Stochastic Processes
15. Distributions of Markov processes

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FDDs for a Markov process

- Markov kernels $\mu_{s,t}$ and transition operator $T_{s,t}$ of $\mathcal{X}$ are

$$\mu_{s,t}(X_s, B) := \mathbf{P}(X_t \in B | X_s) = \mathbf{P}(X_t \in B | \mathcal{F}_s),$$

$$T_{s,t} f(x) := \mathbf{E}[f(X_t) | X_s = x] = \int \mu_{s,t}(x, dy) f(y).$$

$$(\mu \otimes \nu)(x, A \times B) = \int \mu(x, dy)\nu(y, dz) 1_{y \in A, z \in B},$$

$$(\mu\nu)(x, A) = (\mu \otimes \nu)(x, E \times A).$$

FDDs for a Markov process

- Lemma 15.15: $\mathcal{X} = (X_t)_{t \in I}$ Markov with $X_t \sim \nu_t$ for distributions ν_t on E and Markov kernels $(\mu_{s,t})_{s \leq t}$. Then, for $t_0 < \cdots < t_n$

$$(X_{t_0}, ...X_{t_n}) \sim \nu_{t_0} \otimes \mu_{t_0,t_1} \otimes \cdots \otimes \mu_{t_{n-1},t_n}$$

  and

$$\mathbf{P}((X_{t_1}, ..., X_{t_n}) \in \cdot | \mathcal{F}_{t_0}) = (\mu_{t_0,t_1} \otimes \cdots \otimes \mu_{t_{n-1},t_n})(X_{t_0}, \cdot)$$

Chapman-Kolmogorov equations

- $\mathcal{X}$ Markov with $X_t \sim \nu_t$ for distributions ν_t on E, Markov kernels $(\mu_{s,t})_{s\le t}$ and transition operators $(T_{s,t})_{s\le t}$. Then,

$$\mu_{s,t}\mu_{t,u} = \mu_{s,u}, \qquad s \le t \le u$$

$$(T_{s,t}(T_{t,u}f))(X_s) = (T_{s,u}f)(X_s), \qquad f \in \mathcal{B}(E).$$

- Proof: For ν_s-almost all X_s for $A \in \mathcal{B}(E)$ and for $f \in \mathcal{B}(E)$

$$\begin{aligned}\mu_{s,u}(X_s, A) &= \mathbf{P}(X_u \in A|\mathcal{F}_s) = \mathbf{P}((X_t, X_u) \in E \times A|\mathcal{F}_s)\\ &= (\mu_{s,t} \otimes \mu_{t,u})(X_s, E \times A) = (\mu_{s,t}\mu_{t,u})(X_s, A),\\ (T_{s,u}f)(X_s) &= \mathbf{E}[f(X_u)|\mathcal{F}_s] = \mathbf{E}\big[\mathbf{E}[f(X_u)|\mathcal{F}_t]|\mathcal{F}_s\big]\\ &= \mathbf{E}[(T_{t,u}f)(X_t)|\mathcal{F}_s] = (T_{s,t}(T_{t,u}f))(X_s).\end{aligned}$$

Existence of Markov processes

- I index set with $\min I = 0$, $\nu_0 \in \mathcal{P}(I)$. Assume $(\mu_{s,t})_{s\le t}$ is a family of Markov kernels with $\mu_{s,t}\mu_{t,u} = \mu_{s,u}$ ($(T_{s,t})_{s\le t}$ is a family of transition operatorswith with $T_{s,t}T_{t,u} = T_{s,u}$) for all $s \le t \le u$. Then there is a Markov process with ν_0 and kernels $(\mu_{s,t})_{s\le t}$ (and operators $(T_{s,t})_{s\le t}$).
- Proof: It is sufficient to show the first assertion since

  $$(T_{s,t}f)(x) := \int \mu_{s,t}(x,dy)f(y), \qquad \mu_{s,t}(x,A) \quad = (T_{s,t}1_A)(x).$$

  The family $(\nu_{t_1,\dots,t_n})_{\{t_1,\dots,t_n\}\subseteq_f I}$ given by

  $$\nu_{t_1,\dots,t_n} = \nu_0\mu_{0,t_1} \otimes \mu_{t_1,t_2} \otimes \cdots \otimes \mu_{t_{n-1},t_n}.$$

  is projective.

Existence of Markov processes

- I index set with $\min I = 0$, $\nu_0 \in \mathcal{P}(I)$. Assume $(\mu_{s,t})_{s \le t}$ is a family of Markov kernels with $\mu_{s,t}\mu_{t,u} = \mu_{s,u}$ ($(T_{s,t})_{s\le t}$ is a family of transition operatorswith with $T_{s,t}T_{t,u} = T_{s,u}$) for all $s \le t \le u$. Then there is a Markov process with ν_0 and kernels $(\mu_{s,t})_{s\le t}$ (and operators $(T_{s,t})_{s\le t}$).
- To show: $\mathcal{X}$ is Markov. Let $A \in \mathcal{B}(E^J)$ for some $J \subseteq_f I$, $\max J = s \le t$, $B \in \mathcal{B}(E)$. Then,

$$\begin{aligned}\mathbf{P}((X_r)_{r\in J} \in A, X_t \in B) &= \nu_{J\cup\{t\}}(A \times B)\\ &= \mathbf{E}[\mu_{s,t}(X_s, B), (X_r)_{r\in J} \in A].\end{aligned}$$

If $(\mathcal{F}_t)_{t\in I}$ is the filtration generated by $\mathcal{X}$, then for $A \in \mathcal{F}_s$

$$\mathbf{P}(X_t \in B, A) = \mathbf{E}[\mu_{s,t}(X_s, B), A].$$

Distribution of Markov processes

- Corollary 15.18: ν, $(\mu_{s,t})_{s\leq t}$ as in Theorem 15.17. Then, there is a probability distribution $\mathbf{P}_\nu$ on $\mathcal{B}(E)^I$, such that $\mathbf{P}_\nu$ is the distribution of the Markov process with transition kernels $(\mu_{s,t})_{s\leq t}$ and initial distribution ν. Furthermore, $x \mapsto \mathbf{P}_x := \mathbf{P}_{\delta_x}$ defines a transition kernel from E to $\mathcal{B}(E)^I$ and

$$\mathbf{P}_\nu = \int \nu(dx)\mathbf{P}_x.$$