universität freiburg

Measure theory for probabilists

Winter semester 2024

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 9 - $\mathcal{L}^p$-spaces

Exercise 1 (4 Points)**.**
For f in $\mathcal{L}^1[a,b]$, define $||f|| = \int_a^b x^2|f(x)|dx$. Show that this is a norm on $\mathcal{L}^1[a,b]$.

Exercise 2 (4 Points)**.**
For E a measurable set, and functions f in $\mathcal{L}^p(E)$, g in $\mathcal{L}^q(E)$ such that $\frac{1}{p}+\frac{1}{q}=1$, define

$$||f||_p = \left[\int_E |f|^p\right]^{\frac{1}{p}}.$$

Show that if Hölder's Inequality is true for normalized functions, it is true in general.

Exercise 3 (4 Points)**.**
Let $f : \Omega \to \mathbb{R}$ be measurable. Show that the following hold.

- If $\int |f|^p \, d\mu < \infty$ for some $p \in (0,\infty)$, then $||f||_p \xrightarrow{p\to\infty} ||f||_\infty$
- The integrability condition in (i) cannot be waived.

Exercise 4 (4 Points)**.**
Let $p \in (1,\infty)$, $f \in \mathcal{L}^p(\lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$. Let $T : \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1$. Show that

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow{n\to\infty} 0 \quad \text{in } \mathcal{L}^p(\lambda).$$