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Stochastic processes **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 8 - Martingale convergence results

Exercise 1 (4 Points)**.**
Let $(S_t)_{t=0,1,2,...}$ be a simple, symmetrical random walk and let T_K be the hitting time of $(-\infty, -K] \cup [K,\infty)$. Show that for all $t \geq 0$ and $K > 0$

$$\mathbf{P}(T_K \leq t) \leq \frac{\sqrt{t}}{K}.$$

Solution.
The hitting time T_K is defined as $T_K = \inf\{n \geq 0 : |S_n| \geq K\}$. The event $T_k \leq t$ means that at least one of the random walk steps $S_0, S_1, \dots, S_t$ reaches either K or $-K$. Thus, we can express this probability as: $\mathbf{P}(T_k \leq t) = \mathbf{P}(\max_{0\leq n\leq t} |S_n| \geq K)$. According to the maximum inequality (M) and Jensen (J), the following applies:

$$\begin{aligned} K^2 \cdot \mathbf{P}(T_K \leq t)^2 &= \left(K \cdot \mathbf{P}\left(\max_{0\leq n\leq t} |S_n| \geq K\right)\right)^2, \\ &\overset{M}{\leq} \mathbf{E}[|S_t|]^2, \\ &\overset{J}{\leq} \mathbf{E}[S_t^2] = \mathbf{V}[S_t] = \sum_{n=1}^{t} \mathbf{Var}[X_1] = t, \end{aligned}$$

and we are done.

Exercise 2 (2+2 points)**.**
Can you give an example (if it exists!) of a martingale $\mathcal{X} = (X_n)_{n=0,1,2,...}$...

(a) ...with $\lim_{n\to\infty} X_n = \infty$ almost surely?

(b) ... that converges in $\mathcal{L}^1$ but not almost surely?

Solution.

(a) For all n, let

$$\mathbf{P}\big(X_n = -(n^2 - 1)\big) = \frac{1}{n^2} = 1 - \mathbf{P}\big(X_n = 1\big),$$

and $(X_n)_n$ are independent. Then,

$$\begin{aligned} \mathbf{E}[X_n] &= \left(-(n^2-1)\cdot\frac{1}{n^2}\right) + \left(1\cdot\left(1 - \frac{1}{n^2}\right)\right) \\ &= -\frac{n^2-1}{n^2} + \left(1 - \frac{1}{n^2}\right) = -1 + \frac{1}{n^2} + 1 - \frac{1}{n^2} = 0. \end{aligned}$$

Furthermore, since $\mathbf{E}[X_n] = 0$ for all n, then the process $S_t := \sum\limits_{n=1}^{t} X_n$ is indeed a martingale with respect to its natural filtration. Now let us consider the limiting behaviour of S_n. The event $\{X_n < 0\}$ occurs with probability $\frac{1}{n^2}$, which means that for large n, X_n is negative with a decreasing probability. By the Borel-Cantelli lemma, since $\sum_{n=1}^{\infty} \mathbf{P}(X_n < 0) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows that $\{X_n < 0\}$ occurs only finitely often almost surely, so $S_n \to \infty$ almost certainly holds.

(b) No. Let $\mathcal{X} = (X_n)_{n\geq 1}$ be a martingale which converges in $\mathcal{L}^1$. Then $X_n \in \mathcal{L}^1$ applies to all $n \in \mathbb{N}$ and $X_n \xrightarrow{n\to\infty} X_\infty$. However, this means that $(X_n)_{n\geq 1}$ is equally integrable (Theorem 14.32) and therefore $(X_n)_{n\geq 1}$ converges almost surely.

Exercise 3 (2+2=4 Points)**.**
Let $p \in [0,1]$ and let $\mathcal{X} = (X_n)_{n\in\mathbb{N}_0}$ be a stochastic process with values in [0,1]. For each $n \in \mathbb{N}_0$ given $X_0, \ldots, X_n$, we have

$$X_{n+1} = \begin{cases} 1 - p + pX_n & \text{with probability } X_n, \\ pX_n & \text{with probability } 1 - X_n. \end{cases}$$

(a) Show that $\mathcal{X}$ is a martingale.

(b) Show that $\mathcal{X}$ converges almost surely in $\mathcal{L}^1$.

(c) Determine the distribution of the limit value $X := \lim\limits_{n\to\infty} X_n$.

Hint: For (c), consider the process $\big(X_n(1 - X_n)\big)_n$

Solution.

(a) Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Initially it applies that

$$\mathbf{E}[|X_n|] = (1 - p + pX_{n-1})X_{n-1} + pX_{n-1}(1 - X_{n-1}) = 1 < \infty.$$

Further is

$$\mathbf{E}[X_{n+1}|\mathcal{F}_n] = X_n(1 - p + pX_n) + (1 - X_n)pX_n = X_n(1 - p) + pX_n = X_n.$$

(b) (X_n) is non-negative and therefore almost certainly convergent. Furthermore it is bounded, thus equally integrable and therefore $\mathcal{L}^1$-convergent.

(c) The process $\big(X_n(1-X_n)\big)_n$ is a super-martingale because, similar to above, $\mathbf{E}[|X_n(1-X_n)|] < \infty$ and

$$\begin{aligned} \mathbf{E}[X_{n+1}(1 - X_{n+1})|\mathcal{F}_n] &= X_n(1 - p + pX_n)(p - pX_n) + (1 - X_n)pX_n(1 - pX_n) \\ &= pX_n(1 - X_n)(1 - p + pX_n + 1 - pX_n) \\ &= p(2 - p)X_n(1 - X_n) \\ &\leq X_n(1 - X_n) \end{aligned}$$

and just as non-negative, i.e. just as almost certain and $\mathcal{L}^1$-convergent.

Now, consider: $\mathbf{P}_* X_\infty$: Inductively, this results in

$$\mathbf{E}[X_n(1-X_n)] = \big(p(2-p)\big)^n \mathbf{E}[X_0(1-X_0)] \xrightarrow{n\to\infty} 0,$$

since $p(2-p) < 1$ for $p < 1$. Therefore, X_∞ can only assume the values 0 and 1. Since $\mathcal{X}$ is a $\mathcal{L}^1$-convergent martingale, it must be $\mathbf{E}[X_\infty] = \mathbf{E}[X_0]$. This means that $\mathbf{P}(X_\infty = 1) = \mathbf{E}[X_0]$ and $\mathbf{P}(X_\infty = 0) = 1 - \mathbf{E}[X_0]$. In other words,

$$\mathbf{P}_* X_\infty = \mathbf{E}[X_0]\delta_1 + (1 - \mathbf{E}[X_0])\delta_0.$$

(Interestingly, the distribution of the limit value is independent of p.)

Exercise 4 (2+2=4 Points)**.**

(a) Give an example of a non-negative square integrable martingale that converges almost surely but not in $\mathcal{L}^2$.

(b) Show that for $p = 1$, the statement in Theorem in 14.33 may fail. Give an example of a non-negative martingale $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ with $\mathbf{E}[X_n] = 1$ for all $n \in \mathbb{N}$ but such that $X_n \xrightarrow{n\to\infty} 0$ almost surely.

Solution.

(a) Consider a symmetric simple random walk: X_1,$X_2, \ldots$ be iid random variables with $\mathbf{P}(X_1 = -1) = \mathbf{P}(X_1 = 1) = \frac{1}{2}$ and $(S_t)_{t\in\mathbb{N}_0}$ is defined as $S_0 = 1, \quad S_t = \sum_{i=1}^t X_i$. Furthermore, define $T = \inf\{t \geq 0 : S_t = 0\}$ (that is, the process stops when the random walk first returns to zero). Then, by the Optional Sampling Theorem, the stopped process $S^T = (S_{T\wedge t})$ is a non-negative martingale. Since the stopped martingale converges almost surely to zero when $T < \infty$ (the random walk will eventually return to zero with probability 1!), we have $S_{T\wedge t} \to_{as} 0$. However, the stopped process does not converge in $\mathcal{L}^2$ since $\infty > \mathbf{E}[S^2_{T\wedge t}] \geq 0$. (We can easily verify that the stopped process is square integrable!)

(b) Take a painstaking look at the example in (a). $\mathbf{E}[S^T] = \mathbf{E}[S_{T\wedge t}] = \mathbf{E}[S_0] = 1$.

To construct a non-negative martingale $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ such that $\mathbf{E}[X_n] = 1$ for all $n \in \mathbb{N}$ and $X_n \xrightarrow{n\to\infty} 0$ almost surely, we can use a modified version of the stopped random walk example in (a). Let $S_n = S_0 + \sum_{i=1}^n X_i$ be a symmetric simple random walk, where $S_0 = 1$ and X_i are iid random variables:

$$\mathbf{P}(X_i = 1) = \frac{1}{2}, \quad \mathbf{P}(X_i = -1) = \frac{1}{2}.$$

Let $T = \inf\{n \geq 0 : S_n = 0\}$, which is the first time the random walk returns to zero. Define the martingale as:

$$X_n = \begin{cases} S_n & \text{if } n < T \\ 0 & \text{if } n \geq T \end{cases}.$$

Here, X_n is non-negative since $S_n \geq 0$ for $n < T$ and X_n becomes 0 after hitting zero. We need to confirm that $\mathbf{E}[X_n] = 1$ for all n: For $n < T$: $\mathbf{E}[X_n] = \mathbf{E}[S_n] = 1$, since the expected value of a symmetric random walk starting at 1 is always equal to its starting value. For $n \geq T$, $\mathbf{E}[X_n] = 0$, but since T is a stopping time and X_n is defined to be 0 afterwards, we take the limit: $\lim_{n\to\infty} \mathbf{E}[X_n] = 1$. The random walk S_n will eventually hit zero with probability 1. Once the stopping time T is reached, X_n becomes 0 for all $n \geq T$. Therefore, almost surely: $X_n \to 0 \quad$ as $n \to \infty$.