Stochastic Processes
17. Examples of Markov processes

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Generator of a semigroup

- $\mathcal{X} = (X_t)_{t \in I}$ be a time-homogeneous Markov process with operator semigroup $(T_t)_{t \in I}$. The *generator* of $\mathcal{X}$ is defined as
$$(Gf)(x) = \lim_{t \to 0} \frac{\mathbf{E}_x[f(X_t) - f(x)]}{t} = \lim_{t \to 0} \frac{1}{t}((T_t f)(x) - f(x)),$$
for all f for which the limit value exists.
$$\left(f(X_t) - \int_0^t (Gf)(X_s)ds\right)_{t \in I}$$
is a martingale for all $f \in \mathcal{D}(G)$.

Example: ODEs

- $\mathcal{X} = (X_t)_{t\geq 0}$ solution of the ODE

$$\frac{d}{dt}X_t = g(X_t)$$

where g is Lipshitz. Then, the generator of $\mathcal{X}$ is

$$\begin{aligned}(G^{\mathcal{X}}f)(x) &= \lim_{t\to 0}\tfrac{1}{t}(f(X_t) - f(x)) = \frac{d}{dt}(f(X_t))\Big|_{t=0} \\ &= \sum_{i=1}^{d}\frac{\partial f}{\partial x_i}(g(x))\cdot g_i(x) = (\nabla f)(g(x))\cdot g(x).\end{aligned}$$

Example: PPP

- Let $\mathcal{X} \sim \mathrm{PPP}(\lambda)$. Recall

$$Gf(x) = \lambda(f(x+1) - f(x)).$$

Hence,

$$X_t - \int_0^t \lambda(X_s + 1 - X_s)ds = X_t - \lambda t,$$

$$X_t^2 - \int_0^t \lambda(X_s + 1)^2 - X_s^2 ds = X_t^2 - \int_0^t \lambda(2X_s + 1)ds$$

are martingales.

Example: BM

- Let $\mathcal{X} \sim$ BM. Recall

$$G^{\mathcal{X}} f(x) = \tfrac{1}{2} f''(x).$$

Hence,

$$\left(X_t - \tfrac{1}{2}\int_0^t id''(X_s)ds\right)_{t\geq 0} = (X_t)_{t\geq 0},$$

$$\left(X_t^2 - \tfrac{1}{2}\int_0^t (id^2)''(X_s)ds\right)_{t\geq 0} = (X_t^2 - t)_{t\geq 0},$$

$$\left(\exp\left(\mu X_t - \tfrac{1}{2}\int_0^t \frac{(e^{\mu\cdot})''(X_s)}{e^{\mu X_s}} ds\right)\right)_{t\geq 0} = \left(\exp\left(\mu X_t - \tfrac{1}{2}\mu^2 t\right)\right)_{t\geq 0}$$

are martingales.

Example: Markov jump processes

- Given $X_s = x$, jump after $\exp(\lambda(x))$ distributed time according to $\mu(x,.)$.
- If λ is bounded, probability of > 1 jump by time t is $\mathcal{O}(t^2)$, so for $f \in \mathcal{C}_b(E)$

$$
\begin{aligned}
(Gf)(x) &= \lim_{t\to 0} \frac{\mathbf{E}_x[f(X_t) - f(x)]}{t} \\
&= \lim_{t\to 0} \tfrac{1}{t}\Big((e^{-\lambda(x)t} - 1)f(x) + \lambda(x)te^{-\lambda(x)t}\int \mu(x,dy)f(y)\Big) \\
&= \lambda(x)\int \mu(x,dy)\big(f(y) - f(x)\big)\,dy.
\end{aligned}
$$

- If $\lambda(x,dy) = \lambda(x)\mu(x,dy)$ is continuous, $\mathcal{X}$ is Feller.

Example: Markov jump processes

- Assume E is discrete. Using $f(y) = 1_{y=x}$, the master equation

$$\begin{aligned}
\frac{d}{dt}\mathbf{P}(X_t = x) &= \frac{d}{dt}\mathbf{E}[f(X_t)] = \mathbf{E}[(Gf)(X_t)] \\
&= \mathbf{E}\Big[\sum_{y\in E}\lambda(X_t, y)(1_{y=x} - 1_{X_t=x})\Big] \\
&= \sum_{z\in E}\mathbf{P}(X_t = z)\sum_{y\in E}\lambda(z, y)(1_{x=y} - 1_{x=z}) \\
&= \sum_{z\in E}\lambda(z, x)\mathbf{P}(X_t = z) - \lambda(x, z)\mathbf{P}(X_t = x).
\end{aligned}$$

is a linear system of ODEs.

Branching process in continuous time

- Individuals die at rate 1, and are replaced by $Z \sim \mu$ offspring.

$$Gf(x) = x \sum_{n=0}^{\infty} \mu(n)(f(x-1+n) - f(x)).$$

For $f_r(x) = r^x$: $Gf_r(x) = xr^{x-1} \sum_{n=0}^{\infty} \mu(n)(r^n - r)$

$$= xr^{x-1}(g_\mu(r) - r) = (g_\mu(r) - r)\frac{d}{dr} f_r(x).$$

So the function $u : (t, r) \mapsto \mathbf{E}_x[r^{X_t}]$ solves the equation

$$\frac{d}{dt} u(t, r) = (g_\mu(r) - r)\frac{d}{dr} u(t, r)$$

with the boundary conditions $u(0, r) = r^x, u(t, 1) = 1$.

Yule process

- This is the special case $\mu = \delta_2$, i.e. $g_\mu(r) = r^2$;

$$\frac{d}{dt}u(t,r) = -r(1-r)\frac{d}{dr}u(t,r)$$

For $u(0,r) = r, u(t,1) = 1$, the solution is

$$u(t,r) = \frac{e^{-t}r}{1 - r(1-e^{-t})}.$$

The rhs is the generating function of $\text{geo}(e^{-t})$, i.e. $X_t \sim \text{geo}(e^{-t})$. This can also be shown using the master equation

$$\frac{d}{dt}\mathbf{P}(X_t = x) = (x-1)\mathbf{P}(X_t = x-1) - x\mathbf{P}(X_t = x).$$

Extinction probability

- $T := \inf\{t : X_t = 0\}$ extinction time. Then,

  $\mathbf{P}_x(T < \infty) = \mathbf{P}_1(T < \infty)^x$ and $r := \mathbf{P}_1(T < \infty)$

  $$r = (1-h)r + h\sum_{n=0}^{\infty} \mu(n)r^n + o(h),$$

  so

  $$r = g_\mu(r)$$

- For binary branching, $g_\mu(r) = p\delta_0 + (1-p)\delta_2$ the only solution ≤ 1 is

  $$r = \frac{p}{1-p} \wedge 1.$$

Hitting times

- Let $E' \subseteq E$ and $T := T_{E'}$ hitting time of E'.

  Goal: calculate $u : x \mapsto \mathbf{E}_x[T]$.

  Since $u(x) = \mathbf{E}_x[T] = 0$ for $x \in E'$, with $\lambda(x) = \sum_y \lambda(x, y)$

  $$\begin{aligned} \mathbf{E}_x[T] &= (1 - h\lambda(x))\mathbf{E}_x[T + h] + \sum_y \mathbf{E}_x[T | X_h = y] \cdot \mathbf{P}(X_h = y) \\ &= \mathbf{E}_x[T] + h(1 - \lambda(x))\mathbf{E}_x[T] + \sum_y \lambda(x, y)\mathbf{E}_y[T] + O(h^2) \\ &= \mathbf{E}_x[T] + h(1 + G\mathbf{E}_\bullet[T]) + O(h^2). \end{aligned}$$

  Therefore, u must fulfill the equation

  $$\begin{aligned} Gu(x) &= -1, \qquad x \notin E', \\ u(x) &= 0, \qquad x \in E'. \end{aligned}$$

Birth-death processes

- If $E = \mathbb{Z}_+$ and $\lambda(x, y) = 0$ for $|x - y| > 1$, we have a birth-death process. Set

$$\lambda(n, n+1) =: \lambda_n, \qquad \lambda(n, n-1) =: \mu_n,$$

$$Gf(n) = \lambda_n(f(n+1) - f(n)) + \mu_n(f(n-1) - f(n)).$$

For $u(n) := \mathbf{E}_n[T_0], \quad u(0) = 0, \quad u(n) = \sum_{k=1}^{n} \frac{1}{\mu_k \pi_k} \sum_{j=k}^{\infty} \pi_j$

with $\pi_1 = 1$ and $\pi_i = \prod_{j=2}^{i} \frac{\lambda_{j-1}}{\mu_j}$.. Indeed,

$$Gu(n) = \lambda_n \frac{1}{\mu_{n+1}\pi_{n+1}} \sum_{j=n+1}^{\infty} \pi_j - \mu_n \frac{1}{\mu_n \pi_n} \sum_{j=n}^{\infty} \pi_j$$

$$= \frac{1}{\pi_n} \sum_{j=n+1}^{\infty} \pi_j - \frac{1}{\pi_n} \sum_{j=n}^{\infty} \pi_j = -1.$$