universität freiburg

Measure theory for probabilists

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 8 - Measurable functions and the integral II

Exercise 1 (4 Points)**.**
Let $(\Omega,\mathcal{F})$,$(\Omega',\mathcal{F}')$,$(\Omega'',\mathcal{F}'')$ be measurable spaces and $f : \Omega \to \Omega'$ measurable and $Z : \Omega \to \Omega''$. Then, Z is $\sigma(f)$-measurable if and only if there is a $\mathcal{F}'/\mathcal{F}''$-measurable mapping $\varphi : \Omega' \to \Omega''$ with $\varphi \circ f = Z$.

Solution.
'$\Longleftarrow$': Assume there is a there is a $\mathcal{F}'/\mathcal{F}''$-measurable mapping $\varphi : \Omega' \to \Omega''$ with $\varphi\circ f = Z$, let us show that Z is $\sigma(f)$-measurable. That is $\forall\ A \in \mathcal{F}''$, $Z^{-1}(A) \in \sigma(f)$.. Since φ is $\mathcal{F}'/\mathcal{F}''$-measurable, then $\varphi^{-1}(A) \in \mathcal{F}'$. f is $\sigma(f)$-measurable, so $f^{-1}(\varphi^{-1}(A)) \in \sigma(f)$. But $Z^{-1} = (\varphi \circ f)^{-1} = f^{-1} \circ \varphi^{-1}$. Hence, $Z^{-1}(A) \in \sigma(f)$.
'$\Longrightarrow$': It suffices to consider the case $Z \geq 0$; otherwise, we write $Z = Z^+ - Z^-$. First, let $Z = 1_A$ for $A \in \sigma(f)$. Then there is an $A' \in \mathcal{F}'$ with $f^{-1}(A') = A$, i.e. $Z = 1_{f^{-1}(A')} = 1_{A'} \circ f$, i.e. $\varphi = 1_{A'}$ fulfills the statement. Due to linearity, the statement is also true for simple functions, i.e. finite linear combinations of indicator functions. In the general case, there are simple functions $Z_1,Z_2,\dots \geq 0$ with $Z_n \uparrow Z$. In addition, there are $\mathcal{F}'$-measurable functions φ_n with $Z_n = \varphi_n \circ f$. Then $\varphi = \sup_n \varphi_n$ is again $\mathcal{F}'$-measurable and, since $Z \geq 0$, and

$$\varphi \circ f = (\sup_n \varphi_n) \circ f = \sup_n(\varphi_n \circ f) = \sup_n Z_n = Z.$$

Exercise 2 (4 Points)**.**
Prove Theorem 3.25

Solution.
Let $a,b \in \mathbb{R}$ with $a < b$, and $f,f_1,f_2,\ldots : [a,b] \to \mathbb{R}$ be piecewise continuous. If $f_n \xrightarrow{n\to\infty} f$ uniformly, then (using $\int$ for the Riemann integral)

$$\int_a^b f_n(x)dx \xrightarrow{n\to\infty} \int_a^b f(x)dx.$$

We will use the results from Proposition 3.19 here. Since f_n converges uniformly to f on $[a,b]$, for every $\epsilon > 0$, there exists an integer N such that for all $n \geq N$ and for all $x \in [a,b]$:

$$|f_n(x) - f(x)| < \epsilon.$$

We need to show that:

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \to 0 \text{ as } n \to \infty.$$

This can be expressed as:

$$\left| \int_a^b (f_n(x) - f(x))\, dx \right|.$$

By the properties of integrals, we apply the triangle inequality:

$$\left| \int_a^b (f_n(x) - f(x))\, dx \right| \leq \int_a^b |f_n(x) - f(x)|\, dx.$$

Since $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a,b]$ when n is sufficiently large, we can bound the integral:

$$\int_a^b |f_n(x) - f(x)|\, dx < \int_a^b \epsilon\, dx = \epsilon(b-a).$$

Thus, for $n \geq N$:

$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| < \epsilon(b-a).$$

Since $b-a$ is a positive constant, we can say that:

$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| < \delta,$$

for any $\delta > 0$ by choosing ϵ appropriately. Specifically, we can choose $\epsilon = \frac{\delta}{b-a}$. Therefore, we conclude that:

$$\int_a^b f_n(x)\, dx \xrightarrow{n\to\infty} \int_a^b f(x)\, dx.$$

Exercise 3 (4 Points)**.**
Let λ be Lebesgue measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Find $f,f_1,f_2,... \in \mathcal{L}^1(\lambda)$ with $f_n \xrightarrow{n\to\infty} f$ almost everywhere, with $\int f_n d\lambda \xrightarrow{n\to\infty} \int f d\mu$, but the corresponding Riemann integrals do not converge.

Solution.
Let $f_n(x) = \frac{1}{n}$ if $x \in [0,1] \cap \mathbb{Q}$ (the rationals within $[0,1]$) and $f_n(x) = 0$ otherwise. That is,

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(x) = 0$ for all $x \in \mathbb{R}$. For any $x \in [0,1]$: If x is rational, $f_n(x) = \frac{1}{n} \to 0$ as $n \to \infty$; If x is not rational, $f_n(x) = 0$ for all n. Thus, $f_n(x) \to f(x) = 0$ for all $x \in [0,1]$, which means $f_n \to f$ almost everywhere. The Lebesgue integral of f_n:

$$\int_{\mathbb{R}} f_n\, d\lambda = \int_{[0,1]} f_n(x)\, d\lambda = \int_{[0,1]\cap\mathbb{Q}} \frac{1}{n}\, d\lambda = 0,$$

since the set of rational numbers has measure zero. So, $f,f_1,f_2,... \in \mathcal{L}^1(\lambda)$. Therefore, $\int f_n\, d\lambda \to 0 = \int f\, d\lambda$. The function f_n is discontinuous at every point in $[0,1]$: At any rational point, it is $\frac{1}{n}$ (non-zero) but 0 at nearby irrational points. At any irrational point, it is 0, but it takes the value $\frac{1}{n}$ at rational points nearby. Since f_n is discontinuous everywhere on $[0,1]$, it is **not Riemann integrable**.

Exercise 4 (4 Points)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with derivative f'. Show that f' is $\mathcal{B}(\mathbb{R})-\mathbb{B}(\mathbb{R})-$measurable.

Solution.
f is differentiable $\implies$ f is continuous and thus measurable. We consider:

$$f_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}};$$

f_n's are all measurable functions because f is measurable, and $x \mapsto x + \frac{1}{n}$ is also measurable. Recall that $g : (a,b) \to \mathbb{R}$ and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ exists, we say that g is differentiable at x, and we say that the above limit is the derivative of g at x, which we write as $g'(x)$. Moreover,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} = f'(x).$$

Hence, f' is measurable as a pointwise limit of measurable functions.