universität freiburg

Stochastic processes

Winter semester 2024

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 10 - Markov processes II

Exercise 1 (4 Points).
Let $\mathcal{X} = (X_t)_{t=0,1,2,\dots}$ be a stochastic process with state space E and $(\mathcal{F}_t)_{t=0,1,2,\dots}$ its filtration. Show that the following are equivalent:

(a) $\mathcal{X}$ is a Markov process.

(b) For all bounded and measurable functions $f : E \to \mathbb{R}$,

$$f(X_t) - \sum_{k=1}^{t} \mathbf{E}[f(X_s) - f(X_{s-1})|X_{s-1}]$$

is a martingale wrt the filtration $(\mathcal{F}_t)_{t=0,1,2,\dots}$

Exercise 2 (2+2 points).
Let $\lambda > 0$ and ν be a probability measure on $\mathbb{R}$ with $\nu(\{1\}) = \nu(\{-1\}) = \frac{1}{2}$. Furthermore, consider the family of Markov kernels $(P_{s,t})_{s\leq t}$given by:

$$P_{s,t}(x,\{x\}) = \frac{1}{2}\big(1 + e^{-\lambda(t-s)}\big), \quad P_{s,t}(x,\{-x\}) = \frac{1}{2}\big(1 - e^{-\lambda(t-s)}\big), \qquad x \in \{-1-1\}.$$

For $x \neq \pm 1$, it is $P_{s,t}(x,\cdot) = \delta_x$ the Dirac measure on x.

1. Show that the *Chapman-Kolmogorov* equations

$$P_{s,u}(x,A) = \int_{\mathbb{R}} P_{s,t}(x,dz)P_{t,u}(z,A)$$

hold for all $s \leq t \leq u, x \in \mathbb{R}$, and $A \subset \mathbb{R}$.

2. Let $\mathcal{X}$ be a Markov process with Markov kernels $(P_{s,t})_{s\leq t}$. Show that, if $X_0 \sim \nu$, then

$$P(X_t = 1) = P(X_t = -1) = \frac{1}{2}.$$

Exercise 3 (4 Points).
Let $E \subset \mathbb{R}$ be countable and let $\mathcal{X}$ be a Markov chain on E with transition matrix p and with the property that, for any x, there are at most three choices for the next step; that is, there exists a set $A_x \subset E$ of cardinality 3 with $p(x,y) = 0$ for all $y \in E \backslash A_x$. Let $d(x) := \sum_{y\in E}(y - x)p(x - y)$ for all $x \in E$.

(a) Show that $M_n := X_n - \sum_{k=0}^{n-1} d(X_k)$ defines a martingale M with square variation process $\langle M \rangle_n = \sum_{i=0}^{n-1} f(X_i)$ for a unique function $f : E \to [0,\infty)$.

(b) Show that the transition matrix p is uniquely determined by f and d.

Exercise 4 (4 Points)**.**
Let α, $\sigma^2 \in (0,\infty)$. Show that given $K_t(x,\cdot) := N(xe^{-\alpha t}, \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t}))$ for $t > 0$, $K_0(x,\cdot) := \varepsilon_x$ is a semigroup of Markov kernels, i.e:

$$K_{s+t}(x,B) = \int_{\mathbb{R}} K_t(y,B)K_s(x,dy) \qquad \forall (x,B) \in \mathbb{R} \times \mathcal{B} \text{ and } s,t \in \mathbb{R}_+.$$

Hint: The following equation simplifies the calculation:

$$\int_B \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{B-\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx.$$

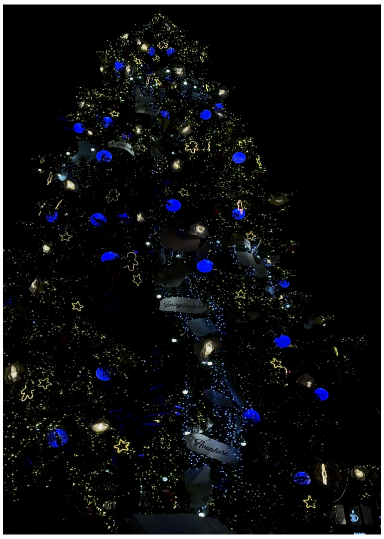

Figure 1: Merry Christmas!