Stochastic Processes
11. The CLT for martingales

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The CLT for Martingales

- Theorem 14.42: $I^n = \{0, 1, 2, ..., t_n\}$, $\mathcal{M}^n = (M_t^n)_{t \in I^n}$ martingale with $M_0^n = 0$. For $X_t^n := M_t^n - M_{t-1}^n$, assume

$$\mathbf{E}[\max_{1 \le s \le t_n} |X_s^n|] \xrightarrow{n\to\infty} 0, \qquad \sum_{s=1}^{t_n} (X_s^n)^2 \xrightarrow{n\to\infty}_p \sigma^2 > 0$$

  Then $M_{t_n}^n \xRightarrow{n\to\infty} X$ with $X \sim N(0, \sigma^2)$.

- Example: $X_1, X_2, ...$ iid with $\mathbf{E}[X_1] = 0$, $\mathbf{V}[X_1] = \sigma^2$ and

$$M_t^n = \frac{1}{\sqrt{n}} \sum_{s=1}^{t} X_t \xRightarrow{n\to\infty} X \sim N(0, \sigma^2).$$

  Then $\int_0^\infty t\mathbf{P}(|X_1| > t)dt < \infty$, $t^2\mathbf{P}(|X_1| > t) \xrightarrow{t\to\infty} 0$,

$$\mathbf{E}[\max_{1 \le s \le n} |X_s|/\sqrt{n}] = \int_0^\infty \mathbf{P}(\max_{1 \le s \le n} |X_s| > t\sqrt{n})dt$$

$$= \int_0^\infty 1 - \Big(1 - \frac{t^2 n \mathbf{P}(|X_1| > t\sqrt{n})}{t^2 n}\Big)^n dt \xrightarrow{n\to\infty} 0.$$

Convergence of Products

- Lemma 14.43: $U_1, U_2, \ldots, T_1, T_2, \ldots$ rvs with
  1. $U_n \xrightarrow{n\to\infty}_p u$,
  2. $(T_n)_{n=1,2,\ldots}$ and $(T_n U_n)_{n=1,2,\ldots}$ ui,
  3. $\mathbf{E}[T_n] \xrightarrow{n\to\infty} 1$,

  Then $\mathbf{E}[T_n U_n] \xrightarrow{n\to\infty} u$.

  Proof: we show $T_n(U_n - u) \xrightarrow{n\to\infty}_p 0$

  (then $\mathbf{E}[T_n(U_n - u)] \xrightarrow{n\to\infty} 0$ due to ui)

  For $\varepsilon > 0$, let K be such that $\sup_n \mathbf{P}(|T_n| > K) \leq \varepsilon$.

  Now $xy > \delta\varepsilon \Rightarrow x > \delta$ or $y > \varepsilon$, therefore

$$\limsup_{n\to\infty} \mathbf{P}(|T_n(U_n - u)| > \varepsilon)$$

$$\leq \limsup_{n\to\infty} \mathbf{P}(|U_n - u| > \varepsilon/K) + \mathbf{P}(|T_n| > K) \leq \varepsilon.$$

An estimate for the exponential function

- Lemma 14.44: There is $C > 0$ and a function r with $|r(x) \leq C|x^3|$, such that

$$exp(ix) = (1 + ix)\exp(-x^2/2 + r(x))$$

for all $x \in \mathbb{R}$. In addition, $|1 + ix| \leq e^{x^2/2}$ for all $x \in \mathbb{R}$.

Proof: For small $|x|$ we write

$$\Big|\exp(ix) - (1 + ix)\exp(-x^2/2)\Big|$$

$$= \Big|1 + ix - x^2/2 - (1 - ix)(1 - x^2/2)\Big| + O(|x|^3) = O(|x|^3).$$

Proof of Theorem 13.43

- Wlog, $\sum_{r=1}^{s}(X_r^n)^2 \leq 2\sigma^2$, $s = 1, 2, \ldots$ For $\lambda > 0$ we show

$$\mathbf{E}[e^{i\lambda M_{t_n}^n}] = \underbrace{\prod_{s=1}^{t_n}(1 + i\lambda X_s^n)}_{=:T_n} \cdot \underbrace{e^{-\frac{\lambda^2}{2}\sum_{s=1}^{t_n}(X_s^n)^2 + \sum_{s=1}^{t_n} r(\lambda X_s^n)}}_{=:U_n \xrightarrow{n\to\infty} u:=e^{-\lambda^2\sigma^2/2}} \xrightarrow{n\to\infty} e^{-i\lambda^2\sigma^2/2}$$

We use lemma 14.43:

1. is clear; for 2. $T_n U_n = e^{i\lambda M_{t_n}^n}$ uniformly integrable. Further

$$|T_n| = \prod_{s=1}^{t_n-1} |1 + i\lambda X_s^n| \cdot |1 + i\lambda X_{t_n}^n| \leq \exp\Big(\frac{\lambda^2}{2}\sum_{s=1}^{t_n-1}(X_s^n)^2\Big)(1 + \lambda|X_{t_n}^n|)$$

$$\leq \exp(\lambda^2\sigma^2) \cdot (1 + |\lambda| \cdot \max_{1\leq s\leq t_n} |X_s^n|) \xrightarrow{n\to\infty}_{L^1} 0,$$

so that (T_n) is ui; for 3.,

$$\mathbf{E}[T_n] = \mathbf{E}\Big[\prod_{s=1}^{t_n-1}(1 + i\lambda X_s^n) \cdot \mathbf{E}[(1 + i\lambda X_{t_n}^n)|\mathcal{F}_{t_n-1}]\Big] = \cdots = 1.$$

Example

- $Y_1, Y_2, \dots$ iid, bounded, $\mathbf{E}[Y_1] = 0, \mathbf{V}[Y_1] = 1$,

$$H_s = \frac{1}{s-1}(Y_1^2 + \dots + Y_{s-1}^2), \qquad M_t^n = \frac{1}{\sqrt{n}} \sum_{s=1}^{t} Y_s.$$

Then

$$(\mathcal{H} \cdot \mathcal{M}^n)_t = \sum_{s=1}^{t} \underbrace{\frac{1}{\sqrt{n}} Y_s \frac{1}{s-1} \sum_{r=1}^{s-1} Y_r^2}_{=:X_s^n}$$

is a martingale. Now $\mathbf{E}[\max_{1\leq s\leq t_n} |X_s^n|] \xrightarrow{n\to\infty} 0$ because of the boundedness of $Y_1, Y_2, \dots$ and

$$\sum_{s=1}^{n} (X_s^n)^2 = \frac{1}{n} \sum_{s=1}^{n} Y_s^2 \Big(\frac{1}{s-1} \sum_{r=1}^{s-1} Y_r^2 \Big)^2 \xrightarrow{n\to\infty} 1,$$

from which $(\mathcal{H} \cdot \mathcal{M}^n)_n \xRightarrow{n\to\infty} X \sim N(0,1)$ follows.