universität freiburg

Stochastic processes

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 9 - Markov processes I

Exercise 1 (4 points)**.**
Let $\mathcal{X} = (X_t)_{t\in[0,\infty)}$ be a stochastic process. Show that $\mathcal{X}$ is Markov if and only if, for all $s \le t \le u$, and all measurable A,

$$\mathbf{P}(X_u \in A|X_s,X_t) = \mathbf{P}(X_u \in A|X_t).$$

Solution.
'$\Longrightarrow$': For $s,t,u \in [0,\infty)$ with $s \le t \le u$ and a measurable A, using the fact that $\sigma(X_t) \subseteq \sigma(X_s,X_t) \subseteq \mathcal{F}_t$ and assuming that $\mathcal{X}$ is a Markov process, then the following applies:

$$\begin{aligned}\mathbf{P}(X_u \in A|X_s,X_t) &= \mathbf{E}[\mathbb{1}_{\{X_u\in A\}}|\sigma(X_s,X_t)]\\ &= \mathbf{E}\left[\mathbf{E}[\mathbb{1}_{\{X_u\in A\}}|\mathcal{F}_t]|\sigma(X_s,X_t)\right]\\ &= \mathbf{E}\left[\mathbf{E}[\mathbb{1}_{\{X_u\in A\}}|\sigma(X_t)]|\sigma(X_s,X_t)\right]\\ &= \mathbf{E}[\mathbb{1}_{\{X_u\in A\}}|\sigma(X_t)]\\ &= \mathbf{P}(X_u \in A|X_t).\end{aligned}$$

'$\Longleftarrow$': For $s \le t \le u$; $s,t,u \in [0.\infty)$ by the law of total probability, for any σ-algebra $\mathcal{F}_t$ generated by (X_s,X_t) it holds that $\mathbf{P}(X_u \in A|X_t) \le \mathbf{P}(X_u \in A|\mathcal{F}_t)$. Also, the following applies:

$$\begin{aligned}\mathbf{P}(X_u \in A|X_t) &= \mathbf{P}(X_u \in A|X_t,X_{i_1}) = \mathbf{P}(X_u \in A|X_t,X_{i_1},X_{i_2})\\ &= \ldots = \mathbf{P}(X_u \in A|X_t,(X_i)_{i\in I}) = \mathbf{P}(X_u \in A|\underbrace{\sigma(X_t,(X_i)_{i\in I})}_{=\mathcal{F}_t})\\ &= \mathbf{P}(X_u \in A|\mathcal{F}_t).\end{aligned}$$

Exercise 2 (2+2=4 points)**.**
Decide and reason if or if not the following stochastic processes are Markov:

(a) $X_t = \phi(B_t), t \ge 0$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and $(B_t)_{t\ge0}$ is Brownian Motion.

(b) For $n = 1,2,...$ let $X_n = Z_n + Z_{n-1}$, where $Z_i, i = 0,1,2,\ldots$ are iid with $\mathbf{P}(Z_1 = 0) = \mathbf{P}(Z_1 = 1) = \frac{1}{2}$.

Solution.

(a) Recall from equation 15.1 that the process $\mathcal{X}$ is called *Markov process* if $\mathcal{F}_s$ is independent of X_t given X_s, $s \le t$. That is,

$$\mathbf{E}(f(X_t)|\mathcal{F}_s) = \mathbf{E}(f(X_t)|X_s)$$

for all measurable and bounded $f : E \to \mathbb{R}$. Since ϕ is strictly increasing, ϕ is continuous and thus measurable. Furthermore, ϕ is bijective $\implies \sigma(B_s) = \sigma(\phi(B_s)) \forall s \in \mathbb{R}^+$. Clearly, $(B_t)_{t\ge 0}$ is a Markov process (see Example 15.4.3) and since $f \circ \phi$ is measurable for all $f \in \mathcal{C}_b(\mathbb{R})$,$f \circ \phi \in \mathcal{C}_b(\mathbb{R})$, the following applies:

$$\begin{aligned}\mathbf{E}[f(X_t)|\mathcal{F}_s] &= \mathbf{E}[f(\phi(B_t))|\mathcal{F}_s] \\ &= \mathbf{E}[(f \circ \phi)(B_t)|\mathcal{F}_s] \\ &= \mathbf{E}[(f \circ \phi)(B_t)|\mathcal{B}_s] \\ &= \mathbf{E}[f(\phi(B_t))|\phi(B_s)] \\ &= \mathbf{E}[f(X_t)|X_s].\end{aligned}$$

(b) We claim that is not a Markov process and we again use equation 15.1. The process $\mathcal{X}$ is called *Markov process* if for $A \in \mathcal{B}(E)$

$$\mathbf{P}(X_t \in A|\mathcal{F}_s) = \mathbf{P}(X_t \in A|X_s)$$

Now, take for instance, on the one hand,

$$\begin{aligned}\mathbf{P}(X_3 \in \{1\}|\mathcal{F}_2) &= \mathbf{P}(X_3 = 1|X_2 = 1, X_1 = 2) \\ &= \mathbf{P}(Z_2 + Z_3 = 1|Z_0 = 1, Z_1 = 1, Z_2 = 0) \\ &= \mathbf{P}(Z_3 = 1) = \frac{1}{2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{P}(X_3 \in \{1\}|X_2 = 1) &= \mathbf{P}(Z_2 + Z_3 = 1| \text{ either } (Z_1 = 1, Z_2 = 0) \text{ or } (Z_1 = 0, Z_2 = 1)) \\ &> \mathbf{P}(Z_2 + Z_3 = 1|Z_1 = 1, Z_2 = 0) = \mathbf{P}(Z_3 = 1) = \frac{1}{2}\end{aligned}$$

That is,

$$\mathbf{P}(X_3 \in \{1\}|\mathcal{F}_2) = \frac{1}{2} \neq \mathbf{P}(X_3 \in \{1\}|X_2).$$

Exercise 3 (2+2=4 Points)**.**
Let $(X_t)_{t\ge 0}$ be a standard Brownian motion and $(Y_t)_{t\ge 0} := e^{-t/2} X_{e^t - 1}$.

(a) Show that $(Y_t)_{t\ge 0}$ is a Gaussian process and a Markov process.

(b) Determine the weak limit of Y_t for $t \to \infty$.

Solution.
We may use Theorem 15.5 here. A Gaussian process $(X_t)_{t\ge 0}$ is a Markov process if and only if $\mathbf{Cov}(X_s, X_u)\mathbf{Var}(X_t) = \mathbf{Cov}(X_s, X_t)\mathbf{Cov}(X_t, X_u)$ applies to all $s \le t \le u$.

(a) Since the time index $t \mapsto e^t - 1$ is strictly monotonically increasing, it is clear that $(Y_t)_{t\geq 0}$ is Gaussian. We show the Markov property as follows for $s \leq t \leq u$:

$$\begin{aligned}
&\mathbf{Cov}\Big(Y_s, Y_u\Big)\mathbf{Var}\Big(X_t\Big) \\
&\qquad = e^{-s/2}e^{-u/2}(e^{-t/2})^2\mathbf{Cov}\Big(X_{e^s-1}, X_{e^u-1}\Big)\mathbf{Var}\Big(X_{e^t-1}\Big) \\
&\qquad = e^{-s/2}e^{-t/2}(e^s - 1)\cdot e^{-t/2}e^{-u/2}(e^t - 1) \\
&\qquad = e^{-s/2}e^{-t/2}\mathbf{Cov}\Big(X_{e^s-1}, X_{e^t-1}\Big)\cdot e^{-t/2}e^{-u/2}\operatorname{Cov}\Big(X_{e^t-1}, X_{e^u-1}\Big) \\
&\qquad = \mathbf{Cov}\Big(Y_s, Y_t\Big)\mathbf{Cov}\Big(Y_t, Y_u\Big).
\end{aligned}$$

(b) For the weak limit, we consider the distribution functions at any point $x \in \mathbb{R}$:

$$\mathbf{P}(Y_t \leq x) = \mathbf{P}\big(X_{e^t-1} \leq xe^{t/2}\big) = \mathbf{P}\Big(X_1 \leq \frac{x}{\sqrt{1-e^{-t}}}\Big) \xrightarrow{t\to\infty} \mathbf{P}(X_1 \leq x).$$

Therefore, the weak limit is standard-normally distributed.

Exercise 4 (2+2=4 Points)**.**

Let $f : [0,\infty) \to [0,\infty)$ be strictly monotonically increasing with $f(0) = 0$, $\mathcal{P} = (P_t)_{t\geq 0}$ a Poisson process with intensity 1 and $\mathcal{M} = (M_n)_{n=0,1,\ldots}$ a Markov chain in discrete time with values in $\mathbb{Z}$ and transition matrix $\Pi = (\pi_{ij})_{i,j\in\mathbb{Z}}$. Furthermore, let $\mathcal{P}$ and $\mathcal{M}$ be stochastically independent.

(a) Show that $\mathcal{X} = (X_t)_{t\geq 0}$ with $X_t := M_{P_{f(t)}}$ is a Markov process with respect to its natural filtration.

(b) Determine its transition kernels and operators.

Solution.

(b) For $K \sim \operatorname{Poi}(f(t) - f(s))$ and $\Pi^k = (\pi_{ij}^{(k)})_{ij}$ it follows that

$$\begin{aligned}
\mu_{s,t}^{\mathcal{X}}(x,\{z\}) &= \mathbf{P}(X_t = z|X_s = x) \\
&= \sum_{k\geq 0}\mathbf{P}(P_{f(t)} - P_{f(s)} = k)\cdot\mathbf{P}(M_k - M_0 = z|M_0 = x) \\
&= \sum_{k\geq 0}\frac{(f(t)-f(s))^k}{k!\exp(f(t)-f(s))}\pi_{xz}^{(k)} = \mathbf{E}[\pi_{xz}^{(K)}] = \mathbf{P}(M_K = z|M_0 = x).
\end{aligned}$$

With K as above and $g : \mathbb{Z} \to \mathbb{R}$ results in

$$\begin{aligned}
\Big(T_{s,t}^{\mathcal{X}}g\Big)(x) &= \mathbf{E}[g(X_t)|X_s = x] = \sum_{z\in\mathbb{Z}} g(z)\mu_{s,t}^{\mathcal{X}}(x,\{z\}) \\
&= \sum_{z\in\mathbb{Z}} g(z)\sum_{k\geq 0}\frac{(f(t)-f(s))^k}{k!\exp(f(t)-f(s))}\pi_{xz}^{(k)} = \mathbf{E}[g(M_K)|M_0 = x].
\end{aligned}$$

(a) We show that the distribution of $\mathcal{X}$ is the distribution of a Markov process. For $s \leq t \leq u$, $x,z \in \mathbb{Z}$, K as above and $L \sim \operatorname{Poi}(f(u) - f(t))$ independent of K follows after monotonic convergence and ...

$$\begin{aligned}
\mu_{s,t}^{\mathcal{X}}\mu_{t,u}^{\mathcal{X}}(x,\{z\}) &= \sum_{y\in\mathbb{Z}}\mathbf{E}[\pi_{xy}^{(K)}]\mathbf{E}[\pi_{yz}^{(L)}] = \mathbf{E}[\sum_{y\in\mathbb{Z}}\pi_{xy}^{(K)}\pi_{yz}^{(L)}] \\
&= \mathbf{E}[\pi_{xz}^{(K+L)}] = \mu_{s,u}^{\mathcal{X}}(x,\{z\}), \text{ since } K + L \sim \operatorname{Poi}(f(u) - f(s)).
\end{aligned}$$