universität freiburg

Stochastic processes **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 7 - Martingales III

Hint: In Exercises 1 and 2, you can use the results from Chapter 14 also for the time-continuous martingale $\mathcal{B}$.

Exercise 1 (4 Points)**.**
This is a follow-up of Exercise 4 in Tutorial 5! Let $\mathcal{B} = (B_t)_{t\geq 0}$ be a Brownian Motion, started in $B_0 = 0$. For a constant $a > 0$ define $T := \inf\{t \geq 0 : B_t \notin (-a,a)\}$. Show that for every $\lambda > 0$,

$$\mathbf{E}[\exp(-\lambda T)] = (\cosh(a\sqrt{2\lambda}))^{-1}$$

Solution.
Since $X_t := \exp\left(-\frac{c^2}{2}t\right)\cosh(cB_t)$ is a martingale, the Optional Sampling Theorem implies

$$1 = \mathbf{E}[X_0] = \mathbf{E}[X_{t\wedge T}] \qquad \text{for all} \quad t \geq 0. \tag{1}$$

Moreover we have $X_{t\wedge T} \to X_T$ almost surely as $t \to \infty$ and it follows from the monotonicity and symmetry of the function $x \mapsto \cosh(x)$ that

$$\begin{aligned} |X_{t\wedge T}| \leq |\cosh(cB_{t\wedge T})| &\leq \cosh(ca) \\ \mathbf{E}[|\cosh(ca)|] = \cosh(ca) &< \infty. \end{aligned}$$

Consequently, Theorem 3.28 (the dominated convergence theorem) implies

$$\mathbf{E}[X_T] = \mathbf{E}[\lim_{t\to\infty} X_{t\wedge T}] = \lim_{t\to\infty} \mathbf{E}[X_{t\wedge T}] = 1, \tag{2}$$

where we use Equation (1). On the other hand, the symmetry of $x \mapsto \cosh(x)$ implies that

$$\begin{aligned} \mathbf{E}[X_T] &= \mathbf{E}\left[e^{-\frac{c^2}{2}T}\cosh(cB_T)\mathbb{1}_{B_T=a}\right] + \mathbf{E}\left[e^{-\frac{c^2}{2}T}\cosh(cB_T)\mathbb{1}_{B_T=-a}\right] \\ &= \mathbf{E}\left[e^{-\frac{c^2}{2}T}\mathbb{1}_{B_T=a}\right]\cosh(ca) + \mathbf{E}\left[e^{-\frac{c^2}{2}T}\mathbb{1}_{B_T=-a}\right]\cosh(-ca) \\ &= \mathbf{E}\left[e^{-\frac{c^2}{2}T}\right]\cosh(ca) \end{aligned}$$

Together with (2) this results in

$$\mathbf{E}[e^{-\lambda T}] = \left(\cosh(\sqrt{2\lambda}a)\right)^{-1}$$

for each $\lambda > 0$.

Exercise 2 (4 points)**.**
For constants a, $b > 0$ define $T := \inf\{t \geq 0 : B_t = a + bt\}$. Show that for each $\lambda > 0$, we have

$$\mathbf{E}[e^{-\lambda T}] = \exp^{-a(b+\sqrt{b^2+2\lambda})}).$$

Hint: Use exercise 1 with $c = b + \sqrt{b^2 + 2\lambda}$.

Solution.
Define $c := b + \sqrt{b^2 + 2\lambda}$ and $X_t := \exp(cB_t - \frac{c^2}{2}t)$ for each $t \geq 0$. We claim that X_t is a martingale! (We can easily show this in class!) Since X_t is a martingale, the Optional Sampling Theorem implies

$$1 = \mathbf{E}[X_0] = \mathbf{E}[X_{t\wedge T}] \qquad \text{for all} \quad t \geq 0. \tag{3}$$

Moreover we have $X_{t\wedge T} \to X_T$ almost surely as $t \to \infty$ and

$$\begin{aligned} |X_{t\wedge T}| &\leq \exp\left(c(a + b(t \wedge T)) - \frac{1}{2}c^2(t \wedge T)\right) \\ &= \exp(ca)\exp\left((cb - \frac{1}{2}c^2)(t \wedge T)\right) \\ &\leq \exp(ca), \end{aligned}$$

because $cb - \frac{1}{2}c^2 = -\lambda < 0$. Since

$$\mathbf{E}[|\exp(ca)|] = \exp(ca) < \infty,$$

Theorem 3.28 (dominated convergence!) implies

$$\mathbf{E}[X_T] = \mathbf{E}[\lim_{t\to\infty} X_{t\wedge T}] = \lim_{t\to\infty} \mathbf{E}[X_{t\wedge T}] = 1, \tag{4}$$

where we use Equation (3). On the other hand, a straightforward calculation yields

$$\mathbf{E}[X_T] = \mathbf{E}[\exp(c(a + bT) - \frac{1}{2}c^2T)] = \mathbf{E}[\exp((cb - \frac{1}{2}c^2)T)]\exp(ca).$$

Together with Equation (4) this results in

$$\mathbf{E}[e^{-\lambda T}] = \exp\left(-a(b + \sqrt{(}b^2 + 2\lambda)\right)$$

for each $\lambda > 0$.
Reflection: Can we solve this exercise for $b < 0$?

Exercise 3 (2+2 points)**.**
Let X_1,X_2,... be iid with $\mathbf{P}(X_1 = \pm 1) = \frac{1}{2}$ and $\mathcal{S} := (S_n)_{n=0,1,...}$ be the simple random walk given by $S_n = \sum_{i=1}^n X_i$. In addition, let $T_k := \min\{n : S_n = k\}$ be the hitting time of k.

(a) For $a \in \mathbb{N}$, let $T := T_a \wedge T_{-a}$. Is the stopped process $\mathcal{S}^T$ uniformly integrable?

(b) For $a \in \mathbb{Z}$, let $T := T_a$. Is the stopped process $\mathcal{S}^T$ uniformly integrable?

Solution.

(a) Define the stopped process:

$$\mathcal{S}_n^T = S_{n\wedge T} = \begin{cases} S_n & \text{for } n < T, \\ S_T & \text{for } n \geq T, \end{cases}$$

and recall that a family $\{Y_n\}$ is uniformly integrable if:

$$\lim_{K\to\infty} \sup_n \mathbb{E}[|Y_n| \cdot \mathbb{1}_{\{|Y_n|>K\}}] = 0.$$

For $K > a$, we have: $\mathbf{P}(|S_T| > K) = 0$, hence, $\mathbf{E}[|S_T| \cdot \mathbb{1}_{\{|S_T|>K\}}] = 0$. For $n < T$, since $|S_n| \leq n$, we also have: $\mathbf{E}[|S_n| \cdot \mathbb{1}_{\{|S_n|>K\}}] = 0$ for large K. Therefore,

$$\lim_{K\to\infty} \sup_n \mathbf{E}[|S_n| \cdot 1_{\{|S_n|>K\}}] = 0.$$

In particular, we recall Example 7.8 here: Let $Y \in \mathcal{L}^1$ and $(X_i)_{i\in I}$ with $\sup_i |X_i| \leq |Y|$. Then $(X_i)_{i\in I}$ is uniformly integrable since

$$\sup_{i\in I} \mathbf{E}[|X_i|; |X_i| > K] \leq \mathbf{E}[|Y|; |Y| > K] \xrightarrow{K\to\infty} 0$$

by dominated convergence. Thus, $\mathcal{S}^T$ is uniformly integrable.

(b) Now consider $T = T_a$ for $a \in \mathbb{Z}$. Here, we stop the process when it first hits the integer a. Similar to part (a), we analyze the properties of the stopped process $\mathcal{S}^T = (S_{T\wedge n})_{n\geq 0}$. The process will remain at S_n until it hits a, at which point it stops. However, unlike in part (a), the random walk can either go to a or continue to $-\infty$ or $+\infty$ without bounds. Here, S_n can take on arbitrarily large values before hitting a. As a result, the values of S_n can become very large, and the expectation of $|S_{T\wedge n}|$ can also become large before stopping. To check for uniform integrability, we find that:

$$\mathbf{E}[|S_{T\wedge n}| \cdot \mathbb{I}_{\{|S_{T\wedge n}|>K\}}] \text{ may not tend to 0 for large } K.$$

Since S_n can take on larger values before hitting a, the stopped process $\mathcal{S}^T$ is not bounded, and the expectations can fail to converge to 0 for large K. Thus, the stopped process is not uniformly integrable.

Exercise 4 (4 points)**.**
A monkey randomly types one of the 26 capital letters on a keyboard every second. Let T be the time at which the monkey typed the word ABRACADABRA for the first time. It is claimed that it takes an average of $\mathbf{E}[T] = 26^{11} + 26^4 + 26$ seconds for the monkey to type the word for the first time. How can this be seen with the help of the Optional Sampling/Stopping Theorems?

Hint: Construct a fair game in which the players have a starting capital of 1€ and in each round bet their entire capital on the monkey typing the next correct letter of the word (each new player initially bets on A).

Solution.
Consider the following game. A player bets 1€ on the letter A. If she wins, she receives 26€ and bets everything on the letter B; if she loses, she stops. If she also wins on the letter B, she receives 26^2€ and bets everything on R and so on. Before each guess, a new player starts and bets on A. The game is over when a player receives 26^{11}€ for the first time. This game is fair because there are exactly 26 letters. This means that the total winnings after a time t of all players is a martingale. Denote this with X_t. Now let T be the time at which the game stops. Initially, according to Borel-Cantelli, this is almost certainly finite, so the optional stopping theorem holds that $X_{t \wedge T}$ is a martingale. It is also certain that

$$X_T = (26^{11} - 1) + (26^4 - 1) + (26 - 1) - 8 - (T - 11)$$

So we get

$$0 = \mathbf{E}[X_T] = -\mathbf{E}[T] + 26^{11} + 26^4 + 26$$

Overall, we get the assertion.