universität freiburg

Probability Theory **Summer semester 2024**

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https://pfaffelh.github.io/hp/2024ss_wtheorie.html
https://www.stochastik.uni-freiburg.de/

Tutorial 5 - Borel-Cantelli-Lemma and Kolmogorov's 0-1-Law

Exercise 1 (4 Points).
Let $X_1, X_2, \ldots$ be $\mathbb{Z}_+$-valued, independent, and identically distributed. Show that

$$|\{n : X_n \geq n| < \infty \text{ almost surely} \iff \mathbf{E}[X_1] < \infty$$

Exercise 2 (2+2 Points +2 bonus points).
Give one example and one counterexample of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ and a set $A \in \mathcal{B}(\mathbb{R})$ for the following identities:

(a) $\mathbf{P}(\{\limsup\limits_{n\to\infty} X_n \in A\}) = \mathbf{P}(\limsup\limits_{n\to\infty}\{X_n \in A\})$.

(b) $\mathbf{P}(\limsup\limits_{n\to\infty}\{X_n \in A\}) = \limsup\limits_{n\to\infty} \mathbf{P}(\{X_n \in A\})$.

(c) $\limsup\limits_{n\to\infty} \mathbf{P}(\{X_n \in A\}) = \mathbf{P}(\{\limsup\limits_{n\to\infty} X_n \in A\})$.

Exercise 3 (2+2 Points).
Let I by arbitrary and $p_i \in [0,1], i \in I$. In addition, let $X_i \sim B(1,p_i)$ for $i \in I$.

(a) Show that

$$(X_i)_{i=1,\ldots,n} \text{ independent} \iff \mathbf{P}(X_i = 1, i \in J) = \prod_{i\in J} \mathbf{P}(X_i = 1) \text{ for all } J \subseteq_f I.$$

(b) Show that

$$(X_i)_{i=1,\ldots,n} \text{ pairwise uncorrelated} \iff (X_i)_{i=1,\ldots,n} \text{ independent.}$$

Exercise 4 (2+2 Points).
On a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ let there be a sequence of independent random variables $(X_n)_{n\in\mathbb{N}}$ which are exponentially distributed identically to the parameter $\alpha > 0$. Show that

(a) $\mathbf{P}(\limsup_{n\to\infty} \frac{X_n}{\ln n} = \frac{1}{\alpha}) = 1$,

(b) $\mathbf{P}(\liminf_{n\to\infty} \frac{X_n}{\ln n} = 0) = 1$.

Hint: Use the Borel-Cantelli lemma. It states that $\limsup_{n\to\infty} \frac{X_n}{\ln n} \leq \frac{1}{\alpha}$ *if and only if for all* $\varepsilon > 0$ *only at most finitely many of the events* $\{\frac{X_n}{\ln n} \geq \frac{1}{\alpha} + \varepsilon\}$ *occur.*