Measure theory for probabilists

Winter semester 2024

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Tutorial 13 - Product spaces I

Exercise 1 (4 Points)**.**
Prove Lemma 5.19 (Convolution of distributions with densities): let λ be a measure on $\mathcal{B}(\mathbb{R})$, $\mu = f_\mu \cdot \lambda$ and $\nu = f_\nu \cdot \lambda$ for measurable densities $f_\mu, f_\nu : \mathbb{R} \to \mathbb{R}_+$. Then $\mu * \nu = f_{\mu*\nu} \cdot \lambda$ with

$$f_{\mu*\nu}(t) = \int f_\mu(s) f_\nu(t-s)\lambda(ds).$$

Hint: Theorem 5.13 (Fubini's theorem)!

Exercise 2 (4 Points)**.**
Show the following convolution formulas.

(a) Let $f_{\Gamma(\theta,r)}$ and $f_{\Gamma(\theta,s)}$ be the density functions of two gamma distributions, where θ is the scale parameter and r and s are the shape parameters, with $\theta > 0$ and $r,s > 0$. Show that

$$f_{\Gamma(\theta,r)} * f_{\Gamma(\theta,s)} = f_{\Gamma(\theta,r+s)}.$$

(b) Let $f_{\mathrm{Cau}(r)}$ and $f_{\mathrm{Cau}(s)}$ be the density functions of two Cauchy distributions, where r and s are the scale parameters, with $r,s > 0$. Show that

$$f_{\mathrm{Cau}(r)} * f_{\mathrm{Cau}(s)} = f_{\mathrm{Cau}(r+s)}.$$

Note: The density function of a Gamma distribution and a Cauchy distribution is given (respectively) by

$$f_{\Gamma(\theta,r)}(x) = \frac{1}{\Gamma(r)\theta^r} x^{r-1} e^{-x/\theta} \quad \text{for } x > 0, \quad \text{and} \quad f_{\mathrm{Cau}(r)}(x) = \frac{1}{\pi r \left(1 + \left(\frac{x}{r}\right)^2\right)} \quad \text{for } x \in \mathbb{R}.$$

Exercise 3 (4 Points)**.**
Let μ_i, $i = 1,2$ be two σ-finite measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $D_i = \{x : \mu_i(\{x\}) > 0\}$, $i = 1,2$.

(a) Show that $D_1 \cup D_2$ is countable.

(b) Let $\varphi_i(x) = \mu_i(\{x\})$ for $x \in \mathbb{R}$, $i = 1,2$. Show that φ_i is Borel measurable for $i = 1,2$.

(c) Show that

$$\int \varphi_1 \, d\mu_2 = \sum_{z \in D_1 \cap D_2} \varphi_1(z)\varphi_2(z).$$

Exercise 4 (4 Points)**.**
Let X be the unit interval [0,1] with Lebesgue measure λ (and the Lebesgue σ-algebra $\mathcal{L}([0,1])$). Let Y be the unit interval [0,1] with counting measure μ (and the discrete σ-algebra $2^{[0,1]}$). Let $f := 1_E$ be the indicator function of the diagonal $E := \{(x,x) : x \in [0,1]\}$.

(a) Show that f is measurable in the product σ-algebra.

(b) Show that
$$\int_X \left(\int_Y f(x,y)\, d\mu(y) \right) d\lambda(x) = 1.$$

(c) Show that
$$\int_Y \left(\int_X f(x,y)\, d\lambda(x) \right) d\mu(y) = 0.$$

(d) Discuss why the results obtained in (b) and (c) do not contradict Fubini's theorem.