universität freiburg

Stochastic processes

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 6 - The stochastic integral as a martingale

Exercise 1 (4 points).
Let $Y_1, Y_2, \dots$ be a sequence of i.i.d random variables with $\mathbf{P}(Y_1 = 1) = \mathbf{P}(Y_1 = -1) + \frac{1}{2}$, and let $X_0 = 0$ and for all $n \in N$,

$$X_n = \begin{cases} Y_1 & \text{if} \quad X_{n-1} = 0 \\ X_{n-1} + Y_n & \text{otherwise,} \end{cases}$$

which means $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ behave like a random walk as long as it does not hit 0, and always jump as Y_1 just after hitting 0. We also define $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

(a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ adapted process with $\mathbf{E}[X_{n+1}|X_n] = X_n$ for all $n \in \mathbb{N}_0$.

(b) Show that $(X_n)_{n\in\mathbb{N}_0}$ is not an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale.

(c) Find a Doob decomposition for $(X_n)_{n\in\mathbb{N}_0}$.

Exercise 2 (2+2 points).

(a) For some $N \in \mathbb{N}$, let $(X_n)_{n=0,1,2,\dots}$ be the Markov chain with transition matrix $p_{xy} = \binom{N}{y}(\frac{x}{N})^y(1-\frac{x}{N})^{N-y}$, i.e. given $X_n = x$, it is $X_{n+1} \sim B(N, x/N)$.
  (i) Show that $\mathcal{X}$ is a bounded martingale.
  (ii) Compute the quadratic variation of $\mathcal{X}$.

(b) Let $(X_n)_{n=0,1,2,\dots}$ be the Markov chain with transition matrix $p_{xy} = e^{-x}\frac{x^y}{y!}$, starting in $X_0 \in \mathbb{N}_0$, i.e. given $X_n = x$, it is $X_{n+1} \sim \text{Poi}(x)$.
  (i) Show that $\mathcal{X}$ is a critical branching process and determine its offspring distribution.
  (ii) Show that $\mathcal{X}$ is a martingale and compute its quadratic variation.

Exercise 3 (4 points).
Let $(X_i)_{i=1,2,\dots}$ be i.i.d. random variables with

$$\mathbf{P}(X_1 = -1) = \mathbf{P}(X_1 = 1) = \frac{1}{2} \quad \text{and} \quad S_n := \sum_{i=1}^n X_i.$$

Thus $\mathcal{S} = (S_n)_{n\geq 0}$ is a martingale. Furthermore, let $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ be its filtration, $T := \inf\{i \geq 1 \mid X_i = 1\}$ and the process $\mathcal{H} = (H_i)_{i\geq 0}$ be given by

$$H_1 := 1, \qquad H_n := 2 \cdot H_{n-1}\mathbb{1}_{\{X_{n-1}=-1\}}.$$

Show that $\mathcal{H}$ is previsible and calculate $\mathbf{E}[\mathcal{H} \cdot \mathcal{S})_n]$ and $\mathbf{E}[(\mathcal{H} \cdot \mathcal{S})_T]$.

Exercise 4 (2+2=4 Points)**.**

Let $\mathcal{Y} = (Y_t)_{t\in I}$ be a stochastic process. A stopped stochastic process is given by $\mathcal{Y}^T := (Y_{T\wedge t})_{t\in I}$, where T is an I-valued random variable. Suppose that $\mathcal{X} = (X_n)_{n\geq 0}$ is a martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$, T an $\mathcal{F}$-stopping time, $\mathcal{X}^T$ the process stopped at T and $\mathcal{H} = (H_n)_{n\geq 0}$ is previsible. Show that

(a) $(\mathcal{H} \cdot (\mathcal{X}^T))_n = ((\mathcal{H} \cdot \mathcal{X})_n^T$ and

(b) $\langle \mathcal{X}^T \rangle_n = \langle \mathcal{X} \rangle_n^T$.