universität freiburg

Measure theory for probabilists — Winter semester 2024

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 7 - Measurable functions and the integral I

Exercise 1 (4 Points).
Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$. Show that a Borel measurable map $g : \mathbb{R} \to \mathbb{R}$ is $\sigma(f) = f^{-1}(\mathcal{B}(\mathbb{R}))-$measurable if and only if g is even.

Exercise 2 (4 Points).
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{-x} 1_{[0,\infty)}(x)$, and let λ be the Lebesgue measure on $\mathbb{R}$.

(a) Find a sequence (f_n) of elementary functions such that $f_n \uparrow f$.

(b) Compute $\int f_n d\lambda$ and determine $\int f d\lambda$ as a limit of integrals.

Exercise 3 (4 Points).
Let $(\Omega,\mathcal{F})$,$(\Omega',\mathcal{F}')$ be measurable spaces and $f : \Omega \to \Omega'$. If there are $\mathcal{C} \subseteq \mathcal{F}$ and $\mathcal{C}' \subseteq \mathcal{F}'$ with $\sigma(\mathcal{C}) = \mathcal{F}$ and $\sigma(\mathcal{C}') = \mathcal{F}'$ and $f^{-1}(\mathcal{C}') \subseteq \mathcal{C}$, then f is $\mathcal{F}/\mathcal{F}'$-measurable.

Exercise 4 (4 Points).
Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?

Exercise 5 (*Bonus question!* 3 Points).
Let $\Omega = \{1,2,3,4,5\}$.

(a) Find the smallest σ- algebra $\mathcal{F}_1$ containing

$$\mathcal{F}_2 := \{\{1,2,3\},\{3,4,5\}\}.$$

(b) Is the function $f : \Omega \to \mathbb{R}$ defined by

$$f(1) = f(2) = 0, \quad f(3) = 10, \quad f(4) = f(5) = 1$$

measurable with respect to $\mathcal{F}_1$?

(c) Find the σ-algebra $\mathcal{F}_3$ generated by $g : \Omega \to \mathbb{R}$ and defined by

$$g(1) = 0, \quad g(2) = g(3) = g(4) = g(5) = 1.$$