universität freiburg

Stochastic processes **Winter semester 2024**

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 8 - Martingale convergence results

Exercise 1 (4 Points)**.**
Let $(S_t)_{t=0,1,2,\ldots}$ be a simple, symmetrical random walk and let T_K be the hitting time of $(-\infty,-K]\cup[K,\infty)$. Show that for all $t\geq 0$ and $K>0$

$$\mathbf{P}(T_K \leq t) \leq \frac{\sqrt{t}}{K}.$$

Exercise 2 (2+2 points)**.**
Can you give an example (if it exists!) of a martingale $\mathcal{X}=(X_n)_{n=0,1,2,\ldots}\cdots$

(a) ...with $\lim\limits_{n\to\infty} X_n = \infty$ almost surely?

(b) ... that converges in $\mathcal{L}^1$ but not almost surely?

Exercise 3 (2+2=4 Points)**.**
Let $p\in[0,1]$ and let $\mathcal{X}=(X_n)_{n\in\mathbb{N}_0}$ be a stochastic process with values in $[0,1]$. For each $n\in\mathbb{N}_0$ given $X_0,\ldots,X_n$, we have

$$X_{n+1}=\begin{cases} 1-p+pX_n & \text{with probability } X_n,\\ pX_n & \text{with probability } 1-X_n.\end{cases}$$

(a) Show that $\mathcal{X}$ is a martingale.

(b) Show that $\mathcal{X}$ converges almost surely in $\mathcal{L}^1$.

(c) Determine the distribution of the limit value $X:=\lim\limits_{n\to\infty} X_n$.

Hint: For (c), consider the process $\big(X_n(1-X_n)\big)_n$

Exercise 4 (2+2=4 Points)**.**

(a) Give an example of a non-negative square integrable martingale that converges almost surely but not in $\mathcal{L}^2$.

(b) Show that for $p=1$, the statement in Theorem in 14.33 may fail. Give an example of a non-negative martingale $\mathcal{X}=(X_n)_{n\in\mathbb{N}}$ with $\mathbf{E}[X_n]=1$ for all $n\in\mathbb{N}$ but such that $X_n \xrightarrow{n\to\infty} 0$ almost surely.