universität freiburg

Stochastic processes **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 11 - Markov processes III

Exercise 1 (4 Points)**.**
Let E_1 be countable, $(E_2,\mathcal{B}(E_2))$ a measurable space and $f : E_1 \times E_2 \to E_1$ measurable. Furthermore, let X_0 be an E_1-valued random variable and $Z_0,Z_1,...$ iid and E_2-valued. Show: If $X_0, Z_0,Z_1,Z_2,...$ are independent, then $(X_t)_{t=0,1,2,...}$ given by $(X_{t+1} := f(X_t,Z_{t+1}), t = 0,1,2,...$ is a time-homogeneous Markov chain.

Exercise 2 (4 points)**.**
Let (E,r) be a complete and separable metric space, $\mathcal{X} = (X_t)_{t\geq 0}$ an E-valued Markov process with generator G, which has domain $\mathcal{D} \subseteq \mathcal{C}_b(E)$. Show that, for $f \in \mathcal{D}$ with $Gf \in \mathcal{C}_b(E)$ and $\lambda \geq 0$

$$\Big(e^{-\lambda t} f(X_t) + \int_0^t e^{-\lambda s}(\lambda f(X_s) - Gf(X_s))ds\Big)_{t\geq}$$

is a martingale.

Exercise 3 (4 Points)**.**
The semigroup of the Ornstein-Uhlenbeck process is given by:

$$(T_t f)(x) = \int f\left(e^{-tx} + \sqrt{1-e^{-2ty}}\right) e^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} dy,$$

for $f \in \mathcal{C}_0(\mathbb{R})$. Show that the infinitesimal generator of the process is given by

$$Gf(x) = f''(x) - xf'(x).$$

Exercise 4 (4 Points)**.**
Let $(B_t)_{t\in\mathbb{R}^+}$ be a Brownian motion and $X_t = |B_t|$ for $t \geq 0$. Assume that the initial distribution of X is $P_0(x,\cdot) = \delta_x(\cdot)$, where δ_x denotes the Dirac measure for all $x \in \mathbb{R}^+$.

(a) Prove that X is a Markov process with transition function given by the density

$$\frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{1}{2t}(y-x)^2\right) + \exp\left(-\frac{1}{2t}(y+x)^2\right)\right] \mathbb{1}_{\{y\geq 0\}}.$$

(b) Is the associated semigroup P_t a Feller semigroup?