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Stochastic processes — Winter semester 2024

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
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Tutorial 10 - Markov processes II

Exercise 1 (4 Points)**.**
Let $\mathcal{X} = (X_t)_{t=0,1,2,\dots}$ be a stochastic process with state space E and $(\mathcal{F}_t)_{t=0,1,2,\dots}$ its filtration. Show that the following are equivalent:

(a) $\mathcal{X}$ is a Markov process.

(b) For all bounded and measurable functions $f : E \to \mathbb{R}$,

$$f(X_t) - \sum_{k=1}^{t} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}]$$

is a martingale wrt the filtration $(\mathcal{F}_t)_{t=0,1,2,\dots}$

Solution.

(a) '$\Longrightarrow$': Let us define $M_t = f(X_t) - \sum_{k=1}^{t} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}]$. Obviously, $M_{t+1} = f(X_{t+1}) - \sum_{k=1}^{t+1} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}]$. It is sufficient to verify the martingale property!

$$\begin{aligned}\mathbf{E}[M_{t+1}|\mathcal{F}_t] &= \mathbf{E}[f(X_{t+1}) - \sum_{k=1}^{t+1} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}]|\mathcal{F}_t] \\ &= \mathbf{E}[f(X_{t+1})|\mathcal{F}_t] - \sum_{k=1}^{t} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}] - \mathbf{E}[f(X_{t+1}) - f(X_t)|X_t].\end{aligned}$$

With the Markov property, we have that,

$$\begin{aligned}\mathbf{E}[M_{t+1}|\mathcal{F}_t] &= \mathbf{E}[f(X_{t+1})|X_t] - \sum_{k=1}^{t} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}] - \mathbf{E}[f(X_{t+1}) - f(X_t)|X_t]. \\ &= f(X_t) - \sum_{k=1}^{t} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}] = M_t\end{aligned}$$

(b) '$\Longleftarrow$': Now, we assume that M_t as defined above is a martingale with respect to the filtration $(\mathcal{F}_t)_{t=0,1,2,\dots}$, and we will verify the Markov property.

$$\begin{aligned}\mathbf{E}[f(X_{t+1})|\mathcal{F}_t] &= \mathbf{E}[M_{t+1} + \sum_{k=1}^{t+1} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}]|\mathcal{F}_t] \\ &= \mathbf{E}[M_{t+1}|\mathcal{F}_t] + \sum_{k=1}^{t+1} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}].\end{aligned}$$

With the martingale property, and by the definition of M_t, it follows that

$$\begin{aligned}\mathbf{E}[f(X_{t+1})|\mathcal{F}_t] &= M_t + \sum_{k=1}^{t+1} \mathbf{E}[f(X_k) - f(X_{k-1})|X_{k-1}] \\ &= f(X_t) + \mathbf{E}[f(X_{t+1}) - f(X_t)|X_t] \\ &= \mathbf{E}[f(X_{t+1})|X_t].\end{aligned}$$

Exercise 2 (2+2 points)**.**
Let $\lambda > 0$ and ν be a probability measure on $\mathbb{R}$ with $\nu(\{1\}) = \nu(\{-1\}) = \frac{1}{2}$. Furthermore, consider the family of Markov kernels $(P_{s,t})_{s\leq t}$given by:

$$P_{s,t}(x,\{x\}) = \frac{1}{2}\big(1 + e^{-\lambda(t-s)}\big), \quad P_{s,t}(x,\{-x\}) = \frac{1}{2}\big(1 - e^{-\lambda(t-s)}\big), \qquad x \in \{-1-1\}.$$

For $x \neq \pm 1$, it is $P_{s,t}(x,\cdot) = \delta_x$ the Dirac measure on x.

(a) Show that the *Chapman-Kolmogorov* equations

$$P_{s,u}(x,A) = \int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A)$$

hold for all $s \leq t \leq u, x \in \mathbb{R}$, and $A \subset \mathbb{R}$.

(b) Let $\mathcal{X}$ be a Markov process with Markov kernels $(P_{s,t})_{s\leq t}$. Show that, if $X_0 \sim \nu$, then

$$\mathbf{P}(X_t = 1) = \mathbf{P}(X_t = -1) = \frac{1}{2}.$$

Solution.

(a) To show that the *Chapman-Kolmogorov* equations

$$P_{s,u}(x,A) = \int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A)$$

hold for all $s \leq t \leq u, x \in \mathbb{R}$, and $A = \{x\}$; on the one hand, for $x = \pm 1$, we have

$$\begin{aligned}\int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A) &= P_{s,t}(x,\{x\})P_{t,u}(x,A) + P_{s,t}(x,\{-x\})P_{t,u}(-x,A) \\ &= \frac{1}{2}(1 + e^{-\lambda(t-s)}) \cdot \frac{1}{2}(1 + e^{-\lambda(u-t)}) + \frac{1}{2}(1 - e^{-\lambda(t-s)}) \cdot \frac{1}{2}(1 - e^{-\lambda(u-s)}) \\ &= \frac{1}{4}(2 + 2e^{-\lambda(t-s)e^{-}\lambda(u-t)}) = \frac{1}{2}(1 + e^{-\lambda(u-s)}) = P_{s,u}(x,\{x\}).\end{aligned}$$

Similarly, $P_{s,u}(x,\{-x\}) = \frac{1}{2}(1 - e^{-\lambda(u-s)})$. In general, all $A \subset \mathbb{R}$ can be written as a disjoint union $A = A_1 \uplus A_2$ with $x \in A_1$ and $-x \in A_2$. Hence,

$$\begin{aligned}P_{s,u}(x,A) &= P_{s,u}(x,A_1) + P_{s,u}(x,A_2) \\ &= \int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A_1) + \int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A_2) \\ &= \int_{\mathbb{R}} P_{s,t}(x,dz)\left(P_{t,u}(z,A_1) + P_{t,u}(z,A_2)\right) \\ &= \int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A)\end{aligned}$$

On the other hand, for $x \neq \pm 1$,

$$\int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A) = \int_{\mathbb{R}} \delta_x P_{t,u}(z,A) = P_{t,u}(x,A).$$

Therefore, the *Chapman-Kolmogorov* equations hold for all $s \leq t \leq u, x \in \mathbb{R}$, and $A \subset \mathbb{R}$.

(b) With the Markov property, it holds,

$$\mathbf{P}(X_t = A) = \int_{\mathbb{R}} P_{0,t}(x,\{A\})\nu(dx).$$

Since $\nu(\{1\}) = \nu(\{-1\}) = \frac{1}{2}$, then

$$\begin{aligned}
\mathbf{P}(X_t = 1) &= \int_{\{-1,1\}} P_{0,t}(x,\{1\})\nu(dx) \\
&= \nu(\{1\}) \cdot P_{0,t}(1,\{1\}) + \nu(\{-1\}) \cdot P_{0,t}(-1,\{1\}) \\
&= \frac{1}{2}(1 - e^{-\lambda(t-0)}) \cdot \frac{1}{2} + \frac{1}{2}(1 + e^{-\lambda(t-0)}) \cdot \frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Similarly, $\mathbf{P}(X_t = -1) = \frac{1}{2}$ or use the fact that $\mathbf{P}(X_t = -1) = 1 - \mathbf{P}(X_t = 1) = \frac{1}{2}$.

Exercise 3 (4 Points)**.**
Let $E \subset \mathbb{R}$ be countable and let $\mathcal{X}$ be a Markov chain on E with transition matrix p and with the property that, for any x, there are at most three choices for the next step; that is, there exists a set $A_x \subset E$ of cardinality 3 with $p(x,y) = 0$ for all $y \in E \backslash A_x$. Let $d(x) := \sum_{y \in E}(y - x)p(x,y)$ for all $x \in E$.

(a) Show that $M_n := X_n - \sum_{k=0}^{n-1} d(X_k)$ defines a martingale M with square variation process $\langle M \rangle_n = \sum_{i=0}^{n-1} f(X_i)$ for a unique function $f : E \to [0,\infty)$.

(b) Show that the transition matrix p is uniquely determined by f and d.

Exercise 4 (4 Points)**.**
Let α, $\sigma^2 \in (0,\infty)$. Show that given $K_t(x,\cdot) := N(xe^{-\alpha t}, \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t}))$ for $t > 0$, $K_0(x,\cdot) := \varepsilon_x$ is a semigroup of Markov kernels, i.e:

$$K_{s+t}(x,B) = \int_{\mathbb{R}} K_t(y,B)K_s(x,dy) \qquad \forall (x,B) \in \mathbb{R} \times \mathcal{B} \text{ and } s,t \in \mathbb{R}_+.$$

Hint: The following equation simplifies the calculation:

$$\int_B \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{B-\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx.$$

Solution.
We show the claim in general for $f : \mathbb{R} \to \overline{\mathbb{R}}_+$ measurable, then the claim follows in

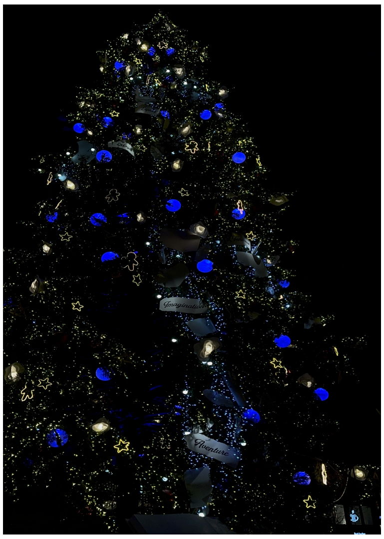

Figure 1: Merry Christmas!

particular also for $f = \mathbb{1}_B$. Further set $k_s = (1-e^{-2\alpha s})$, $k_t = (1-e^{-2\alpha t})$, $k_{s+t} = (1-e^{-2\alpha(s+t)})$ and $\tilde{\sigma}^2 = \frac{\sigma^2}{2\alpha}$. Then the following applies

$$
\begin{aligned}
&(K_t \circ K_s)f(x) = \int\int f(z)K_s(y,dz)K_t(x,dy)\\
&= \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 k_s}}\int\int f(z+ye^{-\alpha s})e^{\frac{-z^2}{2\tilde{\sigma}^2 k_s}}dz\ K_t(x,dy)\\
&= \frac{1}{2\pi\tilde{\sigma}^2\sqrt{k_s k_t}}\int\int f(z+(y+xe^{-\alpha t})e^{-\alpha s})e^{\frac{-z^2}{2\tilde{\sigma}^2 k_s}}e^{\frac{-y^2}{2\tilde{\sigma}^2 k_t}}dz\ dy\\
&= \frac{1}{2\pi\tilde{\sigma}^2\sqrt{k_s k_t}}\int f(u+xe^{-\alpha(t+s)})\int \exp\left(-\frac{1}{2\tilde{\sigma}^2}(y^2(\frac{1}{k_t}+\frac{e^{-2\alpha s}}{k_s})-2y\frac{e^{-\alpha s}u}{k_s}+\frac{u^2}{k_s})\right)dy\ du\\
&= \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 k_{s+t}}}\int f(u+xe^{-\alpha(t+s)})\frac{1}{\sqrt{2\pi\tilde{\sigma}^2\frac{k_s k_t}{k_{s+t}}}}\int \exp\left(-\frac{1}{2\tilde{\sigma}^2\frac{k_s k_t}{k_{s+t}}}(y-\frac{e^{-\alpha s}k_t u}{k_{s+t}})^2\right)dy\\
&\qquad\qquad \exp\left(\frac{-u^2}{2\tilde{\sigma}^2}(\frac{1}{k_s}+\frac{e^{-2\alpha s}k_t}{k_s k_{s+t}})\right)du\\
&= \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 k_{s+t}}}\int f(u+xe^{-\alpha(t+s)})e^{-\frac{u^2}{2\tilde{\sigma}^2 k_{s+t}}}du\\
&= K_{s+t}f(x).
\end{aligned}
$$