Measure theory for probabilists — Winter semester 2024

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 6 - Set functions

Exercise 1 (4 Points)**.**
Let λ be the Lebesgue-measure.

(a) Show that $\lambda(A) = 0$ where A is any finite set.

(b) Show that $\lambda(\mathbb{Q}) = 0$ (In general, the Lebesgue-measure of any countable set is 0).

(c) Let A be the Cantor set from Example 1.10. Compute $\lambda(A)$.

Solution.
A set, A, has measure zero in $\mathbb{R}$ if, given $\varepsilon > 0$ there are countable intervals I_k that cover A, such that $A \subseteq \cup_{k=1}^{\infty} I_k$ where $\sum_{n=1}^{\infty} \lambda(I_k) < \varepsilon$.

(a) Every finite set has measure zero. If we let $A = \{x_i\}_{i=1}^{n} < \infty$ and I_i be a set of intervals such that, $I_i = \left[x_i - \frac{\varepsilon}{2n}, x_i + \frac{\varepsilon}{2n}\right] \implies A \subseteq \cup_{i=1}^{\infty} I_i$. If we compute $\lambda(I_i)$ for some $1 \leq i \leq n$ we see that $\lambda(I_i) = \lambda\left(\left[x_i - \frac{\varepsilon}{2n}, x_i + \frac{\varepsilon}{2n}\right]\right) = x_i + \frac{\varepsilon}{2n} - \left(x_i - \frac{\varepsilon}{2n}\right) = \frac{2\cdot\varepsilon}{2n} = \frac{\varepsilon}{n}$ and now summing over all n intervals we see that $\lambda\left(\{x_i\}_{i=1}^{n}\right) \leq \sum_{i=1}^{n} \lambda(I_i) = \sum_{i=1}^{n} \frac{\varepsilon}{n} = \varepsilon$, so $\lambda(A) = 0$.

(b) Let A be a countable subset of $\mathbb{R}$. Note that every point has measure zero. Let $x \in \mathbb{R}$ then $x \in \left[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right] \implies \lambda(\{x\}) \leq \lambda\left(\left[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right] = \varepsilon\right)$. Thus $\lambda(\{x\}$ has measure zero $\forall x \in \mathbb{R}$. Let $A = \cup_{n=1}^{\infty}\{x_n\}$ be any countable set, then $\lambda(A) = \lambda\left(\cup_{n=1}^{\infty}\{x_n\}\right) = \sum_{n=1}^{\infty} \lambda(\{x\}) = 0$. Since $\mathbb{Q}$ is countable, then we are done. *Alternatively:* We can show as in (a) that if we cover each $\{x_n\}$ by I_n such that $x_n \in I_n \iff I_n = \left[x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right] \implies A \subseteq \cup_{i=1}^{\infty} I_n$. We will see that $\lambda(I_n) = \frac{2\varepsilon}{2^{n+1}}$, so $\sum_{n=1}^{\infty} I_n = \varepsilon \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon$. So $\lambda(A) < \varepsilon$ and we are done.

(c) Quite simply, the Cantor set $\mathbf{C}$ is constructed by starting with the interval $[0,1] \subset \mathbb{R}$, then dividing it into three intervals of equal length and removing the middle interval, where the process of division and removal is repeated $\mathbf{C} = \cap_{n=1}^{\infty} C_n$ (see Example 1.10!). One could verify the following properties of Cantor set: C_n has 2^n intervals; the length of each sub-interval of C_n is $\left(\frac{1}{3}\right)^n$. Observe that we remove $\frac{1}{3}$ of each interval at each step, which means that at step $n-1$, a length of $\left(\frac{1}{3}\right)^n$ is removed 2^{n-1} times, so we remove a total length of

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$

We know that $\lambda([0,1]) = 1$. Now we will consider the pieces removed from the Cantor set. At a step N, we have removed a total length of $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$ where the geometric series converge to 1 as shown above. Given $\varepsilon > 0$, there exists N large enough such that $\sum_{n=1}^{N} \frac{2^{n-1}}{3^n} > 1 - \varepsilon$. Let I_k be the intervals corresponding to this sum. Then taking the complement $(\cup_k I_k)^c$ we have a cover for **C** and we already know that this cover has a sum of lengths less than epsilon. Hence, we are done.

Note: A quick method to establish (b) from (a) is to write any countable set as the union of singleton sets and show that the measure of a singleton set is zero!

Exercise 2 (4 Points)**.**

(a) let (Ω,r) is a metric space. Show that if a set $E \subseteq \Omega$ has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

(b) Show that if E_1 and E_2 are measurable, then

$$\mu^*(E_1 \cup E_2) + \mu^*(E_1 \cap E_2) = \mu^*(E_1) + \mu^*(E_2).$$

Solution.

(a) *Approach: Assume that every bounded subset of E has measure zero, then establish that the measure of E is consequently zero.*

Let $x_0 \in E$ and consider $\mathcal{B}(x_0,r)$, $r \in \mathcal{Q}^+$ balls in Ω. Then $\mathcal{B}(x_0,r) \cap E$ is a bounded subset of E such that $E \subseteq \bigcup_{r \in \mathcal{Q}^+} \mathcal{B}(x_0,r) \cap E$. By monotonicity and $\sigma-$subadditivity of μ^*,

$$\mu^*(E) \leq \mu^* \left(\bigcup_{r \in \mathcal{Q}^+} \mathcal{B}(x_0,r) \cap E \right) \leq \sum_{r \in \mathcal{Q}^+} \underbrace{\mu^* \left(\mathcal{B}(x_0,r) \cap E \right)}_{0}$$

Hence, $\mu^*(E) = 0$. Thus, if a set $E \subseteq \Omega$ has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

(b) Since E_2 is measurable, we have

$$\mu^*(E_1 \cup E_2) = \mu^* \underbrace{((E_1 \cup E_2) \cap E_2)}_{E_2} + \mu^* \underbrace{((E_1 \cup E_2) \cap E_2^c)}_{E_1 \setminus E_2}. \tag{1}$$

Again, by the measurablility of E_2,

$$\mu^*(E_1) = \mu^*(E_1 \cap E_2) + \mu^* \underbrace{(E_1 \cap E_2^c)}_{E_1 \setminus E_2}. \tag{2}$$

Combining (1) and (2), we have

$$\mu^*(E_1 \cup E_2) + \mu^*(E_1 \cap E_2) = \mu^*(E_1) + \mu^*(E_2).$$

Exercise 3 (4 Points)**.**

(a) Let $\mu = \mu_{B(n,p)}$ be the binomial distribution with n trials and success probability p. Let $f : [0,n] \to [0,n]$ be defined by $f(k) = n - k$. Prove that $f_*\mu = \mu_{B(n,1-p)}$. Can you formulate a similar statement for the hypergeometric distribution?

(b) Let $f : \mathbb{R}_+ \to \mathbb{N}$ be given by $f(x) = \lceil x \rceil := \min\{n \in \mathbb{N} : n \geq x\}$, and $\mu = \mu_{\exp(\lambda)}$. Show that $f_*\mu$ is a geometric distribution and compute its success probability.

Solution.

(a) Let $x \in [0,n]$, then,

$$f_*\mu([0,n]) = f_*\mu(\{x\}) = \mu(f^{-1}([0,n])) = \mu(f^{-1}(\{x\})) = \mu(\{n-x\}).$$

That is, we have:

$$f_*\mu([0,n]) = \binom{n}{n-x} p^{n-x}(1-p)^x = \binom{n}{x}(1-(1-p))^{n-x}(1-p)^x = \mu_{B(n,1-p)}.$$

For hypergeometric distribution,

$$f_*\mu_{\mathrm{Hyp}(N,K,n)}(\{k\}) = \mu_{\mathrm{Hyp}(N,K,n)} f^{-1}(\{k\}) = \mu(\{n-k\}).$$

Hence,

$$f_*\mu = \frac{\binom{K}{n-k}\binom{N-K}{n-(n-k)}}{\binom{N}{n}} = \frac{\binom{K}{n-k}\binom{N-K}{k}}{\binom{N}{n}} = \mu_{\mathrm{Hyp}(N,N-K,n)}.$$

That is the hypergeometric distribution finds the probability of obtaining $n-k$ failures.

(b) For any $k \in \mathbb{N}$:

$$f_*\mu(\{k\} = \mu(f^{-1}(\{k\})) = \mu((k-1,k].$$

And since $\mu = \mu_{\exp(\lambda)}$, we have:

$$f_*\mu = \mu((k-1,k]) = \int_{k-1}^{k} \lambda e^{-\lambda x}\,\mathrm{d}x = -\left[e^{-\lambda x}\right]_{k-1}^{k} = (e^{-\lambda})^k(1-e^{-\lambda}),$$

which is the geometric distribution whose probability of success p is clearly seen as $1-e^{-\lambda}$ and $(1-p)^k = (e^{-\lambda})^k$.

Exercise 4 (4 Points)**.**

1. Prove that if $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(B)$.
2. Let (Ω,r) be a metric space, and μ^* the outer measure from Proposition 2.15, where $\mathcal{F}$ is the topology generated from (Ω,r). In addition, let A and B be bounded sets for which there is an $\alpha > 0$ such that $r(a,b) \geq \alpha$ for all $a \in A, b \in B$. Prove that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Solution.

(a) By the monotonicity of μ^*, we have that:

$$\mu^*(B) \leq \mu^*(A \cup B) \quad (\text{since } \ A \subseteq A \cup B). \tag{3}$$

Also, by $\sigma-$subadditivity of μ^*, we have:

$$\mu^*(A \cup B) \leq \underbrace{\mu^*(A)}_{0} + \mu^*(B). \tag{4}$$

From (3) and (4), we establish that $\mu^*(A \cup B) = \mu^*(B)$.

(b) By the $\sigma-$subadditivity of μ^*, we know that

$$\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B).$$

Hence we only need to show that the reverse inequality holds. Now fix $\varepsilon > 0$. Since A and B are bounded, $A \cup B$ is bounded; and $\mu^*(A \cup B)$ is finite. We can therefore find a countable collection of non-empty, open, bounded intervals $\{I_k\}_{k=1}^{\infty}$ which covers $A \cup B$ (such that $A \cup B \subseteq \bigcup_{k=1}^{\infty} I_k$) and satisfies:

$$\mu^*(A \cup B) > \sum_{k=1}^{\infty} l(I_k) - \varepsilon$$

Without loss of generality, assume the length of each interval in the collection is less than $\frac{\alpha}{2}$ (the intervals can be subdivided until this condition holds). Then by construction, each interval only intersect either A or B. Define,

$$\mathcal{A} = \{k : I_k \cap A \neq \emptyset\} \quad \text{and} \quad \mathcal{B} = \{k : I_k \cap B \neq \emptyset\}.$$

Since $\{I_k\}_{k \in \mathcal{A}}$ and $\{I_k\}_{k \in \mathcal{B}}$ form open covers of A and B respectively, we can conclude:

$$\mu^*(A \cup B) > \sum_{k \in \mathcal{A}} l(I_k) + \sum_{k \in \mathcal{B}} l(I_k) - \epsilon \geq \mu^*(A) + \mu^*(B) - \varepsilon.$$

This expression holds for all $\varepsilon > 0$, so we must have $\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$. Therefore, $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Observe that we can equivalently write $(A \cup B) \subseteq \bigcup_{G \in \mathcal{G}_{A \cup B}} G$ where $A \subseteq \bigcup_{G \in \mathcal{G}_A} G$ and $B \subseteq \bigcup_{G \in \mathcal{G}_B} G$, $\mathcal{G}_A$ and $\mathcal{G}_B$ being minimal covers of A and B respectively. And we can then write:

$$\begin{aligned}
\mu^*(A \cup B) &= \inf_{\mathcal{G} \in \mathcal{U}(A \cup B)} \sum_{G \in \mathcal{G}} \mu(G) = \sum_{G \in \mathcal{G}_{(A \cup B)}} \mu(G) \geq \sum_{\substack{G \in \mathcal{G}_A \\ G \in \mathcal{G}_B}} \mu(G) \\
&= \inf_{\mathcal{G} \in \mathcal{U}(A)} \sum_{G \in \mathcal{G}} \mu(G) + \inf_{\mathcal{G} \in \mathcal{U}(B)} \sum_{G \in \mathcal{G}} \mu(G) \\
&= \mu^*(A) + \mu^*(B)
\end{aligned}$$