Stochastic Processes
21. Donsker's Theorem

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Random walk $\Rightarrow$ BM

- $Y_1, Y_2, \ldots$ iid with $E[Y_1] = 0$ and $V[Y_1] = \sigma^2$. Set
$$\widetilde{X}_{n,t} := \frac{Y_1 + \cdots + Y_{\lfloor nt \rfloor}}{\sqrt{n\sigma^2}}$$
for $t \geq 0$. From the CLT, for $0 < t_1 < \cdots < t_k < \infty$ and $\mathcal{X}$ BM
$$(\widetilde{X}_{n,t_2} - \widetilde{X}_{n,t_1}, \ldots, \widetilde{X}_{n,t_k} - \widetilde{X}_{n,t_{k-1}}) \xRightarrow{n\to\infty} (X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}),$$
i.e. $\widetilde{X}_n \xRightarrow{n\to\infty} X$ wrt the finite dimensional distributions.
$$\text{This does not mean } \widetilde{\mathcal{X}}_n \xRightarrow{n\to\infty} \mathcal{X}!$$
- Define
$$X_{n,t} := \widetilde{X}_{n,t} + (nt - \lfloor nt \rfloor)\frac{Y_{\lfloor nt \rfloor + 1}}{\sqrt{n\sigma^2}}.$$
Is it true that
$$\mathcal{X}_n \xRightarrow{n\to\infty} \mathcal{X}?$$

Continuous functions and compact convergence

- Definition 16.13: (E, r) a metric space. For $f, f_1, f_2, ... \in \mathcal{C}_E([0,\infty))$ let $f_n \xrightarrow{n\to\infty} f$ *uniform on compacta* if and only if $\sup_{0\leq s\leq t} r(f_n(s), f(s)) \xrightarrow{n\to\infty} 0$ for all $t > 0$.
- Lemma 16.14: E Polish with complete metric r. Then, the topology of uniform convergence on compacta on $\mathcal{C}_E([0,\infty))$ is separable. Moreover,

$$r_{\mathcal{C}}(f, g) := \int_0^\infty e^{-t} \cdot (1 \wedge \sup_{0\leq s\leq t} |r(f(s), g(s))|)dt$$

is a complete metric on $\mathcal{C}_E([0,\infty))$, which induces this topology. In particular, $\mathcal{C}_E([0,\infty))$ is Polish.

Convergence of stochastic processes

Definition 16.15: $\mathcal{X}$ stochastic processes with state space E.

1. If, for each $t_1, ..., t_k, k = 1, 2, ...,$

$$(X^n_{t_1}, ..., X^n_{t_k}) \xrightarrow{n\to\infty} (X_{t_1}, ..., X_{t_k}),$$

   we write

$$\mathcal{X}^n \xrightarrow{n\to\infty}_{fdd} \mathcal{X}.$$

2. If $\mathcal{X}, \mathcal{X}^1, \mathcal{X}^2, ...$ have paths in $\mathcal{C}_E([0,\infty))$ and

$$\mathcal{X}^n \xrightarrow{n\to\infty} \mathcal{X},$$

   where $\mathcal{X}, \mathcal{X}^1, \mathcal{X}^2, ...$ are random variables in $\mathcal{C}_E([0,\infty))$, we say that $\mathcal{X}^1, \mathcal{X}^2, ...$ converges in distribution to $\mathcal{X}$.

Weak and fdd convergence

- Proposition 16.16: Equivalent are:
  1. $\mathcal{X}^n \xRightarrow{n\to\infty} \mathcal{X}$.
  2. $\mathcal{X}^n \xRightarrow{n\to\infty}_{fdd} \mathcal{X}$ and $\{\mathcal{X}^n : n = 1, 2, ...\}$ is tight in $\mathcal{C}_E([0,\infty))$.
- Proof: '1.⇒2.': Tightness follows as in Corollary 9.18. In addition, $f \mapsto (f(t_1), ..., f(t_k))$ is continuous.

  '2.⇒1.': Define

  $$\mathcal{M} := \{f \mapsto \varphi(f(t_1), ..., f(t_k)) : \varphi \in \mathcal{C}_b(E^k)\} \subseteq \mathcal{C}_b(\mathcal{C}_E([0,\infty))).$$

  $\mathcal{M}$ is an algebra and separates points, according to Theorem 9.24 is therefore separating. Now the weak convergence follows from Proposition 9.27.

Arzela Ascoli

- Definition 16.17: For $f \in \mathcal{C}_E([0,\infty))$ we define the *modulus of continuity*

$$w(f,\tau,h) := \sup\{r(f(s),f(t)) : s,t \leq \tau, |t-s| \leq h\}.$$

- Theorem 16.18: $A \subseteq \mathcal{C}_E([0,\infty))$ is relatively compact $\iff$ $\{f(t) : f \in A\}$ for all $t \in \mathbb{Q}_+ := [0,\infty) \cap \mathbb{Q}$ is relatively compact and for all $\tau > 0$

$$\lim_{h\to 0} \sup_{f\in A} w(f,\tau,h) = 0.$$

Tightness in $\mathcal{C}_{\mathbb{E}}([0,\infty))$

- Theorem 16.19: $\mathcal{X}, \mathcal{X}^1, \mathcal{X}^2, \dots$ rvs with values in $\mathcal{C}_E([0,\infty))$.

  Then $\mathcal{X}^n \xrightarrow{n\to\infty} \mathcal{X}$ iff $\mathcal{X}^n \xrightarrow{n\to\infty}_{fdd} \mathcal{X}$ and

$$\lim_{h\to 0} \limsup_{n\to\infty} \mathrm{E}[w(\mathcal{X}^n, \tau, h) \wedge 1] = 0 \text{ for all } \tau > 0. \qquad (*)$$

- Proof: to show: $(*) \iff$ tightness of $(\mathcal{X}^n)_{n=1,2,\dots}$.

  $\Leftarrow$: Let $\tau > 0, \varepsilon > 0$ and $K \subseteq \mathcal{C}_E([0,\infty))$ compact with $\limsup_{n\to\infty} \mathrm{P}(\mathcal{X}^n \notin K) \le \varepsilon$. By Arzela-Ascoli choose h so that $w(f, \tau, h) \le \varepsilon$ for $f \in K$. Hence,

$$\limsup_{n\to\infty} \mathrm{E}[w(\mathcal{X}^n, \tau, h) \wedge 1] \le \varepsilon + \sup_{n=1,2,\dots} \mathrm{P}[w(\mathcal{X}^n, \tau, h) > \varepsilon] \le 2\varepsilon.$$

Tightness in $\mathcal{C}_{\mathbb{E}}([0,\infty))$

- Theorem 16.19: $\mathcal{X}, \mathcal{X}^1, \mathcal{X}^2, \ldots$ rvs with values in $\mathcal{C}_E([0,\infty))$. Then $\mathcal{X}^n \xrightarrow{n\to\infty} \mathcal{X}$ iff $\mathcal{X}^n \xrightarrow{n\to\infty}_{fdd} \mathcal{X}$ and
$$\lim_{h\to 0} \limsup_{n\to\infty} \mathrm{E}[w(\mathcal{X}^n, \tau, h) \wedge 1] = 0 \text{ for all } \tau > 0. \qquad (*)$$
- Proof: to show: $(*) \iff$ tightness of $(\mathcal{X}^n)_{n=1,2,\ldots}$.
$\Rightarrow$ with $\mathcal{X}^n \xrightarrow{n\to\infty}_{fdd} \mathcal{X}$: w is increasing in h, so
$$(*) \iff \lim_{h\to 0} \sup_{n=1,2,\ldots} \mathrm{P}[w(\mathcal{X}^n, \tau, h) > \varepsilon] = 0, \varepsilon > 0, \tau > 0$$
$$\iff \forall \varepsilon > 0, \exists h_n \downarrow 0, \sup_{n=1,2,\ldots} \mathrm{P}(w(\mathcal{X}^n, k, h_k) > 2^{-k}) \le 2^{-(k+1)}\varepsilon$$
Arzela-Ascoli $\Rightarrow B := \bigcap_{k=1}^\infty \left\{f : w(f, \tau_k, h_k) \le 2^{-k}\right\}$ is relatively compact and $\sup_{n=1,2,\ldots} \mathrm{P}(\mathcal{X}^n \notin B) \le \varepsilon$.

The main estimate

- Lemma 16.20: $Y_1, Y_2, \ldots$ iid with $\mathrm{E}[Y_1] = 0$ and $\mathrm{V}[Y_1] = \sigma^2 > 0$ and $S_n := Y_1 + \cdots + Y_n$. Then, for $r > 1$,
$$\mathrm{P}(\max_{1\le k\le n} S_k > 2r\sqrt{n}) \le \frac{\mathrm{P}(|S_n| > r\sqrt{n})}{1 - \sigma^2 r^{-2}}.$$
- Proof: Define $T := \inf\{k : |S_k| > 2r\sqrt{n}\}$. Then,
$$\begin{aligned}\mathrm{P}(|S_n| > r\sqrt{n}) &\ge \mathrm{P}(|S_n| > r\sqrt{n}, \max_{1\le k\le n} S_k > 2r\sqrt{n}) \\ &\ge \mathrm{P}(T \le n, |S_n - S_T| \le r\sqrt{n}) \\ &\ge \mathrm{P}(\max_{1\le k\le n} S_k > 2r\sqrt{n}) \cdot \min_{1\le k\le n} \mathrm{P}(|S_k| \le r\sqrt{n}).\end{aligned}$$
From Chebychev's inequality,
$$\min_{1\le k\le n} \mathrm{P}(|S_k| \le r\sqrt{n}) \ge \min_{1\le k\le n} 1 - \frac{\sigma^2 k}{r^2 n} = 1 - \frac{\sigma^2}{r^2}.$$

Donsker's Theorem

- Theorem 16.21: Setting as above and

$$X_{n,t} := \frac{1}{\sqrt{n\sigma^2}}\big(Y_1 + \cdots + Y_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Y_{\lfloor nt \rfloor + 1}\big)$$

  Then,
$$\mathcal{X}_n \xRightarrow{n\to\infty} \mathcal{X} \text{ BM.}$$

- Proof: Wlog $\sigma^2 = 1$. It remains to show $(*)$ from Theorem 16.19. Write $S_n := Y_1 + \cdots + Y_n$ and

$$\lim_{h\to 0} \frac{1}{h} \limsup_{n\to\infty} \mathrm{P}\Big(\sup_{0\le s\le h} |X_{n,t+s} - X_{n,t}| > \varepsilon \Big)$$
$$\le \lim_{h\to 0} \frac{1}{h} \limsup_{n\to\infty} \mathrm{P}\Big(\max_{k=1,\dots,\lceil nh\rceil} |S_k| > \varepsilon\sqrt{n} \Big)$$
$$\le \lim_{h\to 0} \frac{1}{h} \limsup_{n\to\infty} \mathrm{P}\Big(\frac{|S_{\lceil nh\rceil}|}{\sqrt{nh}} > \frac{\varepsilon}{2\sqrt{h}} \Big) \le \lim_{h\to 0} \frac{2}{h} \int_{\varepsilon/(2\sqrt{h})}^{\infty} \varphi(x)dx$$
$$= \lim_{h\to 0} \frac{2}{h} \frac{2\sqrt{h}}{\varepsilon} \varphi(\varepsilon/(2\sqrt{h})) = 0$$

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Then,

$$\mathcal{X}_n \xRightarrow{n\to\infty} \mathcal{X} \text{ BM}.$$

- Proof: Now let $\delta > 0$ and h be small enough for

$$\limsup_{n\to\infty} \mathrm{P}\Big(\sup_{0\le s\le h} |X_{n,t+s} - X_{n,t}| > \varepsilon\Big) \le \delta h.$$

$$\begin{aligned}
&\limsup_{n\to\infty} \mathrm{P}(w(\mathcal{X}_n, \tau, h) > 2\varepsilon) \\
&\le \limsup_{n\to\infty} \mathrm{P}(\sup_{k\le[\tau/h], 0\le s\le h} \{|X_{n,kh+s} - X_{n,kh}|\} > \varepsilon) \\
&\le \sum_{k=0}^{[\tau/h]} \limsup_{n\to\infty} \mathrm{P}(\sup_{0\le s\le h} \{|X_{n,kh+s} - X_{n,kh}|\} > \varepsilon) \le [\tau/h]\delta h \xrightarrow{h\to 0} \tau\delta.
\end{aligned}$$

Kolmogorov-Chentsov criterion for tightness

- Theorem 16.22: $\mathcal{X}_1, \mathcal{X}_2, \ldots$ spes with continuous paths. If $\{X_n(0) : n \in \mathbb{N}\}$ is tight and $\forall \tau > 0, \exists \alpha, \beta, C > 0$:
$$\sup_n \mathbf{E}[r(X_n(s), X_n(t))^\alpha] \leq C|t-s|^{1+\beta}, \qquad 0 \leq s, t \leq \tau.$$
Then $\{\mathcal{X}_n : n \in \mathbb{N}\}$ is tight in $\mathcal{C}_E([\infty))$.
- Proof: Wlog $\tau = 1$. Let $0 < \gamma < \beta/\alpha$. We use $\xi_{nk} := \max\{r(X_n(s), X_n(t)) : s, t \in D_k, |t-s| = 2^{-k}\}$ such that $w(\mathcal{X}_n, 1, 2^{-m}) \leq \sum_{k=m}^\infty \xi_{nk}$. We calculate
$$\sum_{k=0}^\infty 2^{\alpha\gamma k}\mathbf{E}[\xi_{nk}^\alpha] \leq C \sum_{k=0}^\infty 2^{(\alpha\gamma-\beta)k}.$$
So, there is a C' with $\sup_n \mathbf{E}[\xi_{nk}^\alpha] \leq C' 2^{-\alpha\gamma k}$.

Kolmogorov-Chentsov criterion for tightness

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- Proof: Wlog $\tau = 1$. Let $0 < \gamma < \beta/\alpha$. We use $\xi_{nk} := \max\{r(X_n(s), X_n(t)) : s, t \in D_k, |t-s| = 2^{-k}\}$ such that $w(\mathcal{X}_n, 1, 2^{-m}) \leq \sum_{k=m}^\infty \xi_{nk}$. From this,

$$\sup_n \mathrm{E}[w(\mathcal{X}_n, 1, 2^{-m})^\alpha \wedge 1] \leq \sup_n \mathrm{E}\Big[\Big(\sum_{k=m}^\infty \xi_{nk}\Big)^\alpha\Big]$$

$$\leq \sup_n \Big(\sum_{k=m}^\infty \mathrm{E}[\xi_{nk}^\alpha]^{1/\alpha}\Big)^\alpha \leq C'\Big(\sum_{k=m}^\infty e^{-\gamma k}\Big)^{1/\alpha} \xrightarrow{m\to\infty} 0.$$