Stochastic Processes
18. Blumenthal's 0-1 law and quadratic variation of Brownian Motion

Peter Pfaffelhuber

January 2, 2025

Blumenthal's 0-1-law

All filtrations are assumed to be complete.

- Theorem 16.1: $\mathcal{X}$ BM, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, and $\mathcal{F}_{0+} := \bigcap_{t>0} \sigma(X_s : s \le t)$. Then, $\mathcal{F}_{0+}$ and $\mathcal{T} := \bigcap_{s \ge 0} \sigma(X_t : t \ge s)$ are $\mathbf{P}$-trivial.
- Proof: $(\sigma(X_s : s \le t))_{t \ge 0}$ is right-continuous, so $\mathcal{F}_{0+} = \sigma(X_0) = \sigma(x)$ is $\mathbf{P}$-trivial.

  Furthermore, let $X'_t = tX_{1/t}$ be another BM. Then,

$$\bigcap_{s \ge 0} \sigma(X_t : t \ge s) = \bigcap_{s \ge 0} \sigma(tX_{1/t} : t \le s) = \bigcap_{s \ge 0} \sigma(X'_t : t \le s)$$

  is $\mathbf{P}$-trivial.

Quadratic variation

- Definition 16.2: Let $f : [0, \infty) \to \mathbb{R}$ for $n = 1, 2, \ldots$

  $\zeta_n := \{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,k_n} = t\}$ with

  $\max_k (t_{n,k} - t_{n,k-1}) \xrightarrow{n\to\infty} 0$. Define the ℓ-variation of f with

  respect to $\zeta = (\zeta_n)_{n=1,2,\ldots}$ as

  $$\nu_{\ell,t,\zeta}(f) := \lim_{n\to\infty} \nu^n_{\ell,t,\zeta}(f) \text{ with}$$

  $$\nu^n_{\ell,t,\zeta}(f) = \sum_{k=1}^{k_n} |f(t_{n,k}) - f(t_{n,k-1})|^{\ell}.$$

  If the limit is independent of ζ, this is the ℓ-variation and

  denoted by $\nu_{\ell,t}(f)$.

  In addition, ζ is called *ascending* if $\zeta_n \subseteq \zeta_{n+1}$ for all n.

Elementary properties of the ℓ-variation

- Lemma 16.4: f continuous, $t \geq 0$, ζ as above. Then,

$$\nu_{\ell,t,\zeta}(f) < \infty \Rightarrow \nu_{\ell+1,t,\zeta}(f) = 0,$$

$$\nu_{\ell+1,t,\zeta}(f) > 0 \Rightarrow \nu_{\ell,t,\zeta}(f) = \infty.$$

- Proof: 1. Write

$$0 \leq \lim_{n\to\infty} \sum_{k=1}^{k_n} |f(t_{n,k}) - f(t_{n,k-1})|^{\ell+1}$$

$$\leq \lim_{n\to\infty} \sup_k |f(t_{n,k}) - f(t_{n,k-1})| \cdot \lim_{n\to\infty} \sum_{k=1}^{k_n} |f(t_{n,k}) - f(t_{n,k-1})|^{\ell}$$

$$= 0$$

since f is uniformly continuous on $[0, t]$.

Quadratic variation of BM

- Theorem 16.5: $\mathcal{X}$ BM, ζ as above. Then,

$$\nu^n_{2,t,\zeta} := \nu^n_{2,t,\zeta}(\mathcal{X}) \xrightarrow{n\to\infty}_{L^2} t.$$

  If ζ is ascending, then convergence is also almost surely.

- Proof of L^2-convergence: Write

$$\begin{aligned}
\mathbf{E}[\nu^n_{2,\zeta}] &= \sum_{k=1}^{k_n} \mathbf{E}[(X_{t_{n,k}} - X_{t_{n,k-1}})^2] = \sum_{k=1}^{k_n} (t_{n,k} - t_{n,k-1})\mathbf{E}[X_1^2] \\
&= \sum_{k=1}^{k_n} (t_{n,k} - t_{n,k-1}) = t,
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}[(\nu^n_{2,\zeta} - t)^2] = \mathbf{V}[\nu^n_{2,\zeta}] &= \sum_{k=1}^{k_n} \mathbf{V}[(X_{n,k} - X_{n,k-1})^2] \\
&= \sum_{k=1}^{k_n} (t_{n,k} - t_{n,k-1})^2 \mathbf{E}[X_1^4] \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Quadratic variation of BM

- Theorem 16.5: $\mathcal{X}$ BM, ζ as above. Then,

$$\nu_{2,t,\zeta}^n := \nu_{2,t,\zeta}^n(\mathcal{X}) \xrightarrow{n\to\infty}_{L^2} t.$$

If ζ is ascending, then convergence is also almost surely.

- Proof of almost sure convergence: Wlog $\zeta_n \setminus \zeta_{n-1} = \{t_n\}$.

The key is to show that $(\nu_{2,\zeta}^{-n})_{n=\dots,-2,-1}$ is a (backward) martingale, which converges almost surely:

Take time points $0 = r < s < u$ and

$\alpha := X_s - X_r, \beta := X_u - X_s$, since $\beta^2 = (-\beta)^2$,

$$\begin{aligned}\mathbf{E}[(\alpha+\beta)^2|\alpha^2+\beta^2] &= \tfrac{1}{2}\mathbf{E}[(\alpha+\beta)^2 + (\alpha-\beta)^2|\alpha^2+\beta^2] \\ &= \alpha^2+\beta^2.\end{aligned}$$

BM has nowhere differentiable paths

- A BM $\mathcal{X}$ almost certainly has nowhere differentiable paths.

  This means that

$$\mathbf{P}\Big(\lim_{h\to 0}\frac{X_{t+h}-X_t}{h} \text{ exists for some } t>0\Big)=0.$$