Stochastic Processes
9. Martingale Convergence results I

Peter Pfaffelhuber

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Maximum Inequality

- Lemma 14.25: I countable, $\mathcal{X} = (X_t)_{t \in I}$ sub-martingale, $\lambda > 0$. Then

$$\lambda \mathbf{P}[\sup_{s \le t} X_s \ge \lambda] \le \mathbf{E}[X_t, \sup_{s \le t} X_s \ge \lambda] \le \mathbf{E}[|X_t|, \sup_{s \le t} X_s \ge \lambda].$$

Set

$$T = t \wedge T_{[\lambda;\infty)}.$$

According to the Optional Sampling Theorem,

$$\begin{aligned} \mathbf{E}[X_t] \ge \mathbf{E}[X_T] &= \mathbf{E}[X_T; \sup_{s \le t} X_s \ge \lambda] + \mathbf{E}[X_T; \sup_{s \le t} X_s < \lambda] \\ &\ge \lambda \mathbf{P}[\sup_{s \le t} X_s \ge \lambda] + \mathbf{E}[X_t; \sup_{s \le t} X_s < \lambda]. \end{aligned}$$

Doob's L^p inequality

- Proposition 14.26: I countable, $\mathcal{X}$ martingale or positive sub-martingale.
  1. For $p \geq 1$ and $\lambda > 0$ $\lambda^p \mathbf{P}[\sup_{s \leq t} |X_s| \geq \lambda] \leq \mathbf{E}[|X_t|^p]$.
  2. For $p > 1$ is $\mathbf{E}[|X_t|^p] \leq \mathbf{E}[\sup_{s \leq t} |X_s|^p] \leq \left(\frac{p}{p-1}\right)^p \mathbf{E}[|X_t|^p]$.
- 1. Assumption follows from Lemma 14.25; 2. For $K > 0$,

$$\mathbf{E}[\sup_{s \leq t}(|X_s| \wedge K)^p] = \mathbf{E}\Big[\int_0^{\sup_{s\leq t}|X_s| \wedge K} p\lambda^{p-1} d\lambda\Big]$$

$$= \mathbf{E}\Big[\int_0^K p\lambda^{p-1} 1_{\{\lambda < \sup_{s \leq t}|X_s|\}} d\lambda\Big] = \int_0^K p\lambda^{p-1}\mathbf{P}(\sup_{s \leq t}|X_s| \geq \lambda) d\lambda$$

$$\leq \int_0^K p\lambda^{p-2}\mathbf{E}[|X_t|, \sup_{s \leq t}|X_s| \geq \lambda] d\lambda = \frac{p}{p-1}\mathbf{E}[|X_t|(\sup_{s \leq t}|X_s| \wedge K)^{p-}$$

$$\leq \frac{p}{p-1}\mathbf{E}[\sup_{s \leq t}(|X_s| \wedge K)^p]^{(p-1)/p} \cdot \mathbf{E}[|X_t|^p]^{1/p}.$$

Upcrossings

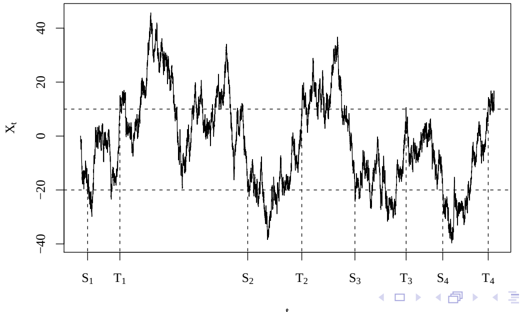
- Definition 14.27: I countable, $\mathcal{X}$ process, $a < b$. Define

  $0 =: T_0 < S_1 < T_1 < S_2 < T_2 < \ldots$ by

  $$S_k := \inf\{t \geq T_{k-1} : X_t \leq a\}, \quad T_k \quad := \inf\{t \geq S_k : X_t \geq b\}$$

  and the number of upcrossings between a and b up to time t

  $$U^t_{a,b} := \sup\{k : T_k \leq t\}$$



Upcrossing lemma

- Lemma 14.28: I countable, $\mathcal{X}$ sub-martingale. Then
$$\mathbf{E}[U_{a,b}^t] \le \frac{\mathbf{E}[(X_t - a)^+]}{b-a}.$$
Wlog $\mathcal{X} \ge 0$ and $a = 0$. Define $\mathcal{H} = (H_t)_{t \in I}$ predvisible by
$$H_t := \sum_{k \ge 1} 1_{\{S_k < t \le T_k\}}.$$
Given $T_k < \infty$, $X_{T_k} - X_{S_k} \ge b$ is obvious. Furthermore,
$$(\mathcal{H} \cdot \mathcal{X})_{T_k} = \sum_{i=1}^{k} \sum_{s=S_i+1}^{T_i} (X_s - X_{s-1}) = \sum_{i=1}^{k} (X_{T_i} - X_{S_i}) \ge kb$$
and
$$\mathbf{E}[X_t] \ge \mathbf{E}[X_t - X_0] = \mathbf{E}[(\mathcal{H} \cdot \mathcal{X})_t + ((1-\mathcal{H}) \cdot \mathcal{X})_t]$$
$$\ge \mathbf{E}[(\mathcal{H} \cdot \mathcal{X})_t] \ge b\mathbf{E}[U_{0,b}^t].$$

First martingale convergence Theorem

- Theorem 14.29: $I = \mathbb{N}_0$, $\mathcal{F}_\infty = \sigma(\bigcup_{t\in I} \mathcal{F}_t)$, $\mathcal{X}$ submartingale with $\sup_{t\in I} \mathbf{E}[X_t^+] < \infty$. Then there exists $X_\infty \in L^1$ with $X_t \xrightarrow{t\to\infty}_{as} X_\infty$.

  Since $\mathbf{P}(U_{a,b}^t < \infty) = 1$ for all a, b, t,

  $N := \bigcup_{\substack{a<b\\ a,b\in\mathbb{Q}}} \{\sup_{t\in I} U_{a,b}^t = \infty\}$ is a null set.

  Furthermore, on N^c $X_\infty := \liminf_{t\to\infty} X_t = \limsup_{t\to\infty} X_t$ and, according to Fatou's lemma,

  $\mathbf{E}[X_\infty^+] \le \sup_{t\in I} \mathbf{E}[X_t^+] < \infty$,

$$\mathbf{E}[X_\infty^-] \le \liminf_{t\to\infty} \mathbf{E}[X_t^-] = \liminf_{t\to\infty} \left(\mathbf{E}[X_t^+] - \mathbf{E}[X_t]\right)$$

$$\le \sup_{t\in I} \mathbf{E}[X_t^+] - \mathbf{E}[X_0] < \infty.$$

Martingale convergence Theorem for positive supermartingales

- Corollary 14.30: $I = \mathbb{N}_0$, $\mathcal{F}_\infty = \sigma(\bigcup_{t\in I} \mathcal{F}_t)$, $\mathcal{X} = (X_t)_{t\in I}$ non-negative super martingale. Then there is $X_\infty \in L^1$ with $\mathbf{E}[X_u] \leq \mathbf{E}[X_0]$ and $X_t \xrightarrow{t\to u}_{as} X_\infty$.

Theorem 13.30, applied to $-\mathcal{X}$ yields X_∞. Using Fatou's lemma,

$$\mathbf{E}[X_u] \leq \liminf_{t\to u} \mathbf{E}[X_t] \leq \mathbf{E}[X_0].$$

Example 4: Branching Process

- $I = \{0, 1, 2, ...\}$, $X_i^{(t)}$ takes values in $\{0, 1, 2, ...\}$, $\mu = \mathbf{E}[X_i^{(t)}]$.

  Set $Z_0 = k$ and
  $$Z_{t+1} = \sum_{i=1}^{Z_t} X_i^{(t)}.$$
- $\mathcal{Z} = (Z_t)_{t\in I}$ martingale $\iff$ $\mu = 1$.

  More generally, $(Z_t/\mu^t)_{t=0,1,2,...}$ is a martingale.
- $\mu = 1$: According to Corollary 3.31, $Z_t \xrightarrow{t\to\infty} 0$.

  More generally, there is Z_∞ with $Z_t/\mu^t \xrightarrow{t\to\infty} Z_\infty$.

Second martingale convergence theorem

- Theorem 14.32: $I = \mathbb{N}_0$, $\mathcal{F}_\infty = \sigma(\bigcup_{\sqcup \in \mathcal{I}} \mathcal{F}_\sqcup)$ and $\mathcal{X}$ submartingale.

  The following are equivalent

  1. $\mathcal{X}$ is uniformly integrable.
  2. There exists X_∞ such that $(X_t)_{t \in I \cup \{\infty\}}$ is a submartingale.
  3. There exists X_∞ such that $X_t \xrightarrow{t \to u}_{as, L^1} X_\infty$.

  2. $\Rightarrow$ 1. ok; 1. $\Rightarrow$ 3. ok; 3. $\Rightarrow$ 2.: Due to the L^1 convergence according to Theorem 12.2.3 ,

  $\mathbf{E}[|\mathbf{E}[X_t|\mathcal{F}_s] - \mathbf{E}[X_\infty|\mathcal{F}_s]|] \xrightarrow{t\to\infty} 0$ and thus for $A \in \mathcal{F}_s$

$$\mathbf{E}[\mathbf{E}[X_\infty|\mathcal{F}_s]; A] = \lim_{t\to\infty} \mathbf{E}[\mathbf{E}[X_t|\mathcal{F}_s]; A] \geq \mathbf{E}[X_s; A],$$

  i.e. $\mathbf{E}[X_\infty|\mathcal{F}_s] \geq X_s$ almost surely.

L^p-bounded martingales

- Theorem 14.33: $I = \mathbb{N}_0$, $\mathcal{F}_\infty = \sigma(\bigcup_{t\in I} \mathcal{F}_t)$, $p > 1$ and $\mathcal{X}$ an L^p-bounded martingale. Then there exists $X_\infty \in L^p$ with $X_t \xrightarrow{t\uparrow u}_{as, L^p} X_\infty$. Furthermore, $(|X_t|^p)_{t\in I}$ is uniformly integrable.

  Proof: $\mathcal{X}$ ui, so there exists X_∞ with $X_t \xrightarrow{t\to\infty}_{as, L^1} X_\infty$. Using Doob's inequality,

  $$\mathbf{E}[\sup_{t\in I} |X_t|^p] = \lim_{t\to\infty} \mathbf{E}[\sup_{s\leq t} |X_s|^p] \leq \lim_{t\uparrow u} \left(\frac{p}{p-1}\right)^p \mathbf{E}[|X_t|^p] < \infty.$$

  Thus, $(|X_t|^p)_{t\in I}$ is ui and L^p-convergence follows.

Branching processes

- $\mathcal{Z}$ branching process with $Z_0 = k$ and $\mathcal{Y}$, $Y_t = Z_t/\mu^t$. Then

$$\langle \mathcal{Y} \rangle_t = \sum_{s=1}^{t} \frac{1}{\mu^{2s}} \mathbf{E}\Big[\Big(\sum_{i=1}^{Z_{s-1}} X_i^{(s-1)} - \mu Z_{s-1}\Big)^2 | \mathcal{F}_{s-1}\Big]$$

$$= \sum_{s=1}^{t} \frac{1}{\mu^{2s}} \mathbf{V}\Big[\sum_{i=1}^{Z_{s-1}} X_i^{(s-1)} | Z_{s-1}\Big]$$

$$= \sum_{s=1}^{t} \frac{1}{\mu^{2s}} Z_{s-1} \cdot \mathbf{V}[X_1^{(1)}].$$

If $\mu > 1$ and $\mathbf{V}[X_1^{(1)}] =: \sigma^2 < \infty$, then

$$\sup_t \mathbf{V}[Y_t] = \sum_{s=1}^{\infty} \frac{1}{\mu^{2s}} \mathbf{E}[Z_s] \cdot \sigma^2 = k\sigma^2 \sum_{s=1}^{\infty} \frac{1}{\mu^s} < \infty.$$

Thus, there is Y_∞, so that $(Y_t)_{t=0,1,2,\dots,\infty}$ martingale.