universität freiburg

Stochastic processes **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 7 - Martingales III

Hint: In Exercises 1 and 2, you can use the results from Chapter 14 also for the time-continuous martingale $\mathcal{B}$.

Exercise 1 (4 Points)**.**
This is a follow-up of Exercise 4 in Tutorial 5! Let $\mathcal{B} = (B_t)_{t\geq 0}$ be a Brownian Motion, started in $B_0 = 0$. For a constant $a > 0$ define $T := \inf\{t \geq 0 : B_t \notin (-a,a)\}$. Show that for every $\lambda > 0$,

$$\mathbf{E}[\exp(-\lambda T)] = (\cosh(a\sqrt{2\lambda}))^{-1}$$

Exercise 2 (4 points)**.**
For constants a, $b > 0$ define $T := \inf\{t \geq 0 : B_t = a + bt\}$. Show that for each $\lambda > 0$, we have

$$\mathbf{E}[e^{-\lambda T}] = \exp^{-a(b+\sqrt{b^2+2\lambda}})}.$$

Hint: Use exercise 1 with $c = b + \sqrt{b^2 + 2\lambda}$.

Exercise 3 (2+2 points)**.**
Let $X_1,X_2,...$ be iid with $\mathbf{P}(X_1 = \pm 1) = \frac{1}{2}$ and $\mathcal{S} := (S_n)_{n=0,1,...}$ be the simple random walk given by $S_n = \sum_{i=1}^n X_i$. In addition, let $T_k := \min\{n : S_n = k\}$ be the hitting time of k.

(a) For $a \in \mathbb{N}$, let $T := T_a \wedge T_{-a}$. Is the stopped process $\mathcal{S}^T$ uniformly integrable?

(b) For $a \in \mathbb{Z}$, let $T := T_a$. Is the stopped process $\mathcal{S}^T$ uniformly integrable?

Exercise 4 (4 points)**.**
A monkey randomly types one of the 26 capital letters on a keyboard every second. Let T be the time at which the monkey typed the word ABRACADABRA for the first time. It is claimed that it takes an average of $\mathbf{E}[T] = 26^{11} + 26^4 + 26$ seconds for the monkey to type the word for the first time. How can this be seen with the help of the Optional Sampling/Stopping Theorems?

Hint: Construct a fair game in which the players have a starting capital of 1€ and in each round bet their entire capital on the monkey typing the next correct letter of the word (each new player initially bets on A).