Stochastic Processes
13. Markov Processes: Definition and examples

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Definition

$(\mathcal{F}_t)_{t\in I}$ filtration, $\mathcal{X} = (X_t)_{t\in I}$ adapted.

- $\mathcal{X}$ is a *Markov process* if $\mathcal{F}_s$ is indep. of X_t given X_s, $s \leq t$,

$$\mathbf{P}(X_t \in A|\mathcal{F}_s) = \mathbf{P}(X_t \in A|X_s) \text{ or}$$

$$\mathbf{E}(f(X_t)|\mathcal{F}_s) = \mathbf{E}(f(X_t)|X_s), \quad f \in \mathcal{C}_b(E).$$

- Markov kernels $\mu^{\mathcal{X}}_{s,t}$ and transition operator $T^{\mathcal{X}}_{s,t}$ of $\mathcal{X}$ are

$$\mu^{\mathcal{X}}_{s,t}(X_s, B) := \mathbf{P}(X_t \in B|X_s) = \mathbf{P}(X_t \in B|\mathcal{F}_s),$$

$$T^{\mathcal{X}}_{s,t}f(x) := \mathbf{E}[f(X_t)|X_s = x] = \int \mu^{\mathcal{X}}_{s,t}(x, dy)f(y).$$

- For Markov kernels μ, ν, define

$$(\mu \otimes \nu)(x, A \times B) = \int \mu(x, dy)\nu(y, dz)1_{y\in A, z\in B},$$

$$(\mu\nu)(x, A) = (\mu \otimes \nu)(x, E \times A).$$

Example: Markov chains

- If E is at most countable, $\mathcal{X}$ is a *Markov chain*. If $I = \{0, 1, 2, \dots\}$, $\mu^{\mathcal{X}}_{t,t+1}$ is given by matrices $(p_{t,t+1}(x,y))_{x,y\in E}$ so that

$$p_{t,t+1}(x,y) = \mathbf{P}(X_{t+1} = y | X_t = x),$$

$$\mu^{\mathcal{X}}_{t,t+1}(x,A) = \sum_{y\in A} p_{t,t+1}(x,y),$$

$$(T^{\mathcal{X}}_{t,t+1}f)(x) = \sum_{y\in E} \mu^{\mathcal{X}}_{t,t+1}(x,dy)f(y) = \sum_{y\in E} p_{t,t+1}(x,y)f(y).$$

Example: Sums

- $X_1, X_2, \dots$ independent. Then, $\mathcal{S} = (S_t)_{t=0,1,2,\dots}$ with $S_t = \sum_{s=1}^t X_s$ is a Markov process with
  $\mathbf{P}(S_{t+1} \in A | \mathcal{F}_t) = \int 1_{S_t \in A-x} \mathbf{P}(X_{t+1} \in dx) = \mathbf{P}(S_{t+1} \in A | S_t).$
  In this case
  $$\mu^{\mathcal{S}}_{t,t+1}(x, A) = \mathbf{P}(X_{t+1} \in A - x)$$
  and
  $$(T^{\mathcal{S}}_{t,t+1} f)(x) = \mathbf{E}[f(x + X_{t+1})].$$

Example: The Poisson Point Process

- $\mathcal{X} = (X_t)_{t \geq 0}$ PPP(λ). Then $(X_t)_{t\geq 0}$ is a Markov process with
$$\mu^{\mathcal{X}}_{s,t}(x, A) = \sum_{k \in A \cap \{x, x+1, \ldots\}} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{k-x}}{(k-x)!},$$
$$(T^{\mathcal{X}}_{s,t} f)(x) = \sum_{k=0}^{\infty} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!} f(x+k) = \mathbf{E}[f(x+P)],$$
where $P \sim \text{Poi}(\lambda(t-s))$.

Example: Brownian Motion

- $\mathcal{X} = (X_t)_{t \geq 0}$ Brownian motion is a Markov process with

$$\mu^{\mathcal{X}}_{s,t}(x, A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A \exp\Big(-\frac{(y-x)^2}{2(t-s)}\Big) dy$$

and the transition operator for $f \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned}(T^{\mathcal{X}}_{s,t} f)(x) &= \frac{1}{\sqrt{2\pi(t-s)}} \int \exp\Big(-\frac{y^2}{2(t-s)}\Big) f(x+y) dy \\ &= \mathbf{E}[f(x + \sqrt{t-s} Z)],\end{aligned}$$

where Z is a $N(0,1)$-distributed random variable.

Gaussian Markov processes

- Theorem 15.5: $\mathcal{X} = (X_t)_{t\geq 0}$ Gaussian. Then, $\mathcal{X}$ is Markov iff

$$\mathbf{COV}(X_s, X_u) \cdot \mathbf{V}(X_t) = \mathbf{COV}(X_s, X_t) \cdot \mathbf{COV}(X_t, X_u), \quad s \leq t \leq u.$$

- Wlog $\mathbf{E}[X_t] = 0$ and $\mathbf{V}(X_t) > 0$. Set

$$X'_u = X_u - \frac{\mathbf{COV}(X_t, X_u)}{\mathbf{V}(X_t)} X_t$$

such that $\mathbf{COV}(X'_u, X_t) = 0$, i.e. X'_u and X_t are independent.

$\Rightarrow$: $X_s \perp X_u$ given X_t, so $X_s \perp X'_u$ given X_t. So,

$$\mathbf{P}(X_s \in A, X'_u \in B) = \mathbf{E}[\mathbf{P}(X_s \in A | X_t) \cdot \mathbf{P}(X'_u \in B | X_t)]$$

$$= \mathbf{E}[\mathbf{P}(X_s \in A | X_t) \cdot \mathbf{P}(X'_u \in B)] = \mathbf{P}(X_s \in A) \cdot \mathbf{P}(X'_u \in B)$$

and therefore $X_s \perp X'_u$. This means that

$$0 = \mathbf{COV}(X_s, X'_u) = \mathbf{COV}(X_s, X_u) - \frac{\mathbf{COV}(X_t, X_u)}{\mathbf{V}(X_t)} \mathbf{COV}(X_s, X_t).$$

Gaussian Markov processes

- Theorem 15.5: $\mathcal{X} = (X_t)_{t\geq 0}$ Gaussian. Then, $\mathcal{X}$ is Markov iff

$$\mathbf{COV}(X_s, X_u)\cdot \mathbf{V}(X_t) = \mathbf{COV}(X_s, X_t)\cdot \mathbf{COV}(X_t, X_u), \quad s\leq t\leq u.$$

- Wlog $\mathbf{E}[X_t] = 0$ and $\mathbf{V}(X_t) > 0$. Set

$$X_u' = X_u - \frac{\mathbf{COV}(X_t, X_u)}{\mathbf{V}(X_t)} X_t$$

such that $\mathbf{COV}(X_u', X_t) = 0$, i.e. X_u' and X_t are independent.

$\Leftarrow$: As seen above, $X_s \perp X_u'$. So, $X_u' \perp \mathcal{F}_t := \sigma((X_s)_{s\leq t})$ and

$$\begin{aligned}\mathbf{P}(X_u\in A|\mathcal{F}_t) &= \int \mathbf{P}\Big(X_u'\in dx, \tfrac{\mathbf{COV}(X_t,X_u)}{\mathbf{V}(X_t)}X_t \in A - x|\mathcal{F}_t\Big) \\ &= \int \mathbf{P}\Big(X_u'\in dx, \tfrac{\mathbf{COV}(X_t,X_u)}{\mathbf{V}(X_t)}X_t \in A - x|X_t\Big) \\ &= \mathbf{P}(X_u\in A|X_t).\end{aligned}$$

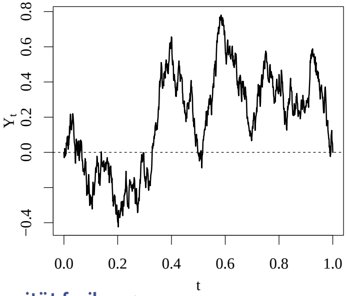
Examples

- BM is a Markov process.
- $(X_t)_{t\geq 0}$ BM, then $(Y_t := X_t - tX_1)_{t\geq 0}$ is the Brownian Bridge. For $s \leq t \leq u$,

  $$\mathbf{COV}(Y_s, Y_t) = \mathbf{COV}(X_s - sX_1, X_t - tX_1) = s - 2st + st = s(1-t),$$

  so $$\mathbf{COV}(Y_s, Y_u) \cdot \mathbf{V}(Y_t) = s(1-u)t(1-t)$$

  $$= \mathbf{COV}(Y_s, Y_t) \cdot \mathbf{COV}(Y_t, Y_u),$$



Spatially homogeneous Markov processes

- Definition 15.8: E Abelian group.
  1. A Markov kernel from E to E is called *homogeneous* if $\mu(x, B) = \mu(0, B - x)$ for all $x \in E$ and $B \in \mathcal{B}(E)$.
  2. A Markov process $\mathcal{X}$ is called *spatially homogeneous*, if the Markov kernels $\mu^{\mathcal{X}}_{s,t}$ are homogeneous, $s \leq t$.
  3. A Markov process $\mathcal{X} = (X_t)_{t \geq 0}$ has independent increments if $X_t - X_s$ is independent of $\mathcal{F}_s$, $s \leq t$.

Homogeneity and independent increments

- Lemma 15.9: $\mathcal{X} = (X_t)_{t\in I}$ Markov process. It has independent increments if and only if $\mathcal{X}$ is spatially homogeneous.
- $\Leftarrow$: $\mu^{\mathcal{X}}_{s,t}(x, B) = \mu^{\mathcal{X}}_{s,t}(0, B - x)$ for all $x \in E$ and $B \in \mathcal{B}(E)$.

$$\mathbf{P}(X_t - X_s \in B|\mathcal{F}_s) = \mu_{s,t}(X_s, X_s + B) = \mu^{\mathcal{X}}_{s,t}(0, B).$$

$\Rightarrow$: $(X_t - X_s)_{t\geq s}$ is a Markov process with

$$\begin{aligned}\mu^{\mathcal{X}}_{s,t}(X_s, B) &= \mathbf{P}(X_t \in B|\mathcal{F}_s) = \mathbf{P}(X_t - X_s \in B - X_s|\mathcal{F}_s)\\ &= \mu^{\mathcal{X}}_{s,t}(0, B - X_s).\end{aligned}$$