Stochastic Processes
10. Martingale convergence 2

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Martingale convergence theorems

- $I = \mathbb{N}_0$, $\mathcal{F}_\infty = \sigma(\bigcup_{t\in I} \mathcal{F}_t)$, $\mathcal{X}$ submartingale
- Theorem 14.29: Let $\sup_{t\in I} \mathbf{E}[X_t^+] < \infty$.

  Then there exists $X_\infty \in L^1$ with $X_t \xrightarrow{t\to\infty}_{fs} X_\infty$.
- Theorem 14.32: The following are equivalent:
  1. $\mathcal{X}$ is uniformly integrable.
  2. There exists X_∞ such that $(X_t)_{t\in I\cup\{\infty\}}$ is a sub-martingale.
  3. There exists X_∞ such that $X_t \xrightarrow{t\to u}_{fs,L^1} X_\infty$.
- Theorem 14.33: Let $\mathcal{X}$ be an L^p-bounded martingale.

  Then there exists $X_\infty \in L^p$ with $X_t \xrightarrow{t\uparrow u}_{fs,L^p} X_\infty$.

  Furthermore, $(|X_t|^p)_{t\in I}$ is uniformly integrable.

Example: Product of random variables

- $X_1, X_2, \ldots \geq 0$ independent with $\mathbf{E}[X_t] = 1$, $t = 1, 2, \ldots$ and $S_t := \prod_{s=1}^t X_s$. There is S_∞ with $S_t \xrightarrow{t\to\infty}_{as} S_\infty$.

  Claim: $\{S_t : t \in I\}$ ui $\quad \iff \quad \displaystyle\prod_{t=1}^\infty a_t > 0$ for $a_t := \mathbf{E}[\sqrt{X_t}]$.

  Indeed: $W_t := \prod_{s=1}^t \frac{\sqrt{X_s}}{a_s}$ is a martingale with $W_t \to_{fs} W_\infty$.

  '$\Leftarrow$': Because $a_t^2 = (\mathbf{E}[\sqrt{X_t}])^2 \leq \mathbf{E}[X_t] = 1$,

$$\sup_{t\in I} \mathbf{E}[W_t^2] = \sup_{t\in I} \mathbf{E}\Big[\prod_{s=1}^t \frac{X_s}{a_s^2}\Big] = \sup_{t\in I} \prod_{s=1}^t \frac{\mathbf{E}[X_s]}{a_s^2} \leq \frac{1}{\Big(\prod_{s=1}^\infty a_s\Big)^2} < \infty.$$

  So, $(W_t)_{t\in I}$ is L^2-bounded, hence ui, and thus also $(S_t)_{t\in I}$

  '$\Rightarrow$': Ang $\prod_{s=1}^t a_s \to 0$. Then $S_t = W_t^2 \Big(\prod_{s=1}^t a_s\Big)^2 \to 0$. If $\{S_t : t \in I\}$ ggi, $0 = \mathbf{E}[S_\infty] = \lim_{t\to\infty} \mathbf{E}[S_t] = 1$, Wds

Convergence of conditional expectations

- Theorem 14.36: $I = \mathbb{N}_0$, $\mathcal{F}_\infty = \sigma(\bigcup_{t \in I} \mathcal{F}_t)$. Then for $X \in \mathcal{L}^1$,

$$\mathbf{E}[X|\mathcal{F}_t] \xrightarrow{t \uparrow u}_{fs, L^1} \mathbf{E}[X|\mathcal{F}_\infty].$$

Proof: We consider the (ui) martingale $(\mathbf{E}[X|\mathcal{F}_t])_{t=0,1,\dots}$.

There is X_∞ with $\mathbf{E}[X|\mathcal{F}_t] \to_{as, L^1} X_\infty$.

Now, for $A \in \mathcal{F}_s$,

$$\mathbf{E}[X_\infty, A] = \lim_{t \to \infty} \mathbf{E}[\mathbf{E}[X|\mathcal{F}_t], A] = \mathbf{E}[X, A],$$

i.e. using $s \to \infty$, we have $X_\infty = \mathbf{E}[X|\mathcal{F}_\infty]$ for all $A \in \mathcal{F}_\infty$.

Martingale convergence theorem for backward martingales

- Theorem 14.37: $I = -\mathbb{N}_0$ and $\mathcal{F}_{-\infty} = \bigcap_{t\in I} \mathcal{F}_t$, $\mathcal{X}$ submartingale. The following are equivalent:
  1. There is $X_{-\infty} \in L^1$ with $X_t \xrightarrow{t\to-\infty}_{\text{as}, L^1} X_{-\infty}$
  2. $\inf_{t\in I} \mathbf{E}[X_t] > -\infty$.

  Then $(X_t)_{t\in I\cup\{-\infty\}}$ is also a submartingale.

  '1. $\Rightarrow$ 2.': $\inf\limits_{t\in I} \mathbf{E}[X_t] = \lim\limits_{t\to-\infty} \mathbf{E}[X_t] = \mathbf{E}[X_{-\infty}] > -\infty$.

  '2. $\Rightarrow$ 1.': as-convergence as in Thm 14.29 with $\sup_{t\in I} \mathbf{E}[X_t^-] < \infty$. With $Y_t := \mathbf{E}[X_t - X_{t-1}|\mathcal{F}_{t-1}] \geq 0$, $\mathbf{E}\Big[\sum_{t=0}^{-\infty} Y_t\Big] = \mathbf{E}[X_0] - \inf_{t\in I} \mathbf{E}[X_t] < \infty$, so $\sum_{t=0}^{-\infty} Y_t < \infty$. It is $A_t = \sum_{s\leq t} Y_s$ ui because $\mathbf{E}[A_0] < \infty$ and $(M_t)_{t\in I}$ for $M_t = X_t - A_t$ also ui. $\Rightarrow \mathcal{X}$ is ui.

The law of large numbers

- Example 14.38: $X_1, X_2, \ldots \in L^1$ iid, $I := \{\ldots, -2, -1\}$ and

$$S_t := \frac{1}{|t|} \sum_{s=1}^{|t|} X_s, \quad \mathcal{F}_t = \sigma(\ldots, S_{t-1}, S_t) = \sigma(S_t, X_{t+1}, X_{t+2}, \ldots).$$

$(S_t)_{t \in I}$ is a backward martingale, hence $S_t \xrightarrow{t \to -\infty}_{as, L^1} S_{-\infty}$. Since $S_{-\infty}$ is measurable with respect to $\mathcal{T}(X_1, X_2, \ldots)$, $S_{-\infty}$ must almost surely be constant. Since $(S_t)_{t \in I \cup \{-\infty\}}$ is a martingale, it follows that

$$\frac{1}{|t|} \sum_{s=1}^{|t|} X_s = S_t \xrightarrow{t \to -\infty}_{as, L^1} S_{-\infty} = \mathbf{E}[S_{-\infty}] = \mathbf{E}[S_{-1}] = \mathbf{E}[X_1].$$

Convergence and increasing processes

- Lemma 14.39: $\mathcal{M}$ an L^2-integrable martingale with $|M_t - M_{t-1}| \leq K \in \mathbb{R}_+$ for $t = 1, 2, \dots$ Then there exists a null set N such that

$$\{\langle \mathcal{M} \rangle_\infty < \infty\} \subseteq \{\lim_{t\to\infty} M_t \text{ exists}\} \cup N,$$

$$\{\langle \mathcal{M} \rangle_\infty = \infty\} \subseteq \{\lim_{t\to\infty} M_t / \langle \mathcal{M} \rangle_t = 0\} \cup N.$$

- Let

$$T_k := \inf\{t : \langle \mathcal{M} \rangle_t > k\}, \text{ thus } \{\langle \mathcal{M} \rangle_\infty < \infty\} = \bigcup_{k=1}^{\infty} \{T_k = \infty\}.$$

$$\sup_t \mathbf{E}[M^2_{T_k \wedge t}] = \sup_t \mathbf{E}[\langle \mathcal{M}^{T_k} \rangle_t] \leq k + K^2 \Rightarrow \lim_{t\to\infty} M_{T_k \wedge t} \text{ exists}$$

$$\{T_k = \infty\} \cap \{\lim_{t\to\infty} M_{T_k \wedge t} \text{ exists}\} = \{T_k = \infty\} \cap \{\lim_{t\to\infty} M_t \text{ exists}\}$$

Extension of the Borel-Cantelli lemma

- Theorem 14.40: Let $A_t \in \mathcal{F}_t$, $t = 0, 1, 2, \ldots$ and

  $X_s := \mathbf{P}(A_s | \mathcal{F}_{s-1})$. Then

$$\left\{ \sum_{t=1}^{\infty} X_t < \infty \right\} \subseteq \left\{ \sum_{t=1}^{\infty} 1_{A_t} < \infty \right\}.$$

- Because: Let $M_t = \sum_{s=1}^{t} 1_{A_s} - X_s$. It holds that

$$\langle \mathcal{M} \rangle_t = \sum_{s=1}^{t} \mathbf{E}[1_{A_s}^2 - X_s^2 | \mathcal{F}_{s-1}] = \sum_{s=1}^{t} X_s(1 - X_s) \leq \sum_{s=1}^{t} X_s$$

  and thus for $A := \left\{ \sum_{t=1}^{\infty} X_t < \infty \right\}$

$$A = A \cap \left\{ \langle \mathcal{M} \rangle_\infty < \infty \right\}$$

$$= A \cap \{ \lim_{t \to \infty} M_t \text{ existiert} \} \subseteq \left\{ \sum_{t=1}^{\infty} 1_{A_t} < \infty \right\}.$$