

The background of the slide features a large, faint watermark of the University of Basel seal. The seal is circular and contains a central figure, likely a saint or scholar, seated and holding a book. Above the figure are three smaller figures in niches. The entire seal is surrounded by a Latin inscription in a circular border.

Stochastic Processes

14. The strong Markov property

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Definition

- ▶ $\mathcal{X} = (X_t)_{t \in I}$ progressively measurable Markov process. \mathcal{X} has the *strong Markov property at S* if for all stopping times S

$$P(X_{S+t} \in A | \mathcal{F}_S) = \mu_{S, S+t}^{\mathcal{X}}(X_S, A), \quad A \in \mathcal{B}(E)$$

or equivalently

$$E[f(X_{S+t}) | \mathcal{F}_S] = (T_{S, S+t}^{\mathcal{X}} f)(X_S), \quad f \in \mathcal{B}(E).$$

Discrete time and the strong Markov property

- ▶ Proposition 15.11: If I is discrete, then every Markov process $\mathcal{X} = (X_t)_{t \in I}$ has the strong Markov property.
- ▶ We write

$$\begin{aligned} \mathbb{E}[f(X_{S+t}), A] &= \sum_s \mathbb{E}[f(X_{s+t}), A \cap \{S = s\}] \\ &= \sum_s \mathbb{E}[\mathbb{E}[f(X_{s+t}) | \mathcal{F}_s], A \cap \{S = s\}] \\ &= \sum_s \mathbb{E}[(T_{s,s+t}f)(X_s), A \cap \{S = s\}] \\ &= \sum_s \mathbb{E}[(T_{S,S+t}f)(X_S), A \cap \{S = s\}] \\ &= \mathbb{E}[(T_{S,S+t}f)(X_S), A]. \end{aligned}$$

Strong Markov with continuous transition operator

- ▶ Theorem 15.12: $\mathcal{X} = (X_t)_{t \in I}$ Markov process with right-continuous paths. If $T_{s,t}^{\mathcal{X}} f$ is continuous for $f \in \mathcal{C}_b(E)$ and $s \mapsto T_{s,s+t}^{\mathcal{X}} f$ continuous for all $f \in \mathcal{C}_b(E)$ (with respect to the supremum norm on $\mathcal{C}_b(E)$), then \mathcal{X} is strong Markov.
- ▶ Proof: Let S_1, S_2, \dots stopping times with $S_n \downarrow S$ so that S_n only takes discrete values, $n = 1, 2, \dots$. Then, $X_{S_n} \xrightarrow{n \rightarrow \infty} X_S$ and

$$\begin{aligned} \mathbb{E}[f(X_{S+t}) | \mathcal{F}_S] &= \lim_{n \rightarrow \infty} \mathbb{E}[\mathbb{E}[f(X_{S_n+t}) | \mathcal{F}_{S_n}] | \mathcal{F}_S] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[(T_{S_n, S_n+t}^{\mathcal{X}} f)(X_{S_n}) | \mathcal{F}_S] \\ &= \mathbb{E}[(T_{S, S+t}^{\mathcal{X}} f)(X_S) | \mathcal{F}_S] = (T_{S, S+t}^{\mathcal{X}} f)(X_S). \end{aligned}$$

PPP and BM are strong Markov

- ▶ For the PPP(λ) \mathcal{X} , we have $(T_{s,t}^{\mathcal{X}}f)(x) = \mathbb{E}[f(x + P)]$, where $P \sim \text{Poi}(\lambda(t - s))$.
- ▶ For the BM \mathcal{X} , $(T_{s,t}^{\mathcal{X}}f)(x) = \mathbb{E}[f(x + \sqrt{t - s}Z)]$, where $Z \sim N(0, 1)$.

A Markov process which is not strong Markov

- ▶ $T \sim \exp(1)$. Define $\mathcal{X} = (X_t)_{t \geq 0}$ via

$$X_t = (t - T)^+$$

and completion of the canonical filtration $(\mathcal{F}_t)_{t \geq 0}$. Then,

$$(T_{S, S+t}^{\mathcal{X}})f(x) = 1_{x=0}E[f((t-T)^+)] + 1_{x>0}f(x+t), \quad f \in \mathcal{B}(\mathbb{R}).$$

Now consider the option time $S = T$. Since $\{T = t\}$ is a nullset and \mathcal{F}_t is complete, S is a stopping time. Now,

$$E[f(X_{S+t})|\mathcal{F}_S] = f(t).$$

On the other hand, $X_S = 0$ and therefore

$$(T_{S, S+t}^{\mathcal{X}}f)(X_S) = (T_{S, S+t}^{\mathcal{X}}f)(0) = E[f((t-T)^+)].$$