universität freiburg

Measure theory for probabilists — Winter semester 2024

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 10 - Theorem of Radon-Nikodym

Exercise 1 (4 Points).
Let λ, μ and ν be measures on $(\Omega, \mathcal{A})$. Show that:

(a) If for all $\varepsilon > 0$ there exists an $A \in \mathcal{A}$ with $\mu(A) < \varepsilon$ and $\nu(A^c) < \varepsilon$, then $\mu \perp \nu$.

(b) If $\lambda \ll \mu$ and $\mu \perp \nu$, then also $\lambda \perp \nu$.

(c) If $\mu \ll \nu$ and $\mu \perp \nu$, then $\mu \equiv 0$.

Exercise 2 (4 Points).
Let μ and ν be two measures on the measure space $(\Omega, \mathcal{A})$ and let ν be finite. Show that the following statements are equivalent:

(a) $\nu \ll \mu$.

(b) For every $\varepsilon > 0$ there is a $\delta > 0$, such that for all $A \in \mathcal{A}$ with $\mu(A) \leq \delta$, also $\nu(A) \leq \varepsilon$.

Exercise 3 (4 Points).
Let $\mathcal{F} = \{(a,b) \cap \mathbb{Q} : a,\, b \in \mathbb{R},\, a \leq b\}$. Define $\mu : \mathcal{F} \to [0,\infty)$ by $\mu\left((a,b) \cap \mathbb{Q}\right) = b - a$. Clearly, as in Example 1.3, $\mathcal{F}$ is a semi-ring. Show that μ is a content on $\mathcal{F}$ that is continuous from below and from above but is not σ-additive. (See Remark 2.3!)

Exercise 4 (4 Points).
Let $(\Omega,\mathcal{F},\mu)$ be a measure space.

(a) Let $(\Omega_1,\mathcal{F}_1)$ and $(\Omega_2,\mathcal{F}_2)$ be measurable spaces and $f_1 : \Omega \to \Omega_1$ a $\mathcal{F}/\mathcal{F}_1$-measurable function, and $f_2 : \Omega_1 \to \Omega_2$ a $\mathcal{F}_1/\mathcal{F}_2$-measurable function. Show that

$$(f_2 \circ f_1) * \mu = f_2 * (f_1 * \mu).$$

(b) Let $\Omega = \mathbb{Z}^2$ and $\mathcal{F} = \mathcal{P}(\mathbb{Z}^2)$, and let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be defined by $f(x,y) = x + y$. Determine $f * \mu$.