universität freiburg

Stochastic processes   **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 12 - Brownian motion I

Exercise 1 (4 Points)**.**
Let X be a normally distributed random variable, that is, $X \sim N(0,\sigma^2)$ for some $\sigma^2 > 0$. Show that

$$\mathbf{E}[X^k] = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \frac{k!}{2^{k/2}(k/2)!}\sigma^k, & \text{if } k \text{ is even.} \end{cases}$$

Hint: Apply partial integration and then induction.
Note: The above guarantees that a Brownian motion satisfies for each $\alpha \in \mathbb{N}$,

$$\mathbf{E}[r(X_s,X_t)^{2\alpha}] \leq C|t-s|^\alpha \qquad \text{for all } 0 \leq s \leq t,$$

for a constant $C > 0$. Theorem 13.8 (*Continuous modifications; Kolmogorov, Chentsov*) implies that there exists a modification of the Brownian motion with Hölder-continuous paths of every order $\beta < \frac{1}{2}$. (*See* Proposition 13.17!)

Exercise 2 (4 points)**.**
Let $(B_t)_{t\geq 0}$ be a Brownian motion with $B_0 \sim N(0,1)$ and consider the Ornstein-Uhlenbeck diffusion $(X_t)_{t\in\mathbb{R}}$ given by

$$X_t := e^{-t}B_{e^{2t}}, \qquad \forall t \in \mathbb{R}.$$

(a) Show that $X_t \sim N(0,1), \quad \forall t \in \mathbb{R}$.

(b) Show that the process $(X_t)_{t\in\mathbb{R}}$ is time reversible, that is, $(X_t)_{t\geq 0} \stackrel{d}{=} (X_{-t})_{t\geq 0}$.

Exercise 3 (4 Points)**.**
Let $\mathcal{X} = (X_t)_{t\geq 0}$ be a real-valued stochastic process with continuous paths. Show that, for all $0 \leq a \leq b$, the map $\omega \mapsto \int_a^b X_t(\omega)dt$ is measurable. Further, let B be a Brownian motion and let λ be the Lebesgue measure on $[0,\infty)$.

(a) Compute the expectation and variance of $\int_0^1 B_s ds$.

(b) Show that almost surely $\lambda\big(\{t : B_t = 0\}\big) = 0$.

(c) Compute the expectation and variance of

$$\int_0^1 \left(B_t - \int_0^1 B_s ds\right)^2 dt.$$

Exercise 4 (4 Points)**.**
Let B be a Brownian motion, $a < 0 < b$. Define the stopping time

$$T_{a,b} = \inf\{t \geq 0 : B_t \in \{a,b\}\}.$$

(a) Show that almost surely $T_{a,b} < \infty$ and that $\mathbf{P}(B_{T_{a,b}} = b) = -\frac{a}{b-a}$.

(b) From Example 14.17, $(B_t^2 - t)_{t\geq 0}$ is a martingale. Using this fact, show that $\mathbf{E}[T_{a,b}] = -ab$.

Exercise 5 (4 Points)**.**
Let $\mathbf{P}_x$ be the distribution of Brownian motion started at $x \in \mathbb{R}$. Let $a > 0$ and $\tau = \inf\{t \geq 0 : B_t \in \{0,a\}\}$. Use the reflection principle to show that, for every $x \in (0,a)$,

$$\mathbf{P}_x(\tau > T) = \sum_{n=-\infty}^{\infty} (-1)^n \mathbf{P}_x(B_T \in [na,(n+1)a]).$$