Stochastic Processes
14. The strong Markov property

Peter Pfaffelhuber

December 4, 2024

Definition

- $\mathcal{X} = (X_t)_{t \in I}$ progressively measurable Markov process. $\mathcal{X}$ has the *strong Markov property at S* if for all stopping times S

$$P(X_{S+t} \in A | \mathcal{F}_S) = \mu^{\mathcal{X}}_{S,S+t}(X_S, A), \quad A \in \mathcal{B}(E)$$

or equivalently

$$E[f(X_{S+t}) | \mathcal{F}_S] = (T^{\mathcal{X}}_{S,S+t} f)(X_S), \qquad f \in \mathcal{B}(E).$$

Discrete time and the strong Markov property

- Proposition 15.11: If I is discrete, then every Markov process $\mathcal{X} = (X_t)_{t\in I}$ has the strong Markov property.
- We write

$$
\begin{aligned}
\mathrm{E}[f(X_{S+t}), A] &= \sum_s \mathrm{E}[f(X_{s+t}), A \cap \{S = s\}] \\
&= \sum_s \mathrm{E}\big[\mathrm{E}[f(X_{s+t})|X_s], A \cap \{S = s\}\big] \\
&= \sum_s \mathrm{E}[(T_{s,s+t}f)(X_s), A \cap \{S = s\}\big] \\
&= \sum_s \mathrm{E}[(T_{S,S+t}f)(X_S), A \cap \{S = s\}\big] \\
&= \mathrm{E}[(T_{S,S+t}f)(X_S), A\big].
\end{aligned}
$$

Strong Markov with continuous transition operator

- Theorem 15.12: $\mathcal{X} = (X_t)_{t\in I}$ Markov process with right-continuous paths. If $T^{\mathcal{X}}_{s,t} f$ is continuous for $f \in \mathcal{C}_b(E)$ and $s \mapsto T^{\mathcal{X}}_{s,s+t} f$ continuous for all $f \in \mathcal{C}_b(E)$ (with respect to the supremum norm on $\mathcal{C}_b(E)$), then $\mathcal{X}$ is strong Markov.
- Proof: Let $S_1, S_2, ...$ stopping times with $S_n \downarrow S$ so that S_n only takes discrete values, $n = 1, 2, ...$ Then, $X_{S_n} \xrightarrow{n\to\infty} X_S$ and

$$\begin{aligned} \mathrm{E}[f(X_{S+t})|\mathcal{F}_S] &= \lim_{n\to\infty} \mathrm{E}[\mathrm{E}[f(X_{S_n+t})|\mathcal{F}_{S_n}]|\mathcal{F}_S] \\ &= \lim_{n\to\infty} \mathrm{E}[(T^{\mathcal{X}}_{S_n,S_n+t} f)(X_{S_n})|\mathcal{F}_S] \\ &= \mathrm{E}[(T^{\mathcal{X}}_{S,S+t} f)(X_S)|\mathcal{F}_S] = (T^{\mathcal{X}}_{S,S+t} f)(X_S). \end{aligned}$$

PPP and BM are strong Markov

- For the PPP(λ) $\mathcal{X}$, we have $(T^{\mathcal{X}}_{s,t}f)(x) = \mathrm{E}[f(x+P)]$, where $P \sim \mathrm{Poi}(\lambda(t-s))$.
- For the BM $\mathcal{X}$, $(T^{\mathcal{X}}_{s,t}f)(x) = \mathrm{E}[f(x+\sqrt{t-s}Z)]$, where $Z \sim N(0,1)$.

A Markov process which is not strong Markov

- $T \sim \exp(1)$. Define $\mathcal{X} = (X_t)_{t\geq 0}$ via

$$X_t = (t - T)^+$$

and completion of the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$. Then,

$$(T^{\mathcal{X}}_{s,s+t})f(x) = 1_{x=0}\mathrm{E}[f((t-T)^+)]+1_{x>0}f(x+t), \qquad f \in \mathcal{B}(\mathbb{R}).$$

Now consider the option time $S = T$. Since $\{T = t\}$ is a nullset and $\mathcal{F}_t$ is complete, S is a stopping time. Now,

$$\mathrm{E}[f(X_{S+t})|\mathcal{F}_S] = f(t).$$

On the other hand, $X_S = 0$ and therefore

$$(T^{\mathcal{X}}_{S,S+t}f)(X_S) = (T^{\mathcal{X}}_{S,S+t}f)(0) = \mathrm{E}[f((t - T)^+)].$$