universität freiburg

Stochastic processes — Winter semester 2024

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 4 - Filtrations, stopping times, measurability

In all exercises, we have a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 1 (4 points)**.**
Verify that
$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t, t \in I\}$$
is a σ-algebra.

Solution.
Clearly, the empty set $\emptyset$ is in $\mathcal{F}_T$ since $\emptyset \cap \{T \leq t\} = \emptyset \in \mathcal{F}_T$ for all $t \in I$. Now if $B \in \mathcal{F}_t$ it follows that
$$B^c \cap \{T \leq t\} = \{T \leq t\} \backslash (B \cap \{T \leq t\}) \in \mathcal{F}_T \quad \text{for all} \quad t \in I,$$
since both $\{T \leq t\}$ and $B \cap \{T \leq t\}$ are in $\mathcal{F}_T$ for each $t \in I$. Thus $B^c \in \mathcal{F}_T$. Lastly, for $B_1, B_2, \ldots \in \mathcal{F}_T$ we obtain
$$\left(\bigcup_{k=1}^{\infty} B_k\right) \cap \{T \leq t\} = \bigcup_{k=1}^{\infty} (B_k \cap \{T \leq t\}) \in \mathcal{F}_T \quad \text{for all} \quad t \in I.$$
Thus, $\bigcup_{k=1}^{\infty} B_k \in \mathcal{F}_T$.

Exercise 2 (4 Points)**.**
Let $\Omega = [0,1]$ and $\mathcal{F} = \mathcal{B}([0,1])$. Define a stochastic process $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ by
$$X_n(\omega) := 2\omega \mathbb{1}_{[0,1-\frac{1}{n}]}(\omega).$$
Show that the generated filtration $(\mathcal{F}_n^{\mathcal{X}})_{n\in\mathbb{N}}$ is given by
$$\mathcal{F}_n^{\mathcal{X}} = \left\{A \cup B : A \in \mathcal{B}((0,1-\frac{1}{n}]), B \in \{\emptyset, \{0\} \cup (1-\frac{1}{n},1]\}\right\}$$

Solution.
For every $n \in \mathbb{N}$ and $C \in \mathcal{B}(\mathbb{R})$ the definition of X_n yields
$$\begin{aligned}(X_n(B))^{-1} &= \begin{cases} \frac{1}{2}C \cap [0,1-\frac{1}{n}], & \text{if } 0 \notin C, \\ \left(\frac{1}{2}C \cap [0,1-\frac{1}{n}]\right) \bigcup (1-\frac{1}{n},1], & \text{if } 0 \in C, \end{cases} \\ &= \begin{cases} \frac{1}{2}C \cap (0,1-\frac{1}{n}], & \text{if } 0 \notin C, \\ \left(\frac{1}{2}C \cap (0,1-\frac{1}{n}]\right) \bigcup (1-\frac{1}{n},1] \bigcup \{0\}, & \text{if } 0 \in C, \end{cases}\end{aligned}$$

where we have use the shorthand notation $\frac{1}{2}C := \{\frac{1}{2}y : y \in C\}$. Hence, the σ-field $\mathcal{F}_n^{\mathcal{X}}$ must contain at least

$$\mathcal{C} := \left\{ \frac{1}{2}C \cap (0{,}1-\frac{1}{n}],\ \frac{1}{2}C \cap [0{,}1-\frac{1}{n}] \cup (1-\frac{1}{n}{,}1] \cup \{0\} \text{ for all } C \in \mathcal{B}(\mathbb{R}) \right\}.$$

Since $\{\frac{1}{2}C \cap (0{,}1-\frac{1}{n}] : C \in \mathcal{B}(\mathbb{R}) = \mathcal{B}((0{,}1-\frac{1}{n}])\}$ this can be written in the form

$$\mathcal{C} = \left\{ A \cup B : A \in \mathcal{B}((0{,}1-\frac{1}{n}]),\ B = \emptyset \text{ or } B = (1-\frac{1}{n}{,}1] \cup \{0\} \right\}$$

$\mathcal{C}$ is already a σ-field (this we can verify!) and $\mathcal{F}_n^{\mathcal{X}}$ is the smallest σ-field which contains $\mathcal{C}$ it follows that $\mathcal{F}_n^{\mathcal{X}} = \mathcal{C}$.

Exercise 3 (4 points)**.**
Given an optional time T of the filtration $(\mathcal{F}_t)_{t\in I}$, consider the sequence $(T_n)_{n\geq 1}$ of random times given by

$$T_n(\omega) = \begin{cases} T(\omega); & \text{on } \{\omega;\ T(\omega) = +\infty\} \\ \\ \frac{k}{2^n}; & \text{on } \left\{\omega;\ \frac{k-1}{2^n} \leq T(\omega) < \frac{k}{2^n}\right\} \end{cases}$$

for $n \geq 1, k \geq 1$. Clearly, $T_n \geq T_{n+1} \geq T$, for $n \geq 1$. Show that each T_n is a stopping time, and that $\lim_{n\to\infty} T_n = T$.

Solution.
We will first show that each T_n is a stopping time. For all $n \geq 1$

$$\{\omega \in \{\omega : T(\omega) = +\infty\} : T_n(\omega) \leq t\} = \{\omega \in \{\omega : T(\omega) = +\infty\} : \infty \leq t\} = \emptyset.$$

It holds that for every $t \in I$, $\exists\ k \in \mathbb{N}$ with $\frac{k}{2^n} \leq t \leq \frac{k+1}{2^n}$ such that

$$\begin{aligned} \{T_n \leq t\} = \left\{T_n \leq \frac{k}{2^n}\right\} &= \bigcup_{1\leq m\leq k} \left\{T_n = \frac{m}{2^n}\right\} \\ &= \bigcup_{1\leq m\leq k} \left\{\frac{m-1}{2^n} \leq T \leq \frac{m}{2^n}\right\} \\ &= \left\{T < \frac{k}{2^n}\right\} \in \mathcal{F}_{\frac{k}{2^n}} \subseteq \mathcal{F}_t. \end{aligned}$$

Thus, each T_n is a stopping time. Now to show that $\lim_{n\to\infty} T_n = T$, first, if

$$\omega \in \{\omega : T(\omega) = +\infty\}, \quad \text{then} \quad \lim_{n\to\infty} T_n(\omega) = \lim_{n\to\infty} T(\omega) = T(\omega).$$

Otherwise, if $\omega \notin \{\omega : T(\omega) = +\infty\}$, then there exists $k \in \mathbb{N}$ such that $T_n(\omega) = \frac{k}{2^n}$ and $T(\omega) \geq \frac{k-1}{2^n}$.

$$\implies |T_n(\omega) - T(\omega)| = \left|\frac{k}{2^n} - T(\omega)\right| \leq \left|\frac{k}{2^n} - \frac{k-1}{2^n}\right| = \frac{1}{2^n} \xrightarrow{n\to\infty} 0.$$

In both cases, $\lim_{n\to\infty} T_n = T$.

Exercise 4 (2+2 Points)**.**

1. Let $(X_t)_{t\in[0,\infty)}$ be a stochastic process, such that, for all $t \geq 0$,

$$Y_t : \begin{cases} I \cap [0,t] \times \Omega & \to E \\ (s,\omega) & \mapsto X_s(\omega) \end{cases}$$

is measurable with respect to $I \cap \mathcal{B}([0,t]) \otimes \mathcal{F}_t/\mathcal{B}(E)$. Show that $(X_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$.

We call $(X_t)_{t\geq 0}$ measurable if

$$\begin{cases} I \times \Omega & \to E \\ (s,\omega) & \mapsto X_s(\omega) \end{cases}$$

is measurable with respect to $\mathcal{B}([0,\infty)) \otimes \mathcal{F}/\mathcal{B}(E)$. Show that every progressively measurable process is measurable, but need not even be adapted if it is only measurable.

2. Let $(\mathcal{X}_t)_{t\geq 0}$ be a stochastic process, which is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and T be an $(\mathcal{F}_t)_{t\geq 0}$ stopping time. Show that the stopped process $(X_{T\wedge t})_{t\geq 0}$ is also progressively measurable.

Solution.
First, for adaptiveness, note that progressive measurability implies by measurability of projections that for all t, the map $\omega \mapsto Y_t|_{\{t\}\times\Omega}$ is measurable. However, this restriction is the same as measurability of $\omega \mapsto X_t(\omega)$ with respect to $\mathcal{F}_t$. That is, since $(X_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$, we have for every $t \geq 0$ and $A \in \mathcal{B}(E)$:

$$Y_t^{-1}(A) = \{(s,\omega) \in [0,t] \times \Omega : X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t.$$

We therefore obtain for every $s \in [0,t]$, $\{\omega \in \Omega : (s,\omega) \in Y_t^{-1}(A)\} \in \mathcal{F}_t$. This holds in particular for $t \in [0,t]$ and so

$$\{\omega \in \Omega : (t,\omega) \in Y_t^{-1}(A)\} = \{\omega \in \Omega : X_t(\omega) \in A\} = X_t^{-1}(A) \in \mathcal{F}_t.$$

That is for every $t \geq 0$, $A \in \mathcal{B}(E)$, $X_t^{-1}(A) \in \mathcal{F}_t$. Thus, X_t is $\mathcal{F}_t/\mathcal{B}(E)$-measurable.

Second, $\{I \cap \mathcal{B}([0,t]) : t \geq 0\}$ generates $\mathcal{B}([0,\infty)$. The first claim then follows from $\mathcal{F}_t \subseteq \mathcal{F}$. That is if we fix $A \in \mathcal{B}(E)$, and we let $(X_t)_{t\in[0,\infty)}$ be progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$, that is for every $t \in [0,\infty)$, $\{(s,\omega) \in [0,t] \times \Omega : X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$. Since $[0,t] \in \mathcal{B}([0,\infty])$, then $\mathcal{B}([0,t]) \subseteq \mathcal{B}([0,\infty))\ \forall\, t \in [0,\infty)$. And since $\mathcal{F}_t \subset \mathcal{F}\ \forall\, t \in [0,\infty)$ (by the definition of filtration!) we get $\mathcal{B}([0,t]) \otimes \mathcal{F}_t \subseteq \mathcal{B}([0,\infty)) \otimes \mathcal{F}$. Because

$$\{B_1 \times B_2 : B_1 \in \mathcal{B}([0,t]),\, B_2 \in \mathcal{F}_t\} \subseteq \{C_1 \times C_2 : C_1 \in \mathcal{B}([0,\infty)),\, C_2 \in \mathcal{F}\}$$

and then the same holds for the σ-algebras generated by those two sets. So we get

$$\{(s,\omega) \in [0,t] \times \Omega : X_s(\omega)|inA\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}$$

and since this holds for any $t \in [0,\infty)$ we also have

$$\{(s,\omega) \in [0,\infty) \times \Omega : X_s(\omega) \in A\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}.$$

Since $A \in \mathcal{B}(E)$ was arbitrarily chosen, we are done. $(X_t)_{t\in[0,\infty)}$ is measurable.

For the third claim, note that the argument from the first claim only gives measurability of X_t with respect to $\mathcal{F}$, which does not suffice for adaptivity of X. We further construct the following counterexample to show this clearly. Let $\Omega = \{a,b\}$, $\mathcal{F} = \{\emptyset,\Omega,\{a\},\{b\}\}$. For an arbitrary $T > 0$, we set the filtration $(\mathcal{F})_{t\geq 0}$

$$\mathcal{F}_t = \begin{cases} \{\emptyset,\Omega\}, & \text{if} \quad t \leq T; \\ \mathcal{F} & \text{if} \quad t > T. \end{cases}$$

We define a stochastic process $\mathcal{X} = (X_t)_{t\geq 0}$ thus:

$$X_t(a) = 1, \quad X_t(b) = 0 \quad \text{for all} \quad t \geq 0.$$

We claim that $\mathcal{X}$ is measurable since for every $A \in \mathcal{B}(\mathbb{R})$:

$$C := \{(s,\omega) \in [0,\infty) \times \Omega : X_s(\omega) \in A\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}.$$

$$\begin{aligned} 1 \in A, 0 \notin A: &\quad C = [0,\infty) \times \{a\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}, \\ 1 \notin A, 0 \in A: &\quad C = [0,\infty) \times \{b\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}, \\ 1 \in A, 0 \in A: &\quad C = [0,\infty) \times \{\Omega\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}, \\ 1 \notin A, 0 \notin A: &\quad C = \emptyset \times \emptyset \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}. \end{aligned}$$

But $\mathcal{X}$ is not adapted to $\mathcal{F}_{t\geq 0}$: For an arbitrary $s \leq T$ and $\{1\} \in \mathcal{B}(\mathbb{R})$, $X_s^{-1}(\{1\}) = \{a\} \notin \mathcal{F}_s$.

To show that the stopped process $(X_{T\wedge t})_{t\geq 0}$ is also progressively measurable, we have to show that the map

$$Y_{t\wedge T} : \begin{cases} [0,t] \times \Omega & \to E \\ (s,\omega) & \mapsto X_{s\wedge T(\omega)}(\omega) \end{cases}$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t/\mathcal{B}(E)$-measurable for all $t \in [0,\infty)$. Fix $t \geq 0$ and write $Y_{t\wedge T} = \varphi_t \circ \Psi_t$ with

$$\varphi_t : \begin{cases} [0,t] \times \Omega & \to [0,t] \times \Omega \\ (s,w) & \mapsto (s \wedge T(\omega),\omega) \end{cases}$$

and

$$\Psi_t : \begin{cases} [0,t] \times \Omega & \to E \\ (s,w) & \mapsto X_s(\omega). \end{cases}$$

By assumption, φ_t is $\mathcal{B}([0,t])\otimes\mathcal{F}_t/\mathcal{B}(E)$-measurable. And since the composition of measurable functions is again measurable, we just need to show that Ψ_t is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t/\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable. $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$ is generated by

$$\mathcal{C} := \{A \times B : A \in \mathcal{B}([0,t]) \; B \in \mathcal{F}_t\},$$

so we have to check $\Psi_(^{-1}t)(A \times B)$ for $A \times B \in \mathcal{C}$. We compute

$$\begin{aligned} \Psi_t^{-1}(A \times B) &= \{(s,\omega) \in [0,t] \times \Omega : s \wedge T(\omega) \in A, \, \omega \in B\} \\ &= \{(s,\omega) \in [0,t] \times \Omega : s \wedge T(\omega) \leq r, \, \omega \in B \cap (\{T \leq r\} \cup \{T > r\})\} \\ &= \{(s,\omega) : s \wedge T(\omega) \leq r, \, \omega \in B \cap \{T \leq r\}\} \cup \{(s,\omega) : s \wedge T(\omega) \leq r, \, \omega \in B \cap \{T > r\}\} \\ &= (\underbrace{[0,t]}_{\in\mathcal{B}([0,t])} \times B \cap \underbrace{\{T \leq r\}}_{\in\mathcal{F}_t}) \cup (\underbrace{[0,r]}_{\in\mathcal{B}([0,t])} \times B \cap \underbrace{\{T > r\}}_{\in\mathcal{F}_t}). \end{aligned}$$

Both sets in this union are in the generator of $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$, then they are in $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$. Thus, Ψ_t is measurable.