\left| \frac{1}{n} \sum_{k=1}^n f(\xi_k) - \int_{[0,1]^d} f(x) \lambda_d(dx) \right| \leq \|\varphi\|_{L^1(\lambda_{d|[0,1]^d})} D_n^*(\xi_1, \ldots, \xi_n).$$

(*b*) Show that if φ is also in $L^p([0,1]^d, \lambda_d)$, for an exponent $p \in [1, +\infty]$ with an Hölder conjugate q, then

$$\left| \frac{1}{n} \sum_{k=1}^n f(\xi_k) - \int_{[0,1]^d} f(x) \lambda_d(dx) \right| \leq \|\varphi\|_{L^p(\lambda_{d|[0,1]^d})} D_n^{(q)}(\xi_1, \ldots, \xi_n)$$

where

$$D_n^{(q)}(\xi_1, \ldots, \xi_n) = \left(\int_{[0,1]^d} \left| \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{[\![0,x]\!]}(\xi_k) - \prod_{i=1}^d x^i \right|^q \lambda_d(dx) \right)^{\frac{1}{q}}.$$

This modulus is called the L^q-discrepancy of $(\xi_1, \ldots, \xi_n)$.

3. *Other forms of finite variation and Koksma-Hlawka inequality.* Let $: [0,1]^d \to \mathbb{R}$ defined as by

$$f(x) = \nu([\![0,x]\!]), \qquad x \in [0,1]^d$$

where ν is a signed measure on $[0,1]^d$. Show that

$$\left| \frac{1}{n} \sum_{k=1}^n f(\xi_k) - \int_{[0,1]^d} f(x) \lambda_d(dx) \right| \leq |\nu|([0,1]^d) D_n^\infty(\xi_1, \ldots, \xi_n).$$

---

[5]In fact it has not finite variation in the Hardy & Krause sense either.

4. L^1-*discrepancy in* 1-*dimension.* Let $d = 1$ and $q = 1$. Show that the L^1-discrepancy satisfies

$$D_n^{(1)}(\xi_1, \ldots, \xi_n) = \sum_{k=0}^{n} \int_{\xi_k^{(n)}}^{\xi_{k+1}^{(n)}} \left| \frac{k}{n} - u \right| du$$

where $\xi_0^{(n)} = 0$, $\xi_{n+1}^{(n)} = 1$ and $\xi_1^{(n)} \leq \cdots \leq \xi_n^{(n)}$ is the reordering of $(\xi_1, \ldots, \xi_n)$.

4.3 Sequences with low discrepancy: definition(s) and examples

4.3.1 Back again to Monte Carlo method on $[0,1]^d$

Let $(U_n)_{n \geq 1}$ be an i.i.d. sequence of random vectors uniformly distributed over $[0,1]^d$ defined in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We saw that

$$\mathbb{P}(d\omega)\text{-a.s.}, \quad (U_n(\omega))_{n \geq 1} \text{ is uniformly distributed.}$$

So, it is natural to evaluate its (random) discrepancy $D_n^*\big((U_k(\omega))_{k \geq 1}\big)$ as a measure of its uniform distribution and one may wonder at which rate it goes to zero: to be more precise is there a kind of transfer of the Central Limit Theorem (CLT) and the Law of the Iterated Logarithm (LIL) – which rule the weak and strong rate of convergence in the $SLLN$ – to this discrepancy modulus. The answer is positive since $D_n^*((U_n)_{n \geq 1})$ satisfies both a CLT and a LLI. These results are due to Chung (see *e.g.* [33]).

▷ **Chung's CLT for the star discrepancy.**

$$\sqrt{n}\, D_n^*\big((U_k)_{k \geq 1}\big) \xrightarrow{\mathcal{L}} \sup_{x \in [0,1]^d} |Z_x^d| \quad \text{and} \quad \mathbb{E}\big(D_n^*((U_k)_{k \geq 1})\big) \sim \frac{\mathbb{E}(\sup_{x \in [0,1]^d} |Z_x^d|)}{\sqrt{n}} \quad \text{as } n \to +\infty$$

where $(Z_x^d)_{x \in [0,1]^d}$ denotes the centered Gaussian multi-index process (or "bridged hyper-sheet") with covariance structure given by

$$\forall\, x = (x^1, \ldots, x^d),\ y = (y^1, \ldots, y^d) \in [0,1]^d, \quad \mathrm{Cov}(Z_x^d, Z_y^d) = \prod_{i=1}^{d} x^i \wedge y^i - \Big(\prod_{i=1}^{d} x^i\Big)\Big(\prod_{i=1}^{d} y^i\Big).$$

When $d = 1$, Z^1 is simply the Brownian bridge over the unit interval $[0,1]$ and the distribution of its sup-norm is given, using its tail (or survival) function, by

$$\forall\, z \in \mathbb{R}_+, \qquad \mathbb{P}\big(\sup_{x \in [0,1]} |Z_x| \geq z\big) = 2 \sum_{k \geq 1} (-1)^{k+1} e^{-2k^2 z^2}$$

(see [40] or [24], chapter 2).

▷ **Chung's** *LIL* **for the star discrepancy.**

$$\limsup_n \sqrt{\frac{2n}{\log(\log n)}} D_n^*\big((U_k(\omega))_{k\geq 1}\big) = 1 \qquad \mathbb{P}(d\omega)\text{-}a.s.$$

At this stage, one can provide a preliminary definition of a *sequence with low discrepancy* on $[0,1]^d$ as a $[0,1]^d$-valued sequence $\xi := (\xi_n)_{n\geq 1}$ such that

$$D_n^*(\xi) = o\left(\sqrt{\frac{\log(\log n)}{n}}\right) \qquad \text{as} \qquad n \to +\infty$$

which means that its implementation with a function with finite variation will speed up the convergence rate of numerical integration by the empirical measure with respect to the worst rate of the Monte Carlo simulation.

▷ **Exercise.** Show, using the standard *LIL*, the easy part of Chung's *LIL*, that is

$$\limsup_n \sqrt{\frac{2n}{\log(\log n)}} D_n^*\big((U_k(\omega))_{k\geq 1}\big) \geq 2 \sup_{x \in [0,1]^d} \sqrt{\lambda_d([\![0,x]\!])(1 - \lambda_d([\![0,x]\!]))} = 1 \qquad \mathbb{P}(d\omega)\text{-}a.s.$$

4.3.2 Röth's lower bounds for the discrepancy

Before providing examples of sequences with low discrepancy, let us first give elements about the known lower bounds for the asymptotics of the (star) discrepancy of a uniformly distributed sequence.

The first results are due to Röth ([143]): there exists a universal constant $c_d \in (0, \infty)$ such that, for any $[0,1]^d$-valued n-tuple $(\xi_1, \ldots, \xi_n)$,

$$D_n^*(\xi_1, \ldots, \xi_n) \geq c_d \frac{(\log n)^{\frac{d-1}{2}}}{n}. \tag{4.5}$$

Furthermore, there exists a real constant $\widetilde{c}_d \in (0, \infty)$ such that, *for every sequence* $\xi = (\xi_n)_{n\geq 1}$,

$$D_n^*(\xi) \geq \widetilde{c}_d \frac{(\log n)^{\frac{d}{2}}}{n} \qquad \text{for infinitely many } n. \tag{4.6}$$

This second lower bound can be derived from the first one, using the Hammersley procedure introduced and analyzed in the next section.

On the other hand, we know (see Section 4.3.3 below) that there exists sequences for which

$$\forall n \geq 1, \quad D_n^*(\xi) = C(\xi)\frac{(\log n)^d}{n} \qquad \text{where} \qquad C(\xi) < +\infty.$$

Based on this, one can derive from the Hammersley procedure (see Section 4.3.4 below) the existence of a real constant $C_d \in (0, +\infty)$ such that

$$\forall n \geq 1, \ \exists (\xi_1, \ldots, \xi_n) \in \big([0,1]^d\big)^n, \ D_n^*(\xi_1, \ldots, \xi_n) \leq C_d\frac{(\log n)^{d-1}}{n}.$$

In spite of more than fifty years of investigations, the gap between these asymptotic lower and upper bounds have not been improved: it has still not been proved whether there exists a sequence for which $C(\xi) = 0$ *i.e.* for which the rate $\frac{(\log n)^d}{n}$ is not optimal.

In fact it is widely shared in the QMC community that in the above lower bounds, $\frac{d-1}{2}$ can be replaced by $d-1$ in (4.5) and $\frac{d}{2}$ by d in (4.6) so that the rate $O\left(\frac{(\log n)^d}{n}\right)$ seems to be the lowest possible for a u.d. sequence. When $d = 1$, Schmidt proved that this conjecture is true.

This leads to a more convincing definition for sequence of a *sequence with low discrepancy.*

Definition 4.5 *A $[0,1]^d$-valued sequence $(\xi_n)_{n\geq 1}$ is a sequence with low discrepancy if*

$$D_n^*(\xi) = O\left(\frac{(\log n)^d}{n}\right) \qquad as \qquad nn \to +\infty.$$

For more insight about the other measures of uniform distribution (L^p-discrepancy $D_n^{(p)}(\xi)$, diaphony, etc), we refer *e.g.* to [25].

4.3.3 Examples of sequences

▷ **Van der Corput and Halton sequence.** Let $p_1, \ldots, p_d$ be the first d prime numbers (or simply, d pairwise prime numbers). The d-dimensional Halton sequence is defined, for every $n \geq 1$, by:
$$\xi_n = (\Phi_{p_1}(n), \ldots, \Phi_{p_d}(n))$$
where the so-called "radical inverse functions" Φ_p are defined by

$$\Phi_p(n) = \sum_{k=0}^{r} \frac{a_k}{p^{k+1}} \quad \text{with} \quad n = a_0 + a_1 p + \cdots + a_r p^r, \ a_i \in \{0, \ldots, p-1\}, \ a_r \neq 0,$$

denotes the p-adic expansion of n. Then, for every $n \geq 1$,

$$D_n^*(\xi) \leq \frac{1}{n}\left(1 + \prod_{i=1}^{d}\left((p_i - 1)\left\lfloor\frac{\log(p_i n)}{\log(p_i)}\right\rfloor\right)\right) = O\left(\frac{(\log n)^d}{n}\right) \qquad \text{as} \qquad n \to +\infty. \qquad (4.7)$$

The proof of this bound essentially relies on the the Chinese Remainder Theorem (known as "Théorème chinois" in French). Several improvements of this classical bound have been established: some of numerical interest (see *e.g.* [119]), some more theoretical like those established by Faure (see [46])

$$D_n^*(\xi) \leq \frac{1}{n}\left(d + \prod_{i=1}^{d}\left((p_i - 1)\frac{\log n}{2\log p_i} + \frac{p_i + 2}{2}\right)\right), \quad n \geq 1.$$

Remark. When $d = 1$, the sequence $(\Phi_p(n))_{n\geq 1}$ is called *the p-adic Van der Corput sequence* (and the integer p needs not to be prime).

One easily checks that the first terms of the $VdC(2)$ sequence are as follows

$$\xi_1 = \frac{1}{2}, \ \xi_2 = \frac{1}{4}, \ \xi_3 = \frac{3}{4}, \ \xi_4 = \frac{1}{8}, \ \xi_5 = \frac{5}{8}, \ \xi_6 = \frac{3}{8}, \ \xi_7 = \frac{7}{8} \ldots$$

▷ **Exercise.** Let $\xi = (\xi_n)_{n \geq 1}$ denote the p-adic Van der Corput sequence and let $\xi_1^{(n)} \leq \cdots \leq \xi_n^{(n)}$ be the reordering of the first n terms of ξ.

(a) Show that, for every $k \in \{1, \ldots, n\}$,

$$\xi_k^{(n)} \leq \frac{k}{n+1}.$$

[Hint: Use an induction on q where $n = qp + r$, $0 \leq r \leq p - 1$.]

(b) Derive that, for every $n \geq 1$,

$$\frac{\xi_1 + \cdots + \xi_n}{n} \leq \frac{1}{2}.$$

(c) One considers the p-adic Van der Corput sequence $(\tilde{\xi}_n)_{n \geq 1}$ *starting at 0 i.e.*

$$\tilde{\xi}_1 = 0, \quad \tilde{\xi}_n = \xi_{n-1}, \ n \geq 2$$

where $(\xi_n)_{n \geq 1}$ is the regular p-adic Van der Corput sequence. Show that $\tilde{\xi}_k^{(n+1)} \leq \frac{k-1}{n+1}$, $k = 1, \ldots, n+1$. Deduce that the L^1-discrepancy of the sequence $\tilde{\xi}$ satisfies

$$D_n^{(1)}(\tilde{\xi}) = \frac{1}{2} - \frac{\xi_1 + \cdots + \xi_n}{n}.$$

▷ **The Kakutani sequences.**   This denotes a family of sequences first obtained as a by-product when trying to generate the Halton sequence as the orbit of an ergodic transform (see [90, 92, 120]). This extension is based on the p-adic addition on $[0, 1]$, also known as the Kakutani adding machine, defined on *regular* p-adic expansions of real of $[0, 1]$ as the addition from the left to the right of the regular p-adic expansions with carrying over (the p-adic regular expansion of 1 is conventionally set to $1 = \overline{0.(p-1)(p-1)(p-1)\ldots}^p$). Let $\oplus_p$ denote this addition. Thus, as an example

$$0.12333... \oplus_{10} 0.412777... = 0.535011...$$

In a more formal way if x, $y \colon in[0, 1]$ have $x = \overline{0, x_1 x_2 \cdots x_k \cdots}$ and $y = \overline{0, y_1 y_2 \cdots y_k \cdots}$ as regular p-adic expansions, then

$$(x \oplus_p y)_k = (x_k + y_k)\mathbf{1}_{\{x_{k-1} + y_{k-1} \leq p-1\}} + (1 + x_k + y_k)\mathbf{1}_{\{x_{k-1} + y_{k-1} \geq p\}}.$$

with the convention $x_1 = y_1 = 0$.

Then, for every $y \in [0, 1]$, one defines the associated *p-adic rotation* with angle y by

$$T_{p,y}(x) := x \oplus_p y.$$

Proposition 4.4 *Let $p_1, \ldots, p_d$ denote the first d prime numbers, $y_1, \ldots, y_d \in (0, 1)$, where y_i is a p_i-adic rational number satisfying $y_i \geq 1/p_i$, $i = 1, \ldots, d$ and $x_1, \ldots, x_d \in [0, 1]$. Then the sequence $(\xi)_{n \geq 1}$ defined by*

$$\xi_n := (T_{p_i, y_i}^{n-1}(x_i))_{1 \leq i \leq d}, \quad n \geq 1.$$

has a discrepancy at the origin $D_n^*(\xi)$ satisfying the same upper-bound (4.7) as the Halton sequence, see [92, 120].

Remark. Note that if $y_i = x_i = 1/p_i = \overline{0.1}^p$, $i = 1, \ldots, d$, the sequence ξ is simply the regular Halton sequence.

One asset of this approach is to provide an easy recursive form for the computation of ξ_n since

$$\xi_n = \xi_{n-1} \oplus_p (y_1, \ldots, y_d)$$

where, with a slight abuse of notation, $\oplus_p$ denotes here the componentwise pseudo-addition.

Appropriate choices of the starting vector and the "angle" can significantly reduce the discrepancy at least at a finite range (see remark below).

Practitioner's corner. Heuristic arguments not developed here suggest that a good choice for the "angles" y_i and the starting values x_i is

$$y_i = 1/p_i, \ x_i = 1/5, \ i \neq 3, 4, \ x_i = \frac{2p_i - 1 - \sqrt{(p_i + 2)^2 + 4p_i}}{3}, \ i = 3, 4.$$

This specified Kakutani sequence is much easier to implement than the Sobol' sequences and behaves quite well up to medium dimensions d, say $1 \leq d \leq 20$ (see [132, 27]).

▷ **The Faure sequences.** These sequences were introduced in [47]. Let p be the smallest *prime* integer not lower than d (*i.e.* $p \geq d$). The d-dimensional Faure sequence is defined for every $n \geq 1$, by

$$\xi_n = (\Phi_p(n-1), C_p(\Phi_p(n-1)), \cdots, C_p^{d-1}(\Phi_p(n-1)))$$

where Φ_p still denotes the p-adic radical inverse function and, for every p-adic rational number u with (regular) p-adic expansion $u = \sum_{k \geq 0} u_k p^{-(k+1)} \in [0, 1]$

$$C_p(u) = \sum_{k \geq 0} \underbrace{\left(\sum_{j \geq k} \binom{j}{k} u_j \bmod p \right)}_{\in \{0, \ldots, p-1\}} p^{-(k+1)}.$$

These sequences their discrepancy at the origin satisfies (see [47])

$$D_n^*(\xi) \leq \frac{1}{n} \left(\frac{1}{d!} \left(\frac{p-1}{2 \log p} \right)^d (\log n)^d + O((\log n)^{d-1}) \right). \tag{4.8}$$

It has been shown later on in [117] that they share the so-called "$P_{p,d}$" property

Proposition 4.5 *Let $m \in \mathbb{N}$ and let $\ell \in \mathbb{N}^*$. For any $r_1, \ldots, r_d \in \mathbb{N}$ such that $r_1 + \cdots + r_d = m$ and every $x_1, \ldots, x_d \in \mathbb{N}$, there is exactly one term in the sequence $\xi_{\ell p^m + i}$, $i = 0, \ldots, p^m - 1$ lying in the hyper-cube $\prod_{k=1}^{d} [x_k p^{-r_k}, (x_k + 1)p^{-r_k})$.*

The prominent feature of Faure's estimate (4.8) is that the coefficient of the leading error term (in the $\log n$-scale) satisfies

$$\lim_{d} \frac{1}{d!} \left(\frac{p-1}{2 \log p} \right)^d = 0$$

which seems to suggest that the rate is asymptotically better than $O\left(\frac{(\log n)^d}{n}\right)$.

A non-asymptotic upper-bound is provided in [132] (due to Y.J. Xiao in his PHD thesis [157]): for every $n \geq 1$,

$$D_n^*(\xi) \leq \frac{1}{n} \left(\frac{1}{d!} \left(\frac{p-1}{2} \right)^d \left(\left\lfloor \frac{\log(2n)}{\log p} \right\rfloor + d + 1 \right)^d \right).$$

Note that this bound has the same coefficient in its leading term than the asymptotic error bound obtained by Faure although, from a numerical point of view, it becomes efficient only for very large n: thus if $d = p = 5$ and $n = 1\,000$,

$$D_n^*(\xi) \leq \frac{1}{n} \left(\frac{1}{d!} \left(\frac{p-1}{2} \right)^d \left(\left\lceil \frac{\log(2n)}{\log p} \right\rceil + d + 1 \right)^d \right) \approx 1.18$$

which is of little interest if one keeps in mind that, by construction, the discrepancy takes its values in $[0, 1]$. This can be explained by the form of the "constant" term (in the $\log n$-scale) since

$$\lim_{dn \to +\infty} \frac{(d+1)^d}{d!} \left(\frac{p-1}{2} \right)^d = +\infty$$

A better bound is provided in Xiao's PHD thesis, provided $n \geq p^{d+2}/2$ (of less interest for applications when d increases since $p^{d+2}/2 \geq d^{d+2}/2$). Furthermore, it is still suffering from the same drawback mentioned above.

▷ **The Sobol' sequence.**   Sobol's sequence, although pioneering and striking contribution to sequences with low discrepancy, appears now as a specified sequence in the family of Niederreiter's sequences defined below. The usual implemented sequence is a variant due to Antonov and Saleev (see [3]). For recent developments on that topic, see *e.g.* [155].

▷ **The Niederreiter sequences.**   These sequences were designed as generalizations of Faure and Sobol' sequences (see [117]).

Let $q \geq d$ be the smallest primary integer not lower than d (a primary integer reads $q = p^r$ with p prime). The $(0, d)$-Niederreiter sequence is defined for every integer $n \geq 1$, by

$$\xi_n = (\Psi_{q,1}(n-1), \Psi_{q,2}(n-1), \cdots, \Psi_{q,d}(n-1))$$

where

$$\Psi_{q,i}(n) := \sum_j \psi^{-1} \left(\sum_k C^{(i)}_{(j,k)} \Psi(a_k) \right) q^{-j}$$

and $\Psi : \{0, \ldots, q-1\} \to \mathbb{F}_q$ is a one-to-one correspondence between $\{0, \ldots, q-1\}$ and the finite field $\mathbb{F}_q$ with cardinal q satisfying $\Psi(0) = 0$ and

$$C^{(i)}_{(j,k)} = \binom{k}{j-1} \Psi(i-1).$$

These sequences coincide with the Faure sequence when q is the lowest prime number not lower than d. When $q = 2^r$, with $2^{r-1} < d \leq 2^r$, the sequence coincides with the Sobol' sequence (in its original form).

The sequences of this family all share the $P_{p,d}$ property and have consequently discrepancy satisfying an upper bound with a structure similar to that of the Faure sequence.

4.3.4 The Hammersley procedure

The Hammersley procedure is a canonical method to design a $[0,1]^d$-valued n-tuple from a $[0,1]^{d-1}$ one with a discrepancy ruled by the $(d-1)$-dimensional one.

Proposition 4.6 *Let $d \geq 2$. Let $(\zeta_1, \dots, \zeta_n)$ be a $[0,1]^{d-1}$-valued n-tuple. Then the $[0,1]^d$-valued n-tuple defined by*

$$(\xi_k)_{1 \leq k \leq n} = \left(\zeta_k, \frac{k}{n} \right)_{1 \leq k \leq n}$$

satisfies

$$\frac{\max_{1 \leq k \leq n} k D_k^*(\zeta_1, \dots, \zeta_k)}{n} \leq D_n^* \left((\xi_k)_{1 \leq k \leq n} \right) \leq \frac{1 + \max_{1 \leq k \leq n} k D_k^*(\zeta_1, \dots, \zeta_k)}{n}$$

Proof. It follows from the very definition of the discrepancy at the origin

$$
\begin{aligned}
D_n^*((\xi_k)_{1 \leq k \leq n}) &= \sup_{(x,y) \in [0,1]^{d-1} \times [0,1]} \left| \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{\zeta_k \in [\![0,x]\!], \frac{k}{n} \leq y\}} - \left(\prod_{k=1}^{d-1} x^i \right) y \right| \\
&= \sup_{x \in [0,1]^{d-1}} \sup_{y \in [0,1]} |\cdots| \\
&= \max_{1 \leq k \leq n} \left(\sup_{x \in [0,1]^{d-1}} \left| \frac{1}{n} \sum_{\ell=1}^k \mathbf{1}_{\{\zeta_\ell \in [\![0,x]\!]\}} - \frac{k}{n} \prod_{i=1}^{d-1} x^i \right| \vee \sup_{x \in [0,1]^{d-1}} \left| \frac{1}{n} \sum_{\ell=1}^{k-1} \mathbf{1}_{\{\zeta_\ell \in [\![0,x]\!]\}} - \frac{k}{n} \prod_{i=1}^{d-1} x^i \right| \right)
\end{aligned}
$$

since a function $y \mapsto \frac{1}{n} \sum_{k=1}^n a_k \mathbf{1}_{\{\frac{k}{n} \leq y\}} - b\,y$ $(a_k, b \geq 0)$ reaches its supremum at $y = \frac{k}{n}$ or at its left limit "$\frac{k}{n}-$" for an index $k \in \{1, \dots, n\}$. Consequently

$$
\begin{aligned}
D_n^*((\xi_k)_{1 \leq k \leq n}) &= \frac{1}{n} \max_{1 \leq k \leq n} \left(k\, D_k^*(\zeta_1, \dots, \zeta_k) \vee (k-1) \sup_{x \in [0,1]^{d-1}} \left| \frac{1}{k-1} \sum_{\ell=1}^{k-1} \mathbf{1}_{\{\zeta_\ell \in [\![0,x]\!]\}} - \frac{k}{k-1} \prod_{i=1}^{d-1} x^i \right| \right) \quad (4.9) \\
&\leq \frac{1}{n} \max_{1 \leq k \leq n} \left(k\, D_k^*(\zeta_1, \dots, \zeta_k) \vee \left((k-1) D_{k-1}^*(\zeta_1, \dots, \zeta_{k-1}) + 1 \right) \right) \\
&\leq \frac{1 + \max_{1 \leq k \leq n} k D_k^*(\zeta_1, \dots, \zeta_k)}{n}.
\end{aligned}
$$

The lower bound is obvious from (4.9). ◇

Corollary 4.2 *Let $d \geq 1$. There exists a real constant $C_d \in (0, \infty)$ such that, for every $n \geq 1$, there exists an n-tuple $(\xi_1^n, \dots, \xi_n^n) \in ([0,1]^d)^n$ satisfying*

$$D_n^*(\xi_1^n, \dots, \xi_n^n) \leq C_d \frac{1 + (\log n)^{d-1}}{n}.$$

Proof. If $d = 1$, a solution is given for a fixed integer $n \geq 1$ by setting $\xi_k = \frac{k}{n}$, $k = 1, \ldots, n$ (or $\xi_k = \frac{2k-1}{2n}$, $k = 1, \ldots, n$, etc). If $d \geq 2$, it suffices to apply for every $n \geq 1$ the Hammersley procedure to any $(d-1)$-dimensional sequence $\zeta = (\zeta_n)_{n \geq 1}$ with low discrepancy in the sense of Definition 4.5. If so, the constant C_d can be taken equal to $2\, c_{d-1}(\zeta)$ where $D_n^*(\zeta) \leq c_{d-1}(\zeta)(1 + (\log n)^{d-1})/n$ (for every $n \geq 1$). $\diamond$

The main drawback of this procedure is that if one starts from a *sequence* with low discrepancy (often defined recursively), one looses the "telescopic" feature of such a sequence. If one wishes, for a given function f defined on $[0,1]^d$, to increase n in order to improve the accuracy of the approximation, all the terms of the sum in the empirical mean have to be re-computed

$\triangleright$ **Exercise.** Derive the theoretical lower bound (4.6) for infinite sequences from the one for (4.5).

4.3.5   Pros and Cons of sequences with low discrepancy

The use of sequences with low discrepancy to compute integrals instead of the Monte Carlo method (using pseudo-random numbers) is known as the *Quasi-Monte Carlo* method (QMC). This terminology extends to so-called *good lattice points* not described here (see [117]).

THE PROS. $\triangleright$ The main attracting feature of sequences with low discrepancy is undoubtedly the Koksma-Hlawka inequality combined with the rate of decay of the discrepancy which suggests that the QMC method is almost dimension free. Unfortunately, in practice, stander bounds for discrepancy do not allow for the sue of this inequality to provide (100%-)"confidence intervals"

$\triangleright$ Furthermore, one can show that for smoother functions on $[0,1]$ the integration rate can be $O(1/n)$ when the sequence ξ can be obtained as the orbit $\xi_n = T^{n-1}(\xi_1)$, $n \geq 1$, of a(n ergodic) transform $T : [0,1]^d \to [0,1]^d$ (^{6}) and f is a *coboundary i.e.* can be written

$$f - \int_{[0,1]^d} f(u)du = g - g \circ T$$

---

^{6}Let $(X, \mathcal{X}, \mu)$ be a probability space. A mapping $T : (X, \mathcal{X}) \to (X, \mathcal{X})$ is ergodic if

$$(i) \quad \mu \circ T^{-1} = \mu \; \mu \text{ is invariant for } T$$

$$(ii) \quad \forall A \in \mathcal{X},\ T^{-1}(A) = A \Longrightarrow \mu(A) = 0 \text{ or } 1.$$

Then the Birkhoff's pointwise ergodic Theorem (see [85]) implies that, for every $f \in L^1(\mu)$

$$\mu(dx)\text{-a.s.} \qquad \frac{1}{n}\sum_{k=1}^{n} f(T^{k-1}(x)) \longrightarrow \int f\, d\mu.$$

The mapping T is *uniquely ergodic* if μ is the only measure satisfying T. If X is a topological space, $\mathcal{X} = \mathcal{B}or(X)$ and T is continuous, then, for any continuous function $f : X \to \mathbb{R}$,

$$\sup_{x \in X} \left| \frac{1}{n}\sum_{k=1}^{n} f(T^{k-1}(x)) \longrightarrow \int_X f d\mu \right|.$$

In particular it shows that any orbit $(T^{n-1}(x))_{n \geq 1}$ is μ-distributed. When $X = [0,1]^d$ and $\mu = \lambda_d$, one retrieves the notion of uniformly distributed sequence. This provides a powerful tool to devise uniformly distributed sequences (like Kakutani sequences).

where g is a bounded Borel function (see *e.g.* [120]). As a matter of fact

$$\frac{1}{n}\sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^d} f(u)du = \frac{g(\xi_1) - g(\xi_{n+1})}{n} = O\Big(\frac{1}{n}\Big).$$

The Kakutani transforms (rotations with respect to $\oplus_p$) and the automorphisms of the torus are typical examples of ergodic transforms, the first ones being uniquely ergodic (keep in mind that the p-adic Vander Corput sequence is an orbit of the transform $T_{p,1/p}$). The fact that the Kakutani transforms are not continuous – or to be precise their representation on $[0,1]^d$ – can be circumvent (see [120]) and one can take advantage of this unique ergodicity to characterize their coboundaries. Easy-to-check criterions based on the rate of decay of trh Fourier coefficients $c_p(f)$, $p = (p^1, \dots, p^d) \in \mathbb{N}$ of a function f as $\|p\| := p^1 \times \cdots \times p^d$ goes to infinity have been provided (see [120, 157, 158]) for the p-adic Van der Corput sequences in 1-dimension and other orbits of the Kakutani transforms. These criterions are based

Extensive numerical tests on problems involving some smooth (periodic) functions on $[0,1]^d$, $d \geq 2$, have been carried out, see *e.g.* [132, 27]. They suggest that this improvement still holds in higher dimension, at least partially. These are for the pros.

THE CONS. ▷ As concerns the cons, the first one is that all the non asymptotic available non asymptotic bounds are very poor from a numerical point of view. We still refer to [132] for some examples which show that these bounds cannot be used to provide any kind of (deterministic) error intervals. By contrast, one must always have in mind that the regular Monte Carlo method *automatically* provides a confidence interval.

▷ The second drawback concerns the family of functions for which the QMC speeds up the convergence through the Koksma-Hlawka Inequality. This family – mainly the functions with some kind of finite variations – somehow becomes smaller and smaller as the dimension d increases since the requested condition becomes more and more stringent. If one is concerned with *the usual regularity of functions like Lipschitz continuous continuity*, the following striking theorem due to Proïnov ([138]) shows that the *curse of dimensionality* comes in the game without possible escape.

Theorem 4.2 (Proïnov) *Assume $\mathbb{R}^d$ is equipped with the ℓ^∞-norm defined by $|(x^1, \dots, x^d)|_\infty := \max_{1 \leq i \leq d} |x^i|$. Let*

$$w(f, \delta) := \sup_{x,y \in [0,1]^d, \, |x-y|_\infty \leq \delta} |f(x) - f(y)|, \quad \delta \in (0,1).$$

denote the uniform continuity modulus of f (with respect to the ℓ^∞-norm)

(a) Let $(\xi_1, \dots, \xi_n) \in ([0,1]^d)^n$. For every continuous function $f : [0,1]^d \to \mathbb{R}$,

$$\left| \int_{[0,1]^d} f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(\xi_k) \right| \leq C_d \, w\big(f, D_n^*(\xi_1, \dots, \xi_n)^{\frac{1}{d}}\big)$$

where $C_d \in (0, \infty)$ is a universal optimal real constant only depending on d.

In particular if f is Lipschitz continuous continuous with coefficient $[f]_{\mathrm{Lip}} := \sup_{x,y \in [0,1]^d} \frac{|f(x) - f(y)|}{|x-y|_\infty}$, then

$$\left| \int_{[0,1]^d} f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(\xi_k) \right| \leq C_d \, [f]_{\mathrm{Lip}} D_n^*(\xi_1, \dots, \xi_n)^{\frac{1}{d}}.$$

(b) If $d = 1$, $C_d = 1$ and if $d \geq 2$, $C_d \in [1, 4]$.

▷ **Exercise.** Show using the Van der Corput sequences starting at 0 (defined in Section 4.3.3, see the exercise in the paragraph devoted to Van der Corput and Halton sequences) and the function $f(x) = x$ on $[0,1]$ that the above Proïnov Inequality cannot be improved for Lipschitz continuous functions even in 1-dimension. [Hint: Reformulate some results of the Exercise in Section 4.3.3.]

▷ A third drawback of QMC for practical numerical integration is that all functions need to be defined on the unit hypercube. One way to get partially rid of that may be to consider integration on some domains $C \subset [0,1]^d$ having a regular boundary in the Jordan sense ([7]). Then a Koksma-Hlawka like inequality holds true :

$$\left| \int_C f(x)\lambda_d(dx) - \frac{1}{n}\sum_{k=1}^n \mathbf{1}_{\{x_k \in C\}} f(\xi_k) \right| \le V(f) D_n^\infty(\xi)^{\frac{1}{d}}$$

where $V(f)$ denotes the variations of f (in the Hardy & Krause or measure sense) and $D_n^\infty(\xi)$ denotes the extreme discrepancy of $(\xi_1, \ldots, \xi_n)$ (see again [117]). The simple fact to integrate over such a set annihilates the low discrepancy effect (at least from a theoretical point of view).

▷ **Exercise.** Prove Proïnov's Theorem when $d = 1$ [Hint: read the next chapter and compare discrepancy and quantization error].

This suggests that the rate of numerical integration in d-dimension by a sequence with low discrepancy of Lipschitz continuous functions is $O\left(\frac{\log n}{n^{\frac{1}{d}}}\right)$ as $n \to \infty$ (and $O\left(\frac{(\log n)^{\frac{d-1}{d}}}{n^{\frac{1}{d}}}\right)$ when considering a n-tuple if one relies on the Hammersley method). This emphasizes that sequences with low discrepancy are not spared by the curse of dimensionality when implemented on functions with standard regularity...

▷ **WARNING** the dimensional trap! A fourth drawback, undoubtedly the most dangerous for beginners, is of course that *a given (one-dimensional) sequence $(\xi_n)_{n \ge 1}$ does not "simulate" independence* as emphasized by the classical exercise below.

▷ **Exercise.** Let $\xi = (\xi_n)_{n \ge 1}$ denote the dyadic Van der Corput sequence. Show that, for every $n \ge 0$

$$\xi_{2n+1} = \xi_{2n} + \frac{1}{2} \qquad \text{and} \qquad \xi_{2n} = \frac{\xi_n}{2}$$

with the convention $\xi_0 = 0$. Deduce that

$$\lim_n \frac{1}{n}\sum_{k=1}^n \xi_{2k}\xi_{2k+1} = \frac{5}{24}.$$

Compare with $\mathbb{E}(UV)$, where U, V are independent with uniform distribution over $[0,1]$. Conclude.

In fact this phenomenon is typical of the price to be paid for "filling up the gaps" faster than random numbers do. This is the reason why *it is absolutely mandatory to use d-dimensional sequences with low discrepancy to perform QMC computations related to a random vector X of the*

---

[7]Namely that for every $\varepsilon > 0$, $\lambda_d(\{u \in [0,1]^d \,|\, \text{dist}(u, \partial C) < \varepsilon\}) \le \kappa_C \varepsilon$.

form $X = \Psi(U_d)$, $U_d \overset{d}{=} U([0,1]^d)$ (d is sometimes called the *structural dimension* of the simulation). The d components of these d-dimensional sequences do simulate independence.

This has important consequences on very standard simulation methods as illustrated below.

APPLICATION TO THE "*QMC* BOX-MÜLLER METHOD": to adapt the Box-Müller method of simulation of a normal distribution $\mathcal{N}(0;1)$ introduced in Corollary 1.3, we proceed as follows: let $\xi = (\xi_n^1, \xi_n^2)_{n \geq 1}$ be a u.d. sequence over $[0,1]^2$ (in practice chosen with low discrepancy). We set for every $n \geq 1$,

$$\zeta_n = (\zeta_n^1, \zeta_n^2) := \left(\sqrt{-\frac{1}{2}\log(\xi_n^1)}\sin\left(2\pi\xi_n^2\right), \sqrt{-\frac{1}{2}\log(\xi_n^1)}\cos\left(2\pi\xi_n^2\right) \right), \; n \geq 1.$$

Then, for every bounded continuous function $f_1 : \mathbb{R} \to \mathbb{R}$,

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f_1(\zeta_n^1) \longrightarrow \mathbb{E}\, f(Z_1), \; Z_1 \overset{d}{=} \mathcal{N}(0;1)$$

and, for every bounded continuous function $f_2 : \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f_1(\zeta_n) \longrightarrow \mathbb{E}\, f(Z), \; Z \overset{d}{=} \mathcal{N}(0;I_2).$$

These continuity assumption on f_1 and f_2 can be relaxed, *e.g.* for f_2, into he function $\widetilde{f}_2$ defined on $]01]^2$ by

$$(\xi^1, \xi^2) \longmapsto \widetilde{f}_2(\xi^1, \xi^2) := f_2\left(\sqrt{-\frac{1}{2}\log(\xi^1)}\sin\left(2\pi\xi^2\right), \sqrt{-\frac{1}{2}\log(\xi^1)}\cos\left(2\pi\xi^2\right) \right)$$

is Riemann integrable. Likewise, the Koksma-Hlawka inequality applies provided $\widetilde{f}_2$ has finite variation. The same holds for f_1 and admits a straightforward extension to functions $f(Z)$, $Z \overset{d}{=} \mathcal{N}(0;I_d)$.

The extension of the multivariate Box-Müller method (**??**) should be performed following the same rule of the structural dimension: it requires to consider a u.d. sequence over $[0,1]^d$.

In particular, we will see further on in Chapter 7 that simulating the Euler scheme with step $\frac{T}{m}$ of a d-dimensional diffusion over $[0,T]$ with an underlying q-dimensional Brownian motion consumes m independent $\mathcal{N}(0;I_q)$-distributed random vectors *i.e.* $m \times q$ independent $\mathcal{N}(0;1)$ random variables. To perform a QMC simulation of a function of this Euler scheme at time T, we consequently should consider a sequence with low discrepancy over $[0,1]^{mq}$. Existing error bounds on sequences with low discrepancy and the sparsity of functions with finite variation make essentially meaningless any use of Koksma-Hlawka's inequality or Proïnov's theorem to produce error bounds. Not to say that in the latter case, the curse of dimensionality will lead to extremely poor theoretical bounds for Lipschitz functions (like $\widetilde{f}_2$ in 2-dimension).

▷ **Exercise.** (*a*) Implement the a QMC-adapted Box-Müller simulation method for $\mathcal{N}(0;I_2)$ (based on the sequence with low discrepancy of your choice) and organize a race *MC vs QMC*

to compute various *calls*, say $Call_{BS}(K,T)$ $(T = 1,\ K \in \{95, 96, \ldots, 104, 105\})$ in a Black-Scholes model (with $r = 2\%$, $\sigma = 30\%$, $x = 100$, $T = 1$). To simulate this underlying Black-Scholes risky asset, first use the closed expression

$$X_t^x = xe^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{T}Z}.$$

(b) Anticipating on Chapter 7, implement the Euler scheme (7.3) of the Black-Scholes dynamics

$$dX_t^x = X_t^x(rdt + \sigma dW_t).$$

Consider steps of the form $\frac{T}{m}$ with $m = 10,\ 20,\ 50,\ 100$. What conclusions can be drawn?

▷ However, in practice, even the statistical independence between the coordinates of a d dimensional sequence with low discrepancy is only true asymptotically: thus, in high dimension, for small values of n the coordinates of the Halton sequence remain highly correlated. As a matter of fact, the ith component of the canonical d-dimensional Halton sequence (*i.e.* designed from the d first prime numbers $p_1, \ldots, p_d$) starts as follows

$$\xi_n^i = \frac{n}{p_i}, \ n \in \{1, \ldots, p_i - 1\}, \ \xi_n = \frac{1}{p_i^2} + \frac{n - p_i}{p_i}, \ n \in \{p_i, \ldots, 2p_i - 1\}$$

so it is clear that the ith *and the* $(i + 1)^{th}$ *components will remain highly correlated* if i is close to d and d is large: $p_{80} = 503$ and $P_{81} = 509,\ldots$

   To overcome this correlation observed for (not so) small values of n, the usual method is to *discard* the first values of a sequence.

   As a conclusion to this section, let us emphasize *graphically* in terms of texture the differences between MC and QMC *i.e.* between (pseudo-)randomly generated points (say $60\,000$) and the same number of terms of the Halton sequence (with $p_1 = 2$ and $p_2 = 3$).

[Figures temporarily not reproduced]

4.4   Randomized QMC

The idea is to introduce some randomness in the QMC method in order to produce a confidence interval or, from a dual viewpoint to use the QMC approach as a variance reducer in the Monte Carlo method.

   Let $(\xi_n)_{n\geq 1}$ be a u.d. sequence over $[0, 1]^d$.

Proposition 4.7 (*a*) *Let* $a := (a^1, \ldots, a^d) \in \mathbb{R}^d$, *the sequence* $(\{a + \xi_n\})_{n\geq 1}$ *is u.d. where* $\{x\}$ *denotes the component-wise fractional part of* $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$ *(defined as* $\{x\} = (\{x^1\}, \ldots, \{x^d\})$*).*
(*b*) *Let* U *be a uniformly distributed random variable on* $[0, 1]^d$. *Then, for every* $a \in \mathbb{R}^d$

$$\{a + U\} \overset{d}{=} U.$$

Proof. (a) This follows from the (static) Weyl criterion: let $p \in \mathbb{N}^d \setminus \{0\}$,

$$
\frac{1}{n}\sum_{k=1}^{n} e^{2\tilde{\imath}\pi(p|\{a+\xi_k\})} = \frac{1}{n}\sum_{k=1}^{n} e^{2\tilde{\imath}\pi(p^1(a^1+\xi_k^1)+\cdots+p^d(a^d+\xi_k^d))}
$$

$$
= e^{2\tilde{\imath}\pi(p|a)}\frac{1}{n}\sum_{k=1}^{n} e^{2\tilde{\imath}\pi(p|\xi_k)}
$$

$$
\longrightarrow \quad 0 \quad \text{as} \quad n \to +\infty.
$$

Hence the random variable $\{a + U\}$ (supported by $[0,1]^d$) has the same Fourier transform as U *i.e.* the same distribution. (b) One easily checks that both random variables have the same characteristic function. $\diamond$

Consequently, if U is a uniformly distributed random variable on $[0,1]^d$ and $f : [0,1]^d \to \mathbb{R}$ is a Riemann integrable function, the random variable

$$
\chi = \chi(f,U) := \frac{1}{n}\sum_{k=1}^{n} f(\{U+\xi_k\})
$$

satisfies

$$
\mathbb{E}\,\chi = \frac{1}{n} \times n\,\mathbb{E}\,f(U) = \mathbb{E}\,f(U)
$$

owing to claim (b). Then, starting from an M-sample $U_1,\ldots,U_M$, one defines the Monte Carlo estimator of size M attached to χ, namely

$$
\widehat{I}(f,\xi)_{n,M} := \frac{1}{M}\sum_{m=1}^{M} \chi(f,U_m) = \frac{1}{nM}\sum_{m=1}^{M}\sum_{k=1}^{n} f(\{U_m+\xi_k\})
$$

This estimator has a complexity approximately equal to $\kappa \times nM$ where κ is the unitary complexity induced by the computation of one value of the function f. It straightforwardly satisfies by the Strong Long of large Numbers

$$
\widehat{I}(f,\xi)_{n,M} \xrightarrow{a.s.} \mathbb{E}\,f(U)
$$

from what precedes and a *CLT*

$$
\sqrt{M}\big(\widehat{I}(f,\xi)_{n,M} - \mathbb{E}\,f(U)\big) \xrightarrow{\mathcal{L}} \mathcal{N}\big(0; \sigma_n^2(f,\xi)\big)
$$

a.s. converges toward $\mathbb{E}\,\chi = \mathbb{E}\,f(U)$ by the above claim (a) and satisfies a *CLT* with variance with

$$
\sigma_n^2(f,\xi) = \mathrm{Var}\left(\frac{1}{n}\sum_{k=1}^{n} f(\{U+\xi_k\})\right).
$$

Hence, the specific rate of convergence of the QMC is irremediably lost. So, this hybrid method should be compared to regular Monte Carlo of size nM through their respective variances. It is clear that we will observe a variance reduction if and only if

$$
\frac{\sigma_n^2(f,\xi)}{M} < \frac{\mathrm{Var}(f(U))}{nM}
$$

i.e.

$$\mathrm{Var}\left(\frac{1}{n}\sum_{k=1}^{n}f(\{U+\xi_k\})\right) \leq \frac{\mathrm{Var}(f(U))}{n}.$$

The only natural upper bound for the left hand side of this inequality is

$$\begin{aligned}
\sigma_n^2(f,\xi) \;&=\; \int_{[0,1]^d}\left(\frac{1}{n}\sum_{k=1}^{n}f(\{u+\xi_k\}) - \int_{[0,1]^d}f\,d\lambda_d\right)^2 du \\
&\leq\; \sup_{u\in[0,1]^d}\left|\frac{1}{n}\sum_{k=1}^{n}f(\{u+\xi_k\}) - \int_{[0,1]^d}f\,d\lambda_d\right|^2 .
\end{aligned}$$

One can show that $f_u : v \mapsto f(\{u+v\})$ has finite variation as soon as f has and $\sup_{u\in[0,1]^d}V(f_u) < +\infty$ (more precise results can be established) then

$$\sigma_n^2(f,\xi) \leq \sup_{u\in[0,1]^d} V(f_u)^2 D_n^*(\xi_1,\ldots,\xi_n)^2$$

so that, if $\xi = (\xi_n)_{n\geq 1}$ is sequence with low discrepancy (Halton, Kakutani, Sobol', etc),

$$\sigma_n^2(f,\xi) \leq C_\xi^2 \frac{(\log n)^{2d}}{n^2}, \; n \geq 1.$$

Consequently, in that case, it is clear that randomized QMC provides a very significant variance reduction (for the same complexity) of a magnitude proportional to $\frac{(\log n)^{2d}}{n}$ (with an impact of magnitude $\frac{(\log n)^d}{\sqrt{n}}$ on the confidence interval). But one must have in mind once again that such functions become dramatically sparse as d increases.

In fact, even better bounds can be obtained for some classes of functions whose Fourier coefficients $c_p(f)$, $p = (p^1,\ldots,p^d) \in \mathbb{N}$ satisfy some decreasing rate assumptions as $\|p\| := p^1 \times \cdots \times p^d$ goes to infinity since in that case, one has $\sigma_n^2(f,\xi) \leq \dfrac{C_{f,\xi}^2}{n^2}$ so that the gain in terms of variance then becomes proportional $\frac{1}{n}$ for such functions (a global budget /complexity being prescribed fro the simulation).

By contrast, if we consider the Lipschitz setting, things go radically differently: assume that $f : [0,1]^d \to \mathbb{R}$ is Lipschitz continuous and isotropically periodic, *i.e.* for every $x \in [0,1]^d$ and every vector $e_i = (\delta_{ij})_{1\leq j\leq d}$, $i = 1,\ldots,d$ of the canonical basis of $\mathbb{R}^d$ (δ_{ij} stands for the Kronecker symbol) $f(x+e_i) = f(x)$ as soon as $x+e_i \in [0,1]^d$, then f can be extended as a Lipschitz continuous function on the whole $\mathbb{R}^d$ with the same Lipschitz coefficient, say $[f]_{\mathrm{Lip}}$. Furthermore, it satisfies $f(x) = f(\{x\})$ for every $x \in \mathbb{R}^d$. Then, it follows from Proïnov's Theorem 4.2 that

$$\begin{aligned}
\sup_{u\in[0,1]^d}\left|\frac{1}{n}\sum_{k=1}^{n}f(\{u+\xi_k\}) - \int_{[0,1]^d}f\,d\lambda_d\right|^2 \;&\leq\; [f]_{\mathrm{Lip}}^2 D_n^*(\xi_1,\ldots,\xi_n)^{\frac{2}{d}} \\
&\leq\; C_d^2[f]_{\mathrm{Lip}}^2 C_\xi^{\frac{2}{d}}\frac{(\log n)^2}{n^{\frac{2}{d}}}.
\end{aligned}$$

(where C_d is Proïnov's constant). This time, still for a prescribed budget, the "gain" factor in terms of variance is proportional to $n^{1-\frac{2}{d}}(\log n)^2$, which is not a gain. . . but a loss as soon as $d \geq 2$!

For more results and details, we refer to the survey [152] on randomized QMC and the references therein.

Finally, randomized QMC is a specific (and not so easy to handle) variance reduction method, not a QMC speeding up method. Which suffers from one drawback shared by all QMC-based simulation methods: the sparsity of the class of functions with finite variations and the difficulty to identify them in practice when $d > 1$.

4.5 QMC in unbounded dimension: the acceptance rejection method

If one looks at the remark "The user's viewpoint" in Section 1.4 devoted to Von Neumann's acceptance-rejection method, it is a simple exercise to check that one can replace pseudo-random numbers by a u.d. sequence in the procedure (almost) *mutatis mutandis* except for some more stringent regularity assumptions.

We adopt the notations of this section and assume that the $\mathbb{R}^d$-valued random vectors X and Y have absolutely continuous distributions with respect to a reference σ-finite measure μ on $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$. Assume that $\mathbb{P}_X$ has a density *proportional* to f, that $\mathbb{P}_Y = g.\mu$ and that f and g satisfy

$$f \leq c\,g \quad \mu\text{-a.e.} \qquad \text{and} \qquad g > 0 \quad \mu\text{-a.e.}$$

where c is a positive real constant.

Furthermore we make the assumption that Y can be simulated at a reasonable cost like in the original rejection-acceptation method *i.e.* that

$$Y = \Psi(U),\ U \sim U([0,1]^r)$$

for some $r \in \mathbb{N}^*$ where $\Psi : [0,1]^r \to \mathbb{R}$.

Additional "QMC assumptions":

▷ The first additional assumption in this QMC framework is that we ask Ψ to be a *Riemann integrable* function (*i.e.* Borel, bounded and λ_r-a.s. continuous).

▷ We also assume that the function

$$\mathcal{I} : (u^1, u^2) \mapsto \mathbf{1}_{\{c\,u^1 g(\Psi(u^2)) \leq f(\Psi(u^2))\}} \quad \text{is } \lambda_{r+1}\text{-a.s. continuous on } [0,1]^{r+1} \qquad (4.10)$$

(which also amounts to Riemann integrability since $\mathcal{I}$ is bounded).

▷ Our aim is to compute $\mathbb{E}\,\varphi(X)$, where $\varphi \in \mathcal{L}^1(\mathbb{P}_X)$. Since we will use $\varphi(Y) = \varphi \circ \Psi(Y)$ to perform this integration (see below), we also ask φ to be such that

$$\varphi \circ \Psi \text{ is Riemann integrable.}$$

This is classically holds true if φ is continuous (see *e.g.* [30], Chapter 3).

Let $\xi = (\xi_n^1, \xi_n^2)_{n \geq 1}$ be a $[0,1] \times [0,1]^r$-valued sequence, assumed to be with low discrepancy (or simply uniformly distributed) over $[0,1]^{r+1}$. Hence, $(\xi_n^1)_{n \geq 1}$ and $(\xi_n^2)_{n \geq 1}$ are in particular uniformly distributed over $[0,1]$ and $[0,1]^r$ respectively.

If $(U, V) \sim U([0,1] \times [0,1]^r)$, then $(U, \Psi(V)) \sim (U, Y)$. Consequently, the product of two Riemann integrable functions being Riemann integrable,

$$\frac{\sum_{k=1}^{n} \mathbf{1}_{\{c\,\xi_k^1 g(\Psi(\xi_k^2)) \leq f(\Psi(\xi_k^2))\}} \varphi(\Psi(\xi_k^2))}{\sum_{k=1}^{n} \mathbf{1}_{\{c\,\xi_k^1 g(\Psi(\xi_k^2)) \leq f(\Psi(\xi_k^2))\}}} \xrightarrow{n \to +\infty} \frac{\mathbb{E}(\mathbf{1}_{\{c\,U g(Y) \leq f(Y)\}} \varphi(Y))}{\mathbb{P}(c\,U g(Y) \leq f(Y))}$$

$$= \int_{\mathbb{R}^d} \varphi(x) f(x) dx$$

$$= \mathbb{E}\,\varphi(X) \qquad (4.11)$$

where the last two lines follow from computations carried out in Section 1.4.

The main gap to apply the method in a QMC framework is the *a.s.* continuity assumption (4.10). The following proposition yields an easy and natural criterion.

Proposition 4.8 *If the function $\frac{f}{g} \circ \Psi$ is λ_r-a.s. continuous on $[0,1]^r$, then Assumption (4.10) is satisfied.*

Proof. First we note that

$$\mathrm{Disc}(\mathcal{I}) \subset [0,1] \times \mathrm{Disc}\Big(\frac{f}{g} \circ \Psi\Big) \cup \Big(\Big\{(\xi^1,\xi^2) \in [0,1]^{r+1} \ \text{s.t.} \ c\,\xi^1 = \frac{f}{g} \circ \Psi(\xi^2)\Big\}\Big).$$

where $\mathcal{I}$ denotes the function defined in (4.10). Now, it is clear that

$$\lambda_{r+1}\Big([0,1] \times \mathrm{Disc}\Big(\frac{f}{g} \circ \Psi\Big)\Big) = \lambda_1([0,1]) \times \lambda_r\Big(\mathrm{Disc}(\frac{f}{g} \circ \Psi)\Big) = 1 \times 0 = 0$$

owing to the λ_r-a.s. continuity of $\frac{f}{g} \circ \Psi$. Consequently

$$\lambda_{r+1}(\mathrm{Disc}(\mathcal{I})) = \lambda_{r+1}\Big(\Big\{(\xi^1,\xi^2) \in [0,1]^{r+1} \ \text{s.t.} \ c\,\xi^1 = \frac{f}{g} \circ \Psi(\xi^2)\Big\}\Big).$$

In turn, this subset of $[0,1]^{r+1}$ is negligible for the Lebesgue measure λ_{r+1} since, coming back to the independent random variables U and Y and having in mind that $g(Y) > 0$ $\mathbb{P}$-a.s.,

$$\lambda_{r+1}\Big(\Big\{(\xi^1,\xi^2) \in [0,1]^{r+1} \ \text{s.t.} \ c\,\xi^1 = \frac{f}{g} \circ \Psi(\xi^2)\Big\}\Big) = \mathbb{P}(c\,U g(Y) = f(Y))$$

$$= \mathbb{P}\Big(U = \frac{f}{c\,g}(Y)\Big) = 0$$

where we used (see exercise below) that U and Y are independent by construction and that U has a diffuse distribution (no atom). $\diamond$

Remark. The criterion of the proposition is trivially satisfied when $\frac{f}{g}$ and Ψ are continuous on $\mathbb{R}^d$ and $[0,1]^r$ respectively.

$\triangleright$ **Exercise.** Show that if X and Y are independent and X or Y has no atom then

$$\mathbb{P}(X = Y) = 0.$$

As a conclusion note that in this section we provide *no rate of convergence* for this acceptation-rejection method by quasi-Monte Carlo. In fact there is no such error bound under realistic assumptions on f, g, φ and Ψ. Only empirical evidence can justify its use in practice.

4.6   Quasi-Stochastic Approximation

It is natural to try replacing regular pseudo-random numbers by quasi-random numbers in others procedures where they are commonly implemented. So is the case of stochastic Approximation which can be seen as the stochastic counterpart of recursive zero search or optimization procedures like Newton-Raphson algorithm, etc. These aspects of QMC will be investigated in Chapter 6.

Chapter 5

Optimal Quantization methods (cubatures) I

Optimal Vector Quantization is a method coming from Signal Processing devised to approximate a continuous signal by a discrete one in an optimal way. Originally developed in the 1950's (see [55]), it was introduced as a quadrature formula for numerical integration in the early 1990's (see [121]), and for conditional expectation approximations in the early 2000's, in order to price multi-asset American style options [12, 13, 11]. In this brief chapter, we focus on the cubature formulas for numerical integration with respect to the distribution of a random vector X taking values in $\mathbb{R}^d$.

In view of applications, we only deal with the canonical Euclidean quadratic optimal quantization in $\mathbb{R}^d$ although general theory of optimal vector quantization can be developed in a much more general framework in finite (general norms on both $\mathbb{R}^d$ and the probability space $(\Omega, \mathcal{A}, \mathbb{P})$) and even infinite dimension (so-called *functional quantization*, see [106, 130]).

We recall that the canonical Euclidean norm on the vector space $\mathbb{R}^d$ is denoted $|\,.\,|$.

5.1   Theoretical background on vector quantization

Let X be an $\mathbb{R}^d$-valued random vector defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The purpose of vector quantization is to study the best approximation of X by random vectors taking at most N fixed values $x^1, \ldots, x^N \in \mathbb{R}^d$.

Definition 5.1.1 *Let* $x = (x^1, \ldots, x^N) \in (\mathbb{R}^d)^N$. *A Borel partition* $(C_i(x))_{i=1,\ldots,N}$ *of* $\mathbb{R}^d$ *is a* Voronoi partition *of the N-quantizer x (or* codebook*; the term* grid *being used for the set of values* $\{x^1, \ldots, x^N\}$*) if, for every* $i \in \{1, \ldots, N\}$,

$$C_i(x) \subset \{\xi \in \mathbb{R}^d, |\xi - x^i| \leq \min_{i \neq j} |\xi - x^j|\}.$$

The Borel sets $C_i(x)$ are called Voronoi cells *of the partition induced by x (note that, if x has not pairwise distinct components, then some of the cells $C_i(x)$ are empty).*

Remark. In the above definition $|\,.\,|$ denotes the canonical Euclidean norm. However (see [66] many results in what follows can be established for any norm on $\mathbb{R}^d$, except *e.g.* some differentiability results on the quadratic distortion function (see Proposition 6.3.1 in Section 6.3.6).

Let $\{x^1, \ldots, x^N\}$ be the set of values of the N-tuple x (whose cardinality is at most N). The nearest neighbour projection $\mathrm{Proj}_x : \mathbb{R}^d \to \{x^1, \ldots, x^N\}$ induced by a Voronoi partition is defined by

$$\mathrm{Proj}_x(\xi) := \sum_{i=1}^{N} x_i \mathbf{1}_{C_i(x)}, \ \ \xi \in \mathbb{R}^d.$$

Then, we define an *x-quantization* of X by

$$\hat{X}^x = \mathrm{Proj}_x(X) = \sum_{i=1}^{N} x_i \mathbf{1}_{\{X \in C_i(x)\}}. \tag{5.1}$$

The *pointwise error* induced when replacing X by $\widehat{X}^x$ is given by

$$|X - \widehat{X}^x| = \mathrm{dist}(X, \{x^1, \ldots, x^N\}) = \min_{1 \le i \le N} |X - x^i|.$$

When X has a strongly continuous distribution , *i.e.* $\mathbb{P}(X \in H) = 0$ for any hyperplane H of $\mathbb{R}^d$, any two x-quantizations are $\mathbb{P}$-*a.s.* equal.

Definition 5.1.2 *The* mean quadratic quantization error *induced by the the N-tuple $x \in (\mathbb{R}^d)^N$ is defined as the quadratic norm of the pointwise error i.e.*

$$\|X - \hat{X}^x\|_2 = \left(\mathbb{E} \min_{1 \le i \le N} |X - x^i|^2 \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^d} \min_{1 \le i \le N} |\xi - x^i|^2 \mathbb{P}_X(d\xi) \right)^{\frac{1}{2}}.$$

We briefly recall some classical facts about theoretical and numerical aspects of Optimal Quantization. For further details, we refer *e.g.* to [66, 128, 129, 123, 130].

Theorem 5.1.1   *[66, 121] Let $X \in L^2_{\mathbb{R}^d}(\mathbb{P})$. The* quadratic distortion function *at level N (defined squared mean quadratic quantization error on $(\mathbb{R}^d)^N$)*

$$x = (x^1, \ldots, x^N) \longmapsto \mathbb{E}\left(\min_{1 \le i \le N} |X - x^i|^2 \right) = \|X - \hat{X}^x\|_2^2 \tag{5.2}$$

reaches a minimum at least at one quantizer $x^* \in (\mathbb{R}^d)^N$.

Furthermore, if the distribution $\mathbb{P}_X$ has an infinite support then $x^{*,(N)} = (x^{*,1}, \ldots, x^{*,N})$ has pairwise distinct components and the sequence $N \mapsto \min_{x \in (\mathbb{R}^d)^N} \|X - \hat{X}^x\|_2$ is decreasing to 0 as $N \to +\infty$.

<p style="text-align:center; color:red;">Rajouter la preuve</p>

This leads naturally to the following definition.

Definition 5.1.3 *Any N-tuple solution to the above distortion minimization problem is called an* optimal N-quantizer *or an optimal quantizer at level N.*

Remark. • When $N = 1$, the N-optimal quantizer is always unique, equal to $\mathbb{E}\,X$.

• Optimal quantizers are never unique, at least because the distortion function is symmetric so that any permutation of an optimal quantizer is still optimal. The question of the geometric uniqueness is much more involved. However, in 1-dimension, when the distribution of X is unimodal *i.e.* has a log-concave density, then there is a unique optimal quantizer (up to a permutation).

Proposition 5.1.1 *[121, 123] Any L^2-optimal N-quantizer $x \in (\mathbb{R}^d)^N$ is stationary in the following sense: for every Voronoi quantization $\widehat{X}^{x^*}$ of X,*

$$\mathbb{E}(X|\widehat{X}^x) = \widehat{X}^x. \tag{5.3}$$

Proof. Let x^* be an optimal quadratic quantizer and $\widehat{X}^{x^*}$ an optimal quantization of X given by (5.1) where $(C_i(x^*))_{1 \le i \le N}$ is a Voronoi partition induced by x^*. Then, for every $y \in (\mathbb{R}^d)^N$, $\|X - \widehat{X}^{x^*}\|_2 \le \|X - \widehat{X}^y\|_2$. Now we consider the random variable $\mathbb{E}(X \mid \widehat{X}^{x^*})$. Being measurable with respect to the σ-field $\mathcal{A}^* = \sigma(\{X \in C_i(x^*), i = 1, \ldots, N\})$ spanned by $\widehat{X}^{x^*}$, it reads

$$E(X \mid \widehat{X}^{x^*}) = \sum_{i=1}^N y_i \mathbf{1}_{\{X \in C_i(x)\}}$$

where $y_1, \ldots, y_N \in \mathbb{R}^d$. Let $y = (y_1, \ldots, y_N)$ (with possibly repeated components). Then, by the very definition of a nearest neighbour projection (on $\{y_1, \ldots, y_N\}$)

$$\left\| X - \mathbb{E}(X \mid \widehat{X}^{x^*}) \right\|_2 \ge \left\| X - \widehat{X}^y \right\|_2 \ge \left\| X - \widehat{X}^{x^*} \right\|_2.$$

On the other hand, by the very definition of conditional expectation as an orthogonal projection on $\mathcal{A}^*$ and the fact that $\widehat{X}^{x^*}$ obviously lies in $L^2_{\mathbb{R}^d}(\Omega, \mathcal{A}^*, \mathbb{P})$,

$$\left\| X - \mathbb{E}(X \mid \widehat{X}^{x^*})| \right\|_2 \le \left\| X - \widehat{X}^{x^*} \right\|_2.$$

Combining these two strings of inequalities, yields

$$\left\| X - \widehat{X}^{x^*} \right\|_2 \left\| X - \mathbb{E}(X \mid \widehat{X}^{x^*}) \right\|_2 = \min \left\{ \left\| X - Y \right\|_2, \ Y \ \mathcal{A}^*\text{-measurable} \right\}$$

Uniqueness of conditional expectation given $\widehat{X}^{x^*}$ finally implies $\widehat{X}^{x^*} = \mathbb{E}(X \mid \widehat{X}^{x^*})$ $\mathbb{P}$-*a.s.* $\diamond$

Remark. • It is shown in [66] (Theorem 4.2, p.38) an additional property of optimal quantizers: the boundaries of any of their Voronoi partition are $\mathbb{P}_X$-negligible (even if $\mathbb{P}_X$ do have atoms).

• Let $x \in (\mathbb{R}^d)^N$ be an N-tuple with pairwise distinct components and its Voronoi partitions (all) have a $\mathbb{P}$-negligible boundary *i.e.* $\mathbb{P}\left(\bigcup_{i=1}^N \partial C_i(x) \right) = 0$. Then the Voronoi quantization $\widehat{X}^x$ of X is $\mathbb{P}$-*a.s.* unique and, if x is a local minimum of the quadratic distortion function, then x is a stationary quantizer, still in the sense that

$$\mathbb{E}\big(X \mid \widehat{X}^x\big) = \widehat{X}^x.$$

This is an easy consequence of the differentiability result of the quadratic distortion established further on in Proposition 6.3.1 in Section 6.3.6.

Figure 5.1 shows a quadratic optimal – or at least close to optimality – quantization grid for a bivariate normal distribution $\mathcal{N}(0, I_2)$. The rate of convergence to 0 of the optimal quantization error is ruled by the so-called Zador Theorem.

Figure 5.2 illustrates on the bi-variate normal distribution the intuitive fact that optimal quantization does not produce quantizers whose Voronoi cells all have the same "weights" . In fact, optimizing a quantizer tends to equalize the local inertia of each cell *i.e.* $\mathbb{E}\big(\mathbf{1}_{\{X \in C_i(x)\}} |X - x_i|^2\big)$, $i = 1, \ldots, N$.
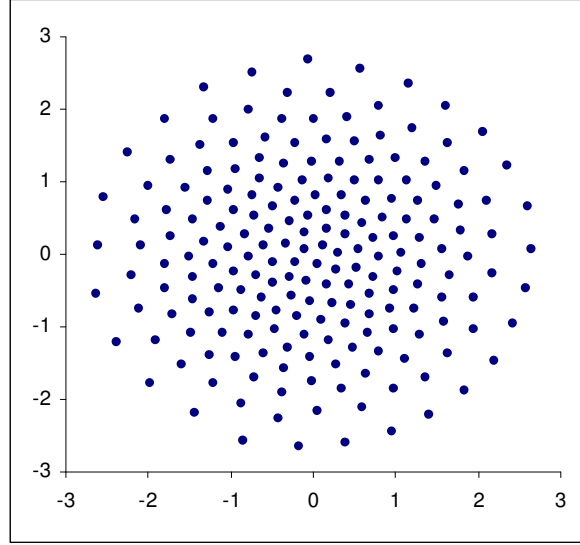
Figure 5.1: *Optimal quadratic quantization of size $N = 200$ of the bi-variate normal distribution $\mathcal{N}(0, I_2)$.*
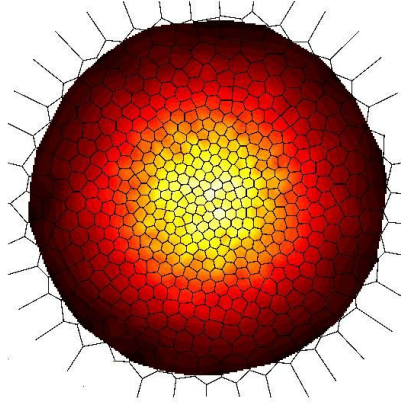
Figure 5.2: *Voronoi Tessellation of an optimal N-quantizer ($N = 500$). Color code: the heaviest the cell is for the normal distribution, the lightest the cell looks.*

Theorem 5.1.2 (Zador's Theorem) *Let $d \geq 1$. (a)* SHARP RATE *(see [66]). Let $X \in L^{2+\delta}_{\mathbb{R}^d}(\mathbb{P})$ for some $\delta > 0$. Assume $\mathbb{P}_X(d\xi) = \varphi(\xi)\lambda_d(d\xi) + \nu(d\xi)$, $\nu \perp \lambda_d$ (λ_d Lebesgue measure on $\mathbb{R}^d$). Then, there is a constant $\widetilde{J}_{2,d} \in (0, \infty)$, such that*

$$\lim_{N \to +\infty} N^{\frac{1}{d}} \min_{x \in (\mathbb{R}^d)^N} \|X - \hat{X}^x\|_2 = \widetilde{J}_{2,d} \left(\int_{\mathbb{R}^d} \varphi^{\frac{d}{d+2}} d\lambda_d \right)^{\frac{1}{2} + \frac{1}{d}}.$$

(b) NON ASYMPTOTIC UPPER BOUND *(see [107]). Let $\delta > 0$. There exists a real constant $C_{d,\delta} \in$
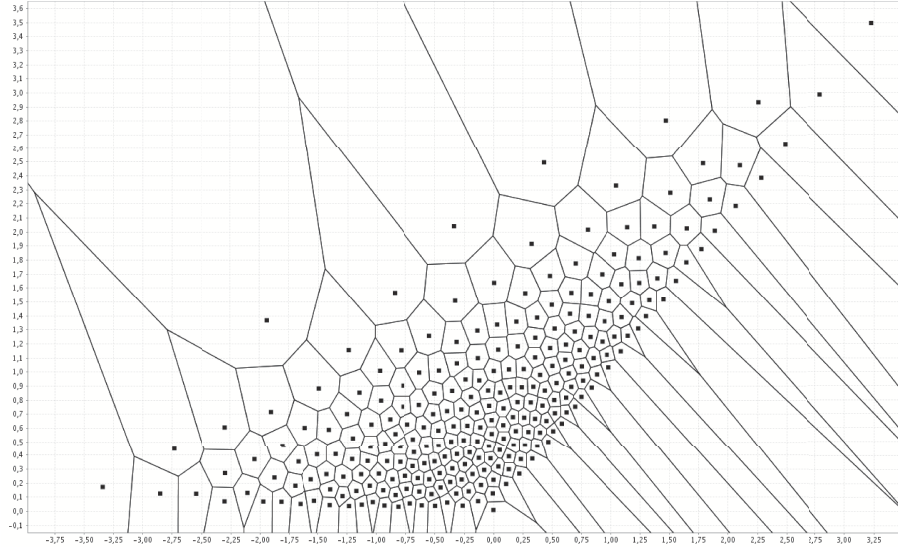
Figure 5.3: *Optimal N-quantization ($N = 500$) of $\left(W_1, \sup_{t\in[0,1]} W_t\right)$ depicted with its Voronoi tessellation, W standard Brownian motion.*

$(0, \infty)$ *such that, for every $\mathbb{R}^d$-valued random vector X,*

$$\forall\, N \geq 1, \qquad \min_{x\in(\mathbb{R}^d)^N} \left\| X - \hat{X}^x \right\|_2 \leq C_{d,\delta}\, \sigma_{2+\delta}(X) N^{-\frac{1}{d}}.$$

where, for any $p \in (0, \infty)$, $\sigma_p(X) = \min_{a\in\mathbb{R}^d} \left\| X - a \right\|_p$.

Remarks. • The $N^{\frac{1}{d}}$ factor is known as *the curse of dimensionality*: this is the optimal rate to "fill" a d-dimensional space by 0-dimensional objects.

• The real constant $\widetilde{J}_{2,d}$ clearly corresponds to the case of the uniform distribution $U([0, 1])^d)$ over the unit hypercube $[0, 1]^d$ for which the slightly more precise statement holds

$$\lim_{N} N^{\frac{1}{d}} \min_{x\in(\mathbb{R}^d)^N} \left\| X - \hat{X}^x \right\|_2 = \inf_{N} N^{\frac{1}{d}} \min_{x\in(\mathbb{R}^d)^N} \left\| X - \hat{X}^x \right\|_2) = \widetilde{J}_{2,d}.$$

One key of the proof is a self-similarity argument "à la Halton" which establishes the theorem for the $U([0, 1])^d)$ distributions.

• Zador's Theorem holds true for any general – possibly non Euclidean – norm on $\mathbb{R}^d$ and the value of $\widetilde{J}_{2,d}$ depends on the reference norm on $\mathbb{R}^d$. When $d = 1$, elementary computations show that $\widetilde{J}_{2,1} = \frac{1}{2\sqrt{3}}$. When $d = 2$, with the canonical Euclidean norm, one shows (see [116] for a proof, see also [66]) that $\widetilde{J}_{2,d} = \sqrt{\frac{5}{18\sqrt{3}}}$. Its exact value is unknown for $d \geq 3$ but, still for the canonical Euclidean norm, one has (see [66]) using some random quantization arguments,

$$\widetilde{J}_{2,d} \sim \sqrt{\frac{d}{2\pi e}} \approx \sqrt{\frac{d}{17,08}} \quad \text{as} \quad d \to +\infty.$$

5.2   Cubature formulas

The random vector $\hat{X}^x$ takes its values in a finite space $\{x^1, \ldots, x^N\}$, so for every continuous functional $f : \mathbb{R}^d \to \mathbb{R}$ with $f(X) \in L^2(\mathbb{P})$, we have

$$\mathbb{E}(f(\hat{X}^x)) = \sum_{i=1}^{N} f(x^i)\mathbb{P}(X \in C_i(x))$$

which is the *quantization-based cubature formula* to approximate $\mathbb{E}(f(X))$ [121, 125]. As $\hat{X}^x$ is close to X, it is natural to estimate $\mathbb{E}(f(X))$ by $\mathbb{E}(f(\hat{X}^x))$ when f is continuous. Furthermore, when f is smooth enough, one can provide an upper bound for the resulting error using the quantization error $\|X - \hat{X}^x\|_2$, or its square (when the quantizer x is stationary).

The same idea can be used to approximate the conditional expectation $\mathbb{E}(f(X)|Y)$ by $\mathbb{E}(f(\hat{X})|\hat{Y})$, but one also needs the transition probabilities:

$$\mathbb{P}(X \in C_j(x) \,|\, Y \in C_i(y)).$$

Numerical computation of $\mathbb{E}(F(\hat{X}^x))$ is possible as soon as $F(\xi)$ can be computed at any $\xi \in \mathbb{R}^d$ and the distribution $(\mathbb{P}(\hat{X} = x_i))_{1 \leq i \leq N}$ of $\hat{X}^x$ is known. The induced quantization error $\|X - \hat{X}^x\|_2$ is used to control the error (see below). These quantities related to the quantizer Γ are also called *companion parameters*.

Likewise, one can consider *a priori* the $\sigma(\hat{X}^x)$-measurable random variable $F(\hat{X}^x)$ as a good approximation of the conditional expectation $\mathbb{E}(F(X)\,|\,\hat{X}^x)$.

5.2.1   Lipschitz continuous functionals

Assume that the functional F is Lipschitz continuous continuous on $\mathbb{R}^d$. Then

$$\left|\mathbb{E}(F(X)\,|\,\hat{X}^x) - F(\hat{X}^x)\right| \leq [F]_{\mathrm{Lip}}\mathbb{E}(|X - \hat{X}^x|\,|\,\hat{X}^x)$$

so that, for every real exponent $r \geq 1$,

$$\|\mathbb{E}(F(X)\,|\,\hat{X}^x) - F(\hat{X}^x)\|_r \leq [F]_{\mathrm{Lip}}\|X - \hat{X}^x\|_r$$

(where we applied conditional Jensen inequality to the convex function $u \mapsto u^r$). In particular, using that $\mathbb{E}\,F(X) = \mathbb{E}(\mathbb{E}(F(X)\,|\,\hat{X}^x))$, one derives (with $r = 1$) that

$$\begin{aligned}\left|\mathbb{E}\,F(X) - \mathbb{E}\,F(\hat{X}^x)\right| &\leq& \|\mathbb{E}(F(X)\,|\,\hat{X}^x) - F(\hat{X}^x)\|_1 \\ &\leq& [F]_{\mathrm{Lip}}\|X - \hat{X}^x\|_1.\end{aligned}$$

Finally, using the monotony of the $L^r(\mathbb{P})$-norms as a function of r yields

$$\left|\mathbb{E}\,F(X) - \mathbb{E}\,F(\hat{X}^x)\right| \leq [F]_{\mathrm{Lip}}\|X - \hat{X}^x\|_1 \leq [F]_{\mathrm{Lip}}\|X - \hat{X}^x\|_2. \tag{5.4}$$

In fact, considering the Lipschitz continuous functional $F(\xi) := d(\xi, x)$, shows that

$$\|X - \hat{X}^x\|_1 = \sup_{[F]_{\mathrm{Lip}} \leq 1} \left|\mathbb{E}\,F(X) - \mathbb{E}\,F(\hat{X}^x)\right|. \tag{5.5}$$

The Lipschitz continuous functionals making up a characterizing family for the weak convergence of probability measures on $\mathbb{R}^d$, one derives that, for any sequence of N-quantizers x^N satisfying $\left\|X - \widehat{X}^{x^N}\right\|_1 \to 0$ as $N \to +\infty$,

$$\sum_{1 \leq i \leq N} \mathbb{P}(\widehat{X}^{x^N} = x_i^N) \, \delta_{x_i^N} \overset{(\mathbb{R}^d)}{\Longrightarrow} \mathbb{P}_X$$

where $\overset{(\mathbb{R}^d)}{\Longrightarrow}$ denotes the weak convergence of probability measures on $\mathbb{R}^d$.

5.2.2   Convex functionals

If F is a convex functional and $\widehat{X}$ is a stationary quantization of X, a straightforward application of Jensen inequality yields

$$\mathbb{E}\left(F(X) \,|\, \widehat{X}\right) \geq F(\widehat{X})$$

so that

$$\mathbb{E}\left(F(\widehat{X})\right) \leq \mathbb{E}\left(F(X)\right).$$

5.2.3   Differentiable functionals with Lipschitz continuous differentials

Assume now that F is differentiable on $\mathbb{R}^d$, with a Lipschitz continuous continuous differential DF, and that the quantizer x is *stationary* (see Equation (5.3)).

A Taylor expansion yields

$$\left|F(X) - F(\widehat{X}^x) - DF(\widehat{X}^x).(X - \widehat{X}^x)\right| \quad \leq \quad [DF]_{\mathrm{Lip}}|X - \widehat{X}^x|^2.$$

Taking conditional expectation given $\widehat{X}^x$ yields

$$\left|\mathbb{E}(F(X)\,|\,\widehat{X}^x) - F(\widehat{X}^x) - \mathbb{E}\left(DF(\widehat{X}^x).(X - \widehat{X}^x)\,|\,\widehat{X}^x\right)\right| \quad \leq \quad [DF]_{\mathrm{Lip}}\mathbb{E}(|X - \widehat{X}^x|^2\,|\,\widehat{X}^x).$$

Now, using that the random variable $DF(\widehat{X}^x)$ is $\sigma(\widehat{X}^x)$-measurable, one has

$$\mathbb{E}\left(DF(\widehat{X}^x).(X - \widehat{X}^x)\right) = \mathbb{E}\left(DF(\widehat{X}^x).\mathbb{E}(X - \widehat{X}^x\,|\,\widehat{X}^x)\right) = 0$$

so that

$$\left|\mathbb{E}(F(X)\,|\,\widehat{X}^x) - F(\widehat{X}^x)\right| \leq [DF]_{\mathrm{Lip}}\mathbb{E}\left(|X - \widehat{X}^x|^2\,|\,\widehat{X}^x\right).$$

Then, for every real exponent $r \geq 1$,

$$\left\|\mathbb{E}(F(X)\,|\,\widehat{X}^x) - F(\widehat{X}^x)\right\|_r \leq [DF]_{\mathrm{Lip}}\|X - \widehat{X}^x\|_{2r}^2. \tag{5.6}$$

In particular, when $r = 1$, one derives like in the former setting

$$\left|\mathbb{E}F(X) - \mathbb{E}F(\widehat{X}^x)\right| \leq [DF]_{\mathrm{Lip}}\|X - \widehat{X}^x\|_2^2. \tag{5.7}$$

In fact, the above inequality holds provided F is $\mathcal{C}^1$ with Lipschitz continuous differential on every Voronoi cell $C_i(x)$. A similar characterization to (5.5) based on these functionals could be established.

Some variant of these cubature formulae can be found in [128] or [67] for functions or functionals F having only some local Lipschitz continuous regularity.

5.2.4　Quantized approximation of $\mathbb{E}(F(X)\,|\,Y)$

Let X and Y be two $\mathbb{R}^d$-valued random vectors defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $F : \mathbb{R}^d \to \mathbb{R}$ be a Borel functional. The natural idea to approximate $\mathbb{E}(F(X)\,|\,Y)$ by quantization is to replace *mutatis mutandis* the random vectors X and Y by their quantizations $\widehat{X}$ and $\widehat{Y}$. The resulting approximation is then

$$\mathbb{E}(F(X)\,|\,Y) \approx \mathbb{E}(F(\widehat{X})\,|\,\widehat{Y}).$$

At this stage a natural question is to look for *a priori* estimates in L^p for the resulting error given the L^p-quantization errors $\|X - \widehat{X}\|_p$ and $\|Y - \widehat{Y}\|_p$.

To this end, we need further assumptions on F. Let $\varphi_F : \mathbb{R}^d \to \mathbb{R}$ be a (Borel) version of the conditional expectation *i.e.* satisfying

$$\mathbb{E}\left(F(X)\,|\,Y\right) = \varphi_F(Y).$$

Usually, no closed form is available for the function φ_F but some regularity property can be established, especially in a (Feller) Markovian framework. Thus assume that both F and φ_F are Lipschitz continuous continuous with Lipschitz continuous coefficients $[F]_{\mathrm{Lip}}$ and $[\varphi_F]_{\mathrm{Lip}}$.

▷ *Quadratic case $p = 2$.* We get

$$
\begin{aligned}
\mathbb{E}\left(F(X)\,|\,Y\right) - \mathbb{E}\left(F(\widehat{X})\,|\,\widehat{Y}\right) &= \mathbb{E}\left(F(X)\,|\,Y\right) - \mathbb{E}\left(F(X)\,|\,\widehat{Y}\right) + \mathbb{E}\left(F(X) - F(\widehat{X})\,|\,\widehat{Y}\right) \\
&= \mathbb{E}\left(F(X)\,|\,Y\right) - \mathbb{E}\left(\mathbb{E}\left(F(X)\,|\,Y\right)\,|\,\widehat{Y}\right) + \mathbb{E}\left(F(X) - F(\widehat{X})\,|\,\widehat{Y}\right)
\end{aligned}
$$

where we used that $\widehat{Y}$ is $\sigma(Y)$-measurable.

Now $\mathbb{E}\left(F(X)\,|\,Y\right) - \mathbb{E}\left(\mathbb{E}\left(F(X)\,|\,Y\right)\,|\,\widehat{Y}\right)$ and $\mathbb{E}\left(F(X) - F(\widehat{X})\,|\,\widehat{Y}\right)$ are clearly orthogonal in $L^2(\mathbb{P})$ so that

$$\left\|\mathbb{E}\left(F(X)\,|\,Y\right) - \mathbb{E}\left(F(\widehat{X})\,|\,\widehat{Y}\right)\right\|_2^2 = \left\|\mathbb{E}\left(F(X)\,|\,Y\right) - \mathbb{E}\left(\mathbb{E}(F(X)\,|\,Y)\,|\,\widehat{Y}\right)\right\|_2^2 + \left\|\mathbb{E}\left(F(X) - F(\widehat{X})\,|\,\widehat{Y}\right)\right\|_2^2.$$

Using this time the very definition of conditional expectation given $\widehat{Y}$ as the best quadratic approximation among $\sigma(\widehat{Y})$-measurable random variables, we get

$$
\begin{aligned}
\left\|\mathbb{E}\left(F(X)\,|\,Y\right) - \mathbb{E}\left(\mathbb{E}\left(F(X)\,|\,Y\right)\,|\,\widehat{Y}\right)\right\|_2 &= \left\|\varphi_F(Y) - \mathbb{E}\left(\varphi_F(Y)|\widehat{Y}\right)\right\|_2 \\
&\leq \left\|\varphi_F(Y) - \varphi_F(\widehat{Y})\right\|_2.
\end{aligned}
$$

On the other hand, still using that $\mathbb{E}(\,.\,|\sigma(\widehat{Y}))$ is an L^2-contraction and this time that F is Lipschitz continuous continuous yields

$$\left\|\mathbb{E}(F(X) - F(\widehat{X})\,|\,\widehat{Y})\right\|_2 \leq \left\|F(X) - F(\widehat{X})\right\|_2 \leq [F]_{\mathrm{Lip}}\left\|X - \widehat{X}\right\|_2.$$

Finally,

$$\left\|\mathbb{E}(F(X)\,|\,Y) - \mathbb{E}\left(F(\widehat{X})\,|\,\widehat{Y}\right)\right\|_2^2 \leq [F]_{\mathrm{Lip}}^2 \left\|X - \widehat{X}\right\|_2^2 + [\varphi_F]_{\mathrm{Lip}}^2 \left\|Y - \widehat{Y}\right\|_2^2.$$

▷ L^p-*case* $p \neq 2$. In the non-quadratic case a counterpart of the above inequality remains valid for the L^p-norm itself, provided $[\varphi_F]_{\mathrm{Lip}}$ is replaced by $2[\varphi_F]_{\mathrm{Lip}}$, namely

$$\left\|\mathbb{E}(F(X)\,|\,Y) - \mathbb{E}(F(\widehat{X})\,|\,\widehat{Y})\right\|_p \leq [F]_{\mathrm{Lip}} \left\|X - \widehat{X}\right\|_p + 2[\varphi_F]_{\mathrm{Lip}} \left\|Y - \widehat{Y}\right\|_p.$$

Exercise. Prove the above L^p-error bound when $p \neq 2$.

5.3 How to get optimal quantization?

This is often considered as the prominent drawback of optimal quantization, at least with respect to Monte Carlo method. Computing optimal or optimized quantization grids (and their weights) is less flexible (and more time consuming) than simulating a random vector. This means that optimal quantization is mainly useful when ones has to compute many integrals (or conditional expectations) with respect to the same probability distribution like *e.g.* the Gaussian distributions. As soon as $d \geq 2$, all procedures to optimize the quantization error are based on some nearest neighbour search. Let us cite the randomized Lloyd I procedure or the Competitive Learning Vector Quantization algorithm. The first one is a fixed point procedure and the second one a recursive stochastic approximation procedure. For further details we refer to [128] and to Section 6.3.6 in Chapter 6. Some optimal/ized grids of the Gaussian distribution are available on the web site

<p align="center">www.quantize.maths-fi.com</p>

as well as several papers dealing with quantization optimization.

Recent implementations of exact or approximate fast nearest neighbour search procedures confirmed that the computation time can be considerably reduced in higher dimension.

5.3.1 Competitive Learning Vector Quantization algorithm

Section 6.3.6 in Chapter 6

5.3.2 Randomized Lloyd's I procedure

Section 6.3.6 in Chapter 6

5.4 Numerical integration (II): Richardson-Romberg extrapolation

The challenge is to fight against the curse of dimensionality to increase the critical dimension beyond which the theoretical rate of convergence of the Monte Carlo method outperform that of optimal quantization. Combining the above cubature formula (5.4), (5.7) and the rate of convergence of the (optimal) quantization error, it seems natural to derive that the critical dimension to use quantization based cubature formulae is $d = 4$ (when dealing with continuously differentiable functions), at least when compared to Monte Carlo

simulation. Several tests have been carried out and reported in [128, 126] and to refine this *a priori* theoretical bound. The benchmark was made of several options on a geometric index on d independent assets in a Black-Scholes model: puts, puts spread and the same in a smoothed version, always without any control variate. Of course, not correlated assets is not a realistic assumption but it is clearly more challenging as far as numerical integration is concerned. Once the dimension d and the number of points N have been chosen, we compared the resulting integration error with a one standard deviation confidence interval of the corresponding Monte Carlo estimator for the same number of integration points N. The last standard deviation is computed thanks to a Monte Carlo simulation carried out using 10^4 trials.

The results turned out to be more favourable to quantization than predicted by theoretical bounds, mainly because we carried out our tests with rather small values of N whereas curse of dimensionality is an asymptotic bound. Until the dimension 4, the larger N is, the more quantization outperforms Monte Carlo simulation. When the dimension $d \geq 5$, quantization always outperforms Monte Carlo (in the above sense) until a critical size $N_c(d)$ which decreases as d increases.

In this section, we provide a method to push ahead theses critical sizes, at least for smooth enough functionals. Let $F : \mathbb{R}^d \to \mathbb{R}$, be a twice differentiable functional with Lipschitz continuous Hessian D^2F. Let $(\widehat{X}^{(N)})_{N\geq 1}$ be a sequence of optimal quadratic quantizations. Then

$$\mathbb{E}(F(X)) = \mathbb{E}(F(\widehat{X}^{(N)})) + \frac{1}{2}\mathbb{E}\left(D^2F(\widehat{X}^{(N)}).(X - \widehat{X}^{(N)})^{\otimes 2}\right) + O\left(\mathbb{E}|X - \widehat{X}|^3\right). \qquad (5.8)$$

Under some assumptions which are satisfied by most usual distributions (including the normal distribution), it is proved in [67] as a special case of a more general result that

$$\mathbb{E}|X - \widehat{X}|^3 = O(N^{-\frac{3}{d}})$$

or at least (in particular when $d = 2$) $\mathbb{E}|X - \widehat{X}|^3 = O(N^{-\frac{3-\varepsilon}{d}})$, $\varepsilon > 0$. If furthermore, we make the conjecture that

$$\mathbb{E}\left(D^2F(\widehat{X}^{(N)}).(X - \widehat{X}^{(N)})^{\otimes 2}\right) = \frac{c_{F,X}}{N^{\frac{2}{d}}} + o\left((\frac{1}{N^{\frac{3}{d}}}\right) \qquad (5.9)$$

one can use a Richardson-Romberg-Richardson extrapolation to compute $\mathbb{E}(F(X))$. Namely, one considers two sizes N_1 and N_2 (in practice one often sets $N_1 = N/2$ and $N_2 = N$). Then combining (5.8) with N_1 and N_2,

$$\mathbb{E}(F(X)) = \frac{N_2^{\frac{2}{d}}\mathbb{E}(F(\widehat{X}^{(N_2)})) - N_1^{\frac{2}{d}}\mathbb{E}(F(\widehat{X}^{(N_1)}))}{N_2^{\frac{2}{d}} - N_1^{\frac{2}{d}}} + O\left(\frac{1}{(N_1 \wedge N_2)^{\frac{1}{d}}(N_2^{\frac{2}{d}} - N_1^{\frac{2}{d}})}\right).$$

▷ NUMERICAL ILLUSTRATION: In order to see the effect of the extrapolation technique described above, numerical computations have been carried out in the case of the regularized version of some Put Spread options on geometric indices in dimension $d = 4, 6, 8, 10$. By "regularized", we mean that the payoff at maturity T has been replaced by its price function at time $T' < T$ (usually, $T' \approx T$). Numerical integration was performed using the Gaussian optimal grids of size $N = 2^k$, $k = 2, \ldots, 12$ (available at the web site www.quantize.maths-fi.com).

We consider again one of the test functions implemented in [128] (pp. 152). These test functions were borrowed from classical option pricing in mathematical finance, namely a Put spread option (on a geometric index which is less classical). Moreover we will use a "regularized" version of the payoff. One considers d independent traded assets $S^1, \ldots, S^d$ following a d-dimensional Black & Scholes dynamics (under its risk neutral probability)

$$S_t^i = s_0^i \exp\left((r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z^{i,t}\right), \qquad i = 1, \ldots, d.$$

where $Z^{i,t} = W_t^i$, $W = (W^1, \ldots, W^d)$ is a d-dimensional standard Brownian motion. Independence is unrealistic but corresponds to the most unfavorable case for numerical experiments. We also assume that $S_0^i = s_0 > 0$, $i = 1, \ldots, d$ and that the d assets share the same volatility $\sigma^i = \sigma > 0$. One considers,

the geometric index $I_t = \left(S_t^1 \ldots S_t^d\right)^{\frac{1}{d}}$. One shows that $e^{-\frac{\sigma^2}{2}(\frac{1}{d}-1)t} I_t$ has itself a risk neutral Black-Scholes dynamics. We want to test the *regularized Put spread option on this geometric index* with strikes $K_1 < K_2$ (at time $T/2$). Let $\psi(s_0, K_1, K_2, r, \sigma, T)$ denote the premium at time 0 of a Put spread on any of the assets S^i.

$$
\begin{aligned}
\psi(x, K_1, K_2, r, \sigma, T) &= \pi(x, K_2, r, \sigma, T) - \pi(x, K_1, r, \sigma, T) \\
\pi(x, K, r, \sigma, T) &= K e^{-rT} \mathrm{erf}(-d_2) - x \, \mathrm{erf}(-d_1),
\end{aligned}
$$

$$
d_1 = \frac{\log(x/K) + (r + \frac{\sigma^2}{2d})T}{\sigma\sqrt{T/d}}, \qquad d_2 = d_1 - \sigma\sqrt{T/d}.
$$

Using the martingale property of the discounted value of the premium of a European option yields that the premium $e^{-rT}\mathbb{E}((K_1 - I_T)_+ - (K_2 - I_T)_+)$ of the Put spread option on I satisfies on the one hand

$$
e^{-rT}\mathbb{E}((K_1 - I_T)_+ - (K_2 - I_T)_+) = \psi(s_0 e^{\frac{\sigma^2}{2}(\frac{1}{d}-1)T}, K_1, K_2, r, \sigma/\sqrt{d}, T)
$$

and, one the other hand

$$
e^{-rT}\mathbb{E}((K_1 - I_T)_+ - (K_2 - I_T)_+) = \mathbb{E}\, g(Z)
$$

where

$$
g(Z) = e^{-rT/2}\psi(e^{\frac{\sigma^2}{2}(\frac{1}{d}-1)\frac{T}{2}} I_{\frac{T}{2}}, K_1, K_2, r, \sigma, T/2)
$$

and $Z = (Z^{1,\frac{T}{2}}, \ldots, Z^{d,\frac{T}{2}}) \overset{d}{=} \mathcal{N}(0; I_d)$. The numerical specifications of the function g are as follows:

$$
s_0 = 100, \qquad K_1 = 98, \qquad K_2 = 102, \qquad r = 5\%, \qquad \sigma = 20\%, \qquad T = 2.
$$

The results are displayed below (see Fig. 5.4) in a log-log-scale for the dimensions $d = 4, 6, 8, 10$.

First, we recover the theoretical rates (namely $-2/d$) of convergence for the error bounds. Indeed, some slopes $\beta(d)$ can be derived (using a regression) for the quantization errors and we found $\beta(4) = -0.48$, $\beta(6) = -0.33$, $\beta(8) = -0.25$ and $\beta(10) = -0.23$ for $d = 10$ (see Fig. 5.4). Theses rates plead for the implementation of Richardson-Romberg extrapolation. Also note that, as already reported in [128], when $d \geq 5$, quantization still outperforms MC simulations (in the above sense) up to a critical number $N_c(d)$ of points ($N_c(6) \sim 5000$, $N_c(7) \sim 1000$, $N_c(8) \sim 500$, etc).

As concerns the Richardson-Romberg extrapolation method itself, note first that it always gives better results than "crude" quantization. As regards now the comparison with Monte Carlo simulation, no critical number of points $N_{Romb}(d)$ comes out beyond which MC simulation outperforms Richardson-Romberg extrapolation. This means that $N_{Romb}(d)$ is greater than the range of use of quantization based cubature formulas in our benchmark, namely $5\,000$.

Romberg extrapolation techniques are commonly known to be unstable, and indeed, it has not been always possible to estimate satisfactorily its rate of convergence on our benchmark. But when a significant slope (in a log-log scale) can be estimated from the Richardson-Romberg errors (like for $d = 8$ and $d = 10$ in Fig. 5.4 (c), (d)), its absolute value is larger than $1/2$, and so, these *extrapolations always outperform the MC method even for large values of N*. As a by-product, our results plead in favour of the conjecture (5.9) and lead to think that Richardson-Romberg extrapolation is a powerful tool to accelerate numerical integration by optimal quantization, even in higher dimension.

5.5 Hybrid quantization-Monte Carlo methods

In this section we explore two aspects of the variance reduction by quantization. First we propose use (optimal) quantization as a control variate, then a stratified sampling method relying on a quantization based stratification. This second method can be seen as a guided Monte Carlo method or a hybrid Quantization/Monte Carlo method. This method has been originally introduced in [128,
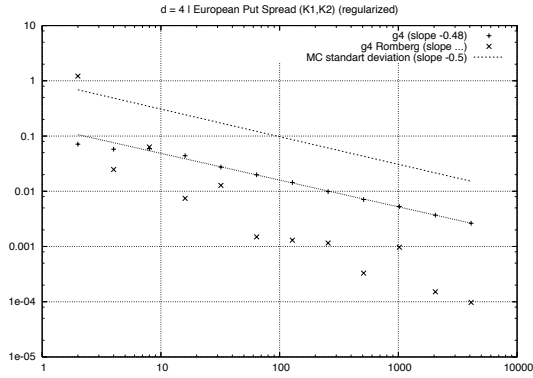
(a)
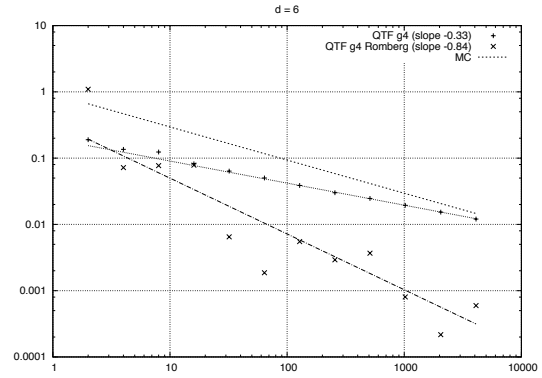
(b)
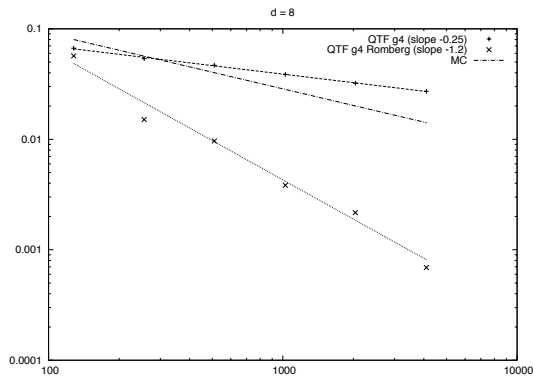
(c)
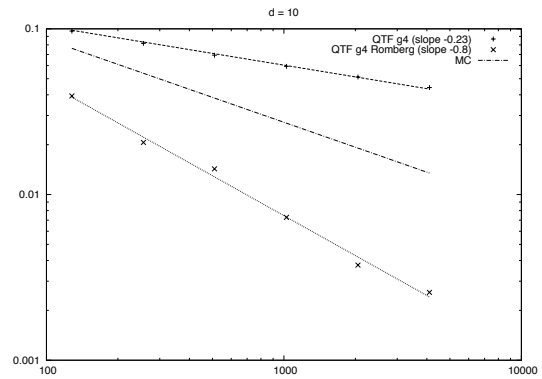
(d)

Figure 5.4: *Errors and standard deviations as functions of the number of points N in a log-log-scale. The quantization error is displayed by the cross $+$ and the Richardson-Romberg extrapolation error by the cross $\times$. The dashed line without crosses denotes the standard deviation of the Monte Carlo estimator. (a) $d = 4$, (b) $d = 6$, (c) $d = 8$, (d) $d = 10$.*

130] to deal with Lipschitz continuous functionals of the Brownian motion. Here, we will deal with a finite dimensional setting.

5.5.1   Optimal quantization as a control variate

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ be square integrable random vector. We assume that we have access to an N-quantizer $x := (x_1, \ldots, x_n) \in (\mathbb{R}^d)^N$ and we denote

$$\widehat{X}^N = \mathrm{Proj}_x(X)$$

(one of) its (Borel) nearest neighbour projection. We also assume that we have access to the numerical values of the "companion" probability distribution of x, that is the distribution of $\widehat{X}^N$ given by

$$\mathbb{P}(\widehat{X}^N = x_i) = \mathbb{P}(X \in C_i(x)), \quad i = 1, \ldots, N,$$

where $(C_i(x))_{i=1,\ldots,N}$ denotes the Voronoi tessellation of the N-quantizer induced by the above nearest neighbour projection.

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz continuous function such that $F(X) \in L^2(\mathbb{P})$. In order to compute $\mathbb{E}(F(X))$, one writes

$$
\begin{aligned}
\mathbb{E}(F(X)) &= \mathbb{E}(F(\widehat{X}^N)) + \mathbb{E}\left(F(X) - F(\widehat{X}^N)\right) \\
&= \underbrace{\mathbb{E}(F(\widehat{X}^N))}_{(a)} + \underbrace{\frac{1}{M} \sum_{m=1}^{M} F(X^{(m)}) - F(\widehat{X^{(m)}}^N)}_{(b)} + R_{N,M}
\end{aligned}
\tag{5.10}
$$

where $X^{(m)}$, $m = 1, \ldots, M$ are M independent copies of X, the symbol $\widehat{}^N$ denotes the nearest neighbour projection on a fixed N-quantizer $x \in (\mathbb{R}^d)^N$ and $R_{N,M}$ is a remainder term defined by (5.10). Term (a) can be computed by quantization and Term (b) can computed by a Monte Carlo simulation. Then,

$$\|R_{N,M}\|_2 = \frac{\sigma(F(X) - F(\widehat{X}^N))}{\sqrt{M}} \leq \frac{\|F(X) - F(\widehat{X}^N)\|_2}{\sqrt{M}} \leq [F]_{\mathrm{Lip}} \frac{\|X - \widehat{X}^N\|_2}{\sqrt{M}}$$

as $M \to +\infty$ where $\sigma(Y)$ denotes the standard deviation of a random variable Y. Furthermore

$$\sqrt{M}\, R_{N,M} \xrightarrow{\mathcal{L}} \mathcal{N}\big(0; \mathrm{Var}(F(X) - F(\widehat{X}^N))\big) \quad \text{as} \quad M \to +\infty.$$

Consequently, if F is simply a Lipschitz continuous function and if $(\widehat{X}^N)_{N \geq 1}$ is a sequence of *optimal quadratic quantizations of X*, then

$$\|R_{N,M}\|_2 \leq \frac{\|F(X) - F(\widehat{X}^N)\|_2}{\sqrt{M}} \leq C_{2,\delta}[F]_{\mathrm{Lip}} \frac{\sigma_{2+\delta}(X)}{\sqrt{M} N^{\frac{1}{d}}}. \tag{5.11}$$

where $C_{2,\delta}$ the constant coming from the non-asymptotic version of Zador's Theorem.

About practical implementation

As concerns practical implementation of this quantization based variance reduction method, the main gap is the *nearest neighbour search* needed at each step to compute $\widehat{X^{(m)}}^{N}$ from $X^{(m)}$.

   In 1-dimension, an (optimal) N-quantizer is usually directly obtained as a sorted N-tuple (with non-decreasing components) and the complexity of a nearest number search on the real line based on a dichotomy procedure is approximately $\frac{\log N}{\log 2}$. Of course this case is of little interest for applications.

   In d dimensions, there exist nearest neighbour search procedure with a $O(\log N)$-complexity, once the N-quantizer has been given an appropriate tree structure (which costs $O(N \log N)$). The most popular tree based procedure for nearest neighbour search is undoubtedly the Kd-tree (see [53]). During the last ten years, several attempts to improve it has been carried out, among them one can mention the Principal Axis Tree algorithm (see [108]). These procedures are efficient for quantizers with a large size N lying in a vector space with medium dimension (say up to 10).

   An alternative to speed up the nearest neighbour search procedure is to restrict to product quantizers whose Voronoi cells are hyper-parallelepipeds. In that case the nearest neighbour search reduces to those on the d marginals with an approximate resulting complexity of $d \frac{\log N}{\log 2}$.

   However this nearest neighbour search procedure slows down the global procedure.

5.5.2   Universal stratified sampling

The main drawback of what precedes is that repeated use of nearest neighbour search procedures. Using a quantization based stratification may be a mean to take advantage of quantization to reduce the variance without having to implement such time consuming procedures. On the other hand, one important drawback of the regular stratification method as described in Section 3.5 is to depend on the function F, at least when concerned by the optimal choice for the allocation parameters q_i. But in fact, one can show stratification has a uniform efficiency among the class of Lipschitz continuous continuous functions. This follows from the easy proposition below where we use some notations already introduced in Section 3.5.

Proposition 5.1 (Universal stratification) *Let* $(A_i)_{i \in I}$ *be a stratification of* $\mathbb{R}^d$. *For every* $i \in I$, *define the local inertia of the random vector* X *by*

$$\sigma_i^2 = \mathbb{E}\left(|X - \mathbb{E}(X|X \in A_i)|^2 | X \in A_i\right).$$

(a) Then, for every Lipschitz continuous continuous function $F : (\mathbb{R}^d, |\,.\,|) \to (\mathbb{R}^d, |\,.\,|)$,

$$\forall\, i \in I, \quad \sup_{[F]_{\mathrm{Lip}} \leq 1} \sigma_{F,i} = \sigma_i \tag{5.12}$$

where, for every $i \in I$, $\sigma_{F,i}$ *is nonnegative and defined by*

$$\sigma_{F,i}^2 = \min_{a \in \mathbb{R}^d} \mathbb{E}\left(|F(X) - a|^2 \,|\, X \in A_i\right) = \mathbb{E}\left(|F(X) - \mathbb{E}\left(X \,|\, X \in A_i\right)|^2 \,|\, X \in A_i\right).$$

(b) Suboptimal choice $(q_i = p_i)$.

$$\sup_{[F]_{\mathrm{Lip}} \leq 1} \left(\sum_{i \in I} p_i \sigma_{F,i}^2\right) = \sum_{i \in I} p_i \sigma_i^2 = \left\| X - \mathbb{E}\left(X | \sigma(\{X \in A_i\}, i \in I)\right) \right\|_2^2. \tag{5.13}$$

(*c*) OPTIMAL CHOICE (OF THE q_i'S).

$$\sup_{[F]_{\mathrm{Lip}}\leq 1} \left(\sum_{i\in I} p_i \sigma_{F,i}\right)^2 = \left(\sum_{i\in I} p_i \sigma_i\right)^2. \tag{5.14}$$

Furthermore

$$\left(\sum_{i\in I} p_i \sigma_i\right)^2 \geq \left\|X - \mathbb{E}\big(X|\sigma(\{X\in A_i\}, i\in I)\big)\right\|_1^2.$$

Remark. Any real-valued Lipschitz continuous continuous function can be seen as an $\mathbb{R}^d$-valued Lipchitz function, but then the above equalities (5.12), (5.13) and (5.14) only hold as inequalities.

Proof. (*a*) In fact

$$
\begin{aligned}
\sigma_{F,i}^2 &= \mathrm{Var}\,(F(X)\,|\,X\in A_i) \\
&= \mathbb{E}\left((F(X) - \mathbb{E}(F(X)|X\in A_i))^2|\,X\in A_i\right) \\
&\leq \mathbb{E}\left((F(X) - F(\mathbb{E}(X|X\in A_i)))^2|X\in A_i\right)
\end{aligned}
$$

owing to the very definition of conditional expectation as a minimizer w.r.t. to the conditional distribution. Now using that F is Lipschitz, it follows that

$$\sigma_{F,i}^2 \leq [F]_{\mathrm{Lip}}^2 \frac{1}{p_i}\mathbb{E}\left(\mathbf{1}_{\{X\in A_i\}}|X - \mathbb{E}(X|X\in A_i)|^2\right) = [F]_{\mathrm{Lip}}^2 \sigma_i^2.$$

Equalities in (*b*) and (*c*) straightforwardly follow from (*a*). Finally, the monotony of the L^p-norms implies

$$\sum_{i=1}^N p_i\sigma_i \;\geq\; \left\|X - \mathbb{E}(X|\sigma(\{X\in A_i\}, i\in I))\right\|_1. \qquad \diamond$$

5.5.3 An (optimal) quantization based universal stratification: a minimax approach

The starting idea is to use the Voronoi diagram of an N-quantizer $x = (x_1, \dots, x_N)$ of X to design the strata in a stratification procedure. Firstly, this amounts to setting $I = \{, \dots, N\}$ and

$$A_i = C_i(x), \quad i\in I.$$

Then, for every $i \in \{1, \dots, N\}$, there exists a Borel function $\varphi(x_i, .) : [0,1]^q \to \mathbb{R}^d$ such that

$$\varphi(x_i, U) \stackrel{d}{=} \mathcal{L}(X\,|\,\widehat{X}^x = x_i) = \frac{\mathbf{1}_{C_i(x)}\mathbb{P}_X(d\xi)}{\mathbb{P}(X\in C_i(x))}.$$

where $U \stackrel{d}{=} U([0,1]^q)$. Note that the dimension q is arbitrary: one may always assume that $q = 1$ by the fundamental theorem of simulation, but in order to obtain some closed forms for $\varphi(x_i, .)$,

we are lead to consider situations where $q \geq 2$ (or even infinite when considering a Von Neumann acceptance-rejection method).

Now let (ξ, U) be a couple of independent random vectors such that $\xi \overset{d}{=} \widehat{X}^x$ and $U \overset{d}{=} U([0,1])^q$. Then, one checks that

$$\varphi(\xi, U) \overset{d}{=} X$$

so that one may assume without loss of generality that $X = \varphi(\xi, U)$ which in turn implies that $\xi = \widehat{X}^x$ i.e.

$$X = \varphi(\widehat{X}^x, U), \quad U \overset{d}{=} U([0,1]^q), \qquad U, \widehat{X}^x \text{ independent.}$$

In terms of implementation, as mentioned above we need a closed formula for the function φ which induces some stringent constraint on the choice of the N-quantizers. In particular there is no reasonable hope to consider true optimal quadratic quantizer for that purpose. A reasonable compromise is to consider some optimal *product* quantization for which the function φ can easily be made explicit (see Section 3.5).

Let $\mathcal{A}_d(N)$ denote the family of all Borel partitions of $\mathbb{R}^d$ having (at most) N elements.

Proposition 5.2 (a) Suboptimal choice *One has*

$$\inf_{(A_i)_{1 \leq i \leq N} \in \mathcal{A}_d(N)} \sup_{[F]_{\text{Lip}} \leq 1} \left(\sum_{i \in I} p_i \sigma_{F,i}^2 \right) = \min_{x \in (\mathbb{R}^d)^N} \|X - \widehat{X}^x\|_2^2.$$

(b) Optimal choice *One has*

$$\inf_{(A_i)_{1 \leq i \leq N} \in \mathcal{A}_d(N)} \sup_{[F]_{\text{Lip}} \leq 1} \left(\sum_{i \in I} p_i \sigma_{F,i} \right) \geq \min_{x \in (\mathbb{R}^d)^N} \|X - \widehat{X}^x\|_1.$$

Proof. First we rewrite (5.13) and (5.14) in terms of quantization *i.e.* with $A_i = C_i(x)$,:

$$\sup_{[F]_{\text{Lip}} \leq 1} \left(\sum_{i \in I} p_i \sigma_{F,i}^2 \right) \leq \sum_{i \in I} p_i \sigma_i^2 = \left\| X - \mathbb{E}(X|\widehat{X}^x) \right\|_2^2. \qquad (5.15)$$

and

$$\sup_{[F]_{\text{Lip}} \leq 1} \left(\sum_{i \in I} p_i \sigma_{F,i} \right)^2 \leq \left(\sum_{i \in I} p_i \sigma_i \right)^2 \geq \left\| X - \mathbb{E}(X|\widehat{X}^x) \right\|_1^2 \qquad (5.16)$$

where we used the obvious fact that $\sigma(\{X \in C_i(x)\}, i \in I) = \sigma(\widehat{X}^x)$.

(a) It follows from (5.13) that

$$\inf_{(A_i)_{1 \leq i \leq N}} \left(\sum_{i \in I} p_i \sigma_i^2 \right) = \inf_{(A_i)_{1 \leq i \leq N}} \|X - \mathbb{E}(X|\sigma(\{X \in A_i\}, i \in I)\|_2.$$

Now (see *e.g.* [66] or [123]),

$$\|X - \widehat{X}^{x^{*N}}\|_2 = \min\left\{\|X - Y\|_2,\ Y : (\omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d,\ |Y(\Omega)| \leq N\right\}$$

and

$$\widehat{X}^{x^{*N}} = \mathbb{E}(X \,|\, \widehat{X}^{x^{*N}}).$$

Consequently, (5.15) completes the proof.

(*b*) It follows from (5.14)

$$\sum_{i \in I} p_i \sigma_i \ \geq \ \|X - \mathbb{E}(X|\sigma(\{X \in A_i\}, i \in I)\|_1$$

$$\geq \ \|\operatorname{dist}(X, x(A))\|_1$$

where $x(A) := \left\{\mathbb{E}\big(X|\sigma(\{X \in A_i\}, i \in I)\big)(\omega),\ \omega \in \Omega\right\}$ has at most N elements. Now $\|d(X, x(A))\|_1 = \mathbb{E}\operatorname{dist}(X, x(A)) = \|X - \widehat{X}^{x(A)}\|_1$, consequently

$$\sum_{i \in I} p_i \sigma_i \ \geq \ \|X - \widehat{X}^{x(A)}\|_1$$

$$= \ \min_{x \in (\mathbb{R}^d)^N} \|X - \widehat{X}^x\|_1. \qquad \diamond$$

As a conclusion, we see that the notion of universal stratification (with respect to Lipschitz continuous functions) and quantization are closely related since the variance reduction factor that can be obtained by such an approach is essentially ruled by the quantization rate of the state space of the random vector X by its distribution.

One dimension In that case the method applies straightforwardly provided both the distribution function $F_X(u) := \mathbb{P}(X \leq u)$ of X (on $\bar{\mathbb{R}}$) and its right continuous (canonical) inverse on $[0, 1]$, denoted F_X^{-1} are computable.

We also need the additional assumption that the N-quantizer $x = (x_1, \ldots, x_N)$ satisfies the following continuity assumption

$$\mathbb{P}(X = x_i) = 0, \qquad i = 1, \ldots, N.$$

Note that so is always the case if X has a density.

Then set
$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2},\ i = 1, \ldots, N-1,\ x_{-\frac{1}{2}} = -\infty,\ x_{N+\frac{1}{2}} = +\infty.$$

Then elementary computations show that, with $q = 1$,

$$\forall\, u \in [0, 1], \quad \varphi_N(x_i, u) = F_X^{-1}\left(F_X(x_{i-\frac{1}{2}}) + (F_X(x_{i+\frac{1}{2}}) - F_X(x_{i-\frac{1}{2}}))u\right),\ i = 1, \ldots, N. \quad (5.17)$$

Higher dimension   We consider a random vector $X = (X^1, \ldots, X^d)$ whose marginals X^i are independent. This may appear as a rather stringent restriction in full generality although it is often possible to " extract" in a model an innovation with that correlation structure. At least in a Gaussian framework, such a reduction is always possible after an orthogonal diagonalization of its covariance matrix. One considers a *product quantizer* (see *e.g.* [122, 123]) defined as follows: for every $\ell \in \{1, \ldots, d\}$, let $x^{N_\ell} = (x_1^{N_\ell}, \ldots, x_{(N_\ell)}^{N_\ell})$ be an N_ℓ-quantizer of the marginal X^ℓ and set $N := N_1 \times \cdots \times N_d$. Then, define for every multi-index $\underline{i} := (i_1, \ldots, i_d) \in I := \prod_{\ell=1}^d \{1, \ldots, N_\ell\}$,

$$x_{\underline{i}} = (x_{i_1}^{(N_1)}, \ldots, x_{i_d}^{(N_d)}).$$

Then, one defines $\varphi_{N,X}(x_{\underline{i}}, u)$ by setting $q = d$ and

$$\varphi_{N,X}(x_{\underline{i}}, (u^1, \ldots, u^d)) = \left(\varphi_{N,X^\ell}(x_{i_\ell}^{(N_\ell)}, u^\ell) \right)_{1 \leq \ell \leq d}.$$

where φ_{N,X_ℓ} is defined by (5.17).

Chapter 6

Stochastic approximation and applications to finance

6.1 Motivation

In Finance, one often faces some optimization problems or zero search problems. The first ones often reduce to the second one since, at least in a convex framework, minimizing a function amounts to finding a zero of its gradient. The most commonly encountered examples are the extraction of implicit parameters (implicit volatility of an option, implicit correlations in the credit markets or for a single best-of-option), the calibration, the optimization of an exogenous parameters for variance reduction (regression, Importance sampling, etc). All these situations share a common feature: the involved functions all have a representation as an expectation, namely they read $h(y) = \mathbb{E}\, H(y, Z)$ where Z is a q-dimensional random vector. The aim of this chapter is to provide a toolbox – stochastic approximation – based on simulation to solve these optimization or zero search problems. It can be seen as an extension of the Monte Carlo method.

Stochastic approximation can be presented as a probabilistic extension of Newton-Raphson like zero search recursive procedures of the form

$$\forall\, n \geq 0, \qquad y_{n+1} = y_n - \gamma_{n+1} h(y_n) \qquad (0 < \gamma_n \leq \gamma_0) \tag{6.1}$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector field satisfying a sub-linear growth assumption at infinity. Under some appropriate *mean-reverting* assumptions, one shows that such a procedure is bounded and eventually converges to a zero y_* of h. As an example if one sets $\gamma_n = (Dh(y_{n-1}))^{-1}$, the above recursion is but the regular Newton-Raphson procedure for zero search of the function h (one can also set $\gamma_n = 1$ and replace h by $(Dh)^{-1} \circ h$).

In one dimension, *mean-reversion* may be obtained by a non-decreasing assumption made on the function h or, more simply by assuming that $h(y)(y - y_*) > 0$ for every $y \neq y_*$: if so, y_n is decreasing as long as $y_n > y_*$ and decreasing whenever $y_n < y_*$. In higher dimension, this assumption becomes $(h(y)\,|\,y - y_*) > 0$, $y \neq y_*$ and will be extensively called upon further on.

More generally mean-reversion may follow from a monotony assumption on h in one (or higher dimension) and more generally follows from he existence of a so-called *Lyapunov function*. To introduce this notion, let us make a (light) connection with Ordinary Differential Equations (ODE).

Let us make a connection with diffferential dynamic systems: thus, when $\gamma_n = \gamma > 0$, Equation (6.1) is but the Euler scheme with step $\gamma > 0$ of the *ODE*

$$ODE_h \equiv \dot{y} = -h(y).$$

A Lyapunov function for ODE_h is a function $L : \mathbb{R}^d \to \mathbb{R}_+$ such that any solution $t \mapsto x(t)$ of the equation satisfies $t \mapsto L(x(t))$ is non-increasing as t increases. If L is differentiable this is mainly equivalent to the condition $(\nabla L | h) \geq 0$ since

$$\frac{d}{dt} L(y(t)) = (\nabla L(y(t)) | \dot{y}(t)) = -(\nabla L | h)(y(t)).$$

If such a Lyapunov function does exist (which is not always the case!), the system is said to be *dissipative*.

Basically one meets two frameworks:

– the function L is identified *a priori*, it is the object of interest for optimization purpose, and one designs a function h *from L e.g.* by setting $h = \nabla L$ (or possibly h proportional to ∇L).

– The function of interest is h and one has to search for a Lyapunov function L (which may not exist). This usually requires a deep understanding of the problem from a dynamical point of view.

This duality also occurs in discrete time Stochastic Approximation Theory from its very beginning in the early 1950' (see [141, 82]).

As concerns the constraints on the Lyapunov function, due to the specificities of the discrete time setting, we will require some further regularity assumption on ∇L, typically ∇L is Lipschitz continuous and $|\nabla L|^2 \leq C(1 + L)$ (essentially quadratic property).

▷ **Exercises. 1.** Show that if a function $h : \mathbb{R}^d \to \mathbb{R}^d$ is non-decreasing in the following sense

$$\forall\, x,\, y \in \mathbb{R}^d, \qquad (h(y) - h(x) | y - x) \geq 0$$

and if $h(y_*) = 0$ then $L(y) = |y - y_*|^2$ is a Lyapunov function for ODE_h.

2. Assume furthermore that $\{h = 0\} = \{y_*\}$ and that h satisfies a sub-linear growth assumption: $|h(y)| \leq C(1 + |y|)$, $y \in \mathbb{R}^d$. Show that the sequence $(y_n)_{n \geq 0}$ defined by (6.1) converges toward y_*.

Now imagine that no straightforward access to numerical values of $h(y)$ is available but that h has an integral representation with respect to an $\mathbb{R}^q$-valued random vector Z, say

$$h(y) = \mathbb{E}\, H(y, Z), \quad H : \mathbb{R}^d \times \mathbb{R}^q \overset{Borel}{\longrightarrow} \mathbb{R}^d, \quad Z \overset{d}{=} \mu, \tag{6.2}$$

(satisfying $\mathbb{E}\, |H(y, Z)| < +\infty$ for every $y \in \mathbb{R}^d$). Assume that

– $H(y, z)$ is easy to compute for any couple (y, z)

– the distribution μ of Z can be simulated at a reasonable cost.

One idea can be to simply "randomize" the above zero search procedure (6.1) by using at each iterate a Monte Carlo simulation to approximate $h(y_n)$.

A more sophisticated idea is to try doing both simultaneously by using on the one hand $H(y_n, Z_{n+1})$ instead of $h(y_n)$ and on the other hand by letting the step γ_n go to 0 to asymptotically

smoothen the chaotic (stochastic...) effect induced by this "local" randomization. However one should not to make γ_n go to 0 too fast so that an averaging effect occurs like in the Monte Carlo method. In fact, one should impose that $\sum_n \gamma_n = +\infty$ to ensure that the initial value of the procedure will be forgotten.

Based on this heuristic analysis, we can reasonably hope that the recursive procedure

$$\forall n \geq 0, \quad Y_{n+1} = Y_n - \gamma_{n+1} H(Y_n, Z_{n+1}) \tag{6.3}$$

where

$$(Z_n)_{n \geq 1} \text{ is an i.i.d. sequence with distribution } \mu \text{ defined on } (\Omega, \mathcal{A}, \mathbb{P})$$

and Y_0 is an $\mathbb{R}^d$-valued random vector (independent of the sequence $(Z_n)_{n \geq 1}$) defined on the same probability space also converges to a zero y_* of h at least under appropriate assumptions, to be specified further on, on both H and the gain sequence $\gamma = (\gamma_n)_{n \geq 1}$.

What precedes can be seen as the "meta-theorem" of stochastic approximation. In this framework, the Lyapunov functions mentioned above are called upon to ensure the stability of the procedure.

▷ *A first toy-example: the Strong Law of Large Numbers.* As a first example note that the sequence of empirical means $(\overline{Z}_n)_{n \geq 1}$ of an i.i.d. sequence (Z_n) of integrable random variable satisfies

$$\overline{Z}_{n+1} = \overline{Z}_n - \frac{1}{n+1}(\overline{Z}_n - Z_{n+1}), \quad n \geq 0, \ \overline{Z}_0 = 0,$$

i.e. a stochastic approximation procedure with $H(y, Z) = y - Z$ and $h(y) := y - z_*$ with $z_* = \mathbb{E}\, Z$ (so that $Y_n = \overline{Z}_n$). Then the procedure converges *a.s.* (and in L^1) to the unique zero z_* of h.

The (weak) rate of convergence of $(\overline{Z}_n)_{n \geq 1}$ is ruled by the *CLT* which may suggest that the generic rate of convergence of this kind of procedure is of the same type. In particular the deterministic counterpart with the same gain parameter, $y_{n+1} = y_n - \frac{1}{n+1}(y_n - z_*)$, converges at a $\frac{1}{n}$-rate to z_* (and this is clearly not the optimal choice for γ_n in this deterministic framework).

However, if we do not know the value of the mean $z_* = \mathbb{E}\, Z$ but if we able to simulate μ-distributed random vectors, the first recursive stochastic procedure can be easily implemented whereas the deterministic one cannot. The stochastic procedure we are speaking about is simply the regular Monte Carlo method!

▷ *A second toy-model: extracting implicit volatility in a Black-Scholes model.* A second toy-example is the extraction of implicit volatility in a Black-Scholes model for a vanilla *Call* or *Put* option. In practice it is carried out by a deterministic Newton procedure (see *e.g.* [109]) since closed forms are available for both the premium and the *vega* of the option. But let us forget about that for a few lines to illustrate the basic principle of Stochastic Approximation. Let x, K, $T \in (0, +\infty)$, let $r \in \mathbb{R}$ and set for every $\sigma \in \mathbb{R}$,

$$X_t^{x,\sigma} = xe^{(r - \frac{\sigma^2}{2})t + \sigma W_t}, \quad t \geq 0.$$

We know that $\sigma \mapsto Put_{BS}(x, K, \sigma, r, T) = e^{-rT}\mathbb{E}(K - X_T^{x,\sigma})_+$ is an even function, increasing on $(0, +\infty)$, continuous with $\lim_{\sigma \to 0} Put_{BS}(x, K, \sigma, r, T) = (e^{-rT}K - x)_+$ and $\lim_{\sigma \to +\infty} Put_{BS}(x, K, \sigma, r, T) = e^{-rT}K$ (these bounds are model-free and can be directly derived by arbitrage arguments). Let

$P_{market}(x, K, r, T) \in [(e^{-rT}K - x)_+, e^{-rT}K]$ be a consistent mark-to-market price for the Put option with maturity T and strike price K. Then the implied volatility $\sigma_{impl} := \sigma_{impl}(x, K, r, T)$ is defined as the unique (positive) solution to the equation

$$Put_{BS}(x, K, \sigma_{impl}, r, T) = P_{market}(x, K, r, T)$$

i.e.

$$\mathbb{E}\left(e^{-rT}(K - X_T^{x,\sigma})_+ - P_{market}(x, K, r, T)\right) = 0.$$

This naturally leads to devise the following stochastic algorithm to solve numerically this equation:

$$\sigma_{n+1} = \sigma_n - \gamma_{n+1}\left(\left(K - xe^{(r-\frac{\sigma_n^2}{2})T+\sigma_n\sqrt{T}Z_{n+1}}\right)_+ - P_{market}(x, K, r, T)\right)$$

where $(Z_n)_{n\geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0;1)$-distributed random variables and the step sequence $\gamma = (\gamma_n)_{n\geq 1}$ is e.g. given by $\gamma_n = \frac{c}{n}$ for some parameter $c > 0$. After a necessary "tuning" of the constant c (try $c = \frac{2}{x+K}$), one observes that

$$\sigma_n \longrightarrow \sigma_{impl} \quad a.s. \quad \text{as} \quad nn \to +\infty.$$

A priori one could imagine that σ_n could converge toward $-\sigma_{impl}$ (which would not be a real problem) but it *a.s.* never happens because this negative solution is repulsive for the related ODE_h and "noisy". This is an important topic often known in the literature as "how stochastic algorithms never fall into noisy traps" (see [29, 96, 135], etc).

To conclude this introductory section, let us briefly come back to the case where $h = \nabla L$ and $L(y) = \mathbb{E}(\Lambda(y, Z))$ so that $\nabla L(y) = \mathbb{E}\, H(y, Z)$ with $H(y, z) := \frac{\partial \Lambda(y,z)}{\partial y}$. The function H is sometimes called a *local gradient* (of L) and the procedure (6.3) is known as a *stochastic gradient* procedure. When Y_n converges to some zero y_* of $h = \nabla L$ at which the algorithm is "noisy enough" – say e.g. $\mathbb{E}(H(y_*, Z)\, H(y_*, Z)^*) > 0$ is a definite symmetric matrix – then y_* is necessarily a *local minimum* of the potential L: y_* cannot be a *trap*. So, if L is strictly convex and $\lim_{|y|n\to+\infty} L(y) = +\infty$, ∇L has a single zero y_* which is but the global minimum of L: the *stochastic gradient* turns out to be a minimization procedure.

However, most recursive stochastic algorithms (6.3) are not stochastic gradients and the Lyapunov function, if any, is not naturally associated to the algorithm: finding a Lyapunov function to "stabilize" the algorithm (by bounding *a.s.* its paths, see Robbins-Siegmund Lemma below) is often a difficult task which requires a deep understanding of the related ODE_h.

As concerns the rate of convergence, one must have in mind that it usually ruled by a CLT at a $1/\sqrt{\gamma_n}$ rate which can reach at most the $\sqrt{n}$-rate of the regular CLT. So, such a "toolbox" is clearly not competitive compared to a deterministic procedure when available but this rate should be compare to that of the Monte Carlo metrhod (*i.e.* the $SLLN$) since their field of application are similar: stochastic approximation is the natural extension of the Monte Carlo method to solve inverse or optimization problems related to functions having a representation as an expectation of simulatable random functionals.

Recently, several contributions (see [4, 101, 102]) draw the attention of the quants world to stochastic approximation as a tool for variance reduction, *implicitation* of parameters, model calibration, risk management... It is also used in other fields of finance like algorithmic trading as an *on line* optimizing device for execution of orders (see e.g. [94]). We will briefly discuss several (toy-)examples of application.

6.2 Typical *a.s.* convergence results

The stochastic approximation provides various theorems that guarantee the *a.s.* and/or L^p convergence of stochastic approximation procedures. We provide below a general (multi-dimensional) preliminary result known as *Robbins-Siegmund Lemma* from which the main convergence results will be easily derived (although, strictly speaking, the lemma does not provide any *a.s.* convergence result).

In what follows the function H and the sequence $(Z_n)_{n \geq 1}$ are defined by (6.2) and h is the vector field from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined by $h(y) = \mathbb{E}\,H(y, Z_1)$.

Theorem 6.1 (Robbins-Siegmund Lemma) *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ and $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ satisfying (6.2). Suppose that there exists a continuously differentiable function $L : \mathbb{R}^d \to \mathbb{R}_+$ satisfying*

$$\nabla L \text{ is Lipschitz continuous continuous and } \quad |\nabla L|^2 \leq C(1 + L) \tag{6.4}$$

such that h satisfies the mean reverting assumption

$$(\nabla L | h) \geq 0. \tag{6.5}$$

Furthermore suppose that H satisfies the following (pseudo-)linear growth assumption

$$\forall\, y \in \mathbb{R}^d, \qquad \|H(y, Z)\|_2 \leq C\sqrt{1 + L(y)} \tag{6.6}$$

(which implies $|h| \leq C\sqrt{1 + L}$).

Let $\gamma = (\gamma_n)_{n \geq 1}$ be a sequence of positive real numbers satisfying the so-called decreasing step assumption

$$\sum_{n \geq 1} \gamma_n = +\infty \qquad and \qquad \sum_{n \geq 1} \gamma_n^2 < +\infty. \tag{6.7}$$

Finally, assume Y_0 is independent of $(Z_n)_{n \geq 1}$ and $\mathbb{E}\,L(Y_0) < +\infty$.

Then, the recursive procedure defined by (6.3) satisfies the following four properties:

(i) $Y_n - Y_{n-1} \overset{\mathbb{P}\,a.s.\&L^2(\mathbb{P})}{\longrightarrow} 0$ *as* $n \to +\infty$,

(ii) the sequence $(L(Y_n))_{n \geq 0}$ is $L^1(\mathbb{P})$-bounded,

(iii) $L(Y_n) \overset{a.s.}{\longrightarrow} L_\infty \in L^1(\mathbb{P})$ *as* $n \to +\infty$,

(iv) $\displaystyle\sum_{n \geq 1} \gamma_n (\nabla L | h)(Y_{n-1}) < +\infty$ *a.s.*

Remarks and terminology. • The sequence $(\gamma_n)_{n \geq 1}$ is called a *step sequence* or a *gain parameter sequence*.

• If the function L satisfies (6.4), (6.5), (6.6) and moreover $\lim_{|y| \to +\infty} L(y) = +\infty$, then L is called a *Lyapunov function* of the system like in Ordinary Differential Equation Theory.

• Note that the assumption (6.4) on L implies that $\nabla\sqrt{1 + L}$ is bounded. Hence $\sqrt{L}$ has at most a linear growth so that L itself has at most a quadratic growth.

• In spite of the standard terminology, the step sequence does not need to be decreasing in Assumption (6.7).

• A careful reading of the proof below shows that the assumption $\sum_{n \geq 1} \gamma_n = +\infty$ is not needed. However we leave it in the statement because it is dramatically useful for *any application* of this Lemma since it implies, combined with (iv) that

$$\liminf_n (\nabla L | h)(Y_{n-1}) = 0.$$

These assumptions are known as "Robbins-Siegmund assumptions".

• When $H(y, z) := h(y)$ (*i.e.* the procedure is noiseless), the above theorem provides a convergence result for the original deterministic procedure (6.1).

The key of the proof is the following convergence theorem for non negative super-martingales (see [115]).

Theorem 6.2 *Let $(S_n)_{n \geq 0}$ be a non-negative super-martingale with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (i.e. for every $n \geq 0$, $S_n \in L^1(\mathbb{P})$ and $\mathbb{E}(S_{n+1} | \mathcal{F}_n) \leq S_n$ a.s.) then, S_n converges $\mathbb{P}$-a.s. to an integrable (non-negative) random variable S_∞.*

For general convergence theorems for sub-, super- and true martingales we refer to any standard course on Probability Theory or, preferably to [115].

Proof. Set $\mathcal{F}_n := \sigma(Y_0, Z_1, \ldots, Z_n)$, $n \geq 1$, and for notational convenience $\Delta Y_n := Y_n - Y_{n-1}$, $n \geq 1$. It follows from the fundamental formula of calculus that there exists $\xi_{n+1} \in (Y_n, Y_{n+1})$ (geometric interval) such that

$$
\begin{aligned}
L(Y_{n+1}) &= L(Y_n) + (\nabla L(\xi_{n+1}) | \Delta Y_{n+1}) \\
&\leq L(Y_n) + (\nabla L(Y_n) | \Delta Y_{n+1}) + [\nabla L]_{\text{Lip}} |\Delta Y_{n+1}|^2 \\
&= L(Y_n) - \gamma_{n+1}(\nabla L(Y_n) | H(Y_n, Z_{n+1})) + [\nabla L]_{\text{Lip}} \gamma_{n+1}^2 |H(Y_n, Z_{n+1})|^2 \qquad (6.8) \\
&\leq L(Y_n) - \gamma_{n+1}(\nabla L(Y_n) | h(Y_n)) - \gamma_{n+1}(\nabla L(Y_n) | \Delta M_{n+1}) \qquad (6.9) \\
&\quad + [\nabla L]_{\text{Lip}} \gamma_{n+1}^2 |H(Y_n, Z_{n+1})|^2
\end{aligned}
$$

where

$$\Delta M_{n+1} = H(Y_n, Z_{n+1}) - h(Y_n).$$

We aim at showing that ΔM_{n+1} is a (square integrable) $\mathcal{F}_n$-martingale increment satisfying $\mathbb{E}(|\Delta M_{n+1}|^2 | \mathcal{F}_n) \leq C(1 + L(Y_n))$ for an appropriate real constant $C > 0$.

First note that $L(Y_n) \in L^1(\mathbb{P})$ and $H(Y_n, Z_{n+1}) \in L^2(\mathbb{P})$ for every $n \geq 0$: this follows from (6.8) and an easy induction since

$$\mathbb{E}|(\nabla L(Y_n) | H(Y_n, Z_{n+1}))| \leq \frac{1}{2}(\mathbb{E}|\nabla L(Y_n)|^2 + \mathbb{E}|H(Y_n, Z_{n+1})|^2) \leq C(1 + \mathbb{E}\, L(Y_n))$$

(where we used that $|(a|b)| \leq \frac{1}{2}(|a|^2 + |b|^2)$, a, $b \in \mathbb{R}^d$).

Now Y_n being $\mathcal{F}_n$-measurable and Z_{n+1} being independent of $\mathcal{F}_n$

$$\mathbb{E}(H(Y_n, Z_{n+1}) | \mathcal{F}_n) = \mathbb{E}(H(y, Z_1))_{|y=Y_n} = h(Y_n).$$

Consequently $\mathbb{E}\,\Delta M_{n+1} | \mathcal{F}_n) = 0$. The inequality for $\mathbb{E}(|\Delta M_{n+1}|^2 | \mathcal{F}_n)$ follows from $|\Delta M_{n+1}|^2 \leq 2(|H(Y_n, Z_{n+1})|^2 + |h(Y_n)|^2)$ and Assumption (6.6).

Now, one derives from the assumptions (6.7) and (6.9) that there exists two positive real constants $C_L = C[\nabla L]_{Lip} > 0$ such that

$$S_n = \frac{L(Y_n) + \sum_{k=0}^{n-1} \gamma_{k+1} (\nabla L(Y_k)|h(Y_k)) + C_L \sum_{k \geq n+1} \gamma_k^2}{\prod_{k=1}^{n} (1 + C_L \gamma_k^2)}$$

is a (non negative) super-martingale with $S_0 = L(Y_0) \in L^1(\mathbb{P})$. This uses that $(\nabla L|h) \geq 0$. Hence $\mathbb{P}$-a.s. converging toward an integrable random variable S_∞. Consequently, using that $\sum_{k \geq n+1} \gamma_k^2 \to 0$, one gets

$$L(Y_n) + \sum_{k=0}^{n-1} \gamma_{k+1} (\nabla L|h)(Y_k) \xrightarrow{a.s.} \widetilde{S}_\infty = S_\infty \prod_{n \geq 1} (1 + C_L \gamma_n^2) \in L^1(\mathbb{P}). \qquad (6.10)$$

The super-martingale $(S_n)_{n \geq 0}$ being $L^1(\mathbb{P})$-bounded, one derives likewise that $(L(Y_n))_{n \geq 0}$ is L^1-bounded since

$$L(Y_n) \leq \left(\prod_{k=1}^{n} (1 + C_L \gamma_k^2) \right) S_n, \quad n \geq 0.$$

Now, a series with non-negative terms which is upper bounded by an (*a.s.*) converging sequence, *a.s.* converges in $\mathbb{R}_+$ so that

$$\sum_{n \geq 0} \gamma_{n+1} (\nabla L|h)(Y_n) < +\infty \qquad \mathbb{P}\text{-}a.s.$$

It follows from (6.10) that, $\mathbb{P}$-*a.s.*, $L(Y_n) \xrightarrow{n \to +\infty} L_\infty$ which is integrable since $(L(Y_n))_{n \geq 0}$ is L^1-bounded.

Finally

$$\sum_{n \geq 1} \mathbb{E}|\Delta Y_n|^2 \leq \sum_{n \geq 1} \gamma_n^2 \mathbb{E}(|H(Y_{n-1}, Z_n)|^2) \leq C \sum_{n \geq 1} \gamma_n^2 (1 + \mathbb{E}L(Y_{n-1})) < +\infty$$

so that $\sum_{n \geq 1} \mathbb{E}(|\Delta Y_n|^2) < +\infty$ *a.s.* which yields $Y_n - Y_{n-1} \to 0$ *a.s.*. $\quad \diamond$

Remark. The same argument which shows that $(L(Y_n))_{n \geq 1}$ is $L^1(\mathbb{P})$-bounded shows that

$$\sum_{n \geq 1} \gamma_n \mathbb{E}(\nabla L|h)(Y_{n-1}) = \mathbb{E}\left(\sum_{n \geq 1} \gamma_n (\nabla L|h)(Y_{n-1}) \right) < +\infty \quad a.s..$$

Corollary 6.1 (*a*) ROBBINS-MONRO ALGORITHM. *Assume that the mean function h of the algorithm is continuous and satisfies*

$$\forall y \in \mathbb{R}^d, \; y \neq y_*, \qquad (y - y_*|h(y)) > 0 \qquad (6.11)$$

(which implies that $\{h = 0\} = \{y_*\}$). Suppose furthermore that $Y_0 \in L^2(\mathbb{P})$ and that H satisfies

$$\forall y \in \mathbb{R}^d, \qquad \left\| H(y, Z) \right\|_2 \leq C(1 + |y|).$$

Assume that the step sequence $(\gamma_n)_{n\geq 1}$ satisfies (6.7). Then

$$Y_n \xrightarrow{a.s.} y_*$$

and in every $L^p(\mathbb{P})$, $p \in (0,2)$.

(b) STOCHASTIC GRADIENT. Let $L : \mathbb{R}^d \to \mathbb{R}_+$ be a differentiable function satisfying (6.4), $\lim_{|y|\to+\infty} L(y) = +\infty$, and $\{\nabla L = 0\} = \{y_*\}$. Assume the mean function of the algorithm is given by $h = \nabla L$ and that H satisfies $\mathbb{E}\left(|H(y,Z)|^2\right) \leq C(1+L(y))$ and that $L(Y_0) \in L^1(\mathbb{P})$. Assume that the step sequence $(\gamma_n)_{n\geq 1}$ satisfies (6.7). Then $L(y_*) = \min_{\mathbb{R}^d} L$ and

$$Y_n \xrightarrow{a.s.} y_*.$$

Moreover $\nabla L(Y_n)$ converges to 0 in every $L^p(\mathbb{P})$, $p \in (0,2)$.

Proof. (a) Assumption (6.11) is but the mean-reverting assumption related to the Lyapunov function $L(y) = |y - y_*|^2$. The assumption on H is clearly the linear growth assumption (6.6) for this function L. Consequently, it follows from the above Robbins-Siegmund Lemma that

$$|Y_n - y_*|^2 \longrightarrow L_\infty \in L^1(\mathbb{P}) \quad \text{and} \quad \sum_{n\geq 1} \gamma_n (h(Y_{n-1})|Y_{n-1} - y_*) < +\infty \quad \mathbb{P} \text{ a.s.}$$

Furthermore, $(|Y_n - y_*|^2)_{n\geq 0}$ is $L^1(\mathbb{P})$-bounded.

Let $\omega \in \Omega$ such that $|Y_n(\omega) - y_*|^2$ converges in $\mathbb{R}_+$ and $\sum_{n\geq 1} \gamma_n (Y_{n-1}(\omega) - y_*|h(Y_{n-1})) < +\infty$. Since $\sum_{n\geq 1} \gamma_n (Y_{n-1}(\omega) - y_*|h(Y_{n-1}(\omega))) < +\infty$, it follows that

$$\liminf_n (Y_{n-1}(\omega) - y_*|h(Y_{n-1}(\omega))) = 0.$$

One may assume the existence of a subsequence $(\phi(n,\omega))_n$ such that

$$(Y_{\phi(n,\omega)}(\omega) - y_*|h(Y_{\phi(n,\omega)}(\omega))) \xrightarrow{n\to+\infty} 0 \quad \text{and} \quad Y_{\phi(n,\omega)}(\omega) \to y_\infty = y_\infty(\omega).$$

If $\liminf_n (Y_{n-1}(\omega) - y_*|h(Y_{n-1}(\omega))) > 0$ the convergence of the series induces a contradiction with $\sum_{n\geq 1} \gamma_n = +\infty$. Now, up to one further extraction, one may assume since $(Y_n(\omega))$ is bounded, that $Y_{\phi(n,\omega)}(\omega) \to y_\infty$. It follows that $(y_\infty - y_*|h(y_\infty)) = 0$ which in turn implies that $y_\infty = y_*$. Now

$$\lim_n |Y_n(\omega) - y_*|^2 = \lim_n |Y_{\phi(n,\omega)}(\omega) - y_*|^2 = 0.$$

Finally, for every $p \in (0,2)$, $(|Y_n - y_*|^p)_{n\geq 0}$ is $L^{\frac{2}{p}}(\mathbb{P})$-bounded, hence uniformly integrable. As a consequence the *a.s.* convergence holds in L^1 *i.e.* $Y_n \to y_*$ converges in $L^p(\mathbb{P})$.

(b) One may apply Robbins-Siegmund's Lemma with L as Lyapunov function since since $(h|\nabla L)(x) = |\nabla L(x)|^2 \geq 0$. The assumption on H is but the quadratic linear growth assumption. As a consequence

$$L(Y_n) \longrightarrow L_\infty \in L^1(\mathbb{P}) \quad \text{and} \quad \sum_{n\geq 1} \gamma_n |\nabla L(Y_{n-1})|^2 < +\infty \quad \mathbb{P}\text{-}a.s.$$

Let $\omega \in \Omega$ such that $L(Y_n(\omega))$ converges in $\mathbb{R}_+$ and $\sum_{n\geq 1}\gamma_n|\nabla L(Y_{n-1}(\omega))|^2 < +\infty$ (and $Y_n(\omega) - Y_{n-1}(\omega) \to 0$). The same argument as above shows that

$$\liminf_n |\nabla L(Y_n(\omega))|^2 = 0.$$

From the convergence of $L(Y_n(\omega))$ toward $L_\infty(\omega)$ and $\lim_{|y|\to\infty} L(y) = +\infty$, one derives the boundedness of $(Y_n(\omega))_{n\geq 0}$. Then there exists a subsequence $(\phi(n,\omega))_{n\geq 1}$ such that $Y_{\phi(n,\omega)} \to \widetilde{y}$ and

$$\lim_n \nabla L(Y_{\phi(n,\omega)}(\omega)) = 0 \quad \text{and} \quad L(Y_{\phi(n,\omega)}(\omega)) \to L_\infty(\omega).$$

Then $\nabla L(\widetilde{y}) = 0$ which implies $\widetilde{y} = y_*$. Then $L_\infty(\omega) = L(y_*)$. The function L being non-negative, differentiable and going to infinity at infinity, it implies that L reaches its unique global minimum at y_*. In particular $\{L = L(y_*)\} = \{\nabla L = 0\} = \{y_*\}$. Consequently the only possible limiting value for the bounded sequence $(Y_n(\omega))_{n\geq 1}$ is y_* *i.e.* $Y_n(\omega)$ converges toward y_*.

The $L^p(\mathbb{P})$-convergence to 0 of $|\nabla L(Y_n)|$, $p \in (0,2)$, follows by the same uniformly integrability argument like in (a). $\diamond$

$\triangleright$ **Exercises. 1.** Show that Claim (a) remains true if one only assume that

$$y \longmapsto (h(y)|y - y_*) \quad \text{is lower semi-continuous.}$$

2 (Non-homogeneous L^2-strong law of large numbers by stochastic approximation). Let $(Z_n)_{n\geq 1}$ be an i.i.d. sequence of square integrable random vectors. Let $(\gamma_n)_{n\geq 1}$ be a sequence of positive real numbers satisfying the decreasing step Assumption (6.7). Show that the recursive procedure defined by

$$Y_{n+1} = Y_n - \gamma_{n+1}(Y_n - Z_{n+1})$$

a.s. converges toward $y_* = \mathbb{E}\, Z_1$.

The above settings are in fact some special cases of a more general result the so-called "pseudo-gradient setting" stated below. However its proof, in particular in a multi-dimensional setting, needs additional arguments, mainly the so-called *ODE* method (for Ordinary Differential Equation) originally introduced by Ljung (see [103]). The underlying idea is to consider that a stochastic algorithm is a perturbed Euler scheme with a decreasing step of the *ODE* $\dot{y} = -h(y)$. For a detailed proof we refer to classical textbooks on Stochastic Approximation like [21, 43, 89].

Theorem 6.3 (PSEUDO-STOCHASTIC GRADIENT) *Assume that L and h and the step sequence $(\gamma_n)_{n\geq 1}$ satisfy all the assumptions of the Robbins-Siegmund Lemma. Assume furthermore that*

$$\lim_{yn\to+\infty} L(y) = +\infty \quad \text{and} \quad (\nabla L|h) \ \text{is lower semi-continuous.}$$

Then, $\mathbb{P}(d\omega)$-a.s. there exists $v = v(\omega) \geq 0$ and a connected component $\mathcal{Y}_\infty(\omega)$ of $\{(\nabla L|h) = 0\} \cap \{L = v\}$ such that

$$\text{dist}(Y_n(\omega), \mathcal{Y}_\infty(\omega)) \longrightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad \text{as } n \to +\infty.$$

In particular, if for every $v \geq 0$, $\{(\nabla L|h) = 0\} \cap \{L = v\}$ is locally finite ([1]), then

$\mathbb{P}$-a.s., there exists ℓ_∞ such that Y_n converges toward a point of $\{(\nabla L|h) = 0\} \cap \{L = \ell_\infty\}$.

---

[1]By *locally finite* we mean "finite on every compact set".

Proof (One-dimensional case). We consider $\omega \in \Omega$ for which all the conclusions of Robbins-Siegmund Lemma are true. Combining $Y_n(\omega) - Y_{n-1}(\omega) \to 0$ with the boundedness of the sequence $(Y_n(\omega))_{n \geq 1}$, one can show that the set $\mathcal{Y}_\infty(\omega)$ of the limiting values of $(Y_n(\omega))_{n \geq 0}$ is a connected compact set ([2]).

On the other hand, $\mathcal{Y}_\infty(\omega) \subset \{L = L_\infty(\omega)\}$ since $L(Y_n(\omega)) \to L_\infty(\omega)$. Furthermore, reasoning like in the proof of claim (b) of the above Corollary shows that there exists a limiting value $y_* \in \mathcal{Y}_\infty(\omega)$ such that $(\nabla L(y_*)|h(y_*)) = 0$ so that $y_* \in \{(\nabla L|h) = 0\} \cap \{L = L_\infty(\omega)\}$.

At this stage, we do assume that $d = 1$. Either $\mathcal{Y}_\infty = \{y_*\}$ and the proof is complete. Or $\mathcal{Y}_\infty(\omega)$ is an interval as a connected subset of $\mathbb{R}$. The function L is constant on this interval, consequently the derivative L' is zero on $\mathcal{Y}_\infty(\omega)$. Hence the conclusion. $\diamond$

6.3   Applications to Finance

6.3.1   Application to recursive variance reduction by importance sampling

This section was originally motivated by the paper [4] but follows the lines of [102] which provides an easier to implement solution. Assume one wants to compute a Gaussian expectation $\mathbb{E}\,\varphi(Z)$, namely

$$\mathbb{E}\,\varphi(Z) = \int_{\mathbb{R}^d} \varphi(z) e^{-\frac{|z|^2}{2}} \frac{dz}{(2\pi)^{\frac{d}{2}}}.$$

In order to deal with a consistent problem, we assume throughout this section that

$$\mathbb{P}(\varphi(Z) > 0) > 0.$$

A typical example is provided by an option pricing in a d-dimensional Black-Scholes model where, with the usual notations

$$\varphi(z) = e^{-rT} \phi\left(\left(x_i\, e^{(r-\frac{\sigma_i^2}{2})T + \sigma_i \sqrt{T}(Az)_i}\right)_{1 \leq i \leq d}\right)$$

where A is a lower triangular matrix such that the covariance matrix $R = AA^*$ has diagonal entries equal to 1 and ϕ is a (non-negative) continuous payoff function. This can also be the Gaussian vector resulting from the Euler scheme of the diffusion dynamics of a risky asset, etc.

Variance reduction by mean translation: first approach (see [4]).

A standard change of variable (Cameron-Martin formula which can be seen here either as a degenerate version of Girsanov change of probability or as an importance sampling procedure) leads to

$$\mathbb{E}\,\varphi(Z) = e^{-\frac{|\theta|^2}{2}} \mathbb{E}\left(\varphi(Z + \theta)e^{-(\theta|Z)}\right). \tag{6.12}$$

---

[2]The method of proof is to first establish that $\mathcal{Y}_\infty(\omega)$ is a "bien enchaîné" set. A subset $A \subset \mathbb{R}^d$ is " bien enchaîné" if for every a, $a' \in A$, every $\varepsilon > 0$, there exists $p \in \mathbb{N}^*$, $b_0, b_1, \ldots, b_p \in A$ such that $b_0 = a$, $b_p = a'$, $|b_i - b_{i-1}| \leq \varepsilon$. Any connected set A is "bien enchaîné" and the converse is true if A is compact.

One natural way to optimize the computation by Monte Carlo simulation of the *Premium* is to choose among the above representations depending on the parameter $\theta \in \mathbb{R}^d$ the one with the lowest variance. This means solving, at least roughly, the following minimization problem

$$\min_{\theta \in \mathbb{R}^d} L(\theta)$$

with

$$L(\theta) = e^{-|\theta|^2} \mathbb{E}\left(\varphi^2(Z+\theta)e^{-2(Z|\theta)}\right)$$

since $\mathrm{Var}(e^{-\frac{|\theta|^2}{2}}\varphi(Z+\theta)e^{-(\theta|Z)}) = L(\theta) - (\mathbb{E}\,\varphi(Z))^2$. A reverse change of variable shows that

$$L(\theta) = e^{\frac{|\theta|^2}{2}} \mathbb{E}\left(\varphi^2(Z)e^{-(Z|\theta)}\right). \tag{6.13}$$

Hence, if $\mathbb{E}\left(\varphi(Z)^2|Z|e^{a|Z|}\right) < +\infty$ for every $a \in (0, \infty)$, one can always differentiate $L(\theta)$ owing to Theorem 2.1(*b*) with

$$\nabla L(\theta) = e^{\frac{|\theta|^2}{2}} \mathbb{E}\left(\varphi^2(Z)e^{-(\theta|Z)}(\theta - Z)\right). \tag{6.14}$$

Rewriting Equation (6.13) as

$$L(\theta) = \mathbb{E}\left(\varphi^2(Z)e^{-\frac{|Z|^2}{2}}e^{\frac{1}{2}|Z-\theta|^2}\right)$$

clearly shows that L is *strictly convex* since $\theta \mapsto e^{\frac{1}{2}|\theta-z|^2}$ is strictly convex for every $z \in \mathbb{R}^d$ (and $\varphi(Z)$ is not identically 0). Furthermore, Fatou's Lemma implies $\lim_{|\theta| \to +\infty} L(\theta) = +\infty$.

Consequently, L has a unique global minimum θ_* which is also local and whence satisfies $\nabla L(\theta_*) = 0$.

We prove now the classical lemma which shows that if L is strictly convex then $\theta \mapsto |\theta - \theta_*|^2$ is a Lyapunov function (in the Robbins-Monro sense defined by (6.11)).

Lemma 6.1 *Let $L : \mathbb{R}^d \to \mathbb{R}_+$ be a differentiable convex function. Then*

$$\forall\, \theta,\, \theta' \in \mathbb{R}^d, \qquad (\nabla L(\theta) - \nabla L(\theta')|\theta - \theta') \geq 0.$$

If furthermore, L is strictly convex, the above inequality is strict as well provided $\theta \neq \theta'$.

Proof. One introduces the differentiable function defined on the unit interval

$$g(t) = L(\theta + t(\theta' - \theta)) - L(\theta), \quad t \in [0, 1].$$

The function g is convex and differentiable. Hence its derivative

$$g'(t) = (\nabla L(\theta + t(\theta' - \theta))|\theta' - \theta)$$

is non-increasing so that $g'(1) \geq g'(0)$ which yields the announced inequality. If L is strictly convex, then $g'(1) > g'(0)$ (otherwise $g'(t) \equiv 0$ which would imply that L is affine on the geometric interval $[\theta, \theta']$). $\diamond$

This suggests (as noted in [4]) to consider the essentially quadratic function W defined by $W(\theta) := |\theta - \theta_*|^2$ as a Lyapunov function instead of L. At least rather than L which is usually not essentially quadratic: as soon as $\varphi(z) \geq \varepsilon_0 > 0$, it is obvious that $L(\theta) \geq \varepsilon_0^2 e^{|\theta|^2}$, $\varepsilon_0 > 0$ but this growth is also observed when φ is bounded away from zero outside a ball. Hence ∇L cannot be Lipschitz continuous either.

If one uses the representation (6.14) of ∇L as an expectation to design a stochastic gradient algorithm (*i.e.* considering the local gradient $H(\theta, z) := \varphi^2(z)e^{-\frac{|z|^2}{2}}\frac{\partial}{\partial\theta}(e^{\frac{1}{2}|z-\theta|^2}))$, the "classical" convergence results like those established in Corollary 6.1 do not apply mainly because the mean linear growth assumption (6.6) is not fulfilled by this choice for H. In fact, this "naive" procedure does explode at almost every implementation as pointed out in [4]. This leads Arouna to introduce in [4] some variants based on repeated re-initializations – the so-called projections "à la Chen" – to force the stabilization of the algorithm and subsequently prevent explosion. An alternative approach has been already explored in [5] where the authors change the function to be minimized, introducing a criterion based on entropy, which turns out to be often close to the original way.

An "unconstrained" approach based on a third change of variable (see [102]).

A new local gradient function: It is often possible to modify this local gradient $H(\theta, z)$ in order to apply the standard results mentioned above. We know and already used above that the Gaussian density is smooth which is not the case of the payoff φ (at least in finance): to differentiate L, we already switched the parameter θ from φ to the Gaussian density by a change of variable (or equivalently an importance sampling procedure). At this stage, we face the converse problem: we know the behaviour of φ at infinity whereas we cannot control efficiently the behaviour of $e^{-(\theta|Z)}$ inside the expectation as θ goes to infinity. The idea is now to cancel this exponential term by plugging back θ in the payoff φ.

The first step is to proceed a new change of variable. Starting from (6.14), one gets

$$\begin{aligned}
\nabla L(\theta) &= e^{\frac{|\theta|^2}{2}} \int_{\mathbb{R}^d} \varphi^2(z)(\theta - z)e^{-(\theta|z)-\frac{|z|^2}{2}}\frac{dz}{(2\pi)^{d/2}} \\
&= e^{|\theta|^2} \int_{\mathbb{R}^d} \varphi^2(\zeta - \theta)(2\theta - \zeta)e^{-\frac{|\zeta|^2}{2}}\frac{d\zeta}{(2\pi)^{d/2}} \quad (z := \zeta - \theta), \\
&= e^{|\theta|^2}\mathbb{E}\left(\varphi^2(Z - \theta)(2\theta - Z)\right).
\end{aligned}$$

Consequently

$$\nabla L(\theta) = 0 \iff \mathbb{E}\left(\varphi^2(Z - \theta)(2\theta - Z)\right) = 0.$$

From now on, we assume that there exist two positive real constants a, $C > 0$ such that

$$0 \leq \varphi(z) \leq Ce^{\frac{a}{2}|z|}, \quad z \in \mathbb{R}^d. \tag{6.15}$$

▷ **Exercises. 1.** Show that under this assumption, $\mathbb{E}\left(\varphi(Z)^2|Z|e^{|\theta||Z|}\right) < +\infty$ for every $\theta \in \mathbb{R}^d$ which implies that (6.14) holds true.

1. Show that in fact $\mathbb{E}\left(\varphi(Z)^2|Z|^m e^{|\theta||Z|}\right) < +\infty$ for every $\theta \in \mathbb{R}^d$ and every $m \geq 1$ which in turn implies that L is $\mathcal{C}^\infty$. In particular, show that for every $\theta \in \mathbb{R}^d$,

$$D^2 L(\theta) = e^{\frac{|\theta|^2}{2}}\mathbb{E}\left(\varphi^2(Z)e^{-(\theta|Z)}\left(I_d + (\theta - Z)(\theta - Z)^*\right)\right)$$

(keep in mind that * stands for transpose). Derive that

$$D^2 L(\theta) \geq e^{\frac{|\theta|^2}{2}} \mathbb{E} \left(\varphi^2(Z) e^{-(\theta|Z)} \right) I_d > 0$$

(in the sense of symmetric matrices) which proves again that L is strictly convex.

Taking this assumption into account, we set

$$H_a(\theta, z) = e^{-a(|\theta|^2+1)^{\frac{1}{2}}} \varphi(z - \theta)^2 (2\theta - z). \tag{6.16}$$

One checks that

$$
\begin{aligned}
\mathbb{E} |H_a(\theta, Z)|^2 &\leq 2C^4 e^{-2a(|\theta|^2+1)^{\frac{1}{2}}} \mathbb{E} \left(e^{2a|Z|+2a|\theta|} (4|\theta|^2 + |Z|^2) \right) \\
&\leq 2C^4 \left(4|\theta|^2 \mathbb{E}(e^{2a|Z|}) + \mathbb{E}(e^{2a|Z|}|Z|^2) \right) \\
&\leq C'(1 + |\theta|^2)
\end{aligned}
$$

since $|Z|$ has a Laplace transform defined on the whole real line which in turn implies $\mathbb{E}(e^{a|Z|}|Z|^m) < +\infty$ for every $a, m > 0$.

On the other hand, it follows that the derived mean function h_a is given by

$$h_a(\theta) = e^{-a(|\theta|^2+1)^{\frac{1}{2}}} \mathbb{E} \left(\varphi(Z - \theta)^2 (2\theta - Z) \right).$$

Note that the function h_a satisfies

$$h_a(\theta) = e^{-a(|\theta|^2+1)^{\frac{1}{2}} - |\theta|^2} \nabla L(\theta)$$

so that h_a is continuous, $(\theta - \theta_* | h_a(\theta)) > 0$ for every $\theta \neq \theta_*$ and $\{h_a = 0\} = \{\theta_*\}$.

Applying Corollary 6.1(a) (called Robbins-Monro Theorem), one derives that for any step sequence $\gamma = (\gamma_n)_{n \geq 1}$ satisfying (6.7), the sequence $(\theta_n)_{n \geq 0}$ defined by

$$\forall n \geq 0, \qquad \theta_{n+1} = \theta_n - \gamma_{n+1} H_a(\theta_n, Z_{n+1}), \tag{6.17}$$

where $(Z_n)_{n \geq 1}$ is an i.i.d. sequence with distribution $\mathcal{N}(0; I_d)$ and θ_0 is an independent $\mathbb{R}^d$-valued random vector, both defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, satisfies

$$\theta_n \xrightarrow{a.s.} \theta_* \qquad \text{as} \quad n \to \infty.$$

Remarks. • The only reason for introducing $\sqrt{|\theta|^2 + 1}$ is that this function behaves like $|\theta|$ (which can be successfully implemented in practice) but is also everywhere differentiable which simplifies the discussion about the rate of convergence detailed further on.

• Note that *no regularity assumption is made on the payoff φ.*

• An alternative based on large deviation principle but which needs some regularity assumption on the payoff φ is developed in [58]. See also [136].

● To prevent a possible "freezing" of the procedure, for example when the step sequence has been misspecified or when the payoff function is too anisotropic, one can replace the above proce-dure (6.17) by the following fully data-driven variant of the algorithm

$$\forall\, n \geq 0, \qquad \widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \gamma_{n+1}\widetilde{H}_a(\widetilde{\theta}_n, Z_{n+1}), \qquad \widetilde{\theta}_0 = \theta_0, \tag{6.18}$$

where

$$\widetilde{H}_a(\theta, z) := \frac{\varphi(z - \theta)^2}{1 + \varphi(-\theta)^2}(2\theta - z).$$

This procedure also converges *a.s.* under sub-multiplicativity assumption on the payoff function φ (see [102]).

● A final – and often crucial – trick to boost the convergence when dealing with rare events as it is often the case with importance sampling is to "drive" a parameter from a "regular" value to the value that makes the event rare. Typically when trying to reduce the variance of a deep-out-of-the-money *Call* option like in the numerical illustrations below, a strategy can be to implement the above algorithm with *a slowly varying strike K_n which goes from $K_0 = x_0$ to the "target" strike K* (see below) during the first iterations.

– RATE OF CONVERGENCE (ABOUT THE): Assume that the step sequence has the following para-metric form $\gamma_n = \dfrac{\alpha}{\beta + n}$, $n \geq 1$, and that $Dh_a(\theta_*)$ is positive in the following sense: all the eigenvalues of $Dh_a(\theta_*)$ have a positive real part. Then, the rate of convergence of θ_n toward θ_* is ruled by a *CLT* (at rate $\sqrt{n}$) if and only if (see 6.4.3)

$$\alpha > \frac{1}{2\Re e(\lambda_{a,\min})} > 0$$

where $\lambda_{a,\min}$ is the eigenvalue of $Dh_a(\theta_*)$ with the lowest real part. Moreover, one can show that the (theoretical...) best choice for α is $\alpha_{opt} := \frac{1}{\Re e(\lambda_{a,\min})}$. (We give in section 6.4.3 some insight about the asymptotic variance.) Let us focus now on $Dh_a(\theta_*)$.

Starting from the expression

$$
\begin{aligned}
h_a(\theta) &= e^{-a(|\theta|^2+1)^{\frac{1}{2}} - |\theta|^2}\nabla L(\theta) \\
&= e^{-a(|\theta|^2+1)^{\frac{1}{2}} - \frac{|\theta|^2}{2}} \times \mathbb{E}\left(\varphi(Z)^2(\theta - Z)e^{-(\theta|Z)}\right) \qquad \text{by } (6.14) \\
&= g_a(\theta) \times \mathbb{E}\left(\varphi(Z)^2(\theta - Z)e^{-(\theta|Z)}\right).
\end{aligned}
$$

Then

$$Dh_a(\theta) = g_a(\theta)\mathbb{E}\left(\varphi(Z)^2 e^{-(\theta|Z)}(I_d + ZZ^t - \theta Z^t)\right) + e^{-\frac{|\theta|^2}{2}}\nabla L(\theta) \otimes \nabla g_a(\theta).$$

(where $u \otimes v = [u_i v_j]_{i,j}$). Using that $\nabla L(\theta_*) = h_a(\theta_*) = 0$ (so that $h_a(\theta_*)(\theta^*)^t = 0$),

$$Dh_a(\theta_*) = g_a(\theta_*)\mathbb{E}\left(\varphi(Z)^2 e^{-(\theta_*|Z)}(I_d + (Z - \theta_*)(Z - \theta_*)^t)\right)$$

Hence $Dh_a(\theta_*)$ is a definite positive symmetric matrix. Its lowest eigenvalue $\lambda_{a,\min}$ satisfies

$$\lambda_{a,\min} \geq g_a(\theta_*)\mathbb{E}\left(\varphi(Z)^2 e^{-(\theta_*|Z)}\right) > 0.$$

These computations show that if the behaviour of the payoff φ at infinity is mis-evaluated, this leads to a bad calibration of the algorithm. Indeed if one considers two real numbers a, a' satisfying (6.15) with $0 < a < a'$, then one checks with obvious notations that

$$\frac{1}{2\lambda_{a,\min}} = \frac{g_{a'}(\theta_*)}{g_a(\theta_*)} \frac{1}{2\lambda_{a',\min}} = e^{(a-a')(|\theta_*|^2+1)^{\frac{1}{2}}} \frac{1}{2\lambda_{a',\min}} < \frac{1}{2\lambda_{a',\min}}.$$

So the condition on α is more stringent with a' than it is with a. Of course in practice, the user does not know these values (since she/he does not know the target θ_*), however she/he will be lead to consider higher values of α than requested which will have as an effect to deteriorate the asymptotic variance (see again 6.4.3).

Implementation in the computation of $\mathbb{E}\,\varphi(Z)$.

At this stage, like for the variance reduction by regression, we may follow two strategies to reduce the variance: "batch" or adaptive.

▷ The *batch strategy* is the simplest and the most elementary one.

Phase 1: One first computes an hopefully good approximation of the optimal variance reducer that we will denote θ_{n_0} for a large enough n_0 (which will remain fixed during the second phase devoted to the computation of $\mathbb{E}\,\varphi(Z)$).

Phase 2: As a second step, one implements a Monte Carlo simulation based on $\varphi(Z+\theta_{n_0})e^{(\theta_{n_0}|Z)-\frac{|\theta_{n_0}|^2}{2}}$ i.e.

$$\mathbb{E}\,\varphi(Z) = \lim_M \frac{1}{M} \sum_{m=n_0+1}^{n_0+M} \varphi(Z_m + \theta_{n_0})e^{-(\theta_{n_0}|Z_m)-\frac{|\theta_{n_0}|^2}{2}}$$

where $(Z_m)_{m \geq n_0+1}$ is an i.i.d. sequence of $\mathcal{N}(0, I_d)$-distributed random vectors. This procedure satisfies a CLT with (conditional) variance $L(\theta_{n_0}) - (\mathbb{E}\varphi(Z))^2$ (given θ_{n_0}).

▷ The *adaptive strategy* is to devise a procedure fully based on the simultaneous computation of the optimal variance reducer and $\mathbb{E}\,\varphi(Z)$ from the same sequence $(Z_n)_{n \geq 1}$ of i.i.d. $\mathcal{N}(0, I_d)$-distributed random vectors used in (6.17). This is approach has been introduced in [4]. To be precise this leads to devise the following adaptive estimator of $\mathbb{E}\,\varphi(Z)$:

$$\frac{1}{M} \sum_{m=1}^{M} \varphi(Z_m + \theta_{m-1})e^{-(\theta_{m-1}|Z_m)-\frac{|\theta_{m-1}|^2}{2}}, \ M \geq 1, \tag{6.19}$$

while the sequence $(\theta_m)_{m \geq 0}$ is obtained by iterating (6.17).

We will briefly prove that the above estimator is unbiased and convergent. Let $\mathcal{F}_m := \sigma(\theta_0, Z_1, \ldots, Z_m)$, $m \geq 0$, be the filtration of the (whole) simulation process. Using that θ_m is $\mathcal{F}_m$-adapted and $Z^{(m+1)}$ is independent of $\mathcal{F}_m$ with the same distribution as Z, one derives classically that

$$\mathbb{E}\left(\varphi(Z^{(m)} + \theta_{m-1})e^{-(\theta_{m-1}|Z^{(m)})-\frac{|\theta_{m-1}|^2}{2}} \,|\, \mathcal{F}_{m-1}\right) = \mathbb{E}\left(\varphi(Z + \theta)e^{-(\theta|Z)-\frac{|\theta_{m-1}|^2}{2}}\right)_{|\theta=\theta_{m-1}} = \mathbb{E}\,\varphi(Z).$$

As a first consequence, the estimator defined by (6.19) is unbiased. Now let us define the $(\mathcal{F}_M)$-martingale

$$N_M := \sum_{m=1}^{M} \frac{\varphi(Z_m + \theta_{m-1})e^{-(\theta_{m-1}|Z_m) - \frac{|\theta_{m-1}|^2}{2}} - \mathbb{E}\,\varphi(Z)}{m} \mathbf{1}_{\{|\theta_{m-1}| \leq m\}}.$$

It is clear that

$$\mathbb{E}\left(\left(\varphi(Z_m + \theta_{m-1})e^{-(\theta_{m-1}|Z_m) - \frac{|\theta_{m-1}|^2}{2}}\right)^2 \Big| \mathcal{F}_{m-1}\right)\mathbf{1}_{\{|\theta_{m-1}| \leq m\}} = L(\theta_{m-1})\mathbf{1}_{\{|\theta_{m-1}| \leq m\}} \xrightarrow{a.s.} L(\theta_*)$$

as $m \to +\infty$ which in turn implies that $(N_m)_{m \geq 1}$ has square integrable increments so that $N_m \in L^2(\mathbb{P})$ for very $m \in \mathbb{N}^*$ and

$$\langle N \rangle_\infty \leq \left(\sup_m L(\theta_m)\right) \sum_{m \geq 1} \frac{1}{m^2} < +\infty \quad a.s.$$

Consequently (see Chapter 11, Proposition 11.4), $N_M \to N_\infty$ where N_∞ is an $a.s.$ finite random variable. Finally, Kronecker's Lemma (see again Chapter 11, Lemma 11.1) implies

$$\frac{1}{M} \sum_{m=1}^{M} \left(\varphi(Z_m + \theta_{m-1})e^{-(\theta_{m-1}|Z_m) - \frac{|\theta_{m-1}|^2}{2}} - \mathbb{E}\,\varphi(Z)\right)\mathbf{1}_{\{|\theta_{m-1}| \leq m\}} \longrightarrow 0 \quad \text{as} \quad M \to +\infty.$$

Since $\theta \to \theta_*$ as $mn \to +\infty$, $\mathbf{1}_{\{|\theta_{m-1}| \leq m\}} = 1$ for large enough m so that

$$\frac{1}{M} \sum_{m=1}^{M} \varphi(Z_m + \theta_{m-1})e^{-(\theta_{m-1}|Z_m) - \frac{|\theta_{m-1}|^2}{2}} \longrightarrow \mathbb{E}\,\varphi(Z) \quad \text{as} \quad M \to +\infty.$$

One can show, using the CLT theorem for triangular arrays of martingale increments (see [70] and Chapter 11, Theorem **??**) that

$$\sqrt{M}\left(\frac{1}{M} \sum_{m=1}^{M} \varphi(Z_m + \theta_{m-1})e^{-(\theta_{m-1}|Z_m) - \frac{|\theta_{m-1}|^2}{2}} - \mathbb{E}\,\varphi(Z)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (\sigma_*)^2)$$

where $(\sigma_*)^2 = L(\theta_*) - (\mathbb{E}\,\varphi(Z))^2$ is the minimal variance.

As set, this second approach seems more performing owing to the asymptotic minimal variance. For practical use, the verdict is more balanced and the batch approach turns out to be quite satisfactory.

NUMERICAL ILLUSTRATIONS. (*a*) *At-the-money Black-Scholes setting.* We consider a vanilla *Call* in *B-S* model

$$X_T = x_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z}, \quad Z \stackrel{d}{=} \mathcal{N}(0; 1)$$

with the following parameters: $T = 1$, $r = 0.10$, $\sigma = 0.5$, $x_0 = 100$, $K = 100$. The Black-Scholes reference price of the Vanilla Call is 23.93.

The recursive optimization of θ has been achieved by running *the data driven version* (6.18) with a sample $(Z_n)_n$ of size 10 000. A *first renormalization has been made prior to the computation*:
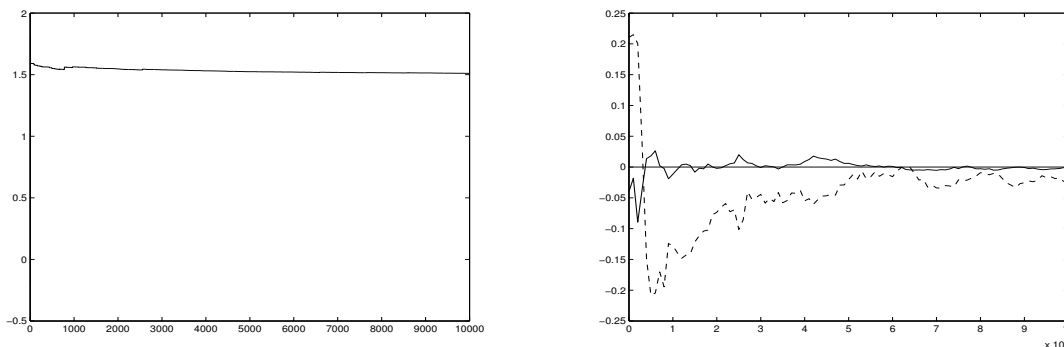
Figure 6.1: *B-S* VANILLA CALL OPTION.$T = 1$, $r = 0.10$, $\sigma = 0.5$, $X_0 = 100$, $K = 100$. *Left: convergence toward θ_* (up to $n = 10\,000$). Right: Monte Carlo simulation of size $M = 10^6$; dotted line: $\theta = 0$, solid line: $\theta = \theta_{10\,000} \approx \theta_*$.*

we considered the equivalent problem (as far as variance reduction si concerned) where the starting values of the asset is 1 and the strike is the *moneyness* K/X_0. The procedure was initialized at $\theta_0 = 1$. (Using (3.9) would have led to set $\theta_0 = -0.2$).

We did not try optimizing the choice of the step γ_n according to the results on the rate of convergence but applied the heuristic rule that if the function H (here H_a) takes its (reasonable) values within a few units, then choosing $\gamma_n = c \frac{20}{20+n}$ with $c \approx 1$ (say *e.g.* $c \in [1/2, 2]$) leads to satisfactory perfomances for the algorithm.

The resulting value $\theta_{10\,000}$ has been used in a standard Monte Carlo simulation of size $M = 1\,000\,000$ based on (6.12) and compared to a regular Monte Carlo simulation (*i.e.* with $\theta = 0$). The numerical results are as follows:

- $\theta = 0$: Confidence interval (95 %) = $[23.92, 24.11]$ (pointwise estimate: 24.02).

- $\theta = \theta_{10\,000} \approx 1.51$: Confidence interval (95 %) = $[23.919, 23.967]$ (pointwise estimate: 23.94).

The gain ratio in terms of standard deviations is $\frac{42.69}{11.01} = 3,88 \approx 4$. This is observed on most simulations we made although to convergence of θ_n may be more chaotic than what may be observed in the figure (the convergence is almost instantaneous). The behaviour of the optimization of θ and the Monte Carlo simulations are depicted in Figure 6.1. The alternative original "parametrized" version of the algorithm ($H_a(\theta, z)$) with $a = 2\sigma\sqrt{T}$ yields quite similar results (when implemented with the same step and the same starting value.

FURTHER COMMENTS: As developed in [123], all what precedes can be extended to non-Gaussian random vectors Z provided their distribution have a log-concave probability density p satisfying for some positive ρ,

$$\log(p) + \rho|\,.\,|^2 \text{ is convex.}$$

One can also replace the mean translation by other importance sampling procedures like those based on the Esscher transform. This has applications *e.g.* when $Z = X_T$ is the value at time T of a process belonging to the family of subordinated (to Brownian motion) Lévy processes *i.e.* of the form $Z_t = W_{Y_t}$ where Y is an increasing Lévy process independent of the standard Brownian motion W (see [22, 146] for more insight on that topic).

6.3.2   Application to implicit correlation search

One considers a 2-dim B-S "toy" model as defined by (2.1) *i.e.* $X_t^0 = e^{rt}$ (riskless asset) and

$$X_t^i = x_0^i e^{(r - \frac{\sigma_i^2}{2})t + \sigma_i W_t^i}, \quad x_0^i > 0, \quad i = 1, 2,$$

for the two risky assets where $\langle W^1, W^2 \rangle_t = \rho t$, $\rho \in [-1, 1]$ denotes the correlation between W^1 and W^2 (that is the correlation between the yields of the risky assets X^1 and X^2).

In this market, we consider a best-of call option characterized by its payoff

$$\left(\max(X_T^1, X_T^2) - K \right)_+ .$$

A market of such best-of calls is a market of the correlation ρ (the respective volatilities being obtained from the markets of vanilla options on each asset as implicit volatilities). In this 2-dimensional B-S setting there is a closed formula for the premium involving the bi-variate standard normal distribution (see [77]), but what follows can be applied as soon as the asset dynamics – or their time discretization – can be simulated (at a reasonable cost, as usual. . .).

We will use a stochastic recursive procedure to solve the inverse problem in ρ

$$P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) = P_{market} \quad \text{[Mark-to-market premium]} \tag{6.20}$$

where

$$
\begin{aligned}
P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) \;\; &:= \;\; e^{-rT} \mathbb{E}\left(\left(\max(X_T^1, X_T^2) - K \right)_+ \right) \\
&= \;\; e^{-rT} \mathbb{E}\left(\left(\max\left(x_0^1 e^{\mu_1 T + \sigma_1 \sqrt{T} Z_1}, x_0^2 e^{\mu_2 T + \sigma_2 \sqrt{T}(\rho Z^1 + \sqrt{1-\rho^2} Z^2)} \right) - K \right)_+ \right)
\end{aligned}
$$

where $\mu_i = r - \frac{\sigma_i^2}{2}$, $i = 1, 2$, $Z = (Z^1, Z^2) \overset{d}{=} \mathcal{N}(0; I_2)$.

We assume from now on that the mark-to-market premium P_{market} is consistent *i.e.* that Equation (6.20) in ρ has at least one solution, say ρ^*. Once again, this is a toy example since in this very setting, more efficient *deterministic* procedures can be called upon, based on the closed form for the option premium. On the other hand, what we propose below is a universal approach.

The most convenient way to prevent edge effects due to the fact that $\rho \in [-1, 1]$ is to use a trigonometric parametrization of the correlation by setting

$$\rho = \cos\theta, \; \theta \in \mathbb{R}.$$

At this stage, note that

$$\sqrt{1 - \rho^2}\, Z^2 = |\sin\theta| Z^2 \overset{d}{=} \sin\theta\, Z^2$$

since $Z^2 \overset{d}{=} -Z^2$. Consequently, as soon as $\rho = \cos\theta$,

$$\rho Z^1 + \sqrt{1 - \rho^2} Z^2 \overset{d}{=} \cos\theta\, Z^1 + \sin\theta\, Z^2$$

owing to the independence of Z^1 and Z^2.

This introduces an over-parametrization (inside $[0, 2\pi]$) since θ and $2\pi - \theta$ yield the same solution, but this is not at all a significant problem for practical implementation: a more careful

examination would show that one equilibrium of these two equilibrium points is "repulsive" and one is "attractive". From now on, for convenience, we will just mention the dependence of the premium function in the variable θ, namely

$$\theta \longmapsto P_{BoC}(\theta) := e^{-rT}\mathbb{E}\left(\left(\max\left(x_0^1 e^{\mu_1 T + \sigma_1 \sqrt{T}Z_1}, x_0^2 e^{\mu_2 T + \sigma_2 \sqrt{T}(\cos\theta\, Z^1 + \sin\theta\, Z^2)}\right) - K\right)_+\right).$$

The function P_{BoC} is a 2π-periodic continuous function. Extracting the implicit correlation from the market amounts to solving

$$P_{BoC}(\theta) = P_{market} \qquad (\text{with } \rho = \cos\theta)$$

where P_{market} is the quoted premium of the option (mark-to-market). We need an additional assumption (which is in fact necessary with almost any zero search procedure): we assume that

$$P_{market} \in (\min_\theta P_{BoC}(\theta), \max_\theta P_{BoC}(\theta))$$

i.e. that the (consistent) value P_{market} is not an extremal value of P_{BoC}. So we are looking for a zero of the function h defined on $\mathbb{R}$ by

$$h(\theta) = P_{BoC}(\theta) - P_{market}.$$

This function admits a representation as an expectation given by

$$h(\theta) = \mathbb{E}\, H(\theta, Z)$$

where $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is defined for every $\theta \in \mathbb{R}$ and every $z = (z^1, z^2) \in \mathbb{R}^2$ by

$$H(\theta, z) = e^{-rT}\left(\max\left(x_0^1 e^{\mu_1 T + \sigma_1 \sqrt{T}z^1}, x_0^2 e^{\mu_2 T + \sigma_2 \sqrt{T}(z^1 \cos\theta + z^2 \sin\theta)}\right) - K\right)_+ - P_0^{market}$$

and $Z = (Z^1, Z^2) \overset{d}{=} \mathcal{N}(0; I_2)$.

Proposition 6.1 *Under the above assumptions made on the P_{market} and the function P_{BoC} and if, moreover, the equation $P_{BoC}(\theta) = P_{market}$ has finitely many solutions on $[0, 2\pi]$, the stochastic zero search recursive procedure defined by*

$$\theta_{n+1} = \theta_n - \gamma_{n+1}H(\theta_n, Z_{n+1}), \ \theta_0 \in \mathbb{R}, \qquad \text{with} \qquad (Z_n)_{n\geq 1} \quad i.i.d. \ \mathcal{N}(0; I_2)$$

with a step sequence satisfying the decreasing step assumption (6.7) a.s. converges toward θ_* solution to $P_{BoC}(\theta) = P_{market}$.

Proof. For every $z \in \mathbb{R}^2$, $\theta \mapsto H(\theta, z)$ is continuous, 2π-periodic and dominated by a function $g(z)$ such that $g(Z) \in L^1(\mathbb{P})$ (g is obtained by replacing $z^1 \cos\theta + z^2 \sin\theta$ by $|z^1| + |z^2|$ in the above formula for H). One derives that the mean function h and $\theta \mapsto \mathbb{E}\, H^2(\theta, Z)$ are both continuous and 2π-periodic as well (hence bounded).

The main difficulty to apply the Robbins-Siegmund Lemma is to find out the appropriate Lyapunov function.

The quoted value P_{market} is not an extremum of the function P, hence $\int_0^{2\pi} h^{\pm}(\theta)d\theta > 0$ where $h^{\pm} := \max(\pm h, 0)$. We consider θ_0 any (fixed) solution to the equation $h(\theta) = 0$ and two real numbers $\beta^{\pm}$ such that

$$0 < \beta^+ < \frac{\int_0^{2\pi} h^+(\theta)d\theta}{\int_0^{2\pi} h^-(\theta)d\theta} < \beta^-$$

and we set, for every $\theta \in \mathbb{R}$,

$$\ell(\theta) := h^+(\theta) - \beta^+ h^-(\theta)\mathbf{1}_{\{\theta \geq \theta_0\}} - \beta^- h^-(\theta)\mathbf{1}_{\{\theta \leq \theta_0\}}.$$

The function ℓ is clearly continuous and is 2π-periodic "on the right" on $[\theta_0, +\infty)$ and "on the left" on $(-\infty, \theta_0]$. In particular it is a bounded function. Furthermore, owing to the definition of $\beta^{\pm}$,

$$\int_{\theta_0}^{\theta_0 2 + \pi} \ell(\theta)d\theta > 0 \quad \text{and} \quad \int_{\theta_0 - 2\pi}^{\theta_0} \ell(\theta)d\theta < 0$$

so that

$$\lim_{\theta \to \pm\infty} \int_{\theta_0}^{\theta} \ell(u)du = +\infty.$$

As aconsequence, there exists a real constant $C > 0$ such that the function

$$L(\theta) = \int_0^{\theta} \ell(u)du + C$$

is non-negative. Its derivative is given by $L' = \ell$ so that

$$L'h \geq (h^+)^2 + \beta^+(h^-)^2 \geq 0 \quad \text{and} \quad \{L'h = 0\} = \{h = 0\}.$$

It remains to prove that $L' = \ell$ is Lipschitz continuous continuous. Calling upon the usual arguments to interchange expectation and differentiation (*i.e.* Theorem 2.1(b)), one shows that the function P_{BoC} is differentiable at every $\theta\mathbb{R} \setminus 2\pi\mathbb{Z}$ with

$$P'_{BoC}(\theta) = \sigma_2\sqrt{T}\,\mathbb{E}\left(\mathbf{1}_{\{X_T^2 > \max(X_T^1, K)\}}X_T^2(\cos(\theta)Z^2 - \sin(\theta)Z^1)\right).$$

Furthermore

$$|P'_{BoC}(\theta)| \leq \sigma_2\sqrt{T}\,\mathbb{E}\left(x_0^2 e^{\mu_2 T + \sigma_2\sqrt{T}(|Z^1|+|Z^2|)}(|Z^2| + |Z^1|)\right) < +\infty$$

so that P_{Boc} is clearaly Lipschitz continuous continuous on the interval $[0, 2\pi]$ hence on the whole real line by periodicity. Consequently h and $h^{\pm}$ are Lipschitz continuous continuous which implies in turn that ℓ is Lipschitz continuous as well.

Moreover, one can show that the equation $P_{BoC}(\theta) = P_{market}$ has finitely many solutions on every interval of length 2π.

One may apply Theorem 6.3 (for which we provide a self-contained proof in one dimension) to derive that θ_n will converge toward a solution θ_* of the equation $P_{BoC}(\theta) = P_{market}$. $\diamond$
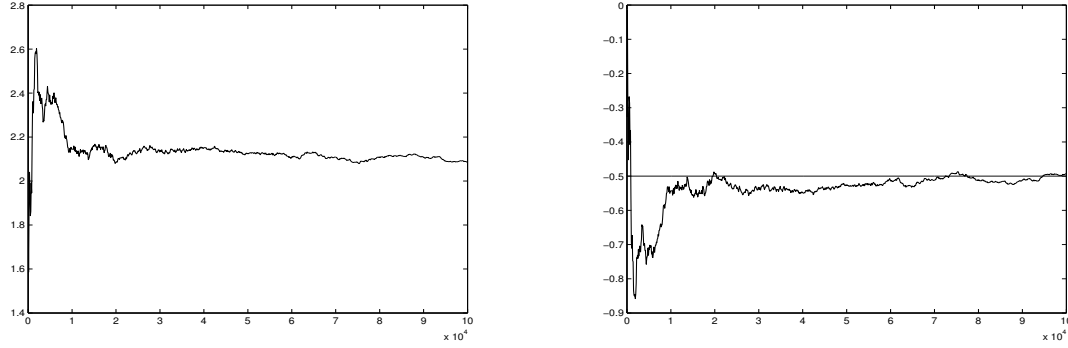
Figure 6.2: *B-S* BEST-OF-CALL OPTION. $T = 1$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.30$, $X_0^1 = X_0^2 = 100$, $K = 100$. *Left: convergence of θ_n toward a θ_* (up to $n = 100\,000$). Right: convergence of $\rho_n := \cos(\theta_n)$ toward -0.5*

Exercises. 1. Show that if $x_/^1 = x_0^2$ or $\sigma_1 \neq \sigma_2$, then P_{BoC} is continuously differentiable on the whole real line.

2. Extend what precedes to any payoff $\varphi(X_T^1, X_T^2)$ where $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a Lipschitz continuous function. In particular, show without the help of differentiation that the corresponding function $\theta \mapsto P(\theta)$ is Lipschitz continuous continuous.

NUMERICAL EXPERIMENT. We set the model parameters to the following values

$$x_0^1 = x_0^2 = 100, \quad r = 0.10, \quad \sigma_1 = \sigma_2 = 0.30, \quad \rho = -0.50$$

and the payoff parameters

$$T = 1, \ K = 100.$$

The reference "Black-Scholes" price 30.75 is used as a market price so that the target of the stochastic algorithm is $\theta_* \in \text{Arccos}(-0.5)$. The stochastic approximation procedure parameters are

$$\theta_0 = 0, \quad n = 10^5.$$

The choice of θ_0 is "blind" on purpose. Finally we set $\gamma_n = \frac{0.5}{n}$. No re-scaling of the procedure has been made in the below example.

n	$\rho_n := cos(\theta_n)$
1000	-0.5606
10000	-0.5429
25000	-0.5197
50000	-0.5305
75000	-0.4929
100000	-0.4952

▷ **Exercise (another toy-example: extracting *B-S*-implied volatility by stochastic approximation).** Devise a similar procedure to compute the implied volatility in a standard Black-Scholes model (starting at $x > 0$ at $t = 0$ with interest rate r and maturity T).

(*a*) Show that the B-S premium $C_{BS}(\sigma)$ is even, increasing on $[0, +\infty)$ and continuous as a function of the volatility. Show that $\lim_{\sigma \to 0} C_{BS}(\sigma) = (x - e^{-rT}K)_+$ and $\lim_{\sigma \to +\infty} C_{BS}(\sigma) = x$.

(*b*) Derive from (*a*) that for any market price $P_{market} \in [(x - e^{-rT}K)_+, x]$ there is a unique B-S implicit volatility for this price.

(*b*) Consider, for every $\sigma \in \mathbb{R}$,

$$H(\sigma, z) = \chi(\sigma) \left(x e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}z} - K e^{-rT} \right)_+$$

where $\chi(\sigma) = (1 + |\sigma|)e^{-\frac{\sigma^2}{2}T}$. Justify carefully this choice for H and implement the algorithm with $x = K = 100$, $r = 0.1$ and a market price equal to 16.73. Choose the step parameter of the form $\gamma_n = \frac{c}{x}\frac{1}{n}$ with $c \in [0.5, 2]$ (this is simply a suggestion).

Warning. The above exercise is definitely a toy exercise! More efficient methods for extracting standard implied volatility are available (see *e.g.* [109] which is based on a Newton algorithm; a dichotomy approach is also very efficient).

▷ **Exercise (Extension to more general asset dynamics).** One considers now a couple of risky assets following two correlated local volatility models,

$$dX_t^i = X_t^i(rdt + \sigma_i(X_t)dW_t^i), \ X_0^1 = x^1 > 0, \ i = 1, 2,$$

where the functions $\sigma_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ are Lipschitz continuous continuous and bounded and the Brownian motions W^1 and W^2 are correlated with correlation $\rho \in [-1, 1]$ so that

$$d\langle W^1, W^2\rangle_t = \rho dt.$$

(This ensures the existence and uniqueness of strong solutions for this SDE, see Chapter 7.)

Assume that we know how to simulate (X_T^1, X_T^2), either exactly, or at least as an approximation by an Euler scheme from a d-dimensional normal vector $Z = (Z^1, \ldots, Z^d) \stackrel{d}{=} \mathcal{N}(0; I_d)$.

Show that the above approach can be extended *mutatis mutandis*.

6.3.3   Application to correlation search (II): reducing variance using higher order Richardson-Romberg extrapolation

This section requires the reading of Section 7.7, in which we introduced the higher order Richardson-Romberg extrapolation (of order $R = 3$). We showed that considering consistent increments in the (three) Euler schemes made possible to control the variance of the Richardson-Romberg estimator of $\mathbb{E}(f(X_T))$. However, we mentioned, as emphasized in [122], that this choice cannot be shown to be optimal like at the order $R = 2$ corresponding to standard Richardson-Romberg extrapolation.

Stochastic approximation can be a way to search for the optimal correlation structure between the Brownian increments. For the sake of simplicity, we will start from the continuous time diffusion dynamics. Let

$$dX_t^i = b(X_t^i)\,dt + \sigma(X_t^i)\,dB_t^i, \quad i = 1, 2, 3,$$

where $B = (B^1, B^2, B^3)$ is a correlated Brownian motion with (symmetric) correlation matrix $R = [r_{ij}]_{1 \leq i,j \leq 3}$ ($r_{ii} = 1$). We consider the 3rd-order Richardson-Romberg weights α_i, $i = 1, 2, 3$. The expectation $\mathbb{E} f(X_T)$ is estimated by $\mathbb{E}\left(\alpha_1 f(\bar{X}_T^1) + \alpha_2 f(\bar{X}_T^2) + \alpha_3 f(\bar{X}_T^3)\right)$ using a Monte Carlo simulation. So our objective is then to minimize the variance of this estimator *i.e.* solving the problem

$$\min_R \mathbb{E}\left(\alpha_1 f(\bar{X}_T^1) + \alpha_2 f(\bar{X}_T^2) + \alpha_3 f(\bar{X}_T^3)\right)^2$$

which converges asymptotically (*e.g.* if f is continuous) toward the problem

$$\min_R \mathbb{E}\left(\alpha_1 f(X_T^1) + \alpha_2 f(X_T^2) + \alpha_3 f(X_T^3)\right)^2 \tag{6.21}$$

or equivalently since $\mathbb{E}(f(X_Y^i)^2)$ does not depend on R to

$$\min_R \mathbb{E}\left(\sum_{1 \leq i < j \leq 3} \alpha_i \alpha_j f(X_T^i) f(X_T^j)\right).$$

The starting idea is to use the trigonometric representation of covariance matrices derived from the Cholesky decomposition (see *e.g.* [118]):

$$R = T T^*, \quad T \text{ lower triangular},$$

so that $B = T W$ where W is a standard 3-dimensional Brownian motion.

The entries t_{ij} of T satisfy $\sum_{1 \leq j \leq i} t_{ij}^2 = r_{ii} = 1$, $i = 1, 2, 3$, so that one may write for $i = 2, 3$,

$$t_{ij} = \cos(\theta_{ij}) \prod_{k=1}^{j-1} \sin(\theta_{ik}), \ 1 \leq j \leq i - 1, \ t_{ii} = \prod_{k=1}^{i-1} \sin(\theta_{ik}).$$

From now, we will denote $T = T(\theta)$ where $\theta = [\theta_{ij}]_{1 \leq j < i \leq 2} \in \mathbb{R}^3$.

We assume that b and σ are smooth enough (say $\mathcal{C}_b^{1+}$ ([3]), see [87], Theorem 3.1) to ensure the existence of the θ-sensitivity process $\left[\frac{\partial X_T}{\partial \theta_{ij}}\right]_{1 \leq j < i \leq 2}$ satisfying the following linear stochastic differential system: for every $i = 2, 3$ and every $j \in \{1, \ldots, i-1\}$,

$$d\left(\frac{\partial X_t^i(\theta)}{\partial \theta_{ij}}\right) = b_x'(X_t^i(\theta))\frac{\partial X_t^i(\theta)}{\partial \theta_{ij}}dt + \sigma_x'(X_t^i(\theta))\frac{\partial X_t^i(\theta)}{\partial \theta_{ij}}(T(\theta)dW_t)^i + \sigma(X_t^i(\theta)) \sum_{1 \leq j' \leq i-1} \frac{\partial T_{ij'}(\theta_{ij})}{\partial \theta}dW_t^{j'}.$$

In practice at that stage one switches to the Euler schemes $\bar{X}^i$ of X^i with steps $\frac{T}{in}$ $i = 1, 2, 3$ respectively. We proceed likewise with the θ-sensitivity processes $(\frac{\partial X_t^i}{\partial \theta_{ij}})_{1 \leq j \leq i-1}$ which Euler scheme is denoted $(\frac{\partial \bar{X}_t^i}{\partial \theta_{ij}})_{1 \leq j \leq i-1}$.

Suppose now that $f : \mathbb{R} \to \mathbb{R}$ is a smooth enough (say differentiable or only λ-a.s. if $\mathcal{L}(\bar{X}_T)$ has a density). Then, setting $\Phi_f(x) = (\sum_{i=1}^3 \alpha_i f(x^i))^2$, we define the *potential function* $L : \mathbb{R}^3 \to \mathbb{R}$ by

$$L(\theta) := \mathbb{E}(\Phi_f(X_T(\theta))).$$

---

[3]This means that the derivatives of the function are bounded and α-Hölder for some $\alpha > 0$.

Our aim is to minimize this potential function or at least to exhibit values of θ such that

$$L(\theta) < L\big((0,0,0)\big)$$

i.e. to do better than considering Euler schemes with consistent increments. Theoretical results from [122] strongly suggest that such θ do exist as well as empirical evidence.

The function L is differentiable and its gradient at θ is given by

$$\nabla L(\theta) := \mathbb{E}\left[\nabla\Phi_f(\bar{X}_T(\theta))^t\,\frac{\partial\bar{X}_T}{\partial\theta}\right]_{1\le j < i \le 3}.$$

Then, one devises the resulting stochastic gradient procedure $(\Theta_m)_{m\ge 0}$ recursively defiened by

$$\Theta_{m+1} = \Theta_n - \gamma_{m+1}\nabla\Phi_f(\bar{X}_T^{(m)}(\Theta_m))^t\,\frac{\partial\bar{X}_T^{(m)}}{\partial\theta}(\Theta_m), \quad \Theta_0 \in (0,2\pi)^3$$

where $\left(\bar{X}_T^{(m)}, \frac{\partial\bar{X}_T^{(m)}}{\partial\theta}\right)$, $m \ge 1$ are independent copies of the joint Euler scheme at time T, computed using the covariance matrix Θ_n.

By contruction the function L plays the role of a Lyapunov function for the algorithm (except for its behaviour at infinity...).

Under natural assumptions, one shows that Θ_n *a.s.* converges toward a random vector Θ^* which takes values in the zero of ∇L. Although the function L is bounded, one can prove this convergence by taking advantage of the periodicity of L by applying *e.g*; results from [50]. Then relying on results on traps (see [29, 96], etc) in the subset of local minima of L. Hopefully it may converge to the global minimum of L or at least induce a significant variance reduction with respect to the Richardson-Romberg-Richardson extrapolation carried out with consistent Brownian increment (i.e. simulated form a unique underlying Brownian motion).

NUMERICAL ILLUSTRATION: This is illustrated by the numerical experiments carried out by A. Grangé-Cabane in [68].

[In progress...]

6.3.4   The paradigm of model calibration by simulation

Let $\Theta \subset \mathbb{R}^d$ be an open convex set of $\mathbb{R}^d$. Let

$$\begin{aligned}Y : (\Theta \times \Omega, \mathcal{B}or(\Theta) \otimes \mathcal{A}) &\longrightarrow (\mathbb{R}^p, \mathcal{B}or(\mathbb{R}^p)) \\ (\theta, \omega) &\longmapsto Y_\theta(\omega) = (Y_\theta^1(\omega), \ldots, Y_\theta^p(\omega))\end{aligned}$$

be a random vector representative of p payoffs, "re-centered" by their mark-to-market price (see examples below). In particular, for every $i \in \{1, \ldots, p\}$, $\mathbb{E}\, Y_\theta^i$ is representative of the error between the "theoretical price obtained with parameter θ and the quoted price. To make the problem consistent we assume throughout this section that

$$\forall\,\theta \in \Theta, \quad Y_\theta \in L^1_{\mathbb{R}^p}(\Omega, \mathcal{A}, \mathbb{P}).$$

Let $S \in \mathcal{S}_+(p, \mathbb{R}) \cap GL(p, \mathbb{R})$ be a (positive definite) matrix. The resulting inner product is defined by

$$\forall\, u,\, v \in \mathbb{R}^p, \quad \langle u|v \rangle_S := u^* S v$$

and the associated Euclidean norm $|\,.\,|_S$ by $|u|_S := \sqrt{\langle u|u \rangle_S}$.

A natural choice for the matrix S can be a simple diagonal matrix $S = \mathrm{Diag}(w_1, \ldots, w_p)$ with "weights" $w_i > 0$, $i = 1, \ldots, p$.

The *paradigm* of model calibration is to find the parameter θ_* that minimizes the "aggregated error" with respect to the $|\,.\,|_S$-norm. This leads to the following minimization problem

$$(C) \quad \equiv \quad \mathrm{argmin}_\theta |\mathbb{E}\, Y_\theta|_S = \mathrm{argmin}_\theta \frac{1}{2} |\mathbb{E}\, Y_\theta|_S^2.$$

Here are two simple examples to illustrate this somewhat abstract definition.

Examples. 1. *Black-Scholes model.* Let, for any $x, \sigma > 0$, $r \in \mathbb{R}$,

$$X_t^{x,\sigma} = x e^{(r - \frac{\sigma^2}{2})t + \sigma W_t}, \ t \geq 0,$$

where W is a standard Brownian motion. Then let $(K_i, T_i)_{i=1,\ldots,p}$ be p couples "maturity-strike price". Set

$$\theta := \sigma, \qquad \Theta := (0, \infty)$$

and

$$Y_\theta := \left(e^{-rT_i}(X_{T_i}^{x,\sigma} - K_i)_+ - P_{market}(T_i, K_i) \right)_{i=1,\ldots,p}$$

where $P_{market}(T_i, K_i)$ is the mark-to-market price of the option with maturity $T_i > 0$ and strike price K_i.

2. *Merton model (mini-krach).* Now, for every $x, \sigma, \lambda > 0$, $a \in (0,1)$, set

$$X_t^{x,\sigma,\lambda,a} = x\, e^{(r - \frac{\sigma^2}{2} + \lambda a)t + \sigma W_t}(1-a)^{N_t}, \ t \geq 0$$

where W is as above and $N = (N_t)_{t \geq 0}$ is a standard Poisson process with jump intensity λ. Set

$$\theta = (\sigma, \lambda, a), \qquad \Theta = (0, +\infty)^2 \times (0,1),$$

and

$$Y_\theta = \left(e^{-rT_i}(X_{T_i}^{x,\sigma,\lambda,a} - K_i)_+ - P_{market}(T_i, K_i) \right)_{i=1\ldots,p}.$$

We will also have to make simulability assumptions on Y_θ and if necessary its derivatives with respect to θ (see below) otherwise our simulation based approach would be meaningless.

At this stage, basically two approaches can be considered as far as solving this problem by simulation is concerned:

- A Robbins-Siegmund zero search approach of ∇L which needs to have access to a *representation of the gradient* – supposed to exist –as an expectation of the function L.

- A more direct treatment based on the so-called *Kiefer-Wolfowitz procedure* which is a kind counterpart of the Robbins-Siegmund approach based on a finite difference method (with decreasing step) which does not require the existence of a representation of ∇L as an expectation.

The Robbins-Siegmund approach

We make the following assumptions: for every $\theta_0 \in \Theta$,

(i) $\mathbb{P}(d\omega)$-a.s. $\theta \longmapsto Y_\theta(\omega)$ is differentiable at θ_0 with Jacobian $\partial_{\theta_0} Y_\theta(\omega)$,

(ii) $\exists\, U_{\theta_0}$, neighbourhood of θ_0 in Θ, such that $\left(\dfrac{Y_\theta - Y_{\theta_0}}{|\theta - \theta_0|} \right)_{\theta \in U_{\theta_0} \setminus \{\theta_0\}}$ is uniformly integrable.

One checks – using the exercise "Extension to uniform integrability" that follows Theorem 2.1 – that $\theta \longmapsto \mathbb{E} Y_\theta$ is differentiable and that its Jacobian is given by

$$\partial_\theta \mathbb{E}\, Y_\theta = \mathbb{E}\, \partial_\theta Y_\theta.$$

Then, the function L is differentiable everywhere on Θ and its gradient (with respect to the canonical Euclidean norm) is given by

$$\forall\, \theta \in \Theta, \quad \nabla L(\theta) = \mathbb{E}\, (\partial_\theta Y_\theta)^* \, S \, \mathbb{E}\, Y_\theta = \mathbb{E}\big((\partial_\theta Y_\theta)^*\big)\, S \, \mathbb{E}\, Y_\theta.$$

At this stage we need a representation of $\nabla L(\theta)$ as an expectation. To this end, we construct for every $\theta \in \Theta$, an independent copy $\widetilde{Y}_\theta$ of Y_θ defined as follows: we consider the product probability space $(\Omega^2, \mathcal{A}^{\otimes 2}, \mathbb{P}^{\otimes 2})$ and set, for every $(\omega, \widetilde{\omega}) \in \Omega^2$, $Y_\theta(\omega, \widetilde{\omega}) = Y_\theta(\omega)$ (extension of Y_θ on Ω^2 still denoted Y_θ) and $\widetilde{Y}_\theta(\omega, \widetilde{\omega}) = Y_\theta(\widetilde{\omega})$. It is straightforward by the product measure Theorem that the two families $(Y_\theta)_{\theta \in \Theta}$ and $(\widetilde{Y}_\theta)_{\theta \in \Theta}$ are independent with the same distribution. From now on we will make the usual abuse of notations consisting in assuming that these two independent copies live on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Now, one can write

$$\forall\, \theta \in \Theta, \quad \nabla L(\theta) \;=\; \mathbb{E}\big((\partial_\theta Y_\theta)^*\big)\, S \, \mathbb{E}\, \widetilde{Y}_\theta$$
$$=\; \mathbb{E}\Big((\partial_\theta Y_\theta)^* \, S \, \widetilde{Y}_\theta\Big).$$

The standard situation, as announced above, is that Y_θ is a vector of payoffs written on d traded risky assets, recentered by their respective quoted prices. The model dynamics of the d risky assets depends on the parameter θ, say

$$Y_\theta = \big(F_i(\theta, X_{T_i}(\theta))\big)_{i=1,\dots,p}$$

where the price dynamics $(X(\theta)_t)_{t \geq 0}$, of the d traded assets is driven by a parametrized diffusion process

$$dX(\theta)_t = b(\theta, t, X_t(\theta))\, dt + \sigma(\theta, t, X_t(\theta)).dW_t, \quad X_0(\theta) = x_0(\theta) \in \mathbb{R}^d,$$

where W is an $\mathbb{R}^q$-valued standard Brownian motion defined ona probability space $(\Omega, \mathcal{A}, \mathbb{P})$, b is an $\mathbb{R}^d$-valued vector field defined on $\Theta \times [0, T] \times \mathbb{R}^d$ and σ is an $\mathcal{M}(d, q)$-valued field defined on the same product space, both satisfying appropriate regularity assumptions.

The pathwise differentiability of Y_θ in θ needs that of $X_t(\theta)$ with respect to θ. This question is closely related to the θ-tangent process $\left(\dfrac{\partial X_t(\theta)}{\partial \theta} \right)_{t \geq 0}$ of $X(\theta)$. A precise statement is provided in Section 9.2.2 which ensures that if b and σ are smooth enough with respect to the variable θ,

then such a θ-tangent process does exist and is solution to a linear SDE (involving $X(\theta)$ in its coefficients).

Some differentiability properties are also required on the functions F_i in order to fulfill the above differentiability Assumption (i). As model calibrations on vanilla derivative products performed in Finance, F_i is never everywhere differentiable – typically $F_i(y) := e^{-rT_i}(y-K_i)_+ - P_{market}(T_i, K_i)$ – but, if $X_t(\theta)$ has an absolutely continuous distribution (*i.e.* a probability density) for every time $t > 0$ and every $\theta \in \Theta$, then F_i only needs to be differentiable outside a Lebesgue negligible subset of $\mathbb{R}_+$. Finally, we can write formally

$$H(\theta, W(\omega)) := (\partial_\theta Y_\theta(\omega))^* \, S \, \widetilde{Y}_\theta(\omega)$$

where W stands for an abstract random *innovation* taking values in an appropriate space. We denote by the capital letter W the innovation because when the underlying dynamics is a Brownian diffusion or its Euler-Maruyama scheme, it refers to a finite-dimensional functional of (two independent copies of) the $\mathbb{R}^q$-valued standard Brownian motion on an interval $[0, T]$: either (two independent copies of) $(W_{T_1}, \ldots, W_{T_p})$ or (two independent copies of) the sequence $(\Delta W_{\frac{kT}{n}})_{1 \le k \le n}$ of Brownian increments with step T/n over the interval $[0, T]$. Thus, these increments naturally appear in the simulation of the Euler scheme $(\bar{X}^n_{\frac{kT}{n}}(\theta))_{0 \le k \le n}$ of the process $(X(\theta)_t)_{t \in [0,T]}$ when the latter cannot be simulated dirctly (see Chapter 7 entirely devoted to the Euler scheme of Brownian diffusions). Of course other situations may occur especially when dealing with with jump diffusions where W usually becomes the increment process of the driving Lévy process.

Whatsoever, we make the following reasonable meta-assumption that

– the process W is simulatable,

– the functional $H(\theta, w)$ can be easily computed for any input (θ, w).

Then, one may define recursively the following zero search algorithm for $\nabla L(\theta) = \mathbb{E}\, H(\theta, W)$, by setting

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, W^{n+1})$$

where $(W^n)_{n \ge 1}$ is an i.i.d. sequence of copies of W and $(\gamma_n)_{n \ge 1}$ is a sequence of steps satisfying the usual decreasing step assumptions

$$\sum_n \gamma_n = +\infty \qquad \text{and} \qquad \sum_n \gamma_n^2 < +\infty.$$

In such a general framework, of course, one cannot ensure that the function s L and H will satisfy the basic assumptions needed to make stochastic gradient algorithms converge, typically

$$\nabla L \text{ is Lipschitz continuous continuous and} \quad \forall\, \theta \in \Theta, \quad \|H(\theta, .)\|_2 \le C(1 + \sqrt{L(\theta)})$$

or one of their numerous variants (see *e.g.* [21] for a large overview of possible assumptions). However, in many situations, one can make the problem fit into a converging setting by an appropriate change of variable on θ or by modifying the function L and introducing an appropriate explicit (strictly) positive "weight function " $\chi(\theta)$ that makes the product $\chi(\theta)H(\theta, W(\omega))$ fit with these requirements.

Even though, the topological structure of the set $\{\nabla L = 0\}$ can be nontrivial, in particular disconnected. Nonetheless, as seen in Proposition 6.3, one can show, under natural assumptions, that

θ_n converges to a connected component of $\{\chi|\nabla L|^2 = 0\} = \{\nabla L = 0\}$.

The next step is that if ∇L has several zeros, they cannot all be local minima of L especially when there are more than two of them (this is a consequence of the well-known Mountain-Pass Lemma, see [79]). Some are local maxima or saddle points of various kinds. These equilibrium points which are not local minima are called *traps*. An important fact is that, under some non-degeneracy assumptions on H at such a parasitic equilibrium point θ_∞ (typically $\mathbb{E}\, H(\theta_\infty, W)^* H(\theta_\infty, W)$ is positive definite at least in the direction of an unstable manifold of h at θ_∞), the algorithm will *a.s.* never converge toward such a "trap". This question has been extensively investigated in the literature in various settings for many years (see [96, 29, 135, 20, 51]).

A final problem may arise due to the incompatibility between the geometry of the parameter set Θ and the above recursive algorithm: to be really defined by the above recursion, we need Θ to be left stable by (almost) all the mappings $\theta \mapsto \theta - \gamma H(\theta, w)$ at least for γ small enough. If not the case, we need to introduce some constraints on the algorithm by projecting it on Θ whenever θ_n skips outside Θ. This question has been originally investigated in [31] when Θ is a convex set.

Once all these technical questions have been circumvented, we may state the following meta-theorem which says that θ_n *a.s.* converges toward a local minimum of L.

At this stage it is clear that calibration looks like quite a generic problem for stochastic optimization and that almost all difficulties arising in the field of Stochastic Approximation can be encountered when implementing such a (pseudo-)stochastic gradient to solve it.

The Kieffer-Wolfowitz approach

Practical implementations of the Robbins-Siegmund approach point out a specific technical difficulty: the random functions $\theta \mapsto Y_\theta(\omega)$ are not always pathwise differentiable (nor in the $L^r(\mathbb{P})$-sense which could be enough). More important in some way, even if one shows that $\theta \mapsto \mathbb{E}\, Y(\theta)$ is differentiable, possibly by calling upon other techniques (log-likelihood method, Malliavin weights, etc) the resulting representation for $\partial_\theta Y(\theta)$ may turn out to be difficult to simulate, requiring ing much programming care, whereas the random vectors Y_θ can be simulated in a standard way. In such a setting, an alternative is provided by the Kiefer-Wolfowitz algorithm (K-W) which combines the recursive stochastic approximation principle with a finite difference approach to differentiation. The idea is simply to approximate the gradient ∇L by

$$\frac{\partial L}{\partial \theta_i}(\theta) \;\approx\; \frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta^i}, \quad 1 \le i \le p,$$

where $(e_i)_{1 \le i \le p}$ denotes the canonical basis of $\mathbb{R}^p$ and $\eta = (\eta^i)_{1 \le i \le p}$. This finite difference term has an integral representation given by

$$\frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta_i} \;=\; \mathbb{E}\left(\frac{\Lambda(\theta + \eta^i, W) - \Lambda(\theta - \eta^i, W)}{2\eta^i}\right)$$

where, with obvious temporary notations,

$$\Lambda(\theta, W) := \langle Y(\theta, W), Y(\theta, W)\rangle_S = Y(\theta, W)^* S Y(\theta, W)$$

$(Y(\theta, W)$ is related to the innovation W). Starting form this representation, we may derive a recursive updating formula for θ_n as follows

$$\theta_{n+1}^i = \theta_n^i - \gamma_{n+1} \frac{\Lambda(\theta_n + \eta_{n+1}^i, W^{n+1}) - \Lambda(\theta_n - \eta_{n+1}^i, W^{n+1})}{2\eta_{n+1}^i}, \quad 1 \le i \le p.$$

We reproduce below a typical convergence result for K-W procedures (see [21]) which is the natural counterpart of the stochastic gradient framework.

Theorem 6.4 *Assume that the function $\theta \mapsto L(\theta)$ is twice differentiable with a Lipschitz continuous Hessian. We assume that*

$$\theta \mapsto \|\Lambda(\theta, W)\|_2 \text{ has (at most) linear growth}$$

and that the two step sequences respectively satisfy

$$\sum_{n\ge 1} \gamma_n = \sum_{n\ge 1} \eta_n^i = +\infty, \qquad \sum_{n\ge 1} \gamma_n^2 < +\infty, \quad \eta_n \to 0, \quad \sum_{n\ge 1} \left(\frac{\gamma_n}{\eta_n^i}\right)^2 < +\infty.$$

then, θ_n a.s. converges to a connected component of a $\{L = \ell\} \cap \{\nabla L = 0\}$ for some level $\ell \ge 0$.

A special case of this procedure in a linear framework is proposed in Chapter 9.1.2: the decreasing step finite difference method for greeks computation. The *traps* problem for the K-W algorithm (convergence toward a *local minimum* of L) has been more specifically investigated in [96].

Users must keep in mind that this procedure needs some care in the tuning of the step parameters γ_n and η_n. This may need some preliminary numerical experiments. Of course, all the recommendations made for the R-Z procedures remain valid. For more details on the K-W procedure we refer to [21].

NUMERICAL IMPLEMENTATION: In progress.

6.3.5 Recursive computation of the $V@R$ and the $CV@R$ (I)

For an introduction to Conditional Value-at-Risk (CV@R), see *e.g.* [142].

$\triangleright$ **Theoretical background.** Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be a random variable *representative of a loss* (i.e. $X \ge 0$ stands for a loss equal to X).

Definition 6.1 *The* Value at Risk *at level $\alpha \in (0, 1)$ is the (lowest) α-quantile of the distribution of X i.e.*

$$V@R_\alpha(X) := \inf\{\xi \,|\, \mathbb{P}(X \le \xi) \ge \alpha\}. \tag{6.22}$$

The Value at Risk does exist since $\lim_{\xi \to +\infty} \mathbb{P}(X \le \xi) = 1$ and satisfies

$$\mathbb{P}(X < V@R_\alpha(X)) \le \alpha \le \mathbb{P}(X \le V@R_\alpha(X)).$$

As soon as the distribution function of X is continuous (the distribution of X has no atom), the value at risk satisfies

$$\mathbb{P}(X \le V@R_\alpha(X)) = \alpha$$

and if the distribution function F_X of X is also increasing (strictly) then, it is the unique solution of the above Equation (otherwise it is the lowest one). In that we will say that the *Value-at-Risk* is unique.

Roughly speaking *the random variable X represents a loss* (*i.e. X is a loss when non-negative*) and α represents *a confidence level*, typically 0.95 or 0.99.

However this measure of risk is not consistent, for several reasons which are discussed *e.g.* by Föllmer & Schied in [49].

When $X \in L^1(\mathbb{P})$ with a continuous distribution (no atom), a consistent measure of risk is provided by the *Conditional Value-at- Risk* (at level α).

Definition 6.2 *Let $X \in L^1(\mathbb{P})$ with an atomless distribution. The* Conditional Value-at-Risk *(at level α) is defined by*

$$CV@R_\alpha(X) := \mathbb{E}\left(X \mid X \geq V@R_\alpha(X)\right). \tag{6.23}$$

Remark. Note that in case of non-uniqueness, the Conditional Value-at-risk is still well-defined since the above conditional expectation does not depend upon the choice of the above α-quantile.

▷ **Exercise.** Assume that the distribution of X has no atom. Show that

$$CV@R_\alpha(X) = V@R_\alpha(X) + \frac{1}{1-\alpha}\int_{V@R_\alpha(X)}^{+\infty} \mathbb{P}(X > u)\,du.$$

[Hint: use that if the r.v. Y is non-negative, $\mathbb{E}\,Y = \int_0^{+\infty}\mathbb{P}(Y \geq y)\,dy.$]

The following formulation of the $V@R_\alpha$ and $CV@R_\alpha$ as solutions to an optimization problem is due to Rockafellar and Uryasev in [142].

Proposition 6.2 (Rockafellar and Uryasev's representation formula) *Let $X \in L^1(\mathbb{P})$ with an atomless distribution. The function $L : \mathbb{R} \to \mathbb{R}$ defined by*

$$L(\xi) = \xi + \frac{1}{1-\alpha}\mathbb{E}\left(X - \xi\right)_+$$

is convex and $\lim_{|\xi|\to+\infty} L(\xi) = +\infty$. Furthermore, L attains a minimum

$$CV@R_\alpha(X) = \min_{\xi\in\mathbb{R}} L(\xi) \geq \mathbb{E}\,X$$

at

$$V@R_\alpha(X) = \inf\,\text{argmin}_{\xi\in\mathbb{R}}L(\xi).$$

Proof. The function L is clearly convex and 1-Lipschitz continuous since the functions $\xi \mapsto (x-\xi)_+$ are convex and 1-Lipschitz continuous for every $x \in \mathbb{R}$. If X has no atom, then, calling upon Theorem 2.1(a), the function L is also differentiable on the whole real line with a derivative given for every $\xi \in \mathbb{R}$ by

$$L'(\xi) = 1 - \frac{1}{1-\alpha}\mathbb{P}(X > \xi) = \frac{1}{1-\alpha}\left(\mathbb{P}(X \leq \xi) - \alpha\right).$$

This follows from the interchange of differentiation and expectation allowed by Theorem 2.1(a) since $\xi \mapsto \xi + \frac{1}{1-\alpha}(X - \xi)_+$ is differentiable at a given ξ_0 on the event $\{X = \xi_0\}$, *i.e* $\mathbb{P}$-*a.s.* since X is atomless and, on the other hand, is Lipschitz continuous continuous with Lipschitz continuous ratio $\frac{X}{1-\alpha} \in L^1(\mathbb{P})$. The second equality is obvious. Then L attains an absolute minimum, if any, at any solution ξ_α of the equation $\mathbb{P}(X > \xi_\alpha) = 1 - \alpha$ *i.e.* $\mathbb{P}(X \le \xi_\alpha) = \alpha$. Hence, L does attain a minimum at the value-at-risk (which is the lowest solution of this equation). Furthermore

$$
\begin{aligned}
L(\xi_\alpha) &= \xi_\alpha + \frac{\mathbb{E}\left((X - \xi_\alpha)_+\right)}{\mathbb{P}(X > \xi_\alpha)} \\
&= \frac{\xi_\alpha \mathbb{E}\mathbf{1}_{\{X>\xi_\alpha\}} + \mathbb{E}\left((X - \xi_\alpha)\mathbf{1}_{\{X>\xi_\alpha\}}\right)}{\mathbb{P}(X > \xi_\alpha)} \\
&= \frac{\mathbb{E}\left(X\mathbf{1}_{\{X>\xi_\alpha\}}\right)}{\mathbb{P}(X > \xi_\alpha)} \\
&= \mathbb{E}\left(X \,|\, \{X > \xi_\alpha\}\right).
\end{aligned}
$$

Finally, by Jensen's Inequality

$$ L(\xi) \ge \xi + \frac{1}{1-\alpha}\left(X - \xi\right)_+ $$

The function L satisfies

$$ \lim_{\xi \to +\infty} \frac{L(\xi)}{\xi} = \lim_{\xi \to +\infty} \left(1 + \frac{1}{1-\alpha}\mathbb{E}(X/\xi - 1)_+\right) = 1 $$

and

$$ \lim_{\xi \to +\infty} \frac{L(-\xi)}{\xi} = \lim_{\xi \to +\infty} \left(-1 + \frac{1}{1-\alpha}\mathbb{E}(X/\xi + 1)_+\right) = -1 + \frac{1}{1-\alpha} = \frac{\alpha}{1-\alpha}. $$

Hence $\lim_{\xi \to \pm\infty} \widetilde{L}(\xi) = +\infty$.

One checks that the function in the right hand side of the above inequality atteins its minimum at its only break of monotony *i.e.* when $\xi = \mathbb{E}\,X$. This completes the proof. ◇

Exercise. Show that the conditional value-at-risk $CV@R_\alpha(X)$ is a consistent measure of risk *i.e.* that it satisfies the following three properties

- $\forall\, \lambda > 0, \quad CV@R_\alpha(\lambda\,X) = \lambda CV@R_\alpha(X).$

- $\forall\, a \in \mathbb{R}, \quad CV@R_\alpha(X + a) = CV@R_\alpha(X) + a.$

- Let $X, Y \in L^1(\mathbb{P}), \ CV@R_\alpha(X + Y) \le CV@R_\alpha(X) + CV@R_\alpha(Y).$

▷ **First step: a stochastic gradient to compute the Value-at-risk.** The Rockafeller-Uryasev representation suggests to implement a stochastic gradient descent since the function L has a representation as an expectation

$$ L(\xi) = \mathbb{E}\left(\xi + \frac{1}{1-\alpha}(X - \xi)_+\right). $$

Furthermore, if the distribution $\mathbb{P}_W$ has no atom, we know that the function L being convex and differentiable, it satisfies

$$\forall \xi, \xi' \in \mathbb{R}, \quad (L'(\xi) - L'(\xi'))(\xi - \xi') = \frac{1}{1-\alpha}\big(F(\xi) - F(\xi')\big)(\xi - \xi') \geq 0$$

and if the Value-at-Risk $CV@R_\alpha(X)$ is unique solution to $F(\xi) = \alpha$,

$$\forall \xi \in \mathbb{R}, \ \xi \neq V@R_\alpha(X), \quad L'(\xi)\big(\xi - V@R_\alpha(X)\big) = \frac{1}{1-\alpha}\big(F(\xi) - \alpha\big)(\xi - V@R_\alpha(X)) > 0.$$

Proposition 6.3 *Assume that $X \in L^1(\mathbb{P})$ with a unique Value-at-Risk $V@R_\alpha(X)$. Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of random variables with the same distribution as X, let $\xi_0 \in L^1(\mathbb{P})$, independent of $(X_n)_{n\geq 1}$ and let $(\gamma_n)_{n\geq 1}$ be a positive a sequence of real numbers satisfying the decreasing step assumption (6.7). The stochastic algorithm $(\xi_n)_{n\geq 0}$ defined by*

$$\xi_{n+1} = \xi_n - \gamma_{n+1} H(\xi_n, X_{n+1}), \quad n \geq 0, \tag{6.24}$$

where

$$H(\xi, x) := 1 - \frac{1}{1-\alpha}\mathbf{1}_{\{x \geq \xi\}} = \frac{1}{1-\alpha}\big(\mathbf{1}_{\{X < \xi\}} - \alpha\big), \tag{6.25}$$

a.s. converges toward the Value-at-Risk i.e.

$$\xi_n \xrightarrow{a.s.} V@R_\alpha(X).$$

Furthermore the sequence $(L(\xi_n))_{n\geq 0}$ is L^1-bounded so that $L(\xi_n) \to CV@R_\alpha(X)$ a.s. and in every $L^p(\mathbb{P})$, $p \in (0, 1]$.

Proof. First assume that $\xi_0 \in L^2(\mathbb{P})$. The sequence $(\xi_n)_{n\geq 0}$ defined by (6.24) is the stochastic gradient related to the Lyapunov function $\widetilde{L}(\xi) = L(\xi) - \mathbb{E}\,X$, but owing to the convexity of L, it is more convenient to rely on Corollary 6.1(a) (Robbins-Monro algorithm) since it is clear that the function $(\xi, x) \mapsto H(\xi, x)$ is bounded by $\frac{\alpha}{1-\alpha}$ so that $\xi \mapsto \|H(\xi, X)\|_2$ is bounded as well. The conclusion directly follows from the Robbins-Monro setting.

In the general case – $\xi_0 \in L^1(\mathbb{P})$ – one introduces the Lyapunov function $\tilde{L}(\xi) = \frac{(\xi - \xi_\alpha)^2}{\sqrt{1+(\xi-\xi_\alpha)^2}}$ where we set $\xi = \xi_\alpha$ for convenience. The derivative of $\tilde{L}$ is given by $\tilde{L}'(\xi) = \frac{(\xi-\xi_\alpha)(2+(\xi-\xi_\alpha)^2)}{(1+(\xi-\xi_\alpha)^2)^{\frac{3}{2}}}$). One checks on the one hand that $\tilde{L}'$ is Lipschitz continuous continuous over the real line (*e.g.* because $\tilde{L}''$ is bounded) and, on the other hand, $\{\tilde{L}' = 0\} \cap \{L' = 0\} = \{L' = 0\} = \{V@R_\alpha(X)\}$. Then Theorem 6.3 (pseudo-Stochastic Gradient) applies and yields the announced conclusion. $\diamond$

▷ **Exercises. 1.** Show that if X has a bounded density f_X, then a direct application of the stochastic gradient convergence result (Corollary 6.1(b)) yields the announced result under the assumption $\xi_0 \in L^1(\mathbb{P})$ [Hint: show that L' is Lipschitz continuous continuous].
2. Under the additional assumption of Exercise 1, show that the mean function h satisfies $h'(x) = L''(x) = \frac{f_X(x)}{1-\alpha}$. Deduce a way to optimize the step sequence of the algorithm based on the CLT for stochastic algorithms stated further on in Section 6.4.3.

Exercise 2 is inspired by a simpler "α-quantile" approach. It leads to a more general *a.s.* convergence result for our algorithm, stated in the proposition below.

Proposition 6.4 *If $X \in L^p(\mathbb{P})$ for a $p > 0$ and is atomless with a unique value-at-risk and if $\xi_0 \in L^p(\mathbb{P})$, then the algorithm (6.24) a.s. converges toward $V@R_\alpha(X)$.*

Remark. The uniqueness of the value-at-risk can also be relaxed. The conclusion becomes that ξ_n a.s.converges to a random variable taking values in the "$V@R_\alpha(X)$ set": $\{\xi \in \mathbb{R} \mid \mathbb{P}(X \leq \xi) = \alpha\}$ (see [19] for a statement in that direction).

$\triangleright$ **Second step: adaptive computation of $CV@R_\alpha(X)$.** The main aim of this section was to compute the $CV@R_\alpha(X)$. How can we proceed? The idea is to devise a *companion procedure* of the above stochastic gradient. Still set temporarily $\xi_\alpha = V@R_\alpha(X)$ for convenience. It follows from what precedes that $\dfrac{L(\xi_0) + \cdots + L(\xi_{n-1})}{n} \to CV@R_\alpha(X)$ a.s. owing to Césaro principle. On the other hand, we know that, for every $\xi \in \mathbb{R}$,

$$L(\xi) = \mathbb{E}\,\Lambda(\xi, X) \quad \text{where} \quad \Lambda(\xi, x) = \xi + \frac{(x - \xi)_+}{1 - \alpha}.$$

Using that X_n and $(\xi_0, \xi_1, \ldots, \xi_n)$ are independent, on can re-write the above Césaro converge as

$$\mathbb{E}\left(\frac{\Lambda(\xi_0, X_1) + \cdots + \Lambda(\xi_{n-1}, X_n)}{n}\right) \longrightarrow CV@R_\alpha(X)$$

which naturally suggests to consider the sequence $(C_n)_{n \geq 0}$ defined by $C_0 = 0$ and

$$C_n = \frac{1}{n} \sum_{k=0}^{n-1} \Lambda(\xi_k, X_{k+1}), \quad n \geq 1,$$

is a candidate to be an estimator of $CV@R_\alpha(X)$. This sequence can clearly be recursively defined since, for every $n \geq 0$,

$$C_{n+1} = C_n - \frac{1}{n+1}\big(C_n - \Lambda(\xi_n, X_{n+1})\big) \tag{6.26}$$

Proposition 6.5 *Assume that $X \in L^{1+\rho}(\mathbb{P})$ for $\rho \in (0, 1]$ and that $\xi_n \longrightarrow V@R_\alpha(X)$. Then*

$$C_n \xrightarrow{a.s.} CV@R_\alpha(X) \quad as \quad n \to +\infty.$$

Proof. We will prove this claim in details in the quadratic case $\rho = 1$. The proof in the general case relies on the Chow Theorem (see [43] or the second exercise right after the proof). First, one decomposes

$$C_n - L(\xi_\alpha) = \frac{1}{n} \sum_{k=0}^{n-1} L(\xi_k) - L(\xi_\alpha) + \frac{1}{n} \sum_{k=1}^{n} Y_k$$

with $Y_k := \Lambda(\xi_{k-1}, X_k) - L(\xi_{k-1})$, $k \geq 1$. It is clear that $\dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} L(\xi_k) - L(\xi_\alpha) \to 0$ as $n \to +\infty$ by Césaro's principle.

As concerns the second term, we first note that

$$\Lambda(\xi, x) - L(\xi) = \frac{1}{1 - \alpha}\big((x - \xi)_+ - \mathbb{E}(X - \xi)_+\big)$$

so that, $x \mapsto x_+$ being 1-Lipschitz continuous continuous

$$|\Lambda(\xi, x) - L(\xi)| \leq \frac{1}{1 - \alpha} \mathbb{E}\, |X - \xi| \leq \frac{1}{1 - \alpha} \big(\mathbb{E}\, |X| + |\xi|\big).$$

Consequently, for every $k \geq 1$,

$$\mathbb{E}\, Y_k^2 \leq \frac{2}{(1 - \alpha)^2} \big((\mathbb{E}\, X)^2 + \mathbb{E}\, X^2\big).$$

We consider the natural filtration of the algorithm $\mathcal{F}_n := \sigma(\xi_0, X_1, \ldots, X_n)$. One checks that, for every $k \geq 1$,

$$\mathbb{E}\, (Y_k \,|\, \mathcal{F}_{k-1}) = \mathbb{E}\, (\Lambda(\xi_{k-1}, X_k) \,|\, \mathcal{F}_{k-1}) - L(\xi_{k-1}) = L(\xi_{k-1}) - L(\xi_{k-1}) = 0.$$

We consider the martingale defined by $N_0 = 0$ and

$$N_n := \sum_{k=1}^{n} \frac{Y_k}{k}, \quad n \geq 1.$$

This martingale is in $L^2(\mathbb{P})$ for every n and its predictable bracket process is given by

$$\langle N \rangle_n = \sum_{k=1}^{n} \frac{\mathbb{E}(Y_k^2 \,|\, \mathcal{F}_{k-1})}{k^2}$$

so that

$$\mathbb{E}\, \langle N \rangle_\infty \leq \sup_n \mathbb{E}\, Y_n^2 \times \sum_{k \geq 1} \frac{1}{k^2} < +\infty.$$

Consequently $N_n \to N_\infty$ *a.s.* and in L^2 as $n \to +\infty$. Then the Kronecker Lemma (see Lemma 11.1) implies that

$$\frac{1}{n} \sum_{k=1}^{n} Y_k \overset{n \to +\infty}{\longrightarrow} 0$$

which finally implies that

$$C_n \overset{n \to +\infty}{\longrightarrow} CV@R_\alpha(X). \qquad \diamond$$

Remark. For practical implementation one may prefer estimating first the $V@R_\alpha(X)$ and, once it is done, use a regular Monte Carlo procedure to evaluate the $CV@R_\alpha(X)$.

▷ **Exercises. 1.** Show that an alternative method to compute $CV@R_\alpha(X)$ is to design the following recursive procedure

$$C_{n+1} = C_n - \gamma_{n+1}(C_n - \Lambda(\xi_n, X_{n+1})), \quad n \geq 0, \quad C_0 = 0. \tag{6.27}$$

where $(\gamma_n)_{n \geq 1}$ is the step sequence implemented in the algorithm (6.24) which computes $V@R_\alpha(X)$.

2 (Proof of Proposition 6.5). Show that the conclusion of Proposition 6.5 remains valid if $X \in L^{1+\rho}(\mathbb{P})$. [Hint: rely on the Chow theorem ([4]).]

▷ WARNING!... TOWARD AN OPERATING PROCEDURE As it is presented, what precedes is essentially a toy exercise for the following reason: in practice, the convergence of the above algorithm will be very slow and chaotic... since $\mathbb{P}(X > V@R_\alpha(X)) = 1 - \alpha$ is close to 0. For a practical implementation on real life portfolios the above procedure must be combined with importance sampling to recenter the simulation where things do happen... A more realistic procedure is developed and analyzed in [19].

6.3.6 Numerical methods for Optimal Quantization

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ be a random vector and let N be a non zero integer. We want to produce an optimal quadratic quantization of X at *level N i.e.* to produce an N-quantizer which minimizes the quadratic quantization error as introduced in Chapter 5.

Everything starts from the fact that any optimal (or even locally optimal) quantizer $x = (x^1, \ldots, x^N)$ with N-components satisfies the stationarity equation as briefly recalled below.

Competitive Learning Vector Quantization

This procedure is a stochastic gradient descent derived form the squared quantization error modulus (see below). Un fortunately it does not fulfill any of the usual assumption needed on a potential/Lyapunov function to ensure the *a.s.* convergence of such a procedure. In particular, the potential does not go to infinity when the norm of the N-quantizer goes to infinity. Except in one dimension – where it is of little interest since deterministic Newton's like procedures can be implemented in general – very partial results are known about the asymptotic behaviour of this procedure (see *e.g.* [121, 21]).

However, the theoretical gaps (in particular the possible asymptotic "merge" of components) is never observed on practical implementations It is recommended to get sharp results when N is small and d not too high.

The quadratic distortion function is defined as the squared quadratic mean-quantization error *i.e.*

$$\forall x = (x^1, \ldots, x^N) \in (\mathbb{R}^d)^N, \quad D_N^X(x) := \|X - \widehat{X}^x\|_2^2 = \mathbb{E}\left(\mathrm{dis}_{loc,N}(\Gamma, X)\right)$$

with $(x, \xi) \mapsto \mathrm{dis}_{loc,N}(x, \xi)$ is a *local potential* defined by

$$\mathrm{dis}_{loc,N}(x, \xi) = \min_{1 \le i \le N} |\xi - x^i|^2 = \mathrm{dist}\left(\xi, \{x^1, \ldots, x^N\}\right)^2.$$

---

[4]Let $(M_n)_{n \ge 0}$ be an $(\mathcal{F}_n, \mathbb{P})$-martingale null at 0 and let $\rho \in (0, 1]$; then

$$M_n \xrightarrow{a.s.} M_\infty \quad \text{on} \quad \left\{ \sum_{n \ge 1} \mathbb{E}\left(|\Delta M_n|^{1+\rho} \mid \mathcal{F}_{n-1}\right) < +\infty \right\}.$$

Proposition 6.3.1 *Let $N \in \mathbb{N}^*$.   The distortion function D_N^X is continuously differentiable at N-tuples $x \in (\mathbb{R}^d)^N$ with pairwise distinct components as a consequence of the local Lebesgue differentiation theorem (Theorem 2.1(a)) and one easily checks that*

$$\frac{\partial D_N^X}{\partial x^i}(x) \; := \; \mathbb{E} \, \frac{\partial \mathrm{dis}_{loc,N}}{\partial x^i}(x,X) = \int_{\mathbb{R}^d} \frac{\partial \mathrm{dis}_{loc,N}}{\partial x^i}(x,\xi)\mathbb{P}_X(d\xi),$$

with a local gradient given by

$$\frac{\partial \mathrm{dis}_{loc,N}}{\partial x^i}(x,\xi) \; := \; 2(x^i - \xi)\mathbf{1}_{\{\mathrm{Proj}_x(\xi)=x^i\}}, \qquad 1 \le i \le N$$

where Proj_x *denotes a (Borel) projection following the nearest neighbour rule on the* grid $\{x^1,\ldots,x^N\}$.

As emphasized in the introduction of this chapter, the gradient ∇D_N^X having an integral representation, it formally possible to minimize D_N^X using a stochastic gradient descent.

Unfortunately, it is easy to check that $\liminf\limits_{|x|\to+\infty} D_N^X(x) < +\infty$ (though $\liminf\limits_{\min_i |x^i|\to+\infty} D_N^X(x) = +\infty$. Consequently, it is hopeless to apply the standard convergence theorem "à la Robbins-Siegmund" for stochastic gradient procedure like that established in Corollary 6.1(b). But of course, we can still write it down formally and implement it...

▷ INGREDIENTS:

    – A sequence $\xi^1,\ldots,\xi^t,\ldots$ of (simulated) independent copies of X,

    – A step sequence $\gamma_1,\ldots,\gamma_t,\ldots$.

One usually choose the step in the parametric families $\gamma_t = \dfrac{A}{B+t} \searrow 0$ (decreasing step) or $\gamma_t = \eta \approx 0$ (small constant step).

▷ STOCHASTIC GRADIENT DESCENT FORMULA.

• The procedure formally reads

$$x(t) = x(t-1) - \gamma_{t+1}\nabla_x \mathrm{dis}_{loc,N}(\Gamma(t-1),\xi^t), \quad x(0) \in (\mathbb{R}^d)^N$$

where $x(0) = (x(0)^1,\ldots,x(0)^N)$ has pairwise distinct components $x(0)^i$ in $\mathbb{R}^d$.

• QUANTIZER UPDATING: $(t \rightsquigarrow t+1)$: $x(t) := \{x^1(t),\ldots,x^N(t)\}$,

    Competition: winner selection $i(t+1) \in \mathrm{argmin}_i |x^i(t) - \xi^{t+1}|$
                (nearest neighbour search)

$$\textit{Learning:} \quad \begin{cases} x^{i(t+1)}(t+1) := \mathrm{Homothety}(\xi^{t+1}, 1-\gamma_{t+1})(x^{i(t+1)}(t)) \\[2mm] x^i(t+1) \quad := x^i(t), \ i \ne i(t+1). \end{cases}$$

One can easily check that if $x(t)$ has pairwise distinct components (in $\mathbb{R}^d$, this is preserved by the "learning" phase. So that the above procedure is well-defined (up to the convention to be made in case of conflict between several components $x(t)^j$ in the "Competitive" phase.

The name of the procedure – Competitive Learning Vector Quantization algorithm – is of course inspired from these two phases.

▷ HEURISTICS: $x(t) \longrightarrow x^* \in \mathrm{argmin(loc)}_{x \in (\mathbb{R}^d)^N} D_N^X(x)$ as $tn \to +\infty$.

▷ ON LINE COMPUTATION OF THE "COMPANION PARAMETERS":

• Weights $\pi^{i,*} = \mathbb{P}(\widehat{X}^{x^*} = x^{i,*})$, $i = 1, \ldots, N$:

$$\pi^{i,t+1} := (1 - \gamma_{t+1})\pi^{i,t} + \gamma_{t+1}\mathbf{1}_{\{i=i(t+1)\}} \xrightarrow{a.s.} \pi^{i,*} = \mathbb{P}(\widehat{X}^{x^*} = x^{i,*}).$$

• (Quadratic) Quantization error $D_N^X(\Gamma^*) = \|X - \widehat{X}^{\Gamma^*}\|_2$:

$$D_N^{X,t+1} := (1 - \gamma_{t+1})D_N^{X,t} + \gamma_{t+1}|x^{i(t+1),t} - \xi^{t+1}|^2 \xrightarrow{a.s.} D_N^X(\Gamma^*).$$

Note that, since the ingredients involved in the above computations are those used in the competition learning phase, there is (almost) no extra C.P.U. time cost induced from these additional terms, especially if one has in mind (see below) that the costfull part of the algorithm (as well as that of the Lloyd procedure below) is the "competition phase" (Winner selection) wince it amounts to a *nearest neighbour search.*

In some way the CLVQ algorithm can be seen as a Non Linear Monte Carlo Simulation devised to design an optimal skeleton of the distribution of X.

For (partial) theoretical results on the convergence of the CLVQ algorithm, we refer to [121] when X has a compactly supported distribution. No theoretical convergence results are known to us when X has an inbounded support. As for the convergence of the online adaptive companion procedures, which rely on classical martingale arguments, we refer again to [121], but also to [12].

Randomized Lloyd's I procedure

This randomized fixed point procedure is recommended to get good results when N is large and d is medium. We start from the fact that a local minimizer of a differentiable function is a zero of its gradient. One checks that a local minimizer of the quadratic distortion function D_N^X must have pairwise distinct components (see [66, 121] among others). Consequently, if $x = (x^1, \ldots, x^N)$ denotes a local minimum, the gradient of the squared quantization error at x must be zero *i.e.*

$$\frac{\partial D_N^X}{\partial x^i}(x, \xi) = 2\mathbb{E}\big((x^i - X)\mathbf{1}_{\{\mathrm{Proj}_x(X)=x^i\}}\big) = 0, \ 1 \le i \le N,$$

or equivalently

$$x^i = \mathbb{E}\big(X \,|\, \{\widehat{X}^x = x^i\}\big), \ 1 \le i \le N. \tag{6.28}$$

This fixed point identity can also be rewritten in term of conditional expectation as follows:

$$\widehat{X}^x = \mathbb{E}\big(X \,|\, \widehat{X}^x\big)$$

since, by the characterization of the conditional expectation of a discrete random vector, $\mathbb{E}\big(X \,|\, \widehat{X}^x\big) =$
$$\sum_{i=1}^{N} \mathbb{E}\big(X \,|\, \{\widehat{X}^x = x^i\}\big) \mathbf{1}_{\{\widehat{X}^x = x^i\}}.$$

▷ REGULAR LLOYD'S PROCEDURE. The Lloyd procedure is simply the recursive procedure associated to the fixed point identity (6.28)

$$x^i(t+1) = \mathbb{E}\big(X \,|\, \{\widehat{X}^{x(t)} = x^i\}\big),\ 1 \le i \le N,\quad t \in \mathbb{N},\ x(0) \in (\mathbb{R}^d)^N,$$

where $x(0)$ has pairwise distinct components in $\mathbb{R}^d$. We leave as an exercise to show that this procedure is entirely determined by the *distribution* of the random vector X.

▷ **Exercise.** Prove that this recursive procedure only involves the distribution $\mu = \mathbb{P}_X$ of the random vector X.

Lloyd's algorithm can be viewed as two step procedure acting on random vectors as follows

$$(i)\text{Grid updating} \quad \widetilde{X}(t+1) = \mathbb{E}\big(X \,|\, \widehat{X}^{x(t)}\big),\ x(t+1) = \widetilde{X}(t+1)(\Omega), (6.29)$$
$$(ii)\text{Distribution/weight updating} \quad \widehat{X}^{x(t+1)} \leftarrow \widetilde{X}(t+1). \tag{6.30}$$

The first step updates the grid, the second step re-assigns to each element of the grid its Voronoi cell which can be viewed as a *weight* updating through the

Proposition 6.6 *The Lloyd's algorithm makes the quadratic quantization error decrease i.e.*

$$t \longmapsto \big\|X - \widehat{X}^{x(t)}\big\|_2 \quad \textit{is non-increasing.}$$

Proof. It follows the the above decomposition of the procedure and the very definitions of nearest neighbour projection and conditional expectation as an orthogonal projector in $L^2(\mathbb{P})$ that, for every $t \in \mathbb{N}$,

$$
\begin{aligned}
\big\|X - \widehat{X}^{x(t+1)}\big\|_2 &= \big\|\text{dist}\big(X, x(t+1)\big)\big\|_2 \\
&\le \big\|X - \widetilde{X}^{x(t+1)}\big\|_2 = \big\|X - \mathbb{E}\big(X \,|\, \widehat{X}^{x(t)}\big)\big\|_2 \\
&\le \big\|X - \widehat{X}^{x(t)}\big\|_2 \qquad \diamond
\end{aligned}
$$

▷ RANDOMIZED LLOYD'S PROCEDURE. It relies on the computation of $\mathbb{E}(X \,|\, \{\widehat{X}^x = x^i\})$, $1 \le i \le N$, by a Monte Carlo simulation. This means that we have independent copies $\xi_1,\dots,\xi_M,\dots$ of X that

$$\mathbb{E}\big(X \,|\, \widehat{X}^{x(t)} = x^i(t)\big) \approx \frac{\sum_{m=1}^{M} \xi_m \mathbf{1}_{\{\widehat{\xi}_m^{x(t)} = x^i(t)\}}}{\big|\{1 \le m \le M,\ \widehat{\xi}_m^{x(t)} = x^i(t)\}\big|},$$

having in mind that the convergence holds when $M \to +\infty$. This amounts to set at every iteration

$$x^i(t+1) = \frac{\sum_{m=1}^{M} \xi_m \mathbf{1}_{\{\widehat{\xi}_m^{x(t)} = x^i(t)\}}}{\big|\{1 \le m \le M,\ \widehat{\xi}_m^{x(t)} = x^i(t)\}\big|},\ i = 1,\dots,N,\ x(0) \in (\mathbb{R}^d)^N$$

where $x(0)$ has pairwise distinct components in $\mathbb{R}^d$.

This "randomized procedure" amounts to replacing the distribution of X by the empirical measure

$$\frac{1}{m} \sum_{m=1}^{M} \delta_{\xi_m}.$$

In particular, if we use the same sample at each iteration, we still have the property that the procedure makes decrease a quantization error modulus (at level N) related to the distribution μ.

This suggests that the random i.i.d. sample $(\xi_m)_{m \geq 1}$ can also be replaced by deterministic copies obtained through a QMC procedure based on a representation of X of the form $X = \psi(U)$, $U \sim U([0,1]^r)$.

When computing larger and larger quantizer of the same distribution, a significant improvement of the method is to initialize the (randomized Lloyd's procedure at "level" $N + 1$ be adding one component to the N-quantizer resulting from the procedure applied at level N, namely to start from the $N + 1$-tuple $(x^{*,(N)}, \xi)$ where $x^{*,(N)}$ denotes the limiting value of the procedure at level N (assumed to exist, which is the case in practice).

Fast nearest neighbour procedure in $\mathbb{R}^d$

This is the key step in all stochastic procedure imagined to compute optimal (or at least "good") quantizers in higher dimension. To speed it up, especially when d increases is one of the major challenge of computer science.

$\triangleright$ The Partial Distance Search paradigm (Chen, 1970).
   We want to check whether $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$ is closer to 0 for the canonical Euclidean distance than a given former "minimal record distance" $\delta_{\min} > 0$. The "trick" is the following

$$(x^1)^2 \geq \delta_{\min}^2 \implies |x| \geq \delta_{\min}$$
$$\vdots$$
$$(x^1)^2 + \cdots + (x^\ell)^2 \geq \delta_{\min}^2 \implies |x| \geq \delta_{\min}$$
$$\vdots$$

This is the simplest idea and the easiest idea to implement but it seems that it is also the only one that still works as d increases.

$\triangleright$ The K-d tree (Friedmann, Bentley, Finkel, 1977, see [53]): store the N points of $\mathbb{R}^d$ in a tree of depth $O(\log(N))$ based on their coordinates on the canonical basis of $\mathbb{R}^d$.

$\triangleright$ Further improvements are due to Mc Names (see [108]): the idea is to perform a pre-processing of the dataset of N point using a Principal Component Axis (PCA) analysis and then implement the K-d-tree method in the new orthogonal basis induced by the PCA.

Numerical optimization of quantizers for the normal distributions $\mathcal{N}(0; I_d)$ on $\mathbb{R}^d$, $d \geq 1$

The procedures that minimizes the quantization error are usually stochastic (except in 1-dimension). The most famous ones are undoubtedly the so-called *Competitive Leaning Vector Quantization*
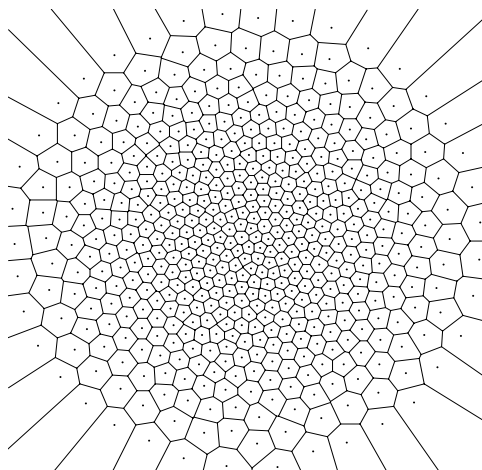
Figure 6.3: *An optimal quantization of the bi-variate normal distribution with size $N = 500$*

algorithm (see [128] or [126]) and the Lloyd I procedure (see [128, 123, 56]) which have been just described and briefly analyzed above. More algorithmic details are also made available on the web site

<div align="center">www.quantize.maths-fi.com</div>

For normal distributions a large scale optimization have been carried out based on a mixed $CLVQ$-Lloyd procedure. To be precise, grids have been computed for For $d = 1$ up to 10 and $N = 1 \le N \le 5\,000$. Furthermore several companion parameters have also been computed (still by simulation): weight, L^1 quantization error, (squared) L^2-quantization error (also known as distortion), local $L^1 \& L^2$-pseudo-inertia of each Voronoi cell. All these grids can be downloaded on the above website.

Thus Figure 6.3.6 depicts an optimal quadratic N-quantization of the bi-variate normal distribution $\mathcal{N}(0; I_2)$ with $N = 500$.

6.4   Further results on Stochastic Approximation

This section can be skipped at a first reading. It deals with the connection between the mean function h in Stochastic Approximation with and the underlying Ordinary Differential Equation $\dot{x} = -h(x)$ as mentioned in the introduction. The second part of this section is devoted to the rate of convergence of stochastic algorithms.

6.4.1   The ODE method

Toward the ODE

The starting idea (due to Ljung in [103]) of the ODE method is to consider the dynamics of a stochastic algorithm as a perturbed Euler scheme with decreasing step of an Ordinary Differentia Equation. In this textbook we mainly deal with algorithms having a Markov representation with

an i.i.d. sequence of innovations of the form (6.3), that is

$$Y_{n+1} = Y_n - \gamma_{n+1} H(Y_n, Z_{n+1})$$

where $(Z_n)_{n \geq 1}$ is an i.i.d. innovation sequence of $\mathbb{R}^q$-valued random vectors, Y_0 is independent of the innovation sequence, $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ is a Borel function such that... and $(\gamma_n)_{n \geq 1}$ is sequence of step parameters.

We saw that this algorithm can be represented in a canonical way as follows:

$$Y_{n+1} = Y_n - \gamma_{n+1} h(Y_n) + \gamma_{n+1} \Delta M_{n+1} \tag{6.31}$$

where $h(y) = \mathbb{E}\, H(y, Z_1)$ is the mean filed or mean function of the algorithm and $\Delta M_{n+1} = h(Y_n) - \mathbb{E}(H(Y_n, Z_{n+1})\,|\,\mathcal{F}_n)$, $n \geq 0$, is a sequence of martingale increments with respect to the filtration $\mathcal{F}_n = \sigma(Y_0, Z_1, \ldots, Z_n)$, $n \geq 0$.

In what precedes we established criterions (see Robins-Siegmund's Lemma) based on the existence of Lyapunov functions which ensure that the sequence $(Y_n)_{n \geq 1}$ is *a.s.* bounded and that the martingale

$$M_n^\gamma = \sum_{k \geq 1} \gamma_k \Delta M_k \quad \text{is } a.s. \text{ convergent in } \mathbb{R}^d.$$

To derive the *a.s.* convergence of the algorithm itself, we used *pathwise* arguments based on elementary topology and functional analysis. The main improvement provided by the *ODE* method is to study the asymptotics of the sequence $(Y_n(\omega))_{n \geq 0}$, assumed *a priori*) to be bounded, through the sequence $(Y_k(\omega))_{k \geq n}$, $n \geq 1$ represented as sequence of function in the scale of the cumulative function of the step. We will also need an assumption on the paths of the martingale $(M_n^\gamma)_{n \geq 1}$, however significantly lighter than the above *a.s.* convergence property. Let us be more specific: first we consider a discrete time dynamics

$$y_{n+1} = y_n - \gamma_{n+1}\big(h(y_n) + \pi_{n+1}\big), \quad y_0 \in \mathbb{R}^d \tag{6.32}$$

where $(\pi_n)_{n \geq 1}$ is a sequence of $\mathbb{R}^d$-valued vectors and $h : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous Borel function.

We set $\Gamma_0 = 0$ and, for every integer $n \geq 1$,

$$\Gamma_n = \sum_{k=1}^{n} \gamma_k.$$

Then we define the stepwise constant càdlàg function $(Y_t^{(0)})_{t \in \mathbb{R}_+}$ by

$$y_t^{(0)} = y_n \quad \text{if} \quad t \in [\Gamma_n, \Gamma_{n+1})$$

and the sequence of time shifted function defined by

$$y_t^{(n)} = Y_{\Gamma_n + t}^{(0)}, \ t \in \mathbb{R}_+.$$

Finally, we set for every $u\mathbb{R}_+$,

$$N(t) = \max\big\{k : \Gamma_k \leq t\big\} = \min\big\{k : \Gamma_{k+1} > t\big\}$$

so that $N(t) = n$ if and only if $t \in [\Gamma_n, \Gamma_{n+1})$ (in particular $N(\Gamma_n) = n$).

Developing the recursive Equation (6.32), we get

$$y_n = y_0 - \sum_{k=1}^{n} \gamma_k h(y_{k-1}) - \sum_{k=1}^{n} \gamma_k \pi_k$$

which can be rewritten, of every $t \in [\Gamma_n, \Gamma_{n+1})$,

$$
\begin{aligned}
y_t^{(0)} &= y_0^{(0)} - \sum_{k=1}^{n} \int_{\Gamma_{k-1}}^{\Gamma_k} h\big(\underbrace{y_s^{(0)}}_{=y_k}\big) ds - \sum_{k=1}^{n} \gamma_k \pi_k \\
&= y_0^{(0)} - \int_0^{\Gamma_n} h(y_s^{(0)}) ds - \sum_{k=1}^{n} \gamma_k \pi_k.
\end{aligned}
$$

As a consequence, for every $t \in \mathbb{R}_+$,

$$
\begin{aligned}
y_t^{(0)} &= y_0^{(0)} - \int_0^{\Gamma_{N(t)}} h(y_s^{(0)}) ds - \sum_{k=1}^{N(t)} \gamma_k \pi_k \\
&= y_0^{(0)} - \int_0^t h(y_s^{(0)}) ds + \int_{\Gamma_{N(t)}}^t h(y_s^{(0)}) ds - \sum_{k=1}^{N(t)} \gamma_k \pi_k \qquad (6.33)
\end{aligned}
$$

Then, by the very definition of the shifted function $y^{(n)}$ and taking advantage of the fact that $\Gamma_{N(\Gamma_n)} = \Gamma_n$, we derive by subtracting (6.33) at times $\Gamma_n + t$ and Γ_n, that for every $t \in \mathbb{R}_+$,

$$
\begin{aligned}
y_t^{(n)} &= y_{\Gamma_n}^{(0)} + y_{\Gamma_n+t}^{(0)} - y_{\Gamma_n}^{(0)} \\
&= y_{\Gamma_n}^{(0)} - \int_{\Gamma_n}^{\Gamma_n+t} h(y_s^{(0)}) ds + \int_{\Gamma_{N(\Gamma_n+t)}}^{\Gamma_n+u} - \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \pi_k.
\end{aligned}
$$

where we keep in mind that $y_{\Gamma_n}^{(0)} = y_n$. The term $\displaystyle\int_{\Gamma_{N(\Gamma_n+t)}}^{\Gamma_n+u} - \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \pi_k$ is candidate to be a

remainder term as n goes to infinity. Our aim is to make a connection between the asymptotic behavior of the sequence of vectors $(y_n)_{n\geq 10}$ and that of the sequence of functions $y^{(n)}$, $n \geq 0$.

Proposition 6.7 (*ODE* method I) *Assume that*

- $H_1 \equiv$ *Both sequences $(y_n)_{n\geq 0}$ and $(h(y_n))_{n\geq 0}$ are bounded,*

- $H_2 \equiv \forall n \geq 0$, $\gamma_n > 0$, $\lim_n \gamma_n = 0$ *and* $\sum_{n\geq 1} \gamma_n = +\infty$,

- $H_3 \equiv \forall T \in (0, +\infty)$, $\displaystyle\lim_{n\to+\infty} \sup_{t\in[0,T]} \left| \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \pi_k \right| = 0$.

(a) The set $\mathcal{Y}^\infty := \{$limiting points of $(y_n)_{n\geq 0}\}$ is a compact connected set.

(b) The sequence $(y^{(n)})_{n\geq 1}$ is sequentially relatively compact ([5]) for the topology of the uniform convergence on compacts sets on the space $\mathcal{B}(\mathbb{R}_+, \mathbb{R}^d)$ of bounded functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ ([6]) and all its limiting points lie in $\mathcal{C}(\mathbb{R}_+, \mathcal{Y}^\infty)$.

Proof. Let $L = \sup_{n\in\mathbb{N}} |h(y_n)|$.

(a) Let $T_0 = \sup_{n\in\mathbb{N}} \gamma_n < +\infty$. The it follows from H_2

$$|\gamma_{n+1}\pi_{n+1}| \leq \sup_{t\in[0,T_0]} \left| \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k\pi_k \right| = 0.$$

It follows form (6.32)

$$|y_{n+1} - y_n| \leq \gamma_{n+1}L + |\gamma_{n+1}\pi_{n+1}| \to 0 \quad \text{as} \quad n \to +\infty.$$

As a consequence (see [6] for details) the set $\mathcal{Y}^\infty$ is compact and *bien enchaîné* ([7]), hence connected.

(b) The sequence $\left(\int_0^\cdot h(y_s^{(n)})ds \right)_{n\geq 0}$ is uniformly Lipschitz continuous with Lipschitz continuous coefficient L since, for every $s, t \in \mathbb{R}_+$, $s \leq t$,

$$\left| \int_0^t h(y_u^{(n)})du - \int_0^s h(y_u^{(n)})du \right| \leq \int_s^t |h(y_u^{(n)})|du \leq L(t - s).$$

hence, it follows form Arzela-Ascoli's Theorem that $(y^{(n)})_{n\geq 0}$ is relatively compact in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ endowed with the topology, denoted U_K, of the uniform convergence on compact sets. On the other hand, for every $T \in (0, +\infty)$,

$$\sup_{t\in[0,T]} \left| y_t^{(n)} - y_0^{(n)} + \int_0^t h(y_s^{(n)})ds \right| \leq \sup_{k\geq n+1} \gamma_k L + \sup_{t\in[0,T]} \left| \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k\pi_k \right| \longrightarrow 0 \quad \text{as} \quad n \to +\infty$$

owing to H_2 and H_3, *i.e.*

$$y^{(n)} - y_0^{(n)} + \int_0^\cdot h(y_s^{(n)})ds \xrightarrow{U_K} 0 \quad \text{as} \quad n \to +\infty. \tag{6.34}$$

The sequence $(y_n)_{n\geq 0}$ is bounded and, as a consequence, the sequence $(y^{(n)})_{n\geq 0}$ is U_K-relatively compact with the same U_K-limiting values as $\left(y_0^{(n)} - \int_0^\cdot h(y_s^{(n)})ds \right)_{n\geq 0}$. Hence, these limiting values are continuous functions having values in $\mathcal{Y}^\infty$. $\diamond$

---

[5]In a metric space (E, d), a sequence $(x_n)_{n\geq 0}$ is sequentially relatively compact if from any subsequence on evan extract a converging subsequence for the distant d.

[6]This topology is defined by the metric $d(f, g) = \sum_{k\geq 1} \frac{\min(\sup_{t\in[0,k]} |f(t)-g(t)|, 1)}{2^n}$.

[7]A set A in a metric space (E, d) is "bien enchaîné" if for every $a, a': in A$, and every $\varepsilon > 0$, there exists an integer $n \geq 1$ and $a_0, \ldots, a_n$ such that $a_0 = a$, $a_n = a'$ and $d(a_i, a_{i+1}) \leq \varepsilon$ for every $i = 0, \ldots, n-1$. Any connected set C in E is *bien enchaîné*. The converse is true if C is compact.

Connection algorithm-ODE

To state the first theorem on the so-called ODE method theorem, we introduce the *reverse* differential equation

$$ODE^* \equiv \dot{y} = h(y).$$

Theorem 6.5 (ODE II) *Assume H_1-H_3 hold and that the mean field function h is continuous.*
(a) Any limiting function of the sequence $(y^{(n)})_{n \geq 0}$ is an $\mathcal{Y}^\infty$-valued solution of
$ODE \equiv \dot{y} = -h(y)$.
(b) Assume that ODE satisfies the following uniqueness property: for every $y_0 \in \mathcal{Y}^\infty$, ODE admits a unique $\mathcal{Y}^\infty$-valued solution $(\Phi_t(y_0))_{t \in \mathbb{R}_+}$ starting at $\Phi_0(y_0) = y_0$. Also assume uniqueness for ODE^* (defined likewise). Then, the set $\mathcal{Y}^\infty$ is a compact, connected set, and flow-invariant for both ODE and ODE^*.

Proof: (a) Given the above proposition, one has to check that any limiting function $y^{(\infty)} = U_K - \lim_{n \to +\infty} y^{(\varphi(n))}$ is solution the ODE. For every $t \in \mathbb{R}_+$, $y_t^{(\varphi(n))} \to y_t^{(\infty)}$, hence $h(y_t^{(\varphi(n))}) \to h(y_t^{(\infty)})$ since the mean field function h is continuous. Then by the Lebesgue dominated convergence theorem, one derives that for every $t \in \mathbb{R}_+$,

$$\int_0^t h(y_s^{(\varphi(n))})ds \longrightarrow \int_0^t h(y_s^{(\infty)})ds.$$

One also has $y_0^{(\varphi(n))} \to y_0^{(\infty)}$ so that finally, letting $\varphi(n) \to +\infty$ in (6.34), we obtain

$$y_t^{(\infty)} = y_0^{(\infty)} - \int_0^t h(y_s^{(\infty)})ds.$$

(b) Any $y_0 \in \mathcal{Y}^\infty$ is the limit of a sequence $y_{\varphi(n)}$. Up to a new extraction, still denoted $\varphi(n)$ for convenience, we may assume that $y^{(\varphi(n))} \to y^{(\infty)}$ as $n \to \infty$, uniformly on compact sets of $\mathbb{R}_+$. The function $y^{(\infty)}$ is a $\mathcal{Y}^\infty$-valued solution to ODE and $y^{(\infty)} = \Phi(y_0^{(\infty)})$ owing to the uniqueness assumption which implies the invariance of $\mathcal{Y}^\infty$ under the flow of ODE.

For every $p \in \mathbb{N}$, we consider for large enough n, say $n \geq n_p$, the sequence $(y_{N(\Gamma_{\varphi(n)}-p)})_{n \geq n_p}$. It is clear by mimicking the proof of Proposition 6.7 that all sequences of functions $(y^{(N(\Gamma_{\varphi(n)}-p))})_{n \geq n_p}$ are U_K-relatively compact. By a diagonal extraction procedure, we may assume that, for every $p \in \mathbb{N}$,

$$y^{(N(\Gamma_{\varphi(n)}-p)} \xrightarrow{U_K} y^{(\infty),p} \quad \text{as} \quad n \to +\infty.$$

Since $y_{t+1}^{(N(\Gamma_{\varphi(n)}-p-1))} = y_t^{(N(\Gamma_{\varphi(n)}-p))}$ for every $t \in \mathbb{R}_+$ and $n \geq n_{p+1}$, one has

$$\forall p \in \mathbb{N}, \quad \forall t \in \mathbb{R}_+, \quad y_{t+1}^{(\infty),p+1} = y_t^{(\infty),p}.$$

Furthermore, it follows form (a) that the functions $y^{(\infty),p}$ are $\mathcal{Y}^\infty$-valued solutions to ODE. One defines

$$\tilde{y}_t^{(\infty)} = y_{p-t}^{(\infty),p}, \ t \in [p-1,p]$$

which satisfies in fact, for every $p \in \mathbb{N}$, $\tilde{y}_t^{(\infty)} = y_{p-t}^{(\infty),p}$, $t \in [0, p]$. This implies that $\tilde{y}^{(\infty)}$ is an $\mathcal{Y}^{\infty}$-valued solution to ODE^* starting from y_0 on $\cup_{p \geq 0}[0, p] = \mathbb{R}_+$. Uniqueness implies that $\tilde{y}^{(\infty)} = \Phi_t^*(y_0)$ which completes the proof. $\diamond$

Remark. If uniqueness fails either for ODE or for ODE^*, one still has that $\mathcal{Y}^{\infty}$ is left invariant by ODE and ODE^* in the sense that, from every $y_0 \in \mathcal{Y}^{\infty}$, there exists $\mathcal{Y}^{\infty}$-valued solutions of ODE and ODE^*.

This property is the first step of the deep connection between the the asymptotic behavior of recursive stochastic algorithm and its associated mean field ODE. Item (b) can be seen a first a criterion to direct possible candidates to a set of limiting values of the algorithm. This any zero y^* of h, or equivalently equilibrium points of ODE satisfies the requested invariance condition since $\Phi_t(y^*) = \Phi_t(y^*) = y^*$ for every $t \in \mathbb{R}_+$. No other single point can satisfy this invariance property. More generally,we have the following result

Corollary 6.2 *If the $ODE \equiv \dot{y} = -h(y)$ has finitely many compact connected two-sided invariant sets $\mathcal{X}_i$, $i \in I$ (I finite), then the sequence $(y_n)_{n \geq 1}$ converges toward one of these sets.*

As an elementary example let us consider the ODE

$$\dot{y} = (1 - |y|)y + \varsigma y^{\perp}, \ y_0 \in \mathbb{R}^2$$

where $y = (y_1, y_2)$ and $y^{\perp} = (y_2, y_1)$. Then the unit circle $C(0; 1)$ is clearly a connect compact set invariant by ODE and ODE^*. The singleton $\{0\}$ also satisfies this invariant property. In fact $C(0; 1)$ is an attractor of ODE and $\{0\}$ is a repeller. So, we know that any stochastic algorithm which satisfies H_1-H_3 with the above mean field function $h(y_1, y_2) = (1 - |y|)y + \varsigma y^{\perp}$ will converter either toward $C(0, 1)$ or 0.

Sharper characterizations of the possible set of limiting points of the sequence $(y_n)_{n \geq 0}$ have been established in close connection with the theory of perturbed dynamical systems (see [20], see also [50] when uniqueness fails and the ODE has no flow). To be slightly more precise it has been shown that the set of limiting points of the sequence $(y_n)_{n \geq 0}$ is *internally chain recurrent* or, equivalently, contains no strict attractor for the dynamics of the ODE *i.e.* as subset $A \subset \mathcal{Y}^{\infty}$, $A \neq \mathcal{Y}^{\infty}$ such that $\phi_t(y)$ converges to A uniformly in $y \in \mathcal{X}^{\infty}$.

Unfortunately, the above results are not able to discriminate between these two sets though it seems more likely that the algorithm converges toward the unit circle, like the flow of the ODE does (except when starting from 0). This intuition can be confirmed under additional assumptions on the fact that 0 is a noisy for the algorithm, *e.g.* if it is at the origin a Markovian algorithm of the form (6.3), that the symmetric nonnegative matrix

$$\mathbb{E} \, H(0, Z_1)^* H(0, Z_1) \neq 0.$$

Practical aspects of assumption H_3 To make the connection with the original form of stochastic algorithms, we come back to hypothesis H_3 in the following proposition. In particular, it emphasizes that this condition is less stringent than a standard convergence assumption on the series.

Proposition 6.8 *Assumption H_3 is satisfied in tow situations (or their "additive" combination):*

(a) Remainder term: If $\pi_n \to 0$ and $\gamma_n \to 0$ as $n \to +\infty$, then

$$\sup_{t \in [0,T]} \left| \sum_{n+1}^{N(\Gamma_n + t)} \gamma_k \pi_k \right| \leq \sup_{k \geq n+1} |\pi_k| (\Gamma_{N(\Gamma_n + T)} - \Gamma_n) \to 0 \quad as \quad n \to +\infty.$$

(a) Converging martingale term: The series $M_n^\gamma = \sum_{n \geq 1} \gamma_n \Delta M_n$ converges in $\mathbb{R}^d$. Consequently it satisfies a Cauchy condition so that

$$\sup_{t \in [0,T]} \left| \sum_{k=n+1}^{N(\Gamma_n + t)} \gamma_k \pi_k \right| \leq \sup_{\ell \geq n+1} \left| \sum_{\ell \geq n+1} \gamma_k \pi_k \right| \quad as \quad n \to +\infty.$$

In practical situations, one meets the combination of these situations:

$$\pi_n = \Delta M_n + r_n$$

where r_n is a remainder term which goes to 0 a.s. and M_n^γ is an a.s. convergent martingale.

This convergence property can even be relaxed for the martingale term. Thus we have the following classical results where H_3 is satisfied while the martingale M_n^γ may diverge.

Proposition 6.9 *(a) [Métivier-Priouret, [?]] Let $(\Delta M_n)_{n \geq 1}$ be a sequence of martingale incre-ments and and $(\gamma_n)_{n \geq 1}$ be a sequence of nonnegative steps satisfying $\sum_n \gamma_n = +\infty$. Then, H_3 is a.s. satisfied (with $\pi_n = \Delta M_n$) as soon as there exists a couple of Hölder conjugate exponents $(p,q) \in [1, +\infty)^2$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$) such that*

$$\sup_n \mathbb{E} |\Delta M_n|^p < +\infty \quad and \quad \sum_n \gamma_n^{1 + \frac{q}{2}} < +\infty$$

This allows for the use of steps of the form $\gamma_n \sim c_1 n^{-a}$, $a > \frac{2}{2+q} = \frac{2(p-1)}{3(p-1)+2}$.

(b) Assume that there exist a real number $c > 0$ such that,

$$\forall \lambda \in \mathbb{R}, \quad \mathbb{E} e^{\lambda \delta M_n} \leq e^{c \frac{\lambda^2}{2}}.$$

Then for every sequence $(\gamma_n)_{n \geq 0}$ such that $\sum_{n \geq 1} e^{-\frac{c}{\gamma_n}} < +\infty$, Assumption H_3 is satisfied with $\pi_n = \Delta M_n$. This allows for the use of steps of the form $\gamma_n \sim c_1 n^{-a}$, $a > 0$, and $\gamma_n = c_1 (\log n)^{-1+a}$, $a > 0$.

EXAMPLES: Typical examples where the sub-Gaussian assumption is satisfied are the following:

- $|\Delta M_n| \leq K \in \mathbb{R}_+$ since, owing to Hoeffding Inequality, $\mathbb{E} e^{\lambda \Delta M_n} \leq e^{\frac{\lambda^2}{8} K^2}$.

- $\Delta M_n \sim \mathcal{N}(0; \sigma_n^2)$ with $\sigma_n \leq K$, then $\mathbb{E} e^{\lambda \Delta M_n} \leq e^{\frac{\lambda^2}{2} K^2}$

The first case is very important since in many situations the perturbation term is a martingale term and is structurally bounded.

Application to an extended Lyapunov approach

By Relying on claim (a) in Proposition 6.7, one can also derive directly *a.s.* convergence results for an algorithm.

Proposition 6.10 (G-Lemma, see [50]) *Assume H_1-H_3. Let $G : \mathbb{R}^d \to \mathbb{R}_+$ be a function satisfying*

$$(\mathcal{G}) \equiv \left(l|im_{n \to +\infty} y_n = y_\infty \quad and \quad \lim_{n \to +\infty} G(y_n) = 0 \right) \Longrightarrow G(y_\infty) = 0.$$

Assume that the sequence $(y_n)_{n \geq 0}$ satisfies

$$\sum_{n \geq 0} \gamma_{n+1} G(y_n) < +\infty. \tag{6.35}$$

Then, there exists a connected component $\mathcal{X}^*$ of the set $\{G = 0\}$ such that $\mathrm{diet}(y_n, \mathcal{X}^*) = 0$.

Proof. First, it follows from Proposition (6.7) that the sequence $(y_n^{(n)})_{n \geq 0}$ is U_K-relatively compact with limiting functions lying in $\mathcal{C}(\mathbb{R}_+, \mathcal{Y}^\infty)$ where $\mathcal{Y}^\infty$ still denotes the compact connected set of limiting values of $(y_n)_{n \geq 0}$.

Set, for every $y \in \mathbb{R}^d$, $\widetilde{G}(y) = \liminf_{x \to y} G(x) = \inf \left\{ \liminf_n G(x_n), \ x_n \to y \right\}$ so that $0 \leq \widetilde{G} \leq G$.

The function $\widetilde{G}$ is the l.s.c. envelope of the function G *i.e.* the highest l.s.c. function not greater than G. In particular, under Assumption $(\mathcal{G})$

$$\{G = 0\} = \{\widetilde{G} = 0\} \ \text{ is closed.}$$

Assumption (6.35) reads

$$\int_0^{+\infty} G(y_s^{(0)}) ds < +\infty.$$

Let $y_\infty \in \mathcal{Y}^\infty$. Up to at most two extractions, one may assume that $y^{(\varphi(n))} \to y^{(\infty)}$ for the U_K topology where $y_0^{(\infty)} = y_\infty$. It follows from Fatou's Lemma that

$$
\begin{aligned}
0 \leq \int_0^{+\infty} \widetilde{G}(y_s^{(\infty)}) ds \ &= \ \int_0^{+\infty} \widetilde{G}(\lim_n y_s^{(\varphi(n))}) ds \\
&\leq \ \int_0^{+\infty} \lim_n \widetilde{G}(y_s^{(\varphi(n))}) ds \quad \text{(since } \widetilde{G} \text{ is l.s.c.)} \\
&\leq \ \lim_n \int_0^{+\infty} \widetilde{G}(y_s^{(\varphi(n))}) ds \quad \text{(owing to Fatou's lemma)} \\
&\leq \ \lim_n \int_0^{+\infty} G(y_s^{(\varphi(n))}) ds \\
&= \ \lim_n \int_{\Gamma_{\varphi(n)}}^{+\infty} G(y_s^{(0)} ds = 0.
\end{aligned}
$$

Consequently, $\int_0^{+\infty} \widetilde{G}(y_s^{(\infty)}) ds = 0$ which implies that $\widetilde{G}(y_s^{(\infty)}) = 0$ ds-a.s.. Now $s \mapsto y_s^{(\infty)}$ is continuous it follows that $\widetilde{G}(y_0^{(\infty)}) = 0$ since $\widetilde{G}$ is l.s.c. Which in turn implies $G(y_0^\infty) = 0$ *i.e.*

$G(y_\infty) = 0$. As a consequence $\mathcal{Y}^\infty \subset \{G = 0\}$ which yields the result since on the other hand it is a connected set. $\quad \diamond$

Now we are in position to prove the convergence of stochastic and stochastic pseudo-gradient procedures in the multi-dimensional case.

Corollary 6.3

Proof. $\quad \diamond$

6.4.2 L^2-rate of convergence

Proposition 6.11 *Let $(Y_n)_{n \geq 1}$ be a stochastic algorithm defined by (6.3) where the function H satisfies the quadratic linear growth assumption*

$$\forall\, y \in \mathbb{R}^d, \qquad \|H(y, Z)\|_2 \leq C(1 + |y|)$$

and the step sequence $(\gamma_n)_{n \geq 1}$ satisfies the usual decreasing step assumption (6.7). Assume $Y_0 \in L^2(\mathbb{P})$ and Y_0 independent of the i.i.d. sequence $(Z_n)_{n \geq 1}$.

If there exists $y^* \in \mathbb{R}^d$ and $\alpha > 0$ such that both the strong mean-reverting assumption

$$\forall\, y \in \mathbb{R}^d, \qquad (y - y^* | h(y)) > \alpha |y - y^*|^2 \tag{6.36}$$

and the additonal assumption on the step sequence (γ_n)

$$(G)_\alpha \;\equiv\; \limsup_n \left[a_n = \frac{1}{\gamma_{n+1}} \left(\frac{\gamma_n}{\gamma_{n+1}} \left(1 - 2\,\alpha\,\gamma_{n+1}\right) - 1 \right) \right] = -\kappa^* < 0.$$

hold, then

$$Y_n \xrightarrow{a.s.} y^* \quad and \quad \|Y_n - y^*\|_2 = O(\sqrt{\gamma_n}).$$

EXAMPLES. • If $\gamma_n = \dfrac{\gamma}{n^\vartheta}$, $\frac{1}{2} < \vartheta < 1$, then $(G)_\alpha$ is satisfied for any $\alpha > 0$.

• If $\gamma_n = \dfrac{\gamma}{n}$, Condition $(G)_\alpha$ reads $\dfrac{1 - 2\,\alpha\,\gamma}{\gamma} = -\kappa^* < 0$ or equivalently $\gamma > \dfrac{1}{2\,\alpha}$.

Proof: The fact that $Y_n \to y^*$ is a straightforward consequence of Corollary 6.1 (Robbins-Monro). As concerns the quadratic rate of convergence, we re-start from the classical proof of Robbins-Siegmund's Lemma.

$$|Y_{n+1} - y^*|^2 = |Y_n - y^*|^2 - 2\gamma_{n+1}(H(Y_n, Z_{n+1})|Y_n - y^*) + \gamma_{n+1}^2 |H(Y_n, Z_{n+1})|^2.$$

Since $Y_n - y^*$ is $\mathcal{F}_n$-measurable, one has

$$\mathbb{E}\Big((H(Y_n, Z_{n+1})|Y_n - y^*) \Big) = \mathbb{E}\Big((\mathbb{E}(H(Y_n, Z_{n+1})|\mathcal{F}_n^Z)|Y_n - y^*) \Big) = \mathbb{E}(h(Y_n)|Y_n - y^*).$$

Likewise we get

$$\mathbb{E}\,|H(Y_n, Z_{n+1})|^2 \leq C^2(1 + |Y_n|^2).$$

This implies

$$
\begin{aligned}
\mathbb{E}\,|Y_{n+1}-y^*|^2 &= \mathbb{E}\,|Y_n-y^*|^2 - 2\gamma_{n+1}\mathbb{E}(h(Y_n)|Y_n-y^*) + \gamma_{n+1}^2\mathbb{E}\,|H(Y_n,Z_{n+1})|^2 \\
&\le \mathbb{E}\,|Y_n-y^*|^2 - 2\gamma_{n+1}\mathbb{E}\,(h(Y_n)|Y_n-y^*) + \gamma_{n+1}^2 C^2(1+\mathbb{E}\,|Y_n|^2) \\
&\le \mathbb{E}\,|Y_n-y^*|^2 - 2\,\alpha\,\gamma_{n+1}\mathbb{E}\,|Y_n-y^*|^2 + \gamma_{n+1}^2 C'(1+\mathbb{E}\,|Y_n-y^*|^2)
\end{aligned}
$$

owing successively to the linear quadratic growth and the strong mean-reverting assumptions. Finally,

$$
\mathbb{E}\,|Y_{n+1}-y^*|^2 = \mathbb{E}\,|Y_n-y^*|^2\Big(1-2\,\alpha\,\gamma_{n+1}+C'\gamma_{n+1}^2\Big)+C'\gamma_{n+1}^2.
$$

If we set for every $n \ge 1$,

$$
u_n = \frac{\mathbb{E}\,|Y_n-y^*|^2}{\gamma_n}, \quad n \ge 1,
$$

the above inequality can be rewritten using the expression for a_n,

$$
\begin{aligned}
u_{n+1} &\le u_n\frac{\gamma_n}{\gamma_{n+1}}\Big(1-2\,\alpha\,\gamma_{n+1}+C'\gamma_{n+1}^2\Big)+C'\gamma_{n+1} \\
&= u_n\Big(1+\gamma_{n+1}(a_n+C'\gamma_n)\Big)+C'\gamma_{n+1}.
\end{aligned}
$$

Let n_0 be an integer such that, for every $n \ge n_0$, $a_n \le -\frac{3}{4}\kappa^*$ and $C'\gamma_n \le \frac{\kappa^*}{4}$. For these integers n,

$$
u_{n+1} \le u_n\Big(1-\frac{\kappa^*}{2}\gamma_{n+1}\Big)+C'\gamma_{n+1}.
$$

Then, one derives by induction that

$$
\forall\, n \ge n_0, \qquad 0 \le u_n \le \max\Big(u_{n_0},\frac{2C'}{\kappa^*}\Big)
$$

which completes the proof. ◇

▷ **Exercise.** Show a similar result (under appropriate assumptions) for an algorithm of the form

$$
Y_{n+1} = Y_n - \gamma_{n+1}(h(Y_n)+\Delta M_{n+1})
$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is Borel continuous function and $(\Delta M_n)_{n\ge1}$ a sequence of $\mathcal{F}_n$-martingale increments satsifying

$$
|h(y)| \le C(1+|y-y^*|) \ \text{ and } \ \sup_n \mathbb{E}\,\big(|\Delta M_{n+1}|^2\,|\,\mathcal{F}_n\big) < C(1+|Y_n|^2).
$$

6.4.3 Weak rate of convergence: CLT

The CLT for stochastic algorithms. In standard settings, a stochastic algorithm converges to its target at a $\sqrt{\gamma_n}$ rate (which suggests to use some rates $\gamma_n = \frac{c}{n}$). To be precise $\frac{Y_n-y_*}{\sqrt{\gamma_n}}$ converges in distribution to some normal distribution with a dispersion matrix based on $H(y_*,Z)$.

The central Limit theorem has given raise to an extensive literature starting from some pioneering (independent) works by Bouton and Kushner in the early 1980's. We give here a result obtained by Pelletier which has as a main asset to be "local" in the following sense. The CLT holds on the set of convergence of the algorithm to an equilibrium which makes possible a straightforward application to multi-target algorithms. It can probably be of significant help to elucidate the rate of convergence of algorithms with constraints or multiple projections like those recently introduced by Chen. So is the case of Arouna's adaptive variance reduction procedure for which a CLT has been recently established by Lelong ([101]) using a direct approach.

Theorem 6.6 (Pelletier (1995) [134]) *We cosnider the stochastic procedure defined by (6.3). Let y_* be an equilibrium point of $\{h = 0\}$.*

(i) The equilibrium point y_* is an attractor for ODE_h: Assume that y_* is a "strongly" attractor for the $ODE \equiv \dot{y} = -h(y)$ in the following sense ([8]):

> *h is is differentiable at y_* and all the eigenvalues of $Dh(y_*)$ have positive real parts.*

(ii) Regularity and growth control of H: Assume that the function H satisfies the following regularity and growth control property

> *$y \mapsto \mathbb{E}\, H(y, Z)H(y, Z)^t$ is continuous at y^* and $y \mapsto \mathbb{E}\, |H(y, Z)|^{2+\delta}$ is locally bounded on $\mathbb{R}^d$*

for some $\delta > 0$.

(iii) Non-degenerate asymptotic variance: Assume that the covariance matrix of $H(y_*, Z)$

$$\Sigma_H(y_*) := \mathbb{E}\big(H(y_*, Z)(H(y_*, Z))^t\big) \quad \text{is positive definite in } \ \mathcal{S}(d, \mathbb{R}). \tag{6.37}$$

(iv) Specification of the the step sequence:

$$\forall\, n \geq 1, \quad \gamma_n = \frac{c}{b + n^\vartheta}, \quad b,\, c > 0,\ \frac{1}{2} < \vartheta \leq 1$$

with the additional constraint, when $\vartheta = 1$,

$$c > \frac{1}{2\Re e(\lambda_{min})} \tag{6.38}$$

(λ_{min} denotes the eigenvalue of $Dh(y_*)$ with the lowest real part).

> *Then, the a.s. convergence is ruled on the convergence event $\{Y_n \to y_*\}$ by the following Central Limit Theorem*

$$\sqrt{n}\,(Y_n - y_*) \xrightarrow{\mathcal{L}_{stably}} \mathcal{N}(0, \alpha\Sigma), \tag{6.39}$$

with $\ \Sigma := \displaystyle\int_0^{+\infty} \left(e^{-(Dh(y_*) - \frac{I_d}{2c_*})s}\right)^t \Sigma_H(y_*)e^{-(Dh(y_*) - \frac{I_d}{2c_*})s}\, ds,\ c_* = +\infty\ \text{if } \vartheta \neq 1\ \text{and } c_* = c\ \text{if } \vartheta = 1.$

---

[8]This is but a locally uniform attractivity condition for y_* viewed as an equilibrium the $ODE\ \dot{y} = -h(y)$.

The "$\mathcal{L}_{stably}$" *stable convergence in distribution* mentioned in (6.39) means that for every bounded continuous function and every $A \in \mathcal{F}$,

$$\mathbb{E}\left(\mathbf{1}_{\{Y_n \to y_*\} \cap A} \, f(\sqrt{n}(Y_n - y_*))\right) \overset{n \to \infty}{\longrightarrow} \mathbb{E}\left(\mathbf{1}_{\{Y_n \to y_*\} \cap A} \, f(\sqrt{\alpha}\sqrt{\Sigma}\,\zeta)\right), \quad \zeta \sim \mathcal{N}(0; I_d).$$

Remarks. • Other rates can be obtained when $\vartheta = 1$ and $c < \frac{1}{2\Re e(\lambda_{min})}$ or, more generally, when the step go to 0 faster although it satisfies the usual decreasing step Assumption (6.7). So is the case for example when $\gamma_n = \frac{c}{b \log(n+1)n}$. Thus, in the latter case, one shows that there exists an $\mathbb{R}^d$-valued random vector Ξ such that

$$Y_n = y_* + \sqrt{\gamma_n}\,\Xi + o(\sqrt{\gamma_n}) \quad a.s. \text{ as } n \to +\infty.$$

• When $d = 1$, then $\lambda_{\min} = h'(y_*)$, $\Sigma_H(y_*) = \mathrm{Var}(H(y_*, Z))$ and

$$c\,\Sigma = \mathrm{Var}(H(y_*, Z))\frac{c^2}{2c\,h'(y_*) - 1}.$$

which reaches its minimum as a function of c at

$$c_{opt} = \frac{1}{h'(y_*)}$$

with a resulting asymptotic variance

$$\frac{\mathrm{Var}(H(y_*, Z))}{h'(y_*)^2}.$$

One shows that this is the lowest possible variance in such a procedure. Consequently, the best choice for the step sequence (γ_n) is

$$\gamma_n := \frac{1}{h'(y_*)n} \quad \text{or, more generally,} \quad \gamma_n := \frac{1}{h'(y_*)(b+n)}$$

where b can be tuned to "control" the step at the beginning of the simulation (when n is small).

At this stage, one encounters the same difficulties as with deterministic procedures since y_* being unknown, $h'(y_*)$ is "more" known. One can imagine to estimate this quantity as a companion procedure of the algorithm but this turns out to be not very efficient. A more efficient approach, although not completely satisfactory in practice, is to implement the algorithm in its averaged version (see Section 6.4.4 below).

However one must have in mind that this tuning of the step sequence is intended to optimize the rate of convergence of the algorithm during its final convergence phase. In real applications, this class of recursive procedures spends post of its time "exploring" the state space before getting trapped in some attracting basin (which can be the basin of a local minimum in case of multiple critical points). The CLT rate occurs once the algorithm is trapped.

An alternative to these procedures is to design some simulated annealing procedure which "super-excites" the algorithm in order to improve the exploring phase. Thus, when the mean function h is a gradient ($h = \nabla L$), it finally converges – but only *in probability* – to the true minimum of the potential/Lyapunov function L. The "final" convergence rate is worse owing

to the additional exciting noise. Practitioners often use the above Robbins-Monro or stochastic gradient procedure with a sequence of steps $(\gamma_n)_{n\geq 1}$ which decreases to a positive limit $\underline{\gamma}$.

We now prove this CLT in the $1D$-framework when the algorithm *a.s.* converges toward a unique "target" y_*. Our method of proof is the so-called SDE method which heavily relies on weak functional convergence arguments. We will have to admit few important results about weak convergence of processes for which we provide precise references. An alternative proof can be carried out relying on the CLT for triangular arrays of martingale increments (see [70]). Thus, such an alternative proof – in a one dimensional setting – can be found in [105].

We purpose below a partial proof of Theorem 6.6, dealing only withe case where the equilibrium point y^* is unique. The extension to a multi-target algorithm is not really more involved and we refer to the original paper [134].

Proof of Theorem 6.6.   We will not need the Markovian feature of stochastic algorithms we are dealing with in this chapter. In fact it will be more useful to decompose the algorithm in its canonical form

$$Y_{n+1} = Y_n - \gamma_{n+1}\Big(h(Y_n) + \Delta M_{n+1}\Big)$$

where

$$\Delta M_n = H(Y_{n-1}, Z_n) - h(Y_{n-1}), \quad n \geq 1,$$

is a sequence of $\mathcal{F}_n$-martingale increments where $\mathcal{F}_n = \sigma(Y_0, Z_1, \ldots, Z_n)$, $n \geq 0$. The so-called SDE method follows the same principle as the ODE method but with the quantity of interest

$$\Upsilon_n := \frac{Y_n - y_*}{\sqrt{\gamma_n}}, \quad n \geq 1$$

(this normalization is strongly suggested by the above L^2-convergence rate theorem). The underlying idea is to write a recursion on Υ_n which appears as an Euler scheme with decreasing step γ_n of an SDE having a stationary/steady regime.

STEP 1(*Toward the SDE*): So, we assume that

$$Y_n \xrightarrow{a.s.} y^* \in \{h = 0\}.$$

We may assume (up to a change of variable by the translation $y \leftarrow y - y^*$) that

$$y^* = 0.$$

The differentiability of h at $y^* = 0$ reads

$$h(Y_n) = Y_n h'(0) + Y_n \eta(Y_n) \quad \text{with} \quad \lim_{y \to 0} \eta(y) = \eta(y^*) = 0.$$

Moreover the function η is bounded on the real line owing to the linear growth of h. For every

$n \geq 1$, we have

$$
\begin{aligned}
\Upsilon_{n+1} &= \sqrt{\frac{\gamma_n}{\gamma_{n+1}}} \Upsilon_n - \sqrt{\gamma_{n+1}} \Big(h(Y_n) + \Delta M_{n+1} \Big) \\
&= \sqrt{\frac{\gamma_n}{\gamma_{n+1}}} \Upsilon_n - \sqrt{\gamma_{n+1}} Y_n \Big(h'(0) + \eta(Y_n) \Big) + \sqrt{\gamma_{n+1}} \Delta M_{n+1} \\
&= \Upsilon_n - \Upsilon_n + \sqrt{\frac{\gamma_n}{\gamma_{n+1}}} \Upsilon_n - \sqrt{\gamma_{n+1}} \sqrt{\gamma_n} \Upsilon_n \Big(h'(0) + \eta(Y_n) \Big) + \sqrt{\gamma_{n+1}} \Delta M_{n+1} \\
&= \Upsilon_n - \gamma_{n+1} \Upsilon_n \left(\sqrt{\frac{\gamma_n}{\gamma_{n+1}}} \Big(h'(0) + \eta(Y_n) \Big) - \frac{1}{\gamma_{n+1}} \left(\sqrt{\frac{\gamma_n}{\gamma_{n+1}}} - 1 \right) \right) + \sqrt{\gamma_{n+1}} \Delta M_{n+1}.
\end{aligned}
$$

Assume that the sequence $(\gamma_n)_{n \geq 1}$ is such that there exists $c \in (0, +\infty]$ satisfying

$$
\lim_n \left[a'_n = \frac{1}{\gamma_{n+1}} \left(\sqrt{\frac{\gamma_n}{\gamma_{n+1}}} - 1 \right) \right] = \frac{1}{2c}.
$$

Note that this implies $\lim_n \frac{\gamma_n}{\gamma_{n+1}} = 1$. In fact it is satisfied by our two families of step sequences of interest since

- if $\gamma_n = \frac{c}{b+n}$, $c > 0$, $b \geq 0$, then $\lim_n a'_n = \frac{1}{2c} > 0$,
- if $\gamma_n = \frac{c}{n^\vartheta}$, $c > 0$, $\frac{1}{2} < \vartheta < 1$, then $\lim_n a'_n = 0$ *i.e.* $c = +\infty$.

Consequently, for every $n \geq 1$,

$$
\Upsilon_{n+1} = \Upsilon_n - \gamma_{n+1} \Upsilon_n \Big(h'(0) - \frac{1}{2c} + \alpha_n^1 + \alpha_n^2 \eta(Y_n) \Big) + \sqrt{\gamma_{n+1}} \Delta M_{n+1}
$$

where $(\alpha_n^i)_{n \geq 1}$ $i = 1, 2$ are two deterministic sequences such that $\alpha_n^1 \to 0$ and $\alpha_n^2 \to 1$ as $n \to +\infty$ 0.

STEP 2 (*Localisation(s)*): Since $Y_n \to 0$ *a.s.*, one can write the scenarios space Ω as follows

$$
\forall \varepsilon > 0, \qquad \Omega = \bigcup_{N \geq 1} \Omega_{\varepsilon,N} \quad a.s. \quad \text{where} \quad \Omega_{\varepsilon,N} := \Big\{ \sup_{n \geq N} |Y_n| \leq \varepsilon \Big\}.
$$

Let $\varepsilon > 0$ and $N \geq 1$ being temporarily free parameters. We define the function $\widetilde{h} = \widetilde{h}_\varepsilon$ by

$$
\forall y \in \mathbb{R}, \qquad \widetilde{h}(y) = h(y) \mathbf{1}_{\{|y| \leq \varepsilon\}} + K y \mathbf{1}_{\{|y| > \varepsilon\}}
$$

($K = K(\varepsilon)$ is also a parameter to be specified further on) and

$$
\begin{cases}
\widetilde{Y}_N^{\varepsilon,N} = Y_N \mathbf{1}_{\{|Y_N| \leq \varepsilon\}}, \\[2mm]
\widetilde{Y}_{n+1}^{\varepsilon,N} = \widetilde{Y}_n^{\varepsilon,N} - \gamma_{n+1} \Big(\widetilde{h}_\varepsilon(\widetilde{Y}_n^{\varepsilon,N}) + \mathbf{1}_{\{|Y_n| \leq \varepsilon\}} \Delta M_{n+1} \Big), \; n \geq N.
\end{cases}
$$

It is straightforward to show by induction that for every $\omega \in \Omega_{\varepsilon,N}$,

$$
\forall n \geq N, \quad Y_n^{\varepsilon,N}(\omega) = Y_n(\omega).
$$

To alleviate notations we will drop the exponent $^{\varepsilon,N}$ in what follows and denote $\widetilde{Y}_n$ instead of $\widetilde{Y}_n^{\varepsilon,N}$.

The "characteristics" (mean function and $\mathcal{F}_n$-martingale increments) of this new algorithm are

$$h \rightsquigarrow \widetilde{h} \quad \text{and} \quad \Delta\widetilde{M}_{n+1} = \mathbf{1}_{\{|Y_n|\leq\varepsilon\}}\Delta M_{n+1},\ n \geq N$$

which satisfy

$$\sup_{n\geq N} \mathbb{E}(|\Delta\widetilde{M}_{n+1}|^{2+\rho}|\mathcal{F}_n) \leq 2^{1+\rho} \sup_{|\theta|\leq\varepsilon}(\mathbb{E}|H(\theta,X)|^{2+\rho}) \leq A(\varepsilon) < +\infty \quad a.s..$$

In what follows we will study the normalized error defined by

$$\widetilde{\Upsilon}_n := \frac{\widetilde{Y}_n}{\sqrt{\gamma_n}},\ n \geq N$$

STEP 3: (*Specification of ε and $K = K(\varepsilon)$*) Since h is differentiable at 0 (and $h'(0) > 0$),

$$h(y) = y(h'(0) + \eta(y)) \quad \text{with} \quad \lim_0 \eta(y) = \eta(0) = 0.$$

If $\gamma_n = \frac{c}{n+b}$ with $c > \frac{1}{2h'(0)}$ and $b \geq 0$, (as prescribed in the statement of the theorem), we may choose $\rho = \rho(h) > 0$ small enough so that

$$c > \frac{1}{2h'(0)(1-\rho)}.$$

If $\gamma_n = \frac{c}{(n+b)^\vartheta}$, $\frac{1}{2} < \vartheta < 1$ and, more generally, as soon as $\lim_n a' = 0$, any choice of $\rho \in (0,1)$ is possible. Now let $\varepsilon(\rho) > 0$ such that $|y| \leq \varepsilon(\rho)$ implies $|\eta(y)| \leq \rho h'(0)$. It follows that

$$\theta h(y) = y^2(h'(0) + \eta(y)) \geq y^2(1-\rho)h'(0) \quad \text{if} \quad |y| \leq \eta(\rho).$$

Now we specify

$$\varepsilon = \varepsilon(\rho) \qquad \text{and} \qquad K = (1-\rho)h'(0) > 0$$

As a consequence, the function $\widetilde{h}$ satisfies

$$\forall\, y \in \mathbb{R}, \quad y\widetilde{h}(y) \geq Ky^2$$

so that, owing to the L^2-rate Theorem,

$$\|\widetilde{Y}_n\|_2 = O(\sqrt{\gamma_n}).$$

since $(G)_\alpha$ is satisfied with $\alpha = K$ and $\kappa^* = 2K - \frac{1}{c} > 0$.

STEP 4: (The *SDE* method) First we apply Step 1 to our ˜ed framework and we write

$$\widetilde{\Upsilon}_{n+1} = \widetilde{\Upsilon}_n - \gamma_{n+1}\widetilde{\Upsilon}_n\Big(h'(0) - \frac{1}{2c} + \widetilde{\alpha}_n^1 + \widetilde{\alpha}_n^2\widetilde{\eta}(\widetilde{Y}_n)\Big) + \sqrt{\gamma_{n+1}}\Delta\widetilde{M}_{n+1}$$

where $\widetilde{\alpha}_n^1 \to 0$ and $\widetilde{\alpha}_n^2 \to 1$ are two deterministic sequences and $\widetilde{\eta}$ is a bounded function.

At this stage, we re-write the above recursive equation in continuous time exactly like we did for the *ODE* method. Namely, we set

$$\Gamma_n = \gamma_1 + \cdots + \gamma_n \quad \text{and} \quad \widetilde{\Upsilon}^{(0)}_{(\Gamma_n)} = \widetilde{\Upsilon}_n, \quad n \geq N,$$

and

$$\forall\, t \in \mathbb{R}_+, \quad N(t) = \min \left\{ k \,|\, \Gamma_{k+1} \geq t \right\}.$$

Hence, setting $a = h'(0) - \frac{1}{2c} > 0$, we get for every $n \geq N$,

$$\widetilde{\Upsilon}^{(0)}_{(\Gamma_n)} = \widetilde{\Upsilon}_N - \int_{\Gamma_N}^{\Gamma_n} \widetilde{\Upsilon}^{(0)}_{(t)} \left(a + \widetilde{\alpha}^1_{N(t)} + \widetilde{\alpha}^2_{N(t)} \widetilde{\varepsilon}(\widetilde{Y}_{N(t)}) \right) dt + \sum_{k=N}^{n} \sqrt{\gamma_k} \Delta \widetilde{M}_{k+1}.$$

Finally, for every $t \geq \Gamma_N$, we set

$$\widetilde{\Upsilon}^{(0)}_{(t)} = \widetilde{\Upsilon}_N - \underbrace{\int_{\Gamma_N}^{t} \widetilde{\Upsilon}^{(0)}_{(s)} \left(\rho + \widetilde{\alpha}^1_{N(s)} + \widetilde{\alpha}^2_{N(s)} \widetilde{\varepsilon}(\widetilde{Y}_{N(s)}) \right) ds}_{=: \widetilde{A}^{(0)}_{(t)}} + \underbrace{\sum_{k=N}^{N(t)} \sqrt{\gamma_k} \Delta \widetilde{M}_{k+1}}_{=: \widetilde{M}^{(0)}_{(t)}}.$$

As for the *ODE*, we will be interested in the functional asymptotics at infinity of $\widetilde{\Upsilon}^{(0)}_{(t)}$ but this time in a weak sense. We set

$$\widetilde{\Upsilon}^{(n)}_{(t)} = \Upsilon^{(0)}_{(\Gamma_n + t)}, \; t \geq 0, \; n \geq N.$$

and

$$\widetilde{A}^{(n)}_{(t)} := \widetilde{A}^{(0)}_{(\Gamma_n + t)} \quad \text{and} \quad \widetilde{M}^{(n)}_{(t)} = \widetilde{M}^{(0)}_{(\Gamma_n + t)}.$$

At this stage, we need two fundamental results about functional weak convergence. The first one is a criterion which implies the functional tightness of the distributions of a sequence of right continuous left limited (càdlàg) processes $X^{(n)}$ (viewed as probability measures on the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg functions form $\mathbb{R}_+$ to $\mathbb{R}$. The second one is an extension of Donsker's Theorem for sequences of martingales.

We need to introduce the uniform continuity modulus defined for every function $f : \mathbb{R}_+ \to \mathbb{R}$ and $\delta, T > 0$ by

$$w(f, \delta, T) = \sup_{s, t \in [0, T], |s - t| \leq \delta} |f(t) - f(s)|.$$

The terminology comes form the seminal property of this modulus: f is (uniformly) continuous over $[0, T]$ if and only if $\lim_{\delta \to 0} w(f, \delta, T) = 0$.

Theorem 6.7 (A criterion for C-tightness) *([24], Theorem 15.5, p.127) (a) Let $(X^n_{(t)})_{t \geq 0}$, $n \geq 1$, be a sequence of càdlàg processes processes null at $t = 0$. If, for every $T > 0$ and every $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_n \mathbb{P}(w(X^{(n)}, \delta, T) \geq \varepsilon) = 0$$

then the sequence $(X^n)_{n \geq 1}$ is C-tight in the following sense: from any subsequence $(X^{n'})_{n \geq 1}$ one may extract a subsequence $(X^{n''})_{n \geq 1}$ such that X^n converges in distribution toward a process X^∞ for the weak topology on the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ (endowed with the topology of uniform convergence on compact sets) such that $\mathbb{P}(X^\infty \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})) = 1$.

(b) *(See [24], proof of Theorem 8.3 p.56) If, for every $T > 0$ and every $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \sup_{s \in [0,T]} \limsup_{n} \mathbb{P}(\sup_{s \leq t \leq t+\delta} |X_t^{(n)} - X_s^{(n)}| \geq \varepsilon) = 0$$

then the above condition in (a) *is satisfied.*

The second theorem provides a tightness criterion for a sequence of martingales based on the sequence of their bracket processes.

Theorem 6.8 (Weak functional limit of a sequence of martingales) *([73]). Let $(M_{(t)}^n)_{t \geq 0}$, $n \geq 1$, be a sequence of càdlàg (local) martingales, null at 0, C-tight, with (existing) predictable bracket process $\langle M^n \rangle$. If*

$$\forall\, t \geq 0, \qquad \langle M^n \rangle_{(t)} \overset{n \to +\infty}{\longrightarrow} \sigma^2 t, \quad \sigma > 0,$$

then

$$M^n \overset{\mathcal{L}_{\mathbb{D}(\mathbb{R}_+, \mathbb{R})}}{\longrightarrow} \sigma W,$$

where W denotes a standard Brownian motion ([9]).

Now we can apply these results to the processes $\widetilde{A}_{(t)}^{(n)}$ and $\widetilde{M}_{(t)}^{(n)}$.

First we aim at showing that the sequence of continuous processes $(\widetilde{A}^{(n)})_{n \geq 1}$ is C-tight. Since $\sup_{n \geq N, t \geq 0} |\widetilde{\varepsilon}(\widetilde{Y}_{N(t)})| \leq \|\widetilde{\varepsilon}\|_{\sup} < +\infty$, the sequence $(A^{(n)})_{n \geq N}$ of time integrals satisfies

$$\forall\, s \in [\Gamma_N, +\infty), \quad \mathbb{E} \sup_{s \leq t \leq s+\delta} |\widetilde{A}_{(t)}^{(0)} - \widetilde{A}_{(s)}^{(0)}|^2 \leq C_{\|\widetilde{\varepsilon}\|_{\sup}, \|\widetilde{\alpha}^i\|_{\sup}} \sup_{n \geq N} \mathbb{E}|\widetilde{\Upsilon}_n|^2 \times \delta^2.$$

Hence

$$\forall\, n \geq N,\, \forall\, s \in \mathbb{R}_+, \quad \mathbb{P}(\sup_{s \leq t \leq s+\delta} |\widetilde{A}_{(t)}^{(n)} - \widetilde{A}_{(s)}^{(n)}| \geq \varepsilon) \leq \frac{C_{\|\widetilde{\varepsilon}\|_{\sup}, \|\widetilde{\alpha}^i\|_{\sup}} \sup_n \mathbb{E}|\widetilde{\Upsilon}_n|^2 \delta^2}{\varepsilon^2}.$$

and one concludes that the sequence $(\widetilde{A}^{(n)})_{n \geq 1}$ is C-tight by applying the above Theorem **??**.

We can apply also this result to the weak asymptotics of the martingales $(M_{(t)}^{(n)})$. We start from the definition of the martingale $\widetilde{M}^{(0)}$ defined by

$$\widetilde{M}_{(t)}^{(0)} = \sum_{k=1}^{N(t)} \sqrt{\gamma_k} \Delta \widetilde{M}_{k+1}$$

(the related filtration is $\mathcal{F}_t^{(0)} = \mathcal{F}_n$, $t \in [\Gamma_n, \Gamma_{n+1})$). Since we know that

$$\sup_n \mathbb{E}|\Delta \widetilde{M}_{n+1}|^{2+\delta} \leq A(\varepsilon) < +\infty$$

---

[9]This means that for every bounded functional $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$, measurable with respect to σ-field spanned by finite projection $\alpha \mapsto \alpha(t)$, $t \in \mathbb{R}_+$, and continuous at every $\alpha \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, one has $\mathbb{E}F(M^n) \to \mathbb{E}F(\sigma W)$ as $n \to +\infty$. This rermains true in fact for measurable functionals F which are $\mathbb{P}_{\sigma W}(d\alpha)$-a.s. continuous on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, such that $(F(M^n))_{n \geq 1}$ is uniformly integrable.

we get, owing to B.D.G. Inequality, that, for every $\rho > 0$ and every $s \in \mathbb{R}_+$,

$$
\mathbb{E} \sup_{s \le t \le s+\delta} \left| \widetilde{M}_{(t)}^{(0)} - \widetilde{M}_{(s)}^{(0)} \right|^{2+\rho} \le C_\rho \mathbb{E} \left(\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k (\Delta \widetilde{M}_k)^2 \right)^{1+\frac{\rho}{2}}
$$

$$
\le C_\rho \left(\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k \right)^{1+\frac{\rho}{2}} \mathbb{E} \left(\frac{\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k (\Delta \widetilde{M}_k)^2}{\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k} \right)^{1+\frac{\rho}{2}}
$$

$$
\le C_\rho \left(\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k \right)^{1+\frac{\rho}{2}} \mathbb{E} \left(\frac{\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k |\Delta \widetilde{M}_k|^{2+\rho}}{\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k} \right)
$$

$$
\le C_\rho \left(\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k \right)^{\frac{\rho}{2}} \sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k \mathbb{E} |\Delta \widetilde{M}_k|^{2+\rho}
$$

where C_ρ is a positive real constant. One finally derives that, for every $s \in \mathbb{R}_+$,

$$
\begin{aligned}
\mathbb{E} \sup_{s \le t \le t+\delta} \left| M_{(t)}^{(0)} - M_{(s)}^{(0)} \right|^{2+\rho} &\le& C_\rho A(\varepsilon) \left(\sum_{k=N(s)+1}^{N(s+\delta)} \gamma_k \right)^{1+\frac{\rho}{2}} \\
&\le& C_\rho A(\varepsilon) \left(\delta + \sup_{k \ge N(s)+1} \gamma_k \right)^{1+\frac{\rho}{2}}
\end{aligned}
$$

which in turn implies that

$$
\forall n \ge N, \quad \mathbb{E} \sup_{s \le t \le s+\delta} \left| \widetilde{M}_{(t)}^{(n)} - \widetilde{M}_{(s)}^{(n)} \right|^{2+\rho} \le C_\rho' \left(\delta + \sup_{k \ge N(\Gamma_N)+1} \gamma_k \right)^{1+\frac{\rho}{2}}.
$$

Then, by Markov inequality,

$$
\limsup_n \frac{1}{\delta} \mathbb{P} \left(\sup_{s \le t \le t+\delta} \left| \widetilde{M}_{(t)}^{(n)} - \widetilde{M}_{(s)}^{(n)} \right| \ge \varepsilon \right) \le C_\rho' \frac{\delta^{\frac{\rho}{2}}}{\varepsilon^{2+\rho}}.
$$

The C-tightness of the sequence $(\widetilde{M}^{(n)})_{n \ge N}$ follows from Theorem 6.7(b).

Furthermore, for every $n \ge N$,

$$
\begin{aligned}
\langle \widetilde{M}^{(n)} \rangle_{(t)} &=& \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \mathbb{E} \left((\Delta \widetilde{M}_k)^2 | \mathcal{F}_{k-1} \right) \\
&=& \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \left((\mathbb{E} \, H(y, Z_k)^2)_{|y=Y_{k-1}} - h(Y_{k-1})^2 \right) \\
&\sim& \left(\mathbb{E} \, H(0, Z)^2 - h(0)^2 \right) \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k
\end{aligned}
$$

Hence

$$\langle \widetilde{M}^{(n)} \rangle_{(t)} \longrightarrow \mathbb{E}\, H(0, Z)^2 \times t \qquad \text{as} \quad n \to +\infty.$$

where we used here that h and $\theta \mapsto \mathbb{E}\, H(\theta, Z)^2$ are both continuous en 0. Theorem 6.8 then implies

$$\widetilde{M}^{(n)} \overset{\mathcal{L}_{C(\mathbb{R}_+, \mathbb{R})}}{\longrightarrow} \sigma\, W, \; W \text{ standard Brownian motion.}$$

STEP 5 (*Synthesis and conclusion*): The sequence $\Upsilon_{(t)}^{(n)}$ satisfies, for every $n \geq N$,

$$\forall\, t \geq 0, \quad \Upsilon_{(t)}^{(n)} = \widetilde{\Upsilon}_n - \widetilde{A}_{(t)}^{(n)} - \widetilde{M}_{(t)}^{(n)}.$$

Consequently, the sequence $(\widetilde{\Upsilon}^{(n)})$ is C-tight since C-tightness is additive and $\widetilde{\Upsilon}_n$ is tight (by L^2-boundedness) and $(\widetilde{A}_{(t)}^{(n)})_{n \geq 1}$, $(\widetilde{M}^{(n)})_{n \geq 1}$ are both C-tight.

Consequently, the sequence of processes $(\widetilde{\Upsilon}^{(n)}, \widetilde{M}^{(n)})$ is C-tight as well.

Let $(\widetilde{\Upsilon}_{(t)}^{(\infty)}, \widetilde{M}_{(t)}^{(\infty)})$ be a weak functional limiting value. It will solve the Ornstein-Uhlenbeck (O.U.) *SDE*

$$d\widetilde{\Upsilon}_{(t)}^{(\infty)} = -a\widetilde{\Upsilon}_{(t)}^{(\infty)} dt + \sigma\, dW_t \tag{6.40}$$

starting from a random variable $\widetilde{Y}^{\infty} \in L^2$ such that

$$\|\widetilde{\Upsilon}^{\infty}\|_2 \leq \sup_n \|\widetilde{\Upsilon}_n\|_2.$$

Let $\nu_0 := \mathcal{L}\text{-}\lim_n \widetilde{\Upsilon}_{\varphi(n)}$ be a weak (functional) limiting value of $(\widetilde{\Upsilon}_{\varphi(n)})_{n \geq 1}$.

For every $t > 0$, one considers the sequence of integers $\psi_t(n)$ uniquely defined by

$$\Gamma_{\psi_t(n)} := \Gamma_{\varphi(n)} - t.$$

Up to an extraction, we may assume that we also have the convergence of

$$\widetilde{\Upsilon}^{(\varphi(n))} \overset{\mathcal{L}}{\longrightarrow} \widetilde{\Upsilon}^{(\infty,0)} \text{ starting from } \widetilde{\Upsilon}_{(0)}^{(\infty,0)} \sim \nu_0$$

and

$$\widetilde{\Upsilon}^{(\psi_t(n))} \overset{\mathcal{L}}{\longrightarrow} \widetilde{\Upsilon}^{(\infty,-t)} \text{ starting from } \widetilde{\Upsilon}_{(0)}^{(\infty,-t)} \sim \nu_{-t}$$

One checks by strong uniqueness of solutions of the above Ornstein-Uhlenbeck *SDE* that

$$\widetilde{\Upsilon}_{(t)}^{(\infty,-t)} = \widetilde{\Upsilon}_{(0)}^{(\infty,0)}.$$

Now let $(P^t)_{t \geq 0}$ denote the semi-group of the Ornstein-Uhlenbeck process. From what precedes, for every $t \geq 0$,

$$\nu_0 = \nu_{-t} P^t$$

Moreover, $(\nu_{-t})_{t \geq 0}$ is tight since it is L^2-bounded. Let $\nu_{-\infty}$ be a weak limiting value of ν_{-t} as $t \to +\infty$.

Let $\Upsilon^{\mu}_{(t)}$ denote a solution to (6.40) starting from a μ-distributed random variable. We know by the confluence property of O.U. paths that

$$|\Upsilon^{\mu}_t - \Upsilon^{\mu'}_t| \le |\Upsilon^{\mu}_0 - \Upsilon^{\mu'}_0|e^{-at}.$$

For every Lipschitz continuous continuous function f with compact support,

$$
\begin{aligned}
|\nu_{-\infty}P^t(f) - \nu_{-t}P^t(f)| &= |\mathbb{E}f(\Upsilon^{\nu_{-\infty}}_{(t)}) - \mathbb{E}f(\Upsilon^{\nu_{-t}}_{(t)})| \\
&\le [f]_{\mathrm{Lip}}\mathbb{E}|\Upsilon^{\nu_{-\infty}}_{(t)} - \Upsilon^{\nu_{-t}}_{(t)}| \\
&\le [f]_{\mathrm{Lip}}e^{-at}\mathbb{E}|\Upsilon^{\nu_{-\infty}}_{(0)} - \Upsilon^{\nu_{-t}}_{(0)}| \\
&\le [f]_{\mathrm{Lip}}e^{-at}\underbrace{\|\Upsilon^{\nu_{-\infty}}_{(0)} - \Upsilon^{\nu_{-t}}_{(0)}\|_2} \\
&\le 2\,[f]_{\mathrm{Lip}}e^{-at}\sup_n\|\widetilde{\Upsilon}_n\|_2 < +\infty \\
&\longrightarrow 0 \quad \text{as} \quad t \to +\infty.
\end{aligned}
$$

Consequently

$$\nu_0 = \lim_{t\to+\infty}\nu_{-\infty}P^t = \mathcal{N}\left(0;\frac{\sigma^2}{2a}\right).$$

We have just proved that the distribution $\mathcal{N}\left(0;\frac{\sigma^2}{2a}\right)$ is the only possible limiting value hence

$$\widetilde{\Upsilon}_n \xrightarrow{\mathcal{L}} \mathcal{N}\left(0;\frac{\sigma^2}{2a}\right).$$

Now we come back to Υ_n (prior to the localization). We have just proved that for $\varepsilon = \varepsilon(\rho)$ and for every $N \ge 1$,

$$\widetilde{\Upsilon}^{\varepsilon,N}_n \xrightarrow{\mathcal{L}} \mathcal{N}\left(0;\frac{\sigma^2}{2a}\right) \quad \text{as } n \to +\infty. \tag{6.41}$$

On the other hand, we already saw that $Y_n \to 0$ *a.s.* implies that $\Omega = \bigcup_{N\ge1}\Omega_{\varepsilon,N}$ *a.s.* where $\Omega_{\varepsilon,N} = \{Y^{\varepsilon,N} = Y_n, n \ge N\} = \{\widetilde{\Upsilon}^{\varepsilon,N} = \Upsilon_n, n \ge N\}$. Moreover the events $\Omega_{\varepsilon,N}$ ar non-decreasing as N increases so that

$$\lim_{N\to\infty}\mathbb{P}(\Omega_{\varepsilon,N}) = 1.$$

Owing to the localization principle, for every Borel bounded function f,

$$\forall\,n \ge N, \qquad \mathbb{E}|f(\Upsilon_n) - f(\Upsilon^{\varepsilon,N}_n)| \le 2\|f\|_\infty\mathbb{P}(\Omega^c_{\varepsilon,N}).$$

Combined with (6.41), if f is continuous and bounded, we get, for every $N \ge 1$,

$$\limsup_n\left|\mathbb{E}f(\Upsilon_n) - \mathbb{E}f\left(\frac{\sigma}{\sqrt{2a}}\Xi\right)\right| \le 2\|f\|_\infty\mathbb{P}\left(\Omega^c_{\varepsilon,N}\right)$$

where $\Xi \sim \mathcal{N}(0;1)$. One concludes by letting N go to infinity that for every bounded continuous function f

$$\lim_n\mathbb{E}f(\Upsilon_n) = \mathbb{E}\,f\left(\frac{\sigma}{\sqrt{2a}}\Xi\right)$$

i.e. $\Upsilon_n \xrightarrow{\mathcal{L}} \mathcal{N}\left(0;\frac{\sigma^2}{2a}\right).$ $\quad\diamond$

▷ **Exercise.** V@R CV@R, etc. A tirer de ...(enlever les commentaires du script).

6.4.4   The averaging principle for stochastic approximation

Practical implementations of recursive stochastic algorithms show the the convergence, although ruled by a CLT, is chaotic even in the final convergence phase, except if the step is optimized to produce the lowest asymptotic variance. Of course, this optimal choice is not realistic in practice.

   The original motivation to introduce the averaging principle was to "smoothen" the behaviour of a converging stochastic algorithm by considering the arithmetic mean of the past values *up to the nth iteration* rather than the computed value at the n^h iteration. In fact, if this averaging procedure is combined with the use of "slowly decreasing" step parameter γ_n one reaches for free the best possible rate of convergence!

   To be precise: let $(\gamma_n)_{n\geq 1}$ be step sequence satisfying

$$\gamma_n \sim \left(\frac{\alpha}{\beta + n}\right)^{\vartheta}, \quad \vartheta \in (1/2, 1).$$

Then, we implement the standard procedure (6.3) and set

$$\bar{Y}_n := \frac{Y_0 + \cdots + Y_n}{n + 1}.$$

Note that, of course, this empirical mean itself satisfies a recursive formulation:

$$\forall\, n \geq 0, \quad \bar{Y}_{n+1} = \bar{Y}_n - \frac{1}{n+1}(\bar{Y}_n - Y_n), \ \ \bar{Y}_0 = 0.$$

Under some natural assumptions (see [134]), one shows in the different cases investigates above (stochastic gradient, Robbins-Monro, pseudo-stochastic gradient) that

$$\bar{Y}_n \xrightarrow{a.s.} y_*$$

where y_* is the target of the algorithm and, furthermore,

$$\sqrt{n}(\bar{Y}_n - y_*) \xrightarrow{\mathcal{L}} \mathcal{N}(0; \Sigma^*)$$

where Σ^* is the lowest possible variance-covariance matrix: thus if $d = 1$, $\Sigma^* = \frac{\mathrm{Var}(H(y_*, Z))}{h'(y_*)^2}$. Like for the *CLT* of the algorithm itself, we again state the CLT for the averaged procedure in the framework of Markovian algorithms, although it can be done for general recursive procedure of the form $Y_{n+1} = Y_n - \gamma_{n+1}\Big(h(Y_n) + \Delta M_{n+1}\Big)$ where $(\Delta M_n)_{n\geq 1}$ is a sequence of martingale increments.

Theorem 6.9 (Ruppert & Poliak, see [43, 145, 137]) *Let $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ a Borel function and let $(Z_n)_{n\geq 1}$ be sequence of i.i.d. $\mathbb{R}^q$-valued random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that, for every $y\in \mathbb{R}^d$, $H(y, Z)\in L^2(\mathbb{P})$. Then the recursively defined procedure*

$$Y_{n+1} = Y_n - \gamma_{n+1}H(Y_n, Z_{n+1})$$

and the mean function $h(y) = \mathbb{E}\, H(y, Z)$ are well-defined. We make the following assumptions that:
(i) The function h has a unique zero a y_* and is "fast" differentiable at y_* in the sense that

$$\forall y\, \in \mathbb{R}^d, \quad h(y) = A(y - y_*) + O(|y - y_*|^2)$$

where all eigenvalues of the Jacobian matrix $A = J_h(y_*)$ of h at y_* have a positive real part.

(ii) The algorithm Y_n converges toward y_* with positive probability.

(iii) There exists an exponent $c > 2$ and a real constant $C > 0$ such that

$$\forall K > 0, \ \sup_{|y| \le K} \mathbb{E}(|H(y, Z)|^c) < +\infty \ \ and \ \ y \mapsto \mathbb{E}(H(y, Z)H(y, Z)^t) \ is \ continuous \ at \ y_* \quad (6.42)$$

Then, if $\gamma_n = \frac{\gamma_0}{n^a + b}$, $n \ge 1$, where $1/2 < a < 1$ and $b \ge 0$, the empirical mean sequence defined by

$$\bar{Y}_n = \frac{Y_0 + \cdots + Y_{n-1}}{n}$$

satisfies the CLT with the optimal asymptotic variance, on the event $\{Y_n \to y_*\}$, namely

$$\sqrt{n} \, (\bar{Y}_n - y_*) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0; A^{-1}\Gamma^* A^{-1}\right) \quad on \quad \{Y_n \to y_*\}.$$

We will prove this result in a more restrictive setting and refer *e.g.* to [43] for the general case. Fist we assume (only for convenience) that $d = 1$. Then, we assume that h satisfies the coercivity assumption (6.36) from Proposition 6.11, has a Lipschitz continuous derivative . Finally, we assume that $Y_n \to y_*$ *a.s.*. Note that the step sequences under consideration all satisfy the Condition $(G)_\alpha$ of Proposition 6.11.

Proof (one-dimensional case). We consider the case of a scalar algorithm $(d = 1)$. We assume without loss of generality that $y_* = 0$. We start from the canonical Markov representation

$$\forall n \ge 0, \ Y_{n+1} = Y_n - \gamma_{n+1}h(Y_n) - \gamma_{n+1}\Delta M_{n+1} \quad where \quad \Delta M_{n+1} = H(Y_n, Z_{n+1}) - h(Y_n)$$

is a sequence of $\mathcal{F}_n^Z$-martingale increments. Then, as $Y_n \to 0$ *a.s.*, $h(Y_n) = h'(0)Y_n + O(|Y_n|^2)$ where $y \mapsto O(|y|^2)/|y|^2$ is bounded by $[h']_{\mathrm{Lip}}$ so that, of every $n \ge 0$,

$$h'(0)Y_n = \frac{Y_n - Y_{n+1}}{\gamma_{n+1}} - \Delta M_{n+1} + O(|Y_n|^2)$$

which in turn implies, by summing up from 0 to $n - 1$,

$$h'(0)\sqrt{n}\,\bar{Y}_n = -\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{Y_k - Y_{k-1}}{\gamma_k} - \frac{1}{\sqrt{n}}M_n - \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} O(|Y_k|^2).$$

We will inspect successively the three sums on the right hand side of the equation. First, by an Abel transform, we get

$$\sum_{k=1}^{n} \frac{Y_k - Y_{k-1}}{\gamma_k} = \frac{Y_n}{\gamma_n} - \frac{Y_0}{\gamma_1} + \sum_{k=2}^{n} Y_{k-1}\left(\frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}}\right).$$

Hence, using that the sequence $(\frac{1}{\gamma_n})_{n\geq 1}$ is non-decreasing, we derive

$$\left|\sum_{k=1}^{n}\frac{Y_k - Y_{k-1}}{\gamma_k}\right| \leq \frac{|Y_n|}{\gamma_n} + \frac{|Y_0|}{\gamma_1} + \sum_{k=2}^{n}|Y_{k-1}|\left(\frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}}\right)$$

$$\leq \frac{|Y_n|}{\gamma_n} + \frac{|Y_0|}{\gamma_1} + \sup_{k\geq 0}|Y_k|\sum_{k=2}^{n}\frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}}$$

$$= \frac{|Y_n|}{\gamma_n} + \frac{|Y_0|}{\gamma_1} + \sup_{k\geq 0}|Y_k|\left(\frac{1}{\gamma_n} - \frac{1}{\gamma_1}\right).$$

$$\leq 3\sup_{k\geq 0}|Y_k|\frac{1}{\gamma_n} \quad a.s.$$

As a consequence, it follows from the assumption $\lim_n \sqrt{n}\gamma_n = +\infty$, that

$$\lim_{n}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{Y_k - Y_{k-1}}{\gamma_k} = 0$$

As for the second (martingale) term, it is straightforward that, if we set $Psi(y) = \mathbb{E}\big(H(y,Z) - h(y)\big)^2$, then

$$\frac{\langle M\rangle_n}{n} = \frac{1}{n}\sum_{k=1}^{n}\Psi(Y_n) \xrightarrow{a.s.} \Psi(y_*).$$

It follows from the assumptions made on the function H and Lindeberg's CLT (Theorem 11.7 in the Miscellany chapter 11) that

$$\frac{1}{h'(0)}\frac{M_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}\left(0; \frac{\Psi(y_*)}{(h'(0))^2}\right).$$

The third term is handled as follows. Under the Assumption (6.36) of Proposition 6.11, we know that $\mathbb{E}\,Y_n^2 \leq C\gamma_n$ since the class of steps we consider satisfy the condition $(G)_\alpha$ (see remark below Proposition 6.11). On the other hand, it follows from the assumption made on the step sequence that

$$\sum_{k=1}^{+\infty}\frac{\mathbb{E}\,O(|Y_k|^2)}{\sqrt{k}} \leq [h']_{\mathrm{Lip}}\sum_{k=1}^{n}\frac{\mathbb{E}\,Y_k^2}{\sqrt{k}}$$

$$\leq C\,[h']_{\mathrm{Lip}}\sum_{k=1}^{+\infty}\frac{\gamma_k}{\sqrt{k}}.$$

One concludes by Kronecker's Lemma that

$$\lim_{n}\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}O(|Y_k|^2) = 0 \quad a.s.$$

Slutsky's Lemma completes the proof.   ◇

Remark. As far as the step sequence is concerned, we only used that (γ_n) is decreasing, satisfies $(G)_\alpha$ and

$$\sum_n \frac{\gamma_k}{\sqrt{k}} < +\infty \quad \text{and} \quad \lim_n \sqrt{n}\gamma_n = +\infty.$$

Indeed, we have seen in the former section that this variance is the *lowest possible asymptotic variance* in the CLT when specifying the step parameter in an optimal way ($\gamma_n = \frac{c_{opt}}{n+b}$). In fact this discussion and its conclusions can be easily extended to higher dimensions (if one considers some matrix-valued step sequences) as emphasized *e.g.* in [43].

So, the Ruppert & Poliak principle performs as the regular stochastic algorithm with the lowest asymptotic variance for free!

▷ **Exercise.** Test the above averaging principle on the former exercises and "numerical illustrations" by considering $\gamma_n = \alpha\, n^{-\frac{3}{4}}$, $n \geq 1$. Compare with a direct approach with a step $\tilde{\gamma}_n = \frac{\alpha}{\beta+n}$.

PRACTITIONER'S CORNER. In practice, one should not start the averaging at the true beginning of the procedure but rather wait for its stabilization, ideally once the "exploration/search" phase is finished. On the other hand, the compromise consisting in using a moving window (typically of length n after $2n$ iterations does not yield the optimal asymptotic variance as pointed out in [101].

6.4.5 Traps

In presence of multiple equilibrium points *i.e.* of points at which the *mean* function h of the procedure vanishes, some of them can be seen as parasitic equilibrium. This is the case of saddle points local maxima in the framework of stochastic gradient descent (to some extend local minima are parasitic too but this is another story and usual stochastic approximation does not provide satisfactory answers to this "second order problem").

There is a wide literature on this problem which says, roughly speaking, that an excited enough parasitic equilibrium point is *a.s.* not a possible limit point for a stochastic approximation procedure. Although natural and expected such a conclusion is far from being straightforward to establish, as testified by the various works on that topic (see [96, 135, 43, 20, 51], etc).

6.4.6 (Back to) $V@R_\alpha$ and $CV@R_\alpha$ computation (II): weak rate

We can apply both above CLTs to the $V@R_\alpha$ and $CV@R_\alpha(X)$ algorithms (6.24) and (6.26). Since

$$h(\xi) = \frac{1}{1-\alpha}\Big(F(\xi) - \alpha\Big) \quad \text{and} \quad \mathbb{E}H(\xi,X)^2 = \frac{1}{(1-\alpha)^2}\big(F(\xi)(1-\mathcal{F}(\xi)\big),$$

one easily derives from Theorems 6.6 and 6.9 the following results

Theorem 6.10 *Assume that* $\mathbb{P}_X = f(x)dx$ *where* f *is a continuous density function (at least at the* $\xi_\alpha^* = V@R_\alpha(X)$*).*
(a) If $\gamma_n = \frac{\kappa}{n^a+b}$, $\frac{1}{2} < a < 1$, $b \geq 0$ *then*

$$n^{-\frac{a}{2}}\big(\xi_n - \xi_\alpha^*\big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0; \frac{\kappa\alpha(1-\alpha)}{2f(\xi_\alpha^*)}\Big)$$

(b) If $\gamma_n = \frac{\kappa}{n+b}$, $b \geq 0$ and $\kappa > \frac{1-\alpha}{2f(\xi_\alpha^*)}$ then

$$\sqrt{n}\big(\xi_n - \xi_\alpha^*\big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0; \frac{\kappa^2 \alpha}{2\kappa f(\xi_\alpha^*) - (1-\alpha)}\Big)$$

so that the minimal asymptotic variance is attained with $\kappa_\alpha^* = \frac{1-\alpha}{f(\xi_\alpha^*)}$ with an asymptotic variance equal to $\frac{\alpha(1-\alpha)}{f(\xi_\alpha^*)^2}$.

(c) Ruppert & Polyiak's averaging principle: If $\gamma_n = \frac{\kappa}{n^a+b}$, $\frac{1}{2} < a < 1$, $b \geq 0$ then

$$\sqrt{n}\big(\xi_n - \xi_\alpha^*\big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0; \frac{\alpha(1-\alpha)}{f(\xi_\alpha^*)^2}\Big).$$

The algorithm for the $CV@R_\alpha(X)$ satisfies the same kind of CLT. In progress [...]

This result is not satisfactory because we see that in general the asymptotic variance remains huge since $f(\xi_\alpha^*)$ is usually very close to 0. Thus if X has a normal distribution $\mathcal{N}(0;1)$, then it is clear that $\xi_\alpha^* \to +\infty$ as $\alpha \to 1$. Consequently

$$1 - \alpha = \mathbb{P}(X \geq \xi_\alpha^*) \sim \frac{f(\xi_\alpha^*)}{\xi_\alpha^*} \quad \text{as} \quad \alpha \to 1$$

so that

$$\frac{\alpha(1-\alpha)}{f(\xi_\alpha^*)^2} \sim \frac{1}{\xi_\alpha^* f(\xi_\alpha^*)} \to +\infty \quad \text{as} \quad \alpha \to 1.$$

This is simply an illustration of the "rare event" effect which implies that when α is close to 1 and the event $\{X_{n+1} > \xi_n\}$ is are especially when ξ_n gets close to its limit $\xi_\alpha^* = V@R_\alpha(X)$.

The way out is to add an importance sapling procedure to somewhat "re-center" the distribution around its $V@R_\alpha(X)$. To proceed, we will take advantage of our recursive variance reduction by importance sampling described and analyzed in Section 6.3.1. This is the object of the next section.

6.4.7 $V@R_\alpha$ and $CV@R_\alpha$ computation (III)

In progress [...]

As emphasized in the previous section, the asymptotic variance of our "naive" algorithms for $V@R_\alpha$ and $CV@R_\alpha$ computation are not satisfactory, in particular when α is close to 1. To improve them, the idea is to mix the recursive data-driven variance reduction procedure introduced in Section 6.3.1 with the above algorithms.

First we make the (not so) restrictive assumption that the r.v. X, representative of a loss, can be represented as a function of a Gaussian normal vector $Z \stackrel{d}{=} \mathcal{N}(0; I_d)$, namely

$$X = \chi(Z), \quad \varphi : \mathbb{R}^d \to \mathbb{R}, \text{ Borel function.}$$

Hence, for a level $\alpha \in (0,1]$, in a (temporarily) static framework (i.e. fixed $\xi \in \mathbb{R}$), the function of interest for variance reduction id defined by

$$\varphi_{\alpha,\xi}(z) = \frac{1}{1-\alpha}\Big(\mathbf{1}_{\{\varphi(z) \leq \xi\}} - \alpha\Big), \quad z \in \mathbb{R}^d.$$

So, still following Section 6.3.1 and taking advantage of the fact that $\varphi_{\alpha,\xi}$ is bounded, we design the following data driven procedure for the variance reducer (with the notations of this section),

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\big(\varphi_{\alpha,\xi}(Z_{n+1} - \theta_n)^2(2\theta_n - Z_{n+1})\big)$$

so that $\mathbb{E}\varphi_{\alpha,\xi}(Z)$ can be computed adaptively by

$$\mathbb{E}\,\varphi_{\alpha,\xi}(Z) = e^{-\frac{|\theta|^2}{2}}\mathbb{E}\Big(\varphi_{\alpha,\xi}(Z)e^{-(\theta|Z)}\Big) = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}e^{-\frac{|\theta_{k-1}|^2}{2}}\varphi_{\alpha,\xi}(Z_k + \theta_{k-1})e^{-(\theta_{k-1}|Z_k)}.$$

In progress [...]

Considering now a dynamic version of these procedure (*i.e.* which adapts recursively ξ leads to design the following procedure

$$\begin{aligned}
\widetilde{\xi}_{n+1} &= \widetilde{\xi}_n - \frac{\gamma_{n+1}}{1-\alpha}e^{-\frac{|\theta_n|^2}{2}}e^{-(\theta_n|Z_k)}\big(\mathbf{1}_{\{\chi(Z_{n+1}+\theta_n)\leq\widetilde{\xi}_n\}} - \alpha\big) \\
\widetilde{\theta}_{n+1} &= \widetilde{\theta}_n - \gamma_{n+1}\big(\mathbf{1}_{\{\chi(Z_{n+1})\leq\widetilde{\xi}_n\}} - \alpha\big)^2(2\widetilde{\theta}_n - Z_{n+1})\big).
\end{aligned}$$

This procedure is *a.s.* converging toward it target $(\theta_\alpha^*, \xi_\alpha)$ and satisfies the averaged component $(\overline{\widetilde{\xi}}_n)_{n\geq 0}$ of $(\xi_n)_{n\geq 0}$ satisfies a CLT.

Theorem 6.11 *(see [19]) (a) If the step sequence satisfies the decreasing step assumption (6.7), then*

$$(\widetilde{\xi}_n, \widetilde{\theta}_n) \stackrel{n\to+\infty}{\longrightarrow} (\xi_\alpha, \theta_\alpha^*) \quad with \quad \xi_\alpha = V@R_\alpha(X).$$

(b) If the step sequence satisfies $\gamma_n = \frac{\kappa}{n^a+b}$, $\frac{1}{2} < a < 1$, $b \geq 0$, $\kappa > 0$, then

$$\sqrt{n}\big(\overline{\widetilde{\xi}}_n - \xi_\alpha\big) \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{N}\left(0, \frac{V_{\alpha,\xi_\alpha}(\theta_\alpha^*)}{f(\xi_\alpha)^2}\right)$$

where

$$V_{\alpha,\xi}(\theta) = e^{-|\theta|^2}\mathbb{E}\left(\big(\mathbf{1}_{\{\chi(Z_{n+1}+\theta_n)\leq\widetilde{\xi}_n\}} - \alpha\big)^2 e^{-2(\theta|Z)}\right).$$

Note that

$$V_{\alpha,\xi}(0) = F(\xi)\big(1 - F(\xi)\big).$$

6.5 From Quasi-Monte Carlo to Quasi-Stochastic Approximation

Plugging quasi-random numbers into a recursive stochastic approximation procedure instead of pseudo-random numbers is a rather natural idea given the performances of QMC methods for numerical integration. It goes back, to our knowledge, to the early 1990's. As expected, several numerical tests showed that it may significantly accelerate the convergence of the procedure like it does in Monte Carlo simulations.

In [93],this question is investigated from a theoretical viewpoint. These papers are based on an extension of uniformly distributed sequences on unit hypercubes called *averaging systems*. The two main results are based on the one hand on a contraction assumption, on the other hand on

a monotone assumption which requires some stringent assumption on the function H. In the first setting some *a priori* error bounds emphasize that *quasi-stochastic approximation* does accelerate the convergence rate of the procedure. Both results are one-dimensional although the first setting can easily be extended to multi-dimensional state variables.

In this section, we will simply give the counterpart of the Robbins-Siegmund Lemma established in Section 6.2. It relies on a *pathwise Lyapunov function* which is of course a rather restrictive assumption. It also emphasizes what kind of assumption is needed to establish theoretical results when using deterministic uniformly distributed sequences. An extended version including several examples of applications are developed in [95].

Theorem 6.12 *(Robbins-Siegmund Lemma in a QMC framework) (a) Let $h : \mathbb{R}^d \to \mathbb{R}^d$ and $H : \mathbb{R}^d \times [0,1]^q \overset{Borel}{\longrightarrow} \mathbb{R}^d$ satisfying*

$$h(y) = \mathbb{E}\big(H(y,U)\big), \qquad y \in \mathbb{R}^d, \qquad U \overset{d}{=} U([0,1]^q).$$

Suppose that

$$\{h = 0\} = \{y_*\}, \qquad y_* \in \mathbb{R}^d$$

and that there exists a continuously differentiable function $L : \mathbb{R}^d \to \mathbb{R}_+$ with a Lipschitz continuous continuous gradient ∇L satisfying

$$|\nabla L| \le C_L \sqrt{1+L}$$

such that H satisfies the following pathwise *mean reverting assumption: the function Φ^H defined by*

$$\forall\, y \in \mathbb{R}^d, \ \Phi^H(y) := \inf_{u \in [0,1]^q} (\nabla L(y) | H(y,u) - H(y_*,u)) \quad \text{is l.s.c. and positive on } \mathbb{R}^d \setminus \{y_*\}. \quad (6.43)$$

Furthermore, assume that

$$\forall\, y \in \mathbb{R}^d, \ \forall\, u \in [0,1]^q, \qquad |H(y,u)| \le C_H (1 + L(y))^{\frac{1}{2}} \qquad (6.44)$$

(which implies that h is bounded) and that the function

$$u \mapsto H(y_*, u) \quad \text{has finite variation in the measure sense.}$$

Let $\xi := (\xi_n)_{n \ge 1}$ be a uniformly distributed sequence over $[0,1]^q$ with low discrepancy, namely

$$\ell_n := \max_{1 \le k \le n} (k D_k^*(\xi)) = O((\log n)^q).$$

Let $\gamma = (\gamma_n)_{n \ge 1}$ be a non-increasing sequence of gain parameters satisfying

$$\sum_{n \ge 1} \gamma_n = +\infty, \quad \gamma_n (\log n)^q \to 0 \quad \text{and} \quad \sum_{n \ge 1} \max(\gamma_n - \gamma_{n+1}, \gamma_n^2)(\log n)^q < +\infty. \qquad (6.45)$$

Then, the recursive procedure defined by

$$\forall\, n \ge 0, \quad y_{n+1} = y_n - \gamma_{n+1} H(y_n, \xi_{n+1}), \quad y_0 \in \mathbb{R}^d$$

satisfies:

$$y_n \longrightarrow y_* \qquad as \qquad n \to +\infty.$$

(b) If $(y, u) \mapsto H(y, u)$ is continuous, then Assumption (6.43) reads

$$\forall\, y \in \mathbb{R}^d \setminus \{y_*\}, \ \forall\, u \in [0, 1]^q, \quad (\nabla L(y) | H(y, u) - H(y_*, u)) > 0. \tag{6.46}$$

Proof. (*a*) STEP 1 (*The regular part*): The beginning of the proof is rather similar to the "regular" stochastic case except that we will use as a Lyapounov function

$$\Lambda = \sqrt{1 + L}.$$

First note that $\nabla \Lambda = \frac{\nabla L}{2\sqrt{1+L}}$ is bounded (by the constant C_L) so that Λ is C-Lipschitz continuous continuous. Furthermore, for every x, $y \in \mathbb{R}^d$,

$$
\begin{aligned}
|\nabla \Lambda(y) - \nabla \Lambda(y')| &\leq \frac{|\nabla L(y) - \nabla L(y')|}{\sqrt{1 + L(y)}} + |\nabla L(y')| \left| \frac{1}{\sqrt{1 + L(y)}} - \frac{1}{\sqrt{1 + L(y')}} \right| \tag{6.47} \\
&\leq [\nabla L]_{\mathrm{Lip}} \frac{|y - y'|}{\sqrt{1 + L(y)}} + \frac{C_L}{\sqrt{1 + L(y)}} |\sqrt{1 + L(y)} - \sqrt{1 + L(y')}| \\
&\leq [\nabla L]_{\mathrm{Lip}} \frac{|y - y'|}{\sqrt{1 + L(y)}} + \frac{C_L^2}{\sqrt{1 + L(y)}} |y - y'| \\
&\leq C_\Lambda \frac{|y - y'|}{\sqrt{1 + L(y)}} \tag{6.48}
\end{aligned}
$$

where $C_\Lambda = [\nabla L]_{\mathrm{Lip}} + C_L^2$.

It followsby using successively the fundamental Theorem of Calculus applied to Λ between y_n and y_{n+1} and Hölder's Inequality that there exists $\zeta_{n+1} \in (y_n, y_{n+1})$ (geometric interval) such that

$$
\begin{aligned}
\Lambda(y_{n+1}) &= \Lambda(y_n) - \gamma_{n+1}(\nabla \Lambda(y_n) \,|\, H(y_n, \xi_{n+1})) + \gamma_{n+1}\big(\nabla \Lambda(y_n) - \nabla \Lambda(\zeta_{n+1}) | H(y_n, \xi_{n+1})\big) \\
&\leq \Lambda(y_n) - \gamma_{n+1}(\nabla \Lambda(y_n) \,|\, H(y_n, \xi_{n+1})) + \gamma_{n+1}\big|\nabla \Lambda(y_n) - \nabla \Lambda(\zeta_{n+1})\big|\big|H(y_n, \xi_{n+1}).\big|
\end{aligned}
$$

Now, the above inequality (6.48) applied with with $y = y_n$ and $y' = \zeta_{n+1}$ yields, knowing that $|\zeta_{n+1} - y_n| \leq |y_{n+1} - y_n|$,

$$\leq \Lambda(y_n) - \gamma_{n+1}(\nabla \Lambda(y_n) \,|\, H(y_n, \xi_{n+1})) + \gamma_{n+1}^2 \frac{C_\Lambda}{\sqrt{1 + L(y_n)}} |H(y_n, \xi_{n+1})|^2$$

$$\Lambda(y_{n+1}) \leq \Lambda(y_n) - \gamma_{n+1}(\nabla \Lambda(y_n) \,|\, H(y_n, \xi_{n+1}) - H(y_*, \xi_{n+1})) - \gamma_{n+1}(\nabla \Lambda(y_n) \,|\, H(y_*, \xi_{n+1}))$$

$$+ \gamma_{n+1}^2 C_\Lambda \Lambda(y_n).$$

Then, using (6.44), it follows

$$\Lambda(y_{n+1}) \;\le\; \Lambda(y_n)(1 + C_\Lambda\gamma_{n+1}^2) - \gamma_{n+1}\Phi^H(y_n) - \gamma_{n+1}(\nabla\Lambda(y_n)\,|\,H(y_*,\xi_{n+1})). \qquad (6.49)$$

Set for every $n \ge 0$

$$s_n := \frac{\Lambda(y_n) + \sum_{k=1}^n \gamma_k\Phi^H(y_{k-1})}{\prod_{k=1}^n(1 + C_\Lambda\gamma_k^2)}$$

with the usual convention $\sum_\emptyset = 0$. It follows from (6.43) that the sequence $(s_n)_{n\ge0}$ is non-negative since all the terms involved in its numerator are non-negative.

Now (6.49) reads

$$\forall\,n \ge 0, \qquad 0 \le s_{n+1} \le s_n - \widetilde{\gamma}_{n+1}(\nabla\Lambda(y_n)\,|\,H(y_*,\xi_{n+1})). \qquad (6.50)$$

where $\widetilde{\gamma}_n = \dfrac{\gamma_n}{\prod_{k=1}^n(1 + C_\Lambda\gamma_k^2)}$, $n \ge 1$.

STEP 2 (*The "QMC part"*). Set for every $n \ge 1$,

$$m_n := \sum_{k=1}^n \gamma_k(\nabla\Lambda(y_{k-1})\,|\,H(y_*,\xi_k)) \qquad \text{and} \qquad S_n^* = \sum_{k=1}^n H(y_*,\xi_k).$$

First note that (6.45) combined with the Koksma-Hlawka Inequality imply

$$|S_n^*| \le C_\xi V(H(y_*,.))(\log n)^q \qquad (6.51)$$

where $V(H(y_*,.))$ denotes the variation in the measure sense of $H(y_*,.)$. An Abel transform yields (with the convention $S_0^* = 0$)

$$
\begin{aligned}
m_n \;=\;& \widetilde{\gamma}_n(\nabla\Lambda(y_{n-1})\,|\,S_n^*) - \sum_{k=1}^{n-1}(\widetilde{\gamma}_{k+1}\nabla\Lambda(y_k) - \widetilde{\gamma}_k\nabla\Lambda(y_{k-1})\,|\,S_k^*) \\[2mm]
\;=\;& \underbrace{\widetilde{\gamma}_n(\nabla\Lambda(y_{n-1})\,|\,S_n^*)}_{(a)} - \underbrace{\sum_{k=1}^{n-1}\widetilde{\gamma}_k\,(\nabla\Lambda(y_k) - \nabla\Lambda(y_{k-1})\,|\,S_k^*)}_{(b)} \\[2mm]
& - \underbrace{\sum_{k=1}^{n-1}\Delta\widetilde{\gamma}_{k+1}\,(\nabla\Lambda(y_k)\,|\,S_k^*)}_{(c)}.
\end{aligned}
$$

We aim at showing that m_n converges in $\mathbb{R}$ toward a finite limit by inspecting the above three terms.

One gets, using that $\gamma_n \le \widetilde{\gamma}_n$,

$$|(a)| \le \gamma_n\|\nabla\Lambda\|_\infty O((\log n)^q) = O(\gamma_n(\log n)^q) \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Owing to (6.48), the partial sum (b) satisfies

$$
\begin{aligned}
\sum_{k=1}^{n-1} \widetilde{\gamma}_k \left| \left(\nabla \Lambda(y_k) - \nabla \Lambda(y_{k-1}) \,|\, S_k^* \right) \right| \;\; &\leq \;\; C_\Lambda \widetilde{\gamma}_k \gamma_k \frac{|H(y_{k-1}, \xi_k)|}{\sqrt{1 + L(y_{k-1})}} |S_k^*| \\
&\leq \;\; C_\Lambda C_H V(H(y_*, .)) \gamma_k^2 (\log k)^q
\end{aligned}
$$

where we used (6.51) in the second inequality.

Consequently the series $\sum_{k \geq 1} \widetilde{\gamma}_k \left(\nabla L(y_k) - \nabla L(y_{k-1}) \,|\, S_k^* \right)$ is (absolutely) converging owing to Assumption (6.45).

Finally, one deals with term (c). Notice that

$$
|\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n| \leq |\gamma_{n+1} - \gamma_n| + C_\Lambda \gamma_{n+1}^2 \gamma_n \leq C_\Lambda' \max(\gamma_n^2, |\gamma_{n+1} - \gamma_n|).
$$

One checks that the series (c) is also (absolutely) converging owing to the boundedness of ∇L, Assumption (6.45) and the upper-bound (6.51) for S_n^*.

Then m_n converges toward a finite limit m_∞. This induces that the sequence $(s_n + m_n)$ is lower bounded since (s_n) is non-negative. Now we know from (6.50) that $(s_n + m_n)$ is also non-increasing, hence converging in $\mathbb{R}$ which in turn implies that the sequence $(s_n)_{n \geq 0}$ itself is converging toward a finite limit. The same arguments as in the regular stochastic case yield

$$
L(y_n) \overset{n \to +\infty}{\longrightarrow} L_\infty \quad \text{and} \quad \sum_{n \geq 1} \gamma_n \Phi^H(y_{n-1}) < +\infty
$$

Once again, one concludes like in the stochasic case that (y_n) is bounded and eventually converges toward the unique zero of Φ^H *i.e.* y_*.

(b) is obvious.   ◇

COMMENTS FOR PRACTICAL IMPLEMENTATION ● The step assumption (6.45) includes all the step $\gamma_n = \frac{c}{n^\alpha}$, $\alpha \in (0, 1]$. Note that as soon as $q \geq 2$, the condition $\gamma_n (\log n)^q \to 0$ is redundant (it follows from the convergence of the series on the right owing to an Abel transform).

● One can replace the (slightly unrealistic) assumption on $H(y_*, .)$ by a Lipschitz continuous assumption provided one strengthens the step assumption (6.45) into

$$
\sum_{n \geq 1} \gamma_n = +\infty, \qquad \gamma_n (\log n) n^{1 - \frac{1}{q}} \to 0 \quad \text{and} \quad \sum_{n \geq 1} \max(\gamma_n - \gamma_{n+1}, \gamma_n^2)(\log n) n^{1 - \frac{1}{q}} < +\infty. \tag{6.52}
$$

This is a straightforward consequence of Proïnov's Theorem (Theorem 4.2) which says that

$$
|S_n^*| \leq C (\log n) \, n^{1 - \frac{1}{q}}.
$$

Note the above assumptions are satisfied by the step sequences $\gamma_n = \frac{c}{n^\rho}$, $1 - \frac{1}{q} < \rho \leq 1$.

● It is clear that the Lyapunov assumptions on H is much more stringent in this QMC setting.

● It remains that the theoretical spectrum of application of the above theorem is dramatically more narrow than the original one. From a practical viewpoint, one observes on simulations a very
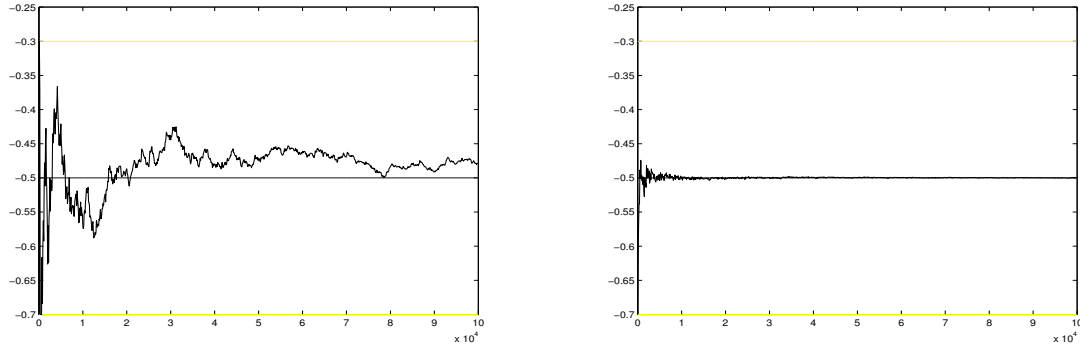
Figure 6.4: *B-S* BEST-OF-CALL OPTION.$T = 1$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.30$, $X_0^1 = X_0^2 = 100$, $K = 100$. *Convergence of $\rho_n = \cos(\theta_n)$ toward a $\rho^* = toward - 0.5$ (up to $n = 100\,000$). Left: MC implementation. Right: QMC implementation.*

satisfactory behaviour of such quasi-stochastic procedures, including the improvement of the rate of convergence with respect to the regular *MC* implementation.

▷ **Exercise.**   We assume now that the recursive procedure satisfied by the sequence $(y_n)_{n\geq 0}$ is given by

$$\forall\, n \geq 0, \quad y_{n+1} = y_n - \gamma_{n+1}(H(y_n, \xi_{n+1}) + r_{n+1}), \quad y_0 \in \mathbb{R}^d$$

where the sequence $(r_n)_{n\geq 1}$ is a disturbance term. Show that if, $\sum_{n\geq 1}\gamma_n r_n$ is a converging series, then the conclusion of the above theorem remains true.

NUMERICAL EXPERIMENT: We reproduced here (without even trying to check any kind of assumption, indeed) the implicit correlation search recursive procedure tested in Section 6.3.2 implemented this time with a sequence of some quasi-random normal numbers, namely

$$(\zeta_n^1, \zeta_n^2) = \left(\sqrt{-2\log(\xi_n^1)}\,\sin(2\pi\xi_n^2),\ \sqrt{-2\log(\xi_n^1)}\,\cos(2\pi\xi_n^2)\right), \qquad n \geq 1,$$

where $\xi_n = (\xi_n^1, \xi_n^2)$, $n \geq 1$, is simply a regular 2-dimensional Halton sequence.

n	$\rho_n := \cos(\theta_n)$
1000	-0.4964
10000	-0.4995
25000	-0.4995
50000	-0.4994
75000	-0.4996
100000	-0.4998

6.5.1   Further readings

From a probabilistic viewpoint, many other moment, regularity assumptions on the Lyapunov function entail some *a.s.* convergence results. From the dynamical point of view, stochastic approximation is rather closely connected to the dynamics of the autonomous Ordinary Differential

Equation (ODE) of its mean function, namely $\dot{x} = -h(x)$. However, the analysis of a stochastic algorithm cannot be really "reduced" to that of its mean ODE as emphasized by several authors ([20, 50]).

There is a huge literature about stochastic approximation, motivated by several fields: optimization, statistics, automatic learning, robotics (?), artificial neural networks, self-organization, etc. For further insight on stochastic approximation, the main textbooks are probably [89, 43, 21] for prominently probabilistic aspects. One may read [20] for dynamical system oriented point of view. For an occupation measure approach, one may also see [51].

Chapter 7

Discretization scheme(s) of a Brownian diffusion

One considers a d-dimensional Brownian diffusion process $(X_t)_{t \in [0,T]}$ solution to the following Stochastic Differential Equation (SDE)

$$(SDE) \quad \equiv \quad dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{7.1}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}(d, q, \mathbb{R})$ are continuous functions, $(W_t)_{t \in [0,T]}$ denotes a q-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $X_0 : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ is a random vector, independent of W. We assume that b and σ are Lipschitz continuous continuous in x uniformly with respect to $t \in [0,T]$ i.e., if $|\,.\,|$ denotes any norm on $\mathbb{R}^d$ and $\|\,.\,\|$ any norm on the matrix space,

$$\forall\, t \in [0,T],\ \forall\, x,\, y \in \mathbb{R}^d, \quad |b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \le K|x-y|. \tag{7.2}$$

We consider the so-called *augmented* filtration generated by X_0 and $\sigma(W_s,\, 0 \le s \le t)$ i.e.

$$\forall\, t \in [0,T], \quad \mathcal{F}_t := \sigma(X_0, \mathcal{N}_{\mathbb{P}}, W_s,\, 0 \le s \le t)$$

where $\mathcal{N}_{\mathbb{P}}$ denotes the class of $\mathbb{P}$-negligible sets of $\mathcal{A}$ (*i.e.* all negligible sets if the σ-algebra $\mathcal{A}$ is supposed to be $\mathbb{P}$-complete). One shows using the Kolmogorov 0-1 law that this completed filtration is right continuous *i.e.* $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$ for every $t \in [0,T)$. Such a combination of completeness and right continuity of a filtration is also known as "usual conditions".

Theorem 7.1 *(see e.g. [78], theorem 2.9, p.289) Under the above assumptions on b, σ, X_0 and W, the above SDE has a unique $(\mathcal{F}_t)$-adapted solution $X = (X_t)_{t \in [0,T]}$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, starting from X_0 at time 0, in the following sense:*

$$\mathbb{P}\text{-a.s.} \qquad \forall\, t \in [0,T], \quad X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

This solution has $\mathbb{P}$-a.s. continuous paths.

NOTATION. When $X_0 = x \in \mathbb{R}^d$, one denotes the solution of (SDE) on $[0, T]$ by X^x or $(X_t^x)_{t \in [0,T]}$.

Remark. • A solution as described in the above theorem is known as a *strong* solution in the sense that it is defined on the probability space on which W lives.

• The continuity assumption on b and σ can be relaxed into Borel measurability, if we add the linear growth assumption

$$\forall t \in [0, T], \ \forall x \in \mathbb{R}^d, \quad |b(t,x)| + \|\sigma(t,x)\| \le K'(1 + |x|).$$

In fact if b and σ are continuous this condition follows from (7.2) applied with (t,x) and $(t,0)$, given the fact that $t \mapsto b(t,0)$ is bounded on $[0, T]$.

• By adding the 0th component t to X *i.e.* be setting $Y_t := (t, X_t)$ one may always assume that the (SDE) is homogeneous *i.e.* that the coefficients b and σ only depend on the space variable. This is often enough for applications although it induces some too stringent assumptions on the time variable in many theoretical results. Furthermore, when some *ellipticity* assumptions are required, this way of considering the equation no longer works since the equation $dt = 1dt + 0dW_t$ is completely degenerate.

7.1 Euler-Maruyama schemes

Except for some very specific equations, it is impossible to devise an exact simulation of the process X even at a fixed time T (by exact simulation, we mean writing $X_T = \chi(U)$, $U \sim U([0,1])$) (nevertheless, when $d = 1$ and $\sigma \equiv 1$, see [23]). Consequently, to approximate $\mathbb{E}(f(X_T))$ by a Monte Carlo method, one needs to approximate X by a process that can be simulated (at least at a fixed number of instants). To this end, we will introduce three types of Euler schemes associated to the SDE: the *discrete time* Euler scheme $\bar{X} = (\bar{X}_{\frac{kT}{n}})_{0 \le k \le n}$ with step $\frac{T}{n}$, its càdlàg stepwise constant extension known as the *stepwise constant* (Brownian) Euler scheme and the *continuous* (Brownian) Euler scheme.

7.1.1 Discrete time and stepwise constant Euler scheme(s)

▷ The *discrete time Euler scheme* is defined by

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + \frac{T}{n} b(t_k^n, \bar{X}_{t_k^n}) + \sigma(t_k^n, \bar{X}_{t_k^n}) \sqrt{\frac{T}{n}} U_{k+1}, \quad \bar{X}_0 = X_0, \quad k = 0, \ldots, n-1, \qquad (7.3)$$

where $t_k^n = \frac{kT}{n}$, $k = 0, \ldots, n$ and $(U_k)_{1 \le k \le n}$ denotes a sequence of i.i.d. $\mathcal{N}(0; I_q)$-distributed random vectors given by

$$U_k := \sqrt{\frac{n}{T}} (W_{t_k^n} - W_{t_{k-1}^n}), \quad k = 1, \ldots, n.$$

(Strictly speaking we should write U_k^n rather than U_k.)

NOTATION: For convenience, we denote from now on

$$\underline{t} := t_k^n \quad \text{if } t \in [t_k^n, t_{k+1}^n).$$

▷ The *stepwise constant Euler scheme*, denoted $(\widetilde{X}_t)_{t \in [0,T]}$ for convenience, is defined by

$$\widetilde{X}_t = \bar{X}_{\underline{t}}, \qquad t \in [0, T]. \qquad (7.4)$$

7.1.2 Genuine (continuous) continuous Euler scheme

At this stage it is natural to extend the definition (7.3) of the Euler scheme at every instant $t \in [0, T]$ by interpolating the drift with respect to time and the diffusion coefficient with respect to the Brownian motion, namely

$$\forall\, k \in \{0, \dots, n-1\}, \ \forall\, t \in [t_k^n, t_{k+1}^n), \quad \bar{X}_t = \bar{X}_{\underline{t}} + (t - \underline{t}) b(\underline{t}, \bar{X}_{\underline{t}}) + \sigma(\underline{t}, \bar{X}_{\underline{t}})(W_t - W_{\underline{t}}). \tag{7.5}$$

It is clear that $\lim_{t \to t_{k+1}^n, t < t_{k+1}^n} \bar{X}_t = \bar{X}_{t_{k+1}^n}$ since W has continuous paths. Consequently, so defined, $(\bar{X}_t)_{t \in [0,T]}$ is an $\mathcal{F}_t^W$-adapted process with continuous paths.

The following proposition is the key property of the genuine (or continuous) Euler scheme.

Proposition 7.1 *Assume that b and σ are continuous functions on $[0, T] \times \mathbb{R}^d$. The genuine Euler scheme is a (continuous) Itô process satisfying the pseudo-SDE with frozen coefficients*

$$d\bar{X}_t = b(\underline{t}, \bar{X}_{\underline{t}})dt + \sigma(\underline{t}, \bar{X}_{\underline{t}})dW_t, \quad \bar{X}_0 = X_0$$

that is

$$\bar{X}_t = X_0 + \int_0^t b(\underline{s}, \bar{X}_{\underline{s}})ds + \int_0^t \sigma(\underline{s}, \bar{X}_{\underline{s}})dW_s. \tag{7.6}$$

Proof. It is clear from (7.5), the recursive definition (7.3) at the discretization dates t_k^n and the continuity of b and σ that $\bar{X}_t \to \bar{X}_{t_{k+1}^n}$ as $t \to t_{k+1}^n$. Consequently, for every $t \in [t_k^n, t_{k+1}^n]$,

$$\bar{X}_t = \bar{X}_{t_k^n} + \int_{t_k^n}^t b(\underline{s}, \bar{X}_{\underline{s}})ds + \int_{t_k^n}^t \sigma(\underline{s}, \bar{X}_{\underline{s}})dW_s$$

so that the conclusion follows by just concatenate the above identities between 0 and t_1^n, ..., t_k^n and t. $\diamond$

NOTATION: In the main statements, we will write $\bar{X}^n$ instead of $\bar{X}$ to recall the dependence of the Euler scheme in its step T/n. Idem for $\widetilde{X}$, etc.

Then, the main (classical) result is that under the assumptions on the coefficients b and σ mentioned above, $\sup_{t \in [0,T]} |X_t - \bar{X}_t|$ goes to zero in every $L^p(\mathbb{P})$, $0 < p < \infty$ as $n \to +\infty$. Let us be more specific on that topic by providing error rates under slightly more stringent assumptions.

How to use this continuous scheme for practical simulation seems not obvious, at least not as obvious as the stepwise constant Euler scheme. However this turns out to be an important method to improve the convergence rate of MC simulations *e.g.* for option pricing. Using this scheme in simulation relies on the so-called diffusion bridge method and will be detailed further on.

7.2 Strong error rate and polynomial moments (I)

7.2.1 Main results and comments

We consider the SDE and its Euler-Maruyama scheme(s) as defined by (7.1) and (7.3), (7.5). A first version of Theorem 7.2 below (including the second remark that follows) is mainly due to O. Faure in his PhD thesis (see [48]).

▷ *Polynomial moment control.* It is often useful to have at hand the following uniform bounds for the solution(s) of (SDE) and its Euler schemes which first appears as a step of the proof of the rate but has many other applications: thus it is an important step to prove the existence of global strong solutions to (SDE).

Proposition 7.2 *Assume that the coefficients b and σ of the SDE (7.1) are Borel functions that simply satisfy the following linear growth assumption:*

$$\forall\, t \in [0,T],\ \forall\, x \in \mathbb{R}^d, \quad |b(t,x)| + \|\sigma(t,x)\| \le C(1 + |x|) \tag{7.7}$$

for some real constant $C > 0$ and a "horizon" $T > 0$. Then, for every $p \in (0, +\infty)$, there exists a universal positive real constant κ_p such that every strong solution $(X_t)_{t\in[0,T]}$ (if any) satisfies

$$\left\| \sup_{t\in[0,T]} |X_t| \right\|_p \le 2 e^{\kappa_p C T}(1 + \|X_0\|_p)$$

and, for every $n \ge 1$, the Euler scheme with step T/n satisfies

$$\left\| \sup_{t\in[0,T]} |\bar{X}_t^n| \right\|_p \le 2 e^{\kappa_p C T}(1 + \|X_0\|_p).$$

One noticeable consequence of this proposition is that, if b and σ satisfy (7.7) with the same real constant C for every $T > 0$, then the conclusion holds true for every $T > 0$, providing a "rough" exponential control in T of any solution.

▷ *Uniform convergence rate in $L^p(\mathbb{P})$.* First we introduce the following condition (H_T^β) which strengthens Assumption (7.2) by adding a time regularity assumption of the Hölder type:

$$(H_T^\beta) \equiv \begin{cases} \exists\, \beta \in [0,1],\ \exists\, C_{b,\sigma,T} > 0 \ \text{ such that } \forall s,t \in [0,T],\ \forall x,y \in \mathbb{R}^d, \\[2mm] |b(t,x) - b(s,y)| + \|\sigma(t,x) - \sigma(s,y)\| \le C_{b,\sigma,T}(|t-s|^\beta + |y-x|). \end{cases} \tag{7.8}$$

Theorem 7.2 (Strong Rate for the Euler scheme) *(a)* CONTINUOUS EULER SCHEME. *Suppose the coefficients b and σ of the SDE (7.1) satisfy the above regularity condition (H_T^β) for a real constant $C_{b,\sigma,T} > 0$ and an exponent $\beta \in (0,1]$. Then the continuous Euler scheme $(\bar{X}^n)_{t\in[0,T]}$ converges toward $(X_t)_{t\in[0,T]}$ in every $L^p(\mathbb{P})$, $p > 0$, such that $X_0 \in L^p$, at a $O(n^{-(\frac12 \wedge \beta)})$-rate. To be precise, there exists a universal constant $\kappa_p > 0$ only depending on p such that, for every $n \ge T$,*

$$\left\| \sup_{t\in[0,T]} |X_t - \bar{X}_t^n| \right\|_p \le K(p,b,\sigma,T)(1 + \|X_0\|_p)\left(\frac{T}{n}\right)^{\beta \wedge \frac12} \tag{7.9}$$

where

$$K(p,b,\sigma,T) = 2\kappa_p (C'_{b,\sigma,T})^2 e^{\kappa_p(1+C'_{b,\sigma,T})T}$$

and

$$C'_{b,\sigma,T} = C_{b,\sigma,T} + \sup_{t\in[0,T]} |b(t,0)| + \sup_{t\in[0,T]} \|\sigma(t,0)\| < +\infty. \tag{7.10}$$

In particular (7.9) is satisfied when the supremum is restricted to discretization instants, namely

$$\left\| \sup_{0 \leq k \leq n} |X_{t_k} - \bar{X}_{t_k}^n| \right\|_p \leq K(p, b, \sigma, T)(1 + \|X_0\|_p) \left(\frac{T}{n} \right)^{\beta \wedge \frac{1}{2}}. \tag{7.11}$$

(a') If b and σ are defined on the whole $\mathbb{R}_+ \times \mathbb{R}^d$ and satisfy (H_T^β) with the same real constant $C_{b,\sigma}$ not depending on T and if $b(\,.\,,0)$ and $\sigma(\,.\,,0)$ are bounded on $\mathbb{R}_+$, then $C'_{b,\sigma,T}$ does not depend on T.

So will be the case in the homogeneous case i.e. if $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$ with b and σ Lipschitz continuous continuous on $\mathbb{R}^d$.

(b) STEPWISE CONSTANT EULER SCHEME. $\triangleright$ As soon as b and σ satisfy the linear growth assumption (7.7) with a real constant $L_{b,\sigma,T} > 0$, then, for every $p \in (0, +\infty)$ and every $n \geq T$,

$$\left\| \sup_{t \in [0,T]} |\bar{X}_t^n - \bar{X}_{\underline{t}}^n| \right\|_p \leq \tilde{\kappa}_p e^{\tilde{\kappa}_p L_{b,\sigma,T} T} (1 + \|X_0\|_p) \sqrt{\frac{T(1 + \log n)}{n}} = O\left(\sqrt{\frac{1 + \log n}{n}} \right)$$

where $\tilde{\kappa}_p > 0$ is a positive real constant only depending on p (and increasing in p).

$\triangleright$ In particular if b and σ satisfy the assumptions of item (a) then the stepwise constant Euler scheme $(\widetilde{X}^n)_{t \in [0,T]}$ converges toward $(X_t)_{t \in [0,T]}$ in every $L^p(\mathbb{P})$, $p > 0$, such that $X_0 \in L^p$ and for every $n \geq T$,

$$\begin{aligned} \left\| \sup_{t \in [0,T]} |X_t - \widetilde{X}_t^n| \right\|_p &\leq \widetilde{K}(p, b, \sigma, T)(1 + \|X_0\|_p)\left(\sqrt{\frac{T(1 + \log n)}{n}} + \left(\frac{T}{n} \right)^{\beta \wedge \frac{1}{2}} \right) \\ &= O\left(\left(\frac{1}{n} \right)^\beta + \sqrt{\frac{1 + \log n}{n}} \right) \end{aligned}$$

where

$$\widetilde{K}(p, b, \sigma, T) = \tilde{\kappa}'_p (1 + C'_{b,\sigma,T})^2 e^{\tilde{\kappa}'_p (1 + C'_{b,\sigma,T}) T},$$

$\tilde{\kappa}'_p > 0$ is a positive real constant only depending on p (increasing in p) and $C'_{b,\sigma,T}$ is given by (7.10).

WARNING! The complete and detailed proof of this theorem in its full generality, *i.e.* including the management of the constants, is postponed to Section 7.8. It makes use of stochastic calculus. A first approach to the proof in the one-dimensional quadratic case is proposed in Section 7.2.2. However, owing to its importance for applications, the optimality of the upper bound for the stepwise constant Euler scheme will be discussed right after the remarks below.

Remarks. • When $n \leq T$, the above explicit bounds still hold true with the same constants provided one replaces Note that as soon as $\frac{T}{n} \leq 1$,

$$2\left(\frac{T}{n} \right)^{\beta \wedge \frac{1}{2}} \text{ by } \frac{1}{2}\left(\left(\frac{T}{n} \right)^\beta + \left(\frac{T}{n} \right)^{\frac{1}{2}} \right) \quad \text{and} \quad \sqrt{\frac{T}{n}(1 + \log n)} \text{ by } \frac{1}{2}\left(\sqrt{\frac{T}{n}(1 + \log n)} + \frac{T}{n} \right)$$

which significantly simplifies the formulation of the error bound in item (b) of the above theorem.

• As a consequence, note that the time regularity exponent β rules the convergence rate of the scheme as soon as $\beta < 1/2$. In fact, the method of proof itself will emphasize this fact: the idea is to use a Gronwall Lemma to upper-bound the error $X - \bar{X}$ in $L^p(\mathbb{P})$ by the $L^p(\mathbb{P})$-norm of the increments of $X_s - X_{\underline{s}}$.

• If $b(t, x)$ and $\sigma(t, x)$ are globally Lipschitz continuous continuous on $\mathbb{R}_+ \times \mathbb{R}^d$ with Lipschitz continuous coefficient $C_{b,\sigma}$, one may consider time t as a $(d+1)^{th}$ spatial component composante of X and apply directly item (a') of the above theorem.

The following corollary is a straightforward consequence of claims (a) of the theorem (to be precise (7.11)): it yields a (first) convergence rate for the pricing of "vanilla" European options (payoff $\varphi(X_T)$).

Corollary 7.1 *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an α-Hölder function for an exponent $\alpha \in (0, 1]$, i.e. a function such that $[\varphi]_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|^\alpha}{|x-y|} < +\infty$. Then, there exists a real constant $C_{b,\sigma,T} \in (0, \infty)$ such that, for every $n \geq 1$,*

$$|\mathbb{E}\,\varphi(X_T) - \mathbb{E}\,\varphi(\bar{X}_T^n)| \leq \mathbb{E}\,|\varphi(X_T) - \varphi(\bar{X}_T^n)| \leq C_{b,\sigma,T}[\varphi]_\alpha \left(\frac{T}{n}\right)^{\frac{\alpha}{2}}.$$

We will see further on that this rate can be considerably improved when b, σ and φ share higher regularity properties.

▷ *About the universality of the rate for the stepwise constant Euler scheme ($p \geq 2$).* Note that the rate in claim (b) of the theorem is universal since it holds as a sharp rate for the Brownian motion itself (here we deal with the case $d = 1$). Indeed, since W is its own continuous Euler scheme,

$$\| \sup_{t \in [0,T]} |W_t - W_{\underline{t}}| \|_p = \left\| \max_{k=1,\ldots,n} \sup_{t \in [t_{k-1}^n, t_k^n)} |W_t - W_{t_{k-1}^n}| \right\|_p$$

$$= \sqrt{\frac{T}{n}} \left\| \max_{k=1,\ldots,n} \sup_{t \in [k-1,k)} |\widetilde{W}_t - \widetilde{W}_{k-1}| \right\|_p$$

where $\widetilde{W}_t := \sqrt{\frac{n}{T}} W_{\frac{T}{n}t}$ is a standard Brownian motion owing to the scaling property. Hence

$$\left\| \sup_{t \in [0,T]} |W_t - W_{\underline{t}}| \right\|_p = \sqrt{\frac{T}{n}} \left\| \max_{k=1,\ldots,n} \zeta_k \right\|_p$$

where the random variables $\zeta_k := \sup_{t \in [k-1,k)} |W_t - W_{k-1}|$ are i.i.d.

Lower bound. Note that, for every $k \geq 1$,

$$\zeta_k \geq Z_k := |W_k - W_{k-1}|.$$

since the Brownian motion W_t has continuous paths. The sequence $(Z_k)_{k \geq 1}$ is i.i.d. as well, with the same distribution as $|W_1|$. Hence, the random variables Z_k^2 are still i.i.d. with a $\chi^2(1)$-distribution so that (see item (b) of the exercises below)

$$\forall p \geq 2, \qquad \left\| \max_{k=1,\ldots,n} |Z_k| \right\|_p = \sqrt{\| \max_{k=1,\ldots,n} Z_k^2 \|_{p/2}} \geq c_p \sqrt{\log n}.$$

so that, finally,

$$\forall\, p \geq 2, \qquad \Big\| \sup_{t \in [0,T]} |W_t - W_{\underline{t}}| \Big\|_p \geq c_p \sqrt{\frac{T}{n}} \sqrt{\log n}.$$

Upper bound. To establish the upper-bound, we proceed as follows. First, note that

$$\zeta_1 = \max\big(\sup_{t \in [0,1)} W_t, \ \sup_{t \in [0,1)} (-W_t) \big).$$

We also know that

$$\sup_{t \in [0,1)} W_t \overset{d}{=} |W_1|$$

(see *e.g.* [140], Reflection principle, p.105). Hence using using that for every a, $b \geq 0$, $e^{(a \vee b)^2} \leq e^{a^2} + e^{b^2}$, that $\sup_{t \in [0,1)} W_t \geq 0$ and that $-W$ is also a standard Brownian motion we derive that

$$\mathbb{E}\, e^{\theta \zeta_1^2} \ \leq \ \mathbb{E}\, e^{\theta (\sup_{t \in [0,1)} W_t)^2} + \mathbb{E}\, e^{\theta (\sup_{t \in [0,1)} (-W_t))^2}$$

$$= \ 2\, \mathbb{E}\, e^{\theta (\sup_{t \in [0,1)} W_t)^2}$$

$$= \ 2\, \mathbb{E}\, e^{\theta W_1^2} = 2 \int_{\mathbb{R}} \exp\left(-\frac{u^2}{2\left(\frac{1}{\sqrt{1-2\theta}}\right)^2} \right) \frac{du}{\sqrt{2\pi}} = \frac{2}{\sqrt{1-2\theta}} < +\infty$$

as long as $\theta \in (0, \frac{1}{2})$. Consequently, it follows from Lemma 7.1 below applied with the sequence $(\zeta_n^2)_{n \geq 1}$ that

$$\Big\| \max_{k=1,\ldots,n} \zeta_k \Big\|_p = \sqrt{ \Big\| \max_{k=1,\ldots,n} \zeta_k^2 \Big\|_{p/2} } \leq C_p \sqrt{1 + \log n}$$

i.e., for every $p \in (0, \infty)$,

$$\Big\| \sup_{t \in [0,T]} |W_t - W_{\underline{t}}| \Big\|_p \leq C_{W,p} \sqrt{\frac{T}{n}(1 + \log n)}. \tag{7.12}$$

Lemma 7.1 *Let $Y_1, \ldots Y_n$ be non-negative random variables with the same distribution satisfying $\mathbb{E}(e^{\lambda Y_1}) < +\infty$ for some $\lambda > 0$. Then,*

$$\forall\, p \in (0, +\infty), \quad \| \max(Y_1, \ldots Y_n) \|_p \leq \frac{1}{\lambda} (\log n + C_{p, Y_1, \lambda}).$$

Proof : We may assume without loss of generality that $p \geq 1$ since the $\| \cdot \|_p$-norm is non-decreasing. First, assume $\lambda = 1$. Let $p \geq 1$. One sets

$$\varphi_p(x) = (\log(e^{p-1} + x))^p - (p-1)^p, \quad x > 0.$$

The function φ_p is continuous, increasing, concave and one-to-one from $\mathbb{R}_+$ onto $\mathbb{R}_+$ (the term e^{p-1} is introduced to ensure the concavity). It follows that $\varphi_p^{-1}(y) = e^{((p-1)^p + y)^{1/p}} - e^{p-1} \leq e^{y^{1/p}}$ for every $y \geq 0$ (since $(u+v)^{\frac{1}{p}} \leq u^{\frac{1}{p}} + v^{\frac{1}{p}}$, u, $v \geq 0$) so that

$$\mathbb{E} \max_{k=1,\ldots,n} Y_k^p \ = \ \mathbb{E}\left(\max_{k=1,\ldots,n} \left(\varphi_p \circ \varphi_p^{-1}(Y_k^p) \right) \right)$$

$$= \ \mathbb{E}\left(\varphi_p\left(\max \left(\varphi_p^{-1}(Y_1^p), \ldots, \varphi_p^{-1}(Y_n^p) \right) \right) \right)$$

since φ_p si non-decreasing. Then Jensen's Inequality implies

$$
\begin{aligned}
\mathbb{E} \max_{k=1,\ldots,n} Y_k^p &\leq \varphi_p \left(\mathbb{E} \max_{k=1,\ldots,n} \varphi_p^{-1}(Y_k^p) \right) \\[2mm]
&\leq \varphi_p \left(\sum_{k=0}^{n} \mathbb{E}\, \varphi_p^{-1}(Y_k^p) \right) \\[2mm]
&= \varphi_p \left(n\, \mathbb{E}\, \varphi_p^{-1}(Y_1^p) \right) \\[2mm]
&\leq \varphi_p \left(n\, \mathbb{E}\, e^{Y_1} \right) \\[2mm]
&\leq \left(\log \left(e^{p-1} + n\, \mathbb{E}\, e^{Y_1} \right) \right)^p.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left\| \max_{k=1,\ldots,n} Y_k \right\|_p &\leq \log \left(e^{p-1} + n\, \mathbb{E}\, e^{Y_1} \right) \\[2mm]
&= \log n + \log \left(\mathbb{E}\, e^{Y_1} + \frac{e^{p-1}}{n} \right) \\[2mm]
&\leq \log n + C_{p,Y_1}
\end{aligned}
$$

where $C_{p,Y_1} = \log \left(\mathbb{E}\, e^{Y_1} + e^{p-1} \right)$.

Let us come back to the general case *i.e.* $\mathbb{E}\, e^{\lambda Y_1} < +\infty$ for a $\lambda > 0$. Then

$$
\begin{aligned}
\left\| \max(Y_1,\ldots,Y_n) \right\|_p &= \frac{1}{\lambda} \left\| \max(\lambda Y_1,\ldots,\lambda Y_n) \right\|_p \\[2mm]
&\leq \frac{1}{\lambda} (\log n + C_{p,\lambda,Z_1}). \qquad \diamond
\end{aligned}
$$

▷ **Exercises.** (*a*) Let Z be a non-negative random variable with distribution function $F(z) = \mathbb{P}(Z \leq z)$ and a continuous probability density function f. Show that if the survival function $\bar{F}(z) := \mathbb{P}(Z > z)$ satisfies

$$
\forall z \geq a > 0, \qquad \bar{F}(z) \geq c\, f(z)
$$

then, if $(Z_n)_{n \geq 1}$ is i.i.d. with distribution $\mathbb{P}_Z(dz)$,

$$
\forall p \geq 1, \qquad \left\| \max(Z_1,\ldots,Z_n) \right\|_p \geq c \sum_{k=1}^{n} \frac{1 - F^k(a)}{k} = c(\log n + \log(1 - F(a))) + C + \varepsilon_n
$$

with $\lim_n \varepsilon_n = 0$.

[Hint: one may assume $p = 1$. Then use the classical representation formula

$$
\mathbb{E}\, U = \int_0^{+\infty} \mathbb{P}(U \geq u)\, du
$$

for any non-negative random variable U and some basic facts about Stieljès integral like $dF(z) = f(z)dz$, etc.]

(*b*) Show that the $\chi^2(1)$ distribution defined by $f(u) := \frac{e^{-\frac{u}{2}}}{\sqrt{2\pi u}}\mathbf{1}_{\{u>0\}}du$ satisfies the above inequality for any $\eta > 0$ [Hint: use an integration by parts and usual comparison theorems on integrals to show that

$$\bar{F}(z) = \frac{2e^{-\frac{z}{2}}}{\sqrt{2\pi z}} - \int_z^{+\infty} \frac{e^{-\frac{u}{2}}}{u\sqrt{2\pi u}}du \sim \frac{2e^{-\frac{z}{2}}}{\sqrt{2\pi z}} \qquad \text{as} \qquad z \to +\infty \ .]$$

▷ *A.s. convergence rate(s).* The last important result of this section is devoted to the *a.s.* convergence of the Euler schemes toward the diffusion process with a first (elementary) approach to its rate of convergence.

Theorem 7.3 *If b and σ satisfy (H_T^β) for a $\beta \in (0,1]$ and if X_0 is a.s. finite, the continuous Euler scheme $\bar{X}^n = (\bar{X}_t^n)_{t\in[0,T]}$ a.s. converges toward the diffusion X for the* sup*-norm over $[0,T]$. Furthermore, for every $\alpha \in [0, \beta \wedge \frac{1}{2})$,*

$$n^\alpha \sup_{t\in[0,T]} |X_t - \bar{X}_t^n| \xrightarrow{a.s.} 0.$$

The proof follows from the L^p-convergence theorem by an approach "à la Borel-Cantelli". The details are deferred to Section 7.8.6.

7.2.2 Proofs in the quadratic Lipschitz case for homogeneous diffusions

We provide below a proof of both Proposition 7.2 and Theorem 7.2 in a restricted 1-dimensional, homogeneous and quadratic ($p = 2$) setting. This means that $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$ *are defined as Lipschitz continuous continuous functions on the real line.* Then (SDE) admits a unique strong solution starting from X_0 on every interval $[0,T]$ which means that there exists a unique strong solution $(X_t)_{t\geq0}$ starting from X_0.

Furthermore, we will not care about the structure of the real constants that come out, in particular no control in T is provided in this simplified version of the proof. The complete and detailed proof is postponed to Section 7.8.

Lemma 7.2 *(Gronwall Lemma) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel non-negative locally bounded function and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying*

$$(G) \quad \equiv \quad \forall\, t \geq 0, \quad f(t) \leq \alpha \int_0^t f(s)\,ds + \psi(t)$$

for a real constant $\alpha > 0$. Then

$$\forall\, t \geq 0, \qquad \sup_{0\leq s\leq t} f(s) \leq e^{\alpha t}\psi(t).$$

Proof. It is clear that the non-decreasing (finite) function $\varphi(t) := \sup_{0 \le s \le t} f(s)$ satisfies (G) instead of f. Now the function $e^{-\alpha t} \int_0^t \varphi(s)\, ds$ has a right derivative at every $t \ge 0$ and that

$$\left(e^{-\alpha t} \int_0^t \varphi(s) ds \right)'_r = e^{-\alpha t} \left(\varphi(t+) - \alpha \int_0^t \varphi(s)\, ds \right)$$

$$\le e^{-\alpha t} \psi(t+)$$

where $\varphi(t+)$ and $\psi(t+)$ denote right limits of φ and ψ at t. Then, it follows from the fundamental theorem of calculus that

$$e^{-\alpha t} \int_0^t \varphi(s) ds - \int_0^t e^{-\alpha s} \psi(s+)\, ds \quad \text{is non-increasing.}$$

Hence, applying that between 0 and t yields

$$\int_0^t \varphi(s) \le e^{\alpha t} \int_0^t e^{-\alpha s} \psi(s+) ds.$$

Plugging this in the above inequality implies

$$\varphi(t) \le \alpha e^{\alpha t} \int_0^t e^{-\alpha s} \psi(s+) ds + \psi(t)$$

$$= \alpha e^{\alpha t} \int_0^t e^{-\alpha s} \psi(s) ds + \psi(t)$$

$$\le e^{\alpha t} \psi(t)$$

where we used successively that a monotone function is ds-$a.s.$ continuous and that ψ is non-decreasing. $\diamond$

Now we recall the classical Doob's Inequality that is needed to carry out the proof (instead of the more sophisticated Burkhölder-Davis-Gundy Inequality which is necessary in the non-quadratic case).

Doob's Inequality. (see *e.g.* [91]) (a) Let $M = (M_t)_{t \ge 0}$ be a continuous martingale with $M_0 = 0$. Then, for every $T > 0$,

$$\mathbb{E} \left(\sup_{t \in [0,T]} M_t^2 \right) \le 4\, \mathbb{E}\, M_T^2 = 4\, \mathbb{E} \langle M \rangle_T.$$

(b) If M is simply a continuous *local* martingale with $M_0 = 0$, then, for every $T > 0$,

$$\mathbb{E} \left(\sup_{t \in [0,T]} M_t^2 \right) \le 4\, \mathbb{E} \langle M \rangle_T.$$

Proof of Proposition 7.2 (A first partial). We may assume without loss of generality that $\mathbb{E}\, X_0^2 < +\infty$ (otherwise the inequality is trivially fulfilled). Let $\tau_L := \min\{t : |X_t - X_0| \ge L\}$,

$L \in \mathbb{N} \setminus \{0\}$ (with the usual convention $\min \emptyset = +\infty$) . It is a positive $\mathcal{F}$-stopping time as the hitting time of a closed set by a process with continuous paths. Furthermore, for every $t \in [0, T]$,

$$|X_t^{\tau_L}| \leq L + |X_0|, \quad t \in [0, \infty).$$

In particular this implies that

$$\mathbb{E} \sup_{t \in [0,T]} |X_t^{\tau_L}|^2 \leq 2(L^2 + \mathbb{E} X_0^2) < +\infty.$$

Then,

$$
\begin{aligned}
X_t^{\tau_L} &= X_0 + \int_0^{t \wedge \tau_L} b(X_s)ds + \int_0^{t \wedge \tau_L} \sigma(X_s)dW_s \\
&= X_0 + \int_0^{t \wedge \tau_L} b(X_s^{\tau_L})ds + \int_0^{t \wedge \tau_L} \sigma(X_s^{\tau_L})dW_s
\end{aligned}
$$

owing to the local feature of (standard and) stochastic integral(s). The stochastic integral

$$M_t^{(L)} := \int_0^{t \wedge \tau_L} \sigma(X_s^{\tau_L})dW_s$$

is a continuous local martingale null at zero with bracket process defined by

$$\langle M^{(L)} \rangle_t = \int_0^{t \wedge \tau_L} \sigma^2(X_s^{\tau_L})ds.$$

Now, using that $t \wedge \tau_L \leq t$, we derive that

$$|X_t^{\tau_L}| \leq |X_0| + \int_0^t |b(X_s^{\tau_L})|ds + \sup_{s \in [0,t]} |M_s^{(L)}|$$

which in turn immediately implies that

$$\sup_{s \in [0,T]} |X_s^{\tau_L}| \leq |X_0| + \int_0^t |b(X_s^{\tau_L})|ds + \sup_{s \in [0,t]} |M_s^{(L)}|.$$

The elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ $(a, b, c \geq 0)$, combined with the Schwarz Inequality successively yields

$$
\begin{aligned}
\sup_{s \in [0,t]} (X_s^{\tau_L})^2 &\leq 3\left(X_0^2 + \left(\int_0^t |b(X_s^{\tau_L})|ds \right)^2 + \sup_{s \in [0,t]} |M_s^{(L)}|^2 \right) \\
&\leq 3\left(X_0^2 + t \int_0^t |b(X_s^{\tau_L})|^2 ds + \sup_{s \in [0,t]} |M_s^{(L)}|^2 \right).
\end{aligned}
$$

We know that the functions b and σ satisfy a linear growth assumption

$$|b(x)| + |\sigma(x)| \leq C_{b,\sigma}(1 + |x|), \quad x \in \mathbb{R},$$

as Lipschitz continuous continuous functions. Then, taking expectation and using Doob Inequality for the local martingale $M^{(L)}$ yield for an appropriate real constant $C_{b,\sigma,T} > 0$ (that may vary from line to line)

$$
\begin{aligned}
\mathbb{E}(\sup_{s\in[0,t]} (X_s^{\tau_L})^2) &\leq 3\left(\mathbb{E}X_0^2 + TC_{b,\sigma}\int_0^t (1+\mathbb{E}|X_s^{\tau_L}|^2)ds + \mathbb{E}\int_0^{t\wedge\tau_L} \sigma^2(X_s^{\tau_L})ds \right) \\
&\leq C_{b,\sigma,T}\left(\mathbb{E}X_0^2 + \int_0^t (1+\mathbb{E}|X_s^{\tau_L}|^2)ds + \mathbb{E}\int_0^t (1+|X_s^{\tau_L}|)^2 ds \right) \\
&= C_{b,\sigma,T}\left(\mathbb{E}X_0^2 + \int_0^t (1+\mathbb{E}|X_s^{\tau_L}|^2)ds \right)
\end{aligned}
$$

where we used again (in the first inequality) that $\tau_L \wedge t \leq t$. Finally, this can be rewritten

$$
\mathbb{E}(\sup_{s\in[0,t]} (X_s^{\tau_L})^2) \leq C_{b,\sigma,T}\left(1 + \mathbb{E}X_0^2 + \int_0^t \mathbb{E}(|X_s^{\tau_L}|^2)ds \right)
$$

for a new real constant $C_{b,\sigma,T}$. Then the Gronwall Lemma applied to the bounded function $f_L(t) := \mathbb{E}(\sup_{s\in[0,t]}(X_s^{\tau_L})^2)$ (at time $t=T$) implies

$$
\mathbb{E}(\sup_{s\in[0,T]} (X_s^{\tau_L})^2) \leq C_{b,\sigma,T}(1+\mathbb{E}X_0^2)e^{C_{b,\sigma,T}T}.
$$

This holds or every $L \geq 1$. Now $\tau_L \uparrow +\infty$ a.s. as $L \uparrow +\infty$ since $\sup_{0\leq s\leq t}|X_t| < +\infty$ for every $t \geq 0$ a.s. Consequently,

$$
\lim_{L\to+\infty} \sup_{s\in[0,T]} |X_s^{\tau_L}| = \sup_{s\in[0,T]} |X_s|.
$$

Then Fatou's Lemma implies

$$
\mathbb{E}(\sup_{s\in[0,T]} X_s^2) \leq C_{b,\sigma,T}(1+\mathbb{E}X_0^2)e^{C_{b,\sigma,T}T} = C'_{b,\sigma,T}(1+\mathbb{E}X_0^2).
$$

As for the Euler scheme the same proof works perfectly if we introduce the stopping time

$$
\bar{\tau}_L = \min\left\{ t \ : \ |\bar{X}_t - X_0| \geq L \ big \right\}
$$

and if we note that, for every $s\in[0,T]$, $\sup_{u\in[0,s]} |\bar{X}_{\underline{u}}| \leq \sup_{u\in[0,s]} |\bar{X}_u|$. Then one shows that

$$
\sup_{n\geq 1}\mathbb{E}\left(\sup_{s\in[0,T]} (\bar{X}_s^n)^2 \right) \leq C_{b,\sigma,T}(1+\mathbb{E}X_0^2)e^{C_{b,\sigma,T}T}. \quad \diamond
$$

Proof of Theorem 7.2 (partial) (a) *(Convergence rate of the continuous Euler scheme)*. Combining the equations satisfied by X and its (continuous) Euler scheme yields

$$
X_t - \bar{X}_t = \int_0^t (b(X_s) - b(\bar{X}_{\underline{s}}))ds + \int_0^t (\sigma(X_s) - \sigma(\bar{X}_{\underline{s}}))dW_s.
$$

Consequently, using that b and σ are Lipschitz, Schwartz and Doob Inequality lead to

$$
\begin{aligned}
\mathbb{E} \sup_{s\in[0,t]} |X_s - \bar{X}_s|^2 &\leq 2\,\mathbb{E}\left(\int_0^t [b]_{Lip}|X_s - \bar{X}_{\underline{s}}|ds\right)^2 + 2\,\mathbb{E}\sup_{s\in[0,t]}\left(\int_0^s (\sigma(X_u) - \sigma(\bar{X}_{\underline{u}}))dW_u\right)^2 \\
&\leq 2\,\mathbb{E}\left(\int_0^t [b]_{Lip}|X_s - \bar{X}_{\underline{s}}|ds\right)^2 + 8\,\mathbb{E}\int_0^t (\sigma(X_u) - \sigma(\bar{X}_{\underline{u}}))^2 du \\
&\leq 2T[b]_{Lip}^2\,\mathbb{E}\int_0^t |X_s - \bar{X}_{\underline{s}}|^2 ds + 8\,[\sigma]_{Lip}^2 \int_0^t \mathbb{E}|X_u - \bar{X}_{\underline{u}}|^2 du \\
&\leq C_{b,\sigma,T}\int_0^t \mathbb{E}|X_s - \bar{X}_{\underline{s}}|^2 ds + 8[\sigma]_{Lip}^2 \int_0^t \mathbb{E}|X_u - \bar{X}_{\underline{u}}|^2 du \\
&\leq C_{b,\sigma,T}\int_0^t \mathbb{E}\sup_{u\in[0,s]}|X_u - \bar{X}_u|^2 ds + C_{b,\sigma,T}\int_0^t \mathbb{E}|\bar{X}_s - \bar{X}_{\underline{s}}|^2 ds.
\end{aligned}
$$

The function $f(t) := \mathbb{E}\sup_{s\in[0,t]}|X_s - \bar{X}_s|^2$ is locally bounded owing to Step 1. Consequently, it follows from Gronwall Lemma (at $t = T$) that

$$
\mathbb{E}\sup_{s\in[0,T]} |X_s - \bar{X}_s|^2 \leq C_{b,\sigma,T}\int_0^T \mathbb{E}|\bar{X}_s - \bar{X}_{\underline{s}}|^2 ds\, e^{C_{b,\sigma,T}T}.
$$

Now

$$
\bar{X}_s - \bar{X}_{\underline{s}} = b(\bar{X}_{\underline{s}})(s - \underline{s}) + \sigma(\bar{X}_{\underline{s}})(W_s - W_{\underline{s}}) \tag{7.13}
$$

so that, using Step 1 (for the Euler scheme) and the fact that $W_s - W_{\underline{s}}$ and $\bar{X}_{\underline{s}}$ are independent

$$
\begin{aligned}
\mathbb{E}|\bar{X}_s - \bar{X}_{\underline{s}}|^2 &\leq C_{b,\sigma}\left(\left(\frac{T}{n}\right)^2 \mathbb{E}\,b^2(\bar{X}_{\underline{s}}) + \mathbb{E}\,\sigma^2(\bar{X}_{\underline{s}})\,\mathbb{E}(W_s - W_{\underline{s}})^2\right) \\
&\leq C_{b,\sigma}(1 + \mathbb{E}\sup_{t\in[0,T]} |\bar{X}_t|^2)\left(\left(\frac{T}{n}\right)^2 + \frac{T}{n}\right) \\
&= C_{b,\sigma}\left(1 + \mathbb{E}\,X_0^2\right)\frac{T}{n}.
\end{aligned}
$$

(b) Stepwise constant Euler scheme. We assume here – for pure convenience – that $X_0 \in L^4$. One derives from (7.13) and the linear growth assumption satisfied by b and σ (since they are Lipschitz continuous continuous) that

$$
\sup_{t\in[0,T]} |\bar{X}_t - \bar{X}_{\underline{t}}| \leq C_{b,\sigma}\left(1 + \sup_{t\in[0,T]} |\bar{X}_t|\right)\left(\frac{T}{n} + \sup_{t\in[0,T]} |W_t - W_{\underline{t}}|\right)
$$

so that,

$$
\|\sup_{t\in[0,T]} |\bar{X}_t - \bar{X}_{\underline{t}}|\|_2 \leq C_{b,\sigma}\left\|\left(1 + \sup_{t\in[0,T]} |\bar{X}_t|\right)\left(\frac{T}{n} + \sup_{t\in[0,T]} |W_t - W_{\underline{t}}|\right)\right\|_2.
$$

Now, if U and V are real valued random variables, Schwarz Inequality implies

$$
\|U\,V\|_2 = \sqrt{\|U^2 V^2\|_1} \leq \sqrt{\|U^2\|_2 \|V^2\|_2} = \|U\|_4 \|V\|_4.
$$

Consequently

$$\| \sup_{t\in[0,T]} |\bar{X}_t - \bar{X}_{\underline{t}}| \|_2 \leq C_{b,\sigma}(1 + \| \sup_{t\in[0,T]} |\bar{X}_t| \|_4)(T/n + \| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \|_4).$$

Now, as already mentioned in the first remark that follows Theorem 7.2,

$$\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \|_4 \leq C_W \sqrt{\frac{T}{n}} \sqrt{1 + \log n}$$

which completes the proof (if we admit that $\| \sup_{t\in[0,T]} |\bar{X}_t| \|_4 \leq C_{b,\sigma,T}(1 + \|X_0\|_4)$). $\diamond$

Remarks. • The proof in the general L^p framework follows exactly the same lines, except that one replaces Doob's Inequality for continuous (local) martingale $(M_t)_{t\geq 0}$ by the so-called Burkh "older-Davis-Gundy Inequality (see *e.g.* [140]) which holds for every exponent $p > 0$ (only in the continuous setting)

$$\forall\, t \geq 0, \qquad \| \sup_{s\in[0,t]} |M_s| \|_p \leq C_p \, \|\langle M \rangle_t\|_{\frac{p}{2}}^2$$

where C_p is a positive real constant only depending on p. This general setting is developed in full details in Section 7.8 (in the one dimensional case to alleviate notations).

• In some so-called mean-reverting situations one may even get boundedness over $t \in (0,\infty)$.

7.3   Non asymptotic deviation inequalities for the Euler scheme

It will be convenient to introduce in this section temporary notations.

• We denote $x_{1:d} = (x_1,\ldots,x_d) \in \mathbb{R}^d$. Let $|x_{1:d}|^2 = x_1^2 + \cdots + x_d^2$ define the canonical Euclidean norm on $\mathbb{R}^d$.
• Let $\|A\| = \left(\mathrm{Tr}(AA^*)\right)^{\frac{1}{2}}$ denote the Euclidean norm of $A \in \mathcal{M}_{d,q}(\mathbb{R})$ and let $|||A||| = \sup_{|x|=1} |Ax|$ be the operator norm of (with respect to the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{R}^q$. Note that $|||A||| \leq \|A\|$.

We still consider a Brownian diffusion process solution with drift b and diffusion coefficient σ, starting at X_0, solution to the SDE (7.1). Furthermore, we assume that $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}_{d,q}(\mathbb{R})$ are Borel functions satisfying a Lipschitz continuous assumption in x uniformly in $t \in [0,T]$ *i.e.*

$$[b]_{\mathrm{Lip}} = \sup_{t\in[0,T],x\neq y} \frac{|b(t,x) - b(t,y)|}{|x - y|} < +\infty \quad \text{and} \quad [\sigma]_{\mathrm{Lip}} = \sup_{t\in[0,T],x\neq y} \frac{\|\sigma(t,x) - \sigma(t,y)\|}{|x - y|} < +\infty.$$

The (augmented) natural filtration of the driving standard q-dimensional Brownian motion W defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ will be denoted $(\mathcal{F}_t^W)_{t\in[0,T]}$.

The definition of the Brownian Euler scheme with step T/n (starting at X_0), is unchanged but to alleviate notations we will use the notation $\bar{X}_k^n$ (rather than $\bar{X}_{t_k^n}^n$). So we have: $\bar{X}_0^n = X_0$ and

$$\bar{X}_{k+1}^n = \bar{X}_k^n + \frac{T}{n}b(t_k^n, \bar{X}_k^n) + \sigma(t_k^n, \bar{X}_k^n)\sqrt{\frac{T}{n}}Z_{k+1}, \ k = 0,\ldots,n-1$$

where $t_k^n = \frac{kT}{n}$, $k = 0, \ldots, n$ and $Z_k = \sqrt{\frac{n}{T}}(W_{t_k^n} - W_{t_{k-1}^n})$, $k = 1, \ldots, n$ is an i.i.d. sequence of $\mathcal{N}(0, I_q)$ random vectors. When $X_0 = x \in \mathbb{R}^d$, we may denote occasionally by $(\bar{X}_k^{n,x})_{0 \le k \le n}$ the Euler scheme starting at x.

The Euler scheme defines a Markov chain with transitions $P_{k,k+1}(x, dy)$, $k = 0, \ldots, n-1$, reading on bounded or non-negative Borel functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$P_{k,k+1}(f)(x) = \mathbb{E}\Big(f(\bar{X}_{k+1}) \mid \bar{X}_k = x\Big) = \mathbb{E}f\Big(x + \frac{T}{n}b(t_k^n, x) + \sigma(t_k^n, x)\sqrt{\frac{T}{n}}Z\Big), \ k = 0, \ldots, n-1.$$

Then, for every $k, \ell \in \{0, \ldots, n-1\}$, $k \le \ell$, we set

$$P_{k,\ell} = P_{k,k+1} \circ \cdots \circ P_{\ell-1,\ell}$$

so that we have, still for bounded or nonnegative Borel functions f,

$$P_{k,\ell}(f)(x) = \mathbb{E}\Big(f(\bar{X}_\ell) \mid \bar{X}_k = x\Big).$$

Still for simplicity we write P_k instead of $P_{k,k+1}$.

Lemma 7.3 *Under the above assumptions on b and σ, the transitions P_k satisfy*

$$[P_k f]_{\mathrm{Lip}} \le [P_k]_{\mathrm{Lip}}[f]_{\mathrm{Lip}}, \quad k = 0, \ldots, n-1,$$

with
$$
\begin{aligned}
[P_k]_{\mathrm{Lip}} &= \Big(\big[1 + \frac{T}{n}b(t_k^n, .)\big]_{\mathrm{Lip}}^2 + \frac{T}{n}[\sigma(t_k^n, .)]_{\mathrm{Lip}}^2\Big)^{\frac{1}{2}} \\
&\le \Big(1 + \frac{T}{n}\big(C_{b,\sigma} + \frac{T}{n}\kappa_b\big)\Big)^{\frac{1}{2}}
\end{aligned}
$$

where $C_{b,\sigma} = 2[b]_{\mathrm{Lip}} + [\sigma]_{\mathrm{Lip}}^2$ and $\kappa_b = [b]_{\mathrm{Lip}}^2$.

Proof. Straightforward consequence of the fact that

$$
\begin{aligned}
|P_k f(x) - P_k f(y)| &\le [f]_{\mathrm{Lip}}\Big\|(x-y) + \frac{T}{n}(b(t_k^n, x) - b(t_k^n, y)) + \sqrt{\frac{T}{n}}(\sigma(t_k^n, x) - \sigma(t_k^n, y))Z\Big\|_1 \\
&\le [f]_{\mathrm{Lip}}\Big\|(x-y) + \frac{T}{n}(b(t_k^n, x) - b(t_k^n, y)) + \sqrt{\frac{T}{n}}(\sigma(t_k^n, x) - \sigma(t_k^n, y))Z\Big\|_2.
\end{aligned}
$$

The key property is the following classical exponential inequality for the Gaussian measure for which we refer to [100].

Proposition 7.3 *Let $f : \mathbb{R}^q \to \mathbb{R}$ be Lipschitz continuous continuous function (with respect to the canonical Euclidean measure) and let Z be an $\mathcal{N}(0; I_q)$ distributed random vector. Then*

$$\forall \lambda > 0, \quad \mathbb{E}\Big(e^{\lambda\big(f(Z) - \mathbb{E}f(Z)\big)}\Big) \le e^{\frac{\lambda^2}{2}[f]_{\mathrm{Lip}}^2}. \tag{7.14}$$

Proof. Step 1 (*Preliminaries*): We consider a standard Orstein-Uhlenbeck process starting at $x \in \mathbb{R}^q$ solution to the Stochastic Differential equation

$$d\, \Xi_t^x = -\frac{1}{2}\, \Xi_t^x dt + dW_t, \ \Xi_0^x = x$$

where W is a standard q-dimensional Brownian motion. This equations has a (unique) explicit solution on the whole real line given by

$$\Xi_t^x = xe^{-\frac{t}{2}} + e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} dW_s, \quad t \in \mathbb{R}_+.$$

This shows that $(\Xi_t^x)_{t \geq 0}$ is a Gaussian process

$$\mathbb{E}\, \Xi_t^x = xe^{-\frac{t}{2}}$$

and, using the Wiener isometry, we derive the covariance matrix $\Sigma_{\Xi_t^x}$ of Ξ_t^x given by

$$
\begin{aligned}
\Sigma_{\Xi_t^x} &= e^t \left[\mathbb{E}\left(\int_0^t e^{\frac{s}{2}} dW_s^k \int_0^t e^{\frac{s}{2}} dW_s^\ell \right) \right]_{1 \leq k, \ell \leq q} \\
&= e^{-t} \int_0^t e^s ds I_q \\
&= (1 - e^{-t}) I_q.
\end{aligned}
$$

(The time covariance structure of the process can also be computed easily but is of no use in this proof). As a consequence, for every Borel function $g : \mathbb{R}^q \to \mathbb{R}$ with polynomial growth

$$Q_t g(x) := \mathbb{E}\, g(\Xi_t^x) = \mathbb{E}\, g\left(x\, e^{-t/2} + \sqrt{1 - e^{-t}} Z \right) \quad \text{where} \quad Z \overset{\mathcal{L}}{\sim} \mathcal{N}(0; I_q).$$

Hence, owing to the Lebesgue dominated convergence theorem,

$$\lim_{t \to +\infty} Q_t g(x) = \mathbb{E} g(Z).$$

Moreover, if g is differentiable with a derivative g' with polynomial growth,

$$\frac{d}{dx} Q_t g(x) = e^{-\frac{t}{2}} \mathbb{E}\, g'(\Xi_t^x). \tag{7.15}$$

Now, if g is twice continuously differentiable with a Hessian $D^2 g$ having polynomial growth, it follows from Itô's formula and Fubini's Theorem that

$$Q_t g(x) = g(x) + \int_0^t \mathbb{E}\, (Lg)(\Xi_s^x) ds$$

where L is the infinitesimal generator of the above equation (known as the Ornstein-Uhlenbeck operator) which maps g to

$$\xi \longmapsto Lg(\xi) = \frac{1}{2}\big(\Delta g(\xi) - (\xi | \nabla g(\xi)) \big).$$

where $\Delta g(\xi) = \sum_{i=1}^{q} g''_{x_i^2}(\xi)$ denotes the Laplacian of g and $\nabla g(\xi) = \begin{bmatrix} g'_{x_i}(\xi) \\ \vdots \\ g'_{x_q}(\xi) \end{bmatrix}$. Now, if h and g are both twice differentiable, one has

$$\mathbb{E}\big((\nabla g(Z)|\nabla h(Z)\big) = \sum_{k=1}^{q} \int_{\mathbb{R}^q} g'_{x_k}(z) h'_{x_k}(z) e^{-\frac{|z|^2}{2}} \frac{dz}{(2\pi)^{\frac{d}{2}}}$$

An integration by parts in the kth integral, after noting that

$$\frac{\partial}{\partial z_k}\left(h'_{x_k}(z) e^{-\frac{|z|^2}{2}}\right) = e^{-\frac{|z|^2}{2}}\left(h''_{x_k^2}(z) - z_k h'_{x_k}(z)\right),$$

yields

$$\mathbb{E}\big((\nabla g(Z)|\nabla h(Z)\big) - 2\mathbb{E}\big(G(Z)Lh(Z)\big). \tag{7.16}$$

since h and all its existing partial derivative have at lost polynomial growth. One also derive from the above identity and the continuity of $s \mapsto \mathbb{E}(Lg)(\Xi_s^x)$ that

$$\frac{d}{dt}Q_t g(x) = \mathbb{E}\, Lg(\Xi_t^x) = Q_t Lg(x).$$

We temporarily admit the classical fact that $LQ_t g(x = Q_t Lg(x)$

Step 2 (*The smooth case*): Let $f : \mathbb{R}^q \to \mathbb{R}$ be a continuously twice differentiable function with bounded existing partial derivatives and such that $\mathbb{E}\, f(Z) = 0$. Let $\lambda \in \mathbb{R}_+$ be fixed. We define the function $H : \mathbb{R} \to \mathbb{R}$ by

$$H_\lambda(t) = \mathbb{E}\, e^{\lambda Q_t g(Z)}.$$

It follows from what precedes that $|Q_t f(Z) \le e^{-t/2}\|f'\|_{\sup}|Z| + |f(Z)|\varepsilon|f(0)| + (1 + e^{-\frac{t}{2}})\|f'\|_{\sup}$ which ensures the existence of H_λ since $\mathbb{E}e^{a|Z|} < +\infty$ for every $a \ge 0$. Moreover, $Q_t f(Z) - f(Z)| \le e^{-t/2}\|f'\|_{\sup}|Z|$ so that $Q_t f(Z) \to f(Z)$ as $t \to 0$ and, by the Lebesgue dominated convergence theorem,

$$\lim_{t \to +\infty} H_\lambda(t) = e^{\mathbb{E}\, f(Z)} = e^0 = 1.$$

Furthermore, one shows using the same arguments that H_λ is differentiable over the whole real line and

$$\begin{aligned}
H'_\lambda(t) &= \lambda\, \mathbb{E}\big(e^{\lambda Q_t f(Z)} Q_t Lf(Z)\big) \\
&= \lambda\, \mathbb{E}\big(e^{\lambda Q_t f(Z)} LQ_t f(Z)\big) \\
&= -\frac{1}{2}\lambda \mathbb{E}\Big(\big(\nabla_z(e^{\lambda Q_t f(z)})_{|z=Z}|Q_t f(Z)\big)\Big) \\
&= -\frac{\lambda^2}{2}e^{-t}\mathbb{E}\Big(e^{\lambda Q_t f(Z)}|Q_t\nabla f(Z)|^2\Big)
\end{aligned}$$

where we used successively (7.16) and (7.15) in the last two lines.

Consequently, as $\lim_{t\to+\infty} H_\lambda(t)$,

$$
\begin{aligned}
H_\lambda(t) &= 1 - \int_t^{+\infty} H'_\lambda(s)ds \\
&\leq 1 + \frac{\lambda^2}{2}\|f'\|_{\sup} \int_t^{+\infty} e^{-s}\mathbb{E}\Big(e^{\lambda Q_s f(Z)}\Big)ds \\
&= 1 + K\int_t^{+\infty} e^{-s}H_\lambda(s)ds \quad \text{were} \quad K = \frac{\lambda^2}{2}\|\nabla f\|_{\sup}^2.
\end{aligned}
$$

where $\|\nabla f\|_{\sup} = \sup_{\xi\in\mathbb{R}^q} |\nabla f(\xi)|$ ($|\,.\,|$ denotes the Euclidean norm). One derives by induction (using that H_λ is non-increasing since its derivative is negative) that, for very integer $m\in\mathbb{N}^*$,

$$
H_\lambda(t) \leq \sum_{k=0}^{m} C^k \frac{e^{-kt}}{k!} + H_\lambda(0)K^{m+1}\frac{e^{-(m+1)t}}{(m+1)!}.
$$

Letting $m \to +\infty$ finally yields

$$
H_\lambda(t) \leq e^K = e^{\frac{\lambda^2}{2}\|\nabla f\|_{\sup}^2}.
$$

On completes this step of the proof the proof by applying the above inequality to the function $f - \mathbb{E}\,f(Z)$.

STEP 3 *(The Lipschitz continuous case)*: This step relies on an approximation technique which is closely related with sensitivity computation for options attached to non-regular payoffs (but in a situation where the Brownian motion is the pseudo-asset). Let $f : \mathbb{R}^q \to \mathbb{R}^q$ be a Lipschitz continuous function with Lipschitz coefficient $[f]_{\mathrm{Lip}}$ and $\zeta \overset{d}{=} \mathcal{N}\big(0;\frac{I_q}{q}\big)$. One considers for every $\varepsilon > 0$,

$$
f_\varepsilon(z) = \mathbb{E}f(z + \sqrt{\varepsilon}\zeta) = \int_{\mathbb{R}^q} f(u)e^{-\frac{|u-z|^2}{2\varepsilon q}}\frac{du}{(2\pi\varepsilon q)^{\frac{q}{2}}}.
$$

It is clear that $f_\varepsilon \to f$ since $|f_\varepsilon(z)-f(z)| \leq \sqrt{\varepsilon}\,\mathbb{E}\,|\zeta|$ for every $z\in\mathbb{R}^q$ so that f_ε uniformly converges on $\mathbb{R}^d$ toward f. One checks that f_ε is differentiable with a gradient given by

$$
\begin{aligned}
\nabla f_\varepsilon(z) &= -\frac{1}{\varepsilon q}\int_{\mathbb{R}^q} f(u)e^{-\frac{|u-z|^2}{2\varepsilon q}}(u-z)\frac{du}{(2\pi\varepsilon q)^{\frac{q}{2}}} \\
&= \frac{1}{q\sqrt{\varepsilon}}\mathbb{E}\big(f(z+\sqrt{\varepsilon}\zeta)\zeta\big) \\
&= \frac{1}{q\sqrt{\varepsilon}}\mathbb{E}\big((f(z+\sqrt{\varepsilon}\zeta)-f(z))\zeta\big)
\end{aligned}
$$

since $\mathbb{E}\,\zeta = 0$. Consequently,

$$
|\nabla f_\varepsilon(z)| \leq \frac{[f]_{\mathrm{Lip}}}{q}\mathbb{E}\,|\zeta|^2 = [f]_{\mathrm{Lip}}
$$

The Hessian of f_ε is also bounded (by a constant depending on ε but this has no consequence on the inequality of interest). One concludes by Fatou's lemma

$$
\mathbb{E}\,e^{\lambda f(Z)} \leq \liminf_{\varepsilon\to 0}\mathbb{E}\,e^{\lambda f_\varepsilon(Z)} \leq \liminf_{\varepsilon\to 0} e^{\frac{\lambda^2}{2}\|\nabla f_\varepsilon\|_{\sup}^2} \leq e^{\frac{\lambda^2}{2}q[f]_{\mathrm{Lip}}^2}. \quad \diamond
$$

We are now in position to state the main result of this section and its application to the design of confidence intervals (see also [54]).

Theorem 7.4 *Assume* $|||\sigma|||_{\sup} := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |||\sigma(t,x)||| < +\infty$. *Then, for every integer* $n_0 \geq 1$, *there exists a real constant* $K(b,\sigma,T,n_0) \in (0,+\infty)$ *such that, for every* $n \geq n_0$ *and Lipschitz continuous continuous function* $f : \mathbb{R}^d \to \mathbb{R}$,

$$\forall\,\lambda > 0, \quad \mathbb{E}\left(e^{\lambda\left(f(\bar{X}_n^n) - \mathbb{E}f(\bar{X}_n^n)\right)}\right) \leq e^{\frac{\lambda^2}{2}|||\sigma|||_{\sup}^2 [f]_{\mathrm{Lip}}^2 K(b,\sigma,T,n)}. \tag{7.17}$$

Moreover $\lim_n K(b,\sigma,T,n) = \dfrac{1}{C_{b,\sigma}} e^{C_{b,\sigma}T}$. *The choice* $K(b,\sigma,T,n) = \dfrac{1}{C_{b,\sigma}} e^{(C_{b,\sigma}+\kappa_b\frac{T}{n_0})T}$ *is admissible.*

APPLICATION. Let us briefly recall that such exponential inequalities yield concentration inequalities in the strong law of large numbers. Let $(\bar{X}^{n,\ell})_{\ell\geq 1}$ be independent copies of the Euler scheme $\bar{X}^n = (\bar{X}_k^n)_{0\leq k\leq n}$. Then, for every $\varepsilon > 0$, Markov inequality and independence imply, for every integer $n \geq n_0$,

$$\mathbb{P}\left(\frac{1}{M}\sum_{\ell=1}^M f(\bar{X}_n^{n,\ell}) - \mathbb{E}f(\bar{X}_n^n) > \varepsilon\right) \leq \inf_{\lambda>0} e^{-\lambda M\varepsilon + M\frac{\lambda^2}{2}|||\sigma|||_{\sup}^2[f]_{\mathrm{Lip}}^2 K(b,\sigma,T,n_0)}$$

$$= e^{-\frac{\varepsilon^2 M}{2|||\sigma|||_{\sup}^2[f]_{\mathrm{Lip}}^2 K(b,\sigma,T,n_0)}}$$

so that by symmetry,

$$\mathbb{P}\left(\left|\frac{1}{M}\sum_{\ell=1}^M f(\bar{X}_n^{n,\ell}) - \mathbb{E}f(\bar{X}_n^n)\right| > \varepsilon\right) \leq 2e^{-\frac{\varepsilon^2 M}{2|||\sigma|||_{\sup}^2[f]_{\mathrm{Lip}}^2 K(b,\sigma,T,n_0)}}.$$

The crucial facts in the above inequality, beyond the fact that it holds for possibly unbounded Lipschitz continuous functions f, is that the right hand upper-bound does not depend on the time step $\frac{T}{n}$ of the Euler scheme. Consequently we can design confidence intervals for Monte Carlo simulations based on the Euler schemes *uniformly in the time discretization step* $\frac{T}{n}$.

Doing so, we can design non asymptotic confidence interval when computing $\mathbb{E}f(X_T)$ by a Monte Carlo simulation. We know that the bias is due to the discretization scheme and only depends on the step $\frac{T}{n}$: usually (see Section 7.6 for the (expansion of the) weak error), one has

$$\mathbb{E}f(\bar{X}_T^n) = \mathbb{E}f(X_T) + \frac{c}{n^\alpha} + o(1/n^\alpha) \quad \text{with } \alpha = \tfrac{1}{2} \text{ or } 1.$$

Remark. Under the assumptions we make on b and σ, the Euler scheme converges *a.s.* and in every L^p spaces (provided X_0 lies in L^p for the sup norm over $[0,T]$ toward the diffusion process X, so that we retrieve the result for (independent copies of) the diffusion itself, namely

$$\mathbb{P}\left(\left|\frac{1}{M}\sum_{\ell=1}^M f(X_T^\ell) - \mathbb{E}f(X_T)\right| > \varepsilon\right) \leq 2e^{-\frac{\varepsilon^2 M}{2|||\sigma|||_{\sup}^2[f]_{\mathrm{Lip}}^2 K(b,\sigma,T)}}$$

with $K(b,\sigma,T) = \dfrac{e^{C_{b,\sigma}T}}{C_{b,\sigma}}$ (with $C_{b,\sigma} = 2[b]_{\mathrm{Lip}} + [\sigma]_{\mathrm{Lip}}^2$).

Proof of Theorem 7.4. The starting point is the following: let $k \in \{0, \ldots, n-1\}$, and let

$$\mathcal{E}_k(x, z) = x + \frac{T}{n}b(t_k^n, x) + \sigma(t_k^n, x)\sqrt{\frac{T}{n}}z, \ x \in \mathbb{R}^d, \ z \in \mathbb{R}^q, \ k = 0, \ldots, n-1,$$

denote the Euler scheme operator. Then, it follows form Proposition 7.3, that for every Lipschitz continuous continuous function $f : \mathbb{R}^d \to \mathbb{R}^q$

$$\forall \lambda > 0, \quad \mathbb{E}\left(e^{\lambda\left(f(\mathcal{E}_k(x,Z)) - P_k f(x)\right)}\right) \leq e^{\frac{\lambda^2}{2}\left(\sqrt{\frac{T}{n}}|||\sigma(t_k^n,x)|||\right)^2 [f]_{\mathrm{Lip}}^2}.$$

since $z \mapsto f(\mathcal{E}(x, z))$ is Lipschitz continuous continuous with respect to the canonical Euclidean norm with a coefficient upper bounded by the trace norm of $\sigma(t_k^n, x)$.

Consequently,

$$\forall \lambda > 0, \quad \mathbb{E}\left(e^{\lambda f(\bar{X}_{k+1}^n)} \mid \mathcal{F}_{t_k^n}^W\right) \leq e^{\lambda P_k f(\bar{X}_k^n) + \frac{\lambda^2}{2}\frac{T}{n}|||\sigma|||_{\mathrm{sup}}^2 [f]_{\mathrm{Lip}}^2}.$$

where $|||\sigma|||_{\mathrm{sup}} := \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |||\sigma(t, x)|||$.

Applying this inequality to $P_{k+1,n}f$ and taking expectation on both sides then yields

$$\forall \lambda > 0, \quad \mathbb{E}\left(e^{\lambda P_{k+1,n}f(\bar{X}_{k+1}^n)}\right) \leq \mathbb{E}\left(e^{\lambda P_{k,n}f(\bar{X}_k^n)}\right) e^{\frac{\lambda^2}{2}\frac{T}{n}|||\sigma|||_{\mathrm{sup}}^2 [P_{k+1,n}f]_{\mathrm{Lip}}^2}.$$

By a straightforward backward induction from $k = n-1$ down to $k = 0$, combined with the fact that $P_{0,n}f(X_0) = \mathbb{E}(f(\bar{X}_T)|X_0)$,

$$\forall \lambda > 0, \quad \mathbb{E}\left(e^{\lambda f(\bar{X}_n^n)}\right) \leq \mathbb{E}\left(e^{\lambda \mathbb{E}(f(\bar{X}_T)|X_0)}\right) e^{\frac{\lambda^2}{2}|||\sigma|||_{\mathrm{sup}}^2 \frac{T}{n} \sum_{k=0}^{n-1} [P_{k+1,n}f]_{\mathrm{Lip}}^2}.$$

First note that by Jensen's Inequality

$$\mathbb{E}\left(e^{\lambda \mathbb{E}(f(\bar{X}_T)|X_0)}\right) \leq \mathbb{E}\left(\mathbb{E}(e^{\lambda f(\bar{X}_T)}|X_0)\right) = \mathbb{E}\,e^{\lambda f(\bar{X}_T)}.$$

On the other hand, it is clear from their very definition that

$$[P_{k,n}]_{\mathrm{Lip}} \leq \prod_{\ell=k}^{n-1} [P_\ell]_{\mathrm{Lip}}$$

(with the consistent convention that an empty product is equal to 1). Hence, owing to Lemma 7.3,

$$[P_{k,n}]_{\mathrm{Lip}} \leq \left(1 + C_{b,\sigma}^{(n)}\frac{T}{n}\right)^{\frac{(n-k)}{2}} \quad \text{with} \quad C_{b,\sigma,T}^{(n)} = C_{b,\sigma} + \frac{T}{n}\kappa_b$$

and

$$\begin{aligned}
\frac{T}{n}\sum_{k=1}^{n}[P_{k,n}f]_{\mathrm{Lip}}^2 &= \frac{T}{n}\sum_{k=0}^{n-1}[P_{n-k,n}f]_{\mathrm{Lip}}^2 \\
&= \frac{(1 + C_{b,\sigma}^{(n)}\frac{T}{n})^n - 1}{C_{b,\sigma}^{(n)}} \\
&\leq \frac{1}{C_{b,\sigma}^{(n)}}e^{(C_{b,\sigma} + \kappa_b\frac{T}{n})T}[f]_{\mathrm{Lip}}^2. \\
&\leq \frac{1}{C_{b,\sigma}}e^{(C_{b,\sigma} + \kappa_b\frac{T}{n})T}[f]_{\mathrm{Lip}}^2
\end{aligned}$$

whenever $n \geq n_0$. $\diamond$

FURTHER COMMENTS. It seems hopeless to get a concentration inequality for the supremum of the Monte Carlo error over all Lipschitz continuous functions (with Lipschitz continuous coefficients bounded by 1) as emphasized below. Let $\mathrm{Lip}_1(\mathbb{R}^d, \mathbb{R})$ be the set of Lipschitz continuous continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ with a Lipschitz continuous coefficient $[f]_{\mathrm{Lip}} \leq 1$.

Let $X_\ell : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$, $\ell \geq 1$, be independent copies of an integrable random vector X with distribution denoted $\mathbb{P}_X$. For every $\omega \in \Omega$ and every $M \geq 1$, the function defined for every $\xi \in \mathbb{R}^d$ by

$$f_{\omega,M}(\xi) = \min_{1 \leq \ell \leq M} |X_\ell(\omega) - \xi|.$$

It is clear form its very definition that $f_{\omega,M} \in \mathrm{Lip}_1(\mathbb{R}^d, \mathbb{R})$. Then, for every $\omega \in \Omega$,

$$
\begin{aligned}
\sup_{f \in \mathrm{Lip}_1(\mathbb{R}^d, \mathbb{R})} \left| \frac{1}{M} \sum_{k=1}^{M} f(X_\ell(\omega)) - \int_{\mathbb{R}^d} f(\xi)\mathbb{P}_X(d\xi) \right| &\geq \left| \frac{1}{M} \sum_{k=1}^{M} \underbrace{f_{\omega,M}(X_\ell(\omega))}_{=0} - \int_{\mathbb{R}^d} f_{\omega,M}(\xi)\mathbb{P}_X(d\xi) \right| \\
&= \int_{\mathbb{R}^d} f_{\omega,M}(\xi)\mathbb{P}_X(d\xi) \\
&\geq e_{1,M}(X) := \inf_{x_1,\dots,x_M \in \mathbb{R}^d} \mathbb{E} \min_{x_1,\dots,x_M} |X - x_i|.
\end{aligned}
$$

The lower bound in the last line is but the optimal L^1-quantization error of the (distribution of the) random vector X. It follows from Zador's Theorem (see *e.g.* [66]) that

$$\liminf_M M^{-\frac{1}{d}} e_{1,M}(X) \geq J_{1,d} \|\varphi_X\|_{\frac{d}{d+1}}$$

where φ_X denotes the density – possibly equal to 0 – of the nonsingular part of the distribution $\mathbb{P}_X$ of X with respect to the Lebesgue measure, $J_{1,d} \in (0, \infty)$ is a universal constant and the pseudo-norm $\|\varphi_X\|_{\frac{d}{d+1}}$ is finite as soon as $X \in L^{1+} = \cup_{\eta>0} L^{p+\eta}$. Furthermore, it is clear that as soon as the support of $\mathbb{P}_X$ is infinite, $e_{1,M}(X) > 0$. Consequently, for non-singular random vector distributions, we have

$$M^{-\frac{1}{d}} \sup_{f \in \mathrm{Lip}_1(\mathbb{R}^d, \mathbb{R})} \left| \frac{1}{M} \sum_{k=1}^{M} f(X_\ell) - \mathbb{E}f(X) \right| \geq \inf_{M \geq 1} M^{-\frac{1}{d}} e_{1,M}(X) > 0.$$

In fact, a careful reading shows that

$$
\begin{aligned}
\sup_{f \in \mathrm{Lip}_1(\mathbb{R}^d, \mathbb{R})} \left| \frac{1}{M} \sum_{k=1}^{M} f(X_\ell(\omega)) - \mathbb{E} f(X)) \right| &= \left| \frac{1}{M} \sum_{k=1}^{M} f_{\omega,M}(X_\ell(\omega)) - \int_{\mathbb{R}^d} f_{\omega,M}(\xi)\mathbb{P}_X(d\xi) \right| \\
&= \int_{\mathbb{R}^d} f_{\omega,M}(\xi)\mathbb{P}_X(d\xi).
\end{aligned}
$$

If we denote $e_{1,M}(x_1, \dots, x_M, X) = \mathbb{E} \min_{x_1,\dots,x_M} |X - x_i|$ we have

$$\int_{\mathbb{R}^d} f_{\omega,M}(\xi)\mathbb{P}_X(d\xi) = e_{1,M}(X_1(\omega), \dots, X_M(\omega), X).$$

The right hand side of the above equation corresponds to the *self-random quantization* of the distribution $\mathbb{P}_X$. It has been shown in [36, 37] that, under appropriate assumptions on the distributions $\mathbb{P}_X$ (satisfied *e.g.* by the normal distribution), one has, $\mathbb{P}$-*a.s.*,

$$\limsup M^{-\frac{1}{d}}(e_{1,M}(X_1(\omega),\dots,X_M(\omega),X) < +\infty.$$

which shows that, for a wide class of random vector distributions, *a.s.*,

$$\sup_{f\in\mathrm{Lip}_1(\mathbb{R}^d,\mathbb{R})} \left| \frac{1}{M}\sum_{k=1}^{M} f(X_\ell(\omega)) - \mathbb{E}\, f(X)) \right| \asymp C(f)M^{-\frac{1}{d}}$$

This illustrates that the strong law of large numbers/Monte Carlo method is not as "dimension free" as it is commonly admitted.

7.4   Pricing path-dependent options (I) (Lookback, Asian, etc)

Let
$$\mathbb{D}([0,T],\mathbb{R}^d) := \left\{ \xi : [0,T] \to \mathbb{R}^d, \ \text{càdlàg} \right\}.$$

(càdlàg is the French acronym for "right continuous with left limits"). The above result shows that if $F : \mathbb{D}([0,T],\mathbb{R}) \to \mathbb{R}$ is a Lipschitz continuous functional for the sup norm *i.e.* satisfies

$$|F(\xi) - F(\xi')| \le [F]_{\mathrm{Lip}} \sup_{t\in[0,T]} |\xi(t) - \xi'(t)|$$

then

$$|\mathbb{E}(F((X_t)_{t\in[0,T]})) - \mathbb{E}(F(\bar{X}^n_{\underline{t}})_{t\in[0,T]})| \le [F]_{\mathrm{Lip}} C_{b,\sigma,T} \sqrt{\frac{1+\log n}{n}}$$

and

$$|\mathbb{E}(F((X_t)_{t\in[0,T]}) - \mathbb{E}(F(\bar{X}^n_t)_{t\in[0,T]})| \le Cn^{-\frac{1}{2}}.$$

Typical example in option pricing. Assume that a one dimensional diffusion process $X = (X_t)_{t\in[0,T]}$ models the dynamics of a single risky asset (we do not take into account here the consequences on the drift and diffusion coefficient term induced by the preservation of non-negativity and the martingale property under a risk-neutral probability for the discounted asset...).

▷ The (partial) Lookback Lookback payoffs:

$$h_T := \left(X_T - \lambda \min_{t\in[0,T]} X_t \right)_+$$

where $\lambda = 1$ in the regular Lookback case and $\lambda > 1$ in the so-called "partial Lookback" case.

▷ "Vanilla" payoffs on extrema (like Calls and Puts)

$$h_T = \varphi(\sup_{t\in[0,T]} X_t),$$

where φ is Lipschitz continuous on $\mathbb{R}_+$.

▷ Asian payoffs of the form

$$h_T = \varphi \left(\frac{1}{T - T_0} \int_{T_0}^{T} X_s ds \right), \quad 0 \le T_0 < T,$$

where φ is Lipschitz continuous on $\mathbb{R}$. In fact such Asian payoffs are continuous with respect to the pathwise L^2-norm *i.e.* $\|f\|_{L_T^2} := \sqrt{\int_0^T f^2(s) ds}$.

7.5 Milstein scheme (looking for better strong rates...)

Throughout this section we consider a homogeneous diffusion just for notational convenience, but the extension to general non homogeneous diffusions is straightforward (in particular it adds no further terms to the discretization scheme). The Milstein scheme has been designed (see [113]) to produce a $O(1/n)$-error (in L^p) like standard schemes in a deterministic framework. This is a higher order scheme. In 1-dimension, its expression is simple and it can easily be implemented, provided b and σ have enough regularity. In higher dimension, some theoretical and simulation problems make its use more questionable, especially when compared to the results about the weak error in the Euler scheme that will be developed in the next section.

7.5.1 The 1-dimensional setting

Assume that b and σ are smooth functions (say twice differentiable with bounded existing derivatives). The starting idea is the following. Let $X^x = (X_t^x)_{t \in [0,T]}$ denote the (homogeneous) diffusion solution to (7.1) starting at $x \in \mathbb{R}$ at time 0. For small t, one has

$$X_t^x = x + \int_0^t b(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s.$$

We wish to analyze the behaviour of the diffusion process X_t^x for small values of t in order to detect the term of order at most 1 *i.e.* that goes to 0 like t when $t \to 0$ (with respect to the $L^2(\mathbb{P})$-norm). Let us inspect the two integral terms successively. First,

$$\int_0^t b(X_s^x) ds = b(x)t + o(t) \qquad \text{as } t \to 0$$

since $X_t^x \to x$ as $t \to 0$ and b is continuous. Furthermore, as b is Lipschitz continuous and $\mathbb{E} \sup_{s \in [0,t]} |X_s^x - x|^2 \to 0$ as $t \to 0$ (see *e.g.* Proposition 7.6 further on), this expansion holds for the $L^2(\mathbb{P})$-norm as well (*i.e.* $o(t) = o_{L^2}(t)$).

As concerns the stochastic integral term, we have in mind that $\mathbb{E}(W_{t+\Delta t} - W_t)^2 = \Delta t$ so that one may consider heuristically that in a scheme *a Brownian increment* $dW_t := W_{t+dt} - W_t$ *between t and $t + dt$ behaves like* $\sqrt{dt}$. Then by Itô's Lemma

$$\sigma(X_s^x) = \sigma(x) + \int_0^s \left(\sigma'(X_u^x) b(X_u^x) + \frac{1}{2}\sigma''(X_u^x)\sigma^2(X_u^x) \right) du + \int_0^s \sigma'(X_u^x)\sigma(X_u^x) dW_u$$

so that

$$
\begin{aligned}
\int_0^t \sigma(X_s^x)dW_s &= \sigma(x)W_t + \int_0^t\int_0^s \sigma(X_u^x)\sigma'(X_u^x)dW_udW_s + O_{L^2}(t^{3/2}) && (7.18) \\
&= \sigma(x)W_t + \sigma\sigma'(x)\int_0^t W_sdW_s + o_{L^2}(t) + O_{L^2}(t^{3/2}) && (7.19) \\
&= \sigma(x)W_t + \frac{1}{2}\sigma\sigma'(x)(W_t^2 - t) + o_{L^2}(t).
\end{aligned}
$$

The $O_{L^2}(t^{3/2})$ in (7.18) comes from the fact that $u \mapsto \sigma'(X_u^x)b(X_u^x) + \frac{1}{2}\sigma''(X_u^x)\sigma^2(X_u^x)$ is $L^2(\mathbb{P})$-bounded in t in the neighbourhood of 0 (note that b and σ have at most linear growth and use Proposition 7.2). Consequently, using the fundamental isometry property of stochastic integration, Fubini-Tonnelli Theorem and Proposition 7.2,

$$
\mathbb{E}\left(\int_0^t\int_0^s \left(\sigma'(X_u^x)b(X_u^x) + \frac{1}{2}\sigma''(X_u^x)\sigma^2(X_u^x)\right)dudW_s\right)^2 = \mathbb{E}\int_0^t\left(\int_0^s \left(\sigma'(X_u^x)b(X_u^x) + \frac{1}{2}\sigma''(X_u^x)\sigma^2(X_u^x)\right)du\right)^2 ds
$$

$$
\leq C(1+x^4)\int_0^t\left(\int_0^s du\right)^2 ds = C(1+x^4)t^3.
$$

The $o_{L^2}(t)$ in Equation (7.19) also follows from the fundamental isometry property of stochastic integral (twice) and Fubini-Tonnelli Theorem which yields

$$
\mathbb{E}\left(\int_0^t\int_0^s \left(\sigma\sigma'(X_u^x) - \sigma\sigma'(x)\right)dW_udW_s\right)^2 = \int_0^t\int_0^s \varepsilon(u)duds
$$

where $\varepsilon(u) = \mathbb{E}(\sigma\sigma'(X_u^x) - \sigma\sigma'(x))^2$ goes to 0 as $u \to 0$ by the Lebesgue dominated convergence Theorem. Finally note that

$$
\|W_t^2 - t\|_2^2 = \mathbb{E}\,W_t^4 - 2t\,\mathbb{E}\,W_t^2 + t^2 = 3t^2 - 2t^2 + t^2 = 2t^2
$$

so that this term is of order one.

Now we consider the *SDE*
$$
dX_t = b(X_t)dt + \sigma(X_t)dW_t,
$$

starting at a random vector X_0, independent of the standard Brownian motion W. Using the Markov property of the diffusion, one can reproduce the above reasoning on each time step $[t_k^n, t_{k+1}^n)$ *given the value of the scheme at time t_k^n*. This leads to the following *simulatable* scheme:

$$
\widetilde{X}_0^{mil} = X_0,
$$
$$
\widetilde{X}_{t_{k+1}^n}^{mil} = \widetilde{X}_{t_k^n}^{mil} + \left(b(\widetilde{X}_{t_k^n}^{mil}) - \frac{1}{2}\sigma\sigma'(\widetilde{X}_{t_k^n}^{mil})\right)\frac{T}{n} + \sigma(\widetilde{X}_{t_k^n}^{mil})\sqrt{\frac{T}{n}}\,U_{k+1} + \frac{1}{2}\sigma\sigma'(\widetilde{X}_{t_k^n}^{mil})\frac{T}{n}U_{k+1}^2, \quad (7.20)
$$

where $U_k = \sqrt{\frac{n}{T}}(W_{t_k^n} - W_{t_{k-1}^n})$, $k = 1, \dots, n$.

By interpolating the above scheme between the discretization times we define the *continuous Milstein scheme* defined (with our standard notations) by

$$\widetilde{X}_0^{mil} = X_0,$$
$$\widetilde{X}_t^{mil} = \widetilde{X}_{\underline{t}}^{mil} + (b(\widetilde{X}_{\underline{t}}^{mil}) - \frac{1}{2}\sigma\sigma'(\widetilde{X}_{\underline{t}}^{mil}))(t - \underline{t}) + \sigma(\widetilde{X}_{\underline{t}}^{mil})(W_t - W_{\underline{t}}) + \frac{1}{2}\sigma\sigma'(\widetilde{X}_{\underline{t}}^{mil})(W_t - W_{\underline{t}})^2. \quad (7.21)$$

The following theorem gives the rate of strong pathwise convergence of the Milstein scheme.

Theorem 7.5 (Strong rate for the Milstein scheme) *(See e.g. [83]) (a) Assume that b and σ are $\mathcal{C}^1$ on $\mathbb{R}$ with bounded, $\alpha_{b'}$ and $\alpha_{\sigma'}$-Hölder continuous first derivatives respectively, $\alpha_{b'}$, $\alpha_{\sigma'} \in (0, 1]$. Then, for every $p \in (0, \infty)$, there exists a real constant $C_{b,\sigma,T,p} > 0$ such that, for every $X_0 \in L^p(\mathbb{P})$, independent of the Brownian motion W, one has*

$$\left\| \max_{0 \leq k \leq n} |X_{t_k^n} - \widetilde{X}_{t_k^n}^{mil,n}| \right\|_p \leq \left\| \sup_{t \in [0,T]} |X_t - \widetilde{X}_t^{mil,n}| \right\|_p \leq C_{b,\sigma,T,p} \left(\frac{T}{n}\right)^{\frac{1+\alpha}{2}} (1 + \|X_0\|_p)$$

where $\alpha = \min(\alpha_{b'}, \alpha_{\sigma'})$. In particular if b' and σ' are Lipschitz continuous continuous

$$\left\| \max_{0 \leq k \leq n} |X_{t_k^n} - \widetilde{X}_{t_k^n}^{mil,n}| \right\|_p \leq \left\| \sup_{t \in [0,T]} |X_t - \widetilde{X}_t^{mil,n}| \right\|_p \leq C_{p,b,\sigma,T} \frac{T}{n}(1 + \|X_0\|_p).$$

(b) As concerns the stepwise constant Milstein scheme, one has

$$\left\| \sup_{t \in [0,T]} |X_t - \widetilde{X}_{\underline{t}}^{mil,n}| \right\|_p \leq C_{b,\sigma,T}(1 + \|X_0\|_p)\sqrt{\frac{T}{n}(1 + \log n)}.$$

Remarks. • As soon as the derivatives of b' and σ' are Hölder, the (continuous) Milstein scheme converges faster than the Euler scheme.

• The $O(1/n)$-rate obtained here should be compared to the *weak rate* investigated in Section 7.6 which is also $O(1/n)$. Comparing performances of both approaches for the computation of $\mathbb{E} f(X_T)$ should rely on numerical evidences and (may) depend on the specified diffusion or function.

• The second claim of the theorem shows that the stepwise constant Milstein scheme does not converge faster than the stepwise constant Euler scheme. To get convinced of this rate without rigorous proof, one has just to think of the Brownian motion itself: in that case $\sigma' \equiv 0$ so that the stepswise constant Milstein and Euler schemes coincide and subsequently converge at the same rate! As a consequence, since it is the only simulatable version of the Milstein scheme, its use for the approximate computation (by Monte Carlo simulation) of the expectation of *functionals* $\mathbb{E} F((X_t)_{t\in[0,T]})$ of the diffusion should not provide better results than implementing the standard stepwise Euler scheme as briefly described in Section 7.4.

By contrast, it happens that functionals of the continuous Euler scheme can be simulated: this is the purpose of Chapter 8 devoted to diffusion bridges.

A detailed proof in the case $X_0 = x \in \mathbb{R}$ and $p \in [2, \infty)$ is provided in Section 7.8.8.

▷ **Exercise.** Derive from these L^p-rates an *a.s.* rate of convergence for the Milstein scheme.

▷ **Exercise.** We consider a diffusion process $(X^x)_{t\in[0,T]}$ with drift $b : \mathbb{R} \to \mathbb{R}$ and diffusion coefficient $\sigma = \mathbb{R} \to \mathbb{R}$, both assumed continuously differentiable with bounded derivatives. Furthermore, we suppose that σ and σ' are never 0, b and σ^2 are non-decreasing.

(a) Show that, if

$$\forall\,\xi \in \mathbb{R}, \quad b(\xi) - \frac{1}{4}(\sigma^2)'(\xi) \geq 0 \quad \text{and} \quad \frac{\sigma}{\sigma'}(\xi) \leq 2\xi$$

then the Milstein scheme with step $\frac{T}{n}$ starting at $x \geq 0$ satisfies (with the usual notations)

$$\forall\,k \in \{0,\ldots,n\}, \quad \widetilde{X}^{mil}_{t^n_k} \geq 0.$$

(b) Show that, under the assumption of (a), the continuous Milstein scheme also satisfies $\widetilde{X}^{mil}_t \geq 0$ for every $t \in [0,T]$.

(c) Let $Y_t = e^{\kappa t}X_t$, $t \in [0,T]$. Show that y is solution to a stochastic differential equation

$$dY_t = \widetilde{b}(t,Y_t) + \widetilde{\sigma}(t,Y_t)dW_t$$

where $\widetilde{b},\ \widetilde{\sigma} : [0,T] \times \mathbb{R} \to \mathbb{R}$ are functions depending on b, σ and κ. Deduce by mimicking the proof of (a) that the Milstein scheme of Y is non-negative as soon as $\dfrac{b}{(\sigma^2)'} \geq \eta > 0$ on the real line.

7.5.2   Higher dimensional Milstein scheme

In higher dimension when the underlying diffusion process $(X_t)_{t\in[0,T]}$ is d-dimensional or the driving Brownian motion W is q-dimensional which means that the drift is a function $b : \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient $\sigma = [\sigma_{ij}] : \mathbb{R}^d \to \mathcal{M}(d \times q, \mathbb{R})$, the same reasoning as in the 1-dimensional setting leads to the following (discrete time) scheme

$$\widetilde{X}^{mil}_0 = X_0,$$

$$\widetilde{X}^{mil}_{t^n_{k+1}} = \widetilde{X}^{mil}_{t^n_k} + \frac{T}{n}b(\widetilde{X}^{mil}_{t^n_k}) + \sigma(\widetilde{X}^{mil}_{t^n_k})\Delta W_{t^n_{k+1}} + \sum_{1\leq i,j\leq q}\partial\sigma_{.i}\sigma_{.j}(\widetilde{X}^{mil}_{t^n_k}) \times \int_{t^n_k}^{t^n_{k+1}}(W^i_s - W^i_{t^n_k})dW^j_s, \quad (7.22)$$

$$k = 0,\ldots,n-1,$$

where $\Delta W_{t^n_{k+1}} := W_{t^n_{k+1}} - W_{t^n_k}$, $\sigma_{.k}(x)$ denotes the kth column of the matrix σ and

$$\forall\,x = (x_1,\ldots,x_d) \in \mathbb{R}^d, \qquad \partial\sigma_{.i}\,\sigma_{.j}(x) := \sum_{\ell=1}^{d}\frac{\partial\sigma_{.i}}{\partial x_\ell}(x)\sigma_{\ell j}(x) \in \mathbb{R}^d.$$

Having a look at this formula when $d = 1$ and $q = 2$ shows that simulating the Milstein scheme at times t^n_k in a general setting amounts to being able to simulate the joint distribution of the triplet

$$\left(W^1_t, W^2_t, \int_0^t W^1_s dW^2_s\right) \quad \text{at time } t = t^n_1$$

that is the *joint* distribution of two independent Brownian motions and their Lévy area. Then, the simulation of

$$\left(W^1_{t^n_k} - W^1_{t^n_{k-1}}, W^2_{t^n_k} - W^2_{t^n_{k-1}}, \int_{t^n_{k-1}}^{t^n_k}(W^1_s - W^1_{t^n_k})dW^2_s\right), \quad k = 1,\ldots,n.$$

More generally, one has to simulate the joint (q^2-dimensional) distribution of

$$\left(W_t^1, \ldots, W_t^d, \int_0^t W_s^i dW_s^j, 1 \le i,j \le q, \, i \ne j\right)$$

No convincing (*i.e.* efficient) method to achieve that has been proposed so far in the literature at least when $q \ge 3$ or 4.

Proposition 7.4 (*a*) *If the "rectangular" terms commute i.e. if*

$$\forall i \ne j, \qquad \partial \sigma_{.i}\, \sigma_{.j} = \partial \sigma_{.j}\, \sigma_{.i},$$

then the Milstein scheme reduces to

$$
\begin{aligned}
\widetilde{X}_{t_{k+1}^n}^{mil} &= \widetilde{X}_{t_k^n}^{mil} + \frac{T}{n}\left(b(\widetilde{X}_{t_k^n}^{mil}) - \frac{1}{2}\sum_{i=1}^{q}\partial\sigma_{.i}\sigma_{.i}(\widetilde{X}_{t_k^n}^{mil})\right) + \sigma(\widetilde{X}_{t_k^n}^{mil})\Delta W_{t_{k+1}^n} \\
&\quad + \frac{1}{2}\sum_{1 \le i,j \le q}\partial\sigma_{.i}\sigma_{.j}(\widetilde{X}_{t_k^n}^{mil})\Delta W_{t_{k+1}^n}^i \Delta W_{t_{k+1}^n}^j, \quad \widetilde{X}_0^{mil} = X_0.
\end{aligned}
\tag{7.23}
$$

(*b*) *When $q = 1$ the commutation property is trivially satisfied.*

Proof. As a matter of fact if $i \ne j$, an integration by parts shows that

$$\int_{t_k^n}^{t_{k+1}^n}(W_s^i - W_{t_k^n}^i)dW_s^j + \int_{t_k^n}^{t_{k+1}^n}(W_s^j - W_{t_k^n}^j)dW_s^i = \Delta W_{t_{k+1}^n}^i \Delta W_{t_{k+1}^n}^j$$

and

$$\int_{t_k^n}^{t_{k+1}^n}(W_s^i - W_{t_k^n}^i)dW_s^i = \frac{1}{2}\left((\Delta W_{t_{k+1}^n}^i)^2 - \frac{T}{n}\right).$$

The announced form for the scheme follows. $\diamond$

In this very special situation, the scheme (7.23) can still be simulated easily in dimension d since it only involves some Brownian increments.

The rate of convergence of the Milstein scheme is formally the same in higher dimension as it is in 1-dimension: Theorem 7.5 remains true with a d-dimensional diffusion driven by a q-dimensional Brownian motion provided $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathcal{M}(d,q,\mathbb{R})$ are $\mathcal{C}^2$ with bounded existing partial derivatives.

Theorem 7.6 (*See e.g. [83]*) *Assume that b and σ are $\mathcal{C}^2$ on $\mathbb{R}^d$ with bounded existing partial derivatives. Then, for every $p \in (0,\infty)$ there exists a real constant $C_{p,b,\sigma,T} > 0$ such that for every $X_0 \in L^p(\mathbb{P})$, independent of the q-dimensional Brownian motion W, the error bound established in the former Theorem 7.5(a) in the Lipschitz continuous continuous case remains valid.*

However, one must keep in mind that this result has nothing to do with the ability to simulate this scheme.

In a way, one important consequence of the above theorem about the strong rate of convergence of the Milstein scheme concerns the Euler scheme.

Corollary 7.2 *If the drift b is $\mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ with bounded existing partial derivatives and if $\sigma(x) = \Sigma$ is constant, then the Euler scheme and the Milstein scheme coincide at the discretization times t_k^n. As a consequence, the strong rate of convergence of the Euler scheme is, in that very specific case, $O(\frac{1}{n})$. Namely, for every $p \in (0, \infty)$, one has*

$$\left\| \max_{0 \leq k \leq n} |X_{t_k^n} - \bar{X}_{t_k^n}^n| \right\|_p \leq C_{b,\sigma,T} \frac{T}{n} (1 + \|X_0\|_p).$$

7.6   Weak error for the Euler scheme (I)

In many situations, like the pricing of so-called "vanilla" European options, a discretization scheme $\bar{X}^n = (\bar{X}_t^n)_{t \in [0,T]}$ of a d-dimensional diffusion process $X = (X_t)_{t \in [0,T]}$ introduce in order to compute by a Monte Carlo simulation an approximation $\mathbb{E}\, f(\bar{X}_T)$ of $\mathbb{E}\, f(X_T)$, *i.e.* only at a fixed (terminal) time. If one relies on the former *strong* (or *pathwise*) rates of convergence, we get, as soon as $f : \mathbb{R}^d \to \mathbb{R}$ is *Lipschitz continuous continuous* and b, σ satisfy (H_T^β) for $\beta \geq \frac{1}{2}$,

$$\left| \mathbb{E}\, f(X_T) - \mathbb{E}\, f(\bar{X}_T^n) \right| \leq [f]_{\text{Lip}} \mathbb{E} \left| X_T - \bar{X}_T^n \right| \leq [f]_{\text{Lip}} \mathbb{E} \sup_{t \in [0,T]} \left| X_t - \bar{X}_t^n \right| = O\left(\frac{1}{\sqrt{n}} \right).$$

In fact the first inequality in this chain turns out to be highly non optimal since it switches form a *weak* error (the difference only depending on the respective(marginal) distributions of X_T and $\bar{X}_T^n$) to a pathwise approximation $X_T - \bar{X}_T^n$. To improve asymptotically the other two inequalities is hopeless since it has been shown (see the remark and comments in Section 7.8.6 for a brief introduction) that under appropriate assumptions $X_T - \bar{X}_T^n$ satisfy a central limit theorem at rate $\sqrt{n}$ with non-zero asymptotic variance. In fact one even has a functional form of this central limit theorem for the whole process $(X_t - \bar{X}_t^n)_{t \in [0,T]}$ (see [88, 74]). As a consequence a rate faster than $n^{-\frac{1}{2}}$ is possible in a L^1 sense would not be consistent with this central limit behaviour.

Furthermore, numerical experiments confirm that the *weak rate of convergence* between the above two expectations is usually much faster then $n^{-\frac{1}{2}}$. This fact has been known for long and has been extensively investigated in the literature, starting with the two seminal papers [151] by Talay-Tubaro and [9] by Bally-Talay), leading to an expansion of the time discretization error at an arbitrary accuracy when b and σ are smooth enough as functions. Two main settings have drawn attention: the case where f is a smooth functions and the case where the diffusion itself is "regularizing" *i.e.* propagate the regularizing effect of the driving Brownian motion (^{1}) thanks to a non-degeneracy assumption on the diffusion coefficient σ typically like uniform ellipticity for σ (see below) or weaker assumption of Hörmander type.

The same kind of question has been investigated for specific classes of functionals F of the whole path of the diffusion X with some applications to path-dependent option pricing. These (although partial) results show that the resulting *weak rate* is the same as the strong rate that can be obtained with the Milstein scheme for this type of functionals (when Lipschitz continuous with respect to the sup-norm over $[0, T]$).

---

[1] The regularizing property of the Brownian motion should be understood as follows: if f is a Borel bounded function on $\mathbb{R}^d$, then $f_\sigma(x) := \mathbb{E}(f(x + \sigma W))$ is a $\mathcal{C}^\infty$ function for every $\sigma > 0$ and converges towards f as $\sigma \to 0$ in every L^p space, $p > 0$. This result is but a classical convolution result with a Gaussian kernel rewritten in a probabilistic form.

As a second step, we will show how the so-called Richardson-Romberg extrapolation methods provides a systematic procedure to take optimally advantage of these weak rates, including their higher order form.

7.6.1 Main results for $\mathbb{E} f(X_T)$: the Talay-Tubaro and Bally-Talay Theorems

We adopt the notations of the former section 7.1, except that we consider, mostly for convenience an homogeneous SDE starting at $x \in \mathbb{R}^d$,

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0^x = x.$$

The notations $(X_t^x)_{t \in [0,T]}$ and $(\bar{X}_t)_{t \in [0,T]}$ denote the diffusion and the Euler scheme of the diffusion starting at x at time 0 (an exponent n will sometimes be added to emphasize that the step of the scheme is $\frac{T}{n}$).

Our first result is the first result on the weak error, the one which can be obtained with the less stringent assumptions on b and σ.

Theorem 7.7 *(see [151]) Assume b and σ are 4 times continuously differentiable on $\mathbb{R}^d$ with bounded existing partial derivatives (this implies that b and σ are Lipschitz). Assume $f : \mathbb{R}^d \to \mathbb{R}$ is 4 times differentiable with polynomial growth as well as its existing partial derivatives. Then, for every $x \in \mathbb{R}^d$,*

$$\mathbb{E} f(X_T^x) - \mathbb{E} f(\bar{X}_T^{n,x}) = O\left(\frac{1}{n}\right) \qquad \text{as } n \to +\infty. \tag{7.24}$$

Sketch of proof. Assume $d = 1$ for notational convenience. We also assume furthermore $b \equiv 0$, σ bounded and f has bounded existing derivatives, for simplicity. The diffusion $(X_t^x)_{t \geq 0, x \in \mathbb{R}}$ is a homogeneous Markov process with transition semi-group $(P_t)_{t \geq 0}$ given (see *e.g.* [140, 78]) by

$$P_t\, g(x) := \mathbb{E}\, g(X_t^x).$$

On the other hand, the Euler scheme with step $\frac{T}{n}$ starting at $x \in \mathbb{R}$, denoted $(\bar{X}_{t_k^n}^x)_{0 \leq k \leq n}$, is a discrete time homogeneous Markov (indexed by k) chain with transition

$$\bar{P}\, g(x) = \mathbb{E}\, g\left(x + \sigma(x)\sqrt{\frac{T}{n}}\, Z\right), \qquad Z \stackrel{d}{=} \mathcal{N}(0;1).$$

To be more precise this means for the diffusion process that, for any Borel bounded or non-negative test function g

$$\forall\, s,\, t \geq 0, \quad P_t g(x) = \mathbb{E}(g(X_{s+t}\,|\,X_s = x) = \mathbb{E}\, g(X_t^x)$$

and for its Euler scheme (still with $t_k^n = \frac{kT}{n}$)

$$\bar{P}\, g(x) = \mathbb{E}(g(\bar{X}_{t_{k+1}^n}^x)\,|\,\bar{X}_{t_k^n}^x = x) = \mathbb{E}\, g(\bar{X}_{t_1^n}^x), \ k = 0, \ldots, n-1.$$

Now

$$\mathbb{E} f(X_T^x) = P_T f(x) = P_{\frac{T}{n}}^n(f)(x)$$

and

$$\mathbb{E} f(\bar{X}_T^x) = \bar{P}^n(f)(x)$$

so that,

$$
\begin{aligned}
\mathbb{E}\, f(X_T^x) - \mathbb{E}\, f(\bar{X}_T^x) &= \sum_{k=1}^{n} P_{\frac{T}{n}}^k(\bar{P}^{n-k} f)(x) - P_{\frac{T}{n}}^{k-1}(\bar{P}^{n-(k-1)} f)(x) \\
&= \sum_{k=1}^{n} P_{\frac{T}{n}}^{k-1}((P_{\frac{T}{n}} - \bar{P})(\bar{P}^{n-k} f))(x).
\end{aligned}
\tag{7.25}
$$

This "domino" sum suggests that we have two tasks to complete:

– the first one is to estimate precisely the asymptotic behaviour of

$$
P_{\frac{T}{n}} g(x) - \bar{P} g(x),
$$

with respect to the step $\frac{T}{n}$ and the (four) first derivatives of the function g.

– the second one is to control the (four first) derivatives of the functions $\bar{P}^\ell g$ for the sup norm when g is regular, uniformly in $\ell \in \{1, \dots, n\}$ and $n \geq 1$, in order to propagate the above local error bound.

Let us deal with the first task. To be precise, let $g : \mathbb{R} \to \mathbb{R}$ be a four times differentiable function g with bounded existing derivatives. First, Itô's formula yields

$$
P_t g(x) := \mathbb{E}\, g(X_T^x) = g(x) + \underbrace{\mathbb{E} \int_0^t (g'\sigma)(X_s^x) dW_s}_{=0} + \frac{1}{2}\mathbb{E} \int_0^t g''(X_s^x)\sigma^2(X_s^x) ds
$$

where we use that $g'\sigma$ is bounded to ensure that the stochastic integral is a true martingale.

A Taylor expansion of $g(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z)$ at x yields for the transition of the Euler scheme (after taking expectation)

$$
\begin{aligned}
\bar{P} g(x) &= \mathbb{E}\, g(\bar{X}_T^x) \\
&= g(x) + g'(x)\sigma(x)\mathbb{E}\left(\sqrt{\frac{T}{n}}\,Z\right) + \frac{1}{2}(g''\sigma^2)(x)\mathbb{E}\left(\sqrt{\frac{T}{n}}\,Z\right)^2 \\
&\quad + g^{(3)}(x)\frac{\sigma^3(x)}{3!}\mathbb{E}\left(\sqrt{\frac{T}{n}}\,Z\right)^3 + \frac{\sigma^4(x)}{4!}\mathbb{E}\left(g^{(4)}(\xi)\left(\sqrt{\frac{T}{n}}\,Z\right)^4\right) \\
&= g(x) + \frac{T}{2n}(g''\sigma^2)(x) + \frac{\sigma^4(x)}{4!}\mathbb{E}\left(g^{(4)}(\xi)\left(\sqrt{\frac{T}{n}}\,Z\right)^4\right) \\
&= g(x) + \frac{T}{2n}(g''\sigma^2)(x) + \frac{\sigma^4(x)T^2}{4!n^2} c_n(g),
\end{aligned}
$$

where $|c_n(g)| \leq 3\|g^{(4)}\|_\infty$. This follows from the well-known facts that $\mathbb{E} Z = \mathbb{E} Z^3 = 0$ and $\mathbb{E}\, Z^2 = 1$ and $\mathbb{E}\, Z^4 = 3$. Consequently

$$
P_{\frac{T}{n}} g(x) - \bar{P} g(x) = \frac{1}{2}\int_0^{\frac{T}{n}} \mathbb{E}((g''\sigma^2)(X_s^x) - (g''\sigma^2)(x))ds + \frac{\sigma^4(x)T^2}{4!n^2} c_n(g).
\tag{7.26}
$$

Applying again Itô's formula to the $\mathcal{C}^2$ function $\gamma := g''\sigma^2$, yields

$$\mathbb{E}((g''\sigma^2)(X_s^x) - (g''\sigma^2)(x)) = \frac{1}{2}\mathbb{E}\left(\int_0^s \gamma''(X_u^x)\sigma^2(X_u^x)du\right).$$

so that

$$\forall\, s \geq 0, \qquad \sup_{x \in \mathbb{R}} \left|\mathbb{E}((g''\sigma^2)(X_s^x) - (g''\sigma^2)(x))\right| \leq \frac{s}{2}\|\gamma''\sigma^2\|_{\sup}.$$

Elementary computations show that

$$\|\gamma''\|_\infty \leq C_\sigma \max_{k=2,3,4} \|g^{(k)}\|_\infty$$

where C_σ depends on $\|\sigma^{(k)}\|_\infty$, $k = 0, 1, 2$ but not on g (with the standard convention $\sigma^{(0)} = \sigma$).

Consequently, we derive from (7.26) that

$$|P_{\frac{T}{n}}(g)(x) - \bar{P}(g)(x)| \;\leq\; C'_{\sigma,T} \max_{k=2,3,4} \|g^{(k)}\|_\infty \left(\frac{T}{n}\right)^2.$$

The fact that the first derivative g' is not involved in these bounds is simply the consequence of our artificial assumption that $b \equiv 0$.

Now we pass to the second task. In order to plug this estimate in (7.25), we need now to control the first four derivatives of $\bar{P}^\ell f$, $\ell = 1, \ldots, n$, uniformly with respect to k and n. In fact we do not directly need to control the first derivative since $b \equiv 0$ but we will do it as a first example to illustrate the method in a simpler case.

Let us consider again the generic function g and its four bounded derivatives.

$$(\bar{P}g)'(x) = \mathbb{E}\left(g'\left(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z\right)\left(1 + \sigma'(x)\sqrt{\frac{T}{n}}\,Z\right)\right).$$

so that

$$
\begin{aligned}
|(\bar{P}g)'(x)| \;&\leq\; \|g'\|_\infty \left\|1 + \sigma'(x)\sqrt{\frac{T}{n}}\,Z\right\|_1 \\
&\leq\; \|g'\|_\infty \left\|1 + \sigma'(x)\sqrt{\frac{T}{n}}\,Z\right\|_2 \\
&=\; \|g'\|_\infty \sqrt{\mathbb{E}\left(1 + \sigma'(x)\sqrt{\frac{T}{n}}\,Z\right)^2} \\
&=\; \|g'\|_\infty \sqrt{\mathbb{E}\left(1 + 2\sigma'(x)\sqrt{\frac{T}{n}}\,Z + \sigma'(x)^2\frac{T}{n}Z^2\right)} \\
&=\; \|g'\|_\infty \sqrt{1 + (\sigma')^2(x)\frac{T}{n}} \\
&\leq\; \|g'\|_\infty \left(1 + \sigma'(x)^2\frac{T}{2n}\right)
\end{aligned}
$$

since $\sqrt{1+u} \leq 1 + \frac{u}{2}$, $u \geq 0$. Hence, we derive by induction that, for every $n \geq 1$ and every $\ell \in \{1, \ldots, n\}$,

$$\forall x \in \mathbb{R}, \quad |(\bar{P}^\ell f)'(x)| \leq \|f'\|_\infty (1 + (\sigma')^2(x)T/(2n))^\ell \leq \|f'\|_\infty e^{\sigma'(x)^2 T/2}.$$

(where we used that $(1+u)^\ell \leq e^{\ell u}$, $u \geq 0$). This yields

$$\|(\bar{P}^\ell f)'\|_{\sup} \leq \|f'\|_\infty e^{\|\sigma'\|_\infty^2 T/2}.$$

Let us deal now with the second derivative,

$$
\begin{aligned}
(\bar{P}g)''(x) &= \frac{d}{dx}\mathbb{E}\left(g'\left(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z\right)\left(1 + \sigma'(x)\sqrt{\frac{T}{n}}\,Z\right)\right) \\
&= \mathbb{E}\left(g''\left(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z\right)\left(1 + \sigma'(x)\sqrt{\frac{T}{n}}\,Z\right)^2\right) + \mathbb{E}\left(g'\left(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z\right)\sigma''(x)\sqrt{\frac{T}{n}}\,Z\right).
\end{aligned}
$$

Then

$$\left|\mathbb{E}\left(g''\left(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z\right)\left(1 + \sigma'(x)\sqrt{\frac{T}{n}}\,Z\right)^2\right)\right| \leq \|g''\|_\infty \left(1 + \sigma'(x)^2 \frac{T}{n}\right)$$

and, using that $g'\left(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z\right) = g'(x) + g''(\zeta)\sigma(x)\sqrt{\frac{T}{n}}\,Z$ owing to the fundamental formula of Calculus, we get

$$\left|\mathbb{E}\left(g'\left(x + \sigma(x)\sqrt{\frac{T}{n}}\,Z\right)\sigma''(x)\sqrt{\frac{T}{n}}\,Z\right)\right| \leq \|g''\|_\infty \|\sigma\sigma''\|_\infty \mathbb{E}(Z^2)\frac{T}{n}$$

so that

$$\forall x \in \mathbb{R}, \quad |(\bar{P}g)''(x)| \leq \|g''\|_\infty \left(1 + (2\|\sigma\sigma''\|_\infty + \|(\sigma')^2\|_\infty)\frac{T}{n}\right)$$

which implies the boundedness of $|(\bar{P}^\ell f)''(x)|$, $\ell = 0, \ldots, n-1$, $n \geq 1$.

The same reasoning yields the boundedness of all derivatives $(\bar{P}^\ell f)^{(i)}$, $i = 1, 2, 3, 4$, $\ell = 1, \ldots, n$, $n \geq 1$.

Now we can combine our local error bound with the control of the derivatives. Plugging these estimates in each term of (7.25), finally yields

$$|\mathbb{E}\,f(X_T^x) - \mathbb{E}\,f(\bar{X}_T^x)| \leq C'_{\sigma,T} \max_{1 \leq \ell \leq n, i=1,\ldots,4} \|(\bar{P}^\ell f)^{(i)}\|_{\sup} \sum_{k=1}^n \frac{T^2}{n^2} \leq C_{\sigma,T,f} T\frac{T}{n}$$

which completes the proof. $\diamond$

$\triangleright$ **Exercises. 1.** Complete the above proof by inspecting the case of higher order derivatives ($k = 3$, 4).

2. Extend the proof to a (bounded) non zero drift b.

If one assumes more regularity on the coefficients or some uniform ellipticity on the diffusion coefficient σ it is possible to obtain an expansion of the error at any order.

Theorem 7.8 (a) *Talay-Tubaro's Theorem (see [151]). Assume b and σ are infinitely differentiable with bounded partial derivatives. Assume $f : \mathbb{R}^d \to \mathbb{R}$ is infinitely differentiable with partial derivative having polynomial growth. Then, for every integer $R \in \mathbb{N}^*$,*

$$(\mathcal{E}_{R+1}) \equiv \mathbb{E}\, f(X_T^x) - \mathbb{E}\, f(\bar{X}_T^{x,n}) = \sum_{k=1}^{R} \frac{c_k}{n^k} + O(n^{-(R+1)}) \qquad as\ n \to +\infty, \qquad (7.27)$$

where the real valued coefficients c_k depend on f, T, b and σ.

(b) *Bally-Talay's Theorem (see [9]). If b and σ are bounded, infinitely differentiable with bounded partial derivatives and if σ is uniformly elliptic i.e.*

$$\forall\, x \in \mathbb{R}^d, \qquad \sigma\sigma^*(x) \geq \varepsilon_0 I_d \qquad for\ an\ \varepsilon_0 > 0$$

then the conclusion of (a) holds true for any bounded Borel function.

One method of proof for (a) is to rely on the *PDE* method *i.e.* considering the solution of the parabolic equation

$$\left(\frac{\partial}{\partial t} + L \right)(u)(t,x) = 0, \quad u(T, \cdot) = f$$

where

$$(Lg)(x) = g'(x)b(x) + \frac{1}{2}g''(x)\sigma^2(x)$$

denotes the infinitesimal generator of the diffusion. It follows from the Feynmann-Kac formula that (under some appropriate regularity assumptions)

$$u(0, x) = \mathbb{E}\, f(X_T^x).$$

Formally (in one dimension), one assumes that u is regular enough to apply Itô's formula so that

$$f(X_T^x) = u(T, X_T^x) = u(0, x) + \int_0^T \left(\frac{\partial}{\partial t} + L \right)(u)(t, X_t^x)dt + \int_0^T \partial_x u(t, X_t^x)\sigma(X_t^x)dW_t.$$

Taking expectation (and assuming that the local martingale is a true martingale...) and using that u is solution to the above parabolic *PDE* yields the announced result.

Then, one uses a domino strategy based on the Euler scheme as follows

$$\begin{aligned}
\mathbb{E}\, f(X_T^x) - \mathbb{E}\, f(\bar{X}_T^x) &= \mathbb{E}(u(0, x) - u(T, \bar{X}_T^x)) \\
&= -\sum_{k=1}^{n} \mathbb{E}\left(u(t_k^n, \bar{X}_{t_k^n}^x) - u(t_{k-1}^n, \bar{X}_{t_{k-1}^n}) \right).
\end{aligned}$$

The core of the proof consists in applying Itô's formula (to u, b and σ) to show that

$$\mathbb{E}\left(u(t_k^n, \bar{X}_{t_k^n}^x) - u(t_{k-1}^n, \bar{X}_{t_{k-1}^n}) \right) = \frac{\mathbb{E}\,\phi(t_k^n, X_{t_k^n}^x)}{n^2} + o(n^{-2}).$$

for some continuous function ϕ. Then, one derives (after new computations) that

$$\mathbb{E}\, f(X_T^x) - \mathbb{E}\, f(\bar{X}_T^x) = \frac{\mathbb{E}\int_0^T \phi(t, X_t^x)dt}{n} + o(n^{-1}).$$

This approach will be developed in Section 7.8.9

Remark. The last important information about weak error is that the weak error induced by the Milstein scheme has exactly the same order as that of the Euler scheme *i.e.* $O(1/n)$. So the Milstein scheme seems of little interest as long as one wishes to compute $\mathbb{E}\, f(X_T)$ with a reasonable framework like the ones described in the theorem, since, even when it can be implemented without restriction, its complexity is higher than that of the standard Euler scheme. Furthermore, the next paragraph about Richardson-Romberg extrapolation will show that it is possible to take advantage of the higher order time discretization error expansion which become dramatically faster than Milstein scheme.

7.7   Standard and multistep Richardson-Romberg extrapolation

7.7.1   Richardson-Romberg extrapolation with consistent increments

▷ *Bias-variance decomposition of the quadratic error in a Monte Carlo simulation.* Let V be a vector space of continuous functions with linear growth satisfying $(\mathcal{E}_2)$ (the case of non continuous functions is investigated in [122]). Let $f \in V$. For notational convenience, in view of what follows, we set $W^{(1)} = W$ and $X^{(1)} := X$. A regular Monte Carlo simulation based on M independent copies $(\bar{X}_T^{(1)})^m$, $m = 1, \ldots, M$, of the Euler scheme $\bar{X}_T^{(1)}$ with step T/n induces the following global (squared) quadratic error

$$\left\| \mathbb{E}(f(X_T)) - \frac{1}{M}\sum_{m=1}^{M} f((\bar{X}_T^{(1)})^m) \right\|_2^2 = \left(\mathbb{E}\, f(X_T) - \mathbb{E}\, f(\bar{X}_T^{(1)}) \right)^2 + \left\| \mathbb{E}(f(\bar{X}_T^{(1)})) - \frac{1}{M}\sum_{m=1}^{M} f((\bar{X}_T^{(1)})^m) \right\|_2^2$$

$$= \left(\frac{c_1}{n} \right)^2 + \frac{\mathrm{Var}(f(\bar{X}_T^{(1)}))}{M} + O(n^{-3}). \qquad (7.28)$$

The above formula is but the bias-variance decomposition of the approximation error of the Monte Carlo estimator. This quadratic error bound (7.28) does not take full advantage of the above expansion $(\mathcal{E}_2)$.

▷ *Richardson-Romberg extrapolation.* To take advantage of the expansion, we will perform a Richardson-Romberg extrapolation. In that framework (originally introduced in [151]), one considers the strong solution $X^{(2)}$ of a "copy" of Equation (7.1), driven by a second Brownian motion $W^{(2)}$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. One may always choose such a Brownian motion by enlarging Ω if necessary.

Then we consider the Euler scheme with *a twice smaller* step $\frac{T}{2n}$, denoted $\bar{X}^{(2)}$, of the diffusion process $X^{(2)}$.

We assume from now on that $(\mathcal{E}_3)$ (as defined in (7.27)) holds for f to get more precise estimates but the principle would work with a function simply satisfying $(\mathcal{E}_2)$. Then combining the two time discretization error expansions related to $\bar{X}^{(1)}$ and $\bar{X}^{(2)}$, we get

$$\mathbb{E}(f(X_T)) = \mathbb{E}(2f(\bar{X}_T^{(2)}) - f(\bar{X}_T^{(1)})) - \frac{1}{2}\frac{c_2}{n^2} + O(n^{-3}).$$

Then, the new global (squared) quadratic error becomes

$$\left\|\mathbb{E}(f(X_T)) - \frac{1}{M}\sum_{m=1}^{M} 2f((\bar{X}_T^{(2)})^m) - f((\bar{X}_T^{(1)})^m)\right\|_2^2 = \left(\frac{c_2}{2n^2}\right)^2 + \frac{\mathrm{Var}(2f(\bar{X}_T^{(2)}) - f(\bar{X}_T^{(1)}))}{M} + O(n^{-5}).$$

(7.29)

The structure of this quadratic error suggests the following natural question:

Is it possible to reduce the (asymptotic) time discretization error *without increasing the Monte Carlo error (at least asymptotically in n...)?*

Or, put differently, to what extent is it possible to control the variance term $\mathrm{Var}(2f(\bar{X}_T^{(2)}) - f(\bar{X}_T^{(1)}))$?

– **Lazy simulation.** If one adopts a somewhat "lazy" approach by using the pseudo-random number generator in a purely sequentially to simulate the two Euler schemes, this corresponds form a theoretical point of view to consider independent Gaussian white noises $(U_k^{(1)})_k$ and $(U_k^{(2)})_k$ to simulate the Brownian increments in bot schemes or equivalently to assume that $W^{(1)}$ and $W^{(2)}$ are two *independent Brownian motions*. Then

$$\mathrm{Var}(2f(\bar{X}_T^{(2)}) - f(\bar{X}_T^{(1)})) = 4\mathrm{Var}(f(\bar{X}_T^{(2)})) + \mathrm{Var}(f(\bar{X}_T^{(1)})) = 5\,\mathrm{Var}(f(\bar{X}_T)) \stackrel{n\to+\infty}{\longrightarrow} 5\,\mathrm{Var}(f(\bar{X}_T)).$$

In this approach the cost of one order on the time discretization error (switch from n^{-1} to n^{-2}) is the increase of the variance by a factor 5 and of the complexity by a factor (approximately) 3.

– **Consistent simulation (of the Brownian increments).** If $W^{(2)} = W^{(1)} (= W)$ then

$$\mathrm{Var}(2f(\bar{X}_T^{(2)}) - f(\bar{X}_T^{(1)})) \stackrel{n\to+\infty}{\longrightarrow} \mathrm{Var}(2f(X_T) - f(X_T)) = \mathrm{Var}(f(X_T))$$

since Euler schemes $\bar{X}^{(i)}$, $i = 1, 2$, converges in $L^p(\mathbb{P})$ to X.

In this second approach, the same gain in terms of time discretization error has no cost in terms of variance (at least for n large enough). Of course the complexity, remains 3 times higher. Finally the pseudo-random number generator is less requested by a factor $2/3$.

In fact, it is shown in [122], that this choice $W^{(2)} = W^{(1)} = W$, leading to *consistent Brownian increments for the two schemes* is asymptotically optimal among all possible choice of Brownian motions $W^{(1)}$ and $W^{(2)}$. This result can be extended to Borel functions f when the diffusion is uniformly elliptic (and b, σ bounded, infinitely differentiable with bounded partial derivatives, see [122]).

▷ *Practitioner's corner.* From a practical viewpoint, one first simulates an Euler scheme with step $\frac{T}{2n}$ using a white Gaussian noise $(U_k^{(2)})_{k\geq 1}$, then one simulates the Gaussian white noise $U^{(1)}$ of the Euler scheme with step $\frac{T}{n}$ by setting

$$U_k^{(1)} = \frac{U_{2k}^{(2)} + U_{2k-1}^{(2)}}{\sqrt{2}}, \ k \geq 1.$$

▷ **Exercise.** Show that if X and $Y \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ have the same distribution. Then, for every $\alpha \in [1, +\infty)$

$$\mathrm{Var}(\alpha X + (1 - \alpha)Y) \geq \mathrm{Var}(X).$$

7.7.2   Toward a multistep Richardson-Romberg extrapolation

In [122], a more general approach to multistep Richardson-Romberg extrapolation with consistent Brownian increments is developed. Given the state of the technology and the needs, it seems interesting up to $R = 4$. We will sketch here the case $R = 3$ which may also works for path dependent options (see below). Assume $\mathcal{E}_4$ holds. Set

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = -4, \qquad \alpha_3 = \frac{9}{2}.$$

Then, easy computations show that

$$\mathbb{E}\left(\alpha_1 f(\bar{X}_T^{(1)}) + \alpha_2 f(\bar{X}_T^{(2)}) + \alpha_3 f(\bar{X}_T^{(3)})\right) = \frac{c_3 \sum_{1 \leq i \leq 3} \alpha_i / i^3}{n^3} + O(n^{-4}).$$

where $\bar{X}^{(r)}$ denotes the Euler scheme with step $\frac{T}{rn}$, $r = 1, 2, 3$, *with respect to the same Brownian motion W*. Once again this choice induces a control of the variance of the estimator. However, note that this choice is theoretically no longer optimal as it is for the regular Rombeg extrapolation. But it is natural and better choices are not easy to specify *a priori* (keeping in mind that they will depend on the function f).

   In practice the three Gaussian white noises can be simulated by using the following consistency rules: We give below the most efficient way to simulate the three white noises $(U_k^{(r)})_{1 \leq k \leq r}$ on one time step T/n. Let Z_1, Z_2, Z_3, Z_4 be four i.i.d. copies of $\mathcal{N}(0; I_q)$. Set

$$U_1^{(3)} = Z_1, \quad U_2^{(3)} = \frac{Z_2 + Z_3}{\sqrt{2}}, \quad U_3^{(3)} = Z_4,$$

$$U_1^{(2)} = \frac{\sqrt{2}\,Z_1 + Z_2}{\sqrt{3}}, \quad U_2^{(2)} = \frac{Z_3 + \sqrt{2}\,Z_4}{\sqrt{3}},$$

$$U_1^{(1)} = \frac{U_1^{(2)} + U_2^{(2)}}{\sqrt{2}}.$$

   A general formula for the weights $\alpha_r^{(R)}$, $r = 1, \ldots, R$, as well as the consistent increments tables are provided in [122].

   Guided by some complexity considerations, one shows that the parameters of this multistep Richardson-Romberg extrapolation should satisfy some constraints. Typically if M denotes the size of the MC simulation and n the discretization parameter, they should be chosen so that

$$M \propto n^{2R}.$$

For details we refer to [122]. The practical limitation of these results about Richardson-Romberg extrapolation is that the control of the variance is only asymptotic (as $n \to +\infty$) whereas the method is usually implemented for small values of n. However it is efficient up to $R = 4$ for Monte Carlo simulation of sizes $M = 10^6$ to $M = 10^8$.

▷ **Exercise:** We consider the simplest option pricing model, the (risk-neutral) Black-Scholes dynamics, but with unusually high volatility. (We are aware that this model used to price Call options does not fulfill the theoretical assumptions made above). To be precise

$$dX_t = X_t \left(r dt + \sigma dW_t \right),$$

with the following values for the parameters

$$X_0 = 100, \ K = 100, \ r = 0.15, \ \sigma = 1.0, \ T = 1.$$

Note that a volatility $\sigma = 100\%$ per year is equivalent to a 4 year maturity with volatility 50% (or 16 years with volatility 25%). The Black-Scholes reference premium is $C_0^{BS} = 42.96$.

- We consider the Euler scheme with step T/n of this equation.

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} \left(1 + r\frac{T}{n} + \sigma\sqrt{\frac{T}{n}} \, U_{k+1} \right), \qquad \bar{X}_0 = X_0,$$

  where $t_k = \frac{kT}{n}$, $k = 0, \dots, n$. We want to price a vanilla Call option *i.e.* to compute

$$C_0 = e^{-rT}\mathbb{E}((X_T - K)_+)$$

  using a Monte Carlo simulation with M sample paths, $M = 10^4$, $M = 10^6$, etc.

- Test now the standard Richardson-Romberg extrapolation ($R = 2$) based on Euler schemes with steps T/n and $T/(2n)$, $n = 2, 4, 6, 8, 10$, respectively with
  - independent Brownian increments,
  - consistent Brownian increments.

  Compute an estimator of the variance of the estimator.

7.8 Further proofs and results

In this section, to alleviate notations we will drop the exponent n in $t_k^n = \frac{kT}{n}$. Furthermore, for the reader's convenience, we will always consider that $d = q = 1$.

7.8.1 Some useful inequalities

On the non-quadratic case, Doob's Inequality is not sufficient to carry out the proof: we need the more general Burkhölder-Davis-Gundy Inequality. Furthermore, to get some real constants having the announced behaviour as a function of T, we will also need to use the generalized Minkowski Inequality established below.

▷ **Generalized Minkowski Inequality:** For any (bi-measurable) non-negative process $X = (X_t)_{t\geq 0}$, and for every $p \in [1, \infty)$,

$$\forall T \in [0, \infty], \qquad \left\| \int_0^T X_t \, dt \right\|_p \leq \int_0^T \|X_t\|_p \, dt. \tag{7.30}$$

Proof. If $p = 1$ the inequality is obvious. Assume now $p \in (1, \infty)$. Let $T \in (0, \infty)$ and let Y be a non-negative random variable defined on the same probability space as $(X_t)_{t \in [0,T]}$. Let $M > 0$. It follows from Fubini's Theorem and Hölder Inequality that

$$
\begin{aligned}
\mathbb{E}\left(\int_0^T (X_s \wedge M) ds \, Y \right) &= \int_0^T \mathbb{E}((X_s \wedge M)Y) ds \\
&\leq \int_0^T \|X_s \wedge M\|_p \|Y\|_q \, ds \qquad \text{where } q = \frac{p}{p-1} \\
&= \|Y\|_q \int_0^T \|X_s \wedge M\|_p ds.
\end{aligned}
$$

The above inequality applied with $Y := \left(\int_0^T X_s \wedge M \, ds \right)^{p-1}$ where M is a positive real number yields

$$
\mathbb{E}\left(\int_0^T X_s \wedge M \, ds \right)^p \leq \left(\mathbb{E}\left(\int_0^T X_s \wedge M ds \right)^p \right)^{1 - \frac{1}{p}} \int_0^{+\infty} \|X_s\|_p ds.
$$

If $\mathbb{E}\left(\int_0^T X_s \wedge M_n \, ds \right)^p = 0$ for any sequence $M_n \uparrow +\infty$, the inequality is obvious since, by Beppo Levi's monotone convergence Theorem, $\int_0^T X_s ds = 0$ $\mathbb{P}$-a.s.. Otherwise, there is a sequence $M_n \uparrow \infty$ such that all these integrals are non zero (and finite since X is bounded by M and T is finite). Consequently, one can divide both sides of the former inequality to obtain

$$
\forall n \geq 1, \qquad \left(\mathbb{E}\left(\int_0^T X_s \wedge M_n ds \right)^p \right)^{\frac{1}{p}} \leq \int_0^{+\infty} \|X_s\|_p ds.
$$

Now letting $M_n \uparrow +\infty$ yields exactly the expected result owing to two successive applications of Beppo Levi's monotone convergence Theorem, the first with respect to the Lebesgue measure ds, the second with respect to $d\mathbb{P}$. When $T = +\infty$, one concludes by Fatou's Lemma by letting T go to infinity in the inequality obtained for finite T. ◇

▷ **Burkhölder-Davis-Gundy Inequality:** For every $p \in (0, \infty)$, there exists two positive real constants c_p^{BDG} and C_p^{BDG} such that for every non-negative *continuous* local martingale $(X_t)_{t \in [0,T]}$ null at 0,

$$
c_p^{BDG} \left\| \sqrt{\langle X \rangle_T} \right\|_p \leq \left\| \sup_{t \in [0,T]} |X_t| \right\|_p \leq C_p^{BDG} \left\| \sqrt{\langle X \rangle_T} \right\|_p.
$$

For a detailed proof based on a stochastic calculus approach, we refer to [140], p.160.

7.8.2 Polynomial moments (II)

Proposition 7.5 (a) *For every $p \in (0, +\infty)$, there exists a positive real constant $\kappa_p' > 0$ (increasing in p), such that, if b, σ satisfy:*

$$
\forall t \in [0,T], \ \forall x \in \mathbb{R}^d, \quad |b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \tag{7.31}
$$

then, every strong solution of Equation (7.1) starting from the finite random vector X_0 (if any), satisfies

$$
\forall p \in (0, \infty), \qquad \left\| \sup_{s \in [0,T]} |X_s| \right\|_p \leq 2 \, e^{\kappa_p' CT} \left(1 + \|X_0\|_p \right).
$$

(b) The same conclusion holds under the same assumptions for the continuous Euler schemes with step $\frac{T}{n}$, $n \geq 1$, as defined by (7.5) with the same constant κ'_p (which does not depend n) i.e.

$$\forall\, p \in (0, \infty),\ \forall\, n \geq 1, \qquad \left\| \sup_{s \in [0,T]} |\bar{X}^n_s| \right\|_p \leq 2\, e^{\kappa'_p CT} \big(1 + \|X_0\|_p\big).$$

Remarks. • Note that this proposition makes no assumption neither on the existence of strong solutions to (7.1) nor on some (strong) uniqueness assumption on a time interval or the whole real line. Furthermore, the inequality is meaningless when $X_0 \notin L^p(\mathbb{P})$.

• The case $p \in (0, 2)$ will be discussed at the end of the proof.

Proof: To alleviate the notations we assume from now on that $d = q = 1$.

(a) STEP 1*(The process: first reduction)*: Assume $p \in [2, \infty)$. First we introduce for every integer $N \geq 1$ the stopping time $\tau_N := \inf\{t \in [0,T] \mid |X_t - X_0| > N\}$ (convention $\inf \emptyset = +\infty$). This is a stopping time since $\{\tau_N < t\} = \bigcup_{r \in [0,t] \cap \mathbb{Q}} \{|X_r - X_0| > N\} \in \mathcal{F}_t$. Moreover $\{\tau_N \leq t\} = \bigcap_{k \geq k_0} \left\{\tau_N < t + \frac{1}{k}\right\}$ for every $k_0 \geq 1$, hence $\{\tau_N \leq t\} = \bigcap_{k \geq 1} \mathcal{F}_{t+\frac{1}{k_0}} = \mathcal{F}_{t+} = \mathcal{F}_t$ since the filtration is càd ([2]). Furthermore,

$$\sup_{t \in [0,T]} |X^{\tau_N}_t| \leq N + |X_0|$$

so that the non-decreasing function f_N defined by $f_N(t) := \left\| \sup_{s \in [0,t]} |X_{s \wedge \tau_N}| \right\|_p$, $t \in [0,T]$, is bounded by $N + \|X_0\|_p$. On the other hand

$$\sup_{s \in [0,t]} |X_{s \wedge \tau_N}| \leq |X_0| + \int_0^{t \wedge \tau_N} |b(s, X_s)| ds + \sup_{s \in [0,t]} \left| \int_0^{s \wedge \tau_N} \sigma(u, X_u) dW_u \right|.$$

It follows from successive applications of both the regular and the generalized Minkowski Inequalities and of the *BDG* Inequality that

$$
\begin{aligned}
f_N(t) &\leq \|X_0\|_p + \int_0^t \|\mathbf{1}_{\{s \leq \tau_N\}} b(s, X_s)\|_p ds + C_p^{BDG} \left\| \sqrt{\int_0^{t \wedge \tau_N} \sigma(s, X_s)^2 ds} \right\|_p \\
&\leq \|X_0\|_p + \int_0^t \|b(s \wedge \tau_N, X_{s \wedge \tau_N})\|_p ds + C_p^{BDG} \left\| \sqrt{\int_0^t \sigma(s \wedge \tau_N, X_{s \wedge \tau_N})^2 ds} \right\|_p \\
&\leq \|X_0\|_p + C \int_0^t (1 + \|X_{s \wedge \tau_N}\|_p) ds + C_p^{BDG} C \left\| \sqrt{\int_0^t (1 + |X_{s \wedge \tau_N}|)^2 ds} \right\|_p \\
&\leq \|X_0\|_p + C \int_0^t (1 + \|X_{s \wedge \tau_N}\|_p) ds + C_p^{BDG} C \left\| \sqrt{t} + \sqrt{\int_0^t X^2_{s \wedge \tau_N} ds} \right\|_p
\end{aligned}
$$

---

[2]This holds true for any hitting time of an open set by an $\mathcal{F}_t$-adapted càd process.

where we used in the last line that the Minkowski inequality on $L^2[[0,T], dt)$ endowed with its usual Hilbert norm. Hence, the $L^p(\mathbb{P})$-Minkowski Inequality and the obvious identity $\|\sqrt{\cdot}\|_p = \|\sqrt{\cdot}\|_{\frac{p}{2}}^{\frac{1}{2}}$ yield

$$f_N(t) \leq \|X_0\|_p + C\int_0^t (1 + \|X_{s\wedge\tau_N}\|_p)ds + C_p^{BDG}C\left(\sqrt{t} + \left\|\left|\int_0^t X_{s\wedge\tau_N}^2 ds\right|\right\|_{\frac{p}{2}}^{\frac{1}{2}}\right).$$

Now the generalized $L^{\frac{p}{2}}(\mathbb{P})$-Minkowski's Inequality (7.30) (use $\frac{p}{2} \geq 1$) yields

$$\begin{aligned}
f_N(t) &\leq \|X_0\|_p + C\int_0^t (1 + \|X_{s\wedge\tau_N}\|_p)ds + C_p^{BDG}C\left(\sqrt{t} + \left(\int_0^t \|X_{s\wedge\tau_N}^2\|_{\frac{p}{2}} ds\right)^{\frac{1}{2}}\right) \\
&= \|X_0\|_p + C\int_0^t (1 + \|X_{s\wedge\tau_N}\|_p)ds + C_p^{BDG}C\left(\sqrt{t} + \left(\int_0^t \|X_{s\wedge\tau_N}\|_p^2 ds\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Consequently the function f_N satisfies

$$f_N(t) \leq C\left(\int_0^t f_N(s)ds + C_p^{BDG}\left(\int_0^t f_N^2(s)ds\right)^{\frac{1}{2}}\right) + \psi(t)$$

where

$$\psi(t) = \|X_0\|_p + C(t + C_p^{BDG}\sqrt{t}).$$

STEP 2:   "À la Gronwall" Lemma.

Lemma 7.4 *Let $f : [0,T] \to \mathbb{R}_+$ and $\psi : [0,T] \to \mathbb{R}_+$ be two non-negative non-decreasing functions satisfying*

$$\forall t \in [0,T], \qquad f(t) \leq A\int_0^t f(s)ds + B\left(\int_0^t f^2(s)ds\right)^{\frac{1}{2}} + \psi(t)$$

where A, B are two positive real constants. Then

$$\forall t \in [0,T], \qquad f(t) \leq 2\,e^{(2A+B^2)t}\psi(t).$$

Proof. First, it follows from the elementary inequality $\sqrt{xy} \leq \frac{1}{2}(x/B + By)$, $x, y \geq 0$, $B > 0$, that

$$\left(\int_0^t f^2(s)ds\right)^{\frac{1}{2}} \leq \left(f(t)\int_0^t f(s)ds\right)^{\frac{1}{2}} \leq \frac{f(t)}{2B} + \frac{B}{2}\int_0^t f(s)ds.$$

Plugging this in the original inequality yields

$$f(t) \leq (2A + B^2)\int_0^t f(s)ds + 2\,\psi(t).$$

Regular Gronwall's Lemma finally yields the announced result. $\diamond$

STEP 3: Applying the above generalized Gronwall's Lemma to the functions f_N and ψ defined in Step 1, leads to

$$\forall t \in [0,T], \qquad \left\|\sup_{s\in[0,t]}|X_{s\wedge\tau_N}|\right\|_p = f_N(t) \leq 2\,e^{(2+(C_p^{BDG})^2)Ct}\left(\|X_0\|_p + C(t + C_p^{BDG}\sqrt{t})\right).$$

The sequence of stopping times τ_N is non-decreasing and converges toward τ_∞ taking values in $[0,T] \cup \{\infty\}$. On the event $\{\tau_\infty \leq T\}$, $|X_{\tau_N} - X_0| \geq N$ so that $|X_{\tau_\infty} - X_0| = \lim_{N\to+\infty}|X_{\tau_N} - X_0| = +\infty$ since X_t

has continuous paths. This is *a.s.* impossible since the process $(X_t)_{t\geq 0}$ has continuous paths on $[0,T]$. As a consequence $\tau_\infty = \infty$ *a.s.* which in turn implies that

$$\lim_N \sup_{s\in[0,t]} |X_{s\wedge\tau_N}| = \sup_{s\in[0,t]} |X_s| \qquad a.s.$$

Then Fatou's Lemma implies by letting N go to infinity that

$$\forall t \in [0,T], \quad \left\| \sup_{s\in[0,t]} |X_s| \right\|_p \leq \liminf_N \left\| \sup_{s\in[0,t]} |X_{s\wedge\tau_N}| \right\|_p \leq 2\, e^{(2+(C_p^{BDG})^2)Ct}\left(\|X_0\|_p + C(t + C_p^{BDG}\sqrt{t}) \right).$$

which finally yields, using that $\max(\sqrt{u}, u) \leq e^u$, $u \geq 0$,

$$\forall t \in [0,T], \quad \left\| \sup_{s\in[0,t]} |X_s| \right\|_p \leq 2\, e^{(2+(C_p^{BDG})^2)Ct}\left(\|X_0\|_p + e^{Ct} + e^{(C^{BDG_p})^2 t} \right).$$

One derives the existence of a positive real constant $\kappa_p' > 0$, only depending on p, such that

$$\forall t \in [0,T], \quad \left\| \sup_{s\in[0,t]} |X_s| \right\|_p \leq 2e^{\kappa_p' Ct}(1 + \|X_0\|_p).$$

STEP 4 ($p \in (0,2)$). The extension can be carried out as follows: for every $x \in \mathbb{R}^d$, the diffusion process starting at x, denoted $(\bar{X}_t^{n,x})_{t\in[0,T]}$, satisfies the following two obvious facts:

– the process X^x is $\mathcal{F}_t^W$-adapted where $\mathcal{F}_t^W := \overline{\sigma(W_s, \, s \leq t)}^{\mathcal{N}_\mathbb{P}}$.

– If X_0 is an $\mathbb{R}^d$-valued random vector defined on $(\Omega, \mathcal{A}, \mathbb{P})$, independent of W, then the process $X = (X_t)_{t\in[0,T]}$ starting from X_0 satisfies

$$X_t = X_t^{X_0}.$$

Consequently, using that $p \mapsto \|\,.\,\|_p$ is non-decreasing, it follows that

$$\left\| \sup_{s\in[0,t]} |X_s^x| \right\|_p \leq \left\| \sup_{s\in[0,t]} |X_s| \right\|_2 \leq 2\, e^{\kappa_2' Ct}(1 + |x|).$$

Now

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p \right) = \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx)\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^x|^p \right) \leq 2^{(p-1)_+}2^p e^{p\kappa_2' CT}(1 + \mathbb{E}|X_0|^p)$$

(where we used that $(u+v)^p \leq 2^{(p-1)_+}(u^p + v^p)$, u, $v \geq 0$) so that

$$\begin{aligned}
\left\| \sup_{t\in[0,T]} |X_t| \right\|_p &\leq 2^{(1-\frac{1}{p})_+}2\, e^{\kappa_2' CT}2^{(\frac{1}{p}-1)_+}\left(1 + \|X_0\|_p\right) \\
&= 2^{|1-\frac{1}{p}|}2\, e^{\kappa_2' CT}\left(1 + \|X_0\|_p\right)
\end{aligned}$$

As concerns the SDE (7.1) itself, the same reasoning can be carried out only if (7.1) satisfies an existence and uniqueness assumption for any starting value X_0.

(*b*) (*Euler scheme*) The proof follows the same lines as above. One starts from the integral form (7.6) of the continuous Euler scheme and one introduces for every n, $N \geq 1$ the stopping times

$$\bar{\tau}_N = \bar{\tau}_N^n := \inf\{t \in [0,T] \mid |\bar{X}_t^n - X_0| > N\}.$$

At this stage, one notes that b (as well as σ^2) satisfy this type of inequality

$$\forall s \in [0,t], \qquad \left\| \mathbf{1}_{\{s \leq \bar{\tau}_N\}}b(\underline{s}, \bar{X}_{\underline{s}}) \right\|_p \leq C\left(1 + \left\| \sup_{s\in[0,t\wedge\bar{\tau}_N]} |\bar{X}_s| \right\|_p \right).$$

which makes possible to reproduce formally the above proof to the continuous Euler scheme. $\diamond$

7.8.3 L^p-pathwise regularity

Lemma 7.5 *Let $p \geq 1$ and let $(Y_t)_{t \in [0,T]}$ be an Itô process defined on $[0,T]$ by*

$$Y_t = Y_0 + \int_0^t G_s \, ds + \int_0^t H_s \, dW_s$$

where H and G are $(\mathcal{F}_t)$-progressively measurable satisfying $\int_0^T |G_s| + H_s^2 ds < +\infty$ a.s.

(a) For every $p \geq 2$,

$$\forall s, t \in [0,T], \quad \|Y_t - Y_s\|_p \;\leq\; C_p^{BDG} \sup_{t \in [0,T]} \|H_t\|_p |t-s|^{\frac{1}{2}} + \sup_{t \in [0,T]} \|G_t\|_p |t-s|$$

$$\leq\; (C_p^{BDG} \sup_{t \in [0,T]} \|H_t\|_p + \sqrt{T} \sup_{t \in [0,T]} \|G_t\|_p)|t-s|^{\frac{1}{2}}.$$

In particular if $\sup_{t \in [0,T]} \|H_t\|_p + \sup_{t \in [0,T]} \|G_t\|_p < +\infty$, the process $t \mapsto Y_t$ is Hölder with exponent $\frac{1}{2}$ from $[0,T]$ into $L^p(\mathbb{P})$.

(b) If $p \in [1,2)$, then

$$\forall s, t \in [0,T], \quad \|Y_t - Y_s\|_p \;\leq\; C_p^{BDG} \| \sup_{t \in [0,T]} |H_t|\|_p |t-s|^{\frac{1}{2}} + \sup_{t \in [0,T]} \|G_t\|_p |t-s|$$

$$\leq\; (C_p^{BDG} \| \sup_{t \in [0,T]} |H_t|\|_p + \sqrt{T} \sup_{t \in [0,T]} \|G_t\|_p)|t-s|^{\frac{1}{2}}.$$

Proof. *(a)* Let $0 \leq s \leq t \leq T$. It follows from the standard and generalized Minkowski Inequalities and the BDG Inequality (applied to the continuous local martingale $(\int_s^{s+u} H_r dW_r)_{u \geq 0}$) that

$$
\begin{aligned}
\|Y_t - Y_s\|_p &\;\leq\; \left\| \int_s^t |G_u| du \right\|_p + \left\| \int_s^t H_u dW_u \right\|_p \\[2mm]
&\;\leq\; \int_s^t \|G_u\|_p du + C_p^{BDG} \left\| \sqrt{\int_s^t H_u^2 du} \right\|_p \\[2mm]
&\;\leq\; \sup_{t \in [0,T]} \|G_t\|_p (t-s) + C_p^{BDG} \left\| \int_s^t H_u^2 du \right\|_{p/2}^{\frac{1}{2}} \\[2mm]
&\;\leq\; \sup_{t \in [0,T]} \|G_t\|_p (t-s) + C_p^{BDG} \sup_{u \in [0,T]} \|H_u^2\|_{p/2}^{\frac{1}{2}} (t-s)^{\frac{1}{2}} \\[2mm]
&\;=\; \sup_{t \in [0,T]} \|G_t\|_p (t-s) + C_p^{BDG} \sup_{u \in [0,T]} \|H_u\|_p (t-s)^{\frac{1}{2}}.
\end{aligned}
$$

The second inequality simply follows from $|t-s| \leq \sqrt{T}|t-s|^{\frac{1}{2}}$.

(b) If $p \in [1,2]$, one simply uses that

$$\left\| \int_s^t H_u^2 du \right\|_{p/2}^{\frac{1}{2}} \leq |t-s|^{\frac{1}{2}} \left\| \sup_{u \in [0,T]} H_u^2 \right\|_{p/2}^{\frac{1}{2}} = |t-s|^{\frac{1}{2}} \left\| \sup_{u \in [0,T]} |H_u| \right\|_p.$$

and one concludes likewise. $\diamond$

Remark : If H, G and Y are defined on the whole real line $\mathbb{R}_+$ and $\sup\limits_{t \in \mathbb{R}_+} (\|G_u\|_p + \|H_u\|_p) < +\infty$ then $t \mapsto Y_t$ est locally $1/2$-Hölder on $\mathbb{R}_+$. If $H = 0$, the process is in fact Lipschitz continuous on $[0, T]$.

Combining the above result for Itô processes with those of Proposition 7.5 leads to the following result on pathwise regularity of the diffusion solution to (7.1) (when it does exists) and the related Euler schemes.

Proposition 7.6 *If the coefficients b and σ satisfy the linear growth assumption (7.31) over $[0,T] \times \mathbb{R}^d$ with a real constant $C > 0$, then the Euler scheme with step T/n and any strong solution of (7.1) satisfy for every $p \geq 1$,*

$$\forall\, n \geq 1, \quad \forall\, s,\, t \in [0,T], \qquad \|X_t - X_s\|_p + \|\bar{X}_t^n - \bar{X}_s^n\|_p \leq \kappa_p'' C e^{\kappa_p'' C T} (1 + \sqrt{T})(1 + \|X_0\|_p)|t - s|^{\frac{1}{2}}.$$

where $\kappa_p'' \in (0, \infty)$ is real constant only depending on p (increasing in p).

Proof. As concerns the process X, this is a straightforward consequence of the above Lemma 7.5 by setting

$$G_t = b(t, X_t) \quad \text{and} \quad H_t = \sigma(t, X_t)$$

since

$$\max(\|\sup_{t \in [0,T]} |G_t|\|_p, \|\sup_{t \in [0,T]} |H_t|\|_p) \leq C(1 + \|\sup_{t \in [0,T]} |X_t|\|_p).$$

One specifies the real constant κ_p'' using Proposition 7.5.   ◇

7.8.4 L^p-converge rate (II): proof of Theorem 7.2

STEP 1 ($p \geq 2$): One sets

$$\begin{aligned}\varepsilon_t &:= X_t - \bar{X}_t^n,\ t \in [0,T] \\ &= \int_0^t \Big(b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})\Big) ds + \int_0^t \Big(\sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})\Big) dW_s\end{aligned}$$

so that

$$\sup_{s \in [0,t]} |\varepsilon_s| \leq \int_0^t |(b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})| ds + \sup_{s \in [0,t]} \left| \int_0^s (\sigma(u, X_u) - \sigma(\underline{u}, \bar{X}_{\underline{u}})) dW_u \right|.$$

One sets for every $t \in [0, T]$,

$$f(t) := \|\sup_{s \in [0,t]} |\varepsilon_s|\|_p.$$

It follows from regular and generalized Minkowski Inequalities and BDG Inequality that

$$\begin{aligned}f(t) &\leq \int_0^t \|b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})\|_p ds + C_p^{BDG} \left\| \left(\int_0^t \big(\sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})\big)^2 ds \right)^{\frac{1}{2}} \right\|_p \\ &= \int_0^t \|b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})\|_p ds + C_p^{BDG} \left\| \int_0^t (\sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}}))^2 ds \right\|_{\frac{p}{2}}^{\frac{1}{2}} \\ &\leq \int_0^t \|b(s, X_s) - b(\underline{s}, \bar{X}_{\underline{s}})\|_p ds + C_p^{BDG} \left(\int_0^t \|\sigma(s, X_s) - \sigma(\underline{s}, \bar{X}_{\underline{s}})\|_p^2 ds \right)^{\frac{1}{2}} \\ &\leq C_{b,\sigma} \left(\int_0^t ((s - \underline{s})^\beta + \|X_s - \bar{X}_{\underline{s}}\|_p) ds + C_p^{BDG} \left(\int_0^t (s - \underline{s})^{2\beta} + \|X_s - \bar{X}_{\underline{s}}\|_p^2 ds \right)^{\frac{1}{2}} \right) \\ &\leq C_{b,\sigma} \left((\sqrt{t} + C_p^{BDG}) \left(\frac{T}{n} \right)^\beta \sqrt{t} + \int_0^t \|X_s - \bar{X}_{\underline{s}}\|_p ds + C_p^{BDG} \left(\int_0^t \|X_s - \bar{X}_{\underline{s}}\|_p^2 ds \right)^{\frac{1}{2}} \right)\end{aligned}$$

where we we used that $0 \leq s - \underline{s} \leq \frac{T}{n}$ and the elementary inequality $\sqrt{u+v} \leq \sqrt{u} + \sqrt{v}$, u, $v \geq 0$ and $C_{b,\sigma}$ denotes a positive real constant only depending on b and σ. Now, noting that

$$\|X_s - \bar{X}_{\underline{s}}\|_p \quad \leq \quad \|X_s - X_{\underline{s}}\|_p + \|X_{\underline{s}} - \bar{X}_{\underline{s}}\|_p$$

$$= \quad \|X_s - X_{\underline{s}}\|_p + \|\varepsilon_{\underline{s}}\|_p$$

$$\leq \quad \|X_s - X_{\underline{s}}\|_p + f(s),$$

it follows that

$$f(t) \leq C_{b,\sigma}\left(\int_0^t f(s)ds + \sqrt{2}\,C_p^{BDG}\left(\int_0^t f(s)^2 ds\right)^{\frac{1}{2}} + \psi(t)\right) \tag{7.32}$$

where

$$\psi(t) := (\sqrt{t} + C_p^{BDG})\left(\frac{T}{n}\right)^\beta \sqrt{t} + \int_0^t \|X_s - X_{\underline{s}}\|_p\, ds + \sqrt{2}\,C_p^{BDG}\left(\int_0^t \|X_s - X_{\underline{s}}\|_p^2\, ds\right)^{\frac{1}{2}}. \tag{7.33}$$

STEP 2: It follows from Lemma 7.4 that

$$f(t) \leq 2\,C_{b,\sigma}e^{2C_{b,\sigma}(1+C_p^{BDG}/\sqrt{2})t}\psi(t). \tag{7.34}$$

Now, we will use the path regularity of the diffusion process X in $L^p(\mathbb{P})$ obtained in Proposition 7.6 to provide an upper-bound for the function ψ. We first note that since b and σ satisfy (H_T^β) with a positive real constant $C_{b,\sigma}$, they satisfy the linear growth assumption yields (7.31) with

$$C'_{b,\sigma,T} := C_{b,\sigma} + \sup_{t \in [0,T]} (|b(t,0)| + |\sigma(t,0)|) < +\infty$$

($b(.,0)$ and $\sigma(.,0)$ are β-Hölder hence bounded on $[0,T]$). It follows from (7.33) and Proposition 7.6 that

$$\psi(t) \quad \leq \quad (\sqrt{t} + C_p^{BDG})\left(\frac{T}{n}\right)^\beta \sqrt{t} + \kappa_p''C'_{b,\sigma}e^{\kappa_p''C'_{b,\sigma}t}(1 + \|X_0\|_p)(1 + \sqrt{t})\left(\frac{T}{n}\right)^{\frac{1}{2}}(t + \sqrt{2}C_p^{BDG}\sqrt{t})$$

$$\leq \quad C_p^{BDG}(1 + t)\left(\frac{T}{n}\right)^\beta + 2(1 + \sqrt{2}C_p^{BDG})(1 + t)^2\kappa_p''C'_{b,\sigma}e^{\kappa_p''C'_{b,\sigma}t}(1 + \|X_0\|_p)$$

where we used the elementary inequality $\sqrt{u} \leq 1 + u$ which implies that $(\sqrt{t} + C_p^{BDG})\sqrt{t} \leq C_p^{BDG}(1 + t)$ and $(1 + \sqrt{t})(t + \sqrt{2}C_p^{BDG}\sqrt{t}) \leq 2(1 + \sqrt{2}C_p^{BDG})(1 + t)^2$.

Hence, there exists a real constant $\tilde{\kappa}_p > 0$ such that

$$\psi(t) \quad \leq \quad \tilde{\kappa}_p(1 + t)\left(\frac{T}{n}\right)^\beta + \tilde{\kappa}_pC'_{b,\sigma}e^{\tilde{\kappa}_pC'_{b,\sigma}t}(1 + \|X_0\|_p)(1 + t)^2$$

$$\leq \quad \tilde{\kappa}_pe^t\left(\frac{T}{n}\right)^\beta + 2\tilde{\kappa}_pC'_{b,\sigma}e^{\tilde{\kappa}_p(1+C'_{b,\sigma})t}(1 + \|X_0\|_p).$$

since $1 + u \leq e^u$ and $(1 + u)^2 \leq 2e^u$, $u \geq 0$. Finally one plugs this bound in (7.34) (at time T) to get the announced upper-bound by setting the real constant κ_p at an appropriate value.

STEP 2 ($p \in (0,2)$): It remains to deal with the case $p \in [1,2)$. In fact, once noticed that Assumption (H_T^β) ensures global existence and uniqueness of the solution X of (7.1) starting from a given random variable X_0 (independent of W), it can be solved following the approach developed in Step 4 of the proof of Proposition 7.5. We leave the details to the reader. $\diamond$

Corollary 7.3 *(Lipschitz continuous framework) If b and σ satisfy Condition (H_T^1) i.e.*

$$\forall\, s,\, t \in [0,T],\ \forall\, x,\, y \in \mathbb{R}^d, \qquad |b(s,x) - b(t,y)| + |\sigma(s,x) - \sigma(t,y)| \leq C_{b,\sigma,T}(|t-s| + |x-y|)$$

then for every $p \in [1,\infty)$,

$$\forall\, n \geq 1, \qquad \left\| \sup_{t\in[0,T]} |X_t - \bar{X}_t^n| \right\|_p \leq \kappa_p C_{b,\sigma,T} e^{\kappa_p(1+C_{b,\sigma,T})T}(1 + \|X_0\|_p)\sqrt{\frac{T}{n}}.$$

7.8.5 The stepwise constant Euler scheme

The aim of this section is to prove in full generality Claim (b) of Theorem 7.2. We recall that the stepwise constant Euler scheme is defined by

$$\forall\, t \in [0,T], \qquad \widetilde{X}_t := \bar{X}_{\underline{t}}$$

i.e. $\widetilde{X}_t = \tilde{X}_{t_k}$, si $t \in [t_t, t_{k+1})$.

It has been seen in Section 7.2.1 that when $X = W$, a $\log n$ factor comes out in the error rate. One must again have in mind that this question is quite crucial since, at least in higher dimension, since the simulation of the continuous Euler scheme may rise difficult problems whereas the simulation of the stepwise constant Euler scheme remains straightforward in any dimension (provided b and σ are known).

Proof of Theorem 7.2(b). STEP 1 *(Deterministic X_0):* We first assume that $X_0 = x$. Then, one may assume without loss of generality that $p \in [1,\infty)$. Then

$$\bar{X}_t^n - \tilde{X}_t^n = \bar{X}_t^n - \bar{X}_{\underline{t}}^n$$

$$= \int_{\underline{t}}^t b(\underline{s}, \bar{X}_{\underline{s}})ds + \int_{\underline{t}}^t \sigma(\underline{s}, \bar{X}_{\underline{s}})dW_s.$$

One derives that

$$\sup_{t\in[0,T]} |\bar{X}_t^n - \tilde{X}_t^n| \leq \frac{T}{n}\sup_{t\in[0,T]} |b(\underline{t}, \bar{X}_{\underline{t}})| + \sup_{t\in[0,T]} |\sigma(\underline{t}, \bar{X}_{\underline{t}})(W_t - W_{\underline{t}})|. \tag{7.35}$$

Now, it follows from Proposition 7.5(b) that

$$\left\| \sup_{t\in[0,T]} |b(\underline{t}, \bar{X}_t^n)| \right\|_p \leq 2e^{\kappa_p' C_{b,\sigma,T}T}(1 + |x|)$$

On the other hand, using the extended Hölder Inequality: for every $p \in (0,\infty)$,

$$\forall\, r,\, s \geq 1,\ \frac{1}{r} + \frac{1}{s} = 1, \quad \|fg\|_p \leq \|f\|_{rp}\,\|g\|_{sp},$$

with $r = 1 + \eta$ and $s = 1 + 1/\eta$, $\eta > 0$, leads to

$$\left\| \sup_{t\in[0,T]} |\sigma(\underline{t}, \bar{X}_{\underline{t}})(W_t - W_{\underline{t}})| \right\|_p \leq \left\| \sup_{t\in[0,T]} |\sigma(\underline{t}, \bar{X}_{\underline{t}})| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \right\|_p$$

$$\leq \left\| \sup_{t\in[0,T]} |\sigma(\underline{t}, \bar{X}_{\underline{t}})| \right\|_{p(1+\eta)} \left\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \right\|_{p(1+1/\eta)}$$

(for convenience we set $\eta = 1$ in what follows). Now, like for the drift b, one has

$$\left\| \sup_{t \in [0,T]} |\sigma(\underline{t}, \bar{X}_{\underline{t}})| \right\|_{p(1+\eta)} \leq 2 e^{\kappa'_{2p} C_{b,\sigma,T} T} (1 + |x|).$$

As concerns the Brownian term, one has

$$\left\| \sup_{t \in [0,T]} |W_t - W_{\underline{t}}| \right\|_{2p} \leq C_{W,2p} \sqrt{\frac{T}{n}(1 + \log n)}$$

owing to (7.12) in Section 7.2.1. Finally, plugging these estimates in (7.35), yields

$$
\begin{aligned}
\left\| \sup_{t \in [0,T]} |\bar{X}^n_t - \widetilde{X}^n_t| \right\|_p &\leq 2\frac{T}{n} e^{\kappa'_p C_{b,\sigma,T} T}(1 + |x|) \\
&\quad + 2 e^{\kappa'_{2p} C_{b,\sigma,T} T}(1 + |x|) \times C_{W,2p} \sqrt{\frac{T}{n}(1 + \log n)}, \\
&\leq 2(C_{W,2p} + 1) e^{\kappa'_{2p} C_{b,\sigma,T} T}(1 + |x|) \left(\sqrt{\frac{T}{n}(1 + \log n)} + \frac{T}{n} \right).
\end{aligned}
$$

One concludes by noting that $\sqrt{\frac{T}{n}(1 + \log n)} + \frac{T}{n} \leq 2\sqrt{\frac{T}{n}(1 + \log n)}$ for every integer $n \geq 1$ and by setting $\tilde{\kappa}_p := 2 \max(2(C_{W,2p} + 1), \kappa'_{2p})$.

STEP 2 *(Random X_0)*: When X_0 is no longer deterministic one uses that X_0 and W are independent so that, with obvious notations,

$$\mathbb{E} \sup_{t \in [0,T]} |\bar{X}^{n,X_0} - \tilde{X}^{n,X_0}|^p = \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0) \mathbb{E} \sup_{t \in [0,T]} |\bar{X}^{n,x_0} - \tilde{X}^{n,x_0}|^p$$

which yields the announced result.

STEP 3 *Combination of the upper-bounds*: This is a straightforward consequence of Claims (a) and (b). $\diamond$

7.8.6   Application to the *a.s.*-convergence of the Euler schemes and its rate.

One can derive from the above L^p-rate of convergence an *a.s.*-convergence result. The main result is given in the following theorem (which extends Theorem 7.3 stated in the homogeneous Lipschitz continuous continuous case).

Theorem 7.9 *If (H_T^β) holds and if X_0 is a.s. finite, the continuous Euler scheme $\bar{X}^n = (\bar{X}^n_t)_{t \in [0,T]}$ a.s. converges toward the diffusion X for the* sup*-norm over $[0,T]$. Furthermore, for every $\alpha \in [0, \beta \wedge \frac{1}{2})$,*

$$n^\alpha \sup_{t \in [0,T]} |X_t - \bar{X}^n_t| \xrightarrow{a.s.} 0.$$

The same convergence rate holds with the stepwise constant Euler scheme $(\tilde{X}^n_t)_{t \in [0,T]}$.

Proof: We make no *a priori* integrability assumption on X_0. We rely on the localization principle (at the origin). Let $N > 1$; set $X_0^{(N)} := X_0 \mathbf{1}_{\{|X_0| \leq N\}} + N \frac{X_0}{|X_0|} \mathbf{1}_{\{|X_0| > N\}}$. Stochastic integration being a local operator, the solutions $(X_t^{(N)})_{t \in [0,T]}$ and $(X_t)_{t \in [0,T]}$ of the *SDE* (7.1) are equal on $\{X_0 = X_0^{(N)}\}$, namely

on $\{|X_0| \leq N\}$. The same property is obvious for the Euler schemes $\bar{X}^n$ and $\bar{X}^{n,(N)}$ starting from X_0 and $X_0^{(N)}$ respectively. For a fixed N, we know from Theorem 7.2 (a) that, for every $p \geq 1$,

$$\exists\, C_{p,b,\sigma,\beta,T} > 0 \quad \text{such that, } \forall\, n \geq 1, \quad \mathbb{E}\Big(\sup_{t \in [0,T]} |\bar{X}_t^{(N)} - X_t^{(N)}|^p\Big) \leq C_{p,b,\sigma,\beta,T} \left(\frac{T}{n}\right)^{p(\beta \wedge \frac{1}{2})} (1 + \|X_0\|_p)^p.$$

In particular

$$
\begin{aligned}
\mathbb{E}\Big(\mathbf{1}_{\{|X_0| \leq N\}} \sup_{t \in [0,T]} |\bar{X}_t^{n,(N)} - X_t|^p\Big) &= \mathbb{E}\Big(\mathbf{1}_{\{|X_0| \leq N\}} \sup_{t \in [0,T]} |\bar{X}_t^{n,(N)} - X_t^{(N)}|^p\Big) \\
&\leq \mathbb{E}\Big(\sup_{t \in [0,T]} |\bar{X}_t^{n,(N)} - X_t^{(N)}|^p\Big) \\
&\leq C_{p,b,\sigma,\beta,T}(1 + \|X_0^{(N)}\|_p)^p \left(\frac{T}{n}\right)^{p(\beta \wedge \frac{1}{2})}.
\end{aligned}
$$

On chooses $p > \frac{1}{\beta \wedge \frac{1}{2}}$, so that $\sum_{n \geq 1} \dfrac{1}{n^{p(\beta \wedge \frac{1}{2})}} < +\infty$. Consequently Beppo Levi's Theorem for series with non-negative terms implies

$$\mathbb{E}\Big(\mathbf{1}_{\{|X_0| \leq N\}} \sum_{n \geq 1} \sup_{t \in [0,T]} |\bar{X}_t^n - X_t|^p\Big) < +\infty.$$

Hence

$$\sum_{n \geq 1} \sup_{t \in [0,T]} |\bar{X}_t^n - X_t|^p < \infty, \quad \mathbb{P} \ a.s. \quad \text{on} \quad \bigcup_{n \geq 1} \{|X_0| \leq N\} = \{X_0 \in \mathbb{R}^d\} \overset{a.s.}{=} \Omega$$

so that $\sup_{t \in [0,T]} |\bar{X}_t^n - X_t| \overset{n \to \infty}{\longrightarrow} 0 \ \mathbb{P} \ a.s.$

One can improve the above approach to get a much more powerful result: let $\alpha \in]0, \beta \wedge \frac{1}{2}[$, and $p > \frac{1}{\beta \wedge \frac{1}{2} - \alpha}$,

$$
\begin{aligned}
n^{p\alpha} \, \mathbb{E}\Big(\sup_{t \in [0,t]} |\bar{X}_t^{(N)} - X_t^{(N)}|^p\Big) &\leq C_{p,b,\sigma,\beta,T}(1 + \|X_0^{(N)}\|_p)^p \left(\frac{T}{n}\right)^{p(\beta \wedge \frac{1}{2})} n^{p\alpha} \\
&= C'_{p,b,\sigma,\beta,T}(1 + \|X_0^{(N)}\|_p)^p n^{-p(\beta \wedge \frac{1}{2} - \alpha)}
\end{aligned}
$$

Finally, one gets:

$$\mathbb{P}\text{-}a.s. \qquad \sup_{t \in [0,T]} |\bar{X}_t^n - X_t| \overset{n \to \infty}{\longrightarrow} O\left(\frac{1}{n^\alpha}\right).$$

The proof for the stepwise constant Euler scheme follows exactly the same lines since an additional $\log n$ term plays no role in the convergence of the above series. $\qquad \diamond$

Remarks and comments. • The above rate result strongly suggests that the critical index for the *a.s.* rate of convergence is $\beta \wedge \frac{1}{2}$. The question is then: what happens when $\alpha = \beta \wedge \frac{1}{2}$? It is shown in [88, 74] that (when $\beta = 1$), $\sqrt{n}(X_t - \bar{X}_t) \overset{\mathcal{L}}{\longrightarrow} \Xi_t$ where $\Xi = (\Xi_t)_{t \in [0,T]}$ is a diffusion process. This weak convergence holds in a functional sense, namely for the topology of the uniform convergence on $\mathcal{C}([0,T], \mathbb{R}^d)$. This process Ξ is not $\mathbb{P}$-*a.s.* $\equiv 0$ if $\sigma(x) \not\equiv 0.$, even *a.s* non-zero if σ never vanishes. The "weak functional" feature means first that we consider the processes as random variables taking values in their natural path space, namely the separable Banach space $(\mathcal{C}([0,T], \mathbb{R}^d), \|\,.\,\|_{\sup})$. Then, one may consider the weak convergence of probability measures defined on (the Borel σ-field of) this space (see [24] for an introduction). The connection with the above elementary results is the following:

If $Y_n \xrightarrow{\mathcal{L}} Y$, then $\mathbb{P}$-a.s. $\forall \varepsilon > 0$

$$\overline{\lim}_n \, n^\varepsilon \|Y_n\|_{\sup} = +\infty \quad \text{on} \quad \{Y \neq 0\} \qquad \text{and} \qquad Y = 0 \quad \text{on} \quad \{\overline{\lim}_n \, |Y_n\|_{\sup} = 0\}.$$

(This follows either from the Skokhorod representation theorem or from a direct approach)

• When $\beta = 1$ (Lipschitz continuous case), one checks that, for every $\varepsilon > 0$, $\overline{\lim}_n (\sqrt{n})^{1+\varepsilon} \sup_{t \in [0,T]} |X_t - \bar{X}_t^n| = +\infty$ $\mathbb{P}$-a.s. If one changes the time scale, it is of iterated logarithm type.

▷ **Exercise.** One considers the geometric Brownian motion $X_t = e^{-\frac{t}{2} + W_t}$ solution to

$$dX_t = X_t dW_t, \qquad X_0 = 1.$$

(a) Show that for every $n \geq 1$ and every $k \geq 0$,

$$\bar{X}_{t_k^n} = \prod_{\ell=1}^{k} \left(1 + \Delta W_{t_\ell^n}\right) \quad \text{where } t_\ell^n = \frac{\ell T}{n}, \, \Delta W_{t_\ell^n} = W_{t_\ell^n} - W_{t_{\ell-1}^n}, \, \ell \geq 1.$$

(b) Show that

$$\forall \varepsilon > 0, \quad \overline{\lim}_n (\sqrt{n})^{1+\varepsilon} |X_T - \bar{X}_T^n| = \infty \quad \mathbb{P}\text{-a.s.}$$

7.8.7   Flow of an SDE, Lipschitz continuous regularity

If Assumption (7.2) holds, there exists a unique solution to the the SDE (7.1) starting from $x \in \mathbb{R}^d$ defined on $[0,T]$, denoted $(X_t^x)_{t \in [0,T]}$ from now on. The mapping $(x,t) \mapsto X_t^x$ defined on $[0,T] \times \mathbb{R}^d$ is called the *flow* of the SDE (7.1). One defines likewise the flow of the Euler schemes (which always exists). We will now elucidate the regularity of these flows when Assumption (H_T^β) holds.

Theorem 7.10 *There exists a function $\kappa_3' : (\mathbb{R}_+^*)^2 \to \mathbb{R}_+$ such that if the coefficient b and σ of (7.1) satisfy Assumption (H_T^β) for a real constant $C > 0$, then the unique strong solution $(X_t^x)_{t \in [0,T]}$ starting from $x \in \mathbb{R}^d$ sur $[0,T]$ and the continuous Euler scheme $(\bar{X}^{n,x})_{t \in [0,T]}$ satisfy*

$$\forall x, y \in \mathbb{R}^d, \, \forall n \geq 1 \quad \left\| \sup_{t \in [0,T]} |X_t^x - X_t^y| \right\|_p + \left\| \sup_{t \in [0,T]} |\bar{X}_t^{n,x} - \bar{X}_t^{n,y}| \right\|_p \leq \kappa_3'(p,C) e^{\kappa_3'(p,C)T} |x - y|.$$

Proof: We focus on the diffusion process $(X_t)_{t \in [0,T]}$. First note that if the above bound holds for some $p > 0$ then it holds true for any $p' \in (0,p)$ since the $\| . \|_p$-norm is non-decreasing in p. Starting from

$$X_t^x - X_t^y \;=\; (x - y) + \int_0^t (b(s, X_s^x) - b(s, X_s^y)) ds + \int_0^t \left(\sigma(s, X_s^x) - \sigma(s, X_s^y)\right) dW_s$$

one gets

$$\sup_{s \in [0,t]} |X_s^x - X_s^y| \;\leq\; |x - y| + \int_0^t |b(s, X_s^x) - b(s, X_s^y)| ds + \sup_{s \in [0,t]} \left| \int_0^s \left(\sigma(u, X_u^x) - \sigma(u, X_u^y)\right) dW_u \right|.$$

Then setting for every $p \geq 2$, $f(t) := \left\| \sup_{s \in [0,t]} |X_s^x - X_s^y| \right\|_p$, it follows from the generalized Minkowski and the *BDG* Inequalities

$$
\begin{aligned}
f(t) &\leq |x - y| + C \int_0^t \|X_s^x - X_s^y\|_p ds + C_p^{BDG} \left\| \sqrt{\int_0^t \left(\sigma(s, X_s^x) - \sigma(s, X_s^y) \right)^2 ds} \right\|_p \\
&\leq |x - y| + C \int_0^t \|X_s^x - X_s^y\|_p ds + C_p^{BDG} C \left\| \int_0^t |X_s^x - X_s^y|^2 ds \right\|_{\frac{p}{2}}^{\frac{1}{2}} \\
&\leq |x - y| + C \int_0^t \|X_s^x - X_s^y\|_p ds + C_p^{BDG} C \left(\int_0^t \|X_s^x - X_s^y\|^2 ds \right)^{\frac{1}{2}}.
\end{aligned}
$$

Consequently the function f satisfies

$$
f(t) \leq |x - y| + C \left(\int_0^t f(s)ds + C_p^{BDG} + \left(\int_0^t f(s)ds \right)^{\frac{1}{2}} \right).
$$

One concludes like in Step 2 of Theorem 7.2 that

$$
\forall\, t \in [0, T], \qquad f(t) \leq e^{C(2 + C_p^{BDG})t} |x - y|. \qquad \diamond
$$

7.8.8 Strong error rate for the Milstein scheme: proof of Theorem 7.5

In this section, we prove Theorem 7.5 *i.e.* the scalar case $d = q = 1$. Throughout this section $C_{b,\sigma,p,T}$ and $K_{b,\sigma,p,T}$ are positive real constants that may vary from line to line.

First we note that the (interpolated) continuous Milstein scheme as defined by (7.21) can be written in an integral form as follows

$$
\widetilde{X}_t^{mil} = x + \int_0^t b(\widetilde{X}_{\underline{s}}^{mil})ds + \int_0^t \sigma(\widetilde{X}_{\underline{s}}^{mil})dW_s + \int_0^t \int_{\underline{s}}^s (\sigma\sigma')(\widetilde{X}_{\underline{u}}^{mil})dW_u dW_s \tag{7.36}
$$

with our usual notation $\underline{t}$ (note that $\underline{u} = \underline{s}$ on $[\underline{s}, s]$). For notational convenience, we will also drop throughout this section the superscript ^{mil} since we will deal exclusively with the Milstein scheme.

(a) STEP 1 (*Moment control*): Our first aim is to prove that the Milstein scheme has uniformly controlled moments at any order, namely that, for every $p \in (0, \infty)$, there exists s a real constant $C_{p,b,\sigma,T} > 0$ such that

$$
\forall\, n \geq 1, \qquad \sup_{n \geq 1} \left\| \sup_{t \in [0,T]} |\widetilde{X}_t^n| \right\|_p < C_{b,\sigma,T}(1 + \|X_0\|_p). \tag{7.37}
$$

Set

$$
H_s = \sigma(\widetilde{X}_{\underline{s}}) + \int_{\underline{s}}^s (\sigma\sigma')(\widetilde{X}_{\underline{u}})dW_u = \sigma(\widetilde{X}_{\underline{s}}) + (\sigma\sigma')(\widetilde{X}_{\underline{s}})(W_s - W_{\underline{s}})
$$

so that

$$
\widetilde{X}_t = X_0 + \int_0^t b(\widetilde{X}_{\underline{s}})ds + \int_0^t H_s dW_s.
$$

It follows from the boundedness of b' and σ' that b and σ satisfy a linear growth assumption.

We will follow the lines of the proof of Proposition 7.5, the specificity of the Milstein framework being that the diffusion coefficient is replace by the process H_s. So, our task is to control the term

$$
\sup_{s \in [0,t]} \left| \int_0^{s \wedge \widetilde{\tau}_N} H_u dW_u \right|
$$

in L^p where $\tilde{\tau}_N = \tilde{\tau}_N^n := \inf\{t \in [0,T] \mid |\widetilde{X}_s^n - X_0| > N\}$, n, $N \geq 1$.

First assume that $p \in [2, \infty)$. A careful reading of the proof of shows Since $\int_0^{t \wedge \tilde{\tau}_N} H_s dW_s$ is a continuous local martingale, it follows from the BDG Inequality that

$$\left\| \sup_{s \in [0,t]} \left| \int_0^{s \wedge \tilde{\tau}_N} H_u dW_u \right| \right\|_p \leq C_p^{BDG} \left\| \left| \int_0^{t \wedge \tilde{\tau}_N} H_s^2 ds \right| \right\|_{p/2}^{\frac{1}{2}}.$$

Consequently, using the generalized Minkowski Inequality

$$\left\| \sup_{s \in [0,t]} \left| \int_0^{s \wedge \tilde{\tau}_N} H_u dW_u \right| \right\|_p \leq C_p^{BDG} \left(\int_0^t \left\| \mathbf{1}_{\{s \leq \tilde{\tau}_N\}} H_s \right\|_p^2 ds \right)^{\frac{1}{2}}$$

$$= C_p^{BDG} \left(\int_0^t \left\| \mathbf{1}_{\{s \leq \tilde{\tau}_N\}} H_{s \wedge \tilde{\tau}_N} \right\|_p^2 ds \right)^{\frac{1}{2}}.$$

Now, for every $s \in [0,t]$,

$$\left\| \mathbf{1}_{\{s \leq \tilde{\tau}_N\}} H_{s \wedge \tilde{\tau}^N} \right\|_p \leq \left\| \sigma(\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N}) \right\|_p + \left\| \mathbf{1}_{\{s \leq \tilde{\tau}_N\}} (\sigma\sigma')(\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N})(W_{s \wedge \tilde{\tau}^N} - W_{\underline{s} \wedge \tilde{\tau}_N}) \right\|_p$$

$$\leq \left\| \sigma(\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N}) \right\|_p + \left\| (\sigma\sigma')(\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N})(W_s - W_{\underline{s}}) \right\|_p$$

$$= \left\| \sigma(\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N}) \right\|_p + \left\| (\sigma\sigma')(\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N}) \right\|_p \left\| W_s - W_{\underline{s}} \right\|_p$$

where we used that $(\sigma\sigma')(\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N})$ and $W_s - W_{\underline{s}}$ are independent. Consequently, using that σ' is bounded so that σ and $\sigma\sigma'$ have at most linear growth, we get

$$\left\| \mathbf{1}_{\{s \leq \tilde{\tau}_N\}} H_{s \wedge \tilde{\tau}^N} \right\|_p \leq C_{b,\sigma,T}(1 + \|\widetilde{X}_{\underline{s} \wedge \tilde{\tau}_N}\|_p).$$

Finally, following the lines of the step 1 of the proof of Proposition 7.5 leads to

$$\left\| \sup_{s \in [0,t]} \left| \int_0^{s \wedge \tilde{\tau}_N} H_u dW_u \right| \right\|_p \leq C_{b,\sigma,T} C_p^{BDG} \left(\sqrt{t} + \left(\int_0^t \left\| \sup_{u \in [0, s \wedge \tilde{\tau}_N]} |\widetilde{X}_u| \right\|_p^2 ds \right)^{\frac{1}{2}} \right).$$

One concludes still following the lines of the proof Proposition 7.5 (including the step 4 to deal with the case $p \in (0, 2)$).

Furthermore, as a by-product we get that, for every $p > 0$ and every $n \geq 1$,

$$\left\| \sup_{t \in [0,T]} |H_t| \right\|_p \leq K_{b,\sigma,T,p}(1 + \|X_0\|_p) < +\infty \tag{7.38}$$

where $\kappa_{b,\sigma,T,p}$ does not depend on the discretization step n. As a matter of fact, this follows from

$$\sup_{t \in [0,T]} |H_t| \leq C_{b,\sigma}\left(1 + \sup_{t \in [0,T]} |\widetilde{X}_t^n|\right)\left(1 + 2 \sup_{t \in [0,T]} |W_t|\right)$$

so that, by the Schwarz Inequality when $p \geq 1$,

$$\left\| \sup_{t \in [0,T]} |H_t| \right\|_p \leq \frac{C_{b,\sigma}}{2}\left(1 + \sup_{n \geq 1} \left\| \sup_{t \in [0,T]} |\widetilde{X}_t^n| \right\|_{2p}\right)\left(1 + 2\left\| \sup_{t \in [0,T]} |W_t| \right\|_{2p}\right).$$

Similar bound holds when $p \in (0, 1)$.

Now, by Lemma 7.5 devoted to the L^p-regularity of Itô processes, one derives the existence of a real constant $K_{b,\sigma,p,T} \in (0,\infty)$ (not depending on $n \geq 1$) such that

$$\forall t \in [0,T], \ \forall n \geq 1, \quad \|\widetilde{X}_t^n - \widetilde{X}_{\underline{t}}^n\|_p \leq K_{b,\sigma,p,T}\left(1 + \|X_0\|_p\right)\left(\frac{T}{n}\right)^{\frac{1}{2}}. \tag{7.39}$$

STEP 2 (*Decomposition and analysis of the error, $p \in [2,+\infty)$, $X_0 = x \in \mathbb{R}^d$*): Set $\varepsilon_t := X_t - \widetilde{X}_t$, $t \in [0,T]$, and, for every $p \in [1,+\infty)$,

$$f(t) := \|\sup_{s \in [0,t]} |\varepsilon_s|\|_p, \quad t \in [0,T].$$

Using the diffusion equation and the continuous Milstein scheme one gets

$$\begin{aligned}
\varepsilon_t &= \int_0^t (b(X_s) - b(\widetilde{X}_{\underline{s}}))ds + \int_0^t (\sigma(X_s) - \sigma(\widetilde{X}_{\underline{s}}))dW_s - \int_0^t \int_{\underline{s}}^s (\sigma\sigma')(\widetilde{X}_{\underline{u}})dW_u dW_s \\
&= \int_0^t (b(X_s) - b(\widetilde{X}_s))ds + \int_0^t (\sigma(X_s) - \sigma(\widetilde{X}_s))dW_s \\
&\quad + \int_0^t (b(\widetilde{X}_s) - b(\widetilde{X}_{\underline{s}}))ds + \int_0^t \left(\sigma(\widetilde{X}_s) - \sigma(\widetilde{X}_{\underline{s}}) - \sigma\sigma'(\widetilde{X}_{\underline{s}})(W_s - W_{\underline{s}})\right)dW_s.
\end{aligned}$$

First one derives that

$$\begin{aligned}
\sup_{s \in [0,t]} |\varepsilon_s| &\leq \|b'\|_{\sup} \int_0^t \sup_{u \in [0,s]} |\varepsilon_u|ds + \sup_{s \in [0,t]} \left|\int_0^s (\sigma(X_u) - \sigma(\widetilde{X}_u))dW_u\right| \\
&\quad + \sup_{u \in [0,s]} \left|\int_0^u b(\widetilde{X}_u) - b(\widetilde{X}_{\underline{u}})du\right| \\
&\quad + \sup_{u \in [0,s]} \left|\int_0^u \left(\sigma(\widetilde{X}_u) - \sigma(\widetilde{X}_{\underline{u}}) - (\sigma\sigma')(\widetilde{X}_{\underline{u}})(W_u - W_{\underline{u}})\right)dW_u\right|
\end{aligned}$$

so that, using the generalized Minkowski Inequality (twice) and the *BDG* Inequality, one gets classically

$$\begin{aligned}
f(t) &\leq \|b'\|_{\sup} \int_0^t f(s)ds + C_p^{BDG}\|\sigma'\|_{\sup}\sqrt{\int_0^t f(s)^2 ds} \\
&\quad + \underbrace{\left\|\sup_{s \in [0,t]}\left|\int_0^s \left(b(\widetilde{X}_u) - b(\widetilde{X}_{\underline{u}})\right)du\right|\right\|_p}_{B} + \underbrace{\left\|\sup_{s \in [0,t]}\left|\int_0^s \left(\sigma(\widetilde{X}_u) - \sigma(\widetilde{X}_{\underline{u}}) - (\sigma\sigma')(\widetilde{X}_{\underline{u}})(W_u - W_{\underline{u}})\right)dW_u\right|\right\|_p}_{C}.
\end{aligned}$$

Now using that b' is $\alpha_{b'}$-Hölder yields for every $u \in [0,T]$,

$$\begin{aligned}
b(\widetilde{X}_u) - b(\widetilde{X}_{\underline{u}}) &= b'(\widetilde{X}_{\underline{u}})(\widetilde{X}_u - \widetilde{X}_{\underline{u}}) + \rho_b(u)|\widetilde{X}_u - \widetilde{X}_{\underline{u}}|^{1+\alpha_{b'}} \\
&= bb'(\widetilde{X}_{\underline{u}})(u - \underline{u}) + b'(\widetilde{X}_{\underline{u}})\int_{\underline{u}}^u H_v dW_v + \rho_b(u)|\widetilde{X}_u - \widetilde{X}_{\underline{u}}|^{1+\alpha_{b'}}
\end{aligned}$$

where $\rho_b(u)$ is defined by the above equation on the event $\{\widetilde{X}_u \neq \widetilde{X}_{\underline{u}}\}$ and is equal to 0 otherwise. This defines an $(\mathcal{F}_u)$-adapted process, bounded by the Hölder coefficient $[b']_{\alpha_{b'}}$ of b'. Using that for every $x \in \mathbb{R}$, $|bb'|(x) \leq \|b'\|_{\sup}(\|b'\|_{\sup} + |b(0)|)|x|$ and (7.39) yields

$$\begin{aligned}
B &\leq \|b'\|_{\sup}(\|b'\|_{\sup} + |b(0)|)\left\|\sup_{t \in [0,T]}|\widetilde{X}_t|\right\|_p \frac{T}{n} + [b']_{\alpha_{b'}} K_{b,\sigma,p,T}\left(1 + |x|\right)\left(\frac{T}{n}\right)^{\frac{1+\alpha_{b'}}{2}} \\
&\quad + \left\|\sup_{s \in [0,t]}\int_0^s b'(\widetilde{X}_{\underline{u}})\int_{\underline{u}}^u H_v dW_v du\right\|_p.
\end{aligned}$$

The last term in the right hand side of the above equation needs a specific treatment: a naive approach would yield a $\sqrt{\frac{T}{n}}$ term that would make the whole proof crash down. So we will transform the regular Lebesgue integral into a stochastic integral (hence a local martingale). This can be done either by a stochastic Fubini theorem, or in a more elementary way by an integration by parts.

Lemma 7.8.1 *Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-progressively measurable process such that $\int_0^T G_s^2 ds < +\infty$ a.s. For every $t \in [0,T]$,*

$$\int_0^t \left(\int_{\underline{s}}^s G_u dW_u \right) ds = \int_0^t (\bar{u} \wedge t - u) G_u dW_u$$

where $\bar{u} := \frac{kT}{n}$ if $u \in \left[\frac{(k-1)T}{n}, \frac{kT}{n} \right)$.

Proof. For every $k = 1, \dots, n$, an integration by parts yields

$$\int_{\frac{(k-1)T}{n}}^{\frac{kT}{n}} \left(\int_{\underline{s}}^s G_u dW_u \right) ds \quad = \quad \int_{\frac{(k-1)T}{n}}^{\frac{kT}{n}} \left(\int_{\frac{(k-1)T}{n}}^s G_u dW_u \right) ds$$

$$= \quad -\int_{\frac{(k-1)T}{n}}^{\frac{kT}{n}} \left(s - \frac{kT}{n} \right) G_s dW_s.$$

Likewise if $t \in \left[\frac{(\ell-1)T}{n}, \frac{\ell T}{n} \right)$, then

$$\int_{\frac{(\ell-1)T}{n}}^t \left(\int_{\underline{s}}^s G_u dW_u \right) ds = \int_{\frac{(\ell-1)T}{n}}^{\frac{\ell T}{n}} (t - s) G_s dW_s$$

which completes the proof by summing up all the terms. ◇

We apply this lemma to the continuous adapted process $G_t = b'(\widetilde{X}_{\underline{t}}) H_t$. We can derive by standard arguments

$$\left\| \sup_{s \in [0,t]} \int_0^s b'(\widetilde{X}_{\underline{u}}) \int_{\underline{u}}^u H_v dW_v du \right\|_p \leq C_p^{BDG} \left(\int_0^T \| (\bar{t} \wedge T - t) b'(\widetilde{X}_{\underline{t}}) H_t \|_p^2 dt \right)^{\frac{1}{2}}$$

$$\leq C_p^{BDG} \| b' \|_{\sup} \frac{T}{n} \left(\int_0^T \| H_t \|_p^2 dt \right)^{\frac{1}{2}}$$

$$\leq C_{b,\sigma,p,T} \left(1 + \| X_0 \|_p \right) \frac{T}{n}$$

where we used first that $0 \leq \bar{t} \wedge T - t \leq \frac{T}{n}$ and then (7.38). Finally, one gets that

$$B \leq C_{b,\sigma,p,T} \left(1 + |x| \right) \left(\frac{T}{n} \right)^{1 + \alpha_{b'}}.$$

We adopt a similar approach for C. Elementary computations show that

$$\sigma(\widetilde{X}_u) - \sigma(\widetilde{X}_{\underline{u}}) - (\sigma \sigma')(\widetilde{X}_{\underline{u}})(W_u - W_{\underline{u}}) = \sigma' b(\widetilde{X}_{\underline{u}}) \frac{T}{n} + \frac{1}{2} \sigma(\sigma')^2(\widetilde{X}_{\underline{u}}) \Big((W_u - W_{\underline{u}})^2 - (u - \underline{u}) \Big) + \rho_\sigma(s) |\widetilde{X}_u - \widetilde{X}_{\underline{u}}|^{1 + \alpha_{\sigma'}}$$

where $\rho_\sigma(s)$ is an $(\mathcal{F}_u)$-adapted process bounded by the Hölder coefficient $[\sigma']_{\alpha_{\sigma'}}$ of σ'. Consequently for every $p \geq 1$,

$$
\begin{aligned}
\left\| \sigma(\widetilde{X}_u) - \sigma(\widetilde{X}_{\underline{u}}) - (\sigma\sigma')(\widetilde{X}_{\underline{u}})(W_u - W_{\underline{u}}) \right\|_p \; &\leq \; \|\sigma'b(\widetilde{X}_{\underline{u}})\|_p (u - \underline{u}) \\
&\quad + \frac{1}{2}\|\sigma(\sigma')^2(\widetilde{X}_{\underline{u}})\|_p \|(W_u - W_{\underline{u}})^2 - (u - \underline{u})\|_p \\
&\quad + [\sigma']_{\alpha_{\sigma'}} \||\widetilde{X}_u - \widetilde{X}_{\underline{u}}|^{1+\alpha_{\sigma'}}\|_{p(1+\alpha_{\sigma'})}^{1+\alpha_{\sigma'}} \\
&\leq \; C_{b,\sigma,p,T}(1 + |x|)\big((u - \underline{u}) + \|Z^2 - 1\|_p(u - \underline{u}) + [\sigma']_{\alpha_{\sigma'}}(u - \underline{u})^{1+\alpha_{\sigma'}}\big) \\
&\leq \; C_{b,\sigma,p,T}(1 + |x|)\left(\frac{T}{n}\right)^{\frac{1+\alpha_{\sigma'}}{2}} .
\end{aligned}
$$

Now, owing to *BDG* Inequality, we derive that for every $p \geq 2$,

$$
\begin{aligned}
C \; &\leq \; C_p^{BDG}\left(\int_0^t \left\| \sigma(\widetilde{X}_u) - \sigma(\widetilde{X}_{\underline{u}}) - (\sigma\sigma')(\widetilde{X}_{\underline{u}})(W_u - W_{\underline{u}}) \right\|_p^2 du\right)^{\frac{1}{2}} \\
&\leq \; C_p^{BDG}C_{b,\sigma,p,T}(1 + |x|)\left(\frac{T}{n}\right)^{\frac{1+\alpha_{\sigma'}}{2}} .
\end{aligned}
$$

Finally combining the upper bounds for B and C leads to

$$
f(t) \leq \|b'\|_{\sup}\int_0^t f(s)ds + C_p^{BDG}\|\sigma'\|_{\sup}\sqrt{\int_0^t f^2(s)ds} + C_{b,\sigma,p,T}(1 + |x|)\left(\frac{T}{n}\right)^{\frac{1+\alpha_{\sigma'}\vee\alpha_{b'}}{2}}
$$

so that by the "à la Gronwall" Lemma there exists a real constant

$$
f(T) \leq C_{b,\sigma,p,T}(1 + |x|)\left(\frac{T}{n}\right)^{\frac{1+\alpha_{\sigma'}\vee\alpha_{b'}}{2}} .
$$

STEP 2 (*Extension to $p \in (0,2)$ and random starting values X_0*): First one uses that $p \mapsto \|\,.\,\|_p$ is non-decreasing to extend the above bound to $p \in (0,2)$. Then one uses that, if X_0 and W are independent, for any non-negative functional $\Phi : \mathcal{C}([0,T],\mathbb{R}^d)^2 \to \mathbb{R}_+$, one has with obvious notations

$$
\mathbb{E}\Phi(X,\widetilde{X}) = \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx)\,\mathbb{E}\,\Phi(X^x, \widetilde{X}^x).
$$

Applying this identity with $\Phi(x,\widetilde{x}) = \sup_{t\in[0,T]} |x(t) - \widetilde{x}(t)|^p$ completes the proof of this item.

(*b*) This second claim follows from the error bound established for the Brownian motion itself: as concerns the Brownian motion, both stepwise constant and continuous versions of the Milstein and the Euler scheme coincide. So a better convergence rate is hopeless.   ◇

7.8.9   Weak error expansion for the Euler scheme by the *PDE* method

We make the following regularity assumption on b and σ on the one hand and on the function f on the other hand:

$$
(R_\infty) \equiv \; b, \sigma \in \mathcal{C}^\infty([0,T]\times\mathbb{R}) \; \text{ and } \; \forall\, k_1, k_2 \in \mathbb{N},\ k_1 + k_2 \geq 1,\ \sup_{(t,x)\in[0,T]\times\mathbb{R}}\left|\frac{\partial^{k_1+k_2}b}{\partial t^{k_1}\partial x^{k_2}}(t,x)\right| + \left|\frac{\partial^{k_1+k_2}\sigma}{\partial t^{k_1}\partial x^{k_2}}(t,x)\right| < +\infty.
$$

In particular, b et σ are Lipschitz continuous in $(t,x) \in [0,T] \times \mathbb{R}$ since, $e.g.$, for every t, $t' \in [0,T]$, and every x, $x' \in \mathbb{R}$,

$$|b(t',x') - b(t,x)| \leq \sup_{(s,\xi) \in [0,T] \times \mathbb{R}} \left| \frac{\partial b}{\partial x}(s,\xi) \right| |x'-x| + \sup_{(s,\xi) \in [0,T] \times \mathbb{R}} \left| \frac{\partial b}{\partial t}(t,\xi) \right| |t'-t|.$$

Consequently (see $e.g.$ [27]), the SDE (7.1) always has a unique strong solution starting from any $\mathbb{R}^d$-valued random vector X_0 (independent of the Brownian motion W). Furthermore, since $|b(t,x)| + |\sigma(t,x)| \leq \sup_{t \in [0,T]} (|b(t,0)| + |\sigma(t,0)|) + C|x| \leq C(1+|x|)$, any such strong solution $(X_t)_{t \in [0,T]}$ satisfies (see Proposition 7.5):

$$\forall p \geq 1, \qquad X_0 \in L^p(\mathbb{P}) \Longrightarrow \mathbb{E}\Big(\sup_{t \in [0,T]} |X_t|^p + \sup_n \sup_{t \in [0,T]} |\bar{X}_t^n|^p \Big) < +\infty. \tag{7.40}$$

The following regularity and growth assumption is made on f.

$$(F) \equiv \quad f \in \mathcal{C}^\infty(\mathbb{R}) \quad \text{and} \quad \exists\, r \in \mathbb{N}, \ \exists\, C \in \mathbb{R}_+^*, \ |f(x)| \leq C(1+|x|^r).$$

The infinitesimal generator L of the diffusion is defined on every function $g \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ by

$$L(g)(t,x) = b(t,x)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 g}{\partial x^2}.$$

Theorem 7.11 (a) *If both Assumptions* (R_∞) *and* (F) *hold, the parabolic PDE*

$$\frac{\partial u}{\partial t} + Lu = 0, \qquad u(T,x) = f(x) \tag{7.41}$$

has a unique solution $u \in \mathcal{C}^\infty([0,T] \times \mathbb{R}, \mathbb{R})$. *This solution satisfies*

$$\forall\, k \geq 0, \quad \sup_{t \in [0,T]} \left| \frac{\partial^k u}{\partial x^k}(t,x) \right| \leq C_{k,T}(1+|x|^{r(k,T)}).$$

(b) FEYNMANN-KAC'S FORMULA: *The solution* u *admits the following representation*

$$\forall\, t \in [0,T], \quad u(t,x) = \mathbb{E}\big(f(X_T) \,|\, X_t = x \big) = \mathbb{E}\, f(X_T^{x,t})$$

where $(X_s^{x,t})_{s \in [t,T]}$ *denotes the unique solution of the SDE (7.1) starting at* x *at time* t. *If furthermore* $b(t,x) = b(x)$ *and* $\sigma(t,x) = \sigma(x)$ *then*

$$\forall\, t \in [0,T], \quad u(t,x) = \mathbb{E}\, f(X_{T-t}^x).$$

NOTATION: To alleviate notations we will use throughout this section the notations $\partial_x f$, $\partial_t f$, $\partial_{xt}^2 f$, etc, for the partial derivatives instead of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$, $\frac{\partial^2 u}{\partial x \partial t} \ldots$

▷ **Exercise.** Combining the above bound for the spatial partial derivatives with $\partial_t u = -Lu$, show that

$$\forall\, k = (k_1, k_2) \in \mathbb{N}^2, \quad \sup_{t \in [0,T]} \left| \frac{\partial^{k_1 + k_2} u}{\partial t^{k_1} \partial x^{k_2}}(t,x) \right| \leq C_{k,T}(1+|x|^{r(k,T)}).$$

Proof. (a) For this result on PDE's we refer to [].

(b) This representation is the so-called Feynmann-Kac formula. Let u be the solution to the PDE (7.41). It is smooth enough to apply Itô's formula. Let $(X_t)_{t \in [0,T]}$ be the unique (strong) solution to the SDE starting at X_0, random vector independent of the Brownian motion W on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Such a solution does exist since b and σ are Lipschitz continuous in x uniformly with respect to t and continuous in (t, x).

For every $t \in [0, T]$,

$$
\begin{aligned}
u(T, X_T) &= u(t, X_t) + \int_t^T \partial_t u(s, X_s) ds + \int_t^T \partial_x u(s, X_s) dX_s + \frac{1}{2} \int_t^T \partial^2_{x,x} u(s, X_s) d\langle X_s \rangle \\
&= u(t, X_t) + \int_t^T (\partial_t u + Lu)(s, X_s) ds + \int_t^T \partial_x u(s, X_s)\sigma(s, X_s) dW_s \\
&= u(t, X_t) + \int_t^T \partial_x u(s, X_s)\sigma(s, X_s) dW_s
\end{aligned}
$$

since u satisfies the *PDE* (7.41). Now the local martingale $M_t := \int_0^t \partial_x u(s, X_s)\sigma(s, X_s) dW_s$ is a true martingale since

$$
\langle M \rangle_t = \int_0^t (\partial_x u(s, X_s))^2 \sigma^2(s, X_s) ds \leq C\Big(1 + \sup_{t \in [0,T]} |X_t|^\theta\Big) \in L^1(\mathbb{P})
$$

for an exponent $\theta \geq 0$. The above inequality follows from the assumptions on b and σ and the induced growth properties of u. The integrability is a consequence of Proposition 7.31. Consequently $(M_t)_{t \in [0,T]}$ is a true martingale. As a consequence, using the assumption $u(T, .) = f$, one derives that

$$
\forall t \in [0, T], \qquad \mathbb{E}\big(f(X_T) \,|\, \mathcal{F}_t\big) = u(t, X_t).
$$

The announced representation follows form the Markov property satisfied by $(X_t)_{t \geq 0}$ (with respect to the filtration $\mathcal{F}_t = \sigma(X_0) \vee \mathcal{F}_t^W$, $t \geq 0$).

One shows likewise that when b and σ do not depend on t, then (weak) uniqueness of the solutions of (7.1) ([3]) implies that $(X_{t+s}^{x,t})_{s \in [0, T-t]}$ and $X_{s \in [0, T-t]}^x$ have the same distribution. $\diamond$

Theorem 7.12 *(Talay-Tubaro [151]): If (R_∞) and (F) hold, then there exists a non-decreasing function $C_{b,\sigma,f} : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$
\max_{0 \leq k \leq n} \Big| \mathbb{E}(f(\bar{X}_{\frac{kT}{n}}^{n,x})) - \mathbb{E}(f(X_{\frac{kT}{n}}^x)) \Big| \leq C_{b,\sigma,f}(T) \frac{T}{n}.
$$

Remark. The result also holds under some weaker smoothness assumptions on b, σ and f (say $\mathcal{C}_b^5$).

Proof. STEP 1 (*Representing and estimating $\mathbb{E}(f(X_T^x)) - \mathbb{E}(f(\bar{X}_T^x)))$*): To alleviate notations we temporarily drop the exponent ^{x} in X_t^x or $\bar{X}_t^{n,x}$. Noting that

$$
\mathbb{E}\, f(X_T^x) = u(0, x) \qquad \text{and} \qquad \mathbb{E}\, f(\bar{X}_T^n) = \mathbb{E}\, u(T, \bar{X}_T^n),
$$

it follows that

$$
\begin{aligned}
\mathbb{E}\big(f(\bar{X}_T^n)\big) - f(X_T) &= \mathbb{E}\big(u(T, \bar{X}_T^n) - u(0, \bar{X}_0^n)\big) \\
&= \sum_{k=1}^n \mathbb{E}\big(u(t_k, \bar{X}_{t_k}^n) - u(t_{k-1}, \bar{X}_{t_{k-1}}^n)\big).
\end{aligned}
$$

We apply Itô's formula between t_{k-1} and t_k to the function u and use that the Euler scheme satisfies the "frozen" *SDE*

$$
d\bar{X}_t^n = b(\underline{t}, \bar{X}_{\underline{t}}^n) dt + \sigma(\underline{t}, \bar{X}_{\underline{t}}^n) dW_t.
$$

---

[3] which in turns follows form the strong existence and uniqueness for this *SDE*, see *e.g.* [140].

This yields

$$
\begin{aligned}
u(t_k, \bar{X}^n_{t_k}) - u(t_{k-1}, \bar{X}^n_{t_{k-1}}) &= \int_{t_{k-1}}^{t_k} \partial_t u(s, \bar{X}^n_s)ds + \int_{t_{k-1}}^{t_k} \partial_x u(s, \bar{X}^n_s)d\bar{X}^n_s + \frac{1}{2}\int_{t_{k-1}}^{t_k} \partial_{xx} u(s, \bar{X}^n_s)d\langle \bar{X}^n \rangle_s \\
&= \int_{t_{k-1}}^{t_k} (\partial_t + \bar{L})u(s, \underline{s}, \bar{X}^n_s, \bar{X}^n_{\underline{s}})ds + \int_{t_{k-1}}^{t_k} \sigma(\underline{s}, \bar{X}^n_{\underline{s}})\partial_x u(s, \bar{X}^n_s)dW_s
\end{aligned}
$$

where $\bar{L}$ is the "frozen" infinitesimal generator defined by

$$
\bar{L}g(s, \underline{s}, x, \underline{x}) = b(\underline{s}, \underline{x})\partial_x g(s, x) + \frac{1}{2}\sigma^2(\underline{s}, \underline{x})\partial g_{xx}(s, x)
$$

(and $\partial_t u(s, \underline{s}, x, \underline{x}) = \partial_t u(s, x)$). The bracket process of the local martingale $M_t = \int_0^t \partial_x u(s, \bar{X}^n_s)\sigma(\underline{s}, \bar{X}^n_{\underline{s}})dW_s$ is given for every $t \in [0, T]$ by

$$
\langle M \rangle_T = \int_0^T (\partial_x u)^2(s, \bar{X}_s)\sigma^2(\underline{s}, \bar{X}^n_{\underline{s}})ds.
$$

Consequently

$$
\langle M \rangle_T \le C\left(1 + \sup_{t \in [0,T]}|\bar{X}^n_{\underline{t}}|^2 + \sup_{t \in [0,T]}|\bar{X}^n_t|^r\right) \in L^1(\mathbb{P})
$$

so that $(M_t)_{t \in [0,T]}$ is a true martingale. Consequently,

$$
\mathbb{E}(u(t_k, \bar{X}^n_{t_k}) - u(t_{k-1}, \bar{X}^n_{t_{k-1}})) = \mathbb{E}\left(\int_{t_{k-1}}^{t_k} (\partial_t + \bar{L})u(s, \underline{s}, \bar{X}^n_s, \bar{X}^n_{\underline{s}})ds\right)
$$

(the integrability of the integral term follows from $\partial_t u = -Lu$ which ensures the polynomial growth of $(\partial_t + \bar{L})u(s, \underline{s}, x, \underline{x})$). At this stage, the idea is to expand the above expectation into a term $\bar{\phi}(\underline{s}, \bar{X}^n_{\underline{s}})\frac{T}{n} + O(\frac{T^2}{n})$. To this end we will use again Itô's formula to $\partial_t u(s, \bar{X}^n_s)$, $\partial_x u(s, \bar{X}^n_s)$ and $\partial_{xx} u(s, \bar{X}^n_s)$, taking advantage of the regularity of u.

    – *Term 1.* The function $\partial_t u$ being $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$, Itô's formula yields between $\underline{s} = t_{k-1}$ and s

$$
\partial_t u(s, \bar{X}_s) = \partial_t u(\underline{s}, \bar{X}_{\underline{s}}) + \int_{\underline{s}}^s \left(\partial^2_{tt} u(r, \bar{X}_r) + \bar{L}(\partial_t u)(r, \bar{X}_r)\right)dr + \int_{\underline{s}}^s \sigma(\underline{r}, \bar{X}_{\underline{r}})\partial^2_{xt} u(r, X_r)dW_r.
$$

First let us show that the locale martingale term is the increment between $\underline{s}$ and s of a true martingale (denoted $(M_t^{(1)})$ from now on). Note that $\partial_t u = -L u$ so that $\partial^2_{xt} u = -\partial_x L u$ which is clearly a function with polynomial growth in x uniformly in $t \in [0, T]$ since

$$
\left|\partial_x \left(b(t, x)\partial_x u + \frac{1}{2}\sigma^2(t, x)\partial^2_{xx} u\right)(t, x)\right| \le C(1 + |x|^{\theta_0}).
$$

Consequently, $(M_t)_{t \in [0,T]}$ is a true martingale since $\mathbb{E}(\langle M \rangle_T) < +\infty$. On the other hand, using (twice) that $\partial_t u = -L u$, leads to

$$
\begin{aligned}
\partial^2_{tt} u(r, \bar{X}_r) + \bar{L}(\partial_t u)(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}}) &= -\partial_t L(u)(r, \bar{X}_r) - \bar{L} \circ Lu(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}}) \\
&=: \bar{\phi}^{(1)}(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}})
\end{aligned}
$$

where $\bar{\phi}^{(1)}$ satisfies for every $x, y \in \mathbb{R}$ and every $t, \underline{t} \in [0, T]$,

$$
|\bar{\phi}^{(1)}(t, \underline{t}, x, \underline{x})| \le C_1(1 + |x|^{\theta_1} + |\underline{x}|^{\theta_1}).
$$

This follows from the fact that $\bar{\phi}^{(1)}$ is defined as a linear combination products of b, $\partial_t b$, $\partial_x b$, $\partial_{xx} b$, σ, $\partial_t \sigma$, $\partial_x \sigma$, $\partial_{xx}\sigma$, $\partial_x u$, $\partial_{xx} u$ at (t,x) or $(t,\underline{x})$ (with "$x = \bar{X}_r$" and "$\underline{x} = \bar{X}_{\underline{r}}$").

– *Term 2.* The function $\partial_x u$ being $\mathcal{C}^{1,2}$, Itô's formula yields

$$
\partial_x u(s, \bar{X}_s) = \partial_x u(\underline{s}, \bar{X}_{\underline{s}}) + \int_{\underline{s}}^s \left(\partial_{tx}^2 u(r, \bar{X}_r) + \bar{L}(\partial_x u)(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}}) \right) dr
$$
$$
+ \int_{\underline{s}}^s \partial_{xx}^2 u(r, \bar{X}_r)\sigma(\underline{r}, \bar{X}_{\underline{r}}) dW_r
$$

The stochastic integral is the increment of a true martingale (denoted $(M_t^{(2)})$ in what follows) and using that $\partial_{tx}^2 u = \partial_x(-L u)$, one shows likewise that

$$
\partial_{tx}^2 u(r, \bar{X}_r) + \bar{L}(\partial_x u)(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}}) = \left(\bar{L}(\partial_x u) - \partial_x(Lu) \right)(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}})
$$
$$
= \bar{\phi}^{(2)}(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}})
$$

where $(t,\underline{t},x,\underline{x}) \mapsto \bar{\phi}^{(2)}(t,\underline{t},x,\underline{x})$ has a polynomial growth in $(x,\underline{x})$ uniformly in $t,\underline{t} \in [0,T]$.

– *Term 3.* Following the same lines one shows that

$$
\partial_{xx}^2 u(s, \bar{X}_s) = \partial_{xx}^2 u(\underline{s}, \bar{X}_{\underline{s}}) + \int_{\underline{s}}^s \bar{\phi}^{(3)}(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}}) dr + M_s^{(3)} - M_{\underline{s}}^{(3)}
$$

where $(M_t^{(3)})$ is a martingale and $\bar{\phi}^{(3)}$ has a polynomial growth in $(x,\underline{x})$ uniformly in $(t,\underline{t})$.

STEP 2: Collecting all the results obtained in Step 2 yields

$$
u(t_k, \bar{X}_{t_k}) - u(t_{k-1}, \bar{X}_{t_{k-1}}) = (\partial_t + L)(u)(t_{k-1}, \bar{X}_{t_{k-1}})
$$
$$
+ \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s \bar{\phi}(r, \underline{r}, \bar{X}_r, \bar{X}_{\underline{r}}) dr\, ds
$$
$$
+ \int_{t_{k-1}}^{t_k} \left(M_s^{(1)} - M_{t_{k-1}}^{(1)} + b(t_{k-1}, \bar{X}_{t_{k-1}})(M_s^{(2)} - M_{t_{k-1}}^{(2)}) + \frac{1}{2}\sigma^2(t_{k-1}, \bar{X}_{t_{k-1}})(M_s^{(3)} - M_{t_{k-1}}^{(3)}) \right) ds
$$
$$
+ M_{t_k} - M_{t_{k-1}}.
$$

where

$$
\bar{\phi}(r, \underline{r}, x, \underline{x}) = \bar{\phi}^{(1)}(r, x, \underline{x}) + b(\underline{r}, \underline{x})\bar{\phi}^{(2)}(r, x, \underline{x}) + \frac{1}{2}\sigma^2(\underline{r}, \underline{x})\bar{\phi}^{(3)}(r, x, \underline{x}).
$$

Hence the function $\bar{\phi}$ satisfies a polynomial growth assumption

$$
|\bar{\phi}(t, x, y)| \leq C_{\phi}(1 + |x|^{\theta} + |y|^{\theta'}), \qquad \theta, \theta' \in \mathbb{N}.
$$

The first term on the right hand side of the equality vanishes since $\partial_t u + Lu = 0$.

As concerns the third term, we will show that it has a zero expectation. One can use Fubini's Theorem since $\sup_{t\in[0,T]} |\bar{X}_t| \in L^p(\mathbb{P})$ for every $p > 0$ (this ensures the integrability of the integrand). Consequently

$$
\mathbb{E}\left(\int_{t_{k-1}}^{t_k} \left(M_s^{(1)} - M_{t_{k-1}}^{(1)} + b(t_{k-1}, \bar{X}_{t_{k-1}})(M_s^{(2)} - M_{t_{k-1}}^{(2)}) + \frac{1}{2}\sigma^2(t_{k-1}, \bar{X}_{t_{k-1}})(M_s^{(3)} - M_{t_{k-1}}^{(3)}) \right) ds \right) =
$$

$$
\int_{t_{k-1}}^{t_k} \mathbb{E}(M_s^{(1)} - M_{t_{k-1}}^{(1)}) + \mathbb{E}\left(b(t_{k-1}, \bar{X}_{t_{k-1}})(M_s^{(2)} - M_{t_{k-1}}^{(2)}) \right) + \frac{1}{2}\mathbb{E}\left(\sigma^2(t_{k-1}, \bar{X}_{t_{k-1}})(M_s^{(3)} - M_{t_{k-1}}^{(3)}) \right) ds.
$$

Now all the three expectations inside the integral are zero since the $M^{(k)}$ are true martingale. Thus

$$\mathbb{E}\Big(b(t_{k-1}, \bar{X}_{t_{k-1}})(M_s^{(2)} - M_{t_{k-1}}^{(2)})\Big) = \mathbb{E}\Big(\ \underbrace{b(t_{k-1}, \bar{X}_{t_{k-1}})}_{\mathcal{F}_{t_{k-1}}\text{-measurable}}\ \underbrace{\mathbb{E}(M_s^{(2)} - M_{t_{k-1}}^{(2)} \mid \mathcal{F}_{t_{k-1}})}_{=0}\Big) = 0,$$

etc. Finally the original expansion amounts to

$$\mathbb{E}\big(u(t_k, \bar{X}_{t_k}) - u(t_{k-1}, \bar{X}_{t_{k-1}})\big) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \mathbb{E}\,\bar{\phi}(r, \bar{X}_{\underline{r}}, \bar{X}_r)\,dr\,ds \qquad (7.42)$$

so that

$$
\begin{aligned}
\big|\mathbb{E}\big(u(t_k, \bar{X}_{t_k}) - u(t_{k-1}, \bar{X}_{t_{k-1}})\big)\big| &\leq \int_{t_{k-1}}^{t_k} ds \int_{t_{k-1}}^{s} dr\,\mathbb{E}\big(|\bar{\phi}(r, \bar{X}_{\underline{r}}, \bar{X}_r)|\big) \\
&\leq C_\phi\Big(1 + 2\,\mathbb{E}\big(\sup_{t\in[0,T]} |\bar{X}_t^n|^{\theta\vee\theta'}\big)\Big)\frac{(t_k - t_{k-1})^2}{2}. \\
&\leq C_{b,\sigma,f}(T)\left(\frac{T}{n}\right)^2
\end{aligned}
$$

where, owing to Proposition 6.6, the function $C_{b,\sigma,f}(.)$ only depends on T (in a non-decreasing manner). Summing over the terms for $k = 1, \ldots, n$ yields the result at time T.

STEP 3: Let $k \in \{0, \ldots, n-1\}$. It follows from the obvious equalities $\frac{T}{n} = \frac{t_k}{k}$, $k = 1, \ldots, n$, and what precedes that

$$\big|\mathbb{E}(f(\bar{X}_{t_k}^{n,x})) - \mathbb{E}(f(X_{t_k}^x))\big| \leq C_{b,\sigma,f}(t_k)\frac{t_k}{k} \leq C_{b,\sigma,f}(T)\frac{T}{n}$$

which yields the announced result.   ◇

▷ **Exercise.** Compute an explicit (closed) form for the function $\bar{\phi}$.

7.8.10 Toward higher order expansion

To obtain an expansion one must come back to the identity (7.42)

$$\mathbb{E}\big(u(t_k, \bar{X}_{t_k}^n) - u(t_{k-1}, \bar{X}_{t_{k-1}}^n)\big) = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \mathbb{E}\,\bar{\phi}(r, \underline{r}, \bar{X}_r^n, \bar{X}_{\underline{r}}^n)\,dr\,ds.$$

This function $\bar{\phi}$ can be written explicit as a polynomial of b, σ, u and (some of) its partial derivatives. So, under suitable assumptions on b and σ and f (like (R_∞) and (F)) one can show that $\bar{\phi}$ satisfies in turn

(*i*) $\bar{\phi}$ is continuous in $(t, \underline{t}, x, \underline{x})$.

(*ii*) $\left|\partial_{x^m, y^{m'}}^{m+m'} \bar{\phi}(t, \underline{t}, x, \underline{x})\right| \leq C_T(1 + |x|^{\theta(m,T)} + |\underline{x}|^{\theta'(m,T)})$ $t \in [0, T]$.

(*iii*) $\left|\partial_{\underline{t}^m}^{m} \bar{\phi}(t, \underline{t}, x, \underline{x})\right| \leq C_T(1 + |x|^{\theta(m,T)} + |\underline{x}|^{\theta'(m,T)})$ $t \in [0, T]$.

In fact, as above, a $\mathcal{C}^{1,2}$-regularity in (t,x) is sufficient to get a second order expansion. The idea is once again to apply Itô's formula, this time to $\bar{\phi}$. Let $r \in [t_{k-1}, t_k]$ (so that $\underline{r} = t_{k-1}$).

$$
\begin{aligned}
\bar{\phi}(r, \underline{r}, \bar{X}_r^n, \bar{X}_{\underline{r}}^n) &= \bar{\phi}(\underline{r}, \underline{r}, \bar{X}_{\underline{r}}^n, \bar{X}_{\underline{r}}^n) + \int_{t_{k-1}}^r \partial_x \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) d\bar{X}_v^n \\
&\quad + \int_{t_{k-1}}^r \partial_t \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) + \frac{1}{2} \partial_{xx}^2 \phi(v, r, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) \sigma^2(\underline{v}, \bar{X}_{\underline{v}}^n) \, dv \\
&= \bar{\phi}(\underline{r}, \underline{r}, \bar{X}_{\underline{r}}^n, \bar{X}_{\underline{r}}^n) + \int_{t_{k-1}}^r \Big(\partial_t \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) + \bar{L}\bar{\phi}(., \underline{r}, ., ., \bar{X}_{\underline{r}}^n)(v, \underline{v}, \bar{X}_v^n, \bar{X}_{\underline{v}}^n) \Big) dv \\
&\quad + \int_{t_{k-1}}^r \partial_x \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) \sigma(\underline{v}, \bar{X}_{\underline{v}}^n) dW_v.
\end{aligned}
$$

The stochastic integral turns out to be a true square integrable martingale (increment) on $[0, T]$ since

$$
\sup_{v,r \in [0,T]} \Big| \partial_x \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) \sigma(\underline{v}, \bar{X}_{\underline{v}}^n) \Big| \leq C(1 + \sup_{v \in [0,T]} |\bar{X}_v^n|^{\alpha_T} + \sup_{r \in [0,T]} |\bar{X}_{\underline{r}}^n|^{\beta_T}) \in L^1(\mathbb{P}).
$$

Then, using Fubini's Theorem,

$$
\begin{aligned}
\mathbb{E}\big(u(t_k, \bar{X}_{t_k}^n) - u(t_{k-1}, \bar{X}_{t_{k-1}}^n) \big) &= \mathbb{E}\Big(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s \bar{\phi}(\underline{r}, \underline{r}, \bar{X}_{\underline{r}}^n, \bar{X}_{\underline{r}}^n) \, dr \, ds \Big) \\
&\quad + \mathbb{E} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s \int_{t_{k-1}}^r \Big(\partial_t \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) + \bar{L}\bar{\phi}(., \underline{r}, ., ., \bar{X}_{\underline{r}}^n)(v, \underline{v}, \bar{X}_v^n, \bar{X}_{\underline{v}}^n) \Big) dv \, dr \, ds \\
&\quad + \underbrace{\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s \mathbb{E}(M_r - M_{t_{k-1}}) dr \, ds}_{=0}.
\end{aligned}
$$

Now,

$$
\mathbb{E} \left(\sup_{v,r \in [0,T]} |\bar{L}\bar{\phi}(., \underline{r}, ., ., \bar{X}_v^n)(v, \underline{v}, \bar{X}_v^n, \bar{X}_{\underline{v}}^n)| \right) < +\infty
$$

owing to (7.40) and the polynomial growth of b, σ, u and its partial derivatives. The same holds for $\partial_{\underline{t}} \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n)$ so that

$$
\left| \mathbb{E} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s \int_{t_{k-1}}^r \Big(\partial_{\underline{t}} \bar{\phi}(v, \underline{r}, \bar{X}_v^n, \bar{X}_{\underline{r}}^n) + \bar{L}\bar{\phi}(., \underline{r}, ., ., \bar{X}_{\underline{r}}^n)(v, \underline{v}, \bar{X}_v^n, \bar{X}_{\underline{v}}^n) \Big) dv \, dr \, ds \right) \right| \leq C_{b,\sigma,f,T} \frac{1}{3} \left(\frac{T}{n} \right)^3.
$$

Summing up from $k = 1$ up to n yields

$$
\begin{aligned}
\mathbb{E}(f(\bar{X}_T^n) - f(X_T)) &= \sum_{k=1}^n \mathbb{E}\big(u(t_k, \bar{X}_{t_k}^n) - u(t_{k-1}, \bar{X}_{t_{k-1}}^n) \big) \\
&= \frac{T}{2n} \mathbb{E}\Big(\int_0^T \bar{\phi}(\underline{s}, \underline{s}, \bar{X}_{\underline{s}}^n, \bar{X}_{\underline{s}}^n) ds \Big) + O\left(\left(\frac{T}{n}\right)^2 \right) \\
&= \frac{T}{2n} \mathbb{E}\Big(\int_0^T \bar{\psi}(\underline{s}, \bar{X}_{\underline{s}}^n) ds \Big) + O\left(\left(\frac{T}{n}\right)^2 \right)
\end{aligned}
$$

where $\bar{\psi}(t,x) := \bar{\phi}(t,t,x,x)$ is at least a $\mathcal{C}^{1,2}$-function. In turn, for every $k \in \{0, \ldots, n\}$, the function $\psi(t_k, .)$ satisfies Assumption (F) with some uniform bounds in k (this follows from the fact that its time partial derivative does exist continuously on $[0,T]$). Consequently

$$
\max_{0 \leq k \leq n} \big| \mathbb{E}(\bar{\psi}(t_k, \bar{X}_{t_k}^{n,x})) - \mathbb{E}(\bar{\psi}(t_k, X_{t_k}^x)) \big| \leq C_{b,\sigma,f}'(T) \frac{T}{n}
$$

so that

$$\left| \mathbb{E}\left(\int_0^T \bar{\psi}(\underline{s}, \bar{X}_{\underline{s}}^n) ds - \int_0^T \bar{\psi}(\underline{s}, X_{\underline{s}}) ds \right) \right| = \left| \int_0^T \left(\mathbb{E}\,\bar{\psi}(\underline{s}, \bar{X}_{\underline{s}}^n) - \mathbb{E}\,\bar{\psi}(\underline{s}, X_{\underline{s}}) \right) ds \right|$$

$$\leq TC'_{b,\sigma,f} \frac{T}{n}.$$

Applying Itô's formula to $\bar{\psi}(u, X_u)$ between $\underline{s}$ and s shows that

$$\bar{\psi}(s, X_s) = \bar{\psi}(\underline{s}, X_{\underline{s}}) + \int_{\underline{s}}^s (\partial_t + L)(\bar{\psi})(r, X_r) dr + \int_{\underline{s}}^s \partial_x \bar{\psi}(r, X_r)\sigma(X_r) dW_r$$

which implies

$$\sup_{s \in [0,T]} |\mathbb{E}\,\bar{\psi}(s, X_s) - \mathbb{E}\,\bar{\psi}(\underline{s}, X_{\underline{s}})| \leq C''_{f,b,\sigma,T} \frac{T}{n}.$$

Hence

$$\left| \mathbb{E}\left(\int_0^T \bar{\psi}(\underline{s}, X_{\underline{s}}) ds \right) - \mathbb{E}\left(\int_0^T \bar{\psi}(s, X_s) ds \right) \right| \leq C''_{b,\sigma,f} \frac{T^2}{n}.$$

Finally, combining all these bounds yields

$$\mathbb{E}(f(\bar{X}_T^{n,x}) - f(X_T^x)) = \frac{T}{2n} \int_0^T \mathbb{E}\,\bar{\psi}(s, X_s^x)\, ds + O\left(\frac{1}{n^2} \right)$$

which completes the proof.   ◇

Remark. One can make $\bar{\psi}$ explicit and by induction we eventually obtain the following Theorem.

Theorem 7.13 *(Talay-Tubaro, 1989) If both (F) and (R_∞) holds then the weak error can be expanded at any order, that is*

$$\forall R \in \mathbb{N}^*, \quad \mathbb{E}(f(\bar{X}_T^n)) - \mathbb{E}(f(X_T)) = \sum_{k=1}^R \frac{a_k}{n^k} + O\left(\frac{1}{n^{R+1}} \right).$$

The main application is of course Richardson-Romberg extrapolation developed in Section 7.7. For more recent results on weak errors we refer to [69].

Remark. What happens when f is no longer smooth? In fact the result still holds provided the diffusion coefficient satisfies a uniform ellipticity condition (or at least a uniform Hormander hypo-ellipticity assumption). This is the purpose of Bally-Talay's Theorem, see [9].

Chapter 8

The diffusion bridge method : application to the pricing of path-dependent options (II)

8.1 Theoretical results about time discretization and path-dependent payoffs

In this section we deal with some "path-dependent" (European) options. Such contracts are characterized by the fact that their payoffs depend on the whole past of the underlying asset(s) between the origin $t = 0$ of the contract and its maturity T. This means that these payoffs are of the form $F((X_t)_{t \in [0,T]})$ where F is a functional usually naturally defined from $\mathbb{D}([0, T], \mathbb{R}^d) \to \mathbb{R}_+$ and $X = (X_t)_{t \in [0,T]}$ stands for the dynamics of the underlying asset. We will assume from now on that $X = (X_t^x)_{t \in [0,T]}$ is still a solution to an SDE of type (7.1), *i.e.*:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x.$$

In the recent past years, several papers provided weak rates of convergence for some families of functionals F. These works were essentially motivated by the pricing of path-dependent (European) options, like Asian or Lookback options in 1-dimension, corresponding to functionals defined for avery $\alpha \in \mathbb{D}([0, T], \mathbb{R}^d)$ by

$$F(\alpha) := f\left(\int_0^T \alpha(s)ds\right), \quad F(x) := f\left(\alpha(T), \sup_{t \in [0,T]} \alpha(t), \inf_{t \in [0,T]} \alpha(t)\right)$$

and, this time possibly in higher dimension, to barrier options with functionals of the form

$$F(\alpha) = f(\alpha(T))\mathbf{1}_{\{\tau_D(\alpha) > T\}}$$

where D is an (open) domain of $\mathbb{R}^d$ and $\tau_D(\alpha) := \inf\{s \in [0, T], \alpha(s) \text{ or } \alpha(s-) \notin D\}$ is the escape time from D by the generic càdlàg path α [1]. In both frameworks f is usually Lipschitz continuous

---

[1] when α is continuous or stepwise constant and càdlàg, $\tau_D(\alpha) := \inf\{s \in [0, T], \alpha(s) \notin D\}$.

continuous. Let us quote two well-known examples of results (in a homogeneous framework *i.e.* $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$):

– In [63] is established the following theorem.

Theorem 8.1 (a) *If the domain D is bounded and has a smooth enough boundary (in fact $\mathcal{C}^3$), if $b \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathcal{C}^3(\mathbb{R}^d, \mathcal{M}(d,q,\mathbb{R}))$, σ uniformly elliptic on D (i.e. $\sigma\sigma^*(x) \geq \varepsilon_0 I_d$, $\varepsilon_0 > 0$), then for every bounded Borel function f vanishing in a neighbourhood of ∂D,*

$$\mathbb{E}\Big(f(X_T)\mathbf{1}_{\{\tau_D(X)>T\}}\Big) - \mathbb{E}\Big(f(\widetilde{X}^n_T)\mathbf{1}_{\{\tau_D(\widetilde{X}^n)>T\}}\Big) = O\left(\frac{1}{\sqrt{n}}\right) \quad as \quad n \to +\infty. \qquad (8.1)$$

(where $\widetilde{X}$ denotes the stepwise constant Euler scheme).

(b) *If furthermore, b and σ are $\mathcal{C}^5$ and D is half-space, then the* continuous Euler scheme *satisfies*

$$\mathbb{E}\Big(f(X_T)\mathbf{1}_{\{\tau_D(X)>T\}}\Big) - \mathbb{E}\Big(f(\bar{X}^n_T)\mathbf{1}_{\{\tau_D(\bar{X}^n)>T\}}\Big) = O\left(\frac{1}{n}\right) \quad as \quad n \to +\infty. \qquad (8.2)$$

Note however that these assumptions are not satisfied by usual barrier options (see below).

– It is suggested in [149] (including a rigorous proof when $X = W$) that if b, $\sigma \in \mathcal{C}^4_b(\mathbb{R}, \mathbb{R})$, σ is uniformly elliptic and $f \in \mathcal{C}^{4,2}_{pol}(\mathbb{R}^2)$ (existing partial derivatives with polynomial growth), then

$$\mathbb{E}\Big(f(X_T, \max_{t\in[0,T]} X_t)\Big) - \mathbb{E}\Big(f(\widetilde{X}^n_T, \max_{0\leq k\leq n} \widetilde{X}^n_{t_k})\Big) = O\left(\frac{1}{\sqrt{n}}\right) \quad as \quad n \to +\infty. \qquad (8.3)$$

A similar improvement – $O(\frac{1}{n})$ rate – as above can be expected (but still holds a conjecture) when replacing $\widetilde{X}$ by the continuous Euler scheme $\bar{X}$, namely

$$\mathbb{E}\Big(f(X_T, \max_{t\in[0,T]} X_t)\Big) - \mathbb{E}\Big(f(\bar{X}^n_T, \max_{t\in[0,T]} \bar{X}^n_{t_k})\Big) = O\left(\frac{1}{n}\right) \quad as \quad n \to +\infty.$$

If we forget about the regularity assumptions, the formal intersection between these two classes of path-dependent options is not empty since the payoff a barrier option with domain $D = (-\infty, L)$ can be written

$$f(X_T)\mathbf{1}_{\{\tau_D(X)>T\}} = g\Big(X_T, \sup_{t\in[0,T]} X_t\Big) \quad with \quad g(x,y) = f(x)\mathbf{1}_{\{y<L\}}.$$

Unfortunately, such a function g is never a smooth function so that if the second result is true it does not solve the first one.

By contrast with the "vanilla case", these results are somewhat disappointing since they point out that the weak error obtained with the stepwise constant Euler scheme is not significantly better than the strong error (the only gain is a $\sqrt{1 + \log n}$ factor). The positive part is that we can reasonably hope that using the continuous Euler scheme yield again the $O(1/n)$-rate and the expansion of the time discretization error, provided we know how to simulate this scheme.

8.2 From Brownian to diffusion bridge: how to simulate functionals of the genuine Euler scheme

To take advantage of the above rates, in fact we do not need to simulate the continuous Euler scheme itself (which is meaningless) but some specific functionals of this scheme like the maximum, the minimum, the time average, etc. between two time discretization instants t_k^n and t_{k+1}^n, *given the (simulated) values of the stepwise constant Euler scheme.* First we deal with the standard Brownian motion itself.

8.2.1 The Brownian bridge method

We denote by $(\mathcal{F}_t^W)_{t \geq 0}$ the (completed) natural filtration of a standard Brownian motion W. Keep in mind that, owing to the Kolmogorov 0-1 law, this filtration coincides with the "naive" natural filtration of W up to the negligible sets of the probability space on which W is defined.

Proposition 8.1 *Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion.*

(a) Let $T > 0$. Then, the so-called standard Brownian bridge on $[0, T]$ *defined by*

$$Y_t^{W,T} := W_t - \frac{t}{T} W_T, \quad t \in [0, T], \tag{8.4}$$

is an $\mathcal{F}_T^W$-measurable centered Gaussian process, independent of $(W_{T+s})_{s \geq 0}$ characterized by its covariance structure

$$\mathbb{E}(Y_s^{W,T} Y_t^{W,T}) = s \wedge t - \frac{st}{T} = \frac{(s \wedge t)(T - s \vee t)}{T}, \qquad 0 \leq s, t \leq T.$$

(b) Let $0 \leq T_0 < T_1 < +\infty$. Then

$$\mathcal{L}\left((W_t)_{t \in [T_0,T_1]} \mid W_s, \, s \notin (T_0, T_1)\right) = \mathcal{L}\left((W_t)_{t \in [T_0,T_1]} \mid W_{T_0}, W_{T_1}\right)$$

so that $(W_t)_{t \in [T_0,T_1]}$ and $(W_s)_{s \notin (T_0,T_1)}$ are independent given (W_{T_0}, W_{T_1}) and this conditional distribution is given by

$$\mathcal{L}\left((W_t)_{t \in [T_0,T_1]} \mid W_{T_0} = x, W_{T_1} = y\right) = \mathcal{L}\left(x + \frac{t - T_0}{T_1 - T_0}(y - x) + (Y_{t-T_0}^{B,T_1-T_0})_{t \in [T_0,T_1]}\right)$$

where B is a generic standard Brownian motion.

Proof. *(a)* The process $Y^{W,T}$ is clearly centered since W is. Elementary computations based on the covariance structure of the standard Brownian Motion

$$\mathbb{E}\, W_s W_t = s \wedge t, \qquad s, t \in [0, T]$$

show that, for every $s, t \in [0, T]$,

$$
\begin{aligned}
\mathbb{E}\,(Y_t^{W,T} Y_s^{W,T}) &= \mathbb{E}\, W_t W_s - \frac{s}{T}\mathbb{E}\, W_t W_T - \frac{t}{T}\mathbb{E}\, W_s W_T + \frac{st}{T^2}\mathbb{E} W_T^2 \\
&= s \wedge t - \frac{st}{T} - \frac{ts}{T} + \frac{ts}{T} \\
&= s \wedge t - \frac{ts}{T} \\
&= \frac{(s \wedge t)(T - s \vee t)}{T}.
\end{aligned}
$$

Likewise one shows that, for every $u \geq T$, $\mathbb{E}(Y_t^{W,T} W_u) = 0$ so that $Y_t^{W,T} \perp \mathrm{span}(W_u, \, u \geq T)$. Now W being a Gaussian process independence and no correlation coincide in the closed subspace of $L^2(\Omega, \mathcal{A}, \mathbb{P})$ spanned by W. The process $Y^{W,T}$ belongs to this space by construction. Consequently $Y^{W,T}$ is independent of $(W_{T+u})_{u \geq 0}$.

(b) First note that for every $t \in [T_0, T_1]$,

$$W_t = W_{T_0} + W_{t-T_0}^{(T_0)}$$

where $W_t^{(T_0)} = W_{T_0+t} - W_{T_0}$, $t \geq 0$, is a standard Brownian motion, independent of $\mathcal{F}_{T_0}^W$. Rewriting (8.4) for $W^{(T_0)}$ leads to

$$W_s^{(T_0)} = \frac{s}{T_1 - T_0} W_{T_1-T_0}^{(T_0)} + Y_s^{W^{(T_0)},T_-T_0}.$$

Plugging this identity in the above equality (at time $s = t - T_0$) leads to

$$W_t = W_{T_0} + \frac{t - T_0}{T_1 - T_0}(W_{T_1} - W_{T_0}) + Y_{t-T_0}^{W^{(T_0)},T_1-T_0}. \tag{8.5}$$

It follows from (a) that the process $\widetilde{Y} := (Y_{t-T_0}^{W^{(T_0)},T_1-T_0})_{t \in [T_0,T_1]}$ is a Gaussian process measurable with respect to $\mathcal{F}_{T_1-T_0}^{W^{(T_0)}}$ by (a), hence it is independent of $\mathcal{F}_{T_0}^W$ since $W^{(T_0)}$ is. Consequently $\widetilde{Y} := (Y_{t-T_0}^{W^{(T_0)},T_1-T_0})_{t \in [T_0,T_1]}$ is independent of $(W_t)_{t \in [0,T_0]}$. Furthermore $Y := (Y_{t-T_0}^{W^{(T_0)},T_1-T_0})_{t \in [T_0,T_1]}$ is independent of $(W_{T_1-T_0+u}^{(T_0)})_{u \geq 0}$ by (a) *i.e.* it is independent of W_{T_1+u}, $u \geq 0$ since $W_{T_1+u} = W_{T_1-T_0+u}^{(T_0)} + W_{T_0}$.

Finally the same argument (no correlation implies independence in the Gaussian space spanned by W) implies the independence of $\widetilde{Y}$ with $\sigma(W_s, \, s \notin (T_0, T_1))$. The end of the proof follows from the above identity (8.5) and the exercises below. $\diamond$

▷ **Exercises. 1.** Let X, Y, Z be three random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in $\mathbb{R}^{k_X}$, $\mathbb{R}^{k_Y}$ and $\mathbb{R}^{k_Z}$ respectively. Assume that Y and (X, Z) are independent. Show, using (8.5) that for every bounded Borel function $f : \mathbb{R}^{k_Y} \to \mathbb{R}$ and every Borel function $g : \mathbb{R}^{k_X} \to \mathbb{R}^{k_Y}$,

$$\mathbb{E}(f(g(X) + Y) \,|\, (X, Z)) = \mathbb{E}(f(g(X) + Y) \,|\, X).$$

Deduce that $\mathcal{L}(g(X) + Y \,|\, (X, Z)) = \mathcal{L}(g(X) + Y \,|\, X)$.

2. Deduce from the previous exercise that

$$\mathcal{L}\big((W_t)_{t \in [T_0,T_1]}) \,|\, (W_t)_{t \notin (T_0,T_1)}\big) = \mathcal{L}((W_t)_{t \in [T_0,T_1]}) \,|\, (W_{T_0}, W_{T_1})\big).$$

[Hint: consider the finite dimensional marginal distributions $(W_{t_1}, \ldots, W_{t_n})$ given $(X, W_{s_1}, \ldots, W_{s_p})$ where $t_i \in [T_0, T_1]$, $i = 1, \ldots, n$, $s_j \in [T_0, T_1]^c$, $j = 1, \ldots, p$, and $X = (W_{T_0}, W_{T_1})$ and use the decomposition (8.5).]

3. The conditional distribution of $(W_t)_{t \in [T_0,T_1]}$ given W_{T_0}, W_{T_1} is that of a Gaussian process hence it can also be characterized by its expectation and covariance structure. Show that they are given respectively by

$$\mathbb{E}(W_t \,|\, W_{T_0} = x, W_{T_1} = y) = \frac{T_1 - t}{T_1 - T_0} x + \frac{t - T_0}{T_1 - T_0} y, \quad t \in [T_0, T_1],$$

and

$$\mathrm{Cov}\left(W_t, W_s \mid W_{T_0} = x, W_{T_1} = y\right) = \frac{(T_1 - t)(s - T_0)}{T_1 - T_0}, \quad s \leq t, \ s, t \in [T_0, T_1].$$

8.2.2 The diffusion bridge (bridge of the Euler scheme)

Now we come back to the (continuous) Euler scheme.

Proposition 8.2 (Bridge of the Euler scheme) *Assume that $\sigma(t, x) \neq 0$ for every $t \in [0, T]$, $x \in \mathbb{R}$.*

(a) The processes $(\bar{X}_t)_{t \in [t_k^n, t_{k+1}^n]}$, $k = 0, \ldots, n-1$, are conditionally independent given the σ-field $\sigma(\bar{X}_{t_k^n}, k = 0, \ldots, n)$.

(b) Furthermore, the conditional distribution

$$\mathcal{L}\left((\bar{X}_t)_{t \in [t_k^n, t_{k+1}^n]} \mid \bar{X}_{t_k^n} = x_k, \bar{X}_{t_{k+1}^n} = x_{k+1}\right) = \mathcal{L}\left(\left(x_k + \frac{t - t_k^n}{t_{k+1}^n - t_k^n}(x_{k+1} - x_k) + \sigma(t_k^n, x_k) Y_{t - t_k^n}^{B, T/n}\right)_{t \in [t_k^n, t_{k+1}^n]}\right)$$

where $(Y_s^{B, T/n})_{s \in [0, T/n]}$ is a Brownian bridge (related to a generic Brownian motion B) as defined by (8.4). The distribution of this Gaussian process (sometimes called a diffusion bridge) is entirely characterized by:

    – its expectation function

$$\left(x_k + \frac{t - t_k^n}{t_{k+1}^n - t_k^n}(x_{k+1} - x_k)\right)_{t \in [t_k^n, t_{k+1}^n]}$$

and

    – its covariance operator

$$\sigma^2(t_k^n, x_k) \frac{(s \wedge t - t_k^n)(t_{k+1}^n - s \vee t)}{t_{k+1}^n - t_k^n}.$$

Proof. Elementary computations show that for every $t \in [t_k^n, t_{k+1}^n]$,

$$\bar{X}_t = \bar{X}_{t_k^n} + \frac{t - t_k^n}{t_{k+1}^n - t_k^n}(\bar{X}_{t_{k+1}^n} - \bar{X}_{t_k^n}) + \sigma(t_k^n, \bar{X}_{t_k^n}) Y_{t - t_k^n}^{W^{(t_k^n)}, T/n}$$

(with in mind that $t_{k+1}^n - t_k^n = T/n$). Consequently the conditional independence claim will follow if the processes $(Y_t^{W^{(t_k^n)}, T/n})_{t \in [0, T/n]}$, $k = 0, \ldots, n-1$, are independent given $\sigma(\bar{X}_{t_\ell^n}, \ell = 0, \ldots, n)$. Now, it follows from the assumption on σ that

$$\sigma(\bar{X}_{t_\ell^n}, \ell = 0, \ldots, n) = \sigma(X_0, W_{t_\ell^n}, \ell = 1, \ldots, n).$$

So we have to establish the conditional independence of the processes $(Y_t^{W^{(t_k^n)}, T/n})_{t \in [0, T/n]}$, $k = 0, \ldots, n-1$, given $\sigma(X_0, W_{t_k^n}, k = 1, \ldots, n)$ or equivalently given $\sigma(W_{t_k^n}, k = 1, \ldots, n)$ since X_0

and W are independent (note all the above bridges are $\mathcal{F}_T^W$-measurable). First note that all the bridges $(Y_t^{W^{(t_k^n)}, T/n})_{t \in [0, T/n]}$, $k = 0, \ldots, n-1$ and W live in a Gaussian space. We know from Proposition 8.1(a) that each bridge $(Y_t^{W^{(t_k^n)}, T/n})_{t \in [0, T/n]}$ is independent of both $\mathcal{F}_{t_k^n}^W$ and $\sigma(W_{t_{k+1}^n + s} - W_{t_k^n}, s \geq 0)$ hence in particular of all $\sigma(\{W_{t_\ell^n}, \ell = 1, \ldots, n\})$ (we use here a specificity of Gaussian processes). On the other hand, all bridges are independent since they are built from independent Brownian motions $(W_t^{(t_k^n)})_{t \in [0, T/n]}$. Hence, the bridges $(Y_t^{W^{(t_k^n)}, T/n})_{t \in [0, T/n]}$, $k = 0, \ldots, n-1$ are i.i.d. and independent of $\sigma(W_{t_k^n}, k = 1, \ldots, n)$.

Now $\bar{X}_{t_k^n}$ is $\sigma(\{W_{t_\ell^n}, \ell = 1, \ldots, k\})$-measurable consequently $\sigma(X_{t_k^n}, W_{t_k^n}, \bar{X}_{t_{k+1}^n}) \subset \sigma(\{W_{t_\ell^n}, \ell = 1, \ldots, n\})$ so that $(Y_t^{W^{(t_k^n)}, T/n})_{t \in [0, T/n]}$ is independent of $(X_{t_k^n}, W_{t_k^n}, \bar{X}_{t_{k+1}^n})$. The conclusion follows. $\diamond$

Now we know the distribution of the continuous Euler scheme between two successive discretization t_k^n and t_{k+1}^n times conditionally to the Euler scheme at its discretization times. Now, we are in position to simulate some functionals of the continuous Euler scheme, namely its supremum.

Proposition 8.3 *The distribution of the supremum of the Brownian bridge starting at 0 and arriving at y at time T, defined by $Y_t^{W,T,y} = \frac{t}{T}y + W_t - \frac{t}{T}W_T$ on $[0,T]$ is given by*

$$
\mathbb{P}\Big(\sup_{t \in [0,T]} Y_t^{W,T,y} \leq z \Big) = \begin{cases} 1 - \exp\left(-\frac{2}{T}z(z-y)\right) & \text{if } z \geq \max(y,0), \\[2mm] = 0 & \text{if } z \leq \max(y,0). \end{cases}
$$

Proof. The key is to have in mind that the distribution of $Y^{W,T,y}$ is that of the conditional distribution of W given $W_T = y$. So, we can derive the result from an expression of the joint distribution of $(\sup_{t \in [0,T]} W_t, W_T)$, *e.g.* from

$$
\mathbb{P}\Big(\sup_{t \in [0,T]} W_t \geq z, \, W_T \leq y \Big).
$$

It is well-known from the symmetry principle that, for every $z \geq \max(y,0)$,

$$
\mathbb{P}\Big(\sup_{t \in [0,T]} W_t \geq z, \, W_T \leq y \Big) = \mathbb{P}(W_T \geq 2z - y).
$$

We briefly reproduce the proof for the reader's convenience. One introduces the hitting time $\tau_z := \inf\{s > 0 \,|\, W_s = z\}$. This random time is in fact a stopping time with respect to the filtration $(\mathcal{F}_t^W)$ of the Brownian motion W since it is the hitting time of the closed set $[z, +\infty)$ by the continuous process W (this uses that $z \geq 0$). Furthermore, τ_z is *a.s.* finite since $\limsup_t W_t = +\infty$ *a.s.* Consequently, still by continuity of its paths, $W_{\tau_z} = z$ *a.s.* and $W_{\tau_z + t} - W_{\tau_z}$ is independent of $\mathcal{F}_{\tau_z}^W$. As a consequence, for every $z \geq \max(y,0)$,

$$
\begin{aligned}
\mathbb{P}\Big(\sup_{t \in [0,T]} W_t \geq z, \, W_T \leq y \Big) &= \mathbb{P}(\tau_z \leq T, W_T - W_{\tau_z} \leq y - z) \\
&= \mathbb{P}(\tau_z \leq T, -(W_T - W_{\tau_z}) \leq y - z) \\
&= \mathbb{P}(\tau_z \leq T, W_T \geq 2z - y) \\
&= \mathbb{P}(W_T \geq 2z - y)
\end{aligned}
$$

since $2z - y \geq z$. Consequently, one may write for every $z \geq \max(y, 0)$,

$$\mathbb{P}\left(\sup_{t\in[0,T]} W_t \geq z, \, W_T \leq y\right) \;\; = \;\; \int_{2z-y}^{+\infty} h(\xi)d\xi$$

where $h(\xi) = \dfrac{e^{-\frac{\xi^2}{2T}}}{\sqrt{2\pi T}}$. Hence, since the involved functions are differentiable one has

$$\begin{aligned}
\mathbb{P}(\sup_{t\in[0,T]} W_t \geq z \,|\, W_T = y) &= \lim_{\eta \to 0} \frac{\Big(\mathbb{P}(W_T \geq 2z - (y+\eta)) - \mathbb{P}(W_T \geq 2z - y)\Big)/\eta}{\Big(\mathbb{P}(W_T \leq y + \eta) - \mathbb{P}(W_T \leq y)\Big)/\eta} \\[2mm]
&= \frac{\frac{\partial \mathbb{P}(W_T \geq 2z - y)}{\partial y}}{\frac{\partial \mathbb{P}(W_T \leq y)}{\partial y}} = \frac{h(2z-y)}{h(y)} \\[2mm]
&= e^{-\frac{(2z-y)^2 - y^2}{2T}} = e^{-\frac{2z(z-y)}{2T}}. \qquad \diamond
\end{aligned}$$

Corollary 8.1 *Let $\lambda > 0$ and let $x, y \in \mathbb{R}$. If $Y^{W,T}$ denotes the standard Brownian bridge related to W between 0 and T, then for every*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left(x + (y-x)\frac{t}{T} + \lambda Y_t^{W,T}\right) \leq z\right) = \begin{cases} 1 - \exp\left(-\frac{2}{T\lambda^2}(z-x)(z-y)\right) & \text{if } z \geq \max(x,y) \\[3mm] 0 & \text{if } z < \max(x,y). \end{cases}$$

$$(8.6)$$

Proof. First note that

$$x + (y-x)\frac{t}{T} + \lambda Y_t^{W,T} = \lambda x' + \lambda\left(y'\frac{t}{T} + Y_t^{W,T}\right) \quad \text{with } x' = x/\lambda, \; y' = (y-x)/\lambda.$$

Then one concludes by the previous proposition, using that for any real-valued random variable ξ, every $\alpha \in \mathbb{R}$, and every $\beta \in (0, \infty)$,

$$\mathbb{P}(\alpha + \beta\,\xi \leq z) = \mathbb{P}\left(\xi \leq \frac{z-\alpha}{\beta}\right) = 1 - \mathbb{P}\left(\xi > \frac{z-\alpha}{\beta}\right). \qquad \diamond$$

▷ **Exercise.** Show using $-W \overset{d}{=} W$ that

$$\mathbb{P}\left(\inf_{t\in[0,T]}(x + (y-x)\frac{t}{T} + \lambda Y_t^{W,T}) \leq z\right) = \begin{cases} \exp\left(-\frac{2}{T\lambda^2}(z-x)(z-y)\right) & \text{if } z \leq \min(x,y) \\[3mm] 1 & \text{if } z > \min(x,y). \end{cases}$$

8.2.3   Application to Lookback like path dependent options

In this section, we focus on Lookback like options (including general barrier options) *i.e.* exotic options related to payoffs of the form $h_T := f\big(X_T, \sup_{t\in[0,T]} X_t\big)$.

We want to compute an approximation of $e^{-rT}\mathbb{E}\,h_T$ using a Monte Carlo simulation based on the continuous time Euler scheme *i.e.* we want to compute $e^{-rT}\mathbb{E}\,f\big(\bar{X}^n_T, \sup_{t\in[0,T]}\bar{X}^n_t\big)$. We first note, owing to the chaining rule for conditional expectation, that

The key is to simulate efficiently the couple $\big(\bar{X}^n_T, \sup_{t\in[0,T]}\bar{X}_t\big)$. We know that

$$
e^{-rT}\mathbb{E}\,f\Big(\bar{X}^n_T, \sup_{t\in[0,T]}\bar{X}^n_t\Big) = e^{-rT}\mathbb{E}\,f\Big(\bar{X}^n_T, \max_{0\le k\le n-1} M^{n,k}_{\bar{X}^n_{t^n_k},\bar{X}^n_{t^n_{k+1}}}\Big)
$$

where

$$
M^{n,k}_{x,y} := \sup_{t\in[0,T/n]}\Big(x + \frac{nt}{T}(y-x) + \sigma(t^n_k,x)Y^{W^{(t^n_k)},T/n}_t\Big).
$$

It follows from Proposition 8.3 that the random variables $M^{n,k}_{\bar{X}^n_{t^n_k},\bar{X}^n_{t^n_{k+1}}}$, $k=0,\ldots,n-1$, are conditionally independent given $\bar{X}_{t^n_k}$, $k=0,\ldots,n$. Following Corollary 8.1, the distribution function $G^{n,k}_{x,y}$ of $M^{n,k}_{x,y}$ is given by

$$
G^{n,k}_{x,y}(z) = \Big[1 - \exp\Big(-\frac{2n}{T\sigma^2(t^n_k,x)}(z-x)(z-y)\Big)\Big]\mathbf{1}_{\{z\ge\max(x,y)\}}, \quad z\in\mathbb{R}.
$$

Then, the inverse distribution simulation rule yields that

$$
\sup_{t\in[0,T/n]}\Big(x + \frac{t}{\frac{T}{n}}(y-x) + \sigma(t^n_k,x)Y^{W^{(t^n_k)},T/n}_t\Big) \;\overset{d}{=}\; (G^{n,k}_{x,y})^{-1}(U), \quad U\overset{d}{=}U([0,1])
$$

$$
\overset{d}{=}\; (G^{n,k}_{x,y})^{-1}(1-U), \tag{8.7}
$$

where we used that $U\overset{d}{=}1-U$. From a computational viewpoint $\zeta := (G^{n,k}_{x,y})^{-1}(1-u)$ is the admissible solution (*i.e.* such that $\ge\max(x,y)$) to the equation

$$
1 - \exp\Big(-\frac{2n}{T\sigma^2(t^n_k,x)}(\zeta-x)(\zeta-y)\Big) = 1 - u
$$

or equivalently to

$$
\zeta^2 - (x+y)\zeta + xy + \frac{T}{2n}\sigma^2(t^n_k,x)\log(u) = 0.
$$

This leads to

$$
(G^n_{x,y})^{-1}(1-u) = \frac{1}{2}\Big(x+y+\sqrt{(x-y)^2 - 2T\sigma^2(t^n_k,x)\log(u)/n}\Big).
$$

Finally

$$
\mathcal{L}\Big(\max_{t\in[0,T]}\bar{X}^n_t \,\big|\, \{\bar{X}^n_{t_k}=x_k,\ k=0,\ldots,n\}\Big) = \mathcal{L}\Big(\max_{0\le k\le n-1}(G^n_{x_k,x_{k+1}})^{-1}(1-U_k)\Big)
$$

where $(U_k)_{0\le k\le n-1}$ are *i.i.d.* and uniformly distributed random variables over the unit interval.

Pseudo-code for Lookback like options.

We assume for the sake of simplicity that the interest rate r is 0. By Lookback like options we mean the class of options whose payoff involve possibly both $\bar{X}_T$ and the maximum of (X_t) over $[0,T]$ *i.e.*

$$\mathbb{E}\,f(\bar{X}_T,\ \sup_{t\in[0,T]}\ \bar{X}_t).$$

Regular Call on maximum is obtained by setting $f(x,y) = (y-K)_+$, the regular maximum Lookback option by setting $f(x,y) = y - x$.

We want to compute a Monte Carlo approximation of $\mathbb{E}\,f(\sup_{t\in[0,T]}\bar{X}_t)$ using the continuous Euler scheme. We reproduce below a pseudo-script to illustrate how to use the above result on the conditional distribution of the maximum of the Brownian bridge.

- Set $S^f = 0$.

`for` $m = 1$ `to` M

- Simulate a path of the discrete time Euler scheme and set $x_k := \bar{X}_{t_k^n}^{(m)}$, $k = 0, \ldots, n$.

- Simulate $\Xi^{(m)} := \max_{0\leq k\leq n-1}(G_{x_k,x_{k+1}}^n)^{-1}(1 - U_k^{(m)})$, where $(U_k^{(m)})_{1\leq k\leq n}$ are i.i.d. with $U([0,1])$-distribution.

- Compute $f(\bar{X}_T^{(m)}, \Xi^{(m)})$.

- Compute $S_m^f := f(\bar{X}_T^{(m)}, \Xi^{(m)}) + S_{m-1}^f$.

`end.`(m)

- Eventually,

$$\mathbb{E}\,f(\ \sup_{t\in[0,T]}\ \bar{X}_t) \approx \frac{S_M^f}{M},$$

at least for large enough M ([2]).

Once one can simulate $\sup_{t\in[0,T]}\bar{X}_t$ (and its minimum, see exercise below), it is easy to price by simulation the exotic options mentioned in the former section (Lookback, options on maximum) but also the barrier options since one can decide whether or not the *continuous* Euler scheme strikes or not a barrier (up or down). Brownian bridge is also involved in the methods designed for pricing Asian options.

▷ **Exercise.** Derive a formula similar to (8.7) for the conditional distribution of the minimum of the continuous Euler scheme using now the inverse distribution functions

$$(F_{x,y}^{n,k})^{-1}(u) = \frac{1}{2}\left(x + y - \sqrt{(x-y)^2 - 2T\sigma^2(t_k^n,x)\log(u)/n}\right), \quad u\in[0,1].$$

---

[2] ... Of course one needs to compute the empirical variance (approximately) given by

$$\frac{1}{M}\sum_{k=1}^{M} f(\Xi^{(m)})^2 - \left(\frac{1}{M}\sum_{k=1}^{M} f(\Xi^{(m)})\right)^2.$$

in order to design a confidence interval without which the method is simply non sense. . . .

of

$$\inf_{t\in[0,T/n]}\left(x+\frac{t}{\frac{T}{n}}(y-x)+\sigma(x)Y_t^{W,T/n}\right).$$

8.2.4 Application to regular barrier options: variance reduction by pre-conditioning

By *regular barrier options* we mean barrier options having a constant level as a barrier. A down-and-out Call is a typical example of such options with a payoff given by

$$h_T=(X_T-K)_+\mathbf{1}_{\{\sup_{t\in[0,T]}X_t\leq L\}}$$

where K denotes the strike price of the option and L $(L>K)$ its barrier.

In practice, the "Call" part is activated at T only if the process (X_t) hits the barrier $L\leq K$ between 0 and T. In fact, as far as simulation is concerned, this "Call part" can be replaced by any Borel function f such that both $f(X_T)$ and $f(\bar{X}_T)$ are integrable (this is always true if f has polynomial growth owing to Proposition 7.2). Note that these so-called barrier options are in fact a sub-class of generalized maximum Lookback options having the specificity that the maximum only shows up through an indicator function.

Then, one may derive a general weighted formula for $\mathbb{E}(f(\bar{X}_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}_t\leq L\}})$ (which is an approximation of $\mathbb{E}(f(X_T)\mathbf{1}_{\{\sup_{t\in[0,T]}X_t\leq L\}}))$.

Proposition 8.4

$$\mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}_t\leq L\}}\right)=\mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\max_{0\leq k\leq n}\bar{X}_{t_k}\leq L\}}\prod_{k=0}^{n-1}\left(1-e^{-\frac{2n}{T}\frac{(\bar{X}_{t_k}-L)(\bar{X}_{t_{k+1}}-L)}{\sigma^2(t_k^n,X_{t_k})}}\right)\right). \quad (8.8)$$

Proof of Equation (8.8). This formula is the typical application of pre-conditioning described in Section 3.4. We start from the chaining identity for conditional expectation:

$$\mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}_t\leq L\}}\right)=\mathbb{E}\left(\mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}_t\leq L\}}\mid\bar{X}_{t_k},\ k=0,\ldots,n\right)\right)$$

Then we use the conditional independence of the Brownian bridges given the values $\bar{X}_{t_k}$, $k=0,\ldots,n$, of the Euler scheme which has been established in Proposition 8.2. It follows

$$
\begin{aligned}
\mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}_t\leq L\}}\right) &= \mathbb{E}\left(f(\bar{X}_T)\mathbb{P}(\sup_{t\in[0,T]}\bar{X}_t\leq L\mid\bar{X}_{t_k},\ k=0,\ldots,n)\right)\\
&= \mathbb{E}\left(f(\bar{X}_T)\prod_{k=1}^{n}G_{\bar{X}_{t_k},\bar{X}_{t_{k+1}}}^n(L)\right)\\
&= \mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\max_{0\leq k\leq n}\bar{X}_{t_k}\leq L\}}\prod_{k=0}^{n-1}\left(1-e^{-\frac{2n}{T}\frac{(\bar{X}_{t_k}-L)(\bar{X}_{t_{k+1}}-L)}{\sigma^2(t_k^n,X_{t_k})}}\right)\right). \quad\diamond
\end{aligned}
$$

Furthermore, we know that *the random variable in the right hand side always has a lower variance* since it is a conditional expectation of the random variable in the left hand side, namely

$$\mathrm{Var}\left(f(\bar{X}_T)\mathbf{1}_{\{\max_{0\leq k\leq n}\bar{X}_{t_k}\leq L\}}\prod_{k=0}^{n-1}\left(1-e^{-\frac{2n}{T}\frac{(\bar{X}_{t_k}-L)(\bar{X}_{t_{k+1}}-L)}{\sigma^2(t_k^n,X_{t_k})}}\right)\right)\leq\mathrm{Var}\left(f(\bar{X}_T)\mathbf{1}_{\{\sup_{t\in[0,T]}\bar{X}_t\leq L\}}\right)$$

▷ **Exercises. 1.** Show likewise that for every Borel function $f \in L^1(\mathbb{P}_{\bar{X}_T})$,

$$\mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\inf_{t\in[0,T]}\bar{X}_t \geq L\}}\right) = \mathbb{E}\left(f(\bar{X}_T)\mathbf{1}_{\{\min_{0\leq k\leq n}\bar{X}_{t_k} \geq L\}} e^{-\frac{2n}{T}\sum_{k=0}^{n-1}\frac{(\bar{X}_{t_k}-L)(\bar{X}_{t_{k+1}}-L)}{\sigma^2(t_k^n, X_{t_k})}}\right) \qquad (8.9)$$

and that the expression in the second expectation has a lower variance.

2. Extend the above results to barriers of the form

$$L(t) := e^{at+b}, \; a, \, b \in \mathbb{R}.$$

8.2.5 Weak errors and Richardson-Romberg extrapolation for path-dependent options

It is clear that all the asymptotic control of the variance obtained in the former section for the estimator $\sum_{r=1}^{R}\alpha_r f(\bar{X}_T^{(r)})$ of $\mathbb{E}(f(X_T))$ when f is continuous can be extended to functionals $F : D([0,T], \mathbb{R}^d) \to \mathbb{R}$ which are $\mathbb{P}_X$-*a.s.* continuous with respect to the sup-norm defined by $\|x\|_{\sup} := \sup_{t\in[0,T]}|x(t)|$ with polynomial growth (*i.e.* $F(x) = O(\|x\|_{\sup}^\ell)$ for some natural integer ℓ as $\|x\|_{\sup} \to \infty$). This simply follows from the fact that the (continuous) Euler scheme $\bar{X}$ (with step T/n) defined by

$$\forall\, t \in [0,T], \qquad \bar{X}_t = x_0 + \int_0^t b(\bar{X}_{\underline{s}})ds + \int_0^t \sigma(\bar{X}_{\underline{s}})dW_s, \qquad \underline{s} = \lfloor ns/T\rfloor T/n$$

converges for the sup-norm toward X in every $L^p(\mathbb{P})$.

Furthermore, this asymptotic control of the variance holds true with any R-tuple $\alpha = (\alpha_r)_{1\leq r\leq R}$ of weights coefficients satisfying $\sum_{1\leq r\leq R}\alpha_r = 1$, so these coefficients can be adapted to the structure of the weak error expansion.

For both classes of functionals (with D as a half-line in 1-dimension in the first setting), the practical implementation of the continuous Euler scheme is known as the *Brownian bridge method*.

At this stage there are two ways to implement the (multistep) Romberg extrapolation with consistent Brownian increments in order to improve the performances of the original (stepwise constant or continuous) Euler schemes. Both rely on natural conjectures about the existence of a higher order expansion of the time discretization error suggested by the above rates of convergence (8.1), (8.2) and (8.3).

● *Stepwise constant Euler scheme:* As concerns the standard Euler scheme, this means the existence of a vector space V (stable by product) of admissible functionals satisfying

$$(\mathcal{E}_R^{\frac{1}{2}, V}) \qquad \equiv \qquad \forall\, F \in V, \qquad \mathbb{E}(F(X)) = \mathbb{E}(F(\widetilde{X})) + \sum_{k=1}^{R-1}\frac{c_k}{n^{\frac{k}{2}}} + O(n^{-\frac{R}{2}}). \qquad (8.10)$$

still for some real constant c_k depending on b, σ, F, etc For small values of R, one checks that

$$R = 2: \quad \alpha_1^{(\frac{1}{2})} = -(1 + \sqrt{2}), \qquad \alpha_2^{(\frac{1}{2})} = \sqrt{2}(1 + \sqrt{2}).$$

$$*[.4em]R = 3: \quad \alpha_1^{(\frac{1}{2})} = \frac{\sqrt{3} - \sqrt{2}}{2\sqrt{2} - \sqrt{3} - 1}, \qquad \alpha_2^{(\frac{1}{2})} = -2\frac{\sqrt{3} - 1}{2\sqrt{2} - \sqrt{3} - 1}, \qquad \alpha_3^{(\frac{1}{2})} = 3\frac{\sqrt{2} - 1}{2\sqrt{2} - \sqrt{3} - 1}.$$

Note that these coefficients have greater absolute values than in the standard case. Thus if $R = 4$, $\sum_{1\leq r\leq 4}(\alpha_r^{(\frac{1}{2})})^2 \approx 10\,900!$ which induces an increase of the variance term for too small values of the time discretization parameter n even when increments are consistently generated. The complexity computations of

the procedure needs to be updated but *grosso modo* the optimal choice for the time discretization parameter n as a function of the MC size M is

$$M \propto n^R.$$

• *The continuous Euler scheme:* The conjecture is simply to assume that the expansion $(\mathcal{E}_R)$ now holds for a vector space V of *functionals* F (with polynomial growth with respect to the sup-norm). The increase of the complexity induced by the Brownian bridge method is difficult to quantize: it amounts to computing $\log(U_k)$ and the inverse distribution functions $F_{x,y}^{-1}$ and $G_{x,y}^{-1}$.

The second difficulty is that simulating (the extrema of) some of continuous Euler schemes using the Brownian bridge in a consistent way is not straightforward at all. However, one can reasonably expect that using independent Brownian bridges "relying" on stepwise constant Euler schemes with consistent Brownian increments will have a small impact on the global variance (although slightly increasing it).

To illustrate and compare these approaches we carried some numerical tests on partial Lookback and barrier options in the Black-Scholes model presented in the previous section.

▷ Partial Lookback options: The partial Lookback Call option is defined by its payoff functional

$$F(x) = e^{-rT} \left(x(T) - \lambda \min_{s \in [0,T]} x(s) \right)_+, \quad x \in \mathcal{C}([0,T], \mathbb{R}),$$

where $\lambda > 0$ (if $\lambda \leq 1$, the $(.)_+$ can be dropped). The premium

$$\mathrm{Call}_0^{Lkb} = e^{-rT} \mathbb{E}((X_T - \lambda \min_{t \in [0,T]} X_t)_+)$$

is given by

$$\mathrm{Call}_0^{Lkb} = X_0 \mathrm{Call}^{BS}(1, \lambda, \sigma, r, T) + \lambda \frac{\sigma^2}{2r} X_0 \mathrm{Put}^{BS}\left(\lambda^{\frac{2r}{\sigma^2}}, 1, \frac{2r}{\sigma}, r, T \right).$$

We took the same values for the B-S parameters as in the former section and set the coefficient λ at $\lambda = 1.1$. For this set of parameters $\mathrm{Call}_0^{Lkb} = 57.475$.

As concerns the MC simulation size, we still set $M = 10^6$. We compared the following three methods for every choice of n:

– A 3-step Richardson-Romberg extrapolation ($R = 3$) of the stepwise constant Euler scheme (for which a $O(n^{-\frac{3}{2}})$-rate can be expected from the conjecture).

– A 3-step Richardson-Romberg extrapolation ($R = 3$) based on the continuous Euler scheme (Brownian bridge method) for which a $O(\frac{1}{n^3})$-rate can be conjectured (see [63]).

– A continuous Euler scheme (Brownian bridge method) of equivalent complexity *i.e.* with discretization parameter $6n$ for which a $O(\frac{1}{n})$-rate can be expected (see [63]).

The three procedures have the same complexity if one neglects the cost of the bridge simulation with respect to that of the diffusion coefficients (note this is very conservative in favour of "bridged schemes").

We do not reproduce the results obtained for the standard stepwise constant Euler scheme which are clearly out of the game (as already emphasized in [63]). In Figure 8.2.5, the abscissas represent the size of Euler scheme with equivalent complexity (*i.e.* $6n$, $n = 2, 4, 6, 8, 10$). Figure 8.2.5(a) (left) shows that both 3-step Richardson-Romberg extrapolation methods converge significantly faster than the "bridged" Euler scheme with equivalent complexity in this high volatility framework. The standard deviations depicted in Figure 8.2.5(a) (right) show that the 3-step Richardson-Romberg extrapolation of the Brownian bridge is controlled even for small values of n. This is not the case with the 3-step Richardson-Romberg extrapolation method of the stepwise constant Euler scheme. Other simulations – not reproduced here – show this is already true for the standard Romberg extrapolation and the bridged Euler scheme. In any case the multistep Romberg extrapolation with $R = 3$ significantly outperforms the bridged Euler scheme.
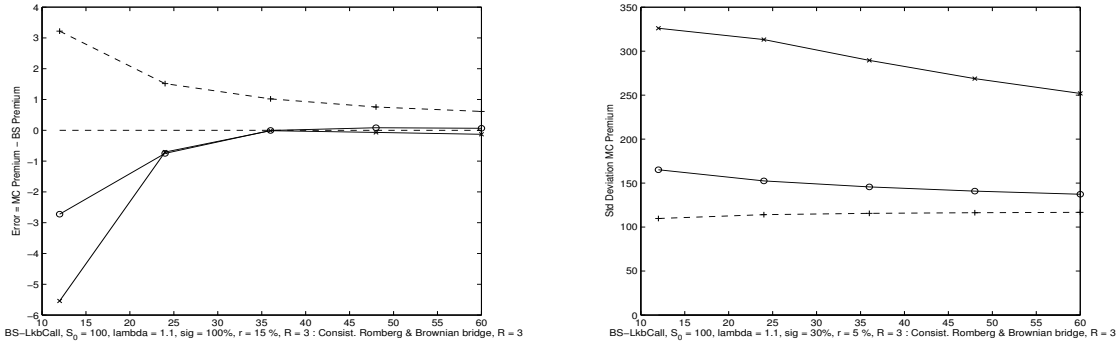
(a)
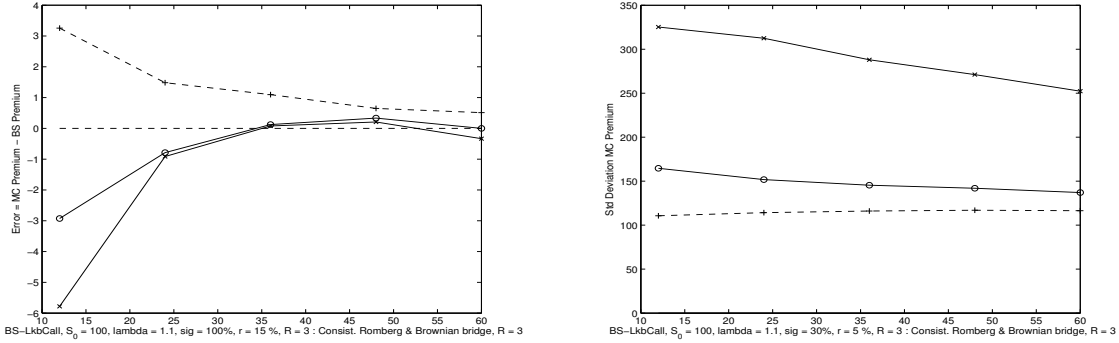
(b)

Figure 8.1: *B-S* EURO PARTIAL LOOKBACK CALL OPTION. *(a)* $M = 10^6$. *Richardson-Romberg extrapolation (R = 3) of the Euler scheme with Brownian bridge: o—o—o. Consistent Richardson-Romberg extrapolation (R = 3): ×—×—×. Euler scheme with Brownian bridge with equivalent complexity: + − − + − − +.* $X_0 = 100$, $\sigma = 100\%$, $r = 15\%$, $\lambda = 1.1$. *Abscissas: 6n, n = 2, 4, 6, 8, 10. Left:* Premia. *Right:* Standard Deviations. *(b) Idem with* $M = 10^8$.

When $M = 10^8$, one verifies (see Figure 8.2.5(b)) that the time discretization error of the 3-step Richardson-Romberg extrapolation vanishes like for the partial Lookback option. In fact for $n = 10$ the 3-step bridged Euler scheme yields a premium equal to 57.480 which corresponds to less than half a cent error, *i.e.* 0.05 % accuracy! This result being obtained without any control variate variable.

The Richardson-Romberg extrapolation of the standard Euler scheme also provides excellent results. In fact it seems difficult to discriminate them with those obtained with the bridged schemes, which is slightly unexpected if one think about the natural conjecture about the time discretization error expansion.

As a theoretical conclusion, these results strongly support both conjectures about the existence of expansion for the weak error in the $(n^{-p/2})_{p \geq 1}$ and $(n^{-p})_{p \geq 1}$ scales respectively.

▷ UP & OUT CALL OPTION: Let $0 \leq K \leq L$. The Up-and-Out Call option with strike K and barrier L is defined by its payoff functional

$$F(x) = e^{-rT} \left(x(T) - K\right)_+ \mathbf{1}_{\{\max_{s \in [0, T]} x(s) \leq L\}}, \quad x \in \mathcal{C}([0, T], \mathbb{R}).$$

It is again classical background, that in a B-S model

$$\text{Call}^{U\&O}(X_0, r, \sigma, T) = \text{Call}^{BS}(X_0, K, r, \sigma, T) - \text{Call}^{BS}(X_0, L, r, \sigma, T) - e^{-rT}(L-K)\Phi(d^-(L))$$
$$- \left(\frac{L}{X_0}\right)^{1+\mu}\left(\text{Call}^{BS}(X_0, K', r, \sigma, T) - \text{Call}^{BS}(X_0, L', r, \sigma, T) - e^{-rT}(L'-K')\Phi(d^-(L'))\right)$$

with

$$K' = K\left(\frac{X_0}{L}\right)^2, \ L' = L\left(\frac{X_0}{L}\right)^2, \ d^-(L) = \frac{\log(X_0/L) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \ \text{and} \ \Phi(x) := \int_{-\infty}^x e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}}$$

and $\mu = \dfrac{2r}{\sigma^2}$.

We took again the same values for the B-S parameters as for the vanilla call. We set the barrier value at $L = 300$. For this set of parameters $C_0^{UO} = 8.54$. We tested the same three schemes. The numerical results are depicted in Figure 8.2.5.

The conclusion (see Figure 8.2.5(a) (left)) is that, at this very high level of volatility, when $M = 10^6$ (which is a standard size given the high volatility setting) the (quasi-)consistent 3-step Romberg extrapolation with Brownian bridge clearly outperforms the continuous Euler scheme (Brownian bridge) of equivalent complexity while the 3-step Richardson-Romberg extrapolation based on the stepwise constant Euler schemes with consistent Brownian increments is not competitive at all: it suffers from both a too high variance (see Figure 8.2.5(a) (right)) for the considered sizes of the Monte Carlo simulation and from its too slow rate of convergence in time.

When $M = 10^8$ (see Figure 8.2.5(b) (left)), one verifies again that the time discretization error of the 3-step Richardson-Romberg extrapolation almost vanishes like for the partial Lookback option. This no longer the case with the 3-step Richardson-Romberg extrapolation of stepwise constant Euler schemes. It seems clear that the discretization time error is more prominent for the barrier option: thus with $n = 10$, the relative error is $\frac{9.09-8.54}{8.54} \approx 6.5\%$ by this first Richardson-Romberg extrapolation whereas, the 3-step Romberg method based on the quasi-consistent "bridged" method yields an approximate premium of 8.58 corresponding to a relative error of $\frac{8.58-8.54}{8.54} \approx 0.4\%$. These specific results (obtained without any control variate) are representative of the global behaviour of the methods as emphasized by Figure 8.2.5(b)(left).

8.2.6 The case of Asian options

The family of Asian options is related to the payoffs of the form

$$h_T := h\left(\int_0^T X_s ds\right).$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative Borel function. This kind of options needs some specific treatments to improve the rate of convergence of its time discretization. This is due to the continuity of the functional $f \mapsto \int_0^T f(s)ds$ in $(L^1([0,T], dt), \| \cdot \|_1)$.

This problem has been extensively investigated (essentially for a Black-Scholes dynamics) by E. Temam in his PHD thesis (see [97]). What follows comes from this work.

APPROXIMATION PHASE: Let

$$X_t^x = x\exp(\mu t + \sigma W_t), \quad \mu = r - \frac{\sigma^2}{2}, \quad x > 0.$$

(a)
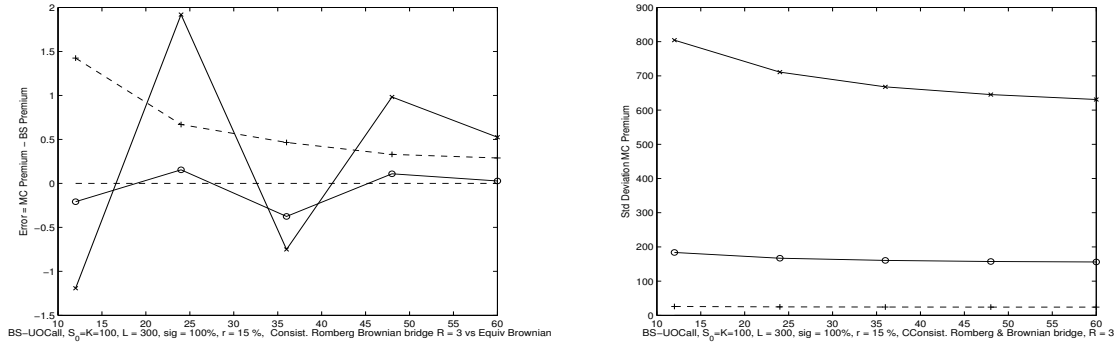
(b)
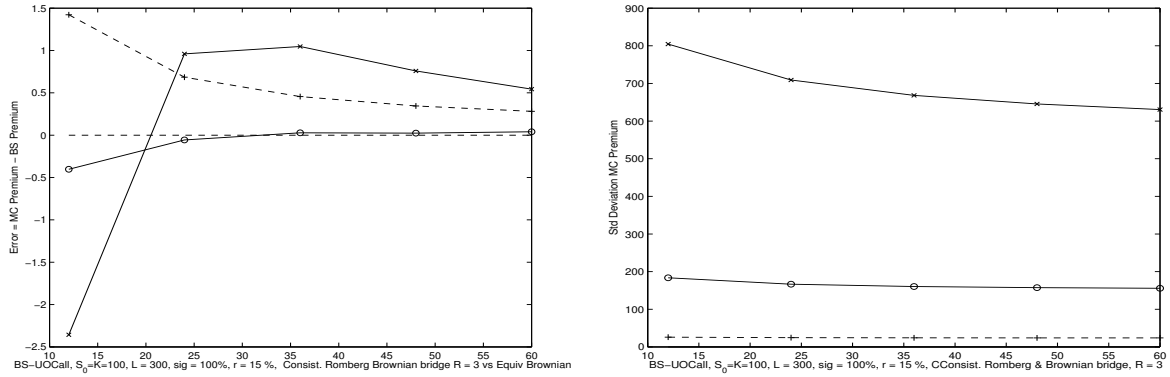
Figure 8.2: *B-S* EURO UP-&-OUT CALL OPTION. (a) $M\!=\!10^6$. *Richardson-Romberg extrapolation ($R\!=\!3$) of the Euler scheme with Brownian bridge:* o—o—o. *Consistent Richardson-Romberg extrapolation ($R\!=\!3$):* —×—×—×—. *Euler scheme with Brownian bridge and equivalent complexity:* $+--+--+$. $X_0\!=\!K\!=\!100$, $L = 300$, $\sigma = 100\%$, $r\!=\!15\%$. *Abscissas:* $6n$, $n = 2, 4, 6, 8, 10$. Left: *Premia.* Right: *Standard Deviations.* (b) *Idem with* $M = 10^8$.

Then

$$
\begin{aligned}
\int_0^T X_s^x ds &= \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} X_s^x ds \\
&= \sum_{k=0}^{n-1} X_{t_k^n}^x \int_0^{T/n} \exp\left(\mu s + \sigma W_s^{(t_k^n)}\right) ds.
\end{aligned}
$$

So, we need to approximate the integral coming out in the above equation. Let B be a standard Brownian motion. Roughly speaking, $\sup_{s\in[0,T/n]} |B_s|$ is proportional to $\sqrt{T/n}$. Although it is not true in the a.s. (owing to a missing $\log\log$ term), it holds true in any $L^p(\mathbb{P})$, $p\in[1,\infty)$ by a simple scaling so that, we may write "almost" rigorously

$$
\forall\, s\in[0,T/n], \qquad \exp\left(\mu s + \sigma B_s\right) = 1 + \mu s + \sigma B_s + \frac{\sigma^2}{2} B_s^2 + \text{``}O(n^{-\frac{3}{2}})\text{''}.
$$

Hence

$$
\begin{aligned}
\int_0^{T/n} \exp\left(\mu s + \sigma B_s\right) ds &= \frac{T}{n} + \frac{\mu}{2}\frac{T^2}{n^2} + \sigma \int_0^{T/n} B_s ds + \frac{\sigma^2}{2}\frac{T^2}{2n^2} + \frac{\sigma^2}{2}\int_0^{T/n} (B_s^2 - s) ds + \text{``}O(n^{-\frac{5}{2}})\text{''} \\
&= \frac{T}{n} + \frac{r}{2}\frac{T^2}{n^2} + \sigma \int_0^{T/n} B_s ds + \frac{\sigma^2}{2}\left(\frac{T}{n}\right)^2 \int_0^1 (\widetilde{B}_u^2 - u) du + \text{``}O(n^{-\frac{5}{2}})\text{''}
\end{aligned}
$$

where $\widetilde{B}_u = \sqrt{\frac{T}{n}} B_{\frac{T}{n}u}$, $u \in [0,1]$, is a standard Brownian motion (on the unit interval) since, combining a scaling and a change of variable,

$$
\int_0^{T/n} (B_s^2 - s) ds = \left(\frac{T}{n}\right)^2 \int_0^1 (\widetilde{B}_u^2 - u) du.
$$

Owing to the fact that the random variable $\int_0^1 (\widetilde{B}_u^2 - u) du$ is centered and that, when B is replaced sucessively by $W^{(t_k^n)}$, $k = 1, \ldots, n$, the resulting random variables are i.i.d., one can in fact consider that the contribution of this term is $O(n^{-\frac{5}{2}})$ ([3]). This leads us to use the following approximation

$$
\int_0^{T/n} \exp\left(\mu s + \sigma W_s^{(t_k^n)}\right) ds \approx I_k^n := \frac{T}{n} + \frac{r}{2}\frac{T^2}{n^2} + \sigma \int_0^{T/n} W_s^{(t_k^n)} ds.
$$

---

[3]To be more precise, the random variable $\int_0^1 ((\widetilde{W}_u^{(t_k^n)})^2 - u)\, du$ is independent of $\mathcal{F}_{t_k^n}^W$, $k = 0, \ldots, n-1$, so that

$$
\begin{aligned}
\left\|\sum_{k=0}^{n-1} X_{t_k^n}^x \int_0^1 ((\widetilde{W}_u^{(t_k^n)})^2 - u)\, du\right\|_2^2 &= \sum_{k=0}^{n-1} \left\|X_{t_k^n}^x \int_0^1 ((\widetilde{W}_u^{(t_k^n)})^2 - u)\, du\right\|_2^2 \\
&= \sum_{k=0}^{n-1} \left\|X_{t_k^n}^x\right\|_2^2 \left\|\int_0^1 ((\widetilde{W}_u^{(t_k^n)})^2 - u)\, du\right\|_2^2 \\
&\leq n\left(\frac{T}{n}\right)^4 \left\|\int_0^1 (B_u^2 - u)\, du\right\|_2^2 \|X_T^x\|_2^2
\end{aligned}
$$

since $(X_t^x)^2$ is a sub-martingale. As a consequence

$$
\left\|\sum_{k=0}^{n-1} X_{t_k^n}^x \int_0^1 ((\widetilde{W}_u^{(t_k^n)})^2 - u)\, du\right\|_2 = O(n^{-\frac{3}{2}})
$$

which justifies to consider "conventionally" the contribution of each term to be $O(n^{-\frac{5}{2}})$ i.e. negligible.

SIMULATION PHASE: Now, it follows from Proposition 8.2 applied to the Brownian motion (which is its own continuous Euler scheme), that the n-tuple of processes $(W_t)_{t \in [t_k^n, t_{k+1}^n]}$, $k = 0, \ldots, n-1$ are independent processes given $\sigma(W_{t_k^n}, k = 1, \ldots, n)$. This also reads as $(W_t^{(t_k^n)})_{t \in [0, T/n]}$, $k = 0, \ldots, n-1$ are independent processes given $\sigma(W_{t_k^n}, k = 1, \ldots, n)$. The same proposition implies that for every $\ell \in \{0, \ldots, n-1\}$,

$$\mathcal{L}\left((W_t^{(t_\ell^n)})_{t \in [0, T/n]} \mid W_{t_k^n} = w_k, k = 1, \ldots, n\right) = \mathcal{L}\left(\left(\frac{nt}{T}(w_{\ell+1} - w_\ell) + Y_t^{W, T/n}\right)_{t \in [0, T/n]}\right)$$

are independent processes given $\sigma(\{W_{t_k^n}, k = 1, \ldots, n\})$. Consequently the random variables $\int_0^{T/n} W_s^{(t_\ell^n)} ds$, $\ell = 1, \ldots, n$, are conditionally i.i.d. given $\sigma(W_{t_k^n} = w_k, k = 1, \ldots, n)$ with a conditional Gaussian distribution with (conditional mean) given by

$$\int_0^{T/n} \frac{nt}{T}(w_{\ell+1} - w_\ell) dt = \frac{T}{2n}(w_{\ell+1} - w_\ell).$$

As concerns the variance, we can use the exercise below Proposition 8.1 but, at this stage, we will detail the computation for a generic Brownian motion, say W, between 0 and T/n.

$$\begin{aligned}
\mathrm{Var}\left(\int_0^{T/n} W_s \, ds \mid W_{\frac{T}{n}}\right) &= \mathbb{E}\left(\left(\int_0^{T/n} Y_s^{W, T/n} ds\right)^2\right) \\
&= \int_{[0, T/n]^2} \mathbb{E}(Y_s^{W, T/n} Y_t^{W, T/n}) \, ds \, dt \\
&= \frac{n}{T} \int_{[0, T/n]^2} (s \wedge t)\left(\frac{T}{n} - (s \vee t)\right) ds \, dt \\
&= \left(\frac{T}{n}\right)^3 \int_{[0,1]^2} (u \wedge v)(1 - (u \vee v)) \, du \, dv \\
&= \frac{1}{12}\left(\frac{T}{n}\right)^3.
\end{aligned}$$

▷ **Exercise.** Use stochastic calculus to show directly that

$$\mathbb{E}\left(\left(\int_0^T Y_s^{W,T} ds\right)^2\right) = \mathbb{E}\left(\left(\int_0^T W_s ds - \frac{T}{2} W_T\right)^2\right) = \int_0^T \left(\frac{T}{2} - s\right)^2 ds = \frac{T^3}{12}.$$

No we can describe the pseudo-code for the pricing of an Asian option with payoff $h\left(\int_0^T X_s^x ds\right)$ by a Monte Carlo simulation.

`for` $m := 1$ `to` M

- Simulate the Brownian increments $\Delta W_{t_{k+1}^n}^{(m)} \overset{d}{:=} \sqrt{\frac{T}{n}} Z_k^{(m)}$, $k = 1, \ldots, n$, $Z_k^{(m)}$ i.i.d. with distribution $\mathcal{N}(0; 1)$; set
  - $w_k^{(m)} := \sqrt{\frac{T}{n}}\left(Z_1^{(m)} + \cdots + Z_n^{(m)}\right)$,
  - $x_k^{(m)} := x \exp\left(\mu t_k^n + \sigma w_k^{(m)}\right)$, $k = 0, \ldots, n$.

- Simulate *independently* $\zeta_k^{(m)}$, $k = 1, \ldots, n$, i.i.d. with distribution $\mathcal{N}(0;1)$ and set

$$I_k^{n,(m)} \overset{d}{:=} \frac{T}{n} + \frac{r}{2}\left(\frac{T}{n}\right)^2 + \sigma\left(\frac{T}{2n}(w_{k+1} - w_k) + \frac{1}{\sqrt{12}}\left(\frac{T}{n}\right)^{\frac{3}{2}}\zeta_k^{(m)}\right), \qquad k = 0, \ldots, n-1.$$

- Compute

$$h_T^{(m)} =: h\left(\sum_{k=0}^{n-1} x_k^{(m)} I_k^{n,(m)}\right)$$

end. (m)

$$\text{Premium} \approx e^{-rT}\frac{1}{M}\sum_{m=1}^{M} h_T^{(m)}.$$

end.

TIME DISCRETIZATION ERROR ESTIMATES: Set

$$A_T = \int_0^T X_s^x ds \qquad \text{and} \qquad \bar{A}_T^n = \sum_{k=0}^{n-1} X_{t_k}^x I_k^n.$$

This scheme induces the following time discretization error.

Proposition 8.5 *(see [97]) For every $p \geq 2$,*

$$\|A_T - \bar{A}_T^n\|_p = O(n^{-\frac{3}{2}})$$

so that, if $g : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous continuous, then

$$\|g(X_T^x, A_T) - g(X_T^x, \bar{A}_T^n)\|_p = O(n^{-\frac{3}{2}}).$$

Proof. In progress. $\diamond$

Remark. The main reason for not considering higher order expansions of $\exp(\mu t + \sigma B_t)$ is that we are not able to simulate at a reasonable cost the triplet $(B_t, \int_0^t B_s ds, \int_0^t B_s^2 ds)$ which is no longer a Gaussian vector and, consequently, $(\int_0^t B_s ds, \int_0^t B_s^2 ds)$ given B_t.

Chapter 9

Back to sensitivity computation

Let $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{E})$ be an E-valued random variable where $(E, \mathcal{E})$ is an abstract measurable space, let I be a nonempty open interval of $\mathbb{R}$ and let $F : I \times E \to \mathbb{R}$ be a $\mathcal{B}or(I) \otimes \mathcal{E}$-measurable function such that, for every $x \in I$, $F(x, Z) \in L^2(\mathbb{P})$ ([1]). Then set

$$f(x) = \mathbb{E}\, F(x, Z).$$

Assume that the function f is regular, at least at some points. Our aim is to devise a method to compute by simulation $f'(x)$ at such points (or higher derivatives $f^{(k)}(x)$, $k \geq 1$). If the functional $F(x, z)$ is differentiable at x (with respect to its first variable) for $\mathbb{P}_Z$ almost every z, if a domination or uniform integrability property holds (like (9.1) or (9.5) below), if the partial derivative $\frac{\partial F}{\partial x}(x, z)$ can be computed at a reasonable cost and Z is a simulatable random vector (still at a reasonable...), it is natural to compute $f'(x)$ using a Monte Carlo simulation based on the representation formula

$$f'(x) = \mathbb{E}\left(\frac{\partial F}{\partial x}(x, Z)\right).$$

This approach has already been introduced in Chapter 2 and will be more deeply developed further on in Section 9.2 mainly devoted to the tangent process.

Otherwise, when $\frac{\partial F}{\partial x}(x, z)$ does not exist or cannot be compute easily (whereas F can), a natural idea is to introduce a stochastic finite difference approach. Other methods based on the introduction of an appropriate weight will be introduced in the last two sections of this chapter.

9.1   Finite difference method(s)

The finite difference method is in some way the most elementary and natural method to compute sensitivity parameters (Greeks) although it is an approximate method in its standard form. It is also known in financial Engineering as the "Bump Method. It can be described in a very general setting (which does correspond to its wide field of application). These methods have been originally investigated in [59, 60, 99].

---

[1] In fact E can be replaced by the probability space Ω itself: Z becomes the canonical variable/process on this probability space (endowed with the distribution $\mathbb{P} = \mathbb{P}_Z$ of the process). In particular Z can be the Brownian motion or any process at time T starting at x or its entire path, etc. The notation is essentially formal and could be replaced by the more general $F(x, \omega)$

9.1.1   The constant step approach

We consider the framework described in the introduction caption. We will distinguish two cases: in the first one – called "regular setting" – the function $x \mapsto F(x, Z(\omega))$ is "not far" from being pathwise differentiable whereas in the second one – called "singular setting" – f remains smooth but F becomes "singular".

The regular setting

Proposition 9.1 *Let $x \in \mathbb{R}$. Assume that F satisfies the following local mean quadratic Lipschitz continuous assumption ("at x")*

$$\exists\, \varepsilon_0 > 0, \ \forall\, x' \in (x - \varepsilon_0, x + \varepsilon_0), \quad \|F(x, Z) - F(x', Z)\|_2 \le C_{F,Z} |x - x'|. \tag{9.1}$$

Assume the function f is twice differentiable with a Lipschitz continuous second derivative on $(x - \varepsilon_0, x + \varepsilon_0)$. Let $(Z_k)_{k \ge 1}$ be a sequence of i.i.d. random vectors with the same distribution as Z, then for every $\varepsilon \in (0, \varepsilon_0)$,

$$\left\| f'(x) - \frac{1}{M} \sum_{k=1}^{M} \frac{F(x + \varepsilon, Z_k) - F(x - \varepsilon, Z_k)}{2\varepsilon} \right\|_2 \le \sqrt{\left([f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2}\right)^2 + \frac{C_{F,Z}^2 - (f'(x) - \frac{\varepsilon^2}{2}[f'']_{\mathrm{Lip}})_+^2}{M}} \tag{9.2}$$

$$\le \sqrt{\left([f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2}\right)^2 + \frac{C_{F,Z}^2}{M}} \tag{9.3}$$

$$\le [f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2} + \frac{C_{F,Z}}{\sqrt{M}}.$$

Furthermore, if f is three times differentiable on $(x - \varepsilon_0, x + \varepsilon_0)$ with a bounded third derivative, then one can replace $[f'']_{\mathrm{Lip}}$ by $\frac{1}{3} \sup_{|\xi - x| \le \varepsilon} |f^{(3)}(\xi)|$.

Remark. In the above sum $[f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2}$ represents the *bias* and $\frac{C_{F,Z}}{\sqrt{M}}$ is the *statistical* error.

Proof. Let $\varepsilon \in (0, \varepsilon_0)$. It follows from the Taylor formula applied to f between x and $x \pm \varepsilon$ respectively that

$$|f'(x) - \frac{f(x + \varepsilon) - f(x - \varepsilon)}{2\varepsilon}| \le [f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2}. \tag{9.4}$$

On the other hand

$$\mathbb{E}\left(\frac{F(x + \varepsilon, Z) - F(x - \varepsilon, Z)}{2\varepsilon}\right) = \frac{f(x + \varepsilon) - f(x - \varepsilon)}{2\varepsilon}$$

and

$$\begin{aligned}
\mathrm{Var}\left(\frac{F(x + \varepsilon, Z) - F(x - \varepsilon, Z)}{2\varepsilon}\right) &= \mathbb{E}\left(\left(\frac{F(x + \varepsilon, Z) - F(x - \varepsilon, Z)}{2\varepsilon}\right)^2\right) - \left(\frac{f(x + \varepsilon) - f(x - \varepsilon)}{2\varepsilon}\right)^2 \\
&= \frac{\mathbb{E}(F(x + \varepsilon, Z) - F(x - \varepsilon, Z))^2}{4\varepsilon^2} - \left(\frac{f(x + \varepsilon) - f(x - \varepsilon)}{2\varepsilon}\right)^2 \\
&\le C_{F,Z}^2 - \left(\frac{f(x + \varepsilon) - f(x - \varepsilon)}{2\varepsilon}\right)^2 \tag{9.5} \\
&\le C_{F,Z}^2. \tag{9.6}
\end{aligned}$$

Using the Huyguens Theorem (^{2}), we get

$$\left\| f'(x) - \frac{1}{M} \sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, Z_k)}{2\varepsilon} \right\|_2^2 = \left(f(x) - \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right)^2$$

$$+ \mathrm{Var}\left(\frac{1}{M} \sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, Z_k)}{2\varepsilon} \right)$$

$$= \left(f(x) - \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right)^2$$

$$+ \frac{1}{M} \mathrm{Var}\left(\frac{F(x+\varepsilon, Z) - F(x-\varepsilon, Z)}{2\varepsilon} \right)$$

$$\leq \left([f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2} \right)^2 + \frac{1}{M} \left(C_{F,Z}^2 - \left(\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right)^2 \right)$$

where we combined (9.4) and (**??**) to derive the last inequality *i.e.* (9.3). To get the improved bound (9.2), we first derive from (9.4) that

$$\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \geq f'(x) - \frac{\varepsilon^2}{2} [f'']_{\mathrm{Lip}}$$

so that

$$\left| \mathbb{E}\left(\frac{F(x+\varepsilon, Z) - F(x-\varepsilon, Z)}{2\varepsilon} \right) \right| = \left| \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right|$$

$$\geq \left(\frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} \right)_+$$

$$\geq \left(f'(x) - \frac{\varepsilon^2}{2} [f'']_{\mathrm{Lip}} \right)_+.$$

Plugging this lower bound into the definition of the variance yields the announced inequality. $\diamond$

NUMERICAL RECOMMENDATIONS. The above result suggests to choose $M = M(\varepsilon)$ (and ε) so that

$$M(\varepsilon) = o\left(\varepsilon^{-4} \right).$$

As a matter of fact, it is useless to pursue the Monte Carlo simulation so that the statistical error becomes smaller than that induced by the approximation of the derivative. From a practical point of view this means that, in order to reduce the error by a factor 2, we need to reduce ε and increase M as follows:

$$\varepsilon \rightsquigarrow \varepsilon/\sqrt{2} \qquad \text{and} \qquad M \rightsquigarrow 4\,M.$$

Remark (what should never be done). Imagine that one uses two *independent* samples $(Z_k)_{k\geq 1}$ and $(\widetilde{Z}_k)_{k\geq 1}$ to simulate $F(x-\varepsilon, Z)$ and $F(x+\varepsilon, Z)$. Then, it is straightforward that

$$\mathrm{Var}\left(\frac{1}{M} \sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, \widetilde{Z}_k)}{2\varepsilon} \right) = \frac{1}{4M\varepsilon^2} \Big(\mathrm{Var}(F(x+\varepsilon, Z)) + \mathrm{Var}(F(x-\varepsilon, Z)) \Big)$$

$$\approx \frac{\mathrm{Var}(F(x, Z))}{2M\varepsilon^2}.$$

---

^{2}which formally simply reads $\mathbb{E}|X - a|^2 = (a - \mathbb{E}\,X)^2 + \mathrm{Var}(X)$.

Then the asymptotic variance of the estimator of $\frac{f(x+\varepsilon)-f(x-\varepsilon)}{2}$ explodes as $\varepsilon \to 0$ and the resulting quadratic error read approximately

$$[f'']_{\text{Lip}} \frac{\varepsilon^2}{2} + \frac{\sigma(F(x,Z))}{\varepsilon\sqrt{2M}}.$$

where $\sigma(F(x,Z)) = \sqrt{\text{Var}(F(x,Z))}$ is the standard deviation of $F(x,Z)$. This leads to consider the unrealistic constraint $M(\varepsilon) \propto \varepsilon^{-6}$ to keep the balance between the bias term and the variance term; or equivalently to

$$\varepsilon \rightsquigarrow \varepsilon/\sqrt{2} \qquad \text{and} \qquad M \rightsquigarrow 8\,M.$$

to reduce the error by a 2 factor.

EXAMPLES: (a) *Black-Scholes model.* The basic fields of applications in finance is the Greeks computation. This corresponds to functions F of the form

$$F(x,z) = e^{-rT} h\left(x\, e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}\right), \quad z \in \mathbb{R},\ x \in (0,\infty)$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is a Borel function with lienar growth. If h is Lipschitz continuous continuous, then

$$|F(x,Z) - F(x',Z)| \le [h]_{\text{Lip}}|x-x'|e^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}Z}$$

so that elementary computations show, using that $Z \stackrel{d}{=} \mathcal{N}(0;1)$,

$$\|F(x,Z) - F(x',Z)\|_2 \le [h]_{\text{Lip}}|x-x'|e^{\frac{\sigma^2}{2}T}.$$

The regularity of f follows from the following easy change of variable

$$f(x) = e^{-rT}\int_{\mathbb{R}} f\left(x\, e^{\mu T+\sigma\sqrt{T}z}\right)e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}} = e^{-rT}\int_{\mathbb{R}} f(y)e^{-\frac{\left(\log(y/x)-\mu T\right)^2}{2\sigma^2 T}}\frac{dy}{y\sigma\sqrt{2\pi}}.$$

where $\mu = r - \frac{\sigma^2}{2}$. This change of variable makes the integral appears as a "log"-convolution with similar regularizing effects as the standard convolution. Under appropriate growth assumption on the function h (say polynomial growth), one shows from the above identity that the function f is in fact infinitely differentiable over $(0,+\infty)$. In particular it is twice differentiable with Lipschitz continuous second derivative over any compact interval included in $(0,+\infty)$.

(b) *Diffusion model with Lipschitz continuous coefficients.* Let $X^x = (X^x_t)_{t\in[0,T]}$ denote the Brownian diffusion solution of the *SDE*

$$dX_t = b(X_t)dt + \vartheta(X_t)dW_t, \quad X_0 = x$$

where b and ϑ are locally Lipschitz continuous functions (on the real line) with at most linear growth (which implies the existence and uniqueness of a strong solution $(X^x_t)_{t\in[0,T]}$ starting from $X^x_0 = x$. In such a case one should rather writing

$$F(x,\omega) = h(X^x_T(\omega)).$$

The Lipschitz continuous continuity of the flow of the above SDE (see Theorem 7.10) shows that

$$\|F(x,.) - F(x',.)\|_2 \leq C_{b,\vartheta}[h]_{\mathrm{Lip}}|x - x'|e^{C_{b\vartheta}T}$$

where $C_{b,\vartheta}$ is a positive constant only depending on the Lipschitz continuous coefficients of b and ϑ. In fact this also holds for multi-dimensional diffusion processes and for path-dependent functionals.

The regularity of the function f is a less straightforward question. But the answer is positive in two situations: either h, b and σ are regular enough to apply results on the flow of the SDE which allows pathwise differentiation of $x \mapsto F(x, \omega)$ (see Theorem 9.1 further on in Section 9.2.2) or ϑ satisfies a uniform ellipticity assumption $\vartheta \geq \varepsilon_0 > 0$.

(c) *Euler scheme of a diffusion model with Lipschitz continuous coefficients.* The same holds for the Euler scheme. Furthermore Assumption (9.1) holds uniformly with respect to n if T/n is the step size of the Euler scheme.

(d) F can also stand for a functional of the whole path of a diffusion provided F is Lipschitz continuous with respect to the sup-norm over $[0, T]$.

As emphasized in the section devoted to the tangent process below, the generic parameter x can be the maturity T (in practice the *residual maturity* $T - t$, also known as *seniority*), or any finite dimensional parameter on which the diffusion coefficient depend since they always can be seen as a component or a starting value of the diffusion.

▷ **Exercises.** **1.** Adapt the results of this section to the case where $f'(x)$ is estimated by its "forward" approximation

$$f'(x) \approx \frac{f(x + \varepsilon) - f(x)}{\varepsilon}.$$

2. Apply the above method to approximate the γ-parameter by setting

$$f''(x) = \frac{f(x + \varepsilon) + f(x - \varepsilon) - 2f(x)}{\varepsilon^2}$$

under suitable assumptions on f and its derivatives.

The singular setting

In the setting described in the above Proposition 9.1, we are close to a framework in which one can interchange derivation and expectation: the (local) Lipschitz continuous assumption on $x' \mapsto F(x', Z)$ implies that $\left(\frac{F(x',Z) - F(x,Z)}{x' - x}\right)_{x' \in (x - \varepsilon_0, x + \varepsilon_0)}$ is a uniformly integrable family. Hence as soon as $x' \mapsto F(x', Z)$ is $\mathbb{P}$-*a.s.* pathwise differentiable at x (or even simply in $L^2(\mathbb{P})$), one has $f'(x) = \mathbb{E} \, F'_x(x, Z)$.

Consequently it is important to investigate the *singular setting* in which f is differentiable at x and $F(., Z)$ is not Lipschitz continuous in L^2. This is the purpose of the next proposition (whose proof is quite similar to that the Lipschitz continuous setting and is subsequently left to the reader as an exercise).

Proposition 9.2 *Let $x \in \mathbb{R}$. Assume that F satisfies in a neighbourhood $(x - \varepsilon_0, x + \varepsilon_0)$, $\varepsilon_0 > 0$, of x the following local mean quadratic θ-Hölder assumption ($\theta \in (0,1]$) assumption ("at x") i.e. there exists a positive real constant $C_{Hol,F,Z}$*

$$\forall\, x', x'' \in (x - \varepsilon_0, x + \varepsilon_0), \quad \|F(x'', Z) - F(x', Z)\|_2 \le C_{Hol,F,Z}|x'' - x'|^\theta.$$

Assume the function f is twice differentiable with a Lipschitz continuous second derivative on $(x - \varepsilon_0, x + \varepsilon_0)$. Let $(Z_k)_{k \ge 1}$ be a sequence of i.i.d. random vectors with the same distribution as Z, then for every $\varepsilon \in (0, \varepsilon_0)$,

$$\left\| f'(x) - \frac{1}{M} \sum_{k=1}^{M} \frac{F(x + \varepsilon, Z_k) - F(x - \varepsilon, Z_k)}{2\varepsilon} \right\|_2 \le \sqrt{\left([f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2}\right)^2 + \frac{C_{Hol,F,Z}^2}{(2\varepsilon)^{2(1-\theta)} M}}. \tag{9.7}$$

$$\le [f'']_{\mathrm{Lip}} \frac{\varepsilon^2}{2} + \frac{C_{Hol,F,Z}}{(2\varepsilon)^{1-\theta}\sqrt{M}}.$$

Once again, the variance of the finite difference estimator explodes as $\varepsilon \to 0$ as soon as $\theta \ne 1$. As a consequence, in such a framework, to divide the quadratic error by a 2 factor, we need to

$$\varepsilon \rightsquigarrow \varepsilon/\sqrt{2} \qquad \text{and} \qquad M \rightsquigarrow 2^{1-\theta} \times 4\, M.$$

A slightly different point of view in this singular case is to (roughly) optimize the parameter $\varepsilon = \varepsilon(M)$, given a simulation size M in order to minimize the quadratic error, or at least its natural upper bounds. Such an optimization performed in (9.7) yields

$$\varepsilon_{opt} = \left(\frac{2^\theta C_{Hol,F,Z}}{[f'']_{\mathrm{Lip}}\sqrt{M}} \right)^{\frac{1}{3-\theta}}.$$

which of course depends on M so that it breaks the recursiveness of the estimator. Moreover its sensitivity to $[f'']_{\mathrm{Lip}}$ makes its use rather unrealistic in practice.

The resulting rate of decay of the quadratic error is $O\left(M^{-\frac{2-\theta}{3-\theta}}\right)$. This rate show that when $\theta \in (0,1)$, the lack of L^2-regularity of $x \mapsto F(x, Z)$ slows down the convergence of the finite difference method by contrast with the Lipschitz continuous case where the standard rate of convergence of the Monte Carlo method is preserved.

EXAMPLE OF THE DIGITAL OPTION. A typical example of such a situation is the pricing of digital options (or equivalently the computation of the δ-hedge of a *Call* or *Put* options).

Let us consider in the still in the standard risk neutral Black-Scholes model a digital *Call* option with strike price $K > 0$ defined by its payoff

$$h(\xi) = \mathbf{1}_{\{\xi \ge K\}}$$

and set $F(x, z) = e^{-rT} h\left(x\, e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}z} \right)$, $z \in \mathbb{R}$, $x \in (0, \infty)$ (r denotes the constant interest rate as usual). We know that the premium of this option is given for every initial price $x > 0$ of the underlying risky asset by

$$f(x) = \mathbb{E}\, F(x, Z) \quad \text{where} \quad Z \stackrel{d}{=} \mathcal{N}(0; 1).$$

Set $\mu = r - \frac{\sigma^2}{2}$. It is clear (using that Z and $-Z$ have the same distribution), that

$$
\begin{aligned}
f(x) &= e^{-rT}\mathbb{P}\Big(x\,e^{\mu T + \sigma\sqrt{T}Z} \ge K\Big) \\
&= e^{-rT}\mathbb{P}\Big(Z \ge -\frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\Big) \\
&= e^{-rT}\Phi_0\Big(\frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\Big)
\end{aligned}
$$

where Φ_0 denotes the distribution function of the $\mathcal{N}(0;1)$ distribution. Hence the function f is infinitely differentiable on $(0,\infty)$ since the probability density of the standard normal distribution is on the real line.

On the other hand, still using that Z and $-Z$ have the same distribution, for every x, $x' \in \mathbb{R}$,

$$
\begin{aligned}
\|F(x,Z) - F(x',Z)\|_2^2 &= e^{-2rT}\left\|\mathbf{1}_{\left\{Z \ge -\frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\right\}} - \mathbf{1}_{\left\{Z \ge -\frac{\log(x'/K)+\mu T}{\sigma\sqrt{T}}\right\}}\right\|_2^2 \\
&= e^{-2rT}\mathbb{E}\left|\mathbf{1}_{\left\{Z \le \frac{\log(x/K)+\mu T}{\sigma\sqrt{T}}\right\}} - \mathbf{1}_{\left\{Z \le \frac{\log(x'/K)+\mu T}{\sigma\sqrt{T}}\right\}}\right|^2 \\
&= e^{-2rT}\left(\Phi_0\Big(\frac{\log(\max(x,x')/K)+\mu T}{\sigma\sqrt{T}}\Big) - \Phi_0\Big(\frac{\log(\min(x,x')/K)+\mu T}{\sigma\sqrt{T}}\Big)\right).
\end{aligned}
$$

Using that that Φ_0' is bounded (by $\kappa_0 = \frac{1}{\sqrt{2\pi}}$), we derive that

$$
\|F(x,Z) - F(x',Z)\|_2^2 \le \frac{\kappa_0 e^{-2rT}}{\sigma\sqrt{T}}\left|\log x - \log x'\right|.
$$

Consequently for every interval $I \subset (0,\infty)$ *bounded away from* 0, there exists a real constant $C_{r,\sigma,T,I} > 0$ such that

$$
\forall\, x,\, x' \in I, \quad \|F(x,Z) - F(x',Z)\|_2 \le C_{r,\sigma,T,I}\sqrt{|x - x'|}
$$

i.e. the functional F is $\frac{1}{2}$-Hölder in $L^2(\mathbb{P})$ and the above proposition applies.

▷ **Exercises. 1.** Prove the above proposition 9.2.

2. *Digital option.* (a) Consider in the risk neutral Black-Scholes model a digital option defined by its payoff

$$
h(\xi) = \mathbf{1}_{\{\xi \ge K\}}
$$

and set $F(x,z) = e^{-rT}h\left(x\,e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}z}\right)$, $z \in \mathbb{R}$, $x \in (0,\infty)$ (r is a constant interest rate as usual). We still consider the computation of $f(x) = \mathbb{E}\,F(x,Z)$ where $Z \overset{d}{=} \mathcal{N}(0;1)$.

(b) Verify on a numerical simulation that the variance of the finite difference estimator introduced in Theorem 9.1 explodes as $\varepsilon \to 0$ at the rate expected from the computations that precede.

(c) Show that, in a fixed neighbourhood of $x \in (0,+\infty)$. Derive from what precedes a way to "synchronize" the step ε and the size M of the simulation.

9.1.2   A recursive approach: finite difference with decreasing step

In the former finite difference method with constant step, the bias never fades. Consequently, increasing the accuracy of the sensitivity computation, requires to resume it from the beginning with a new ε. In fact it is easy to propose a recursive version of the above finite difference procedure by considering some variable steps ε which go to 0. This can be seen as an application of the Kiefer-Wolfowitz principle originally developed for Stochastic Approximation purpose.

We will focus on the "regular setting" (F Lipschitz) in this section, the singular setting is proposed as an exercise. Let $(\varepsilon_k)_{k\geq 1}$ be a sequence of positive real numbers decreasing to 0. With the notations and the assumptions of the former section, consider the estimator

$$\widehat{f'(x)}_M := \frac{1}{M} \sum_{k=1}^{M} \frac{F(x + \varepsilon_k, Z_k) - F(x - \varepsilon_k, Z_k)}{2\varepsilon_k}. \tag{9.8}$$

It can be computed in a recursive way since

$$\widehat{f'(x)}_{M+1} = \widehat{f'(x)}_M + \frac{1}{M+1}\left(\frac{F(x + \varepsilon_{M+1}, Z_{M+1}) - F(x - \varepsilon_{M+1}, Z_{M+1})}{2\varepsilon_{M+1}} - \widehat{f'(x)}_M \right).$$

Elementary computations show that the squared quadratic error satisfy

$$
\begin{aligned}
\left\| f'(x) - \widehat{f'(x)}_M \right\|_2^2 &= \left(f'(x) - \frac{1}{M} \sum_{k=1}^{M} \frac{f(x + \varepsilon_k) - f(x - \varepsilon_k)}{2\varepsilon_k} \right)^2 \\
&\quad + \frac{1}{M^2} \sum_{k=1}^{M} \frac{\mathrm{Var}(F(x + \varepsilon_k, Z_k) - F(x - \varepsilon_k, Z_k))}{4\varepsilon_k^2} \\
&\leq \frac{[f'']_{\mathrm{Lip}}^2}{4M^2} \left(\sum_{k=1}^{M} \varepsilon_k^2 \right)^2 + \frac{C_{F,Z}^2}{M} \\
&= \frac{1}{M} \left(\frac{[f'']_{\mathrm{Lip}}^2}{4M} \left(\sum_{k=1}^{M} \varepsilon_k^2 \right)^2 + C_{F,Z}^2 \right)
\end{aligned}
\tag{9.9}
$$

where we used again (9.4) to get Inequality (9.9).

As a consequence

$$\left\| f'(x) - \widehat{f'(x)}_M \right\|_2 \leq \frac{1}{\sqrt{M}} \sqrt{ \frac{[f'']_{\mathrm{Lip}}^2}{4M} \left(\sum_{k=1}^{M} \varepsilon_k^2 \right)^2 + C_{F,Z}^2 }. \tag{9.10}$$

Standard $\sqrt{M}$-L^2 of convergence   In order to prove a $\frac{1}{\sqrt{M}}$ rate (like in a standard Monte Carlo simulation) we need the sequence $(\varepsilon_m)_{m\geq 1}$ and the size M to satisfy

$$\left(\sum_{k=1}^{M} \varepsilon_k^2 \right)^2 = O(M).$$

This leads to choose ε_k of the form

$$\varepsilon_k = O\big(k^{-\frac{1}{4}}\big) \quad \text{as } k \to +\infty,$$

since

$$\sum_{k=1}^{M} \frac{1}{k^{\frac{1}{2}}} = \sqrt{M} \frac{1}{M} \sum_{k=1}^{M} \left(\frac{k}{M}\right)^{-\frac{1}{2}} \sim \sqrt{M} \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{M} \quad \text{as} \quad M \to +\infty.$$

Erasing the asymptotic bias A more efficient way to take advantage of a decreasing step approach is to erase the bias by considering steps of the form

$$\varepsilon_k = o\big(k^{-\frac{1}{4}}\big) \quad \text{as } k \to +\infty.$$

Then, the first term in the right hand side of (9.10) goes to zero. So the bias asymptotically fades in the scale $\sqrt{M}$

However, the choice of too small steps ε_m may introduce numerical instability in the computations, so we recommend to choose the ε_k of the form

$$\varepsilon_k = o\big(k^{-(\frac{1}{4}+\delta)}\big) \quad \text{with } \delta > 0 \text{ small enough.}$$

One can refine the bound obtained in (9.10): note that

$$\sum_{k=1}^{M} \frac{\operatorname{Var}(F(x+\varepsilon_k, Z_k) - F(x-\varepsilon_k, Z_k))}{4\varepsilon_k^2} \;=\; \sum_{k=1}^{M} \frac{\big\| F(x+\varepsilon_k, Z_k) - F(x-\varepsilon_k, Z_k) \big\|_2^2}{4\varepsilon_k^2}$$

$$- \sum_{k=1}^{M} \left(\frac{f(x+\varepsilon_k) - f(x-\varepsilon_k)}{4\varepsilon_k^2}\right)^2.$$

Now, since $\dfrac{f(x+\varepsilon_k) - f(x-\varepsilon_k)}{2\varepsilon_k} \to f'(x)$ as $k \to +\infty$,

$$\frac{1}{M} \sum_{k=1}^{M} \left(\frac{f(x+\varepsilon_k) - f(x-\varepsilon_k)}{4\varepsilon_k^2}\right)^2 \to f'(x)^2 \quad \text{as } M \to +\infty.$$

Plugging this in the above computations yields the refined asymptotic upper-bound

$$\limsup_{M \to +\infty} \sqrt{M} \left\| f'(x) - \widehat{f'(x)}_M \right\|_2 \le \sqrt{C_{F,Z}^2 - (f'(x))^2}.$$

This approach has the same quadratic rate of convergence as a regular Monte Carlo simulation (*e.g.* a simulation carried out with $\frac{\partial F}{\partial x}(x, Z)$ if it exists).

Now, we show that the estimator $\widehat{f'(x)}_M$ is consistent *i.e.* converging toward $f'(x)$.

Proposition 9.3 *Under the assumptions of Proposition 9.1 and if ε_k goes to zero as k goes to infinity, the estimator $\widehat{f'(x)}_M$ a.s. converges to its target $f'(x)$.*

Proof. It amounts to showing that

$$\frac{1}{M}\sum_{k=1}^{M}\frac{F(x+\varepsilon_k,Z_k)-F(x-\varepsilon_k,Z_k)}{2\varepsilon_k}-\frac{f(x+\varepsilon_k)-f(x-\varepsilon_k)}{2\varepsilon_k}\xrightarrow{a.s.}0.\qquad(9.11)$$

This is (again) a straightforward consequence of the *a.s.* convergence of L^2-bounded martingales combined with the Kronecker Lemma (see Lemma 11.1): first define the martingale

$$L_M=\sum_{k=1}^{M}\frac{1}{k}\frac{F(x+\varepsilon_k,Z_k)-F(x-\varepsilon_k,Z_k)-(f(x+\varepsilon_k)-f(x-\varepsilon_k))}{2\varepsilon_k},\quad M\geq 1.$$

One checks that

$$\begin{aligned}\mathbb{E}\,L_M^2 &= \sum_{k=1}^{M}\mathbb{E}\,(\Delta L_k)^2=\sum_{k=1}^{M}\frac{1}{4k^2\varepsilon_k^2}\mathrm{Var}(F(x+\varepsilon_k,Z_k)-F(x-\varepsilon_k,Z_k))\\ &\leq \sum_{k=1}^{M}\frac{1}{4k^2\varepsilon_k^2}\mathbb{E}\,(F(x+\varepsilon_k,Z_k)-F(x-\varepsilon_k,Z_k))^2\\ &\leq \sum_{k=1}^{M}\frac{1}{4k^2\varepsilon_k^2}C_{F,Z}^2\,4\,\varepsilon_k^2=C_{F,Z}^2\sum_{k=1}^{M}\frac{1}{k^2}\end{aligned}$$

so that

$$\sup_M\mathbb{E}L_M^2<+\infty.$$

Consequently, L_M *a.s.* converges to a square integrable (hence *a.s.* finite) random variable L_∞ as $M\to+\infty$. The announced *a.s.* convergence in (9.11) follows from the Kronecker Lemma (see Lemma 11.1). ◇

▷ **Exercises. 1.** (*Central Limit Theorem*) Assume that $x\mapsto F(x,Z)$ is Lipschitz continuous from $\mathbb{R}$ to $L^{2+\eta}(\mathbb{P})$ for an $\eta>0$. Show that the convergence of the finite difference estimator with decreasing step $\widehat{f'(x)}_M$ defined in (9.8) satisfies the following property: from every subsequence (M') of (M) one may extract a subsequence (M'') such that

$$\sqrt{M''}\left(\widehat{f'(x)}_{M''}-f'(x)\right)\xrightarrow{\mathcal{L}}\mathcal{N}(0;v),\ v\in[0,\bar v],\ \text{as}\ M\to+\infty$$

where $\bar v=C_{F,Z}^2-\left(f'(x)\right)^2$.

[Hint: Note that the sequence $\left(\mathrm{Var}\big(\frac{F(x+\varepsilon_k,Z_k)-F(x-\varepsilon_k,Z_k)}{2\varepsilon_k}\big)\right)$, $k\geq 1$, is bounded and use the following Central Limit Theorem: if $(Y_n)_{n\geq1}$ is a sequence of i.i.d. random variables such that there exists $\eta>0$ satisfying

$$\sup_n\mathbb{E}|Y_n|^{2+\eta}<+\infty\quad\text{and}\quad \exists\,N_n\to+\infty\ \text{with}\ \frac{1}{N_n}\sum_{k=1}^{N_n}\mathrm{Var}(Y_k)\xrightarrow{n\to+\infty}\sigma^2>0$$

then

$$\frac{1}{\sqrt{N_n}} \sum_{k=1}^{N_n} Y_k \xrightarrow{\mathcal{L}} \mathcal{N}(0; \sigma^2) \quad \text{as } n \to +\infty.]$$

2. (*Hölder framework*) Assume that $x \mapsto F(x, Z)$ is only θ-Hölder from $\mathbb{R}$ to $L^2(\mathbb{P})$ with $\theta \in (0, 1)$, like in Proposition 9.2.

(*a*) Show that a natural upper-bound for the quadratic error induced by the symmetric finite difference estimator with decreasing step $\widehat{f'(x)}_M$ defined in (9.8) is given by

$$\frac{1}{M} \sqrt{\frac{[f'']^2_{\text{Lip}}}{4} \left(\sum_{k=1}^{M} \varepsilon_k^2\right)^2 + \frac{C_{F,Z}^2}{2^{2(1-\theta)}} \sum_{k=1}^{M} \frac{1}{\varepsilon_k^{2(1-\theta)}}}.$$

(*b*) Show that the resulting estimator $\widehat{f'(x)}_M$ a.s. converges to its target $f'(x)$ as soon as

$$\sum_{k \geq 1} \frac{1}{k^2 \varepsilon_k^{2(1-\theta)}} < +\infty.$$

(*c*) Assume that $\varepsilon_k = \frac{c}{k^a}$, $k \geq 1$, where c is a positive real constant and $a \in (0, 1)$. Show that the exponent a corresponds to an admissible step iff $a \in (0, \frac{1}{2(1-\theta)})$. Justify the choice of $a^* = \frac{1}{2(3-\theta)}$ for the exponent a and derive that the resulting rate of decay of the quadratic error is $O\left(\sqrt{M}^{-\frac{2}{3-\theta}}\right)$.

The above exercise shows that the (lack of) regularity of $x \mapsto F(x, Z)$ in $L^2(\mathbb{P})$ does impact the rate of convergence of the finite difference method.

9.2 Pathwise differentiation method

9.2.1 (Temporary) abstract point of view

We keep the notations of the former section. We assume that there exists $p \in [1, +\infty)$ such that

$$\forall \xi \in (x - \varepsilon_0, x + \varepsilon_0), \quad F(\xi, Z) \in L^p(\mathbb{P}).$$

Definition 9.1 *The function $\xi \mapsto F(\xi, Z)$ from $(x - \varepsilon_0, x + \varepsilon_0)$ to $L^p(\mathbb{P})$ is L^p-differentiable at x if there exists a random vector denoted $\partial_x F(x, .) \in L^p(\mathbb{P})$ such that*

$$\lim_{\xi \to x, \xi \neq x} \left\| \frac{F(x, Z) - F(\xi, Z)}{x - \xi} - \partial_x F(x, \omega) \right\|_p = 0.$$

Proposition 9.4 *If $\xi \mapsto F(\xi, Z)$ is L^p-differentiable at x, then the function f defined by $f(\xi) = \mathbb{E}\, F(\xi, Z)$ is differentiable at x and*

$$f'(x) = \mathbb{E}\big(\partial_x F(x, \omega)\big).$$

Proof. It is a straightforward consequence of the inequality $|\mathbb{E}\,Y| \leq \|Y\|_p$ (which holds for any $Y \in L^p(\mathbb{P})$) applied to $Y(\omega) = \frac{F(x,Z(\omega)) - F(\xi,Z(\omega))}{x-\xi} - \partial_x F(x,\omega)$ by letting ξ converge to x.   ◇

PRACTICAL APPLICATION. As soon as $\partial_x F(x,\omega)$ is simulatable at a reasonable cost, one may compute $f'(x)$ by a standard Monte Carlo simulation.

The usual criterion to establish L^p-differentiability, especially when the underlying source of randomness comes from a diffusion (see below), is to establish a pathwise differentiability of $\xi \mapsto F(\xi, Z(\omega))$ combined with an L^p-uniform integrability property of the ratio $\frac{F(x,Z) - F(\xi,Z)}{x-\xi}$ (see Theorem 11.2 and the Corollary that follows in Chapter 11 for a short background on uniform integrability).

Usually, this is applied with $p = 2$ since one needs $\partial_x F(x,\omega)$ to be in $L^2(\mathbb{P})$ to ensure that the Central Limit Theorem applies to rule the rate of convergence in the Monte Carlo simulation.

This can be summed up in the following proposition which proof is obvious.

Proposition 9.5 *Let $p \in [1, +\infty)$. If*

(i) there exists a random variable $\partial_x F(x,.)$ such that $\mathbb{P}(d\omega)$-a.s. $\xi \mapsto F(\xi, Z(\omega))$ is differentiable at x with derivative $\partial_x F(x,\omega)$,

(ii) there exists $\varepsilon_0 > 0$ such that the family $\left(\frac{F(x,Z) - F(\xi,Z)}{x-\xi} \right)_{\xi \in (x-\varepsilon_0, x+\varepsilon_0) \setminus \{x\}}$ is L^p-uniformly integrable,

then $\partial_x F(x,.) \in L^p(\mathbb{P})$ and $\xi \mapsto F(\xi, Z)$ is L^p-differentiable at x with derivative $\partial_x F(x,.)$.

9.2.2 Tangent process of a diffusion and application to sensitivity computation

In a diffusion framework, it is important to have in mind that, more or less like in a deterministic setting, the solution of an SDE is usually smooth when viewed as a (random) function of its starting value. This smoothness even holds in a pathwise sense with a an (almost) explicit differential. The main result in that direction is due to Kunita (see [87], Theorem 3.1).

This result must be understood as follows: when a sensitivity (the δ-hedge and the γ parameter but also other "greek parameters" as will be seen further on) related to the premium $\mathbb{E}\,h(X_T^x)$ of a an option cannot be computed by "simply" interchanging differentiation and expectation, this lack of differentiablity comes from the payoff function h. This also holds true for path-dependent options.

Let us come to Kunita's theorem on the regularity of the flow of an SDE.

Theorem 9.1 *(a) Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\vartheta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathcal{M}(d,q,\mathbb{R})$, with regularity $\mathcal{C}_b^1$ with bounded α-Hölder partial derivatives for an $\alpha > 0$. Let $X^x = (X_t^x)_{t \geq 0}$ denote the unique strong solution of the SDE*

$$dX_t = b(t, X_t)dt + \vartheta(t, X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d, \tag{9.12}$$

where $W = (W^1, \ldots, W^q)$ is a q-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then at every $t \in \mathbb{R}_+$, the mapping $x \mapsto X_t^x$ is a.s. continuously differentiable and its

gradient $Y_t(x) := \nabla_x X_t^x = \left[\frac{\partial (X_t^x)^i}{\partial x^j}\right]_{1 \leq i,j \leq d}$ satisfies the linear stochastic differential system

$$\forall t \in \mathbb{R}_+, \quad Y_t^{ij}(x) = \delta_{ij} + \sum_{\ell=1}^{d} \int_0^t \frac{\partial b^i}{\partial y^\ell}(s, X_s^x) Y_s^{\ell j}(x) ds + \sum_{\ell=1}^{d} \sum_{k=1}^{q} \int_0^t \frac{\partial \vartheta_{ik}}{\partial y^\ell}(s, X_s^x) Y_s^{\ell j}(x) dW_s^k, \ 1 \leq i,j \leq d,$$

where δ_{ij} denote the Kronecker symbol.

A verifier!!! (b) Furthermore, the tangent process $Y(x)$ takes values in the set $GL(d, \mathbb{R})$ of invertible square matrices. (see Theorem ?? in [?]).

Remark. One easily derives from the above theorem the slightly more general result about the tangent process to the solution $(X_s^{t,x})_{s \in [t,T]}$ starting from x at time t. This process, denoted $Y(tx)_s)_{\sigma \geq t}$ can be deduced from $Y(x)$ be

$$\forall s \geq t, \quad Y(t,x)_s = Y(x)_s Y^{-1}(x)_t.$$

This is a consequence often unique ness of the solution of a linear SDE.

Remark. • Higher order differentiability properties hold true if b are ϑ are smoother. For a more precise statement, see Section 9.2.2 below.

Example. If $d = q = 1$, the above SDE reads

$$dY_t(x) = Y_t(x)\Big(b_x'(t, X_t)dt + \vartheta_x'(t, X_t^x)dW_t\Big), \qquad Y_0(x) = 1$$

and elementary computations show that

$$Y_t(x) = \exp\left(\int_0^t \left(b_x'(s, X_s^x) - \frac{1}{2}(\vartheta_x'(s, X_s^x))^2\right) ds + \int_0^t \vartheta_x'(s, X_s^x)dW_s\right) \qquad (9.13)$$

so that, in the Black-Scholes model $(b(t,x) = rx, \vartheta(t,x) = \vartheta x)$, one retrieves that

$$\frac{d}{dx}X_t^x = Y_t(x) = \frac{X_t^x}{x}.$$

▷ **Exercise.** Let $d = q = 1$. Show that under the assumptions of Theorem 9.1, the tangent process at x $Y_t(x) = \frac{d}{dx}X_t^x$ satisfies

$$\sup_{s,t \in [0,T]} \frac{Y_t(x)}{Y_s(x)} \in L^p(\mathbb{P}), \quad p > 0.$$

APPLICATIONS TO δ-HEDGING. The tangent process and the δ hedge are closely related. Assume that the interest rate is 0 (for convenience) and that a basket is made up of d risky assets whose price dynamics $(X_t^x)_{t \in [0,T]}$, $X_0^x = x \in (0, +\infty)^d$, $(X_t^x)_{t \in [0,T]}$ is solution to (9.12).

Then the premium of the payoff $h(X_T^x)$ on the basket is given by

$$f(x) := \mathbb{E}\, h(X_T^x).$$

The δ-hedge vector of this option (at time 0 and) at $x = (x^1, \ldots, x^d) \in (0, +\infty)^d$ is given by $\nabla f(x)$.

We have the following proposition that establishes the existence and the representation of $f'(x)$ as an expectation (with in view its computation by a Monte Carlo simulation). It is a straightforward application of Theorem 2.1(b).

Proposition 9.6 *(δ-hedge of vanilla European options) If a Borel function $h : \mathbb{R}^d \to \mathbb{R}$ satisfies the following assumptions:*

(i) A.s. differentiability: $\nabla h(y)$ *exists* $\mathbb{P}_{X_T^x}(dy)$*-a.s.,*

(ii) Uniform integrability property [3]*: there exists a neighbouring interval* $(x - \varepsilon_0, x + \varepsilon_0)$ *($\varepsilon_0 > 0$) of x such that,*

$$\left(\frac{|h(X_T^x) - h(X_T^{x'})|}{|x - x'|} \right)_{|x'-x|<\varepsilon_0, x' \neq x} \qquad \text{is uniformly integrable.}$$

Then f is differentiable at x and $\nabla f(x)$ has the following representation as an expectation:

$$\frac{\partial f}{\partial x^i}(x) = \mathbb{E}\left(\left\langle \nabla h(X_T^x) | \frac{\partial X_T^x}{\partial x^i} \right\rangle \right), \ i = 1, \ldots, d. \tag{9.14}$$

Remark. One can also consider a forward start payoff $h(X_{T_1}^x, \ldots, X_{T_N}^x)$. Then, under similar assumptions its premium $v(x)$ is differentiable and

$$\nabla f(x) = \sum_{j=1}^{N} \mathbb{E}\left(\left\langle \nabla_{y^j} h(X_{T_1}^x, \ldots, X_{T_N}^x) | \nabla_x X_{T_j}^x \right\rangle \right).$$

Computation by simulation

One uses these formulae to compute some sensibility by Monte Carlo simulations since sensitivity parameters are functions of the couple made by the diffusion process X^x solution to the SDE starting at x and its tangent process $\nabla_x X^x$ at x: it suffices to consider the *Euler scheme of this couple* $(X_t^x, \nabla_x X_t^x)$ over $[0, T]$ with step $\frac{T}{n}$.

Assume $d = 1$ for notational convenience and set $Y_t = \nabla_x X_t^x$:

$$\begin{aligned} dX_t^x &= b(X_t^x)dt + \vartheta(X_t^x)dW_t, \qquad X_0^x = x \in \mathbb{R} \\ dY_t &= Y_t(b'(X_t^x)dt + \vartheta'(X_t^x)dW_t), \qquad Y_0 = 1. \end{aligned}$$

In 1-dimension, one can take advantage of the semi-closed formula (9.13) obtained in the above exercise for the tangent process.

Extension to an exogenous parameter θ

All the theoretical results obtained for the δ, *i.e.* for the differentiation of the flow of an SDE with respect to its initial value can be extended to any parameter provided no ellipticity is required. This follows from the remark that if the coefficient(s) b and/or ϑ of a diffusion depend(s) on a parameter θ then the couple (X_t, θ) is still diffusion process, namely

$$\begin{aligned} dX_t &= b(\theta, X_t)dt + \vartheta(\theta, X_t)dW_t, \quad X_0 = x, \\ d\theta_t &= 0, \qquad\qquad\qquad\qquad\qquad \theta_0 = \theta. \end{aligned}$$

---

[3]If h is Lipschitz continuous and X^x is a solution to an SDE with Lipschitz continuous coefficients b and ϑ in the sense of (7.2), this uniform integrability is always satisfied since it follows from Theorem 7.10 applied with $p > 1$.

Set $\widetilde{x} = (x, \theta)$ and $\widetilde{X}_t^{x'} := (X_t^x, \theta_t)$. Thus, following Theorems 3.1 and 3.3 of Section 3 in [87], if $(\theta, x) \mapsto b(\theta, x)$ and $(\theta, x) \mapsto \vartheta(\theta, x)$ are $\mathcal{C}_b^{k+\alpha}$ $(0 < \alpha < 1)$ with respect to x and θ [4] then the solution of the SDE at a given time t will be $\mathcal{C}^{k+\beta}$ $(0 < \beta < \alpha)$ as a function of (x and θ). A more specific approach would show that some regularity in the sole variable θ would be enough but then this result does not follow for free from the general theorem of the differentiability of the flows.

Assume $b = b(\theta, .)$ and $\vartheta = \vartheta(\theta, .)$ and the initial value $x = x(\theta)$, $\theta \in \Theta$, $\Theta \subset \mathbb{R}^q$ (open set). One can also differentiate a SDE with respect to this parameter θ. We can assume that $q = 1$ (by considering a partial derivative if necessary). Then, one gets

$$\frac{\partial X_t(\theta)}{\partial \theta} = \frac{\partial x(\theta)}{\partial \theta} + \int_0^t \left(\frac{\partial b}{\partial \theta}(\theta, X_s(\theta)) + \frac{\partial b}{\partial x}(\theta, X_s(\theta)) \left(\frac{\partial X_s(\theta)}{\partial \theta} \right) \right) ds$$
$$+ \int_0^t \left(\frac{\partial \vartheta}{\partial \theta}(\theta, X_s(\theta)) + \frac{\partial \vartheta}{\partial x}(\theta, X_s(\theta)) \left(\frac{\partial X_s(\theta)}{\partial \theta} \right) \right) dW_s.$$

Toward practical implementation (practitioner's corner)

Once coupled with the original diffusion process X^x, this yields some expressions for the the sensitivity with respect to the parameter θ, possibly closed, but usually computable by *a Monte Carlo simulation of the Euler scheme of the couple* $\left(X_t(\theta), \frac{\partial X_t(\theta)}{\partial \theta} \right)$.

As a conclusion let us mention that this tangent process approach is close to the finite difference method applied to $F(x, \omega) = h(X_T^x(\omega))$: it appears as a limit case of the finite difference method.

9.3 Sensitivity computation for non smooth payoffs

The tangent process based approach needs a smoothness assumption of the payoff function, typically almost everywhere differentiability, as emphasized above (and in Section 2.3). Unfortunately this assumption is not fulfilled by many usual payoffs functions like the digital payoff

$$h_T = h(X_T) \quad \text{with} \quad h(x) := \mathbf{1}_{\{x \geq K\}}$$

whose δ hedge parameter cannot be computed by the tangent process method (in fact $\frac{\partial}{\partial x} \mathbb{E}\, h(X_T^x)$ is but the probability density of X_T^x at the strike K). For similar reasons, so is the case for the γ sensitivity parameter of a vanilla Call option.

We also saw in Section 2.3 that in the Black-Scholes model, this problem can be overcome since integration by parts or differentiating the log-likelihood leads to some sensitivity formulas for non smooth payoffs. Is it possible to extend this idea to more general models?

9.3.1 The log-likelihood approach (II)

A general abstract result

We saw in Section 2.3 that a family of random vectors $X(\theta)$ indexed by a parameter $\theta \in \Theta$, Θ open interval of $\mathbb{R}$, all have a positive probability density $p(\theta, y)$ with respect to a reference nonnengative

---

[4]A function, g has a $\mathcal{C}_b^{k+\alpha}$ regularity if g is $\mathcal{C}^k$ with k-th order partial derivatives globally α-Holder and all partial derivatives up to k-th order bounded.

measure μ on $\mathbb{R}^d$. Assume this density is positive on a domain D of $\mathbb{R}^d$ for every $\theta \in \Theta$. Then one could derive the sensitivity of functions

$$f(\theta) := \mathbb{E}\,\varphi(X(\theta)) = \int_D \varphi(y)p(\theta, y)\mu(dy)$$

(with respect to $\theta \in \Theta$, φ Borel function with appropriate integrability assumptions) provided the density function p is smooth enough as a function of θ, *regardless of the regularity of φ*. In fact, this was briefly developed in a one dimensional Black-Scholes framework but the extension to an abstract framework is straightforward and yields the following result.

Proposition 9.7 *If the probability density $p(\theta, y)$ as a function defined on $\Theta \times D$ satisfies*

(i) $\qquad\qquad\qquad\qquad\qquad \theta \longmapsto p(\theta, y)$ *is differentiable on Θ* $\quad \mu(dy)$-*a.e.*

(ii) $\quad \exists\, g\mathbb{R}^d \to \mathbb{R}, Borel\ such\ that\ g\varphi \in L^1(\mu)\ and\, \forall\, \theta \in \Theta,\ \left(|\partial_\theta p(\theta, y)| \le g(y)\ \mu(dy)\text{-}a.e.\right),$

then

$$\forall\, \theta \in \Theta, \qquad f'(\theta) = \mathbb{E}\left(\varphi\big(X(\theta)\big)\frac{\partial \log p}{\partial \theta}(\theta, X(\theta))\right).$$

The log-likelihood method for the Euler scheme

At a first glance this approach is attractive, unfortunately, in most situations we have no explicitly computable form for the density $p(\theta, y)$ even when its existence is proved. However, if one thinks about diffusion approximation (in a non-degenerate setting) by some discretization schemes like Euler schemes, the application of the log-likelihood method becomes much less unrealistic at least to compute by simulation some proxies of the greek parameters by a Monte Carlo simulation.

As a matter of fact, under slight ellipticity assumption, the (constant step) Euler scheme of a diffusion does have a probability density at each time $t \ge 0$ which can be made explicit (in some way, see below). This is a straightforward consequence of the fact that it is a discrete time Markov process with conditional Gaussian increments. The principle is the following. We consider a diffusion $(X_t(\theta))_{t \in [0,T]}$ depending on a parameter $\theta \in \Theta$, say

$$dX_t(\theta) = b(\theta, X_t(\theta))\,dt + \vartheta(\theta, X_t(\theta))dW_t, \quad X_0(\theta) = x.$$

Let $p_T(\theta, x, y)$ and $\bar{p}_T(\theta, x, y)$ denote the density of $X_T^x(\theta)$ and its Euler scheme $\bar{X}_T^x(\theta)$ (with step size $\frac{T}{n}$). Then one may naturally propose the following naive approximation

$$f'(\theta) = \mathbb{E}\left(\varphi(X_T^x(\theta))\frac{\partial \log p_T}{\partial \theta}(\theta, x, X_T^x(\theta))\right) \approx \mathbb{E}\left(\varphi(\bar{X}_T^x(\theta))\frac{\partial \log \bar{p}_T}{\partial \theta}(\theta, x, \bar{X}_T^x(\theta))\right).$$

In fact the story is not as straightforward because what can be made explicit and tractable is the density of the whole n-tuple $(\bar{X}_{t_1^n}^x, \ldots, \bar{X}_{t_k^n}^x, \ldots, \bar{X}_{t_n^n}^x)$ (with $t_n^n = T$).

Proposition 9.8 *Let $q \ge d$ and $\vartheta\vartheta^*(\theta, x) \in GL(d, \mathbb{R})$ for every $x \in \mathbb{R}^d$, $\theta \in \Theta$.*

(a) *Then the distribution* $\mathbb{P}_{\bar{X}^x_{\frac{T}{n}}}(dy)$ *of* $\bar{X}^x_{\frac{T}{n}}$ *has a probability density given by*

$$\bar{p}_{\frac{T}{n}}(\theta, x, y) = \frac{1}{(2\pi\frac{T}{n})^{\frac{d}{2}}\sqrt{\det \vartheta\vartheta^*(\theta, x)}} e^{-\frac{n}{2T}(y-x-\frac{T}{n}b(\theta,x))^*\,(\vartheta\vartheta^*(\theta,x))^{-1}(y-x-\frac{T}{n}b(\theta,x))}.$$

(b) *The distribution* $\mathbb{P}_{(\bar{X}^x_{t^n_1},\dots,\bar{X}^x_{t^n_k},\dots,\bar{X}^x_{t^n_n})}(dy_1,\dots,dy_n)$ *of the* n-*tuple* $(\bar{X}^x_{t^n_1},\dots,\bar{X}^x_{t^n_k},\dots,\bar{X}^x_{t^n_n})$ *has a probability density given by*

$$\bar{p}_{t^n_1,\dots,t^n_n}(\theta, x, y_1,\dots,y_n) = \prod_{k=1}^{n} \bar{p}_{\frac{T}{n}}(\theta, y_{k-1}, y_k)$$

with the convention $y_0 = x$.

Proof. (a) is a straightforward consequence of the definition of the Euler scheme at time $\frac{T}{n}$ and the formula for the density of a Gaussian vector. Claim (b) follows from an easy induction based on the Markov property satisfied by the Euler scheme [Details in progress, to be continued...]. ◇

The above proposition shows that every marginal $\bar{X}^x_{t^n_k}$ has a density which, unfortunately, cannot be made explicit. So to take advantage of the above closed form for the density of the n-tuple, we can write

$$
\begin{aligned}
f'(\theta) &\approx \mathbb{E}\left(\varphi(\bar{X}^x_T(\theta))\frac{\partial \log \bar{p}_{t^n_1,\dots,t^n_n}}{\partial \theta}(\theta, x, \bar{X}^x_{t^n_1}(\theta),\dots,\bar{X}^x_{t^n_1}(\theta))\right)\\
&= \sum_{k=1}^{n}\mathbb{E}\left(\varphi(\bar{X}^x_T(\theta))\frac{\partial \log \bar{p}_{\frac{T}{n}}}{\partial \theta}(\theta, \bar{X}^x_{t^n_{k-1}}(\theta), \bar{X}^x_{t^n_k}(\theta))\right).
\end{aligned}
$$

At this stage it appears clearly that the method also works for path dependent problems *i.e.* when considering $\Phi((X_t(\theta))_{t\in[0,T]})$ instead of $\varphi(X_T(\theta))$ (at least for specific functionals Φ involving time averaging, a finite number of instants, supremum, infimum, etc). This leads to new difficulties in connection with the Brownian bridge method for diffusions, that need to be encompassed.

Finally, let us mention that evaluating the rate of convergence of these approximations from a theoretical point of view is quite a challenging problems since it involves not only the rate of convergence of the Euler scheme itself but also that of the probability density functions of the scheme toward that of the diffusion (see [10]).

▷ **Exercise.** Apply what precedes to the case $\theta = x$ (starting value) when $d = 1$.

9.4 A flavour of stochastic variational calculus: from Bismut to Malliavin

9.4.1 Bismut's formula

In this section for the sake of simplicity, we assume that $d = 1$ and $q = 1$ (scalar Brownian motion).

Theorem 9.2 *(Bismut formula) Let $W = (W_t)_{t\in[0,T]}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be its augmented (hence càd) natural filtration. Let $X^x = (X_t^x)_{t\in[0,T]}$ be a diffusion process solution to the SDE*

$$dX_t = b(X_t)dt + \vartheta(X_t)dW_t, \quad X_0 = x$$

where b and ϑ are $\mathcal{C}_b^1$ (hence Lipschitz continuous). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that

$$\mathbb{E}\left(f^2(X_T^x) + (f')^2(X_T^x)\right) < +\infty.$$

Let $(H_t)_{t\in[0,T]}$ be an $\mathcal{F}$-progressively measurable ([5]) process lying in $L^2([0,T] \times \Omega, dt \otimes d\mathbb{P})$ i.e. satisfying $\mathbb{E}\int_0^T H_s^2 ds < \infty$. Then

$$\mathbb{E}\left(f(X_T^x)\int_0^T H_s dW_s\right) = \mathbb{E}\left(f'(X_T^x)Y_T\int_0^T \frac{\vartheta(X_s^x)H_s}{Y_s}ds\right)$$

where $Y_t = \frac{dX_t^x}{dx}$ is the tangent process of X^x at x.

Proof (Sketch of). We will assume to simplify the arguments in the proof that $|H_t| \leq C < +\infty$, $C \in \mathbb{R}_+$ and that f and f' are bounded functions. Let $\varepsilon \geq 0$. Set on the probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$,

$$\mathbb{P}_\varepsilon = L_T^{(\varepsilon)}.\mathbb{P}$$

where

$$L_t^{(\varepsilon)} = \exp\left(-\varepsilon \int_0^t H_s dW_s - \frac{\varepsilon^2}{2}\int_0^t H_s^2 ds\right), \quad t \in [0,T]$$

is a $\mathbb{P}$-martingale since H is bounded. It follows from Girsanov's Theorem that

$$\widetilde{W}^\varepsilon := \left(W_t + \varepsilon\int_0^t H_s ds\right)_{t\in[0,T]} \quad \text{is a} \quad \left(\mathbb{P}^{(\varepsilon)}, (\mathcal{F}_t)_{t\in[0,T]}\right)\text{-Brownian motion.}$$

Now it follows from Theorem 2.1 that

$$\mathbb{E}\left(f(X_T)\int_0^T H_s dW_s\right) = -\frac{\partial}{\partial\varepsilon}\mathbb{E}(f(X_T)L_T^{(\varepsilon)})_{|\varepsilon=0}$$

On the other hand

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}\left(f(X_T)L_T^{(\varepsilon)}\right)_{|\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{P}_\varepsilon}(f(X_T))_{|\varepsilon=0}.$$

Now we can rewrite the *SDE* satisfied by X as follows

$$\begin{aligned}
dX_t &= b(X_t)dt + \vartheta(X_t)dW_t \\
&= \left(b(X_t) - \varepsilon H_t\vartheta(X_t)\right)dt + \vartheta(X_t)d\widetilde{W}_t^{(\varepsilon)}.
\end{aligned}$$

Consequently (see [140], Theorem 1.11, p.372), X has the same distribution under $\mathbb{P}^{(\varepsilon)}$ as $X^{(\varepsilon)}$ solution to

$$dX^{(\varepsilon)} = (b(X^{(\varepsilon)}) - \varepsilon H_t\vartheta(X^{(\varepsilon)}))dt + \vartheta(X^{(\varepsilon)})dW_t, \quad X_0^{(\varepsilon)} = x.$$

---

[5]This means that for every $t \in [0,T]$, $(H_s(\omega))_{(s,\omega)\in[0,T]\times\Omega}$ is $\mathcal{B}or([0,t]) \otimes \mathcal{F}_t$-measurable.

Now we can write

$$\mathbb{E}\left(f(X_T)\int_0^T H_s dW_s\right) = -\frac{\partial}{\partial \varepsilon}\mathbb{E}(f(X_T^{(\varepsilon)}))_{|\varepsilon=0}$$

$$= -\mathbb{E}\left(f'(X_T)\left(\frac{\partial X_T^{(\varepsilon)}}{\partial \varepsilon}\right)_{|\varepsilon=0}\right)$$

where we used once again Theorem 2.1 and the obvious fact that $X^{(0)} = X$.

Using the tangent process method with ε as an auxiliary variable, one derives that the process $U_t := \left(\frac{\partial X_t^{(\varepsilon)}}{\partial \varepsilon}\right)_{|\varepsilon=0}$ satisfies

$$dU_t = U_t(b'(X_t)dt + \vartheta'(X_t)dW_t) - H_t\vartheta(X_t)dt.$$

Plugging the regular tangent process Y in this equation yields

$$dU_t = \frac{U_t}{Y_t}dY_t - H_t\vartheta(X_t)dt. \qquad (9.15)$$

We know that Y_t is never 0 so (up to some localization if necessary) we can apply Itô formula to the ratio $\frac{U_t}{Y_t}$: elementary computations of the partial derivatives of the function $(u,y) \mapsto \frac{u}{y}$ on $\mathbb{R} \times (0, +\infty)$ combined with Equation (9.15) show that

$$d\left(\frac{U_t}{Y_t}\right) = \frac{dU_t}{Y_t} - \frac{U_t dY_t}{Y_t^2} + \frac{1}{2}\left(-2\frac{d\langle U, Y\rangle_t}{Y_t^2} + \frac{2U_t d\langle Y\rangle_t}{Y_t^3}\right)$$

$$= -\frac{H_t\vartheta(X_t)}{Y_t}dt + \frac{1}{2}\left(-2\frac{d\langle U, Y\rangle_t}{Y_t^2} + \frac{2U_t d\langle Y\rangle_t}{Y_t^3}\right)$$

Then we derive from (9.15) that

$$d\langle U, Y\rangle_t = \frac{U_t}{Y_t}d\langle Y\rangle_t$$

which yields

$$d\left(\frac{U_t}{Y_t}\right) = -\frac{\vartheta(X_t)H_t}{Y_t}dt.$$

Noting that $U_0 = \frac{dX_0^{(\varepsilon)}}{d\varepsilon} = \frac{dx}{d\varepsilon} = 0$ finally leads to

$$U_t = -Y_t\int_0^t \frac{\vartheta(X_s)H_s}{Y_s}ds, \quad t \in [0, T],$$

which completes this step of the proof.

The extension to more general processes H can be done by introducing for every $n \geq 1$

$$H_t^{(n)}(\omega) := H_t(\omega)\mathbf{1}_{\{|H_t(\omega)| \leq n\}}.$$

It is clear by the Lebesgue dominated convergence Theorem that $H^{(n)}$ converges to H in $L^2([0, T] \times \Omega, dt \otimes d\mathbb{P})$. Then one checks that both sides of Bismut's identity are continuous with respect to this topology (using Hölder's Inequality).

The extension to unbounded functions and derivatives f follows by approximation of f by bounded $\mathcal{C}_b^1$ functions. $\diamond$

APPLICATION TO THE COMPUTATION OF THE δ-PARAMETER: Assume b and ϑ are $\mathcal{C}_b^1$. If f is continuous with polynomial growth and satisfies

$$\mathbb{E}\left(f^2(X_T^x) + \int_0^T \left(\frac{Y_t}{\vartheta(X_t^x)}\right)^2 dt\right) < +\infty,$$

then

$$\frac{\partial}{\partial x}\mathbb{E}\, f(X_T^x) = \mathbb{E}\left(f(X_T^x)\underbrace{\frac{1}{T}\int_0^T \frac{Y_s}{\vartheta(X_s^x)}dW_s}_{\text{weight}}\right). \tag{9.16}$$

Proof. We proceed like we did with the Black-Scholes model in Section 2.3: we first assume that f is regular, namely bounded, differentiable with bounded derivative. Then, using the tangent process method approach

$$\frac{\partial}{\partial x}\mathbb{E}\, f(X_T^x) = \mathbb{E}\left(f'(X_T^x)Y_T\right)$$

still with the notation $Y_t = \frac{dX_t^x}{dx}$. Then, we set

$$H_t = \frac{Y_t}{\vartheta(X_t^x)}.$$

Under the above assumption, we can apply Bismut's formula to get

$$T\mathbb{E}\left(f'(X_T^x)Y_T\right) = \mathbb{E}\left(f(X_T^x)\int_0^T \frac{Y_t}{\vartheta(X_t^x)}dW_t\right)$$

which yields the announced result. The extension to continuous functions with polynomial growth relies on an approximation argument. $\diamond$

Remarks. ● One retrieves in the case of a Black-Scholes model the formula (2.6) obtained for the δ in Section 2.3 by using an elementary integration by parts since $Y_t = \frac{X_t^x}{x}$ and $\vartheta(x) = \sigma x$.

● Note that the assumption

$$\int_0^T \left(\frac{Y_t}{\vartheta(X_t^x)}\right)^2 dt < +\infty$$

is basically an ellipticity assumption. Thus if $\vartheta^2(x) \geq \varepsilon_0 > 0$, one checks that the assumption is always satisfied.

▷ **Exercises. 1.** Apply what precedes to get a formula for the γ-parameter in a general diffusion model. [Hint: Apply the above "derivative free" formula to the δ-formula obtained using the tangent process method].

2. Show that if $b' - b\frac{\vartheta'}{\vartheta} - \frac{1}{2}\vartheta''\vartheta = c \in \mathbb{R}$, then

$$\frac{\partial}{\partial x}\mathbb{E}\, f(X_T^x) = \frac{e^{cT}}{\vartheta(x)}\mathbb{E}\left(f(X_T^x)W_T\right)$$

9.4.2 The Haussman-Clark-Occone formula: toward Malliavin calculus

In this section we state an elementary version of the so-called Haussman-Clark-Occone formula, following the seminal paper by Haussman [71]. We still consider the standard SDE

$$dX_t = b(X_t)dt + \vartheta(X_t)dW_t, \quad X_0 = x, \quad t \in [0, T].$$

with Lipschitz continuous coefficients b and σ. We denote by $X^x = (X_t^x)_{t \in [0,T]}$ its unique solution starting at x and by $(\mathcal{F}_t)_{t \in [0,T]}$ the (augmented) filtration of the Brownian motion W. We state the result in a one-dimensional setting for (at least notational) convenience.

Theorem 9.3 *Let* $F : (\mathcal{C}([0,T], \mathbb{R}), \|\cdot\|_{\sup}) \to \mathbb{R}$ *be a differentiable functional with differential* DF. *Then*

$$F(X^x) = \mathbb{E}\, F(X^x) + \int_0^T \mathbb{E}(DF(X).(\mathbf{1}_{[t,T]}Y_{\cdot}^{(t)}) \,|\, \mathcal{F}_t)\vartheta(X_t)dW_t$$

where $Y^{(t)}$ *is the tangent process of* X^x *at time* t, *solution to*

$$dY_t^{(t)} = Y_s^{(t)}(b'(X_s^x)ds + \vartheta'(X_s^x)dW_s), \quad s \in [t, T], \quad Y_t^{(t)} = 1.$$

This tangent process $Y^{(t)}$ *also reads*

$$Y_u^{(t)} = \frac{Y_u}{Y_t}, \ u \in [t, T] \quad \text{where} \quad Y_y = Y_t^{(0)} \text{ is the tangent process of } X^x \text{ at the origin.}$$

Remarks. • The starting point to understand this formula is to see it as a more explicit version of the classical representation formula of Brownian martingales, namely $M_t = \mathbb{E}(F(X) \,|\, \mathcal{F}_t)$ which admits a formal representation as a Brownian stochastic integral

$$M_t = M_0 + \int_0^T H_t \, dW_t.$$

So, the Clark-Occone-Haussman provides a kind of closed form for the process H.

• The differential $DF(\xi)$ of the functional F at an element $\xi \in \mathcal{C}([0,T], \mathbb{R})$ is a continuous linear form on $\mathcal{C}([0,T], \mathbb{R})$. Hence, following the Riesz representation Theorem (see *e.g.* [30]), it can be represented by a finite signed measure, say $\mu_{DF(\xi)}(ds)$ so that the term $DF(\xi).(\mathbf{1}_{[t,T]}Y_{\cdot}^{(t)})$ reads

$$DF(\xi).(\mathbf{1}_{[t,T]}Y_{\cdot}^{(t)}) = \int_0^T \mathbf{1}_{[t,T]}(s)Y_s^{(t)}\mu_{DF(\xi)}(ds) = \int_t^T Y_s^{(t)}\mu_{DF(\xi)}(ds).$$

Toward Malliavin derivative.

Assume $F(x) = f(x(t_0))$, $x \in \mathcal{C}([0,T], \mathbb{R})$, $f : \mathbb{R} \to \mathbb{R}$ differentiable with derivative f'. Then $\mu_x(ds) = f'(x(t_0))\delta_{t_0}(ds)$ where $\delta_{t_0}(ds)$ denotes the Dirac mass at time t_0. Consequently

$$DF(X).(\mathbf{1}_{[t,T]}Y_{\cdot}^{(t)}) = f'(X_{t_0}^x)Y_{t_0}^{(t)}\mathbf{1}_{[0,t_0]}(t)$$

whence one derives that

$$f(X_{t_0}^x) = \mathbb{E}\, f(X_{t_0}^x) + \int_0^{t_0} \mathbb{E}(f'(X_{t_0})Y_{t_0}^{(t)} \,|\, \mathcal{F}_t)\,\vartheta(X_t^x)\, dW_t. \tag{9.17}$$

This leads to introduce the *Malliavin* derivative $D_t F(X^x)_t$ of $F(X^x)$ at time t which is a derivative with respect to the path of Brownian motion W (viewed at time t) itself by

$$D_t F(X^x) := \begin{cases} DF(X^x).(\mathbf{1}_{[t,T]} Y_.^{(t)}) \vartheta(X_t^x) & \text{if } t \le t_0 \\ 0 & \text{if } t > t_0 \end{cases}.$$

The simplest interpretation is to write the following formal chain rule for differentiation

$$D_t F(X^x) = \frac{\partial F(X^x)}{\partial X_t^x} \times \frac{\partial X_t^x}{\partial W}.$$

If one notes that, for any $s \ge t$, $X_s^x = X_s^{X_t^x, t}$, the first term "$\frac{\partial F(X^x)}{\partial X_t^x}$" in the above product is clearly equal to $DF(X).(\mathbf{1}_{[t,T]} Y_.^{(t)})$ whereas the second term is the result of a formal differentiation of the *SDE* at time t with respect to W, namely $\vartheta(X_t^x)$.

An interesting feature of this derivative is that it satisfies usual chaining rules like $D_t F^2(X^x) = 2F(X^x) D_t F(X^x)$ and more generally $D_t \Phi(F(X^x)) = D\Phi(F(X^x)) D_t F(X^x)$, etc .

What is called Malliavin calculus is a way to extend this notion of differentiation to more general functionals using some functional analysis arguments (closure of operators), etc) using *e.g.* the domain of the operator $D_t F$ (see []).

USING THE HAUSSMAN-CLARK-OCCONE FORMULA TO GET BISMUT'S FORMULA. As a first conclusion we will show that the Haussman-Clark-Occone formula contains the Bismut formula. Let X^x, H, f and T be like in Section 9.4.1. We consider the two true martingales

$$M_t = \int_0^t H_s dW_s \quad \text{and} \quad N_t = \mathbb{E}(f(X_T^x)) + \int_0^t \mathbb{E}(f'(X_T) Y_T^{(s)} \mid \mathcal{F}_s) dW_s \quad t \in [0, T].$$

and perform a (stochastic) integration by parts. Owing to (9.17), we get, under appropriate integrality conditions,

$$
\begin{aligned}
\mathbb{E}\left(f(X_T^x) \int_0^T H_s dW_s \right) &= 0 + \mathbb{E} \int_0^T [\dots] dM_t + \mathbb{E} \int_0^T [\dots] dN_t \\
&\quad + \mathbb{E}\left(\int_0^T \mathbb{E}(f'(X_T) Y_T^{(s)} \mid \mathcal{F}_s) \vartheta(X_s^x) H_s \, ds \right) \\
&= \int_0^T \mathbb{E}\left(\mathbb{E}(f'(X_T) Y_T^{(s)} \mid \mathcal{F}_s) \vartheta(X_s^x) H_s \right) ds
\end{aligned}
$$

owing to Fubini's Theorem. Finally, using the characterization of conditional expectation to get rid of the conditioning, we obtain

$$\mathbb{E}\left(f(X_T^x) \int_0^T H_s dW_s \right) = \int_0^T \mathbb{E}\left(f'(X_T) Y_T^{(s)} \vartheta(X_s^x) H_s \right) ds.$$

Finally, a reverse application of Fubini's Theorem and the identity $Y_T^{(s)} = \frac{Y_T}{Y_s}$ leads to

$$\mathbb{E}\left(f(X_T^x) \int_0^T H_s dW_s \right) = \mathbb{E}\left(f'(X_T) Y_T \int_0^T \frac{\vartheta(X_s^x)}{Y_s} H_s \, ds \right)$$

which is but the Bismut formula (9.2).

▷ **Exercise.** (a) Considers the functional $\xi \mapsto F(\xi) = \varphi\Big(\int_0^T f(\xi(s))ds\Big)$, where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a differentiable function. Show that

$$F(X^x) = \mathbb{E}\, F(X^x) + \int_0^T \mathbb{E}\Big[\varphi'\Big(\int_0^T X_s^x ds\Big)\int_t^T Y_s ds \mid \mathcal{F}_t\Big]\frac{\vartheta(X_t^x)}{Y_t}dW_t$$

(b) Derive using the homogeneity of the Equation (9.12) that

$$\mathbb{E}\Big(\varphi'\Big(\int_0^T X_s^x ds\Big)\int_t^T \frac{Y_s}{Y_t}ds \mid \mathcal{F}_t\Big) = \mathbb{E}\Big(\varphi'\Big(\bar{x} + \int_0^{T-t}X_s^x ds\Big)\int_0^{T-t}Y_s ds\Big)\Big|_{x=X_t^x, \bar{x}=\int_0^t X_s^x ds}$$

$$=: \Phi\Big(T - s, X_t^x, \int_0^t X_s^x ds\Big).$$

9.4.3 Toward practical implementation: the paradigm of localization

For practical implementation, one should be aware that *the weighted estimators often suffer from high variance* compared to formulas derived from the tangent process as this can be measured when two formulas co-exist (the one with a differentiation of the payoff and the weighted one). This is can be seen on the formula for the δ-hedge when the maturity is small. Consequently, weighted formula usually need to be speeded up by variance reduction methods. Consequently the usual approach is to isolate the singular part (where differentiation does not apply) from the smooth part.

Let us illustrate the principle of localization functions on a very simple toy-example ($d = 1$): assume that, for every $\varepsilon > 0$,

$$|F(x, z) - F(x', z)| \leq C_{F,\varepsilon}|x - x'|, \qquad x, x', z \in \mathbb{R}, |x - z|, |x' - z| \geq \varepsilon > 0$$

with $\liminf_{\varepsilon \to 0} C_{F,\varepsilon} = +\infty$. Assume furthermore that,

$$\forall\, x, z \in \mathbb{R}^d,\ x \neq z, \quad F'_x(x, z)\ \text{does exist}$$

(hence bounded by $C_{F,\varepsilon}$ if $|x - z| \geq \varepsilon$).

On the other hand, assume that *e.g.* $F(x, Z) = h(G(x, Z))$ where $G(x, Z)$ has a probability density which is regular in x whereas h is "highly" singular when in the neighbourhood of $\{G(z, z), z \in \mathbb{R}\}$ (think of an indicator function). Then, the function $f(x) := \mathbb{E}\, F(x, Z)$ is differentiable.

Then, one considers $\varphi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ function such that $\varphi \equiv 1$ on $[-\varepsilon, \varepsilon]$ and $\operatorname{supp}(\varphi) \subset [-2\varepsilon, 2\varepsilon]$. Then, one may decompose

$$F(x, Z) = (1 - \varphi(x - Z))F(x, Z) + \varphi(x - Z)F(x, Z) := F_1(x, Z) + F_2(x, Z).$$

Functions φ can be obtained as *mollifiers* in convolution theory but other choices are possible, like simply Lipschitz continuous continuous functions (see the numerical illustration in Section 9.4.4).

Set $f_i(x) = \mathbb{E}\, F_i(x, Z))$ so that $f(x) = f_1(x) + f_2(x)$. Then one may use a direct differentiation to compute

$$f_1'(x) = \mathbb{E}\left(\frac{\partial F_1(x, Z)}{\partial x}\right)$$

(or a finite difference method with constant or decreasing increments). As concerns $f_2'(x)$, since $F_2(x, Z)$ is singular, it is natural to look for a weighted estimator

$$f_2'(x) = \mathbb{E}\left(F_2(x, Z)\Pi\right)$$

obtained *e.g.* by the above described method if we are in a diffusion framework.

When working in a diffusion setting at a fixed time T, the above Bismut formula makes the job. When we work with a functional of the whole trajectory, typically some path-depedent options (like barriers, etc) in local or stochastic volatility models, Malliavin calculus methods is a convenient and powerful tool even if, in most settings, more elementary approaches can often be used to derive these explicit weights... which may lead to different results. For an example of weight computation by means of Malliavin calculus in the case of Lookback or barriers options, we refer to [64].

9.4.4 Numerical illustration: what localization is useful for (with V. Lemaire)

Let us consider in a standard Black-Scholes model $(S_t)_{t \in [0,T]}$ with interest rate $r > 0$ and volatility $\sigma > 0$, two binary options:

- a *digital Call* with strike $K > 0$ and

- an *asset-or-nothing Call* with strike $K > 0$,

defined respectively by their payoff functions

$$h_1(\xi) = \mathbf{1}_{\{\xi \geq K\}} \quad \text{and} \quad h_S(\xi) = \xi \mathbf{1}_{\{\xi \geq K\}},$$

reproduced in figures 9.1 and 9.2 (pay attention to the scales of y-axis in each figure).

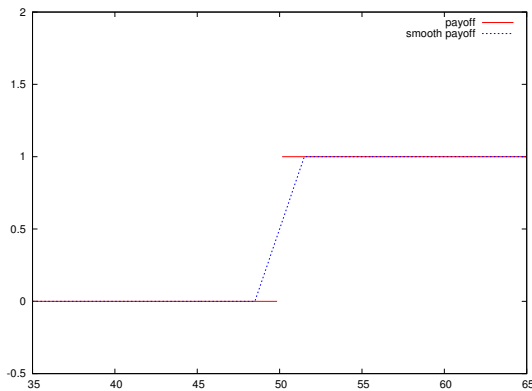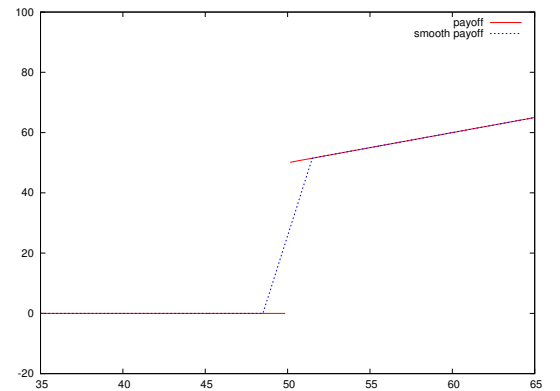Figure 9.1: Payoff h_1 of the *digital Call* with strike $K = 50$.

Figure 9.2: Payoff h_S of the *asset-or-nothing Call* with strike $K = 50$.

Denoting $F(x, z) = e^{-rT}h_1(xe^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}z})$ and $F(x, z) = e^{-rT}h_S(xe^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}z})$ in the digital Call case and in the asset-or-nothing Call respectively, we consider $f(x) = \mathbb{E}\, F(x, Z)$ where

Z is a standard Gaussian variable. With both payoff functions, we are in the singular setting in which $F(., Z)$ is not Lipschitz continuous continuous but only $\frac{1}{2}$-Hölder in L^2. As expected, we are interested in computing the delta of the two options *i.e.* $f'(x)$.

In such a singular case, the variance of the finite difference estimator explodes as $\varepsilon \to 0$ (see Proposition 9.2) and $\xi \mapsto F(\xi, Z)$ is not L^p-differentiable for $p \geq 1$ so that the tangent process approach can not be used (see Section 9.2.2).

We first illustrate the variance explosion in Figures 9.3 and 9.4 where the parameters have been set to $r = 0.04$, $\sigma = 0.1$, $T = 1/12$ (one month), $x_0 = K = 50$ and $M = 10^6$.
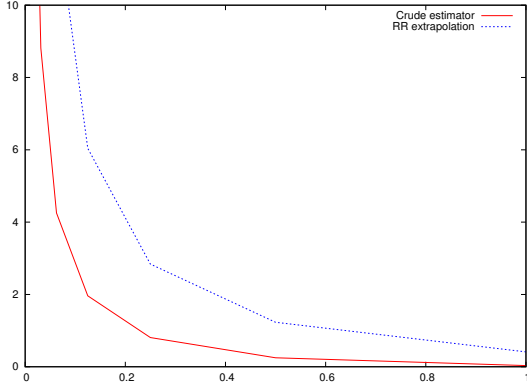

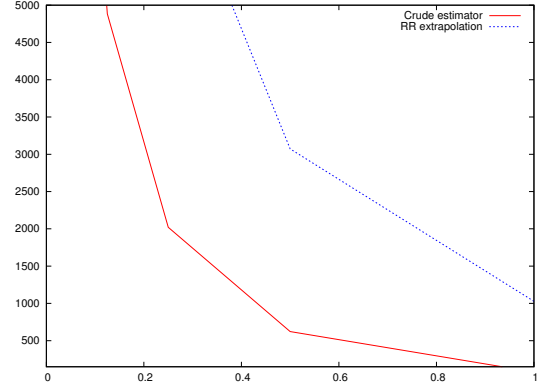Figure 9.3: Variance of the two estimator as a function of ε (digital Call).

Figure 9.4: Variance of the two estimator as a function of ε (asset-or-nothing Call).

To avoid the explosion of the variance one considers a smooth (namely Lipschitz continuous continuous) approximation of both payoffs. Given a small parameter $\eta > 0$ one defines

$$h_{1,\eta}(\xi) = \begin{cases} h_1(\xi) & \text{if } |\xi - K| > \eta, \\ \frac{1}{2\eta}\xi + \frac{1}{2}(1 - \frac{K}{\eta}) & \text{if } |\xi - K| \leq \eta \end{cases} \quad \text{and} \quad h_{\eta,S}(\xi) = \begin{cases} h_S(\xi) & \text{if } |\xi - K| > \eta, \\ \frac{K+\eta}{2\eta}\xi + \frac{K+\eta}{2}(1 - \frac{K}{\eta}) & \text{if } |\xi - K| \leq \eta \end{cases}$$

We define $F_\eta(x, Z)$ and $f_\eta(x)$ similarly as in the singular case.

In this numerical section, we introduce a Richardson-Romberg (RR) extrapolation of the finite difference estimator. The extrapolation is done using a linear combination of the finite difference estimator of step ε and the one of step $\frac{\varepsilon}{2}$. This linear extrapolation allow us to "kill" the first bias term in the expansion of the error. Similarly as in the proof of Propositions 9.1 and 9.2 we then prove that

$$\left\| f'(x) - \left(\frac{4}{3} \widehat{f'(x)}_{\frac{\varepsilon}{2},M} - \frac{1}{3} \widehat{f'(x)}_{\varepsilon,M} \right) \right\|_2 = \mathcal{O}(\varepsilon^3) + \mathcal{O}\left(\frac{1}{\varepsilon\sqrt{M}} \right), \tag{9.18}$$

where $\widehat{f'(x)}_{\varepsilon,M} = \frac{1}{M} \sum_{k=1}^{M} \frac{F(x+\varepsilon, Z_k) - F(x-\varepsilon, Z_k)}{2\varepsilon}$ and

$$\left\| f'_\eta(x) - \left(\frac{4}{3} \widehat{f'_\eta(x)}_{\frac{\varepsilon}{2},M} - \frac{1}{3} \widehat{f'_\eta(x)}_{\varepsilon,M} \right) \right\|_2 = \mathcal{O}(\varepsilon^3) + \mathcal{O}\left(\frac{1}{\sqrt{M}} \right), \tag{9.19}$$

where, as usual, $\widehat{f'_\eta(x)}_{\varepsilon,M} = \frac{1}{M} \sum_{k=1}^{M} \frac{F_\eta(x + \varepsilon, Z_k) - F_\eta(x - \varepsilon, Z_k)}{2\varepsilon}.$

▷ **Exercise.** Prove (9.18) and (9.19).

The control of the variance in the smooth case is illustrated in Figures 9.5 and 9.6 when $\eta = 2$, and in Figures 9.7 and 9.8 when $\eta = 0.5$. The variance increases when η decreases to 0 but does not explode as ε goes to 0.

For a given ε, note that the variance is usually higher using the RR extrapolation. However, in the Lipschitz continuous case the variance of the RR estimator and that of the crude finite difference converge toward the same value when ε goes to 0. Moreover from (9.18) we deduce the choice $\varepsilon = \mathcal{O}(M^{-1/8})$ to keep the balance between the bias term of the RR estimator and the variance term.

As a consequence, for a given level of the L^2-error, we can choose a bigger ε with the RR estimator, which reduces the bias *without increasing the variance.*

Figure 9.5: Variance of the two estimator as a function of ε (digital Call). Smooth payoff with $\eta = 1$.

Figure 9.6: Variance of the two estimator as a function of ε (asset-or-nothing Call). Smooth payoff with $\eta = 1$.
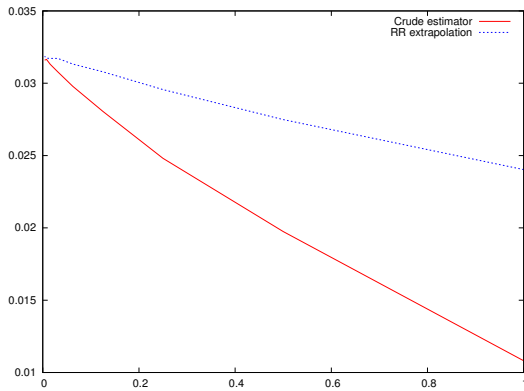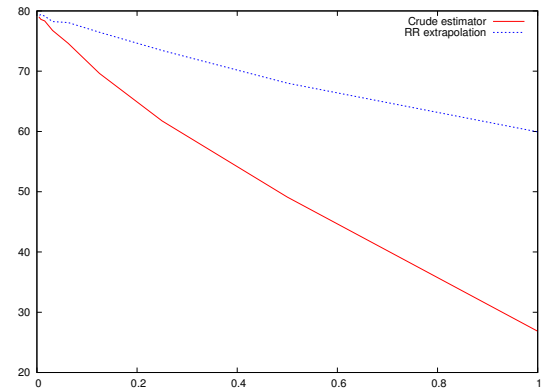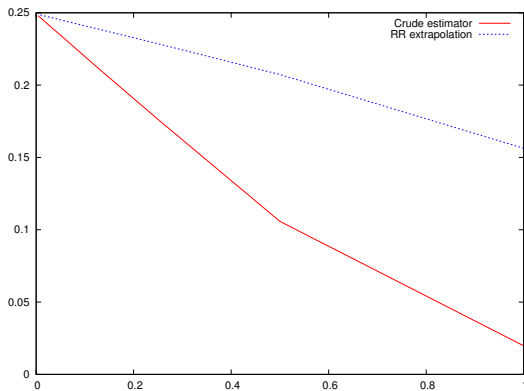

Figure 9.7: Variance of the two estimator as a function of ε (digital Call). Smooth payoff with $\eta = 0.5$.
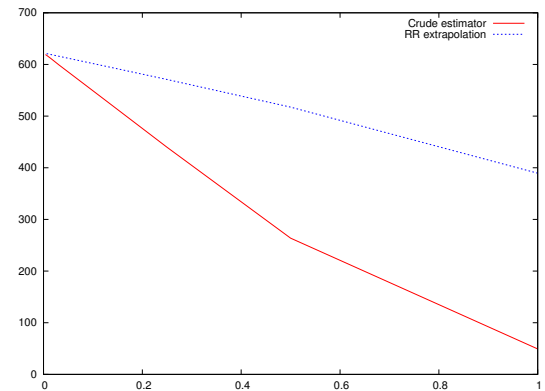
Figure 9.8: Variance of the two estimator as a function of ε (asset-or-nothing Call). Smooth payoff with $\eta = 0.5$.

The parameters of the model are the following: $x_0 = 50$, $K = 50$, $r = 0.04$, $\sigma = 0.1$ and $T = \frac{1}{52}$ (one week) or $T = \frac{1}{12}$ (one month). The number of Monte Carlo simulations is fixed to $M = 10^6$. We now compare the following estimators in the two cases with two different maturities $T = 1/12$ (one month) and $T = 1/52$ (one week):

- Finite difference estimator on the non-smooth payoffs h_1 and h_S with $\varepsilon = M^{-\frac{1}{4}} \simeq 0.03$.

- Finite difference estimator with Richardson-Romberg extrapolation on the non-smooth payoffs with $\varepsilon = M^{-\frac{1}{6}} = 0.1$.

- Crude weighted estimator (with standard Black-Scholes δ-weight) on the non-smooth payoffs h_1 and h_S.

- Localization: Finite difference estimator on the smooth payoffs $h_{1,\eta}$ and $h_{S,\eta}$ with $\eta = 1.5$ and $\varepsilon = M^{-\frac{1}{4}}$ combined with the weighted estimator on the (non-smooth) differences $h_1 - h_{1,\eta}$ and $h_S - h_{S,\eta}$.

- Localization: Finite difference estimator with Richardson-Romberg extrapolation on the smooth payoffs $h_{1,\eta}$ and $h^{S,\eta}$ with $\eta = 1.5$ and $\varepsilon = M^{-\frac{1}{6}}$ combined with the weighted estimator on the (non-smooth) differences $h_1 - h_{1,\eta}$ and $h_S - h_{S,\eta}$.

The results are summarized in the following Tables ?? and ?? for the delta of the digital Call option and in the Tables ?? and ?? for the one of the asset-or-nothing Call option.

[In progress...]
- Multi-dimensional case
- Variance reduction?

Chapter 10

Multi-asset American/Bermuda Options, swing options

10.1 Introduction

In this chapter devoted to numerical methods for multi-asset American and Bermuda options, we will consider a slightly more general framework that the nonengative dynamics traded assets in complete or incomplete markets. This is we will switch from a dynamics $(S_t)_{t \in [0,T]}$ to a more general $\mathbb{R}^d$-valued Brownian diffusion $(X_t)_{t \in [0,T]}$ satisfying the *SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 \in L^1_{\mathbb{R}^d}(\Omega, \mathcal{A}, \mathbb{P}) \tag{10.1}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}(d, q, \mathbb{R})$ are Lipschitz continuous continuous in x, uniformly in $t \in [0,T]$ and W is a q-dimensional standard Brownian motion defined on $(\Omega, \mathcal{A}, \mathbb{P})$, independent of X_0. The filtration of interest is $\mathcal{F}_t = \sigma(X_0, \mathcal{N}_{\mathbb{P}}) \vee \mathcal{F}_t^W$.

Each component X^i, $i = 1, \ldots, d$, of X can still be seen as the price process of a traded risky asset (although it can be negative *a priori* in what follows).

To keep the link with American option pricing we will call *American* "vanilla" payoff any process of the form $(f(t, X_t))_{t \in [0,T]}$ where $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}_+$ (temporarily) is a Borel function. If r denotes the interest rate (supposed to be constant for the sake of simplicity), the "obstacle process" is given by

$$\widetilde{Z}_t = e^{-rt} f(t, X_t).$$

We will always make the following assumptions on the payoff function f

$(Lip) \equiv f$ is Lipschitz continuous continuous in $x \in \mathbb{R}^d$, uniformly in $t \in [0,T]$.

This assumption will be sometimes refined into a semi-convex condition

$(SC) \equiv f$ satisfies (Lip) and is semi-convex in the following sense: there exists $\delta_f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, Borel and bounded, and $\rho \in (0, \infty)$ such that

$$\forall\, t \in [0,T], \quad \forall\, x, y \in \mathbb{R}^d, \quad f(t, y) - f(t, x) \geq \big(\delta_f(t, x)\,|\,y - x\big) - \rho|y - x|^2.$$

Examples. 1. If f is Lipschitz continuous continuous and convex, then f is semi-convex with

$$\delta_f(t,x) = \left[\left(\frac{\partial f}{\partial x_i} \right)_r (t,x) \right]_{1 \le i \le d}$$

where $\left(\frac{\partial f}{\partial x_i} \right)_r (t,x)$ denotes the right derivative with respect to x_i at (t,x).

2. If $f(t,.) \in \mathcal{C}^1(\mathbb{R}^d)$ and, for every $t \in [0,T]$, $\nabla_x h(t,.)$ is Lipschitz continuous in x, uniformly in $t \in [0,T]$, then

$$
\begin{aligned}
h(t,y) - h(t,x) &\ge (\nabla_x h(t,x)|y-x) - [\nabla_x h]_{\mathrm{Lip}}|\xi - x||y-x|, \ \xi \in [x,y] \\
&\ge \langle \nabla_x h(t,x)|y-x \rangle - [\nabla_x h]_{\mathrm{Lip}}|y-x|^2
\end{aligned}
$$

where $[\nabla_x h]_{\mathrm{Lip}} = \sup_{t \in [0,T]} [\nabla_x h(t,.)]_{\mathrm{Lip}}$.

Proposition 10.1 *(see [147]) There exists a continuous function $F : \mathbb{R}^d \to \mathbb{R}_+$ such that*

$$F(t, X_t) = \mathbb{P}\text{-esssup} \left\{ \mathbb{E}\left(e^{-r\tau} f(\tau, X_\tau) \,|\, \mathcal{F}_t \right), \ \tau \in \mathcal{T}_t^{\mathcal{F}} \right\}$$

where $\tau \in \mathcal{T}_t^{\mathcal{F}}$ denotes the set of $(\mathcal{F}_s)_{s \in [0,T]}$-stopping times having values in $[t,T]$.

Note that $F(0,x)$ is but the premium of the American option with payoff $f(t, X_t^x)$, $t \in [0,T]$.

10.2   Time discretization

We will discretize the optimal stopping problem and, if necessary, the diffusion itself to have at hand a simulatable underlying structure process.

First we note that if $t_k^n = \frac{kT}{n}$, $k = 0, \ldots, n$, $(X_{t_k^n})_{0 \le k \le n}$ is an $\mathcal{F}_{t_k^n})_{0 \le k \le n}$-Markov chain with transitions

$$P_k(x, dy) = \mathbb{P}(X_{t_{k+1}^n} \in dy \,|\, X_{t_k^n} = x), \ k = 0, \ldots, n-1.$$

Proposition 10.2 *Let $n \in \mathbb{N}^*$; Set*

$$\mathcal{T}_k^n = \left\{ \tau : (\Omega, \mathcal{A}, \mathbb{P}) \to \{t_\ell^n, \ \ell = k, \ldots, n\}, \ (\mathcal{F}_{t_\ell^n})\text{-stopping time} \right\}$$

and, for every $x \in \mathbb{R}^d$,

$$F_n(t_k^n, x) \sup_{\tau \in \mathcal{T}_k^n} \mathbb{R}\left(e^{-r(t\tau - t_k^n)} f(\tau, X_\tau^{t_k^n, x}) \right)$$

where $(X_s^{t_k^n, x})_{s \in [t_k^n, T]}$ is the unique solution to the (SDE) starting from x at time t_k^n.

(a) $\left(e^{-rt_k^n} F_n(t_k^n, X_{t_k^n}) \right)_{k=0,\ldots,n} = (\mathbb{P}, (\mathcal{F}_{t_k^n}))\text{-}Snell\left(e^{-rt_k^n} f(t_k^n, X_{t_k^n}) \right).$

(b) $\left(F_n(t_k^n, X_{t_k^n}) \right)_{k=0,\ldots,n}$ *satisfies the "pathwise" backward dynamic programming principle (denoted BDPP in the sequel)*

$$F_n(T, X_T = f(T, X_T)$$

and

$$F_n(t_k^n, X_{t_k^n}) = \max\left(f(t_k^n, X_{t_k^n}), e^{-r\frac{T}{n}}\mathbb{E}\left(F_n(t_k^n, X_{t_{k+1}^n}) \,|\, \mathcal{F}_{t_k^n}^W\right)\right), \ k = 0, \dots, n-1.$$

(c) The functions $F_n(t_k^n, .)$ satisfy the "functional" backward dynamic programming principle

$$F_n(T, x) = \bar{f}(T, x) \quad and \quad F_n(t_k^n, x) = \max\left(f(t_k^n, x), e^{-r\frac{T}{n}}P_k\left(F_n(t_k^n, .))(x)\right), \ k = 0, \dots, n-1.$$

Proof. We proceed backward as well. We consider the functions $F_n(t_k^n, .)$, $k = 0, \dots, n$, as defined in (c).

$(c) \Rightarrow (b)$. The result follows from the Markov property which implies for every $k = 0, \dots, n-1$,

$$\mathbb{E}\left(F_n(t^n, X_{t_{k+1}^n}) \,|\, \mathcal{F}_{t_k^n}^W\right) = P_k\left(F_n(t_k^n, .))(x).$$

$(b) \Rightarrow (a)$. This is a trivial consequence of the $(BDPP)$ since $\left(e^{-rt_k^n}F_n(t_k^n, X_{t_k^n})\right)_{k=0,\dots,n}$ is the Snell envelope associated to the obstacle sequence $\left(e^{-rt_k^n}f(t_k^n, X_{t_k^n})\right)_{k=0,\dots,n}$.

Applying what precedes to the case $X_0 = x$, we derive from the general theory on optimal stopping that

$$F_n(0, x) = \sup\left\{\mathbb{E}\left(e^{-r\tau}f(\tau, X_\tau), \ \tau \in \mathcal{T}_{[0,T]}^n\right\}.$$

The extension to times t_k^n, $k = 1, \dots, n$ follows likewise from the same reasoning carried out with $(X_{t_\ell^n}^{t_k^n, x})_{\ell=k,\dots,n}$. $\diamond$

Remark-Exercise. It is straightforward to establish, given that the flow of the SDE is Lipschitz continuous (see Theorem 7.10) from $\mathbb{R}^d$ to any $L^p(\mathbb{P})$, $1 \le p < +\infty$, in the sense

$$\left\| \sup_{t\in[0,T]} |X_t^x - X_t^y| \right\|_p \le C_{b,\sigma,T}|x - y|,$$

that the functions $F_n(t_k^n, .)$ are Lipschitz continuous continuous, uniformly in t_k^n, $k = 0, \dots, n$, $n \ge 1$.

Now we pass to the (discrete time) Euler scheme as defined by Equation (7.3) in Chapter 7. We recall its definition for convenience (with a slight change of notation concerning the Gaussian noise):

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + \frac{T}{n}b(t_k^n, \bar{X}_{t_k^n}) + \sigma(t_k^n, \bar{X}_{t_k^n})\sqrt{\frac{T}{n}} Z_{k+1}, \quad \bar{X}_0 = X_0, \quad k = 0, \dots, n-1, \tag{10.2}$$

where $(Z_k)_{1\le k\le n}$ denotes a sequence of i.i.d. $\mathcal{N}(0; I_q)$-distributed random vectors given by

$$U_k := \sqrt{\frac{n}{T}} \left(W_{t_k^n} - W_{t_{k-1}^n}\right), \quad k = 1, \dots, n.$$

(Strictly speaking we should write Z_k^n rather than Z_k.)

Proposition 10.3 (Euler scheme) *The above proposition remains true when replacing the sequence $(X_{t_k^n})_{0 \le k \le n}$ by its Euler scheme with step $\frac{T}{n}$, still with the filtration $(\sigma(X_0, \mathcal{F}_{t_k^n}^W))_{0 \le k \le n}$. In both cases one just has to replace the transitions $P_k(xdy)$ of the original process by that of its Euler scheme with step $\frac{T}{n}$, namely $\bar{P}_{k,k+1}^n(x, dy)$ defined by*

$$\bar{P}_{k,k+1}^n f(x) = \mathbb{E}\left(f\left(x + \frac{T}{n} b(t_{k-1}^n, x) + \sqrt{\frac{T}{n}} \sigma(t_{k-1}^n, x)Z\right)\right), \quad Z \sim \mathcal{N}(0; I_q),$$

and F_n by $\bar{F}_n$.

Remark. In fact the result holds true as well with the (smaller) natural innovation filtration $\mathcal{F}_k^{X_0, Z} = \sigma(X_0, Z_1, \ldots, Z_k)$, $0 \le k \le n$, since the Euler scheme remains a Markov chain with the same transitions with respect to this filtration

Now we state a convergence rate result for the "réduites" and the value functions. In what follows we consider the value function of the original continuous time stopping time problem denoted $F(t, x)$ and defined by

$$F(t, x) = \sup\left\{\mathbb{E}\left(e^{-r\tau} f(\tau, X_\tau^x)\right), \ \tau : (\Omega, \mathcal{A}) \to [t, T], \ (\mathcal{F}_t^W)\text{-stopping time}\right\} \quad (10.3)$$

$$= \mathbb{E}\left(\mathbb{P}\text{-esssup}\left\{\mathbb{E}\left(e^{-r\tau} f(\tau, X_\tau^x)|\mathcal{F}_t^W\right), \tau : (\Omega, \mathcal{A}) \to [t, T], \ (\mathcal{F}_t^W)\text{-stopping time}\right\}\right). (10.4)$$

Furthermore on can also prove that

$$F(t, X_t^x) = \mathbb{P}\text{-esssup}\left\{\mathbb{E}\left(e^{-r\tau} f(\tau, X_\tau^x)\right), \ \tau : (\Omega, \mathcal{A}) \to [t, T], \ (\mathcal{F}_t^W)\text{-stopping time}\right\}.$$

GENERAL INTERPRETATION. In an optimal stopping framework, $F(t, x)$ is also known as the "réduite" at time t of the optimal stopping problem (with horizon T). The second equality is (true but) not trivial and we refer to [148].

If $(X_t^x)_{t \in [0,T]}$ denotes an underlying (Markov) structure process and $f(t, X_t)$ is the gain you have if you leave the "game" at time t the above right hand side of the above equality, represents the supremum of your mean gains over all "honest" stopping strategies. By honest we mean here non-anticipative with respect to the "history" (or filtration) of the process W. In mathematical terms it reads that admissible strategies τ are stopping times with respect to the filtration $(\mathcal{F}_t^W)$ *i.e.* random variables $\tau(\Omega, \mathcal{A}) \to [0, T]$ satisfying

$$\forall t \in [0, T], \qquad \{\tau \le t\} \in \mathcal{F}_t^W.$$

This means that you decide to leave the game between 0 end t based on your observation of the process W between 0 and t.

Asking $\{\tau = t\}$ to lie in $\mathcal{F}_t^W$ (I decide to stop exactly at time t) seems more natural assumption, but in a continuous time framework this condition turns out to be technically not strong enough to make the theory work.

▷ **Exercise.** Show that for any $\mathcal{F}^W$-stopping time one has, for every $t \in [0, T]$, $\{\tau = t\} \in \mathcal{F}_t^W$ (the converse is not true in general in a continuous time setting).

In practice, the underlying assumption that the Brownian motion W is *observable* by the player is highly unlikely, or at least induces dramatic limitations for the modeling of $(X_t)_{t\in[0,T]}$ which models in Finance the vector of market prices of traded assets.

In fact, since the Brownian diffusion process X is a $\mathcal{F}^W$-Markov process, one shows (see [148]) one can replace in the above definition and characterizations of $F(t,x)$ the natural filtration of the Brownian motion W by that of X. This is clearly more in accordance with usual models in Finance (and elsewhere) since X represents in most models the observable structure process.

INTERPRETATION IN TERMS OF AMERICAN OPTIONS. As far as derivative pricing is concerned, the (non-negative) function f defined on $[0,T] \times \mathbb{R}^d_+$ is an American payoff function. This means that the holder of the contract exercises his/her option at time t, he/she will receive a monetary flow equal to $f(t,x)$ if the vector of market prices is equal to $x \in \mathbb{R}^d_+$ at time t. In a complete market model, one shows that, for hedging purpose, one should price all the American payoffs under the unique *risk-neutral* probability measure *i.e.* the unique probability measure $\mathbb{P}^*$ equivalent to the historical probability that makes the price process $X^x = (X^x_t)_{t\in[0,T]}$ a $\mathbb{P}^*$-martingale (x is here the starting value of the process X^x). However, as far as numerical aspects are concerned, this kind of restriction is of little interest in the sense that it has no impact on the methods or on their performances. This is the reason why, in what follows we will keep a general form for the drift b of our Brownian diffusion dynamics (and still denote by $\mathbb{P}$ the probability measure on $(\Omega, \mathcal{A})$).

Theorem 10.1 (see [13], 2003) *(a)* DISCRETIZATION OF THE STOPPING RULES FOR THE STRUCTURE PROCESS X: *If f satisfies (Lip), i.e. is Lipschitz continuous in x uniformly in $t \in [0,T]$, then so are the value functions $F_n(t^n_k,.)$ and $F(t^n_k,.)$, uniformly with respect to t^n_k, $k = 0,\ldots,n$, $n \geq 1$.*
Furthermore $F(t^n_k,.) \geq F_n(t^n_k,.)$ and

$$\left\| \max_{0\leq k\leq n} \left(F(t^n_k, X_{t^n_k}) - F_n(t^n_k, X_{t^n_k}) \right) \right\|_p \leq \frac{C_{b,\sigma,f,T}}{\sqrt{n}}$$

and, for every compact set $K \subset \mathbb{R}^d$,

$$0 \leq \sup_{x\in K} \left(\max_{0\leq k\leq n} \left(F(t^n_k, x) - F_n(t^n_k, x) \right) \right) \leq \frac{C_{b,\sigma,f,T,K}}{\sqrt{n}}.$$

(b) If f is semi-convex, then there exists real constants $C_{b,\sigma,f,T}$ and $C_{b,\sigma,f,T,K} > 0$ such that

$$\left\| \max_{0\leq k\leq n} \left(F_n(t^n_k, X_{t^n_k}) - F(t^n_k, X_{t^n_k}) \right) \right\|_p \leq \frac{C_{b,\sigma,f,T}}{n}$$

and, for every compact set $K \subset \mathbb{R}^d$,

$$0 \leq \sup_{x\in K} \left(\max_{0\leq k\leq n} \left(F(t^n_k, x) - F_n(t^n_k, x) \right) \right) \leq \frac{C_{b,\sigma,f,T,K}}{n}.$$

(c) EULER APPROXIMATION SCHEME $\bar{X}^n$: *There exists real constants $C_{b,\sigma,f,T}$ and $C_{b,\sigma,f,T,K} > 0$ such that*

$$\left\| \max_{0\leq k\leq n} |F_n(t^n_k, X_{t^n_k}) - \bar{F}_n(t^n_k, X_{t^n_k})| \right\|_p \leq \frac{C_{b,\sigma,f,T}}{\sqrt{n}}$$

and, for every compact set $K \subset \mathbb{R}^d$,

$$0 \leq \sup_{x \in K} \left(\max_{0 \leq k \leq n} \left(F(t_k^n, x) - \bar{F}_n(t_k^n, x) \right) \right) \leq \frac{C_{b,\sigma,f,T,K}}{\sqrt{n}}.$$

HOW TO PROCEED IN PRACTICE?

▷ If the diffusion process X is simulatable at times t_k^n (in an exact way), it is useless to introduce the Euler scheme and it will be possible to take advantage of the semi-convexity of the payoff/obstacle function f to get a time discretization error of the réduite F by F_n at a $O(1/n)$-rate.

A typical example of this situation is provided by the multi-dimensional Black-Scholes model (and its avatars for FX (Garman-Kohlagen) or future (Black)) and more generally by models where the process (X_t^x) can be written at each time $t \in [0, T]$ as an explicit function of W_t, namely

$$\forall\, t \in [0, T], \qquad X_t = \varphi(t, W_t)$$

where $\varphi(t, x)$ can be computed at a very low cost. When $d = 1$ (although of smaller interest for application in view of the available analytical methods based on variational inequalities), one can also rely on the exact simulation method of one dimensional diffusions, see [23]).

▷ In the general case, we will rely on this two successive steps of discretization: one for the optimal stopping rules and one for the underlying process to make it simulatable. However in both cases, as far as numerics are concerned, we will rely on the $BDPP$ which itself requires the computation of conditional expectations.

In both case we have now access to a simulatable Markov chain (either the Euler scheme or the process itself at times t_k^n). This task requires a new discretization phase, making possible the computation of the discrete time Snell envelope (and its réduite).

10.3   A generic discrete time Markov chain model

We consider a standard discrete time Markovian framework: let $(X_k)_{0 \leq k \leq n}$ be an $\mathbb{R}^d$-valued $(\mathcal{F}_k)_{0 \leq k \leq n}$-Markov chain defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_k)_{0 \leq k \leq n} \mathbb{P})$ with transitions

$$P_k(x, dy) = \mathbb{P}\big(X_{k+1} \in dy \,|\, X_k = x\big), \ k = 0, \ldots, n-1,$$

and let $Z = (Z_k)_{0 \leq k \leq n}$ be an $(\mathcal{F}_n)$-adapted obstacle/payoff sequence of non-negative *integrable* random variables of the form

$$0 \leq Z_k = f_k(X_k), \ k = 0, \ldots, n.$$

We want to compute the so-called $(\mathbb{P}, (\mathcal{F}_n)_{0 \leq k \leq n}))$-*Snell envelope* $U = (U_k)_{0 \leq k \leq n}$ defined by

$$U_k = \mathrm{esssup}\Big\{ \mathbb{E}\big(f(\tau, X_\tau) \,|\, \mathcal{F}_k\big), \ \tau : (\Omega, \mathcal{A}) \to \{k, \ldots, n\}, (\mathcal{F}_\ell)_{0 \leq k \leq n} \text{ stopping time}\Big\}$$

and its "réduite", *i.e.* $\mathbb{E}\, U_k$.

Proposition 10.4 *The Snell envelope $(U_k)_{0 \leq k \leq n}$ is solution to the following Backward Dynamic Programing Principle*

$$U_n = Z_n \ \ and \ \ U_k = \max\Big(Z_k, \mathbb{E}\big(U_{k+1} \,|\, \mathcal{F}_k\big)\Big), \ k = 0, \ldots, n-1.$$

Furthermore, for every $k \in \{0, \ldots, n\}$ there exists a Borel function $u_k : \mathbb{R}^d \to \mathbb{R}$ such that

$$U_k = u_k(X_k), \ k = 0, \ldots, n,$$

where

$$u_n = f_n \quad and \quad u_k = \max\left(f_k, P_k u_{k+1}\right), \ k = 0, \ldots, n-1.$$

Proof. Temporarily left as an exercise (see also [115]). $\diamond$

Moreover, general discrete time optimal stopping theory (with finite horizon) ensures that, for every $k \in \{0, \ldots, n\}$, there exists an optimal stopping time τ_k for this problem when starting to play at time k *i.e.* such that

$$U_k = \mathbb{E}\left(Z_{\tau_k} \,|\, \mathcal{F}_k\right).$$

This optimal stopping time may be not unique but

$$\tau_k = \min\left\{\ell \in \{k, \ldots, n\}, \ U_k = Z_k\right\}$$

the lowest one in any case.

An alternative to the above "regular" *BDPP* formula is its dual form involving these optimal stopping times. This second programing principle is used in Longstaff-Schwarz's original paper on regression methods (see next section).

Proposition 10.5 *The sequence of optimal times $(\tau_k)_{0 \le k \le n}$ is defined by*

$$\tau_n = n \quad and \quad \tau_k = k\mathbf{1}_{\{Z_k \ge \mathbb{E}(Z_{\tau_{k+1}}|X_k)\}} + \tau_{k+1}\mathbf{1}_{\{Z_k \ge \mathbb{E}(Z_{\tau_{k+1}}|X_k)\}^c}, \ k = 0, \ldots, n-1.$$

Proof. We proceed by a backward induction on k. If $k = $ the result is obvious. If $U_{k+1} = \mathbb{E}\left(Z_{\tau_{k+1}} \,|\, \mathcal{F}_{k+1}\right)$, then

$$\mathbb{E}\left(Z_{\tau_{k+1}} \,|\, X_k\right) = \mathbb{E}\left(\mathbb{E}\left(Z_{\tau_{k+1}} \,|\, \mathcal{F}_{k+1}\right) \,|\, X_k\right) = \mathbb{E}\left(U_{k+1} \,|\, X_k\right) = \mathbb{E}\left(u_{k+1}(X_{k+1}) \,|\, X_k\right) = \mathbb{E}\left(U_{k+1} \,|\, \mathcal{F}_k\right)$$

by the Markov property, so that

$$
\begin{aligned}
\mathbb{E}\left(Z_{\tau_k} \,|\, \mathcal{F}_k\right) &= Z_k\mathbf{1}_{\{Z_k \ge \mathbb{E}(U_{k+1}|\mathcal{F}_k)\}} + \mathbb{E}\left(Z_{\tau_{k+1}} \,|\, \mathcal{F}_k\right)\mathbf{1}_{\{Z_k \ge \mathbb{E}(U_{k+1}|\mathcal{F}_k)\}^c} \\
&= \max\left(Z_k, \mathbb{E}\left(U_{k+1} \,|\, \mathcal{F}_k\right)\right) \\
&= U_k.
\end{aligned}
$$

Furthermore,

$$\tau_k = k\mathbf{1}_{\{Z_k \ge \mathbb{E}(U_{k+1}|\mathcal{F}_k)\}} + \tau_{k+1}\mathbf{1}_{\{Z_k \ge \mathbb{E}(U_{k+1}|\mathcal{F}_k)\}^c}$$

which implies by a straightforward induction that

$$\tau_k = \min\left\{\ell \in \{k, \ldots, n\}, \ U_k = Z_k\right\}. \quad \diamond$$

10.3.1   Principle of the regression method

These methods are also known as Longstaff-Schwarz method in reference to the paper [104] (see also [32]). Assume that all the random variables Z_k, $k = 0, \dots, n$, are square integrable, then so are Z_{τ_k}s. The idea is to replace the conditional expectation operator $\mathbb{E}(\,.\,|X_k)$ by a linear regression on the first N elements of a Hilbert basis of $(L^2(\Omega, \sigma(X_k), \mathbb{P}), \langle .,.\rangle_{L^2(\mathbb{P})})$. This is a very natural idea to approximate conditional expectation (see *e.g.* [27], chap. for an introduction in a general framework).

Mainly for convenience we will consider a sequence $e_i : \mathbb{R}^d \to \mathbb{R}$, $i \in \mathbb{N}^*$, of Borel functions such that, for every $k \in \{0, \dots, n\}$, $(e_i(X_k))_{i \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(\Omega, \sigma(X_k), \mathbb{P})$, *i.e.*

$$\{e_i(X_k),\ i \in \mathbb{N}^*\}^{\perp} = \{0\}, \quad k = 0, \dots, n.$$

In practice, one may choose different functions at every time k, *i.e.* families $e_{i,k}$ so that $(e_{i,k}(X_k))_{i \geq 1}$ makes up a Hilbert basis of $L^2(\Omega, \sigma(X_k), \mathbb{P})$.

Example. If $X_k = W_{t_k^n}$, $k = 0, \dots, n$, show that Hermite polynomials provide a possible solution (up to an appropriate normalization at every time k). What is its specificity?

META-SCRIPT OF A REGRESSION PROCEDURE.

- **Approximation 1:** Dimension Truncation

   ▷ At every time $k \in \{0, \dots, n\}$, truncate at level N_k

   $$e^{[N_k]}(X_k) := (e_1(X_k), e_2(X_k), \dots, e_{N_k}(X_k)).$$

   and set

   ▷ $\tau_n^{[N_k]} := n,$

   ▷ $\tau_k^{[N_k]} := k \mathbf{1}_{\{Z_k > (\alpha_k^{[N_k]}|e^{[N]}(X_k))\}} + \tau_{k+1}^{[N_k]} \mathbf{1}_{\{Z_k \leq (\alpha_k^{[N_k]}|e^{[N_k]}(X_k))\}}, \quad k = 0, \dots, n-1.$

   where $\alpha_k^{[N_k]} := \operatorname{argmin}\left\{\mathbb{E}(Z_{\tau_{k+1}^{[N_k]}} - (\alpha|e^{[N_k]}(X_k)))^2, \quad \alpha \in \mathbb{R}^{N_k}\right\}.$

   In fact this finite dimensional optimization problem has a well-known solution given by

   $$\alpha_k^{[N]} = \left(\operatorname{Gram}(e^{[N_k]}(X_k))^{-1}\right)\left(\langle Z_{\tau_{k+1}^{[N_k]}} \,|\, e_\ell^{[N_k]}(X_k)\rangle_{L^2(\mathbb{P})}\right)_{1 \leq \ell \leq N_k}$$

   where the so-called Gram matrix of $e^{[N_k]}(X_k)$ defined by

   $$\operatorname{Gram}(e^{[N_k]}(X_k)) = \left[\langle e_\ell^{[N_k]}(X_k) \,|\, e_{\ell'}^{[N_k]}(X_k)\rangle_{L^2(\mathbb{P})}\right]_{1 \leq \ell, \ell' \leq N}.$$

- **Approximation 2 :** Monte Carlo approximation

  This second approximation phase is itself decomposed into two successive phases :

  – a forward MC simulation of the underlying "structure" Markov process $(X_k)_{0 \le k \le n}$

  – followed by a backward approximation of $\tau_k^{[N]}$, $k = 0, \ldots, n$. In a more formal way, the idea is to replace the true distribution of the Markov chain $(X_k)_{0 \le k \le n}$ by the empirical measure of a simulated sample of size M of the chain.

  For notational convenience we will set $N = (N_0, \ldots, N_n) \in \mathbb{N}^{n+1}$ and will denote $e_i^{[N]}(X_k)$ instead of $e_i^{[N_k]}(X_k)$, etc.

  ▷ *Forward Monte Carlo simulation phase:* Simulate (and store) M independent copies $X^{(1)}, \ldots, X^{(m)}, \ldots, X^{(M)}$ of $X = (X_k)_{0 \le k \le n}$ in order to have access to the empirical measure

  $$\frac{1}{m} \sum_{m=1}^{M} \delta_{X^{(\ell)}}.$$

  ▷ *Backward phase:*

  – At time n: For every $m \in \{1, \ldots, M\}$,

  $$\tau_n^{[N,m,M]} := n$$

  – for $k = n - 1$ downto 0.
    Compute

  $$\alpha_k^{[N],M} := \operatorname{argmin}_{\alpha \in \mathbb{R}^N} \left(\frac{1}{M} \sum_{m=1}^{M} Z_{\tau_{k+1}^{[N],m,M}}^{(m)} - \alpha . e^{[N]}(X^{(m)}) \right)^2$$

  using the closed form formula

  $$\alpha_k^{[N],M} = \left(\frac{1}{M} \sum_{m=1}^{M} e_\ell^{[N]}(X_k^{(m)}) e_{\ell'}^{[N]}(X_k^{(m)}) \right)_{1 \le \ell, \ell' \le N_k}^{-1} \left(\frac{1}{M} \sum_{m=1}^{M} Z_{\tau_{k+1}^{[N],m,M}}^{(m)} e_\ell^{[N]}(X_k^{(m)}) \right)_{1 \le \ell \le N_k}.$$

  – For every $m \in \{1, \ldots, M\}$, set

  $$\tau_{k+1}^{[N],m,N} := k \mathbf{1}_{\{Z_k^{(m)} > \alpha_k^{[N],M} . e^{[N]}(X_k)\}} + \tau_{k+1}^{[N],m,N} \mathbf{1}_{\{Z_k^{(m)} \le \alpha_k^{[N],M} . e^{[N]}(X_k)\}}.$$

Finally, the resulting approximation of the mean value at the origin of the Snell envelope, also called "réduite", reads

$$\mathbb{E}\, U_0 = \mathbb{E}(Z_{\tau_0}) \approx \mathbb{E}(Z_{\tau_0^{[N_0]}}) \approx \frac{1}{M} \sum_{m=1}^{M} Z_{\tau_0^{[N],m,M}}^{(m)} \quad \text{as} \quad M \to +\infty.$$

Note that when $\mathcal{F}_0 = \{\emptyset, \Omega\}$, (so that $X_0 = x_0 \in \mathbb{R}^d$), $U_0 = \mathbb{E}\, U_0 = u_0(x_0)$.

Remarks. • One may rewrite formally the second approximation phase by simply replacing the distribution of the chain $(X_k)_{0 \le k \le n}$ by the empirical measure

$$\frac{1}{M} \sum_{m=1}^{M} \delta_{X^{(\ell)}}.$$

• The Gram matrices $[\mathbb{E}\big(e_i^{(N_k)}(X_k)e_j^{(N_k)}(X_k)\big)]_{1 \le i,j \le N_k}$ can be computed "off line" in the sense that they are not "payoff dependent" by contrast with the term $\left(\langle Z_{\tau_{k+1}^{[N_k]}} \, | \, e_\ell^{[N_k]}(X_k) \rangle_{L^2(\mathbb{P})} \right)_{1 \le \ell \le N_k}$.

   In various situations, it is even possible to have some closed form for this Gram matrix, *e.g.* when $e_i(X_k)$, $i \ge 1$, happens to be an orthonormal basis of $L^2(\Omega, \sigma(X_k), \mathbb{P})$. In that case the Gram matrix is reduced to the identity matrix! So is the case for example when $X_k = \sqrt{\frac{n}{kT}} W_{\frac{kT}{n}}$, $k = 0, \ldots, n$ and $e_i = H_{i-1}$ where $(H_i)_{i \ge 0}$ is the basis of Hermite polynomials (see [78], chapter 3, p.167).

• The algorithmic analysis of the above described procedure shows that its implementation requires

   – a *forward* simulation of M paths of the Markov chain,

   – a *backward* nonlinear optimization phase in which all the (stored) paths have to interact through the computation of $\alpha_k^{[N],M}$ which depends on all the values $X_k^{(m)}$, $k \ge 1$.

   However, still in very specific situations, the forward phase can be skipped if a backward simulation simulation method for the Markov chain $(X_k)_{0 \le k \le n}$ is available. So is the case for the Brownian motion at times $\frac{k}{n}T$ using Brownian bridge simulation methods (see Chapter 8).

   The rate of convergence of the Monte Carlo step of the procedure is ruled by a Central Limit Theorem stated below.

Theorem 10.2 *(Clément-Lamberton-Protter (2003), see [34]) The Monte Carlo approximation satisfies a CLT*

$$\sqrt{M} \left(\frac{1}{M} \sum_{m=1}^{M} Z_{\tau_k^{[N],m,M}}^{(m)} - \mathbb{E}\, Z_{\tau_k^{[N]}}, \alpha_k^{[N],M} - \alpha_k^{[N]} \right)_{0 \le k \le n-1} \xrightarrow{\mathcal{L}} \mathcal{N}(0; \Sigma)$$

where Σ is non-degenerate covariance matrix.

Pros & Cons of the regression method

PROS

   • The method is "natural": Approximation of conditional expectation by (affine) regression operator on a truncated basis of $L^2(\sigma(X_k), \mathbb{P})$.

   • The method appears as "flexible": the opportunity to change or adapt the (truncated) basis of $L^2(\sigma(X_k), \mathbb{P}))$ at each step of the procedure (*e.g.* by including the payoff function from time n in the (truncated) basis at each step).

Cons

• From a purely theoretical point of view the regression approach does not provide error bounds or rate of approximation for the convergence of $\mathbb{E}\, Z_{\tau_0^{[N_0]}})$ toward $\mathbb{E}\, Z_{\tau_0} = \mathbb{E}\, U_0$ which is mostly ruled by the rate at which the family $e^{[N_k]}(X_k)$ "fills" $L^2(\Omega, \sigma(X_k), \mathbb{P})$ when N_k goes to infinity. However, in practice this information would be of little interest since, especially in higher dimension, the possible choices for the size N_k are limited by the storing capacity of the used computing device.

• Almost all computations are made *on-line* since they are payoff dependent. However, note that the Gram matrix of $(e_\ell^{[N]}(X_k))_{1 \le \ell \le N}$ can be computed off-line since it only depends on the structure process.

• The **choice of the functions** $e_i(x)$, $i \ge 1$, is crucial and needs much care (and intuition). In practical implementations, it may vary at every times step. Furthermore, it may have a biased effect for options deep in or deep out of the money since the coordinates of the functions $e_i(X_k)$ are computed locally "where things happen most of the time" inducing an effect on the prices at long range through their behaviour at infinity. On the other hand this choice of the functions, if they are smooth, has a smoothing effect which can be interesting to users (if it does not induce hidden arbitrages. . .). To overcome the first problem one may choose local function like indicator functions of a Voronoi diagram (see the next section 10.4 devoted to quantization tree methods or Chapter **??**) with the counterpart that no smoothing effect can be expected any more.

When there is a family of distributions "related" to the underlying Markov structure process, a natural idea can be to consider an orthonormal basis of $L^2(\mu_0)$ where μ_0 is a normalized distribution of the family. A typical example is the sequence of Hermite polynomials for the normal distribution $\mathcal{N}(0, 1)$.

When no simple solution is available, considering the simple basis $(t^\ell)_{\ell \ge 0}$ remains a quite natural and efficient choice in 1-dimension.

In higher dimensions (in fact the only case of interest in practice since one dimensional setting is usually solved by PDE methods!), this choice becomes more and more influenced by the payoff itself.

• Huge RAM capacity are needed to store all the paths of the simulated Markov chain (forward phase) except when a backward simulation procedure is available. This induces a stringent limitation of the size M of the simulation, even with recent devices, to prevent a swapping effect which would dramatically slow down the procedure. By swapping effect we mean that when the quantity of data to be stored becomes too large the computer uses its hard disk to store them but the access to this ROM memory is incredibly slow compared to the access to RAM memory.

• Regression methods are strongly *payoff* dependent in the sense that a significant part of the procedure (product of the inverted Gram matrix by the projection of the payoff at every time k) has to be done for each payoff.

▷ **Exercise** Write a regression algorithm based on the "primal" BDPP.

10.4   Quantization (tree) methods (II)

10.4.1   Approximation by a quantization tree

In this section, we still deal with the simple discrete time Markovian optimal stopping problem introduced in the former section. The underlying idea of quantization tree method is to approximate the whole Markovian dynamics of the chain $(X_k)_{0 \le k \le n}$ using a *skeleton* of the distribution supported by a tree.

For every $k \in \{0, \dots, n\}$, we replace the marginal X_k by a function $\widehat{X}_k$ of X_k taking values in a *grid* Γ_k, namely $\widehat{X}_k = \pi_k(X_k)$, where $\pi_k : \mathbb{R}^d \to \Gamma_k$ is a Borel function. The grid $\Gamma_k = \pi_k(\mathbb{R}^d)$ (also known as a *codebook* in Signal processing or Information Theory) will always be supposed finite in practice, with size $|\Gamma_k| = N_k \in \mathbb{N}^*$ although the error bounds established below still hold if the grids are infinite provided π_k is sublinear $(|\pi_k(x)| \le C(1 + |x|))$ so that $\widehat{X}_k$ has at least as many moments as $\widehat{X}_k$.

We saw in Chapter 5 an optimal way to specify the function π_k (including Γ_k) when trying by minimizing the induced L^p-mean quadratic error $\|X_k - \widehat{X}_k\|_p$. This is the purpose of optimal quantization Theory. We will come back to these aspects further on.

The starting point, being aware that the sequence $(\widehat{X}_k)_{0 \le k \le n}$ has no reason to share a Markov property, is to force this Markov property in the Backward Dynamic Programing Principle. This means defining by induction a quantized pseudo-Snell envelope of $\big(f_k(X_k)\big)_{0 \le k \le n}$ (assumed to lie at least in L^1), namely

$$\widehat{U}_n = f_n(\widehat{X}_n), \qquad \widehat{U}_k = \max\Big(f_k(\widehat{X}_k), \mathbb{E}\big(\widehat{U}_{k+1} \,|\, \widehat{X}_k\big)\Big). \tag{10.5}$$

The forced Markov property results from the conditioning by $\widehat{X}_k$ rather than by the σ-field $\widehat{\mathcal{F}}_k := \sigma(\widehat{X}_\ell), \, 0 \le \ell \le k)$.

It is straightforward by induction that, for every $k \in \{0, \dots, n\}$,

$$\widehat{U}_k = \widehat{u}_k(\widehat{X}_k) \quad \widehat{u}_k : \mathbb{R}^d \to \mathbb{R}_+, \text{ Borel function.}$$

See subsection 10.4.4 for the detailed implementation.

10.4.2   Error bounds

The following theorem establishes the control on the approximation of the true Snell envelope $(U_k)_{0 \le k \le n}$ by the quantized pseudo-Snell envelope $(\widehat{U}_k)_{0 \le k \le n}$ using the L^p-mean approximation errors $\|X_k - \widehat{X}_k\|_p$.

Theorem 10.3 (see [12] (2001), [131] (2011)) *Assume that all functions $f_k : \mathbb{R}^d \to \mathbb{R}_+$ are Lipschitz continuous and that the transitions $P_k(x, dy) = \mathbb{P}(X_{k+1} \in dy \,|\, X_k = x)$ are Lipschitz continuous in the following sense*

$$[P_k]_{\text{Lip}} = \sup_{[g]_{\text{Lip}} \le 1} [P_k g]_{\text{Lip}} < +\infty, \quad k = 0, \dots, n.$$

Set $[P]_{\text{Lip}} = \max_{0 \le k \le n-1} [P_k]_{\text{Lip}}$ and $[f]_{\text{Lip}} = \max_{0 \le k \le n} [f_k]_{\text{Lip}}$.

Let $p \in [1, +\infty)$. We assume that $\sum_{k=1}^{n} \|X_k\|_p + \|\widehat{X}_k\|_p < +\infty$.

(a) For every $k \in \{0, \ldots, n\}$,

$$\|U_k - \widehat{U}_k\|_p \leq 2[f]_{\mathrm{Lip}} \sum_{\ell=k}^{n} \left([P]_{\mathrm{Lip}} \vee 1\right)^{n-\ell} \|X_\ell - \widehat{X}_\ell\|_p.$$

(b) If $p = 2$, for every $k \in \{0, \ldots, n\}$,

$$\|U_k - \widehat{U}_k\|_2 \leq \sqrt{2}[f]_{\mathrm{Lip}} \left(\sum_{\ell=k}^{n} \left([P]_{\mathrm{Lip}} \vee 1\right)^{2(n-\ell)} \|X_\ell - \widehat{X}_\ell\|_2^2 \right)^{\frac{1}{2}}.$$

Proof. STEP 1. First, we control the Lipschitz continuous constants of the functions u_k. It follows form the classical inequality

$$\forall\, a_i,\, b_i \in \mathbb{R}, \quad |\sup_{i \in I} a_i - \sup_{i \in I} b_i| \leq \sup_{i \in I} |a_i - b_i|$$

that

$$
\begin{aligned}
[u_k]_{\mathrm{Lip}} &\leq \max \left([f_k]_{\mathrm{Lip}}, [P_k u_{k+1}]_{\mathrm{Lip}}\right) \\
&\leq \max \left([f]_{\mathrm{Lip}}, [P_k]_{\mathrm{Lip}}[u_{k+1}]_{\mathrm{Lip}}\right)
\end{aligned}
$$

with the convention $[u_{n+1}]_{\mathrm{Lip}} = 0$. An easy backward induction yields $[u_k]_{\mathrm{Lip}} \leq [f]_{\mathrm{Lip}}\left([P]_{\mathrm{Lip}} \vee 1\right)^{n-k}$.

STEP 2. We focus on claim (b) when $p = 2$. First we show

$$\left\|P_1 g(X_1) - \mathbb{E}\big(h(\widehat{X}_2) \,|\, \widehat{X}_1\big)\right\|_2^2 \leq [P_1 g]_{\mathrm{Lip}}^2 \big\|X_1 - \widehat{X}_1\big\|_2^2 + \big\|g(X_2) - h(\widehat{X}_2)\big\|_2^2. \tag{10.6}$$

We have

$$P_1 g(X_1) - \mathbb{E}\big(h(\widehat{X}_2) \,|\, \widehat{X}_1\big) = \mathbb{E}(g(X_2)|X_1) - \mathbb{E}\big(\mathbb{E}(g(X_2)|X_1) \,|\, \widehat{X}_1\big) \overset{+}{\perp} \mathbb{E}(g(X_2)|\widehat{X}_1) - \mathbb{E}\big(h(X_2)|X_1\big)$$

where we used the chaining rule for conditional expectation $\mathbb{E}(\mathbb{E}(\,.\,|X_1)|\widehat{X}_1) = \mathbb{E}(\,.\,|\widehat{X}_1)$ since $\widehat{X}_1$ is $\sigma(X_1)$-measurable. The orthogonality follows from the very definition of conditional expectation as a projection. Consequently

$$
\begin{aligned}
\left\|P_1 g(X_1) - \mathbb{E}\big(h(\widehat{X}_2) \,|\, \widehat{X}_1\big)\right\|_2^2 &\leq \left\|P_1 g(X_1) - \mathbb{E}\big(P_1 g(X_1)|\widehat{X}_1\big)\right\|_2^2 + \left\|\mathbb{E}(g(X_2)|\widehat{X}_1) - \mathbb{E}\big(h(X_2)|X_1\big)\right\|_2^2 \\
&\leq \left\|P_1 g(X_1) - P_1 g(\widehat{X}_1)\right\|_2^2 + \left\|g(X_2) - h(\widehat{X}_2)\right\|_2^2
\end{aligned}
$$

where we used in the last line the very definition of conditional expectation $\mathbb{E}(.|\widehat{X}_1)$ as the best L^2-approximation by a $\sigma(\widehat{X}_1)$-measurable random vector and the fact that conditional expectation operator are projectors (hence contraction with operator norm at most equal to 1).

Now, it follows from both dynamic programing formulas (original and quantized) that

$$|U_k - \widehat{U}_k| \leq \max\left(|f_k(X_k) - f_k(\widehat{X}_k)|, \left|\mathbb{E}(U_{k+1}|X_k) - \mathbb{E}(\widehat{U}_{k+1}|\widehat{X}_k)\right|\right)$$

so that

$$|U_k - \widehat{U}_k|^2 \leq |f_k(X_k) - f_k(\widehat{X}_k)|^2 + \left|\mathbb{E}(U_{k+1}|X_k) - \mathbb{E}(\widehat{U}_{k+1}|\widehat{X}_k)\right|^2.$$

Now, we derive from (10.6) applied with P_k and $g = f_k$ and $h = \widehat{u}_{k+1}$ that

$$\left\|\mathbb{E}(U_{k+1}|X_k) - \mathbb{E}(\widehat{U}_{k+1}|\widehat{X}_k)\right\|_2^2 \leq [P_k u_{k+1}]^2 \|X_k - \widehat{X}_k\|_2^2 + \|u_{k+1}(X_{k+1}) - \widehat{u}_{k+1}(\widehat{X}_{k+1})\|_2^2.$$

Plugging this inequality in the original one yields for every $k \in \{0, \dots, n\}$,

$$\left\|U_k - \widehat{U}_k\right\|_2^2 \leq \left([f]_{\mathrm{Lip}}^2 + [P]_{\mathrm{Lip}}^2 [u_{k+1}]_{\mathrm{Lip}}^2\right) \|X_k - \widehat{X}_k\|_2^2 + \|U_{k+1} - \widehat{U}_{k+1}\|_2^2$$

still with the convention $[u_{n+1}]_{\mathrm{Lip}} = 0$. Now,

$$
\begin{aligned}
[f]_{\mathrm{Lip}}^2 + [P]_{\mathrm{Lip}}^2 [u_{k+1}]_{\mathrm{Lip}}^2 &\leq [f]_{\mathrm{Lip}}^2 + [P]_{\mathrm{Lip}}^2 \left(1 \vee [P_k]_{\mathrm{Lip}}\right)^{2(n-(k+1))} \\
&\leq 2[f]_{\mathrm{Lip}}^2 \left(1 \vee [P_k]_{\mathrm{Lip}}\right)^{2(n-k)}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\left\|U_k - \widehat{U}_k\right\|_2^2 &\leq \sum_{\ell=k}^{n-1} 2[f]_{\mathrm{Lip}}^2 \left(1 \vee [P_k]_{\mathrm{Lip}}\right)^{2(n-k)} \|X_\ell - \widehat{X}_\ell\|_2^2 + [f]_{\mathrm{Lip}}^2 \|X_n - \widehat{X}_n\|_2^2 \\
&\leq 2[f]_{\mathrm{Lip}}^2 \sum_{\ell=k}^{n} \left(1 \vee [P_k]_{\mathrm{Lip}}\right)^{2(n-k)} \|X_\ell - \widehat{X}_\ell\|_2^2
\end{aligned}
$$

which completes the proof. $\diamond$

Remark. The above control emphasizes the interest of minimizing the "quantization" error $\|X_k - \widehat{X}_k\|_p$ at each tim step of the Markov chain to reduce the final resulting error.

▷ **Exercise.** Prove claim (a) starting from the inequality

$$\left\|P_1 g(X_1) - \mathbb{E}(h(\widehat{X}_2) \,|\, \widehat{X}_1)\right\|_p \leq [P_1 g]_{\mathrm{Lip}} \|X_1 - \widehat{X}_1\|_p + \|g(X_2) - h(\widehat{X}_2)\|_p.$$

Example of application: the Euler scheme. Let $(\bar{X}_{t_k^n}^n)_{0 \leq k \leq n}$ be the Euler scheme with step $\frac{T}{n}$ of the d-dimensional diffusion solution to the SDE (10.1). It defines an homogenous Markov chain with transition

$$\bar{P}_k^n g(x) = \mathbb{E}\, g\left(x + \frac{T}{n} b(t_k^n, \bar{X}_{t_k^n}^n) + \sigma(t_k^n, \bar{X}_{t_k^n}^n)\sqrt{\frac{T}{n}} Z\right), \quad Z \overset{\mathcal{L}}{\sim} \mathcal{N}(0, I_q).$$

If f is Lipschitz

$$
\begin{aligned}
\left|\bar{P}_k^n g(x) - \bar{P}_k^n g(x')\right|^2 &\leq [g]_{\mathrm{Lip}}^2 \mathbb{E}\left|x - x' + \frac{T}{n}\left(b(t_k^n, x) - b(t_k^n, x')\right) + \sqrt{\frac{T}{n}}\left(\sigma(t_k^n, x) - \sigma(t_k^n, x')\right)Z\right|^2 \\
&\leq [g]_{\mathrm{Lip}}^2 \left(\left|x - x' + \frac{T}{n}\left(b(t_k^n, x) - b(t_k^n, x')\right)\right|^2 + \left|\sigma(t_k^n, x) - \sigma(t_k^n, x')\right|^2\right) \\
&\leq [g]_{\mathrm{Lip}}^2 |x - x'|^2 \left(1 + \frac{T}{n}[\sigma]_{\mathrm{Lip}}^2 + \frac{2T}{n}[b]_{\mathrm{Lip}} + \frac{T^2}{n^2}[b]_{\mathrm{Lip}}^2\right).
\end{aligned}
$$

As a consequence

$$
[\bar{P}_k^n g]_{\mathrm{Lip}} \leq \left(1 + \frac{C_{b,\sigma,T}}{n}\right) [g]_{\mathrm{Lip}}, \ k = 0, \ldots, n - 1,
$$

i.e.

$$
[\bar{P}^n]_{\mathrm{Lip}} \leq \left(1 + \frac{C_{b,\sigma,T}}{n}\right).
$$

Applying the control established in claim (b) of the above theorem yields with obvious notations

$$
\begin{aligned}
\left\|U_k - \widehat{U}_k\right\|_2 &\leq \sqrt{2}[f]_{\mathrm{Lip}} \left(\sum_{\ell=k}^n \left(1 + \frac{C_{b,\sigma,T}}{n}\right)^{2(n-\ell)} \left\|X_\ell - \widehat{X}_\ell\right\|_2^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2} e^{C_{b,\sigma,T}}[f]_{\mathrm{Lip}} \left(\sum_{\ell=k}^n e^{2C_{b,\sigma,T} t_\ell^n} \left\|X_\ell - \widehat{X}_\ell\right\|_2^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

▷ **Exercise.** Derive a result in the case $p \neq 2$ based on Claim (a) of the theorem.

10.4.3 Backkground on optimal quantization

See (and update if necessary) Chapter 5

10.4.4 Implementation of a quantization tree method

▷ **Quantization tree.** The pathwise Quantized backward Dynamic Programing Principle (10.5) can be rewritten in distribution as follows. Let for every $k \in \{0, \ldots, n\}$,

$$
\Gamma_k = \left\{x_1^k, \ldots, x_{N_k}^k\right\}.
$$

Keeping in mind that $\widehat{U}_k = \widehat{u}_k(\widehat{X}_k)$, $k = 0, \ldots, n$, we first get

$$
\begin{cases}
\widehat{u}_n &= f_n \text{ on } \Gamma_n, \\
\widehat{u}_k(x_i^k) &= \max\left(f_k(x_i^k), \mathbb{E}\left(\widehat{u}_{k+1}(\widehat{X}_{k+1})|\widehat{X}_k = x_i^k\right)\right), \ i = 1, \ldots, N_k, k = 0, \ldots, n-1
\end{cases} \tag{10.7}
$$

which finally leads to

$$
\begin{cases}
\widehat{u}_n(x_i^n) &= f_n(x_i^n), \ i = 1, \ldots, N_n, \\
\widehat{u}_k(x_i^k) &= \max\left(f_k(x_i^k), \sum_{j=1}^{N_{k+1}} p_{ij}^k \widehat{u}_{k+1}(x_j^{k+1})\right), \ i = 1, \ldots, N_k, k = 0, \ldots, n-1
\end{cases} \tag{10.8}
$$

where the transition weight "super-matrix" $[p_{ij}^k]$ is defined by

$$p_{ij}^k = \mathbb{P}\big(\widehat{X}_{k+1} = x_j^{k+1} | \widehat{X}_x = x_i^k\big),\ 1 \le i \le N_k,\ 1 \le j \le N_{k+1},\ k = 0, \dots, n-1. \tag{10.9}$$

Although the above super-matrix defines a family of Markov transitions, the sequence $(\widehat{X}_k)_{0 \le k \le n}$ is definitely not a Markov Chain since there is no reason why $\mathbb{P}\big(\widehat{X}_{k+1} = x_j^{k+1} | \widehat{X}_k = x_i^k\big)$ and $\mathbb{P}\big(\widehat{X}_{k+1} = x_j^{k+1} | X_k = x_i^k, \widehat{X}_\ell x = x_{i_\ell}^\ell, \ell = 0, \dots, i-1\big)$.

In fact one should rather seeing the quantized transitions

$$\widehat{P}_k(x_i^k, dy) = \sum_{j=1}^{N_{k+1}} p_{ij}^k \delta_{x_j^{k+1}},\ x_i^k \in \Gamma_k,\ k = 0 = \dots, n-1,$$

as spatial discretizations of the original transitions $P_k(x, du)$ of the original Markov chain.

Definition 10.1 *The family of grids* (Γ_k), $0 \le k \le n$, *and the transition super-matrix* $[p_{ij}^k]$ *defined by (10.9) defines a* quantization tree *of size* $N = N_0 + \dots + N_n$.

Remark. A quantization tree in the sense of the above definition does not characterize the distribution of the sequence $(\widehat{X}_k)_{0 \le k \le n}$.

▷ **Implementation.** The implementation of the whole quantization tree method relies on the computation of this transition super-matrix. Once the grids (optimal or not) have been specified and the weights of the super-matrix have been computed or, to be more precise have been estimated, the computation of the pseudo-réduite $\mathbb{E}\,\widehat{U}_0$ at the origin amounts to an almost instantaneous "backward descent" of the quantization tree based on (10.8).

If we can simulate M independent copies of the Markov chain $(X_k)_{0 \le k \le n}$ denoted $X^{(1)}, \dots, X^{(M)}$, then the weights p_{ij}^k can be estimated by a standard Monte Carlo estimator

$$p_{ij}^{(M),k} = \frac{\#\big\{m \in \{1, \dots, M\},\ \widehat{X}_{k+1}^{(m)} = x_j^{k+1} \& \widehat{X}_k^{(m)} = x_i^k\big\}}{\#\big\{m \in \{1, \dots, M\},\ \widehat{X}_k^{(m)} = x_i^k\big\}} \xrightarrow{M \to +\infty} p_{ij}^k.$$

Remark. By contrast with the regression methods for which theoretical results are mostly focused on the rate of convergence of the Monte Carlo phase, we will analyze here this part of the procedure for which we refer to [8] where the Monte Carlo simulation procedure to compute the transition super-matrix and its impact on the quantization tree is deeply analyzed.

Application. We can apply what precedes, still within the framework of an Euler scheme, to our original optimal stopping problem. We assume that all random vectors X_k lie in $L^{p'}(\mathbb{P})$, for a real exponent $p' > 2$ and that they have been optimally quantized (in L^2) by grids of size N_k, $k = 0, \dots, n$, then, relying on the non-asymptotic Zador's Theorem (claim (*b*) from Theorem 5.1.2), we get with obvious notations,

$$\|\bar{U}_0^n - \widehat{\bar{U}}_0^n\|_2 \le \sqrt{2} e^{C_{b,\sigma,T}} [f]_{\mathrm{Lip}} C'_{p',d} \left(\sum_{k=0}^n e^{2C_{b,\sigma,T}} \sigma_{p'}(\bar{X}_{t_k}^n)^2 N_k^{-\frac{2}{d}} \right)^{\frac{1}{2}}.$$

SMALL CAPS QUANTIZATION TREE OPTIMAL DESIGN At this stage, one can proceed an optimization of the quantization tee. To be precise, one can optimize the size of the grids Γ_k subject to a "budget" (or total allocation) constraint, typically

$$\min\left\{\sum_{k=0}^{n} e^{2C_{b,\sigma,T}}\sigma_{p'}(\bar{X}_{t_k^n}^n)^2 N_k^{-\frac{2}{d}}, \; N_k \geq 1, \; N_0 + \cdots + N_n = N\right\}.$$

▷ **Exercise.** (*a*) Solve rigorously the constrained optimization problem

$$\min\left\{\sum_{k=0}^{n} e^{2C_{b,\sigma,T}}\sigma_{p'}(\bar{X}_{t_k^n}^n)^2 x_k^{-\frac{2}{d}}, \; x_k \in \mathbb{R}_+, \; x_0 + \cdots + x_n = N\right\}.$$

(*b*) Derive a asymptotically optimal (suboptimal) choice for the grid size allocation (as $N \to +\infty$).

This optimization turns out to have a significant numerical impact, even if, in terms of rate, the uniform choice $N_k = \bar{N} = \frac{N-1}{n}$ (doing so we assume implicitly that $X_0 = x_0 \in \mathbb{R}^d$ so that $\widehat{X}_0 = X_0 = x_0$, $N_0 = 1$ and $\widehat{U}_0 = \widehat{u}_0(x_0)$) leads to a quantization error of the form

$$
\begin{aligned}
|\bar{u}_0^n(x_0) - \widehat{\bar{u}_0^n}(x_0)| \; &\leq \; \left\|\bar{U}_0^n - \widehat{\bar{U}^n}_0\right\|_2 \\
&\leq \; \kappa_{b,\sigma,T}[f]_{\mathrm{Lip}} \max_{0\leq k\leq n} \sigma_{p'}(\bar{X}_{t_k^n}^n) \times \frac{\sqrt{n}}{\bar{N}^{\frac{1}{d}}} \\
&\leq \; [f]_{\mathrm{Lip}}\kappa'_{b,\sigma,T,p'} \frac{\sqrt{n}}{\bar{N}^{\frac{1}{d}}}
\end{aligned}
$$

since we know (see Proposition 7.2 in Chapter 7) that $\sup_n \mathbb{E}\big(\max_{0\leq k\leq n} |\bar{X}_{t_k^n}^n|^{p'}\big) < +\infty$. If we plug that in the global estimate obtained in Theorem 10.1(*b*), we obtain the typical error bound

$$|u_0(x_0) - \widehat{\bar{u}_0^n}(x_0)| \leq C_{b,\sigma,T,f,d}\left(\frac{1}{n^{\frac{1}{2}}} + \frac{n^{\frac{1}{2}}}{\bar{N}^{\frac{1}{d}}}\right). \tag{10.10}$$

Remark. If we can simulate directly the sampled $(X_{t_k^n})_{0\leq k\leq n}$ of the diffusion instead of its Euler scheme and if the obstacle/payoff function is semi-convex in the sense Condition (SC), then we get as a typical error bound

$$|u_0(x_0) - \widehat{\bar{u}_0^n}(x_0)| \leq C_{b,\sigma,T,f,d}\left(\frac{1}{n} + \frac{n^{\frac{1}{2}}}{\bar{N}^{\frac{1}{d}}}\right).$$

Remark. ● The rate of decay $\bar{N}^{-\frac{1}{d}}$ which becomes worse and worse as the dimension d of the structure Markov process increases can be encompassed by such tree methods. It is a consequence of Zador's Theorem and is known as the *curse of dimensionality*.

● These rates can be significantly improved by introducing a Romberg like extrapolation method or/and some martingale corrections to the quantization tree (see [131]).

10.4.5   How to improve convergence performances?

▷ **Tree optimization** *vs BDDP* **complexity**   If, n being fixed, we set

$$N_k = \left\lfloor \frac{(\sigma_{2+\delta}(X_k))^{\frac{2d}{d+2}}}{\sum_{0 \leq \ell \leq n}(\sigma_{2+\delta}(X_k))^{\frac{2d}{d+2}}} N \right\rfloor \vee 1, \ k = 0, \dots, n$$

(see exercise, in the previous section) we (asymptotically) minimize the resulting quantization error induced by the *BDDP* descent. This error reads with $\tilde{N} = N_0 + \cdots + N_n$ (usually $> N$).

EXAMPLES: • Brownian motion $X_k = W_{t_k}$: Then $\widehat{W}_0 = 0$ and

$$\|W_{t_k}\|_{2+\delta} = C_\delta \sqrt{t_k}, \qquad k = 0, \dots, n.$$

Hence $N_0 = 1$ and

$$N_k \approx \frac{2(d+1)}{d+2} \left(\frac{k}{n}\right)^{\frac{d}{2(d+1)}} N, \qquad k = 1, \dots, n;$$

$$|\mathcal{V}_0 - \widehat{v}_0(0)| \leq C_{W,\delta} \left(\frac{2(d+1)}{d+2}\right)^{1-\frac{1}{d}} \frac{n^{1+\frac{1}{d}}}{N^{\frac{1}{d}}} = O\left(\frac{n}{\bar{N}^{\frac{1}{d}}}\right)$$

with $\bar{N} = \frac{N}{n}$. Theoretically this choice may look not crucial since it has no impact on the convergence rate, but in practice, it does influence the numerical performances.

• Stationary process: The process $(X_k)_{0 \leq k \leq n}$ is stationary and $X_0 \in L^{2+\delta}$ for some $\delta > 0$. A typical example in the Gaussian world, is, as expected, the stationary Ornstein-Uhlenbeck process (sampled at times $t_k^n = \frac{kT}{n}$, $k = 0, \dots, n$). The essential feature of such a setting is that the quantization tree only relies on one optimal $\bar{N}$-grid (say L^2 optimal for the distribution of $\widehat{X}_0$) $\Gamma = \Gamma_N^* = \{x^1, \dots, x^{\bar{N}}\}$ and one quantized transition matrix $\left[\mathbb{P}(\widehat{X}_1^\Gamma = x^j \mid \widehat{X}_0^\Gamma = x^i)\right]_{1 \leq i,j \leq n}$.

  – For every $k \in \{0, \dots, n\}$, $\|X_k\|_{2+\delta} = \|X_0\|_{2+\delta}$ hence $N_k = \left\lceil \frac{N}{n+1} \right\rceil$, $k = 0, \dots, n$ and

$$\|\mathcal{V}_0 - \widehat{v}_0(\widehat{X}_0)\|_2 \leq C_{X,\delta} \frac{n^{\frac{1}{2}+\frac{1}{d}}}{N^{\frac{1}{d}}} \leq C_{X,\delta} \frac{n}{\bar{N}^{\frac{1}{d}}} \quad \text{with } \bar{N} = \frac{N}{n+1}.$$

▷ **Richardson-Romberg extrapolation(s)**   Romberg extrapolation in this framework is based on an *heuristic guess*: there exists a "sharp rate" of convergence of the quantization tree method as the total budget N goes to infinity and this rate of convergence is given by (10.10) when replacing $\bar{N}$ by N/n.

  On can proceed a Romberg extrapolation in N for fixed n or even a full Romberg extrapolation involving both the time discretization step n and the size N of the quantization tree.

▷ **Exercise (temporary).** Make the assumption that the error in a quantization scheme admits a first order expansion of the form

$$\mathcal{E}rr(n, N) = \frac{c_1}{n^\alpha} + \frac{c_2 n^{\frac{1}{2}+\frac{1}{d}}}{N^{\frac{1}{d}}}.$$

Devise a Richardson-Romberg extrapolation based on two quantization trees with sizes $N^{(1)}$ and $N^{(2)}$ respectively.

$\triangleright$ **Martingale correction**   When $(X_k)_{0 \leq k \leq n}$, is a martingale, one can force this martingale property on the quantized chain by freezing the transition weigh super-matrix and by moving in a backward way the grids Γ_k so that the resulting grids $\widetilde{\Gamma}_k$ satisfy

$$\mathbb{E}\big(\widehat{X}_k^{\widetilde{\Gamma}_k} \mid \widehat{X}_{k-1}^{\widetilde{\Gamma}_{k-1}}\big) = \widehat{X}_{k-1}^{\widetilde{\Gamma}_{k-1}}.$$

In fact this identity defines by a backward induction the grids $\widehat{\Gamma}_k$. As a final step, one translate the grids so that Γ_0 and $\widetilde{\Gamma}_0$ have the same mean. Of course such a procedure is entirely heuristic.

10.4.6   Numerical optimization of the grids: Gaussian and non-Gaussian vectors

The procedures that minimizes the quantization error are usually stochastic (except in 1-dimension). The most famous ones are undoubtedly the so-called *Competitive Leaning Vector Quantization* algorithm (see [128] or [126]) and the Lloyd procedure (see [128, 123, 56]) which have been described in details and briefly analyzed in Section 6.3.6 of Chapter 6. More algorithmic details can also be found on the website

<div align="center">www.quantize.maths-fi.com</div>

as well as optimal quantizers of the normal distributions for $d = 1$ to to $d = 10$ and $N = 1$ up to 10^5.

10.4.7   The case of normal distribution $\mathcal{N}(0; I_d)$ on $\mathbb{R}^d$, $d \geq 1$

For these distributions a large scale optimization have been carried out based on a mixed $CLVQ$-Lloyd's procedure. To be precise, grids have been computed for For $d = 1$ up to 10 and $N = 1 \leq N \leq 5\,000$. Furthermore several companion parameters have also been computed (still by simulation): weight, L^1 quantization error, (squared) L^2-distortion, local $L^1 \& L^2$-pseudo-inertia of each Voronoi cell. All these grids can be downloaded on the website.

Thus depicting an optimal quadratic N-quantization of the bi-variate normal distribution $\mathcal{N}(0; I_2)$ with $N = 500$ yields

On the above website are also available some (free access) grids for the d-variate normal distribution.

The 1**-dimension. . .**   Although of little interest for applications since other (deterministic methods like PDE approach are available) we propose below for the sake of completeness a Newton-Raphson method to compute optimal quantizers of scalar *unimodal* distributions, *i.e.* absolutely continuous distributions whose density is log-concave. The starting point is the following theorem due to Kieffer [81] which states uniqueness of the optimal quantizer in that setting.

Theorem 10.4 (1*D*-distributions, see [81]) *If $d = 1$ and $\mathbb{P}_X(d\xi) = \varphi(\xi)\,d\xi$ with $\log \varphi$ concave, then there is exactly one stationary N-quantizer (up to the reordering of its component, i.e. as a*
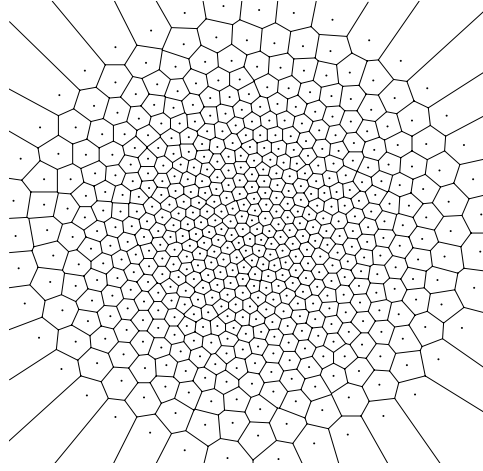
Figure 10.1: *An optimal quantization of the bi-variate normal distribution with size $N = 500$*

set). *This unique stationary quantizer is a global (and local) minimum of the distortion function* i.e.

$$\forall\, N \geq 1, \qquad \mathrm{argmin}_{\mathbb{R}^N}\, D_N^X = \{x^{(N)}\}.$$

Remark. Absolutely continuous distributions on the real line with a log-concave density are sometimes called *unimodal distributions*. The support of such distributions is a (closed) interval $[a, b]$ where $-\infty \leq a \leq b \leq +\infty$.

EXAMPLES OF UNIMODAL DISTRIBUTIONS: The normal and Gaussian distributions, the gamma distributions $\gamma(\alpha, \beta)$ with $\alpha \geq 1$, the Beta $B(a, b)$-distributions with a, $b \geq 1$, etc.

In $1D$-setting, a deterministic optimization approach can be developed by proceeding as follows.

▷ Specification of the Voronoi cells of $x = (x_1, \ldots, x_N)$, $a < x_1 < x_2 < \cdots < x_N < b$ where $[a, b]$ denote the support of $\mathbb{P}_X$ (which is that of φ as well): $C_i(x) = [x^{i-\frac{1}{2}}, x^{i+\frac{1}{2}}[,\quad x^{i+\frac{1}{2}} = \frac{x^{i+1}+x^i}{2},$ $i = 2, \ldots, N-1$, $x^{\frac{1}{2}} = a$, $x^{N+\frac{1}{2}} = b$.

▷ Computation of the gradient Gradient: $\nabla D_N^X(x) = 2 \left(\displaystyle\int_{x^{i-\frac{1}{2}}}^{x^{i+\frac{1}{2}}} (x^i - \xi)\varphi(\xi)d\xi \right)_{1 \leq i \leq N}.$

The Hessian $\nabla^2(D_N^X)(x^1, \ldots, x^N)$ can in turn be computed (at least at N-tuples having components where the density φ is continuous which is the case everywhere, except possibly at the endpoints of its support which is an interval). It reads

$$\nabla^2(D_N^X)(x^1, \ldots, x^N) = \big[\ tobecompleted\ \big]$$

where $\Phi(u) = \displaystyle\int_{-\infty}^{u} \varphi(v)dv$ and $\Psi(u) = \displaystyle\int_{-\infty}^{u} v\varphi(v)dv.$

Thus if $X \sim \mathcal{N}(0;1)$ the function Φ is (tabulated or) computable (see Section 11.3) at low computational cots with high accuracy and

$$\Psi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \ x \in \mathbb{R}.$$

This allows for an almost instant search for the unique optimal N-quantizer using a Newton-Raphson descent on $\mathbb{R}^N$ with the requested accuracy.

$\triangleright$ For the normal distribution $\mathcal{N}(0;1)$ and $N = 1,\ldots,500$, tabulation within 10^{-14} accuracy of both optimal N-quantizers and companion parameters:

$$x^{(N)} = \left(x^{(N),1},\ldots,x^{(N),N}\right)$$

and

$$\mathbb{P}(X \in C_i(x^{(N)})), \ i = 1,\ldots N, \ \text{ and } \ \|X - \widehat{X}^{x^{(N)}}\|_2$$

can also be downloaded at the website www.quantize.maths-fi.com.

Chapter 11

Miscellany

11.1 More on the normal distribution

11.2 Characteristic function

Proposition 11.1 *If* $Z \overset{\mathcal{L}}{\sim} \mathbb{N}(0;1)$, *then*

$$\forall u \in \mathbb{R}, \ \Phi_Z(u) := \int_{-\infty}^{+\infty} e^{iux} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = e^{-\frac{u^2}{2}}.$$

Proof. Differentiating under the integral symbol yields

$$\Phi'_Z(u) = i \int_{-\infty}^{+\infty} e^{iux} e^{-\frac{x^2}{2}} x \frac{dx}{\sqrt{2\pi}}.$$

Now the following integration by parts yields

$$\begin{cases} e^{iux} & \xrightarrow{\ '\ } iue^{iux} \\ xe^{-\frac{x^2}{2}} & \xrightarrow{\ \int\ } -e^{-\frac{x^2}{2}} \end{cases}$$

$$\Phi'_Z(u) = i^2 u \int_{-\infty}^{+\infty} e^{iux} e^{-\frac{x^2}{2}} x \frac{dx}{\sqrt{2\pi}} = -u\Phi_Z(u).$$

so that

$$\Phi_Z(u) = \Phi_Z(0)e^{-\frac{u^2}{2}} = e^{-\frac{u^2}{2}}. \qquad\qquad \diamond$$

11.3 Numerical approximation of the distribution function

To compute the distribution function of the normal distribution, one usually relies on the fast approximation formula obtained by continuous fractions expansion techniques (see [1]).

$$\forall x \in \mathbb{R}_+, \quad \Phi_0(x) = 1 - \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left((a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + O\left(e^{-\frac{x^2}{2}} t^6 \right) \right)$$

where

$$t := \tfrac{1}{1+px}, \qquad\qquad p := 0.231\,6419$$

$$a_1 := 0.319\,381\,530. \quad a_2 := -0.356\,563\,782,$$

$$a_3 := 1.781\,477\,937, \quad a_4 := -1.821\,255\,978,$$

$$a_5 := 1.330\,274\,429$$

and

$$O\left(e^{-\frac{x^2}{2}}t^6\right) \le 7,5\,10^{-8}.$$

11.4   Table of the distribution function of the normal distribution

The distribution function of the $\mathcal{N}(0;1)$ distribution is given for every real number t by

$$\Phi_0(t) := \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}}\,dx.$$

Since the probability density is even, one easily checks that

$$\Phi_0(t) - \Phi_0(-t) = 2\,\Phi_0(t) - 1.$$

The following tables give the values of $\Phi_0(t)$ for $t = x_0, x_1 x_2$ where $x_0 \in \{0, 1, 2\}$, $x_1 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $x_2 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

For example, if $t = 1.23$ (*i.e.* row 1.2 and column 0.03) one has $\Phi_0(t) \approx 0.8907$.

t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6661	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7290	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1,0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1,1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1,2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1,3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1,4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1,5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1,6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1,7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1,8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1,9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2,0	0.9772	0.9779	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2,1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2,2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2,3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2,4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2,5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2,6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2,7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2,8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2,9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

One notes that $\Phi_0(t) = 0.9986$ for $t = 2.99$. This comes from the fact that the mass of the normal distribution is mainly concentrated on the interval $[-3, 3]$ as emphasized by the table of the "large" values hereafter (for instance, one remarks that $\mathbb{P}(\{|X| \leq 4,5\}) \geq 0.99999$!).

t	3,0	3,1	3,2	3,3	3,4	3,5	3,6	3,8	4,0	4,5
$\Phi_0(t)$	.99865	.99904	.99931	.99952	.99966	.99976	.999841	.999928	.999968	.999997

11.5   Uniform integrability as a domination property

In this action, we propose a brief background on uniform integrability for random variables taking values in $\mathbb{R}^d$. Mostly for notational convenience all these random variables are defined on the same probability space $(\Omega, \mathcal{A}_{\mathbb{P}})$ though it is absolutely not mandatory. We leave as an exercise to check in

what follows that each random variable X_i can be defined on its own probability space $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i)$ with a straightforward adaptation of the statements.

Theorem 11.1 (Equivalent definitions of uniform integrability I.) *A family $(X_i)_{i \in I}$ of $\mathbb{R}^d$-valued random vectors, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, is said to be* uniformly integrable *if it satisfies one of the following equivalent properties,*

(i) $\displaystyle\lim_{R \to +\infty} \sup_{i \in I} \mathbb{E}\left(|X_i| \mathbf{1}_{\{|X_i| \geq R\}}\right) = 0.$

(ii) $\begin{cases} (\alpha) \quad \sup_{i \in I} \mathbb{E}\,|X_i| < +\infty, \\[2ex] (\beta) \quad \forall\, \varepsilon > 0,\ \exists\, \eta = \eta(\varepsilon) > 0 \ \ such\ that,\ \ \forall\, A \in \mathcal{A},\ \mathbb{P}(A) \leq \eta \Longrightarrow \sup_{i \in I} \int_A |X_i| d\mathbb{P} \leq \varepsilon. \end{cases}$

Remark. • All norms being strongly equivalent on $\mathbb{R}^d$, claims (i) and (ii) do not depend on the selected norm on $\mathbb{R}^d$.

• L^1-Uniform integrability of a family of probability distribution $(\mu_i)_{i \in I}$ defined on $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$ can be defined accordingly by

$$\lim_{R \to +\infty} \sup_{i \in I} \int_{\{|x| \geq R\}} |x| \mu_i(dx) = 0.$$

All what follows can be straightforwardly "translated" in terms of probability distributions.

More generally a non-zero Borel function $f : \mathbb{R}^d \to \mathbb{R}_+$ such that $\liminf_{|x| \to +\infty} = 0$ being defined, a family of probability distribution $(\mu_i)_{i \in I}$ defined on $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$ is f-uniformly integrable if

$$\lim_{R \to +\infty} \sup_{i \in I} \int_{\{f(x)) \geq R\}} f(x) \mu_i(dx) = 0.$$

Proof. Assume first that (i) holds. First it is clear that

$$\sup_{i \in I} \mathbb{E}|X_i| \leq R + \sup_{i \in I} \mathbb{E}\left(|X_i| \mathbf{1}_{\{|X_i| \geq R\}} < +\infty\right.$$

at least for large enough $R \in (0, \infty)$. Now, for every $i \in I$ and every $A \in \mathcal{A}$,

$$\int_A |X_i| d\mathbb{P} \leq R\mathbb{P}(A) + \int_{\{|X_i| \geq R\}} |X_i| d\mathbb{P}.$$

Owing to (i) there exists real number $R = R(\varepsilon) > 0$ such that $\sup_{i \in I} \int_{\{|X_i| \geq R\}} |X_i| d\mathbb{P} \leq \dfrac{\varepsilon}{2}$. Then setting $\eta = \eta(\varepsilon) = \frac{\varepsilon}{2R}$ yields (ii).

Conversely, for every real number $R > 0$, the Markov Inequality implies

$$\sup_{i \in I} \mathbb{P}(\{|X_i| \geq R\}) \leq \frac{\sup_{i \in I} \mathbb{E}\,|X_i|}{R}.$$

Let $\eta = \eta(\varepsilon)$ given by $(ii)(\beta)$. As soon as $R > \frac{\sup_{i \in I} \mathbb{E}\,|X_i|}{\eta}$, $\sup_{i \in I} \mathbb{P}(\{|X_i| \geq R\}) \leq \eta$ and $(ii)(\beta)$ implies that

$$\sup_{i \in I} \mathbb{E}\left(|X_i|\mathbf{1}_{\{|X_i| \geq R\}}\right) \leq \varepsilon$$

which completes the proof. $\diamond$

As a consequence, one easily derives that

P0. $(X_i)_{i \in I}$ is uniformly integrable iff $(|X_i|)_{i \in I}$ is.

P1. If $X \in L^1(\mathbb{P})$ then the family (X) is uniformly integrable.

P2. If $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ are two families of uniformly integrable $\mathbb{R}^d$-valued random vectors, then $(X_i + Y_i)_{i \in I}$ is uniformly integrable.

P3. If $(X_i)_{i \in I}$ is a family of $\mathbb{R}^d$-valued random vectors dominated by a uniformly integrable family $(Y_i)_{i \in I}$ of random variables in the sense that

$$\forall\, i \in I,\ |X_i| \leq Y_i \quad \mathbb{P}\text{-}a.s.$$

then $(X_i)_{i \in I}$ is uniformly integrable.

The four properties follow from characterization (i). To be precise **P1** is a consequence of the Lebesgue Dominated Convergence Theorem whereas the second one follows from the obvious

$$\mathbb{E}\left(|X_i + Y_i|\mathbf{1}_{\{|X_i+Y_i| \geq R\}}\right) \leq \mathbb{E}\left(|X_i|\mathbf{1}_{\{|X_i| \geq R\}}\right) + \mathbb{E}\left(|Y_i|\mathbf{1}_{\{|Y_i| \geq R\}}\right).$$

Now let us pass to a simple criterion of uniform integrability.

Corollary 11.1 *Let $(X_i)_{i \in I}$ be a family of $\mathbb{R}^d$-valued random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ satisfying $\lim_{|x|n \to +\infty} \dfrac{\Phi(x)}{|x|} = +\infty$. If*

$$\sup_{i \in I} \mathbb{E}\,\Phi(X_i) < +\infty$$

then the family $(X_i)_{i \in I}$ is uniformly integrable.

Theorem 11.2 (Uniform integrability II.) *Let $(X_n)_{n \geq 1}$ a sequence of $\mathbb{R}^d$-valued random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let X be an $\mathbb{R}^d$-valued random vector defined on the same probability space. If*

    (i) *$(X_n)_{n \geq 1}$ is uniformly integrable,*

    (ii) *$X_n \xrightarrow{\mathbb{P}} X$,*

then

$$\mathbb{E}|X_n - X| \longrightarrow 0 \quad \text{as } n \to +\infty.$$

(In particular $\mathbb{E}\,X_n \to \mathbb{E}\,X$.)

Proof. One derives from (ii) the existence of a subsequence $(X_{n'})_{n\geq 1}$ such that $X_{n'} \to X$ *a.s.*. Hence by Fatou's Lemma

$$\mathbb{E}\,|X| \leq \liminf_n \mathbb{E}|X_{n'}| \leq \sup_n \mathbb{E}|X_n| < +\infty.$$

Hence $X \in L^1(\mathbb{P})$ and, owing to **P1** and **P2**, $(X_n - X)_{n\geq 1}$ is a uniformly integrable sequence. Now, for every integer $n \geq 1$ and every $R > 0$,

$$\mathbb{E}\,|X_n - X| \leq \mathbb{E}(|X_n - X| \wedge M) + \mathbb{E}\big(|X_n - X|\mathbf{1}_{\{|X_n-X|\geq M\}}\big).$$

The Lebesgue Dominated Convergence Theorem implies $\lim_n \mathbb{E}(|X_n - X| \wedge M) = 0$ so that

$$\limsup_n \mathbb{E}\,|X_n - X| \leq \lim_{R\to+\infty} \sup_n \mathbb{E}\Big(|X_n - X|\mathbf{1}_{\{|X_n-X|\geq M\}}\Big) = 0. \quad \diamond$$

Corollary 11.2 (L^p-uniform integrability) *Let $p \in [1, +\infty)$. Let $(X_n)_{n\geq 1}$ a sequence of $\mathbb{R}^d$-valued random vectors defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let X be an $\mathbb{R}^d$-valued random vector defined on the same probability space. If*

    (i) *$(X_n)_{n\geq 1}$ is L^p-uniformly integrable (i.e. $(|X_n|^p)_{n\geq 1}$ is uniformly integrable),*

    (ii) *$X_n \xrightarrow{\mathbb{P}} X$,*

then

$$\|X_n - X\|_p \longrightarrow 0 \quad \text{as } n \to +\infty.$$

(In particular $\mathbb{E}\,X_n \to \mathbb{E}\,X$.)

Proof. By the same argument as above $X \in L^p(\mathbb{P})$ so that $(|X_n - X|^p)_{n\geq 1}$ is uniformly integrable by **P2** and **P3**. One concludes by the above theorem since $|X_n - X|^p \to 0$ in probability as $n \to +\infty$. $\quad \diamond$

11.6   Interchanging. . .

Theorem 11.3 (Interchanging continuity and expectation) *(see [30], Chapter 8) (a) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let I be a nontrivial interval of $\mathbb{R}$ and let $\Psi : I \times \Omega \to \mathbb{R}$ be a $\mathcal{B}or(I) \otimes \mathcal{A}$-measurable function. Let $x_0 \in I$. If the function Ψ satisfies:*

    (i) *for every $x \in I$, the random variable $\Psi(x, \,.\,) \in L^1_{\mathbb{R}}(\mathbb{P})$,*

    (ii) *$\mathbb{P}(d\omega)$-a.s., $x \mapsto \Psi(x, \omega)$ is continuous at x_0,*

    (iii) *There exists $Y \in L^1_{\mathbb{R}_+}(\mathbb{P})$ such that for every $x \in I$,*

$$\mathbb{P}(d\omega)\text{-a.s.} \ \ |\Psi(x, \omega)| \leq Y(\omega),$$

then, the function $\psi(x) := \mathbb{E}(\Psi(x, \,.\,))$ is defined at every $x \in I$, continuous at x_0.
(b) The domination property (iii) in the above theorem can be replaced mutatis mutandis *by a uniform integrability assumption on the family $(\Psi(x, .))_{x\in I}$.*

The same extension based on uniform integrability as Claim (b) holds true for the differentiation Theorem 2.1 (see the exercise following the theorem).

11.7 Measure Theory

Theorem 11.4 (Baire σ-field Theorem) *Let (S, d_S) be a metric space. Then*

$$\mathcal{B}or(S, d_S) = \sigma \left(\mathcal{C}(S, R) \right)$$

where $\mathcal{C}(S, R)$ denotes the set of continuous functions from (S, d_S) to $\mathbb{R}$. When S is σ-compact (i.e. is a countable union of compact sets), one may replace the space $\mathcal{C}(S, R)$ by the space $\mathcal{C}_K(S, R)$ of continuous functions with compact support.

Theorem 11.5 (Functional monotone class Theorem) *Let $(S, \mathcal{S})$ be a measurable space. Let V be a vector space of real valued bounded measurable functions defined on $(S, \mathcal{S})$. Let C be a subset of V, stable by the product of two functions. Assume furthermore that V satisfies*

(i) $\mathbf{1} \in V$,

(ii) V is closed under uniform convergence

(iii) V is closed under "bounded non-decreasing convergence": if $\varphi_n \in V$, $n \geq 1$, $\varphi_n \leq \varphi_{n+1}$, $|\varphi_n| \leq K$ (real constant) and $\varphi_n(x) \to \varphi(x)$ for every $x \in S$, then $\varphi \in V$.

Then H contains the vector subspace of all $\sigma(C)$-measurable bounded functions.

We refer to [114] for a proof of this result.

11.8 Weak convergence of probability measures on a Polish space

The main reference for this topic is [24]. See also [133].

The basic result of weak convergence theory is the so-called Portmanteau Theorem stated below (the definition and notation of weak convergence of probability measures are recalled in section 4.1).

Theorem 11.6 *Let $(\mu_n)_{n \geq 1}$ be a sequence of probability measures on a Polish (metric) space (S, δ) equipped with its Borel σ-field $\mathcal{S}$ and let μ be a probability measure on the same space. The following properties are equivalent:*

(i) $\mu_n \overset{\mathcal{S}}{\Longrightarrow} \mu$ *as* $n \to +\infty$,

(ii) For every open set O of (S, δ),

$$\mu(O) \leq \liminf_n \mu_n(O).$$

(ii) For every closed set F of (S, δ),

$$\mu(F) \geq \limsup_n \mu_n(F).$$

(iii) For every Borel set $A \in \mathcal{S}$ such that $\mu(\partial A) = 0$ (where $\partial A = \overline{A} \setminus \mathring{A}$ is the boundary of A),

$$\lim_n \mu_n(A) = \mu(A).$$

(iv) *For every nonnegative lower semi-continuous function $f : S \to \mathbb{R}_+$,*

$$0 \le \int_S f d\mu \le \liminf_n \int_S f d\mu_n$$

(v) *For every bounded Borel function $f : S \to \mathbb{R}$ such that $\mu\big(Disc(f)\big) = 0$,*

$$\lim_n \int_S f d\mu_n = \int_S f d\mu.$$

For a proof we refer to [24] Chapter 1.

When dealing with unbounded functions, there exists a kind of weak Lebesque dominated convergence theorem.

Proposition 11.2 *Let $(\mu_n)_{n \ge 1}$ be a sequence of probability measures on a Polish (metric) space (S, δ) weakly converging to μ.*

(a) Let $g : S \to \mathbb{R}_+$ a (non-negative) μ-integrable Borel function and let $f : S \to \mathbb{R}$ be a μ-a.s. continuous Borel function. If

$$0 \le |f| \le g \quad and \quad \int_S g d\mu_n \longrightarrow \int_S g d\mu \ \ as \ n \to +\infty$$

then $f \in L^1(\mu)$ and $\int_S f d\mu_n \longrightarrow \int_S f d\mu$ as $n \to +\infty$.

(b) The conclusion still holds if $(\mu_n)_{n \ge 1}$ is f-uniformly integrable i.e.

$$\limsup_n \ \int_{\substack{n \ge 1 \\ |f| \ge R}} |f| d\mu_n = 0$$

Proof.*cqfd*

If $S = \mathbb{R}^d$, weak convergence is also characterized by the Fourier transform $\widehat{\mu}$ defined on $\mathbb{R}^d$ by

$$\widehat{\mu}(u) = \int_{\mathbb{R}^d} e^{\tilde{i}(u|x)} \mu(dx), \ u \in \mathbb{R}^d.$$

Proposition 11.3 *Let $(\mu_n)_{n \ge 1}$ be a sequence of probability measures on $(\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d))$ and let μ be a probability measure on the same space. Then*

$$\Big(\mu_n \overset{\mathcal{S}}{\Longrightarrow} \mu\Big) \Longleftrightarrow \Big(\forall\, u \in \mathbb{R}^d, \ \widehat{\mu}_n(u) \longrightarrow \widehat{\mu}(u)\Big).$$

For a proof we refer *e.g.* to [74], or to any textbook on a first course in Probability Theory.

Remark. The convergence in distribution of a sequence $(X_n)_{n \ge 1}$ defined on probability spaces $(\Omega^n, \mathcal{A}^n, \mathbb{P}^n)$ of random variables taking values in a Polish space is defined as *the weak convergence of their distributions on* their distribution $\mu_n = \mathbb{P}^n_{X_n} = \mathbb{P}^n \circ X_n^{-1}$ on $(S, \mathcal{S})$.

11.9 Martingale Theory

Proposition 11.4 *Let $(M_n)_{n \geq 0}$ be a square integrable discrete time $\mathcal{F}_n$-martingale defined on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_n)_{n \geq 1})$.*

(a) Then there exists a unique non-decreasing $\mathcal{F}_n$-predictable process null at 0 denoted $(\langle M \rangle_n)_{n \geq 0}$ such that

$$(M_n - M_0)^2 - \langle M \rangle_n \quad \text{is an } \mathcal{F}_n\text{-martingale.}$$

This process reads

$$\langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}\big((M_k - M_{k-1})^2 \,|\, \mathcal{F}_{k-1}\big).$$

(b) Set $\langle M \rangle_\infty := \lim_n \langle M \rangle_n$. Then,

$$M_n \stackrel{n \to +\infty}{\longrightarrow} M_\infty \quad \text{on the event } \{\langle M \rangle_\infty < +\infty\}$$

where M_∞ is a finite random variable. If furthermore $\mathbb{E}\langle M \rangle_\infty < +\infty$, then $M_\infty \in L^2(\mathbb{P})$.

We refer to [115] for a proof of this result.

Lemma 11.1 (Kronecker Lemma) *Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let $(b_n)_{n \geq 1}$ be a non decreasing sequence of positive real numbers with $\lim_n b_n = +\infty$. Then*

$$\left(\sum_{n \geq 1} \frac{a_n}{b_n} \text{ converges in } \mathbb{R} \text{ as a series} \right) \implies \left(\frac{1}{b_n} \sum_{k=1}^{n} a_k \longrightarrow 0 \text{ as } n \to +\infty \right).$$

Proof. Set $C_n = \sum_{k=1}^{n} \dfrac{a_k}{b_k}$, $n \geq 1$, $C_0 = 0$. The assumption says that $C_n \to C_\infty = \sum_{k \geq 1} \frac{a_k}{b_k} \in \mathbb{R}$. Now, for large enough n $b_n > 0$ and

$$
\begin{aligned}
\frac{1}{b_n} \sum_{k=1}^{n} a_k &= \frac{1}{b_n} \sum_{k=1}^{n} b_k \Delta C_k \\
&= \frac{1}{b_n} \left(b_n C_n - \sum_{k=1}^{n} C_{k-1} \Delta b_k \right) \\
&= C_n - \frac{1}{b_n} \sum_{k=1}^{n} \Delta b_k C_{k-1}
\end{aligned}
$$

where we used Abel Transform for series. One concludes by the extended Césaro Theorem since $\Delta b_n \geq 0$ and $\lim_n b_n = +\infty$. $\diamond$

To establish Central Limit Theorems outside the 'ô.i.d.' setting, we will rely on Lindberg's Central limit theorem for arrays of martingale increments (seeTheorem 3.2 and its corollary 3.1, p.58 in [70]), stated below in a simple form involving only one square integral martingale.

Theorem 11.7 (Lindeberg Central Limit Theorem for martingale increments) *([70]) Let $(M_n)_{n\geq 1}$ be a sequence of square integrable martingales with respect to a filtration $(\mathcal{F}_n)_{n\geq 1}$ and let $(a_n)_{n\geq 1}$ be a non-decreasing sequence of real numbers going to infinity when n goes to infinity.*

If the following conditions hold:

(i) $\dfrac{1}{a_n} \displaystyle\sum_{k=1}^{n} \mathbb{E}\left(|\Delta M_k|^2 \,|\, \mathcal{F}_{n-1}\right) \longrightarrow \sigma^2 \in [0, +\infty)$ *in probability,*

(ii) $\forall\, \varepsilon > 0,\ \dfrac{1}{a_n} \displaystyle\sum_{k=1}^{n} \mathbb{E}\left((\Delta M_k)^2 \mathbf{1}_{\{|\Delta M_k| \geq a_n \varepsilon\}} \,|\, \mathcal{F}_{n-1}\right) \longrightarrow 0$ *in probability,*

then

$$\frac{M_n}{\sqrt{a_n}} \xrightarrow{\ \mathcal{L}\ } \mathcal{N}\left(0; \sigma^2\right).$$

▷ Derive from this result a d-dimensional theorem [Hint: Consider a linear combination of the d-dimensional martingale under consideration.]

Bibliography

[1] M. ABRAMOVICZ, I.A. STEGUN (1964). *Handbook of Mathematical Functions*, National Bureau of Standards, Washington.

[2] A. ALFONSI, B. JOURDAIN B. AND A. KOHATSU-HIGA. Pathwise optimal transport bounds between a one-dimensional diffusion and its Euler scheme, to appear in *The Annals of Applied Probability*, 2013.

[3] I.A. ANTONOV AND V.M. SALEEV (1979). An economic method of computing LP_τ-sequences, *Zh. vȳ chisl. Mat. mat. Fiz.*, **19**, 243-245. English translation: *U.S.S.R. Comput. Maths. Math. Phys.*, **19**, 252-256.

[4] B. AROUNA (2004). Adaptive Monte Carlo method, a variance reduction technique. *Monte Carlo Methods and Appl.*, **10**(1):1-24.

[5] B. AROUNA, O. BARDOU (2004). Efficient variance reduction for functionals of diffusions by relative entropy, CERMICS-ENPC, technical report.

[6] M.D. AŠIĆ, D.D. ADAMOVIĆ (1970): Limit points of sequences in metric spaces, *The American Mathematical Monthly*, **77**(6):613-616.

[7] J. BARRAQUAND, D. MARTINEAU (1995). Numerical valuation of high dimensional multivariate American securities, *Journal of Finance and Quantitative Analysis*, **30**.

[8] V. BALLY (2004). The central limit theorem for a nonlinear algorithm based on quantization, *Proc. of The Royal Soc.*, **460**:221-241.

[9] V. BALLY, D. TALAY (1996). The distribution of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function, *Probab. Theory Related Fields*, **104**(1): 43-60 (1996).

[10] V. BALLY, D. TALAY (1996). The law of the Euler scheme for stochastic differential equations. II. Convergence rate of the density, *Monte Carlo Methods Appl.*, **2**(2):93-128.

[11] V. BALLY, G. PAGÈS AND J. PRINTEMS (2001). A Stochastic quantization method for nonlinear problems, *Monte Carlo Methods and Appl.*, **7**(1):21-34.

[12] V. BALLY, G. PAGÈS (2003). A quantization algorithm for solving discrete time multidimensional optimal stopping problems, *Bernoulli*, **9**(6):1003-1049.

[13] V. BALLY, G. PAGÈS (2003). Error analysis of the quantization algorithm for obstacle problems, *Stochastic Processes & Their Applications*, **106**(1):1-40.

[14] V. BALLY, G. PAGÈS, J. PRINTEMS (2003). First order schemes in the numerical quantization method, *Mathematical Finance* **13**(1):1-16.

[15] V. BALLY, G. PAGÈS, J. PRINTEMS (2005). A quantization tree method for pricing and hedging multidimensional American options, *Mathematical Finance*, **15**(1):119-168.

[16] C. BARRERA-ESTEVE AND F. BERGERET AND C. DOSSAL AND E. GOBET AND A. MEZIOU AND R. MUNOS AND D. REBOUL-SALZE (2006). Numerical methods for the pricing of Swing options: a stochastic control approach, *Methodology and Computing in Applied Probability.*

[17] O. BARDOU, S. BOUTHEMY, G. PAGÈS (2009). Pricing swing options using Optimal Quantization, *Applied Mathematical Finance*, **16**(2):183-217.

[18] O. BARDOU, S. BOUTHEMY, G. PAGÈS (2006). When are Swing options bang-bang?, *International Journal for Theoretical and Applied Finance*, **13**(6):867-899, 2010.

[19] O. BARDOU, N. FRIKHA, G. PAGÈS (2009). Computing VaR and CVaR using Stochastic Approximation and Adaptive Unconstrained Importance Sampling, *Monte Carlo and Applications Journal*, **15**(3):173-210.

[20] M. BENAÏM (1999). Dynamics of Stochastic Approximation Algorithms, *Séminaire de Probabilités XXXIII*, J. Azéma, M. Émery, M. Ledoux, M. Yor éds, Lecture Notes in Mathematics n^0 1709, 1-68.

[21] A. BENVENISTE, M. MÉTIVIER AND P. PRIOURET (1987). *Algorithmes adaptatifs et approximations stochastiques.* Masson, Paris, 1987.

[22] J. BERTOIN, *Lévy processes.* (1996) Cambridge tracts in Mathematics, **121**, Cambridge University Press, 262p.

[23] A. BESKOS, G.O. ROBERTS (2005). Exact simulation of diffusions, *Ann. appl. Prob.*, **15**(4): 2422-2444.

[24] P. BILLINGSLEY (1968). *Convergence of probability measure*, Wiley.

[25] J.-P. BOREL, G. PAGÈS, Y.J. XIAO (1992). Suites à discrépance faible and intégration numérique, *Probabilités Numériques*, N. Bouleau, D. Talay eds., Coll. didactique, INRIA, ISBN-10: 2726107087.

[26] B. BOUCHARD, I. EKELAND, N. TOUZI (2004) On the Malliavin approach to Monte Carlo approximation of conditional expectations. (English summary) *Finance Stoch.*, **8** (1):45-71.

[27] N. BOULEAU, D. LÉPINGLE (1994). *Numerical methods for stochastic processes*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 359 pp. ISBN: 0-471-54641-0.

[28] P. P. BOYLE (1977). Options: A Monte Carlo approach, *Journal of Financial Economics*, **4**(3):323-338

[29] O. BRANDIÈRE, M. DUFLO (1996). Les algorithmes stochastiques contournent-ils les pièges? (French) [Do stochastic algorithms go around traps?], *Ann. Inst. H. Poincaré Probab. Statist.*, **32**(3):395-427.

[30] M. BRIANE, G. PAGÈS, *Théorie de l'intégration*, Cours & exercices, Vuibert, 1999.

[31] BUCHE R., KUSHNER H.J. (2002). Rate of convergence for constrained stochastic approximation algorithm, *SIAM J. Control Optim.*, **40**(4):1001-1041.

[32] J.F. CARRIÈRE (1996) Valuation of the early-exercise price for options using simulations and nonparametric regression. *Insurance Math. Econ.*, **19**:1930.

[33] K.L. CHUNG (1949) An estimate concerning the Kolmogorov limit distribution, *Trans. Amer. Math. Soc.*, **67**, 36-50.

[34] E. CLÉMENT, A. KOHATSU-HIGA, D. LAMBERTON (2006). A duality approach for the weak approximation of Stochastic Differential Equations, *Annals of Applied Probability*, **16**(3):449-471.

[35] É. CLÉMENT, P. PROTTER, D. LAMBERTON (2002). An analysis of a least squares regression method for American option pricing, *Finance & Stochastics*, **6**(2):449-47.

[36] P. COHORT (2000), *Sur quelques problèmes de quantification*, thèse de l'Univsersité Paris 6, Paris, 187p,

[37] P. COHORT (2004). Limit theorems for random normalized distortion, *Ann. Appl. Probab.*, **14**(1):118-143.

[38] S. CORLAY, G. PAGÈS (2014). Functional quantization based stratified sampling methods, pre-print LPMA, 1341, to appear in *Monte Carlo and Applications Journal*.

[39] R. CRANLEY, T.N.L. PATTERSON (1976). Randomization of number theoretic methods for multiple integration, *SIAM J. Numer. Anal.*, **13**(6):904-914.

[40] D. DACUNHA-CASTEL, M. DUFLO (1986). *Probabilités, Problèmes à temps mobile*, Masson, Paris.

[41] R.B. DAVIS , D.S. HARTE (1987) Tests for Hurst effect, *Biometrika*, **74**:95-101.

[42] L. DEVROYE (1986). *Non uniform random variate generation*, Springer, New York.

[43] M. DUFLO (1996) (1997). *Algorithmes stochastiques*, coll. Mathématiques & Applications, **23**, Springer-Verlag, Berlin, 319p.

[44] M. DUFLO, *Random iterative models*, translated from the 1990 French original by Stephen S. Wilson and revised by the author. Applications of Mathematics (New York), **34** Springer-Verlag, Berlin, 385 pp.

[45] P. ÉTORÉ, B. JOURDAIN (2010). Adaptive optimal allocation in stratified sampling methods, *Methodol. Comput. Appl. Probab.*, **12**(3):335-360.

[46] H. FAURE (1986). Suite à discrépance faible dans T^s, Technical report, Univ. Limoges (France).

[47] H. FAURE (1982). Discrépance associée à un système de numération (en dimension s), *Acta Arithmetica*, **41**:337-361.

[48] O. FAURE (1992). *Simulation du mouvement brownien and des diffusions*, Thèse de l'ENPC (Marne-la-Vallée, France), 133p.

[49] H. FÖLLMER, A. SCHIED (2002). *Stochastic Finance, An introduction in discrete time*, De Gruyter Studies in Mathematics **27**, De Gruyter, Berlin, 422p, second edition in 2004.

[50] J.C FORT, G. PAGÈS (1996). Convergence of stochastic algorithms: from the Kushner & Clark Theorem to the Lyapunov functional, *Advances in Applied Probability*, **28**:1072-1094.

[51] J.C FORT, G. PAGÈS (2002). Decreasing step stochastic algorithms: *a.s.* behaviour of weighted empirical measures, *Monte Carlo Methods & Applications*, **8**(3):237-270.

[52] A. FRIEDMAN (1975). *Stochastic differential equations and applications. Vol. 1-2. Probability and Mathematical Statistics*, Vol. 28. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 528p.

[53] J. H. FRIEDMAN, J.L. BENTLEY AND R.A. FINKEL (1977). An Algorithm for Finding Best Matches in Logarithmic Expected Time, *ACM Transactions on Mathematical Software*, **3**(3):209-226.

[54] N. FRIKHA, S. MENOZZI (2012). Concentration bounds for Stochastic Approximations, *Electron. Cummun. Proba.*, **183 17**(47):1-15.

[55] A. GERSHO AND R.M. GRAY (1988). Special issue on Quantization, **I-II** (A. Gersho and R.M. Gray eds.), *IEEE Trans. Inform. Theory*, **28**.

[56] A. GERSHO AND R.M. GRAY (1992). *Vector Quantization and Signal Compression*. Kluwer, Boston.

[57] P. GLASSERMAN (2003). Monte Carlo Methods in Financial Engineering, Springer-Verlag, New York, 596p.

[58] P. GLASSERMAN, P. HEIDELBERGER, P. SHAHABUDDIN (1999). Asymptotically optimal importance sampling and stratification for pricing path-dependent options. *Math. Finance* **9**(2):117-152.

[59] P. GLASSERMAN, D.D. YAO (1992). Some guidelines and guarantees for common random numbers, *Manag. Sci.*, **36**(6):884-908.

[60] P. GLYNN (1989). Optimization of stochastic systems via simulation. In *Proceedings of the 1989 Winter simulation Conference*, Society for Computer Simulation, San Diego, 90-105.

[61] E. GOBET, G. PAGÈS, H. PHAM, J. PRINTEMS (2007). Discretization and simulation for a class of SPDE's with applications to Zakai and McKean-Vlasov equations, *SIAM Journal on Numerical Analysis*, **44**(6):2505-2538.

[62] E. GOBET, S. MENOZZI (2004): Exact approximation rate of killed hypo-elliptic diffusions using the discrete Euler scheme, *Stochastic Process. Appl.*, **112**(2):201-223.

[63] E. GOBET (2000): Weak approximation of killed diffusion using Euler schemes, *Stoch. Proc. and their Appl.*, **87**:167-197.

[64] E. GOBET, A. KOHATSU-HIGA (2003). Computation of Greeks for barrier and look-back options using Malliavin calculus, *Electron. Comm. Probab.*, **8**:51-62.

[65] E. GOBET, R. MUNOS (2005). Sensitivity analysis using Itô-Malliavin calculus and martingales. Application to optimal control, *SIAM J. Control Optim.*, **43**(5):1676-1713.

[66] S. GRAF, H. LUSCHGY (2000). *Foundations of quantization for probability distributions*, LNM 1730, Springer, Berlin, 230p.

[67] S. GRAF, H. LUSCHGY, AND G. PAGÈS (2008). Distortion mismatch in the quantization of probability measures, *ESAIM P&S*, **12**:127-154.

[68] A. GRANGÉ-CABANE (2011), *Algorithmes stochastiques, applications à la finance*, Master thesis (supervisIon G. Pagès), 49p.

[69] J. GUYON (2006). Euler scheme and tempered distributions, *Stochastic Processes and their Applications*, **116**(6), 877-904.

[70] P. HALL, C.C. HEYDE (1980). *Martingale Limit Theory and Its Applications*, Academic Press, 308p.

[71] U.G. HAUSSMANN (1979). On the Integral Representation of Functionals of Itô Processes, *???*, **3**:17-27.

[72] S.L. HESTON (1993). A closed-form solution for options with stochastic volatility with applications to bon and currency options, *The review of Financial Studies*, **6**(2):327-343.

[73] J. JACOD, A.N. SHIRYAEV (2003). *Limit theorems for stochastic processes.* Second edition (first edition 1987). Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **288**. Springer-Verlag, Berlin, 2003. xx+661 pp.

[74] J. JACOD, P. PROTTER (1998). Asymptotic error distributions for the Euler method for stochastic differential equations. *Annals of Probability*, **26**(1):267-307.

[75] J. JACOD, P. PROTTER *Probability essentials.* Second edition. Universitext. Springer-Verlag, Berlin, 2003. x+254 pp.

[76] P. JAILLET, D. LAMBERTON, AND B. LAPEYRE (1990). Variational inequalities and the pricing of American options, *Acta Appl. Math.*, **21**:263-289.

[77] H. JOHNSON (1987). Call on the maximum or minimum of several assets, *J. of Financial and Quantitative Analysis*, **22**(3):277-283.

[78] I. KARATZAS, S.E. SHREVE (1998). *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, Springer-Verlag, New York, 470p.

[79] O. KAVIAN (1993). Introduction à la théorie des points critiques et applications aux problèmes elliptiques. (French) [Introduction to critical point theory and applications to elliptic problems]. coll. *Mathématiques & Applications* [Mathematics & Applications], **13**, Springer-Verlag, Paris [Berlin], 325 pp. ISBN: 2-287-00410-6.

[80] A.G. KEMNA, A.C. VORST (1990). A pricing method for options based on average asset value, *J. Banking Finan.*, **14**:113-129.

[81] J.C. KIEFFER (1982) Exponential rate of convergence for Lloyd's method I, *IEEE Trans. on Inform. Theory*, Special issue on quantization, **28**(2):205-210.

[82] J. KIEFER, J. WOLFOWITZ (1952). Stochastic estimation of the maximum of a regression function, *Ann. Math. Stat.*, **23**:462-466.

[83] P.E. KLOEDEN, E. PLATTEN (1992). *Numerical solution of stochastic differential equations. Applications of Mathematics*, **23**, Springer-Verlag, Berlin (New York), 632 pp. ISBN: 3-540-54062-8.

[84] A. KOHATSU-HIGA, R. PETTERSSON (2002). Variance reduction methods for simulation of densities on Wiener space, *SIAM J. Numer. Anal.*, **40**(2):431-450.

[85] U. KRENGEL (1987). *Ergodic Theorems*, De Gruyter Studies in Mathematics, **6**, Berlin, 357p.

[86] L. KUIPERS, H. NIEDERREITER (1974). *Uniform distribution of sequences*, Wiley.

[87] H. KUNITA (1982). Stochastic differential equations and stochastic flows of diffeomorphisms, *Cours d'école d'été de Saint-Flour*, LN-1097, Springer-Verlag, 1982.

[88] T.G. KURTZ, P. PROTTER (1991). Weak limit theorems for stochastic integrals and stochastic differential equations, *Annals of Probability*, **19**:1035-1070.

[89] H.J. KUSHNER, G.G. YIN (2003). Stochastic approximation and recursive algorithms and applications, 2nd edition, *Applications of Mathematics, Stochastic Modelling and Applied Probability*, **35**, Springer-Verlag, New York.

[90] J.P. LAMBERT (1985). Quasi-Monte Carlo, low-discrepancy, and ergodic transformations, *J. Computational and Applied Mathematics*, **13/13**, 419-423.

[91] D. LAMBERTON, B. LAPEYRE (1996). *Introduction to stochastic calculus applied to finance.* Chapman & Hall, London, 185 pp. ISBN: 0-412-71800-6.

[92] B. LAPEYRE, G. PAGÈS (1989). Familles de suites à discrépance faible obtenues par itération d'une transformation de $[0,1]$, *Comptes rendus de l'Académie des Sciences de Paris*, Série I, **308**:507-509.

[93] B. LAPEYRE, G. PAGÈS, K. SAB (1990). Sequences with low discrepancy. Generalization and application to Robbins-Monro algorithm, *Statistics*, **21**(2):251-272.

[94] S. LARUELLE, C.-A. LEHALLE, G. PAGÈS (2009). Optimal split of orders across liquidity pools: a stochastic algorithm approach, pre-pub LPMA 1316, to appear in *SIAM J. on Financial Mathematics*.

[95] S. LARUELLE, G. PAGÈS (2012). Stochastic Approximation with averaging innovation, *Monte Carlo and Appl. J.*. **18**:1-51.

[96] V.A. LAZAREV (1992). Convergence of stochastic approximation procedures in the case of a regression equation with several roots, (transl. from) *Problemy Pederachi Informatsii*, **28**(1): 66-78.

[97] B. LAPEYRE, E. TEMAM (2001). Competitive Monte Carlo methods for the pricing of Asian Options, *J. of Computational Finance*, **5**(1):39-59.

[98] M. LEDOUX, M. TALAGRAND (2011). *Probability in Banach spaces. Isoperimetry and processes.* Reprint of the 1991 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2011. xii+480 pp.

[99] P. L'ECUYER, G. PERRON (1994). On the convergence rates of ipa and fdc derivatives estimators, *Oper. Res.*, **42**:643-656.

[100] M. LEDOUX (1997) Concentration of measure and logarithmic Sobolev inequalities, technical report, http://www.math.duke.edu/ rtd/CPSS2007/Berlin.pdf.

[101] J. LELONG (2007). *Algorithmes Stochastiques and Options parisiennes*, Thèse de l'École Nationale des Ponts and Chaussées.

[102] V. LEMAIRE, G. PAGÈS (2010). Unconstrained recursive importance sampling, *Annals of Applied Probability*, **20**(3):1029-1067.

[103] LJUNG L. (1977). Analysis of recursive stochastic algorithms, *IEEE Trans. on Automatic Control*, AC-**22**(4):551-575.

[104] F.A. LONGSTAFF, E.S. SCHWARZ (2001). Valuing American options by simulation: a simple least-squares approach, *Review of Financial Studies*, **14**:113-148.

[105] H. LUSCHGY (2013). *Martingale die discrete Ziet, Theory und Anwendungen.* Berlin. Springer, 452p.

[106] H. LUSCHGY, G. PAGÈS (2001). Functional quantization of Gaussian processes, *J. of Funct. Analysis*, **196**(2), 486-531.

[107] H. LUSCHGY, G. PAGÈS (2008). Functional quantization rate and mean regularity of processes with an application to Lévy processes, *Annals of Applied Probability*, **18**(2):427-469.

[108] MAC NAMES (2001). A fast nearest-neighbour algorithm based on principal axis search tree. *IEEE T. Pattern. Anal.*, **23**(9):964-976.

[109] S. MANASTER, G. KOEHLER (1982). The calculation of Implied Variance from the Black-Scholes Model: A Note, *The Journal of Finance*, **37**(1):227-230.

[110] G. MARSAGLIA, T.A. BRAY (1964) A convenient method for generating normal variables. *SIAM Rev.*, **6** , 260-264.

[111] M. MATSUMATA, T. NISHIMURA (1998). Mersenne twister: A 623-dimensionally equidistributed uniform pseudorandom number generator, *ACM Trans. in Modeling and Computer Simulations*, **8**(1):3-30.

[112] G. MARSAGLIA, W.W.TSANG (2000). The Ziggurat Method for Generating Random Variables, *Journal of Statistical Software*, **5**(8):363-372.

[113] MILSTEIN G.N., (1976). A method of second order accuracy for stochastic differential equations, *Theor. Probab. Its Apll.* (USSR), **23**:396-401.

[114] J. NEVEU, (1964). *Bases mathématiques du calcul des Probabilités*, Masson, Paris; English translation *Mathematical Foundations of the Calculus of Probability* (1965), Holden Day, San Francisco.

[115] J. NEVEU (1972). *Martingales à temps discret*, Masson, 1972, 218p. English translation: *Discrete-parameter martingales*, North-Holland, New York, 1975, 236p.

[116] D.J. NEWMAN (1982). The Hexagon Theorem, *IEEE Trans. Inform. Theory* (A. Gersho and R.M. Gray eds.), **28**:137-138.

[117] H. NIEDERREITER (1992) *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF regional conference series in Applied mathematics, SIAM, Philadelphia.

[118] NUMERICAL RECIPES: Textbook available at

`www.ma.utexas.edu/documentation/nr/bookcpdf.html`.

See also [139].

[119] G. PAGÈS (1987). *Sur quelques problèmes de convergence*, thèse Univ. Paris 6, Paris.

[120] G. PAGÈS (1992). Van der Corput sequences, Kakutani transform and one-dimensional numerical integration, *J. Computational and Applied Mathematics*, **44**:21-39.

[121] G. PAGÈS (1998). A space vector quantization method for numerical integration, *J. Computational and Applied Mathematics*, **89**, 1-38 (Extended version of "Voronoi Tessellation, space quantization algorithms and numerical integration", in: M. Verleysen (Ed.), Proceedings of the ESANN' 93, Bruxelles, Quorum Editions, (1993), 221-228).

[122] G. PAGÈS (2007). Multistep Richardson-Romberg extrapolation: controlling variance and complexity, *Monte Carlo Methods and Applications*, **13**(1):37-70.

[123] G. PAGÈS (2007). Quadratic optimal functional quantization methods and numerical applications, *Proceedings of MCQMC, Ulm'06*, Springer, Berlin, 101-142.

[124] G. PAGÈS (2013). Functional co-monotony of processes with an application to peacocks and barrier options, *Sémiaire de Probabilités XXVI*, LNM 2078, Springer, Cham, 365:400.

[125] G. PAGÈS (2014). Introduction to optimal quantization for numerics, to appear in *ESAIM Proc. & Surveys*, Paris, France.

[126] G. Pagès, H. Pham, J. Printems (2004). Optimal quantization methods and applications to numerical problems in finance, *Handbook on Numerical Methods in Finance* (S. Rachev, ed.), Birkhauser, Boston, 253-298.

[127] G. Pagès, H. Pham, J. Printems (2004). An Optimal Markovian Quantization Algorithm for Multidimensional Stochastic Control Problems, Stochastics and Dynamics, **4**(4):501-545.

[128] G. Pagès, J. Printems (2003). Optimal quadratic quantization for numerics: the Gaussian case, *Monte Carlo Methods and Appl.* **9**(2):135-165.

[129] G. Pagès, J. Printems (2005). `www.quantize.maths-fi.com`, web site devoted to optimal quantization.

[130] G. Pagès, J. Printems (2009). Optimal quantization for finance: from random vectors to stochastic processes, chapter from *Mathematical Modeling and Numerical Methods in Finance* (special volume) (A. Bensoussan, Q. Zhang guest éds.), coll. *Handbook of Numerical Analysis* (P.G. Ciarlet Editor), North Holland, 595-649.

[131] G. Pagès, B. Wilbertz (2012). Optimal Delaunay et Voronoi quantization methods for pricing American options, pré-pub PMA, 2011, *Numerical methods in Finance* (R. Carmona, P. Hu, P. Del Moral, .N. Oudjane éds.), Springer, New York, 171-217.

[132] G. Pagès, Y.J. Xiao (1988) Sequences with low discrepancy and pseudo-random numbers: theoretical results and numerical tests, *J. of Statist. Comput. Simul.*, **56**:163-188.

[133] K.R. Parthasarathy (1967). *Probability measures on metric spaces.* Probability and Mathematical Statistics, No. 3 Academic Press, Inc., New York-London, xi+276 pp.

[134] M. Pelletier (1998). Weak convergence rates for stochastic approximation with application to multiple targets and simulated annealing, *Ann. Appl. Probab.*, **8**(1):10-44.

[135] R. Pemantle (1990), Nonconvergence to unstable points in urn models and stochastic approximations, *Annals of Probability*, **18**(2):698-712.

[136] H. Pham (2007). Some applications and methods of large deviations in finance and insurance in *Paris-Princeton Lectures on Mathematical Finance 2004*, LNM 1919, New York, Springer, 191-244.

[137] B.T. Polyak (1990) A new method of stochastic approximation type. (Russian) Avtomat. i Telemekh. **7**: 98-107; translation in *Automat. Remote Control*, **51 7**, part 2, 937946 (1991).

[138] P.D. Proïnov (1988). Discrepancy and integration of continuous functions, *J. of Approx. Theory*, **52**:121-131.

[139] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery (2002). *Numerical recipes in C++. The art of scientific computing,* Second edition, updated for C++. Cambridge University Press, Cambridge, 2002. 1002p. ISBN: 0-521-75033-4 65-04.

[140] D. Revuz, M. Yor (1999). *Continuous martingales and Brownian motion.* Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **293**, Springer-Verlag, Berlin, 602 p. ISBN: 3-540-64325-7 60G44 (60G07 60H05).

[141] H. ROBBINS, S. MONRO (1951). A stochastic approximation method, *Ann. Math. Stat.*, **22**:400-407.

[142] R.T. ROCKAFELLAR, S. URYASEV (1999). Optimization of Conditional Value at Risk, pre-print, Univ. Florida, `http://www.ise.ufl.edu/uryasev`.

[143] K.F. ROTH (1954). On irregularities of distributions, *Mathematika*, **1**:73-79.

[144] W. RUDIN (1966). *Real and complex analysis*, McGraw-Hill, New York.

[145] W. RUPPERT ...

[146] K.-I. SATO (1999). *Lévy processes and Infinitely Divisible Distributions*, Cambridge University Press.

[147] A. N. SHYRIAEV (2008). Optimal stopping rules. Translated from the 1976 Russian second edition by A. B. Aries. Reprint of the 1978 translation. *Stochastic Modelling and Applied Probability*, **8**. Springer-Verlag, Berlin, 2008. 217 pp.

[148] A. N. SHYRIAEV (1984). *Probability*, Graduate Tetxts in mathematics, Springer, New York, 577p.

[149] P. SEUMEN-TONOU (1997). *Méthodes numériques probabilistes pour la résolution d'équations du transport and pour l'évaluation d'options exotiques*, Theèse de l'Université de Provence (Marseille, France), 116p.

[150] M. SUSSMAN, W. CRUTCHFIELD, M. PAPAKIPOS (2006). Pseudo-random numbers Generation on GPU, in *Graphic Hardware 2006*, Proceeding of the Eurographics Symposium Vienna, Austria, September 3-4, 2006, M. Olano & P. Slusallek (eds.), A K Peters Ltd., 86-94, 424p., ISBN 1568813546.

[151] D. TALAY, L. TUBARO (1990). Expansion of the global error for numerical schemes solving stochastic differential equations, *Stoch. Anal. Appl.*, **8**:94-120.

[152] B. TUFFIN (2004). Randomization of Quasi-Monte Carlo Methods for error estimation: Survey and Normal Approximation, *Monte Carlo Methods and Applications*, **10**, (3-4):617-628.

[153] S. VILLENEUVE, A. ZANETTE (2002). Parabolic A.D.I. methods for pricing american option on two stocks, *Mathematics of Operation Research*, **27**(1):121-149.

[154] D. S. WATKINS (2010) *Fundamentals of matrix computations* (3rd edition), *Pure and Applied Mathematics (Hoboken)*, John Wiley & Sons, Inc., Hoboken, NJ, xvi+644 pp.

[155] M. WINIARSKI (2006). Quasi-Monte Carlo Derivative valuation and Reduction of simulation bias, Master thesis, Royal Institute of Technology (KTH), Stockholm (Sweden).

[156] A. WOOD, G. CHAN (1994). Simulation of stationary Gaussian processes in $[0, 1]^d$, *Journal ofcomputational and graphical statistics*, **3**(4):409-432.

[157] Y.J. XIAO (1990). *Contributions aux méthodes arithmétiques pour la simulation accélérée*, Thèse de l'ENPC, Paris.

[158] Y.J. XIAO (1990). Suites équiréparties associées aux automorphismes du tore, *C.R. Acad. Sci. Paris* (Série I), **317**:579-582.

[159] P.L. ZADOR (1963). Development and evaluation of procedures for quantizing multivariate distributions. Ph.D. dissertation, Stanford Univ.

[160] P.L. ZADOR (1982). Asymptotic quantization error of continuous signals and the quantization dimension, *IEEE Trans. Inform. Theory*, **28**(2):139-14.