Stochastic Processes
20. The law of the iterated logarithm

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Introduction

- Goal: Find $t \mapsto h_t$ such that

$$0 < \limsup_{t\to\infty} \frac{X_t}{h_t} < \infty.$$

- We know that

$$\frac{X_t}{t} \xrightarrow{t\to\infty} 0.$$

Also,

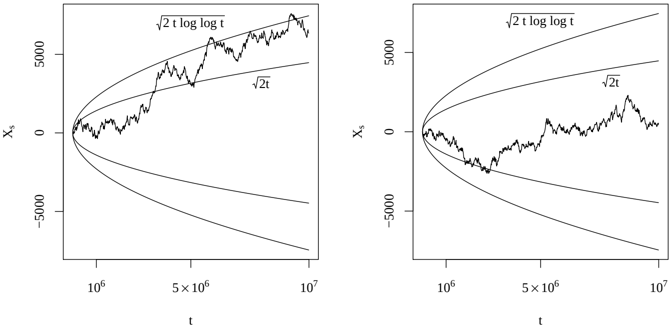
$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{t}} = \infty.$$

Indeed: $\limsup_{t\to\infty} \frac{X_t}{\sqrt{t}}$ is measurable wrt terminal σ-algebra of $\mathcal{X}$, i.e. is as constant. Suppose, $\limsup_{t\to\infty} \frac{X_t}{\sqrt{t}} \xrightarrow{t\to\infty} \gamma$ for a $0 < \gamma < \infty$. Then $\mathrm{P}(\frac{X_t}{\sqrt{t}} > 2\gamma) \xrightarrow{t\to\infty} 0$, in contradiction to the CLT.

The iterated logarithm

- Theorem 16.10: $\mathcal{X}$ BM. Then

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2t \log\log t}} = \limsup_{t\to 0} \frac{X_t}{\sqrt{2t \log\log 1/t}} = 1, \text{ almost surely}$$



The iterated logarithm

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- By symmetry,
$$\liminf_{t\to\infty} \frac{X_t}{\sqrt{2t\log\log t}} = \liminf_{t\to 0} \frac{X_t}{\sqrt{2t\log\log 1/t}} = -1$$
- The second equality follows from the first by using $X'_t = tX_{1/t}$:
$$\frac{X_t}{\sqrt{2t\log\log 1/t}} = \frac{X'_{1/t}}{\sqrt{2\frac{1}{t}\log\log t}} = \frac{tX'_{1/t}}{\sqrt{2t\log\log t}}$$
- We will write
$$h_t := h(t) := \sqrt{2t\log\log t}$$
$$a(x) \approx b(x) \text{ if } \frac{a(x)}{b(x)} \xrightarrow{x\to\infty} 1$$

Estimation of normal distribution

- Lemma: Let φ be the density of $X \sim N(0,1)$. Then,

$$\text{(i) } \mathrm{P}(X > x) \le \tfrac{1}{x}\varphi(x), \qquad \text{(ii) } \mathrm{P}(X > x) \ge \tfrac{x}{1+x^2}\varphi(x).$$

- Proof: Since $\varphi'(y) = -y\varphi(y)$,

$$\varphi(x) = \int_x^\infty y\varphi(y)dy \ge x\int_x^\infty \varphi(y)dy = x \cdot \mathrm{P}(X > x),$$

which shows (i). Quite similarly, $\left(\frac{\varphi(y)}{y}\right)' = -\frac{1+y^2}{y^2}\varphi(y)$, so

$$\begin{aligned}\frac{\varphi(x)}{x} &= \int_x^\infty \frac{1+y^2}{y^2}\varphi(y)dy \le \frac{1+x^2}{x^2}\int_x^\infty \varphi(y)dy \\ &= \frac{1+x^2}{x^2} \cdot \mathrm{P}(X > x).\end{aligned}$$

- We will use, by Theorem 16.8, for $x > 0$

$$\mathrm{P}(\sup_{0\le s\le t} X_s > x\sqrt{t}) = 2 \cdot \mathrm{P}(X_t > x\sqrt{t}) \overset{t\to\infty}{\approx} \tfrac{2}{x}\varphi(x).$$

The iterated logarithm

- Theorem 16.10: $\mathcal{X}$ BM. Then
$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2t\log\log t}} = \limsup_{t\to 0} \frac{X_t}{\sqrt{2t\log\log 1/t}} = 1, \text{ almost surely}$$
- Proof, upper bound: Let $r > 1$. Then
$$h(r^{n-1}) = \sqrt{\frac{2(\log(n-1) + \log\log r)}{r}}\sqrt{r^n} \overset{n\to\infty}{\approx} \sqrt{\frac{2\log n}{r}}\sqrt{r^n}$$
Now for $c > 0$,
$$\mathrm{P}\Big(\sup_{0\le t\le r^n} X_t > ch(r^{n-1})\Big) \overset{n\to\infty}{\approx} 2\cdot \mathrm{P}\left(X_{r^n} > c\sqrt{\frac{2\log n}{r}}\sqrt{r^n}\right)$$
$$\overset{n\to\infty}{\approx} \frac{1}{c}\sqrt{\frac{2r}{\log n}}\varphi(c\sqrt{2\log n^{1/r}}) \overset{n\to\infty}{\approx} \frac{1}{c}\sqrt{\frac{r}{\pi\log n}}\frac{1}{n^{c^2/r}}.$$
For $c > 1$ and $1 < r < c^2$, the right-hand side is summable, so
$$\mathrm{P}\left(\limsup_{t\to\infty} \frac{X_t}{h_t} \ge c\right) \le \mathrm{P}\Big(\sup_{0\le t\le r^n} X_t > ch_{r^{n-1}} \text{ inf. often }\Big) = 0.$$

- Theorem 16.10: $\mathcal{X}$ BM. Then

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2t\log\log t}} = \limsup_{t\to 0} \frac{X_t}{\sqrt{2t\log\log 1/t}} = 1, \text{ almost surely}$$

- Proof, lower bound: Let $r > 1$ (large) and $0 < c < 1$ (close to 1). Define

$$A_n := \{X_{r^n} - X_{r^{n-1}} > ch(r^n - r^{n-1})\}.$$

Since $X_{r^n} - X_{r^{n-1}} \sim N(0, r^n - r^{n-1})$,

$$\begin{aligned} \mathrm{P}(A_n) &= \mathrm{P}\Big(\frac{X_{r^n} - X_{r^{n-1}}}{\sqrt{r^n - r^{n-1}}} > c\frac{h(r^n - r^{n-1})}{\sqrt{r^n - r^{n-1}}}\Big) \\ &= \mathrm{P}\big(X_1 > c\sqrt{2\log\log(r^n - r^{n-1})}\big) \\ &\overset{n\to\infty}{\approx} \frac{1}{c}\frac{1}{\sqrt{4\pi\log\log(r^n - r^{n-1})}} \exp\big(-c^2\log\log(r^n - r^{n-1})\big) \\ &\overset{n\to\infty}{\approx} \frac{1}{c}\frac{1}{\sqrt{4\pi\log n}}\frac{1}{n^{c^2}} \text{ not summable.} \end{aligned}$$

- Theorem 16.10: $\mathcal{X}$ BM. Then

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2t\log\log t}} = \limsup_{t\to 0} \frac{X_t}{\sqrt{2t\log\log 1/t}} = 1, \text{ almost surely}$$

- Proof, lower bound: Since $A_1, A_2, \ldots$ are independent, infinitely many A_n occur, i.e.

$$X_{r^n} > ch(r^n - r^{n-1}) + X_{r^{n-1}}.$$

Upper bound $\Rightarrow X_{r^{n-1}} > -2h(r^{n-1})$ for almost all n, i.e. $\liminf_{n\to\infty} \frac{X_{r^{n-1}}}{h(r^n)} \geq -2\liminf_{n\to\infty} \frac{h(r^{n-1})}{h(r^n)} = -\frac{2}{\sqrt{r}}$ is almost certain. Further, $h(r^n - r^{n-1})/h(r^n) \xrightarrow{n\to\infty} 1$ and thus

$$\limsup_{t\to\infty} \frac{X_t}{h_t} \geq \limsup_{n\to\infty} \frac{X_{r^n}}{h(r^n)} \geq \limsup_{n\to\infty} \frac{X_{r^n} - X_{r^{n-1}}}{h(r^n - r^{n-1})} - \frac{2}{\sqrt{r}} \geq c - \frac{2}{\sqrt{r}}.$$

The result follows with $r \to \infty$ and $c \to 1$.