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Stochastic processes — Winter semester 2024

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
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Tutorial 2 - Definition and existence of stochastic processes

Exercise 1 (1+2+1=4 Points)**.**
Let $\Omega = \{1,2,3,4,5\}$.

(a) Find the smallest σ- algebra $\mathcal{F}_1$ containing

$$\mathcal{F}_2 := \{\{1,2,3\},\{3,4,5\}\}.$$

(b) Is the random variable $X : \Omega \to \mathbb{R}$ defined by

$$X(1) = X(2) = 0, \quad X(3) = 10, \quad X(4) = X(5) = 1$$

measurable with respect to $\mathcal{F}_1$?

(c) Find the σ-algebra $\mathcal{F}_3$ generated by $Y : \Omega \to \mathbb{R}$ and defined by

$$Y(1) = 0, \quad Y(2) = Y(3) = Y(4) = Y(5) = 1.$$

Solution.

(a) $\mathcal{F}_1 = \{\emptyset,\Omega,\{1,2,3\},\{3,4,5\},\{3\},\{1,2,4,5\},\{1,2\},\{4,5\}\}$.

(b) The random variable X is measurable with respect to $\mathcal{F}_1$ since we have for each $A \in \mathcal{B}(\mathbb{R})$:

$$\begin{aligned}
&\text{if} \quad 0 \in A,\, 1,10 \notin A: \quad X^{-1}(A) = \{1,2\} \in \mathcal{F}_1,\\
&\text{if} \quad 1 \in A,\, 0,10 \notin A: \quad X^{-1}(A) = \{4,5\} \in \mathcal{F}_1,\\
&\text{if} \quad 10 \in A,\, 1,10 \notin A: \quad X^{-1}(A) = \{3\} \in \mathcal{F}_1,\\
&\text{if} \quad 0,1,10 \notin A,: \quad X^{-1}(A) = \emptyset \in \mathcal{F}_1,\\
&\text{if} \quad 0,1,10 \in A,: \quad X^{-1}(A) = \Omega \in \mathcal{F}_1.
\end{aligned}$$

where $X^{-1}(A) = \{\omega \in \Omega : X(w) \in A.\}$ We can reduce every other case to these, take for example, if $0,1, \in A$ but $10 \notin A$, then:

$$X^{-1}(A) = X^{-1}(\{0\}) \cup X^{-1}(\{1\}) = \{1,2\} \cup \{4,5\} = \{1,2,3,4,5\} \in \mathcal{F}_1.$$

(c) $\mathcal{F}_3 = \sigma(Y) = \{\Omega,\emptyset,\{1\},\{2,3,4,5\}\}$.

Exercise 2 (2+2 points)**.**

(a) Given an example of two stochastic processes $\mathcal{X}$ and $\mathcal{Y}$ which are versions of each other, but no modifications of each other.

(b) Give an example of a real-valued stochastic process $\mathcal{X}$, such that $\mathbf{V}[X_t] > 0$ for all t and $\mathcal{X} = (X_t)_{t\in I}$ and $\mathcal{X}^2 := (X_t^2)_{t\in I}$ are indistinguishable.

Solution.

(a) Let $\Omega = \{1,2,3\}$ and $\mathcal{F} = \{\{1,2,3\},\emptyset,\{1,2\},\{3\}\}$. Define two stochastic processes $\mathcal{X} = (X_t)_{t=1,2,3,\ldots}$ and $\mathcal{Y} = (Y_t)_{t=1,2,3,\ldots}$ as follows:

$$X_t(w) = 1 \quad \text{for all } t \text{ and } w \in \Omega,$$

$$Y_t(w) = 2 \quad \text{for all } t \text{ and } w \in \Omega.$$

We need to show that X_t and Y_t are versions of each other, which means we need to verify that their distributions are the same for all sets $A \in \mathcal{F}$. Observe the following for each A in $\mathcal{F}$:

$$\begin{aligned}
\mathbf{P}(X_t \in \{1,2,3\}) =&1 = \mathbf{P}(Y_t \in \{1,2,3\},\\
\mathbf{P}(X_t \in \{1,2\}) =&1 = \mathbf{P}(Y_t \in \{1,2\}),\\
\mathbf{P}(X_t \in \{3\}) =&0 = \mathbf{P}(Y_t \in \{3\}),\\
\mathbf{P}(X_t \in \emptyset) =&0 = \mathbf{P}(Y_t \in \emptyset).
\end{aligned}$$

Hence for all $A \in \mathcal{F}$:

$$\mathbf{P}(X_t \in A) = \mathbf{P}(Y_t \in A).$$

Thus, X_t and Y_t have the same distribution with respect to the sigma-algebra $\mathcal{F}$. Therefore, we conclude that X_t and Y_t are versions of each other. However, since $X_t(w) = 1$ and $Y_t(w) = 2$ for all t and for all $w \in \Omega$, it follows that $X_t(w) \neq Y_t(w)$ for all $w \in \Omega$. Thus,

$$\mathbf{P}(X_t = Y_t) = \mathbf{P}(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\}) = \mathbf{P}(\emptyset) = 0 \neq 1 \text{ for all } t.$$

So, $\mathcal{X}$ is not a modification of $\mathcal{Y}$, and vice versa.

(b) Let $\mathcal{X} = (X_t)_{t\in I}$ be defined as:

$$X_t = \begin{cases} 1, & \text{with probability } \frac{1}{2},\\ 0, & \text{with probability } \frac{1}{2}. \end{cases}$$

Clearly,

$$\mathbf{E}[X_t] = \frac{1}{2}, \quad \mathbf{E}[X_t^2] = \frac{1}{2} \implies \mathbf{V}[X_t] = \frac{1}{4} > 0.$$

Furthermore,

$$X_t^2 = \begin{cases} 1, & \text{with probability } \frac{1}{2},\\ 0, & \text{with probability } \frac{1}{2}. \end{cases}$$

Thus, $\mathbf{P}(X_t = X_t^2 \text{ for all } t \in I) = 1$. Hence, $\mathcal{X}$ and $\mathcal{X}^2$ are indistinguishable with $\mathbf{V}[X_t] > 0$ for all t.

Exercise 3 (4 Points)**.**

Let I be some index set, (E,r) be Polish and $(\mathbf{P}_i)_{i\in I}$ a family of probability measures on $\mathcal{B}(E)$. Show that there exists an E-valued stochastic process $(X_t)_{t\in I}$ such that $(X_{t_1},...,X_{t_n}) \sim \otimes_{i=1}^n \mathbf{P}_{t_i}$ for any $t_1,...,t_n \in I$. In other words, $(X_t)_{t\in I}$ is an independent family with $X_t \sim \mathbf{P}_t$.

Solution.

Recall that if $(\Omega,\mathcal{F})$ is a measurable space, I an arbitrary index set and $(\Omega^J,\mathcal{F}^J)_{J\subseteq_f I}$ is a family of measurable product spaces, equipped with the product σ-algebra, as in Definition 5.3. A family of probability measures $(\mathbf{P}_J)_{J\subseteq_f I}$, where $\mathbf{P}_J$ is a probability measure on $\mathcal{F}^J$, is called a projective family if $\mathbf{P}_H = (\pi_H^J)_*\mathbf{P}_J$ for all $H \subseteq J \subseteq_f I$. Also, if for a projective family $(\mathbf{P}_J)_{J\subseteq_f I}$ of probability measures there exists a probability measure $\mathbf{P}_I$ on $\mathcal{F}^I$ with $\mathbf{P}_J = (\pi_J)_*\mathbf{P}_I$ for all $J \subseteq_f I$, then $\mathbf{P}_I$ is called the projective limit of the projective family and we write $\mathbf{P}_I = \varprojlim_{J\subseteq_f I} \mathbf{P}_J$. In the above, we have the following probability spaces: $(E,\mathcal{B}(E),\mathbf{P}_i)_{i\in I}$. Suppose $X_i : E \to E$ represent the random variables which are $\mathcal{B}(E)/\mathcal{B}(E)$ measurable. As in Example 5.22 and Remark 5.23, define: $\mathbf{P}^{\otimes J} := \otimes_{j\in J}(X_j * \mathbf{P}_j)$, $J \subseteq_f I$. We claim that the family $(\mathbf{P}^{\otimes J})_{J\subseteq_f I}$ is projective. If $H \subseteq J \subseteq_f I$, then for $A_j \in \mathcal{B}(E), j \in H$,

$$\begin{aligned}
(\pi_H^J)_*\mathbf{P}^{\otimes J}\big(\bigtimes_{i\in H} A_j\big) &= \mathbf{P}^{\otimes J}\big((\pi_H^J)^{-1}\big(\bigtimes_{j\in H} A_j\big)\big) \\
&= \mathbf{P}^{\otimes J}\big(\bigtimes_{j\in H} A_j \times \bigtimes_{j\in J\setminus H} E\big) \\
&= \prod_{j\in J} \mathbf{P}_j(X_j \in A_j) \cdot \prod_{j\in J\setminus H} (X_j \in E) \\
&= \prod_{j\in H} \mathbf{P}_j(X_j \in A_j) \\
&= \mathbf{P}^{\otimes H}\big(\bigtimes_{j\in H} A_j\big).
\end{aligned}$$

Thus, we claim that the projective limit exists! (See Theorem 5.24 [Existence of processes, Kolmogorov])

$$\mathbf{P}^{\otimes I} = \otimes_{i\in I}(X_i * \mathbf{P}_i) = \varprojlim_{J\subseteq_f I} \otimes_{j\in J}(X_j * \mathbf{P}_j) = \varprojlim_{J\subseteq_f I} \mathbf{P}^{\otimes J}.$$

Thus, for any $j = \{t_1,t_2,\dots,t_n\} \subseteq I$ and for all $A := (A_j)_{j\in J} \in \mathcal{B}(E)$,

$$\begin{aligned}
\mathbf{P}^{\otimes J}(A) = \otimes_{j\in J}\left(X_j * \mathbf{P}_j(A_j)\right) &= \otimes_{j\in J}\Big(\mathbf{P}_j(X_j^{-1}(A_j))\Big) \\
&= \otimes_{j\in J}\left(\mathbf{P}_j(A_j)\right) = \otimes_{j\in J}\mathbf{P}_j(A).
\end{aligned}$$

That is, such $E-$valued stochastic process $\mathcal{X} = (X_t)_{t\in I}$ exists!

Exercise 4 (1+1+2= 4 Points)**.**

Let $(X_k)_{k\in\mathbb{N}_0}$ be a **symmetric** simple random walk, i.e. $X_k = \sum_{i=0}^{k-1} Z_i$ where $Z_1,Z_2,...$ are iid with $\mathbf{P}(Z_1 = \pm 1) = \frac{1}{2}$[1]. For $n \in \mathbb{N}$ define

$$S_n = \sum_{k=1}^n X_k.$$

[1]Recall the convention that $\sum_{i=0}^{-1} a_i = 0$

(a) Show whether or not $(S_n)_{n\in\mathbb{N}_0}$ is a simple random walk (not necessarily symmetric).

(b) Compute the covariance $\mathbf{COV}[X_k,X_l]$ for $k \le l \in \mathbb{N}$.

(c) Compute the variance of S_n for $n \in \mathbb{N}$.

Note: You may need to recall that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{for } n \in \mathbb{N}.$$

Solution.

(a) We start by writing

$$S_n = \sum_{k=1}^{n}\sum_{i=0}^{k-1} Z_i = \sum_{i=0}^{n-1}\sum_{k=i+1}^{n} Z_i = \sum_{i=0}^{n-1}(n-i)Z_i$$

This expression shows that S_n is a linear combination of the Z_i values, where each Z_i is multiplied by the number of times it contributes to the sum S_n. Specifically, Z_i appears in S_n a total of $n-i$ times. To verify whether S_n constitutes a simple random walk, we analyze its increment:

$$S_{n+1} - S_n = X_{n+1}.$$

Since $X_{n+1} = Z_0 + Z_1 + \cdots + Z_n$, we can express:

$$S_{n+1} = S_n + X_{n+1} = S_n + (Z_0 + Z_1 + \cdots + Z_n).$$

Each increment $S_{n+1} - S_n$ depends on the Z values, but it does not result in independent increments because the contribution of each Z_i to S_n depends on their indices. Thus, although S_n is a sum of random variables, it is not a simple random walk because the increments are not independent and identically distributed. Therefore, we conclude that:

The process $(S_n)_{n\in\mathbb{N}_0}$ is not a simple random walk.

As illustration: Consider for $k = 2$:

$$X_2 = Z_0 + Z_1.$$

The possible values of X_2 are:

$$\begin{cases} 2 & \text{if} \quad Z_0 = 1,\, Z_1 = 1, \\ 0 & \text{if} \quad Z_0 = 1,\, Z_1 = -1 \quad \text{or} \quad Z_0 = -1,\, Z_1 = 1, \\ -2 & \text{if} \quad Z_0 = -1,\, Z_1 = -1. \end{cases}$$

The increment $S_{n+1} - S_n = X_{n+1}$ is not independent of previous Z_i. The distribution of X_k changes with k (e.g., X_2 can take values -2,0,2). Note that, $\mathbf{P}(X_2 = \pm 1) = 0$.

(b) We compute as follows:

$$\begin{aligned}\mathbf{COV}[X_k,X_l] &= \mathbf{E}[(X_k - \mathbf{E}[X_k])(X_l - \mathbf{E}[X_l])] \\ &= \mathbf{E}[X_k X_l] = \mathbf{E}\left[\left(\sum_{i=0}^{k-1} Z_i\right)\left(\sum_{j=0}^{l-1} Z_i\right)\right]\end{aligned}$$

When we expand the sum above, we observe that the terms of the form $\mathbf{E}[Z_i Z_j]$, for $i \neq j$ will varnish because Z_i and Z_j are independent for $i \neq j$. The only terms left are the those of the form $\mathbf{E}[Z_i Z_i]$ or $\mathbf{E}[Z_j Z_j]$. Their value is 1, and since there are k of them ($k \leq l$), then

$$\mathbf{COV}[X_k,X_l] = k.$$

(c) We already know that $\mathbf{E}[S_n] = 0$. So, it suffices to compute

$$\mathbf{V}[S_n] = \mathbf{E}[S_n^2] = \mathbf{E}\left[\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{l=1}^{n} X_l\right)\right]$$

Recall the identity of Bienamyé (see Proposition 6.9!)

$$\begin{aligned}\mathbf{V}\Big[\sum_{k=1}^{n} X_k\Big] &= \mathbf{E}\Big[\Big(\sum_{k=1}^{n} X_k\Big)^2\Big] = \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{E}[X_k X_l] = \sum_{k=1}^{n} \mathbf{E}[X_k^2] + 2\sum_{1\leq k<l\leq n} \mathbf{E}[X_k X_l] \\ &= \sum_{k=1}^{n} \mathbf{V}[X_k] + 2\sum_{1\leq k<l\leq n} \mathbf{COV}[X_k X_l].\end{aligned}$$

For each $k = 1,2,\ldots,n$, there are exactly $n-k$ values of l such that $k < l \leq n$. Therefore, $\sum_{1\leq k\leq l\leq n} k = \sum_{k=1}^{n} k(n-k)$, and so

$$\begin{aligned}\mathbf{V}[S_n] &= \sum_{k=1}^{n} k + 2\sum_{k=1}^{n} k(n-k) = (2n+1)\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} k^2 \\ &= (2n+1)\frac{n(n+1)}{2} - 2\frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

Alternatively: We can use the representation $X_k = \sum_{i=1}^{k} Y_k$ so that

$$\begin{aligned}\mathbf{V}[S_n] &= \mathbf{E}\Big[\Big(\sum_{k=1}^{n} X_k\Big)^2\Big] = \mathbf{E}\left[\left(\sum_{k=1}^{n}\sum_{i=1}^{k} Y_i\right)^2\right] \\ &= \mathbf{E}\left[\left(\sum_{i=1}^{n}\sum_{k=1}^{k} Y_i\right)^2\right] = \mathbf{E}\left[\left(\sum_{i=1}^{n}(n-i+1)Y_i\right)^2\right] \\ &= \sum_{i=1}^{n}(n-i+1)^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

where we have used the fact that $\mathbf{E}[Y_i Y_j] = 0$ for $i \neq j$.