universität freiburg

Stochastic processes

Winter semester 2024

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 14 - Brownian motion III

Exercise 1 (4 Points).
Let $\mathbf{P}_x$ be the distribution of Brownian motion started at $x \in \mathbb{R}$. Let $a > 0$ and $\tau = \inf\{t \geq 0 : B_t \in \{0,a\}\}$. Use the reflection principle to show that, for every $x \in (0,a)$,

$$\mathbf{P}_x(\tau > T) = \sum_{n=-\infty}^{\infty} (-1)^n \mathbf{P}_x(B_T \in [na,(n+1)a]).$$

Exercise 2 (4 Points).
Let $(B_t)_{t\geq 0}$ be a standardized Brownian motion. Show that

$$\mathbf{P}(\sup_{s\leq t} B_s > \sqrt{2t \log\log\log t}) \xrightarrow{t\to\infty} 0.$$

Does this contradict the law of the iterated logarithm?

Exercise 3 (4 points).
Let $\mathcal{B} = (B_t)_{t\geq 0}$ and $\mathcal{B}' = (B'_t)_{t\geq 0}$ be Brownian motions and $T := \inf\{t \geq 0 \mid B_t = 0\}$. Now consider $\mathcal{X} = (X_t) := (B_{t\wedge T})$, a Brownian motion stopped at 0, and $\mathcal{Y} = (Y_t) := (|B'_t|)$, a Brownian motion mirrored at 0. Show that for all $t,x,y > 0$

$$\mathbf{P}_x(X_t \leq y) = \mathbf{P}^y(x \leq Y_t),$$

where $\mathbf{P}_x(\ \cdot\ ) := \mathbf{P}(\ \cdot\ |B_0 = x)$ and $\mathbf{P}^y(\ \cdot\ ) := \mathbf{P}(\ \cdot\ |B'_0 = y)$.

Hint: Consider that for a standardized Brownian motion $(W_t)_t$ both sides are identical to $\mathbf{P}(W_t \geq x - y) + \mathbf{P}(W_t \geq x + y)$. The reflection principle can be helpful here.

Exercise 4 (4 points).
Consider an independent and identically distributed sequence of random variables $(X_i)_{i\geq 1}$. Let $S_n = \sum_{i=1}^n X_i$. Using Skohorod's embedding theorem, show that if $\mathbf{E}[X_1] = 0$, and $\mathbf{E}[X_1^2] = 1$, then

$$\frac{S_n}{\sqrt{n}} \xRightarrow{n\to\infty} Z \sim N(0,1).$$