Stochastic Processes
12. Martingales in continuous time

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Martingales derived from the PPP

- $\mathcal{X} = (X_t)_{t\in I}$ PPP(λ). Then,

$$(X_t - \lambda t)_{t\in I} \qquad \text{and} \qquad \left(X_t^2 - \lambda \int_0^t (2X_r + 1)dr\right)_{t\in I}$$

  are martingales.

- Indeed, for $s \leq t$,

$$\mathbf{E}[X_t - \lambda t|\mathcal{F}_s] = \mathbf{E}[X_s + X_t - X_s - \lambda t|\mathcal{F}_s] = X_s - \lambda s,$$

$$\begin{aligned}
&\mathbf{E}\Big[X_t^2 - X_s^2 - \lambda \int_s^t (2X_r + 1)dr|\mathcal{F}_s\Big] \\
&\quad = \mathbf{E}\Big[(X_t - X_s)^2 + 2(X_t - X_s)X_s - \lambda \int_s^t 2(X_s + X_r - X_s) + 1dr|\mathcal{F}
\end{aligned}$$

$$\begin{aligned}
&= \lambda(t-s) + \lambda^2(t-s)^2 + 2\lambda(t-s)X_s \\
&\qquad - \lambda((2X_s + 1)(t-s) - \lambda^2(t-s)^2 = 0.
\end{aligned}$$

Martingales derived from Brownian motion

- $\mathcal{X} = (X_t)_{t\in I}$ BM. The following are martingales for all $\alpha \in \mathbb{R}$:

$$(\alpha X_t)_{t\in I}, \qquad (\alpha X_t^2 - \alpha t)_{t\in I}, \qquad \big(\exp(\alpha X_t - \alpha^2 t/2)\big)_{t\in I}$$

- Indeed: For $s \le t$,

$$\mathbf{E}[\alpha X_t|\mathcal{F}_s] = \mathbf{E}[\alpha X_s + \alpha(X_t - X_s)|\mathcal{F}_s] = \alpha X_s,$$

$$\mathbf{E}\Big[\alpha X_t^2 - \alpha t|\mathcal{F}_s] = \alpha\mathbf{E}[(X_t - X_s)^2 + 2(X_t - X_s)X_s + X_s^2 - t|\mathcal{F}_s]$$

$$= \alpha(t-s) + \alpha X_s^2 - \alpha t = \alpha X_s^2 - \alpha s,$$

$$\mathbf{E}\Big[\exp(\alpha X_t - \alpha^2 t/2)|\mathcal{F}_s] = \exp(\alpha X_s - \alpha^2 t/2)\cdot\mathbf{E}[\exp(\alpha(X_t - X_s))]$$

$$= \exp(\alpha X_s - \alpha^2 t/2 + \alpha^2(t-s)/2) = \exp(\alpha X_s - \alpha^2 s/2).$$

Martingales derived from Brownian motion

- $\big(\exp(\alpha X_t - \alpha^2 t/2)\big)_{t\in I}$ is a non-negative martingale with $\mathbf{E}[\exp(\alpha X_t - \alpha t/2)] = 1$, so it represents a density. So, for $\tau > 0$,
$$\mathbf{Q}_\tau : \begin{cases} \mathcal{B}(\mathbb{R})^{[0,\tau]} & \to [0,1] \\ A & \mapsto \mathbf{E}[\exp(\alpha X_\tau - \alpha^2\tau/2), A] \end{cases}$$
is another probability measure on $\mathcal{B}(\mathbb{R})^{[0,\tau]}$
- For $\mu \in \mathbb{R}$ the process $(X_t + \mu t)_{t\in[0,\infty)}$ is called *Brownian motion with drift* μ. This is a martingale if and only if $\mu = 0$. For $\mu > 0$ it is a sub-martingale and for $\mu < 0$ it is a super-martingale.

Martingales derived from Brownian motion

- $\mathcal{X} = (X_t)_{t\in I}$ BM defined on $(\Omega, \mathcal{F}, \mathbf{P})$, and $\mathcal{Y} = (Y_t)_{t\in I} = (X_t + \mu t)_{t\in I}$, and $\mathbf{Q}$ from above, Then,

$$\mathcal{X}_*\mathbf{Q} = \mathcal{Y}_*\mathbf{P} \qquad \text{and} \qquad \mathcal{Y}_*\mathbf{Q} = \mathcal{X}_*\mathbf{P}.$$

- Let $f \in \mathcal{C}_b(I)$ and $0 \leq s \leq t$. Then,

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}[f(X_t)] &= \mathbf{E}_{\mathbf{P}}[f(X_t)e^{\mu X_t - \mu^2 t/2}] \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\mu^2 t/2} \int f(y) e^{\mu y} e^{-y^2/(2t)} dy \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\mu^2 t/2 + \mu^2 t/2} \int f(y) e^{-(y-\mu t)^2/(2t)} dy \\
&= \frac{1}{\sqrt{2\pi t}} \int f(y + \mu t) e^{-y^2/(2t)} dy \\
&= \mathbf{E}_{\mathbf{P}}[f(X_t + \mu t)].
\end{aligned}$$

Regularization

- $\mathcal{X} = (X_t)_{t\in[0,\infty)}$ sub-martingale, $\mathcal{Y} = (Y_t)_{t\in[0,\infty)\cap\mathbb{Q}}$ with $Y_t = X_t$ for rational t. Then:
  1. There is a null set N such that $Y_t^+ := \lim_{s\downarrow t} Y_t$ for all $t \in I$ outside N exists. The process $\mathcal{Z} = (Z_t)_{t\in I}$ with $Z_t = 1_{N^c} Y_t^+$ is a sub-martingale.
  2. If the filtration is right-continuous, $\mathcal{X}$ has a modification with paths in $\mathcal{D}_{\mathbb{R}}([0,\infty))$ if $t \mapsto \mathbf{E}[X_t]$ is right-continuous.
- Usual conditions for $I = [0,\infty)$.
  1. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and complete.
  2. For each sub-martingale, we take its modification with càdlàg paths.

Porting result to continuous time

- Let the usual conditions hold. Then, Proposition 14.19, Theorem 14.22, Lemma 14.25, Proposition 14.26, Lemma 14.28, Theorem 14.29, Corollary 14.30, Theorem 14.32, Theorem 14.33, Theorem 14.36 and Theorem 14.37 continue to hold for continuous time.