Stochastic Processes
16. Semigroups and generators

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Time-homogeneous Markov processes

- Definition 15.19: A Markov process $\mathcal{X}$ is called *time-homogeneous* if there is a family of Markov kernels $(\mu_t)_{t\in I}$ with $\mu_{s,t} = \mu_{t-s}$, which we call *transition semigroup*. Equivalently, there is a family of transition operators $(T_t)_{t\in I}$ with $T_{s,t} = T_{t-s}$, which we call *operator semigroup*.
- For a time-homogeneous Markov process,

$$(X_{t_0}, \ldots X_{t_n}) \sim \nu_{t_0} \otimes \mu_{t_1-t_0} \otimes \cdots \otimes \mu_{t_n-t_{n-1}},$$

$$\mu_s \mu_t = \mu_{s+t}, \qquad T_s T_t = T_{s+t}.$$

The strong Markov property is in this case

$$\mathbf{P}[X_{S+t} \in A | \mathcal{F}_S] = \mu_t(X_S, A), \qquad \mathbf{E}[f(X_{S+t})|\mathcal{F}_S] = (T_t f)(X_S).$$

Feller semigroup, Feller process

- Definition 15.22: A family of operators $(T_t)_{t \in I}$ is called an *operator semigroup* if $T_t(T_s f) = T_{t+s} f$. Such a semigroup is called
  1. *positive* if $T_t f \geq 0$ if $f \geq 0$ for all $t \in I$,
  2. *contraction* if $0 \leq T_t f \leq 1$ for $0 \leq f \leq 1$ for a
  3. *conservative* if $T_t 1 = 1$ for all $t \in I$,
  4. *strongly continuous* if $||T_t f - f||_\infty \xrightarrow{t \to 0} 0$ for all $f \in \mathcal{C}_b(E)$.
  5. *Feller semigroup* if $T_t f(x) \xrightarrow{t \to 0} f(x)$ for $x \in E$ and $f \in \mathcal{C}_b(E)$ and $T_t f \in \mathcal{C}_b(E)$ for all $f \in \mathcal{C}_b(E)$ and $t \in I$.
- A time-homogeneous Markov process $\mathcal{X} = (X_t)_{t \in \mathcal{I}}$ is called *Feller process* if its operator semigroup $(T_t)_{t \in I}$ is a Feller semigroup.

Probabilistic properties of Feller processes

- Let $(T_t)_{t\in I}$ be the operator semigroup of a Markov process $\mathcal{X} = (X_t)_{t\in I}$, i.e. $T_t f(x) = \mathbf{E}[f(X_t)|X_0 = x]$.
  1. $(T_t)_{t\in I}$ is conservative and a positive contraction.
  2. $T_t f(x) \xrightarrow{t\to 0} f(x)$ for all $f \in \mathcal{C}_b(E) \iff X_t \xrightarrow{t\to 0}_p X_0$.

     '$\Rightarrow$': With $g(y) := r(x,y) \wedge 1$, note

     $\mathbf{E}_x[r(X_0, X_t) \wedge 1] = T_t g(X_0) \xrightarrow{t\to 0} g(X_0) = 0.$

     '$\Leftarrow$': Since $X_t \xRightarrow{t\to 0} X_0 =: x$, for $f \in \mathcal{C}_b(E)$ in particular

     $T_t f(x) = \mathbf{E}_x[f(X_t)] \xrightarrow{t\to\infty} \mathbf{E}_x[f(X_0)] = f(x).$

PPP and BM are Feller

- Lemma 15.24: PPP and BM are Feller processes.
- Proof: Define in both cases $Z_t := X_t - X_0$. Then, for the PPP, $Z_t \sim \text{Poi}(\lambda t)$ and for BM, $Z_t \sim N(0, t)$. In both cases, write (due to independent increments),

$$T_t f(x) = \mathbf{E}[f(x + Z_t)], \qquad f \in \mathcal{C}_b(\mathbb{R}).$$

Since $Z_t \xRightarrow{t \to 0} 0$ in both cases, we find $T_t f(x) \xrightarrow{t \to 0} f(x)$. By dominated convergence, $x \mapsto \mathbf{E}[f(x + Z_t)]$ is continuous.

Generator of a semigroup

- Definition 15.25: $\mathcal{X} = (X_t)_{t \in I}$ be a time-homogeneous Markov process with $(T_t)_{t \in I}$. The *generator* of $\mathcal{X}$ is defined as
$$(Gf)(x) = \lim_{t \to 0} \frac{\mathbf{E}_x[f(X_t) - f(x)]}{t} = \lim_{t \to 0} \frac{1}{t}((T_t f)(x) - f(x)),$$
for all f for which the limit value exists. The set of functions f for which $(Gf)(x)$ exists for all $x \in E$ exists is $\mathcal{D}(G)$.

Generator for the PPP(λ)

- For $\mathcal{X} = \text{PPP}(\lambda)$, and $P_t \sim \text{Poi}(\lambda)$, and $f \in \mathcal{C}_b(\mathbb{R})$,

$$\begin{aligned} Gf(x) &= \lim_{t \to 0} \frac{1}{t}(\mathbf{E}_x[f(x + P_t) - f(x)]) \\ &= \lim_{t \to 0} \frac{1}{t} \sum_{k=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} (f(x+k) - f(x)) \\ &= \lim_{t \to 0} \lambda \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{(k+1)!} (f(x+1+k) - f(x)) \\ &= \lambda(f(x+1) - f(x)) \end{aligned}$$

Generator for BM

- Let $\mathcal{X}$ be BM and $f \in \mathcal{C}_b^2(\mathbb{R})$. Then, with $Z \sim N(0,1)$,

$$\begin{aligned}(Gf)(x) &= \lim_{t\to 0} \frac{1}{t}(\mathbf{E}_x[f(x+\sqrt{t}Z) - f(x)]) \\ &= \lim_{t\to 0} \frac{1}{t}(\mathbf{E}_x[f'(x)\sqrt{t}Z \\ &\qquad + \tfrac{1}{2}f''(x)tZ^2 + \tfrac{1}{2}(f''(x+\sqrt{t}Y) - f''(x))tZ^2]) \\ &= \tfrac{1}{2}f''(x) + \lim_{t\to\infty} \mathbf{E}[\tfrac{1}{2}(f''(x+\sqrt{t}Y) - f''(x))Z^2] \\ &= \tfrac{1}{2}f''(x).\end{aligned}$$

Operator semigroups and generators

- Lemma 15.28: $\mathcal{X}$ Feller process with $(T_t)_{t\in I}$ and generator G and $\mathcal{D} \subseteq \mathcal{D}(G)$ with $G(\mathcal{D}) \subseteq \mathcal{C}_b(E)$. Then,

$$f \in \mathcal{C}_b(E) \Rightarrow \int_0^t (T_s f)ds \in \mathcal{D}(G) \text{ with}$$

$$(T_t f)(x) - f(x) = \Big(G\Big(\int_0^t (T_s f)ds\Big)\Big)(x),$$

$$f \in \mathcal{D} \Rightarrow T_t f \in \mathcal{D}(G) \text{ with } G(T_t f) = T_t(Gf) \text{ and}$$

$$(T_t f)(x) - f(x) = \int_0^t (T_s(Gf))(x)ds.$$

In particular,

$$\mathbf{E}_x[f(X_t)] = f(x) + \int_0^t \mathbf{E}[(Gf)(X_s)]ds.$$

Proof of Lemma 15.28

- For the first equation,

$$\frac{1}{h}\mathbf{E}_x\Big[\int_0^t (T_s f)(X_h) - (T_s f)(x)ds\Big] = \frac{1}{h}\Big(\int_0^t (T_{s+h} f)(x) - (T_s f)(x)ds\Big)$$

$$= \frac{1}{h}\Big(\int_h^{t+h} (T_s f)(x)ds - \int_0^t (T_s f)(x)ds\Big)$$

$$= \frac{1}{h}\int_t^{t+h} (T_s f)(x)ds - \frac{1}{h}\int_0^h (T_s f)(x)ds \xrightarrow{h\to 0} (T_t f)(x) - f(x).$$

Proof of Lemma 15.28

- For the other statements,

$$\frac{d}{dt}\mathbf{E}_x[f(X_t)] = \lim_{h\to 0}\tfrac{1}{h}\mathbf{E}_x[f(X_{t+h}) - f(X_t)]$$
$$= T_t \lim_{h\to 0}\tfrac{1}{h}\mathbf{E}_x[f(X_h) - f(x)] = (T_t(Gf))(x),$$

  and

$$\frac{d}{dt}\mathbf{E}_x[f(X_t)] = \lim_{h\to 0}\tfrac{1}{h}\mathbf{E}_x[f(X_{t+h}) - f(X_t)]$$
$$= \lim_{h\to 0}\tfrac{1}{h}\mathbf{E}_x[(T_t f)(X_h) - (T_t f)(x)] = (G(T_t f))(x)$$

  and since $t \mapsto (T_t(Gf))(x)$ is continuous

$$(T_t f)(x) - f(x) = \int_0^t \frac{d}{ds}\mathbf{E}_x[f(X_s)]ds = \int_0^t (T_s(Gf))(x)ds.$$

Domain is dense

- Corollary 15.29: Assumptions of Lemma 15.28 apply with $\mathcal{D} = \mathcal{C}_b(E)$, and $(T_t)_{t\in I}$ is strongly continuous. Then, $\mathcal{D}(G)$ is dense in $\mathcal{C}_b(E)$.
- Let $f \in \mathcal{C}_b(E)$. Then,

$$\tfrac{1}{t}\int_0^t (T_s f)ds \xrightarrow{t\to 0} f.$$

Since the function on the left-hand side according to Lemma 15.28 lie in $\mathcal{D}(G)$, the assertion is shown.

Martingales derived from Markov processes

- Theorem 15.30: $\mathcal{X} = (X_t)_{t \in I}$ Feller with generator G and domain $\mathcal{D}(G)$, and $f \in \mathcal{D}(G)$ such that $Gf \in \mathcal{C}_b(E)$. Then, the following are martingales:

$$\Big(f(X_t) - \int_0^t (Gf)(X_s)ds\Big)_{t \in I}, \Big(f(X_t) \exp\Big(-\int_0^t \frac{(Gf)(X_s)}{f(X_s)} ds\Big)\Big)_{t \in I}.$$

- For $s \le t$,

$$\begin{aligned}
&\mathbf{E}\Big[f(X_t) - f(X_s) - \int_s^t (Gf)(X_r)dr \Big| \mathcal{F}_s\Big] \\
&= \mathbf{E}\Big[f(X_t) - f(X_s) - \int_s^t (Gf)(X_r)dr \Big| X_s\Big] \\
&= (T_{t-s}f)(X_s) - f(X_s) - \int_s^t (T_{r-s}(Gf))(X_s)dr = 0.
\end{aligned}$$