universitätfreiburg

Stochastic processes

Winter semester 2024

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 6 - The stochastic integral as a martingale

Exercise 1 (4 points)**.**
Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d random variables with $\mathbf{P}(Y_1 = 1) = \mathbf{P}(Y_1 = -1) = \frac{1}{2}$, and let $X_0 = 0$ and for all $n \in N$,

$$X_n = \begin{cases} Y_1 & \text{if} \quad X_{n-1} = 0 \\ X_{n-1} + Y_n & \text{otherwise,} \end{cases}$$

which means $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ behave like a random walk as long as it does not hit 0, and always jump as Y_1 just after hitting 0. We also define $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.

(a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ adapted process with $\mathbf{E}[X_{n+1}|X_n] = X_n$ for all $n \in \mathbb{N}_0$.

(b) Show that $(X_n)_{n\in\mathbb{N}_0}$ is not an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale.

(c) Find a Doob decomposition for $(X_n)_{n\in\mathbb{N}_0}$.

Solution.

(a) We will show by mathematical induction that for all $n \in \mathbb{N}_0$, X_n is $\mathcal{F}_n$-measurable. Clearly, for $n = 0$, we have $X_0 = 0$ so X_0 is $\mathcal{F}_0$-measurable. For $(n-1) \in \mathbb{N}$ suppose that X_{n-1} is $\mathcal{F}_{n-1}$-measurable, since by definition $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$, Y_1, Y_n are $\mathcal{F}_n$-measurable. So, $X_{n-1} + Y_n$ is $\mathcal{F}_n$-measurable, since by implication X_{n-1} is $\mathcal{F}_n$-measurable. For every $A \in \mathcal{B}(\mathbb{R})$,

$$X_n^{-1}(A) = \begin{cases} Y_1^{-1}(A) & \text{if} \quad X_{n-1} = 0 \\ X_{n-1} + Y_n^{-1}(A) & \text{otherwise,} \end{cases} \in \mathcal{F}_n$$

So, $(X_n)_{n\in\mathbb{N}_0}$ is an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ adapted process. Now, we will establish that $\mathbf{E}[X_{n+1}|X_n] = X_n$ for all $n \in \mathbb{N}_0$. For $n = 0$,

$$\mathbf{E}[X_1|X_0] = \mathbf{E}[Y_1|X_0] = \mathbf{E}[Y_1|\sigma(X_0)] = \mathbf{E}[Y_1].$$

The last equality is due to the fact that $X_0 = 0$ is a constant and independent of Y_1. So,

$$\mathbf{E}[X_{n+1}|X_n] = 1 \cdot \mathbf{P}(Y_1 = 1) + (-1) \cdot \mathbf{P}(Y_1 = -1) = 0.$$

Note that $X_1 = Y_1 \implies \sigma(X_1) = \sigma(Y_1) \implies \sigma(X_1) = \mathcal{F}_1 \subseteq \mathcal{F}_n \; \forall \; n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$:

If $X_n = 0$;

$$\mathbf{E}[X_{n+1}|X_n] = \mathbf{E}[Y_1|X_n] = \mathbf{E}[Y_1|X_0] = 0 = X_n,$$

and if $X_n \neq 0$,

$$\mathbf{E}[X_{n+1}|X_n] = \mathbf{E}[X_n + Y_{n+1}|X_n] = X_n + \mathbf{E}[Y_{n+1}|X_n] = X_n.$$

So we are done.

(b) We will show that $(X_n)_{n\in\mathbb{N}_0}$ is not an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale. First, we observe the following: If $X_n = 0$;

$$\mathbf{E}[X_{n+1}|\mathcal{F}_n] = \mathbf{E}[Y_1|\mathcal{F}_n] = Y_1 = X_1,$$

and if $X_n \neq 0$,

$$\mathbf{E}[X_{n+1}|\mathcal{F}_n] = \mathbf{E}[X_n + Y_{n+1}|\mathcal{F}_n] = X_n + \mathbf{E}[Y_{n+1}|X_n] = X_n + \underbrace{\mathbf{E}[Y_{n+1}]}_{0} = X_n.$$

So,

$$\mathbf{E}[X_{n+1}|\mathcal{F}_n] = Y_1 \mathbb{1}_{\{X_n=0\}} + X_n \mathbb{1}_{\{X_n\neq 0\}} = Y_1 \mathbb{1}_{\{X_n=0\}} + X_n.$$

But if we consider an exception which is for $n = 2$, we have,

$$\begin{aligned}
\mathbf{E}[X_3|\mathcal{F}_2] &= Y_1 \mathbb{1}_{\{X_2=0\}} + X_2 = Y_1 \mathbb{1}_{\{X_2=0\}} + (X_1 + Y_2) \\
&= Y_1 \mathbb{1}_{\{X_2=0\}} + Y_1 + Y_2 \\
&= Y_1 \mathbb{1}_{\{Y_1+Y_2=0\}} + Y_1 + Y_2 \neq Y_1 + Y_2 = X_2.
\end{aligned}$$

(c) For every $t \in \mathbb{N}_0$, like in 14.1, define

$$\begin{aligned}
A_t &= \sum_{s=1}^{t} \mathbf{E}[X_s - X_{s-1}|\mathcal{F}_{s-1}] = \sum_{s=1}^{t} \big(\mathbf{E}[Y_1|\mathcal{F}_{s-1}]\mathbb{1}_{\{X_{s-1}=0\}} + \mathbf{E}[Y_s|\mathcal{F}_{s-1}]\mathbb{1}_{\{X_{s-1}\neq 0\}}\big) \\
&= \sum_{s=1}^{t} \left(Y_1 \mathbb{1}_{\{X_{s-1}=0\}} + \underbrace{\mathbf{E}[Y_s]}_{0} \mathbb{1}_{\{X_{s-1}\neq 0\}} \right) = Y_1 \sum_{s=1}^{t} \mathbb{1}_{\{X_{s-1}=0\}}.
\end{aligned}$$

Because of proposition 14.9, the adapted process $\mathcal{X} = (X_t)_{t\in\mathbb{N}_0}$ has an almost surely unique decomposition $\mathcal{X} = \mathcal{M} + \mathcal{A}$, where $\mathcal{M} = (M_t)_{t\in\mathbb{N}_0}$ is a martingale and $\mathcal{A} = (A_t)_{t\in\mathbb{N}_0}$ is previsible. That is,

$$\mathcal{M}_t = X_t - A_t = X_t - Y_1 \sum_{s=1}^{t} \mathbb{1}_{\{X_{s-1}=0\}}$$

Exercise 2 (2+2 points)**.**

(a) For some $N \in \mathbb{N}$, let $(X_n)_{n=0,1,2,\dots}$ be the Markov chain with transition matrix $p_{xy} = \binom{N}{y}(\frac{x}{N})^y(1-\frac{x}{N})^{N-y}$, i.e. given $X_n = x$, it is $X_{n+1} \sim B(N, x/N)$.

   (i) Show that $\mathcal{X}$ is a bounded martingale.

   (ii) Compute the quadratic variation of $\mathcal{X}$.

(b) Let $(X_n)_{n=0,1,2,\dots}$ be the Markov chain with transition matrix $p_{xy} = e^{-x}\frac{x^y}{y!}$, starting in $X_0 \in \mathbb{N}_0$, i.e. given $X_n = x$, it is $X_{n+1} \sim \mathrm{Poi}(x)$.

(i) Show that $\mathcal{X}$ is a critical branching process and determine its offspring distribution.

(ii) Show that $\mathcal{X}$ is a martingale and compute its quadratic variation.

Solution.

(a) Define $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

(i) To show that (X_n) is a martingale, we need to establish that $\mathbf{E}[X_{n+1} \mid X_n] = X_n$. Given $X_n = x$, we have

$$\mathbf{E}[X_{n+1} \mid \mathcal{F}_n] = N \cdot \frac{x}{N} = x = X_n.$$

Since X_n takes values in $\{0,1,\ldots,N\}$, it is bounded. Thus, (X_n) is a bounded martingale.

(ii) The quadratic variation process $\langle \mathcal{X} \rangle = (\langle \mathcal{X} \rangle_t)_{t \in I}$ is defined such that $(X_t^2 - \langle \mathcal{X} \rangle_t)_{t \in I}$ is a martingale. According to proposition 14.11, we have:

$$\langle \mathcal{X} \rangle_t = \sum_{s=1}^{t} \mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}].$$

Calculating $(X_s - X_{s-1})^2$: Given that $X_s \mid X_{s-1} = x$ follows a binomial distribution $B(N,x/N)$, we can calculate the variance of the increments:

$$\mathbf{Var}(X_s \mid X_{s-1} = x) = N \cdot \frac{x}{N} \cdot x \left(1 - \frac{x}{N}\right) = x \left(1 - \frac{x}{N}\right).$$

We also know that:

$$\mathbf{E}[X_s \mid X_{s-1} = x] = x.$$

Therefore, the increment can be expressed as:

$$\mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \mathrm{Var}(X_s \mid X_{s-1}) = X_{s-1} \left(1 - \frac{X_{s-1}}{N}\right).$$

Substituting this back into the expression for the quadratic variation process, we get:

$$\langle \mathcal{X} \rangle_t = \sum_{s=1}^{t} \mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \sum_{s=1}^{t} X_{s-1} \left(1 - \frac{X_{s-1}}{N}\right).$$

Thus, the quadratic variation process of the martingale (X_n) can be expressed as:

$$\langle \mathcal{X} \rangle_t = \sum_{s=1}^{t} X_{s-1} \left(1 - \frac{X_{s-1}}{N}\right).$$

If it is not crystal clear! We can show that

$$\mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \mathbf{Var}(X_s \mid X_{s-1}),$$

consider the definition of variance:

$$\mathbf{Var}(Y) = \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2.$$

For the martingale X_s:

$$\mathbf{Var}(X_s \mid X_{s-1}) = \mathbf{E}[X_s^2 \mid X_{s-1}] - (\mathbf{E}[X_s \mid X_{s-1}])^2.$$

The increment expands as:

$$(X_s - X_{s-1})^2 = X_s^2 - 2X_sX_{s-1} + X_{s-1}^2.$$

Taking the conditional expectation gives:

$$\mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \mathbf{E}[X_s^2 \mid \mathcal{F}_{s-1}] - 2\mathbf{E}[X_s \mid \mathcal{F}_{s-1}]X_{s-1} + X_{s-1}^2.$$

Substituting $\mathbf{E}[X_s \mid \mathcal{F}_{s-1}] = X_{s-1}$, we simplify to:

$$\mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \mathbf{E}[X_s^2 \mid \mathcal{F}_{s-1}] - X_{s-1}^2.$$

Thus, we conclude:

$$\mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \mathrm{Var}(X_s \mid X_{s-1}).$$

(b) We do the following:

(i) A critical branching process occurs when the expected number of offspring per individual is exactly 1. In this case, given $X_n = x$, the next state X_{n+1} follows a Poisson distribution:

$$X_{n+1} \mid X_n = x \sim \mathrm{Poi}(x).$$

The expected value of a Poisson random variable $Y \sim \mathrm{Poi}(\lambda)$ is given by: $\mathbf{E}[Y] = \lambda$. Thus, for our case: $\mathbf{E}[X_{n+1} \mid X_n = x] = x$. To check whether the process is critical, we compute the expected number of offspring:

$$\mathbf{E}[X_{n+1} \mid X_n = x] = \mathbf{E}[\mathrm{Poi}(x)] = x.$$

Since we want the average number of offspring per individual to equal 1, we compute the overall expected number of offspring when starting with $X_0 = k$: $\mathbf{E}[X_{n+1} \mid X_0 = k] = \mathbf{E}[X_0] = k$. Thus, for $\mathcal{X}$ to be critical, we need: $\mathbf{E}[X] = 1$.

The offspring distribution is characterized by the distribution of X_{n+1} given $(X_n = x; X_{n+1} \mid X_n = x \sim \mathrm{Poi}(x)$. For a critical branching process, the offspring distribution is therefore:

$$\mathbf{P}(X_{n+1} = y \mid X_n = x) = e^{-x}\frac{x^y}{y!}, \quad y = 0,1,2,\ldots$$

This confirms that $\mathcal{X}$ is a critical branching process.

(ii) To show that $\mathcal{X} = (X_n)_{n=0,1,2,\ldots}$ is a martingale under the given conditions, and to compute its quadratic variation, we can proceed with the following steps: Given that $X_n = x$, we have:

$$X_{n+1} \mid X_n = x \sim \mathrm{Poi}(x).$$

Thus, the expected value is:

$$\mathbf{E}[X_{n+1} \mid X_n = x] = x = X_n.$$

Since this holds for all n, we conclude that $\mathcal{X}$ is indeed a martingale. The quadratic variation process is given by:

$$\langle \mathcal{X} \rangle_t = \sum_{s=1}^{t} \mathbf{E}[X_s^2 - X_{s-1}^2 | \mathcal{F}_{s-1}] = \sum_{s=1}^{t} \mathbf{E}[(X_s - X_{s-1})^2 | \mathcal{F}_{s-1}]$$

We already know that:

$$\mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \mathbf{Var}(X_s \mid \mathcal{F}_{s-1}),$$

where $\mathbf{Var}(X_s \mid \mathcal{F}_{s-1}) = X_{s-1}$ since $X_s \mid X_{s-1} = x \sim \text{Poi}(x)$ has variance equal to its mean. Thus, we have:

$$\langle \mathcal{X} \rangle_t = \sum_{s=1}^{t} \mathbf{E}[(X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}] = \sum_{s=1}^{t} \text{Var}(X_s \mid \mathcal{F}_{s-1}) = \sum_{s=1}^{t} X_{s-1}.$$

Exercise 3 (4 points)**.**
Let $(X_i)_{i=1,2,...}$ be i.i.d. random variables with

$$\mathbf{P}(X_1 = -1) = \mathbf{P}(X_1 = 1) = \frac{1}{2} \quad \text{and} \quad S_n := \sum_{i=1}^{n} X_i.$$

Thus $\mathcal{S} = (S_n)_{n \geq 0}$ is a martingale. Furthermore, let $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$ be its filtration, $T := \inf\{i \geq 1 \mid X_i = 1\}$ and the process $\mathcal{H} = (H_i)_{i \geq 0}$ be given by

$$H_1 := 1, \qquad H_n := 2 \cdot H_{n-1} \mathbb{1}_{\{X_{n-1} = -1\}}.$$

Show that $\mathcal{H}$ is previsible and calculate $\mathbf{E}[\mathcal{H} \cdot \mathcal{S})_n]$ and $\mathbf{E}[(\mathcal{H} \cdot \mathcal{S})_T]$.

Solution.
Recall that $\mathcal{X}$ is called $(\mathcal{F}_t)_{t \in I}$*-previsible* if $X_0 = 0$ and X_t is $\mathcal{F}_{t-1}$-measurable, $t = 1,2,...$. The previsibility of $\mathcal{H}$ follows simply by induction. Clearly, $H_1 = 1$ is $\mathcal{F}_0$-measurable. Assume that H_{n-1} is $\mathcal{F}_{n-2}$-measurable, which by implication means H_{n-1} is $\mathcal{F}_{n-1}$-measurable. The measurability of X_{n-1} with respect to $\mathcal{F}_{n-1}$ yields

$$\{X_{n-1} = -1\} \in \mathcal{F}_{n-1} \implies \mathbb{1}_{\{X_{n-1} = -1\}} \text{ is } \mathcal{F}_n\text{-measurable.}$$

This implies that $H_n := 2 \cdot H_{n-1} \mathbb{1}_{\{X_{n-1} = -1\}}$ is $\mathcal{F}_{n-1}$-measurable. Now, because of the independence of H_i and X_i for all i, it is true that

$$\mathbf{E}[(\mathcal{H} \cdot \mathcal{S})_n] = \mathbf{E}\Big[\sum_{i=1}^{n} H_i(S_i - S_{i-1})\Big] = \sum_{i=1}^{n} \mathbf{E}[H_i X_i] = \sum_{i=1}^{n} \mathbf{E}[H_i]\mathbf{E}[X_i] = 0.$$

On the other hand, it follows that

$$\mathbf{E}[(\mathcal{H} \cdot \mathcal{S})_T] = \mathbf{E}\Big[\sum_{i=1}^{T} H_i X_i\Big] = \mathbf{E}\Big[\sum_{i=1}^{T-1} 2^{i-1}(-1) + 2^{T-1}\Big] = 1.$$

(When T is the first time $X_i = 1$, the sum of the contributions from H_i when $X_i = -1$ and the final contribution when $X_T = 1$ leads to the correct formulation as seen above.)
Alternatively, by definition 14.13

$$\mathbf{E}[(\mathcal{H} \cdot \mathcal{S})_n] = \mathbf{E}\Big[\sum_{s=1}^{n} H_s(S_s - S_{s-1})\Big] = \mathbf{E}\left[\sum_{s=1}^{n} H_s X_s\right] = \sum_{s=1}^{n} \mathbf{E}[H_s X_s].$$

But

$$HsX_s = \begin{cases} 2^{s-1} \quad \text{with probability} \quad 2^{-s} \\ -2^{s-1} \quad \text{with probability} \quad 2^{-s} \\ 0 \quad \text{with probability} \quad 1-2^{-s+1} \end{cases}$$

Finally, we obtain,

$$\mathbf{E}[(\mathcal{H}\cdot\mathbf{S})_n] = 2^{s-1}\cdot 2^{-s} - 2^{s-1}\cdot 2^{-s} + 0 = 0$$

Similarly,

$$\mathbf{E}[(\mathcal{H}\cdot\mathcal{S})_T] = \mathbf{E}\left[\sum_{s=1}^{T} H_s X_s\right] = \mathbf{E}[H_T X_T] + \sum_{s=1}^{T-1}\mathbf{E}[H_s X_s].$$

For $s \le T$, $X_s = -1$ and $H_s = 2^{s-1}$ so that $H_s X_s = -2^{s-1}$ with probability 1. For T, it is $X_T = 1$ and $H_T = 2^{T-1}$, so that $H_T X_T = 2^{T-1}$. Finally,

$$\mathbf{E}[(\mathcal{H}\cdot\mathcal{S})_T] = 2^{T-1} - \sum_{s=1}^{T-1} 2^{s-1} = 2^{T-1} - (2^{T-1}-1) = 1.$$

Exercise 4 (2+2=4 Points)**.**
Let $\mathcal{Y} = (Y_t)_{t\in I}$ be a stochastic process. A stopped stochastic process is given by $\mathcal{Y}^T := (Y_{T\wedge t})_{t\in I}$, where T is an I-valued random variable. Suppose that $\mathcal{X} = (X_n)_{n\ge 0}$ is a martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\ge 0}$, T an $\mathcal{F}$-stopping time, $\mathcal{X}^T$ the process stopped at T and $\mathcal{H} = (H_n)_{n\ge 0}$ is previsible. Show that

(a) $(\mathcal{H}\cdot(\mathcal{X}^T))_n = ((\mathcal{H}\cdot\mathcal{X})_n^T$ and

(b) $\langle\mathcal{X}^T\rangle_n = \langle\mathcal{X}\rangle_n^T$.

Solution.

(a) It follows that

$$\begin{aligned}(\mathcal{H}\cdot(\mathcal{X}^T))_n &= \sum_{k=1}^{n} H_{k-1}(X_k^T - X_{k-1}^T) = \sum_{k=1}^{n} H_{k-1}(X_k - X_{k-1})\mathbb{1}_{\{k\le T\}} \\ &= \sum_{k=1}^{n\wedge T} H_{k-1}(X_k - X_{k-1}) = (\mathcal{H}\cdot\mathcal{X})_n^T.\end{aligned}$$

(b) Also,

$$\begin{aligned}\langle\mathcal{X}^T\rangle_n &= \sum_{k=1}^{n}\mathbf{E}[(X_k^T - X_{k-1}^T)^2|\mathcal{F}_{k-1}] = \sum_{k=1}^{n}\mathbf{E}[(X_k - X_{k-1})^2\mathbb{1}_{\{T\le k-1\}^c}|\mathcal{F}_{k-1}] \\ &= \sum_{k=1}^{n}\mathbb{1}_{\{k\le T\}}\mathbf{E}[(X_k - X_{k-1})^2|\mathcal{F}_{k-1}] = \sum_{k=1}^{n\wedge T}\mathbf{E}[(X_k - X_{k-1})^2|\mathcal{F}_{k-1}] = \langle\mathcal{X}\rangle_n^T.\end{aligned}$$