Stochastic Processes
8. Stopped Martingales

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Stopping times and stopped processes

- Remark 14.17: Given $\mathcal{X} = (X_t)_{t\in I}$, $(\mathcal{F}_t)_{t\in I}$ filtration.

  A random time is an $\bar{I}$-valued random variable T.

  T is called a stopping time if $\{T \le t\} \in \mathcal{F}_t$ for all $t \in I$.

  T stopping time defines the σ-algebra

  $$\mathcal{F}_T := \{A \in \mathcal{A} : A \cap \{T \le t\} \in \mathcal{F}_t, t \in I\}$$

  of the T-past.

  The hitting time of $B \in \mathcal{B}(E)$ is $T_B := \inf\{t : X_t \in B\}$.

  It is $X_T : \omega \mapsto X_{T(\omega)}(\omega)$ and $\mathcal{X}^T := (X_{T\wedge t})_{t\in I}$.

- $\mathcal{X}$ adapted, I countable: For $B \in \mathcal{B}(E)$, T_B is a stopping time:

  $$\{T_B \le t\} = \bigcup_{s\le t} \underbrace{\{X_s \in B\}}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t} \in \mathcal{F}_t.$$

Optional Stopping Theorem

- Proposition 14.19: $I = \{0, 1, 2, ...\}$, T stopping time

  $\mathcal{X} = (X_t)_{t \in I}$ a (sub, super) martingale

  $\Rightarrow \mathcal{X}^T = (X_{T \wedge t})_{t \in I}$ is (sub-, super-) martingale.

  Proof for $\mathcal{X}$ sub-martingale: For $\{T > t-1\} \in \mathcal{F}_t$ is

$$\begin{aligned}\mathbf{E}[X_{T\wedge t} - X_{T\wedge(t-1)} | \mathcal{F}_{t-1}] &= \mathbf{E}[(X_t - X_{t-1}) 1_{\{T > t-1\}} | \mathcal{F}_{t-1}] \\ &= 1_{\{T>t-1\}} \mathbf{E}[X_t - X_{t-1} | \mathcal{F}_{t-1}] \geq 0,\end{aligned}$$

  i.e. $\mathcal{X}^T$ is a submartingale.

Martingale and $\mathcal{F}_T$

- Lemma 14.20: $I = \{0, 1, 2, ...\}$, $\mathcal{X}$, martingale, $T \leq t$ stopping time. Then $X_T = \mathbf{E}[X_t|\mathcal{F}_T]$.

  Since X_T is measurable with respect to $\mathcal{F}_T \subseteq \mathcal{F}_t$ (see Proposition 13.23), it suffices to show

$$\mathbf{E}[X_t; A] = \mathbf{E}[X_T; A], \qquad A \in \mathcal{F}_T$$

  For $s \in I$, $\{T = s\} \cap A \in \mathcal{F}_s$, hence

$$\begin{aligned}\mathbf{E}[X_T; A] &= \sum_{s=1}^{t} \mathbf{E}[X_s; \{T = s\} \cap A] = \sum_{s=1}^{t} \mathbf{E}[\mathbf{E}[X_t|\mathcal{F}_s]; \{T = s\} \cap A] \\ &= \sum_{s=1}^{t} \mathbf{E}[X_t; \{T = s\} \cap A] = \mathbf{E}[X_t; A].\end{aligned}$$

Uniform integrability

- Lemma 14.21: $I = \{0, 1, 2, ...\}$, $\mathcal{X} = (X_t)_{t\in I}$ martingale.

  Then

  $$\mathcal{X} \text{ ui} \qquad \iff \qquad \{X_T : T \text{ almost surely finite stopping time}\} \text{ ui.}$$

  '$\Leftarrow$': clear. '$\Rightarrow$': Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ convex, $\frac{f(x)}{x} \xrightarrow{x\to\infty} \infty$, $\sup_{t\in I} \mathbf{E}[f(|X_t|)] =: L < \infty$, $T < \infty$ stopping time.

  Because $\mathbf{E}[X_t|\mathcal{F}_{T\wedge t}] = X_{T\wedge t}$ and $\{T \le t\} \in \mathcal{F}_{T\wedge t}$,

  $$\begin{aligned}\mathbf{E}[f(|X_T|), \{T \le t\}] &= \mathbf{E}[f(|X_{T\wedge t}|), \{T \le t\}] \\ &= \mathbf{E}[f(|\mathbf{E}[X_t|\mathcal{F}_{T\wedge t}]|), \{T \le t\}] \\ &\le \mathbf{E}[\mathbf{E}[f(|X_t|)|\mathcal{F}_{T\wedge t}], \{T \le t\}] \\ &= \mathbf{E}[f(|X_t|), \{T \le t\}] \le L.\end{aligned}$$

Optional Sampling Theorem

- Theorem 14.22: $I = \{0, 1, 2, ...\}$, $S \leq T < \infty$ stopping times, $\mathcal{X}$ sub-martingale. If T is bounded or $\mathcal{X}$ is uniformly integrable, then X_T is integrable and $X_S \leq \mathbf{E}[X_T|\mathcal{F}_S]$.

  Proof for $T \leq t$. Let $\mathcal{X} = \mathcal{M} + \mathcal{A}$ Doob decomposition. Then

$$
\begin{aligned}
X_S = M_S + A_S &= \mathbf{E}[M_t + A_S|\mathcal{F}_S] \\
&\leq \mathbf{E}[M_t + A_T|\mathcal{F}_S] \\
&= \mathbf{E}[\mathbf{E}[M_t|\mathcal{F}_T] + A_T|\mathcal{F}_S] \\
&= \mathbf{E}[M_T + A_T|\mathcal{F}_S] \\
&= \mathbf{E}[X_T|\mathcal{F}_S].
\end{aligned}
$$

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  Proof for $\mathcal{X}$ ui martingale, thus $\{X_{S\wedge t}, X_{T\wedge t} : t \in I\}$ ui.

  For $A \in \mathcal{F}_S$, $\{S \leq t\} \cap A \in \mathcal{F}_{S\wedge t}$,

$$\begin{aligned}
\mathbf{E}[X_T, A] &= \lim_{t\to\infty} \mathbf{E}[X_{T\wedge t}, \{S \leq t\} \cap A] \\
&= \lim_{t\to\infty} \mathbf{E}[\mathbf{E}[X_{T\wedge t} | \mathcal{F}_{S\wedge t}], \{S \leq t\} \cap A] \\
&= \lim_{t\to\infty} \mathbf{E}[X_{S\wedge t}, \{S \leq t\} \cap A] = \mathbf{E}[X_S, A].
\end{aligned}$$

Characterization of martingales

- Lemma 14.23: $I = \{0, 1, 2, ...\}$, $\mathcal{X}$ adapted. Then:

  $\mathcal{X}$ martingale $\iff$ $\mathbf{E}[X_S] = \mathbf{E}[X_T]$ for stopping times S, T, which only take two values.

  '$\Rightarrow$': This is clear from the Optional Sampling Theorem. '$\Leftarrow$': Let $s \le t$, $A \in \mathcal{F}_s$, and $T = s1_A + t1_{A^c}$ be a stopping time such that

  $$0 = \mathbf{E}[X_t - X_T] = \mathbf{E}[X_t] - \mathbf{E}[X_s, A] - \mathbf{E}[X_t, A^c] = \mathbf{E}[X_t - X_s, A].$$

  Since A was arbitrary, it follows that $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$, so $\mathcal{X}$ is a martingale.

Wald's Identities

- $X_1, X_2, ... \in \mathcal{L}^1$ independent,$\mu := \mathbf{E}[X_1] = \mathbf{E}[X_2] = ...$, and $S_t := \sum_{s=1}^t X_s$, $T \leq t$ stopping time. Then
$$\mathbf{E}[S_T] = \mathbf{E}[T]\mu.$$
Because $\mathcal{M}$ with $M_0 = 0$, $M_t = S_t - t\mu$ is a martingale, we have
$$0 = \mathbf{E}[M_T] = \mathbf{E}[S_T] - \mathbf{E}[T]\mu.$$
- $X_1, X_2, ... \in L^2$ with $\sigma^2 = \mathbf{V}[X_1] = \mathbf{V}[X_2] = ...$ and T independent, then $\mathbf{V}[S_T] = \mathbf{E}[T]\sigma^2 + \mathbf{V}[T]\mu^2$..
Indeed: $(M_t^2 - \langle M \rangle_t)_{t=0,1,2,...}$ martingale with $\langle M \rangle_t = t\sigma^2$, so
$$\mathbf{E}[T]\sigma^2 = \mathbf{E}[M_T^2] = \mathbf{V}[(S_T - T\mu)^2] = \mathbf{V}[S_T] - \mu^2\mathbf{V}[T].$$

Ruin problem

- $X_1, X_2, \ldots$ iid,

  $\mathbf{P}(X_1 = 1) = 1 - \mathbf{P}(X_1 = -1) = p := 1 - q \neq \frac{1}{2}$,

  $S_0 = k$ and $S_t = S_0 + \sum_{i=1}^t X_i$,

  $T := \inf\{t : S_t \in \{0, N\}\}$ and $p_k := \mathbf{P}(S_T = 0)$. Then

  $$p_k := \mathbf{P}(S_T = 0) = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}.$$

  Indeed:

  $$\mathbf{E}\left[\left(\frac{q}{p}\right)^{X_1}\right] = \frac{q}{p}p + \frac{p}{q}q = 1,$$

  hence $Y_t := \left(\frac{q}{p}\right)^S_t$ is a martingale. Hence

  $$\left(\frac{q}{p}\right)^k = \mathbf{E}[Y_0] = \mathbf{E}[Y_T] = p_k + (1 - p_k)\left(\frac{q}{p}\right)^N.$$