universität freiburg

Measure theory for probabilists — Winter semester 2024

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 11 - Theorem of Radon-Nikodym

Exercise 1 (4 Points)**.**
Give an example of two measures μ,ν with $\nu \ll \mu$ for which there is no density $f\colon \Omega \to \mathbb{R}$ with $d\nu = f d\mu$.

Exercise 2 (4 Points)**.**
Let $(\Omega,\mathcal{B},\nu)$ be a measure space, and let $\Omega = \bigcup_{n=1}^\infty E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(B) = \sum_{n=1}^\infty 2^{-n}\nu(B \cap E_n)/(\nu(E_n)+1)$.

(a) Prove that μ is a finite measure on $(\Omega,\mathcal{B})$

(b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ if and only if $\nu(B) = 0$.

Exercise 3 (4 Points)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Determine all functions $g : \mathbb{R} \to \mathbb{R}$ that are measurable with respect to the σ-algebra generated by f, denoted as $\sigma(f)$, and the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. Provide justification for your answer.

Exercise 4 (4 Points)**.**
Let $(\Omega,\mathcal{F},\mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ a sequence of non-negative, measurable, real functions on Ω, which converges pointwise to a function $f : \Omega \to [0,\infty]$.

(a) Show that in the case $\lim_{n\to\infty} \int f_n\, d\mu = c < \infty$ f is also μ-integrable with $\int f\, d\mu \leq c$.

(b) Furthermore, by means of an example, show that $\int f\, d\mu$ can take any value in $[0,c]$.

Hint: For example, $(\Omega,\mathcal{F},\mu) = (\mathbb{R},\mathcal{B}(\mathbb{R}),\mathbb{1}_{[0,1]} \cdot \lambda)$ and $f := a \cdot \mathbb{1}_{(0,1]}$ for given $a \in [0,c]$.

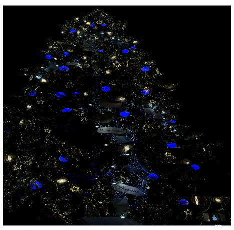

Figure 1: Merry Christmas!