universitätfreiburg

Measure theory for probabilists **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 9 - $\mathcal{L}^p$-spaces

Exercise 1 (4 Points)**.**
For f in $\mathcal{L}^1[a,b]$, define $||f|| = \int_a^b x^2|f(x)|dx$. Show that this is a norm on $\mathcal{L}^1[a,b]$.

Solution.
To show that the mapping $||f|| = \int_a^b x^2|f(x)|\,dx$ defines a norm on the space $\mathcal{L}^1[a,b]$, we need to verify the three properties of a norm (see the footnote on Page 36!):

(i) For any $f \in \mathcal{L}^1[a,b]$,

$$||f|| = \int_a^b x^2|f(x)|\,dx \geq 0$$

because $x^2 \geq 0$ and $|f(x)| \geq 0$. Also, we have $||f|| = 0$ if and only if

$$\int_a^b x^2|f(x)|\,dx = 0.$$

Since x^2 is positive for $x \neq 0$, this integral can only be zero if $|f(x)| = 0$ almost everywhere on $[a,b]$. Therefore, $f = 0$ almost everywhere.

(ii) For any $c \in \mathbb{R}$ and $f \in \mathcal{L}^1[a,b]$,

$$||c \cdot f|| = \int_a^b x^2|c \cdot f(x)|\,dx = |c| \int_a^b x^2|f(x)|\,dx = |c| \cdot ||f||.$$

(iii) For $f,g \in \mathcal{L}^1[a,b]$,

$$||f + g|| = \int_a^b x^2|f(x) + g(x)|\,dx.$$

By the triangle inequality for the absolute value,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|.$$

Thus,

$$||f + g|| \leq \int_a^b x^2(|f(x)| + |g(x)|)\,dx = \int_a^b x^2|f(x)|\,dx + \int_a^b x^2|g(x)|\,dx = ||f|| + ||g||.$$

Since all three properties of a norm are satisfied, we conclude that $||f|| = \int_\alpha^b x^2|f(x)|\,dx$ defines a norm on $\mathcal{L}^1[\alpha,b]$. Thus, $(\mathcal{L}^1[\alpha,b],||\cdot||)$ is a normed space.

Exercise 2 (4 Points)**.**
For E a measurable set, and functions f in $\mathcal{L}^p(E)$, g in $\mathcal{L}^q(E)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, define

$$||f||_p = \left[\int_E |f|^p\right]^{\frac{1}{p}}.$$

Show that if Hölder's Inequality is true for normalized functions, it is true in general.

Solution.
We start by recalling the statement of Hölder's inequality. If f and g are measurable functions and $\frac{1}{p} + \frac{1}{q} = 1$, then Hölder's inequality states:

$$||fg||_{r=1} \leq ||f||_p ||g||_q.$$

For normalized functions, it means that we consider $||f||_p = 1$ and $||g||_q = 1$. Assume that f and g are normalized such that: $||f||_p = \left[\int_E |f|^p\right]^{\frac{1}{p}} = 1$, and $||g||_q = \left[\int_E |g|^q\right]^{\frac{1}{q}} = 1$. By the assumption of the normalized case, we want to show that $||fg||_r \leq ||f||_p ||g||_q$.
Now, let $f \in \mathcal{L}^p(E)$ and $g \in \mathcal{L}^q(E)$ be arbitrary functions. Define the normalized functions:

$$\tilde{f} = \frac{f}{||f||_p} \quad \text{and} \quad \tilde{g} = \frac{g}{||g||_q}.$$

Note that:

$$||\tilde{f}||_p = 1 \quad \text{and} \quad ||\tilde{g}||_q = 1.$$

Now apply Hölder's inequality to $\tilde{f}$ and $\tilde{g}$:

$$||\tilde{f}\tilde{g}||_r \leq ||\tilde{f}||_p ||\tilde{g}||_q = 1 \cdot 1 = 1.$$

Or more appropriately, we can write:

$$\begin{aligned} ||\tilde{f}\tilde{g}||_r \leq ||\tilde{f}||_p ||\tilde{g}||_q &= \left[\int_E \frac{|f|^p}{||f||_p^p}\right]^{\frac{1}{p}} \cdot \left[\int_E \frac{|g|^q}{||g||_q^q}\right]^{\frac{1}{q}} \\ &= \frac{1}{||f||_p}\left[\int_E |f|^p\right]^{\frac{1}{p}} \cdot \frac{1}{||g||_q}\left[\int_E |g|^q\right]^{\frac{1}{q}} \\ &= \frac{1}{||f||_p}||f||_p \cdot \frac{1}{||g||_q}||g||_q = 1 \end{aligned}$$

Finally, we write:

$$||\tilde{f}\tilde{g}||_r = ||\frac{f}{||f||_p} \cdot \frac{g}{||g||_q}||_r = \frac{||fg||_r}{||f||_p ||g||_q} \leq 1..$$

Multiplying both sides by $||f||_p ||g||_q$ gives:

$$||fg||_r \leq ||f||_p ||g||_q.$$

Exercise 3 (4 Points)**.**
Let $f : \Omega \to \mathbb{R}$ be measurable. Show that the following hold.

(a) If $\int |f|^p \, d\mu < \infty$ for some $p \in (0,\infty)$, then $||f||_p \xrightarrow{p\to\infty} ||f||_\infty$

(b) The integrability condition in (a) cannot be waived.

Solution.

(a) For p large enough, using the property of $||f||_\infty$, we know that:

$$|f(x)| \leq ||f||_\infty \quad \text{almost everywhere.}$$

Therefore:

$$||f||_p = \left(\int |f|^p \, d\mu\right)^{1/p} \leq \left(\int ||f||_\infty^p \, d\mu\right)^{1/p} = ||f||_\infty \cdot (\mu(\Omega))^{1/p}.$$

As $p \to \infty$, $(\mu(\Omega))^{1/p} \to 1$, thus: $||f||_p \leq ||f||_\infty$ for sufficiently large p. Now, we establish a lower bound. Let $M = ||f||_\infty$. Define the set:

$$A = \{x \in \Omega : |f(x)| > M - \varepsilon\}.$$

For any $\varepsilon > 0$, if $\mu(A) > 0$, we have:

$$||f||_p \geq \left(\int_A |f|^p \, d\mu\right)^{1/p} \geq \left(\int_A (M-\epsilon)^p \, d\mu\right)^{1/p} = (M-\epsilon)^p \cdot \mu(A)^{1/p}.$$

As $p \to \infty$, $\mu(A)^{1/p} \to 1$, leading to:

$$||f||_p \geq (M - \varepsilon) \quad \text{for sufficiently large } p.$$

Putting together the upper and lower bounds, we have:

$$(M-\varepsilon) \leq ||f||_p \leq ||f||_\infty \quad \text{for sufficiently large } p.$$

Taking the limit as $\epsilon \to 0$:

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

(b) To establish that the integrability condition in part (a) cannot be waived, we need to provide a counterexample where $||f||_p$ does not converge to $||f||_\infty$ when $\int |f|^p \, d\mu$ is not finite. Consider the function defined on the interval $\Omega = [0,1]$:

$$f_n(x) = n \quad \text{for } x \in \left[0, \frac{1}{n}\right], \quad f_n(x) = 0 \quad \text{for } x \in \left(\frac{1}{n}, 1\right].$$

We study the properties of the function as follows. For $p < \infty$:

$$||f_n||_p = \left(\int |f_n|^p \, d\mu\right)^{1/p} = \left(\int_0^{1/n} n^p \, dx\right)^{1/p} = \left(n^p \cdot \frac{1}{n}\right)^{1/p} = n^{(p-1)/p}.$$

As $n \to \infty$, $||f_n||_p \to \infty$ for any fixed $p < \infty$. Furthermore, we have:

$$||f_n||_\infty = \text{ess sup}_{x\in[0,1]} |f_n(x)| = n.$$

As $n \to \infty$: We find that $||f_n||_p \to \infty$ and $||f_n||_\infty \to \infty$, but the L^p norms diverge to infinity while $||f_n||_\infty$ also diverges. If we consider the condition $\int |f_n|^p \, d\mu < \infty$, we see that $\int |f_n|^p \, d\mu = \infty$ for any $p < \infty$ because:

$$\int |f_n|^p \, d\mu = \int_0^{1/n} n^p \, dx = n^p \cdot \frac{1}{n} = n^{p-1} \to \infty \text{ as } n \to \infty.$$

Exercise 4 (4 Points).
Let $p \in (1,\infty)$, $f \in \mathcal{L}^p(\lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$. Let $T : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + 1$. Show that

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow{n\to\infty} 0 \quad \text{in } \mathcal{L}^p(\lambda).$$

Solution.
We can rewrite the expression as:

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(x+k).$$

We want to show that:

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} f(x+k) \right\|_p = 0.$$

Taking the L^p norm, we have:

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} f(x+k) \right\|_p^p = \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{k=0}^{n-1} f(x+k) \right|^p d\lambda(x).$$

Using Fubini's theorem, we can interchange the sum and the integral:

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} f(x+k) \right\|_p^p = \frac{1}{n^p}\sum_{k=0}^{n-1} \int_{\mathbb{R}} |f(x+k)|^p \, d\lambda(x).$$

Using the change of variables $y = x + k$:

$$\int_{\mathbb{R}} |f(x+k)|^p \, d\lambda(x) = \int_{\mathbb{R}} |f(y)|^p \, d\lambda(y) = ||f||_p^p.$$

Thus, we have:

$$\frac{1}{n^p}\sum_{k=0}^{n-1} ||f||_p^p = \frac{1}{n^p} \cdot n \cdot ||f||_p^p = \frac{||f||_p^p}{n^{p-1}}.$$

As $n \to \infty$:

$$\frac{||f||_p^p}{n^{p-1}} \to 0 \quad \text{for } p > 1.$$

Therefore, we conclude that:

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \right\|_p = 0, \implies \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow{n\to\infty} 0 \quad \text{in } \mathcal{L}^p(\lambda).$$