universität freiburg

Measure theory **Winter semester 2025**

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2025WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 2 - Review of metric spaces and topologies II

These exercises are based on Appendix A

Exercise 1 (4 Points).
Let (Ω,r) be a metric space and recall Lemma A.4: the set A' of cluster points of A is a closed set. Furthermore, let B be a subset of Ω.

(a) Show that $\overline{A} = A \cup A'$.

(b) Show that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. Moreover, $(\overline{A \cup B}) = \overline{A} \cup \overline{B}$ and $(\overline{A \cap B}) \subseteq \overline{A} \cap \overline{B}$.

Note: x is a cluster point of A if it is a point in the closure of $A \backslash \{x\}$

Exercise 2 (4 Points).
Consider the cofinite topology $\mathcal{O}$ on $\mathbb{Z}$ defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or O^c is finite. Show that $\mathcal{O}$ is in fact a topology on $\mathbb{Z}$.

Exercise 3 (4 Points).
Suppose that there is a continuous, one-to-one mapping from a metric space (Ω,r) to another metric space (Ω',r') where r' is the discrete metric from sheet 1, exercise 2. Show that every subset of Ω is open.

Exercise 4 (4 Points).
Let (X,r) and (Y,r') be metric spaces and $f : X \to Y$. Show that f is continuous on X if and only if one of the following holds:

1. for every closed set A in Y, $f^{-1}(A)$ is closed in X.
2. for every $x,x_1,x_2,... \in X$ with $r(x_n,x) \xrightarrow{n\to\infty} 0$, it is $r'(f(x_n),f(x)) \xrightarrow{n\to\infty} 0$.