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Stochastic processes — Winter semester 2024

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
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Tutorial 5 - Martingales I

Exercise 1 (2+2=4 Points).

(a) Let $X \sim N(\mu,\sigma^2)$ with $\mu \neq 0$ and $\sigma^2 > 0$. Prove that there is a unique $\theta \neq 0$ such that $\mathbf{E}[e^{\theta X}] = 1$.

(b) Let $(X_i)_{i=1,2,\dots}$ be iid with $X_0 \sim N(\mu,\sigma^2)$ with $\mu \neq 0$ and $\sigma^2 > 0$. Show that $\mathcal{Z} = (Z_n)_{n=0,1,2,\dots}$ with $Z_n = e^{\theta \sum_{j=1}^n X_j}$ is a martingale with θ defined in (a).

Solution.

(a) We can use characteristic functions here and use the results from 6.13(3). So we write:

$$1 = \mathbf{E}[e^{\theta X}] = \mathbf{E}[e^{i(-i\theta)X}] = e^{i(-i\theta)\mu} \cdot e^{-\sigma^2(i\theta)^2/2} = e^{\theta\mu+\sigma^2\theta^2/2} = e^{\theta(\mu+\sigma^2\theta/2)}$$
$$\iff 0 = \theta\left(\mu + \frac{\sigma^2\theta}{2}\right) \iff \theta = 0 \quad \text{or} \quad \theta = -\frac{2\mu}{\sigma^2} \neq 0.$$

(b) We will use 14.4.3 and remark 14.3 here and our aim is to check the property of martingales as in Definition 14.2. First, for $i = 1,2,\dots$, define $Y_i := e^{\theta X_i}$. Then with θ in (a), $\mathbf{E}[Y_i] = \mathbf{E}[e^{\theta X_i}] = 1$. Clearly, $\mathbf{E}[Z_0] = 1$ since $Z_0 = e^{\theta\cdot 0} = 1$. Let $\mathcal{F}_t = \sigma(Z_s : s \le t) = \sigma(X_s : s \le t)$. Since $(X_i)_{i=1,2,\dots}$ are iid, for $t = 1,2,\dots$, we have:

$$\mathbf{E}[Z_t] = \mathbf{E}[e^{\theta\sum_{j=1}^t X_j}] = \mathbf{E}[\prod_{j=1}^t e^{\theta X_j}] = \prod_{j=1}^t \mathbf{E}[e^{\theta X_j}] = \prod_{j=1}^t 1 = \prod_{j=1}^t \mathbf{E}[Y_j] = 1 < \infty.$$

Lastly, since Z_{t-1} is $\mathcal{F}_{t-1}$-measurable, and X_t is independent of $\mathcal{F}_{t-1}$, we can write:

$$\mathbf{E}[Z_t|\mathcal{F}_{t-1}] = \mathbf{E}[Z_{t-1}\cdot e^{\theta X_t}|\mathcal{F}_{t-1}] = Z_{t-1}\cdot\mathbf{E}[e^{\theta X_t}|\mathcal{F}_{t-1}] = Z_{t-1}\cdot\mathbf{E}[e^{\theta X_t}] = Z_{t-1}\cdot 1 = Z_{t-1}.$$

Exercise 2 (4 points).
Let $\mathcal{X} = (X_n)_{n\ge 0}$ be a super-martingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\ge 0}$. Show the following: $\mathcal{X}$ is a martingale iff there exists a sequence $(n_m)_{m\ge 1}$ with $n_m \xrightarrow{m\to\infty} \infty$ and $\mathbf{E}[X_{n_m}] \ge \mathbf{E}[X_0]$.

Solution.
Note that $n \mapsto \mathbf{E}[X_n]$ is non-increasing for $\mathcal{X}$ by assumption.
$\Rightarrow$: If $\mathcal{X}$ is a martingale, $n \mapsto \mathbf{E}[X_n]$ is constant, so choose $m_n = n$.
$\Leftarrow$: Assume that there is an n with $\mathbf{E}[X_n|\mathcal{F}_{n-1}] < X_{n-1}$ on some set of positive probability. Choose m such that $n_m > n$. Then,

$$\mathbf{E}[X_{n_m}] \leq \mathbf{E}[X_n] < \mathbf{E}[X_{n-1}],$$

which contradicts the fact that $n \mapsto \mathbf{E}[X_n]$ is a constant if $\mathcal{X}$ is a martingale.

Solution.
$\Rightarrow$: If $\mathcal{X}$ is a martingale, take a sequence $(n_m)_{m\geq 1}$ defined by $n_m = m$. Then it follows for every $m \geq 1$ that

$$\mathbf{E}[X_{n_m}] = \mathbf{E}[\mathbf{E}[X_{n_m}|\mathcal{F}_0]] = \mathbf{E}[X_0]$$

$\Leftarrow$: Let $s,t \in \mathbb{N}$, $s < t$. Then there is $n \in \mathbb{N}$ such that $n_m > t$:

$$\mathbf{E}[X_0] \leq \mathbf{E}[X_{n_m}] = \mathbf{E}[\mathbf{E}[X_{n_m}|\mathcal{F}_t]] \leq \mathbf{E}[X_t] = \mathbf{E}[\mathbf{E}[X_t|\mathcal{F}_s]] \leq \mathbf{E}[X_s] = \mathbf{E}[\mathbf{E}[X_s|\mathcal{F}_0]] \leq \mathbf{E}[X_0].$$

Thus, $\mathbf{E}[\mathbf{E}[X_t|\mathcal{F}_s]] = \mathbf{E}[X_s]$ which also means $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$. Hence, we are done.

Exercise 3 (2+2=4 Points)**.**
Let $\mathcal{X} = (X_t)_{t\geq 0}$ and $\mathcal{Y} = (Y_t)_{t\geq 0}$ be square integrable stochastic processes, adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. They are said to be conditionally uncorrelated, if

$$\mathbf{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = 0, \qquad 0 \leq s \leq t < \infty.$$

(a) Give an example of non-conditionally uncorrelated $\mathcal{X}$ and $\mathcal{Y}$.

(b) Show that $(X_tY_t)_{t\geq 0}$ is a martingale iff $\mathcal{X}$ and $\mathcal{Y}$ are conditionally uncorrelated.

Solution.

(a) We consider $Z \sim \mathrm{Ber}(p)$ with $p \in (0,1)$. For $t \geq 0$, let $X_t = tZ$ and $Y_t = tZ + 1$. We first note that $\mathbf{E}[X_t^2] = \mathbf{E}[(tZ)^2] = t^2\mathbf{E}[Z^2] = t^2p$. Similarly, $\mathbf{E}[Y_t^2] = 3t^2p + 1$. Define $\mathcal{F}_t = \sigma(Z_s : s \leq t)$. We claim that $\mathcal{X} = (X_t)_{t\geq 0}$ and $\mathcal{Y} = (Y_t)_{t\geq 0}$ are non-conditionally uncorrelated and we show that

$$\mathbf{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] \neq 0, \qquad 0 \leq s \leq t < \infty.$$

Now, $X_t - X_s = (t-s)Z$ and $Y_t - Y_s = (t-s)Z$. Thus,

$$\mathbf{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = \mathbf{E}[(t-s)^2Z^2|\mathcal{F}_s] = (t-s)^2\mathbf{E}[Z^2|\mathcal{F}_s] = (t-s)^2 \cdot p \neq 0.$$

(b) We begin by writing

$$\begin{aligned}
\mathbf{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] &= \mathbf{E}[X_tY_t - X_tY_s - X_sY_t + X_sY_s|\mathcal{F}_s] \\
&= \mathbf{E}[X_tY_t|\mathcal{F}_s] - \mathbf{E}[X_tY_s|\mathcal{F}_s] - \mathbf{E}[X_sY_t|\mathcal{F}_s] + \mathbf{E}[X_sY_s|\mathcal{F}_s] \\
&= \mathbf{E}[X_tY_t|\mathcal{F}_s] - Y_s\mathbf{E}[X_t|\mathcal{F}_s] - X_s\mathbf{E}[Y_t|\mathcal{F}_s] + X_sY_s \\
&= \mathbf{E}[X_tY_t|\mathcal{F}_s] - Y_sX_s - X_sY_s + X_sY_s \\
&= \mathbf{E}[X_tY_t|\mathcal{F}_s] - X_sY_s,
\end{aligned}$$

where we have used the fact that $\mathcal{X}$ and $\mathcal{Y}$ are martingales.

$\Rightarrow$: If $(X_tY_t)_{t\geq 0}$ is a martingale, then $\mathbf{E}[X_tY_t|\mathcal{F}_s] = X_sY_s$ which consequently makes $\mathbf{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = 0$. That is $\mathcal{X}$ and $\mathcal{Y}$ are conditionally uncorrelated.

$\Leftarrow$: If $\mathcal{X}$ and $\mathcal{Y}$ are conditionally uncorrelated, then $0 = \mathbf{E}[X_tY_t|\mathcal{F}_s] - X_sY_s$, which means $\mathbf{E}[X_tY_t|\mathcal{F}_s] = X_sY_s$ that is $(X_tY_t)_{t\geq 0}$ is a martingale. Of course, $(X_tY_t)_{t\geq 0}$ is $\mathcal{F}_t$-measurable since X_t and Y_t are both $\mathcal{F}_t$-measurable and we can also use Proposition 6.5 to verify that the expectation is finite!

$$\mathbf{E}[|X_tY_t|] \leq \mathbf{E}[|X_t|^2]^{1/2} \cdot \mathbf{E}[|Y_t|^2]^{1/2} < \infty \qquad \text{(Cauchy-Schwartz inequality)}$$

Exercise 4 (2+2=4 points)**.**
Let $\mathcal{B} = (B_t)_{t\geq 0}$ be a Brownian Motion, started in $B_0 = 0$. For a constant $a > 0$ define $T := \inf\{t \geq 0 : B_t \notin (-a,a)\}$.

(a) Why is T a stopping time with respect to $\mathcal{F} = \{\mathcal{F}_t^{\mathcal{B}}\}_{t\geq 0}$.

(b) Show that, for all c,

$$X_t := \exp\left(-\frac{c^2}{2}t\right)\cosh(cB_t)$$

defines a martingale $(X_t)_{t\geq 0}$ with respect to $\{\mathcal{F}_t^{\mathcal{B}}\}_{t\geq 0}$.

Solution.

(a) For $A := (-\infty, -a] \cup [a,\infty)$ the random time T obeys the equation

$$T = \inf\{t \geq 0 : B_t \in A\}.$$

Since A is a closed set in $\mathbb{R}$ and $\mathcal{B}$ has continuous path, Proposition 13.30 implies that T is a stopping time.

(b) Recall that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ for all $x \in \mathbb{R}$. We check the property of martingales as in Definition 14.2.

(i) For each $t \geq 0$ the random variable B_t is $\mathcal{F}_t^{\mathcal{B}}$-measurable since it is the image of the continuous function $f : \mathbb{R} \to \mathbb{R}$, $\quad f(x) = e^{-\frac{c^2}{2}t}\cosh(cx)$, applied to B_t.

(ii) For all $t \geq 0$, we obtain

$$\mathbf{E}\left[\left|e^{-\frac{c^2}{2}t}\cosh(cB_t)\right|\right] = \frac{1}{2}e^{-\frac{c^2}{2}t}\mathbf{E}\left[e^{cB_t} + e^{-cB_t}\right] = e^{-\frac{c^2}{2}t} \cdot e^{\frac{c^2}{2}t} = 1 < \infty.$$

Here, we have recalled that $B_t \sim N(0,t)$ and the moment generating function (MGF) of a normally distributed random variable $X \sim N(\mu,\sigma^2)$ is given by: $\mathbf{E}[e^{\lambda X}] = e^{\mu\lambda + \frac{1}{2}\sigma^2\lambda^2}$. In our case, B_t has mean 0 and variance t, we substitute $\mu = 0$, $\sigma^2 = t$, and $\lambda = c$ so that $\mathbf{E}[e^{cB_t}] = e^{\frac{1}{2}tc^2} = \mathbf{E}[^{-cB_t}]$.

(iii) Note that $B_t = B_s + (B_t - B_s)$, where $B_t - B_s \sim N(0,t-s)$ is independent of $\mathcal{F}_s^{\mathcal{B}}$ and B_s is $\mathcal{F}_s^{\mathcal{B}}-$measurable. For each $0 \le s \le t$ it follows that:

$$\begin{aligned}
\mathbf{E}\left[e^{-\frac{c^2}{2}t}\cosh(cB_t)|\mathcal{F}_s^{\mathcal{B}}\right] &= \frac{1}{2}\left(\mathbf{E}\left[e^{-\frac{c^2}{2}t}\cdot(e^{cB_t}+e^{-cB_t}|\mathcal{F}_s^{\mathcal{B}}\right]\right) \\
&= \frac{1}{2}e^{-\frac{c^2}{2}t}\cdot\mathbf{E}\left[e^{c\cdot(B_s+B_t-B_s)}+e^{-c\cdot(B_s+B_t-B_s)}|\mathcal{F}_s^{\mathcal{B}}\right] \\
&= \frac{1}{2}e^{-\frac{c^2}{2}t}\cdot\left(e^{cB_s}\mathbf{E}\left[e^{c\cdot(B_t-B_s)}|\mathcal{F}_s^{\mathcal{B}}\right]+e^{-cB_s}\mathbf{E}\left[e^{-c\cdot(B_t-B_s)}|\mathcal{F}_s^{\mathcal{B}}\right]\right) \\
&= \frac{1}{2}e^{-\frac{c^2}{2}t}\cdot\left(e^{cB_s}\cdot\mathbf{E}\left[e^{c(B_t-B_s)}\right]+e^{-cB_s}\mathbf{E}\left[e^{-c(B_t-B_s)}\right]\right) \\
&= \frac{1}{2}e^{-\frac{c^2}{2}t}\cdot\left(e^{cB_s}\cdot e^{\frac{1}{2}(t-s)c^2}+e^{-cB_s}\cdot e^{\frac{1}{2}(t-s)c^2}\right) \\
&= \frac{1}{2}e^{-\frac{c^2}{2}t}\cdot e^{\frac{c^2}{2}(t-s)}\cdot(e^{cB_s}+e^{-cB_s}) \\
&= \frac{1}{2}e^{-\frac{c^2}{2}s}(e^{cB_s}+e^{-cB_s}) = e^{-\frac{c^2}{2}s}\cosh(cB_s).
\end{aligned}$$