Stochastic Processes
19. The reflection principle

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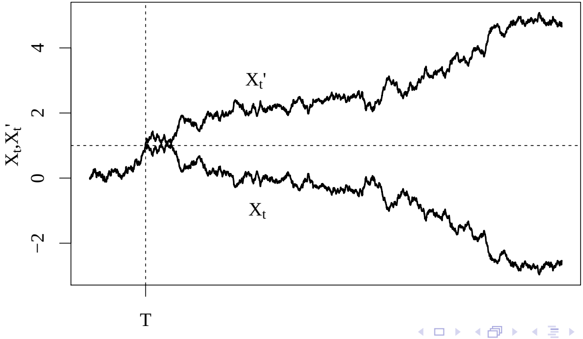
The reflection principle

- Lemma 16.7: $\mathcal{X}$ BM, T stopping time. Then $\mathcal{X}' = (X'_t)_{t\geq 0}$ with

$$X'_t := X_{t\wedge T} - (X_t - X_{t\wedge T}) = \begin{cases} X_t, & t \leq T, \\ 2X_T - X_t, & t > T \end{cases}$$

is also a BM.



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- Define

$$Y_t := X_{t\wedge T}, \qquad Z_t := X_{T+t} - X_T.$$

  $\mathcal{Z}$ is a BM. Hence, $(T, \mathcal{Y}, \mathcal{Z}) \stackrel{d}{=} (T, \mathcal{Y}, -\mathcal{Z})$. So, $(\mathcal{Y}, \mathcal{Z}^T) \stackrel{d}{=} (\mathcal{Y}, -\mathcal{Z}^T)$ with $\mathcal{Z}^T := (Z_t^T)_{t\geq 0}$, $Z_t^T := Z_{(t-T)^+}$.
  From this,

$$\mathcal{X} = \mathcal{Y} + \mathcal{Z}^T \stackrel{d}{=} \mathcal{Y} - \mathcal{Z}^T = \mathcal{X}'.$$

The maximum of BM

- Theorem 16.8: $\mathcal{X}$ BM with $X_0 = 0$. Define $\mathcal{M} = (M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} X_s$. Then, $M_t \stackrel{d}{=} M_t - X_t \stackrel{d}{=} |X_t|$.
- φ_t: density of X_t, $\Rightarrow 2\varphi_t(x)1_{x\geq 0}$: density of $|X_t|$; $T_x = \inf\{s \geq 0 : X_s = x\}$; $(X'_t)_{t\geq 0}$ BM mirrored at T; $y \leq x$,

$$\mathbf{P}(M_t \geq x, X_t \leq y) = \mathbf{P}(X'_t \geq 2x - y) = \int_{2x-y}^{\infty} \varphi_t(z)dz,$$

$$\mathbf{P}(M_t \geq x) = \mathbf{P}(M_t \geq x, X_t \leq x) + \mathbf{P}(X_t \geq x) = 2\int_x^{\infty} \varphi_t(z)dz,$$

so $M_t \stackrel{d}{=} |X_t|$. We further calculate

$$\mathbf{P}(M_t - X_t \geq x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{\infty} \mathbf{P}(z \leq M_t \leq z + \varepsilon, X_t \leq z - x)dz$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{\infty} \mathbf{P}(M_t \geq z, X_t \leq z - x) - \mathbf{P}(M_t \geq z + \varepsilon, X_t \leq z - x)dz$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{\infty} \Big(\int_{z+x}^{\infty} \varphi_t(y)dy - \int_{z+x+2\varepsilon}^{\infty} \varphi_t(y)dy \Big) dz = \int_x^{\infty} 2\varphi(z)dz.$$