universität freiburg

Measure theory **Winter semester 2025**

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2025WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 1 - Review of metric spaces and topologies I

Exercise 1.
Let $X = \{a,b,c,d\}$. Which of the following are topologies for X?

(a) $\{\emptyset, X, \{a\}, \{b\}, \{a,c\}, \{a,b,c\}, \{a,b\}\}$

(b) $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,d\}\}$

(c) $\{\emptyset, X, \{a,c,d\}, \{b,c,d\}\}$

Can you further give an example of two sets A and B of $\mathbb{R}$ such that

$$A \cap B = \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad A \cap \overline{B} \neq \emptyset.$$

Solution.

(i) Yes. Easy to check!

(ii) No! $\{a\} \cup \{b,d\} = \{a,b,d\}$ is in fact not included in the set.

(iii) No! $\{a,c,d\} \cap \{b,c,d\} = \{c,d\}$ is not included in the set.

It holds that $\overline{\mathbb{Q}} = \overline{\mathbb{R}\setminus\mathbb{Q}} = \mathbb{R}$. Thus, $A = \mathbb{Q}$ and $B = \mathbb{R}\setminus\mathbb{Q}$ works. Another example is $A = [0,1) \cup [2,3)$ and $B = [1,2)$.

Exercise 2 (4 Points)**.**
If X is a set and $r : X \times X \to \mathbb{R}_+$ is defined by

$$r(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Show that r is a metric on X.
Note: r is in fact called the discrete metric on X.

Solution.
From Definition A.1 (1), we see clearly that conditions (i) and (ii) are satisfied. To show that the triangle inequality holds (condition (iii)), we consider the possible cases for $x,y,z \in X$ and establish that

$$r(x,z) \leq r(x,y) + r(y,z).$$

Case 1: $x = y$ and $y = z$:

$$r(x,z) = 0, \quad r(x,y) = 0, \quad r(y,z) = 0 \implies 0 \leq 0 + 0.$$

Case 2: $x = y$ and $y \neq z$:

$$r(x,z) = 1, \quad r(x,y) = 0, \quad r(y,z) = 1 \implies 1 \leq 0 + 1.$$

Case 3: $x \neq y$ and $y = z$:

$$r(x,z) = 1, \quad r(x,y) = 1, \quad r(y,z) = 0 \implies 1 \leq 1 + 0.$$

Case 4 $x \neq y$ and $y \neq z$:

$$r(x,z) = 1, \quad r(x,y) = 1, \quad r(y,z) = 1 \implies 1 \leq 1 + 1.$$

Thus, r is a metric on X.

Exercise 3 (4 Points)**.**
Show that every mapping from a metric space (Ω,r) to a metric space (Ω',r') is continuous if r is the discrete metric.

Solution.
Let $f : (\Omega,r) \to (\Omega',r')$ and assume r is the discrete metric. Recall that f is continuous at $x \in \Omega$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in \Omega$,

$$r(x,y) < \delta \implies r'(f(x),f(y)) < \varepsilon.$$

Take any $x \in \Omega$ and $\varepsilon > 0$. Choose $\delta = 1$.

- If $y = x$, then $r(x,y) = 0 < \delta$, and $r'(f(x),f(y)) = r'(f(x),f(x)) = 0 < \varepsilon$.
- If $y \neq x$, then $r(x,y) = 1 \not< \delta$, so the implication is trivially true.

Therefore, f is continuous at x. Since x was arbitrary, f is continuous on Ω.

Exercise 4.
Given a metric space (Ω,r). Consider the topology generated by r and recall the definition of the open set in A.1. Then the following hold:

(a) the whole set and the empty set are open;

(b) the union of any collection of open subsets of Ω is open.

(c) the intersection of any two open subsets of Ω is open;

Solution.

(a) Clear!

(b) Let $\{U_i\}_{i \in I}$ be a collection of open sets in Ω. Define $U = \bigcup_{i \in I} U_i$. Take any $\omega \in U$. Then, there exists $j \in I$ such that $\omega \in U_j$. Since U_j is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(\omega) \subseteq U_j$ for $\omega \in U_j$. Thus, $B_\varepsilon(\omega) \subseteq U$, implying that for every $\omega \in U$, there exists a neighborhood contained in U. Therefore, U is open.

(c) It is already clear that the union of a collection of open sets is open. Let A and B then be open subsets of Ω. If these two sets are disjoint, then the intersection is empty, which is open. Otherwise, let $\omega \in A \cap B$. Since A and B are open, and both contain x, there exists positive ε_1 and ε_2 such that $B_{\varepsilon_1}(\omega) \subseteq A$ and $B_{\varepsilon_2}(\omega) \subseteq B$. Now, define $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$. Then the open ball $B_\varepsilon(\omega)$ is contained in $A \cap B$.

Exercise 5.

Is the set of rational numbers open or closed? Give any two examples of sets that are both open and closed.

Solution.

Note that $\mathbb{Q}$ is dense in $\mathbb{R}$. This means that provided any two real numbers, there exists an element of $\mathbb{Q}$. We can also verify that the irrational numbers are dense in $\mathbb{R}$. Now, the set of rational numbers is not open because every interval around a rational number contains an irrational number. The set of irrational numbers is also not open because every interval around an irrational number contains a rational number. Since a set is open if and only if its complement is closed, the set of rational numbers is also not closed.