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https://pfaffelh.github.io/hp/2024ws_stochproc.html

<https://www.stochastik.uni-freiburg.de/>

Tutorial 3 - Poisson process and Brownian motion

Exercise 1 (2+2 points).

Assume that the service of buses in Freiburg starts at 8 pm and then they arrive according to a Poisson process of intensity $\lambda = 4$ per hour. Franz Kafka starts to wait for a bus at 8pm.

- (a) What is the expected waiting time for the next bus?
- (b) At 8:30pm Kafka is still waiting. What is now the expected waiting time?

Exercise 2 (1+3 Points).

- (a) Let Z be a standard normal random variable. For all $t \geq 0$, let $X_t = \sqrt{t}Z$. The stochastic process $\mathcal{X} = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0, X_t \sim N(0, t)$. Is \mathcal{X} a Brownian motion? Justify!
- (b) Let W_t and \tilde{W}_t be two independent Brownian motion and ρ is a constant contained in the unit interval. For all $t \geq 0$, let $X_t = \rho W_t + \sqrt{1 - \rho^2} \tilde{W}_t$. The stochastic process $\mathcal{X} = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0, X_t \sim N(0, t)$. Is \mathcal{X} a Brownian motion? Justify!

Exercise 3 (2+2=4 Points).

Let $d, k \in \mathbb{N}$, $C \in \mathbb{R}^{d \times d}$ and $X \sim N(0, C)$. (Recall from definition 10.14 from the manuscript of probability theory!)

- (a) If there is $S \in \mathbb{R}^{k \times d}$ with $C = S^\top S$, and $Z \sim N(0, I_k)$ (where I_k is the $k \times k$ identity matrix), show that $SZ^\top \stackrel{d}{=} X$.
- (b) Let $t_1 \leq \dots \leq t_n$ and $Z \sim N(0, I_d)$. Find $S \in \mathbb{R}^{d \times d}$ such that $SZ^\top \sim N(0, C)$ with $C_{ij} := t_i \wedge t_j$ (as in the covariance matrix of Brownian Motion).

Exercise 4 (4 Points).

Let $\mathcal{X} = (X_n)_{n \in \mathbb{N}}$ be a Gaussian process. Show that if \mathcal{X} converges to a random variable X in probability, it also converges in \mathcal{L}^2 to X .