Tutorial 7 - Measurable functions and the integral I

Exercise 1 (4 Points).
Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$. Show that a Borel measurable map $g : \mathbb{R} \to \mathbb{R}$ is $\sigma(f) = f^{-1}(\mathcal{B}(\mathbb{R}))-$measurable if and only if g is even.

Solution.
'$\Leftarrow$': Suppose g is even, let us show that g is $\sigma(f) = f^{-1}(\mathcal{B}(\mathbb{R}))-$measurable. Let $B \subseteq \mathbb{R}$ be a Borelian of $\mathbb{R}$, we shall show that $g^{-1}(B) \in f^{-1}(\mathcal{B}(\mathbb{R}))$ that is $f(g^{-1}(B))$ is a Borelian of $\mathbb{R}$. Now by definition: $y \in f(g^{-1}(B)) \implies \exists\ y_0 \in (g^{-1}(B) \mid y = f(y_0)$, that is $\exists\ y_0 \in (g^{-1}(B) \mid y = |y_0|$, that is $y = |y_0|$ and $g(y_0) \in B$. Since g is even, then $g(y_0) = g(|y_0|)$; $g(y) \in B$; $y \in g^{-1}(B)$. So, $f(g^{-1}(B)) = g^{-1}(B)$. But since g is a Borel-measurable map, $g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ and we are done.
'$\Rightarrow$': Assume g is $\sigma(f)$-measurable and we show that the set $S = \{x \in \mathbb{R} \mid g(x) \neq g(-x)\}$ is empty. Suppose S is not empty, that is, $\exists\ y_0 \in S$ such that $g(y_0) \neq g(-y_0)$; therefore there are two Borelian B_1 and B_2 such that $g(y_0) \in B_1$ and $g(-y_0) \in B_2$ with $B_1 \cap B_2 = \emptyset$. But since g is $\sigma(f)$-measurable, $g^{-1}(B_1)$ and $g^{-1}(B_1) \in f^{-1}(\mathcal{B}(\mathbb{R})$. On the other hand, $y_0 \in g^{-1}(B_1)$ and $-y_0 \in g^{-1}(B_2) \implies y_0 \in f^{-1}(B_1)$ and $-y_0 \in f^{-1}(B_2) \implies \in -y_0 \in f^{-1}(B_1) \cap f^{-1}(B_2)$ (since $f(-y_0) = |y_0| = f(y_0)$. But $f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2) = f^{-1}(\emptyset) = \emptyset$. So, S is empty, and we are done.

Exercise 2 (4 Points).
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{-x} 1_{[0,\infty)}(x)$, and let λ be the Lebesgue measure on $\mathbb{R}$.

(a) Find a sequence (f_n) of elementary functions such that $f_n \uparrow f$.

(b) Compute $\int f_n d\lambda$ and determine $\int f d\lambda$ as a limit of integrals.

Solution.

(a) Consider the sequence of functions $f_n(x)$ defined by

$$f_n(x) = \begin{cases} (1 - \frac{1}{n})e^{-x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

We observe the following: For $x < 0$, $f_n(x) = 0$ for all n. Thus, $f_n(x) \to 0 = f(x)$. For $x = 0$, $f_n(0) = (1 - \frac{1}{n})e^0 = 1 - \frac{1}{n}$. As $n \to \infty$, $f_n(0) \to 1 = f(0)$. For $x > 0$, $f_n(x) = (1 - \frac{1}{n})e^{-x}$. As $n \to \infty$, $(1 - \frac{1}{n})e^{-x} \to e^{-x} = f(x)$. Also, since $\left(1 - \frac{1}{n}\right)$ is an increasing function of n, it follows that $f_n(x)$ is indeed increasing for each $x \geq 0$. Alternately, $\forall x \in \mathbb{R}$,

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}^+} \left| \left(1 - \frac{1}{n}\right) e^{-x} - e^{-x} \right| \leq \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

Then $f_n(x) \to f(x)$.

(b) We clearly see that

$$\begin{aligned} \int f_n d\lambda &= \underbrace{\int_{-\infty}^{0} f_n(x) d\lambda}_{0} + \int_0^\infty f_n(x) d\lambda \\ &= \int_0^\infty \left(1 + \frac{1}{n}\right) e^{-x} d\lambda = \left(1 + \frac{1}{n}\right). \end{aligned}$$

(f_n) is a sequence of positive functions, $fn \to f$, then

$$\int f d\lambda = \lim_{n\to\infty} \int f_n d\lambda = 1.$$

Exercise 3 (4 Points)**.**
Let $(\Omega,\mathcal{F})$,$(\Omega',\mathcal{F}')$ be measurable spaces and $f : \Omega \to \Omega'$. If there are $\mathcal{C} \subseteq \mathcal{F}$ and $\mathcal{C}' \subseteq \mathcal{F}'$ with $\sigma(\mathcal{C}) = \mathcal{F}$ and $\sigma(\mathcal{C}') = \mathcal{F}'$ and $f^{-1}(\mathcal{C}') \subseteq \mathcal{C}$, then f is $\mathcal{F}/\mathcal{F}'$-measurable.

Solution.
With Lemma 3.6.1 it is trivial. With Lemma 3.2, we can write

$$f^{-1}(\mathcal{F}') = f^{-1}(\sigma(\mathcal{C}')) = \sigma(f^{-1}(\mathcal{C}') \subseteq \sigma(\mathcal{C}) = \mathcal{F}.$$

Exercise 4 (4 Points)**.**
Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?

Solution.
The set E_0 can be expressed using the Cauchy criterion as follows:

$$\begin{aligned} E_0 &= \{x \in E : \{f_n(x)\} \text{ converges}\} \\ &= \{x \in E : \text{ for every } \epsilon > 0, \text{ there exists } N \text{ such that } |f_n(x) - f_m(x)| < \epsilon \text{ for all } n,m \geq N\}. \end{aligned}$$

Since $|f_n(x) - f_m(x)|$ is a measurable function, each set of the form $\{x \in E : |f_n(x) - f_m(x) < \frac{1}{k}\}$ is measurable. We can use the fact that a sequence of points converges if and only if it is Cauchy to write

$$E_0 = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m\geq N} \left\{ x \in E; |f_n(x) - f_m(x)| < \frac{1}{k} \right\}.$$

Since the collection of measurable sets is indeed a σ-algebra, E_0 is measurable.

Exercise 5 (*Bonus question!* 3 Points)**.**
Let $\Omega = \{1,2,3,4,5\}$.

(a) Find the smallest σ- algebra $\mathcal{F}_1$ containing

$$\mathcal{F}_2 := \{\{1,2,3\},\{3,4,5\}\}.$$

(b) Is the function $f : \Omega \to \mathbb{R}$ defined by

$$f(1) = f(2) = 0, \quad f(3) = 10, \quad f(4) = f(5) = 1$$

measurable with respect to $\mathcal{F}_1$?

(c) Find the σ-algebra $\mathcal{F}_3$ generated by $g : \Omega \to \mathbb{R}$ and defined by

$$g(1) = 0, \quad g(2) = g(3) = g(4) = g(5) = 1.$$

Solution.

(a) $\mathcal{F}_1 = \{\emptyset,\Omega,\{1,2,3\},\{3,4,5\},\{3\},\{1,2,4,5\},\{1,2\},\{4,5\}\}$.

(b) The random variable f is measurable with respect to $\mathcal{F}_1$ since we have for each $A \in \mathcal{B}(\mathbb{R})$:

$$\begin{array}{lll}
\text{if} & 0 \in A,\, 1{,}10 \notin A: & f^{-1}(A) = \{1,2\} \in \mathcal{F}_1, \\
\text{if} & 1 \in A,\, 0{,}10 \notin A: & f^{-1}(A) = \{4,5\} \in \mathcal{F}_1, \\
\text{if} & 10 \in A,\, 0{,}1 \notin A: & f^{-1}(A) = \{3\} \in \mathcal{F}_1, \\
\text{if} & 0{,}1{,}10 \notin A,: & f^{-1}(A) = \emptyset \in \mathcal{F}_1, \\
\text{if} & 0{,}1{,}10 \in A,: & f^{-1}(A) = \Omega \in \mathcal{F}_1.
\end{array}$$

where $f^{-1}(A) = \{\omega \in \Omega : f(w) \in A.\}$ We can reduce every other case to these, take for example, if $0{,}1, \in A$ but $10 \notin A$, then:

$$f^{-1}(A) = f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = \{1,2\} \cup \{4,5\} = \{1,2,3,4,5\} \in \mathcal{F}_1.$$

(c) $\mathcal{F}_3 = \sigma(g) = \{\Omega,\emptyset,\{1\},\{2,3,4,5\}\}$.