universität freiburg

Stochastic processes

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 9 - Markov processes I

Exercise 1 (4 points)**.**
Let $\mathcal{X} = (X_t)_{t\in[0,\infty)}$ be a stochastic process. Show that $\mathcal{X}$ is Markov if and only if, for all $s \leq t \leq u$, and all measurable A,

$$\mathbf{P}(X_u \in A|X_s,X_t) = \mathbf{P}(X_u \in A|X_t).$$

Exercise 2 (2+2=4 points)**.**
Decide and reason if or if not the following stochastic processes are Markov:

(a) $X_t = \phi(B_t), t \geq 0$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and $(B_t)_{t\geq 0}$ is Brownian Motion.

(b) For $n = 1,2,...$ let $X_n = Z_n + Z_{n-1}$, where $Z_i, i = 0,1,2,\dots$ are iid with $\mathbf{P}(Z_1 = 0) = \mathbf{P}(Z_1 = 1) = \frac{1}{2}$.

Exercise 3 (2+2=4 Points)**.**
Let $(X_t)_{t\geq 0}$ be a standard Brownian motion and $(Y_t)_{t\geq 0} := e^{-t/2}X_{e^t-1}$.

(a) Show that $(Y_t)_{t\geq 0}$ is a Gaussian process and a Markov process.

(b) Determine the weak limit of Y_t for $t \to \infty$.

Exercise 4 (2+2=4 Points)**.**
Let $f : [0,\infty) \to [0,\infty)$ be strictly monotonically increasing with $f(0) = 0$, $\mathcal{P} = (P_t)_{t\geq 0}$ a Poisson process with intensity 1 and $\mathcal{M} = (M_n)_{n=0,1,\dots}$ a Markov chain in discrete time with values in $\mathbb{Z}$ and transition matrix $\Pi = (\pi_{ij})_{i,j\in\mathbb{Z}}$. Furthermore, let $\mathcal{P}$ and $\mathcal{M}$ be stochastically independent.

(a) Show that $\mathcal{X} = (X_t)_{t\geq 0}$ with $X_t := M_{P_{f(t)}}$ is a Markov process with respect to its natural filtration.

(b) Determine transition kernels and operators.