

The background of the slide features a large, faint watermark of the University of Basel seal. The seal is circular and contains a central figure, likely a saint or scholar, seated and holding a book. The figure is surrounded by a decorative border with Latin text. The entire slide has a solid blue background.

Stochastic Processes

19. The reflection principle

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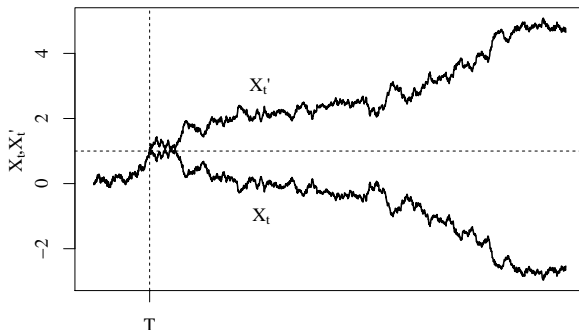
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The reflection principle

- Lemma 16.7: X BM, T stopping time. Then $X' = (X'_t)_{t \geq 0}$ with

$$X'_t := X_{t \wedge T} - (X_t - X_{t \wedge T}) = \begin{cases} X_t, & t \leq T, \\ 2X_T - X_t, & t > T \end{cases}$$

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- Define

$$Y_t := X_{t \wedge T}, \quad Z_t := X_{T+t} - X_T.$$

\mathcal{Z} is a BM. Hence, $(T, \mathcal{Y}, \mathcal{Z}) \stackrel{d}{=} (T, \mathcal{Y}, -\mathcal{Z})$. So,

$(\mathcal{Y}, \mathcal{Z}^T) \stackrel{d}{=} (\mathcal{Y}, -\mathcal{Z}^T)$ with $\mathcal{Z}^T := (Z_t^T)_{t \geq 0}$, $Z_t^T := Z_{(t-T)^+}$.

From this,

$$\mathcal{X} = \mathcal{Y} + \mathcal{Z}^T \stackrel{d}{=} \mathcal{Y} - \mathcal{Z}^T = \mathcal{X}'.$$

The maximum of BM

- ▶ Theorem 16.8: \mathcal{X} BM with $X_0 = 0$. Define $\mathcal{M} = (M_t)_{t \geq 0}$ by $M_t = \sup_{0 \leq s \leq t} X_s$. Then, $M_t \stackrel{d}{=} M_t - X_t \stackrel{d}{=} |X_t|$.
- ▶ φ_t : density of X_t , $\Rightarrow 2\varphi_t(x)1_{x \geq 0}$: density of $|X_t|$;

$T_x = \inf\{s \geq 0 : X_s = x\}$; $(X'_t)_{t \geq 0}$ BM mirrored at T ; $y \leq x$,

$$\mathbf{P}(M_t \geq x, X_t \leq y) = \mathbf{P}(X'_t \geq 2x - y) = \int_{2x-y}^{\infty} \varphi_t(z) dz,$$

$$\mathbf{P}(M_t \geq x) = \mathbf{P}(M_t \geq x, X_t \leq x) + \mathbf{P}(X_t \geq x) = 2 \int_x^{\infty} \varphi_t(z) dz,$$

so $M_t \stackrel{d}{=} |X_t|$. We further calculate

$$\begin{aligned} \mathbf{P}(M_t - X_t \geq x) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^{\infty} \mathbf{P}(z \leq M_t \leq z + \varepsilon, X_t \leq z - x) dz \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^{\infty} \mathbf{P}(M_t \geq z, X_t \leq z - x) - \mathbf{P}(M_t \geq z + \varepsilon, X_t \leq z - x) dz \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^{\infty} \left(\int_{z+x}^{\infty} \varphi_t(y) dy - \int_{z+x+2\varepsilon}^{\infty} \varphi_t(y) dy \right) dz = \int_x^{\infty} 2\varphi(z) dz. \end{aligned}$$