

The background of the slide features a large, faint watermark of the University of Basel seal. The seal is circular and contains a central figure, likely a saint or scholar, seated and holding a book. Above the figure are three smaller figures in niches. The entire seal is surrounded by a Latin inscription in a circular border.

Stochastic Processes

11. The CLT for martingales

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November 29, 2024

The CLT for Martingales

- ▶ Theorem 14.42: $I^n = \{0, 1, 2, \dots, t_n\}$, $\mathcal{M}^n = (M_t^n)_{t \in I^n}$ martingale with $M_0^n = 0$. For $X_t^n := M_t^n - M_{t-1}^n$, assume

$$\mathbf{E}[\max_{1 \leq s \leq t_n} |X_s^n|] \xrightarrow{n \rightarrow \infty} 0, \quad \sum_{s=1}^{t_n} (X_s^n)^2 \xrightarrow{n \rightarrow \infty} \sigma^2 > 0$$

Then $M_{t_n}^n \xrightarrow{n \rightarrow \infty} X$ with $X \sim N(0, \sigma^2)$.

- ▶ Example: X_1, X_2, \dots iid with $\mathbf{E}[X_1] = 0$, $\mathbf{V}[X_1] = \sigma^2$ and

$$M_t^n = \frac{1}{\sqrt{n}} \sum_{s=1}^t X_s \xrightarrow{n \rightarrow \infty} X \sim N(0, \sigma^2).$$

Then $\int_0^\infty t \mathbf{P}(|X_1| > t) dt < \infty$, $t^2 \mathbf{P}(|X_1| > t) \xrightarrow{t \rightarrow \infty} 0$,

$$\begin{aligned} \mathbf{E}[\max_{1 \leq s \leq n} |X_s| / \sqrt{n}] &= \int_0^\infty \mathbf{P}(\max_{1 \leq s \leq n} |X_s| > t \sqrt{n}) dt \\ &= \int_0^\infty 1 - \left(1 - \frac{t^2 n \mathbf{P}(|X_1| > t \sqrt{n})}{t^2 n}\right)^n dt \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Convergence of Products

► Lemma 14.43: $U_1, U_2, \dots, T_1, T_2, \dots$ rvs with

1. $U_n \xrightarrow{n \rightarrow \infty}_p u$,
2. $(T_n)_{n=1,2,\dots}$ and $(T_n U_n)_{n=1,2,\dots}$ ui,
3. $\mathbf{E}[T_n] \xrightarrow{n \rightarrow \infty} 1$,

Then $\mathbf{E}[T_n U_n] \xrightarrow{n \rightarrow \infty} u$.

Proof: we show $T_n(U_n - u) \xrightarrow{n \rightarrow \infty}_p 0$

(then $\mathbf{E}[T_n(U_n - u)] \xrightarrow{n \rightarrow \infty} 0$ due to ui)

For $\varepsilon > 0$, let K be such that $\sup_n \mathbf{P}(|T_n| > K) \leq \varepsilon$.

Now $xy > \delta\varepsilon \Rightarrow x > \delta$ or $y > \varepsilon$, therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{P}(|T_n(U_n - u)| > \varepsilon) \\ \leq \limsup_{n \rightarrow \infty} \mathbf{P}(|U_n - u| > \varepsilon/K) + \mathbf{P}(|T_n| > K) \leq \varepsilon. \end{aligned}$$

An estimate for the exponential function

- Lemma 14.44: There is $C > 0$ and a function r with $|r(x)| \leq C|x^3|$, such that

$$\exp(ix) = (1 + ix) \exp(-x^2/2 + r(x))$$

for all $x \in \mathbb{R}$. In addition, $|1 + ix| \leq e^{x^2/2}$ for all $x \in \mathbb{R}$.

Proof: For small $|x|$ we write

$$\begin{aligned} & \left| \exp(ix) - (1 + ix) \exp(-x^2/2) \right| \\ &= \left| 1 + ix - x^2/2 - (1 - ix)(1 - x^2/2) \right| + O(|x|^3) = O(|x|^3). \end{aligned}$$

Proof of Theorem 13.43

- Wlog, $\sum_{r=1}^s (X_r^n)^2 \leq 2\sigma^2$, $s = 1, 2, \dots$. For $\lambda > 0$ we show

$$\mathbf{E}[e^{i\lambda M_{t_n}^n}] = \underbrace{\prod_{s=1}^{t_n} (1 + i\lambda X_s^n)}_{=: T_n} \cdot \underbrace{e^{-\frac{\lambda^2}{2} \sum_{s=1}^{t_n} (X_s^n)^2 + \sum_{s=1}^{t_n} r(\lambda X_s^n)}}_{=: U_n \xrightarrow{n \rightarrow \infty} u := e^{-\lambda^2 \sigma^2 / 2}} \xrightarrow{n \rightarrow \infty} e^{-i\lambda^2 \sigma^2 / 2}$$

We use lemma 14.43:

1. is clear; for 2. $T_n U_n = e^{i\lambda M_{t_n}^n}$ uniformly integrable. Further

$$\begin{aligned} |T_n| &= \prod_{s=1}^{t_n-1} |1 + i\lambda X_s^n| \cdot |1 + i\lambda X_{t_n}^n| \leq \exp\left(\frac{\lambda^2}{2} \sum_{s=1}^{t_n-1} (X_s^n)^2\right) (1 + \lambda |X_{t_n}^n|) \\ &\leq \exp(\lambda^2 \sigma^2) \cdot (1 + |\lambda| \cdot \max_{1 \leq s \leq t_n} |X_s^n|) \xrightarrow{n \rightarrow \infty}_{L^1} 0, \end{aligned}$$

so that (T_n) is ui; for 3.,

$$\mathbf{E}[T_n] = \mathbf{E}\left[\prod_{s=1}^{t_n-1} (1 + i\lambda X_s^n) \cdot \mathbf{E}[(1 + i\lambda X_{t_n}^n) | \mathcal{F}_{t_n-1}]\right] = \dots = 1.$$

Example

- Y_1, Y_2, \dots iid, bounded, $\mathbf{E}[Y_1] = 0$, $\mathbf{V}[Y_1] = 1$,

$$H_s = \frac{1}{s-1}(Y_1^2 + \dots + Y_{s-1}^2), \quad M_t^n = \frac{1}{\sqrt{n}} \sum_{s=1}^t Y_s.$$

Then

$$(\mathcal{H} \cdot \mathcal{M}^n)_t = \sum_{s=1}^t \underbrace{\frac{1}{\sqrt{n}} Y_s \frac{1}{s-1} \sum_{r=1}^{s-1} Y_r^2}_{=: X_s^n}$$

is a martingale. Now $\mathbf{E}[\max_{1 \leq s \leq t_n} |X_s^n|] \xrightarrow{n \rightarrow \infty} 0$ because of the boundedness of Y_1, Y_2, \dots and

$$\sum_{s=1}^n (X_s^n)^2 = \frac{1}{n} \sum_{s=1}^n Y_s^2 \left(\frac{1}{s-1} \sum_{r=1}^{s-1} Y_r^2 \right)^2 \xrightarrow{n \rightarrow \infty} 1,$$

from which $(\mathcal{H} \cdot \mathcal{M}^n)_n \xrightarrow{n \rightarrow \infty} X \sim N(0, 1)$ follows.