universität freiburg

Measure theory for probabilists **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 12 - $\mathcal{L}^p$-spaces II

In Exercise 1 and 2, we define a maximum norm $||f||_{\max} = \max_{x\in[a,b]}|f(x)|$. Also, we write $\mathcal{L}^p(E) = \left\{f : E \to \mathbb{R} \text{ measurable } \mid \int_E |f(x)|^p \, d\mu < \infty\right\}, \quad 1 \le p < \infty$.

Exercise 1 (4 Points)**.**
For f in $\mathcal{C}[a,b]$, the space of continuous real-valued functions on $[a,b]$, define

$$||f||_1 = \int_a^b |f|.$$

(a) Show that this is a norm on $\mathcal{C}[a,b]$.

(b) Show that there is no $c \ge 0$ for which

$$||f||_{\max} \le c||f||_1 \text{ for all } f \text{ in } \mathcal{C}[a,b].$$

(c) Show that there is a $c \ge 0$ for which

$$||f||_1 \le c||f||_{\max} \text{ for all } f \text{ in } \mathcal{C}[a,b].$$

Exercise 2 (4 Points)**.**
For f in $\mathcal{L}^\infty[a,b]$,

(a) Show that

$$||f||_\infty = \min\left\{M \mid m\left\{x \text{ in } [a,b] \mid f(x)| > M\right\} = 0\right\}.$$

(b) Furthermore, if f is continuous on $[a,b]$, show that

$$||f||_\infty = ||f||_{\max}.$$

Exercise 3 (6 Points)**.**

(a) Show that if $f(x) = \ln(1/x)$ for $x \in (0,1]$, then f belongs to $\mathcal{L}^p(0,1]$ for all $1 \le p < \infty$ but does not belong to $\mathcal{L}^\infty(0,1]$.

(b) Define $f(x) = \frac{x^{-1/2}}{1+|\ln x|}$ and $E = (0,\infty)$. Show that f belongs to $\mathcal{L}^p(E)$ if and only if $p = 2$.

Exercise 4 (4 Points)**.**
For $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} n^{\frac{1}{p}} & \text{if} \quad 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{if} \quad \frac{1}{n} \leq x \leq 1. \end{cases}$$

Let f be identically zero on [0,1]. Show that $\{f_n\}$ converges pointwise to f but does not converge in $\mathcal{L}^p$.