Separating classes of functions

May 7, 2024

Our goal is to show that any algebra of functions (defined on a Polish space) which separates points is separating.

Remark 0.1 (Notation)**.** We will write (E, r) for some extended pseudo-metric space, $\mathcal{P}(E)$ for the set of probability measures on the Borel σ-algebra on E, $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$, and $\mathcal{C}_b(E, \Bbbk)$ the set of $\Bbbk$-valued bounded continuous functions on E. For some $\mathbf{P} \in \mathcal{P}(E)$ and $f \in \mathcal{C}_b(E, \Bbbk)$, we let $\mathbf{P}[f] := \int f(x)\mathbf{P}(dx) \in \Bbbk$ be the expectation.

1 Bounded pointwise convergence

The following is a simple consequence of dominated convergence, and often needed in probability theory.

Definition 1.1. *Let E be some set and $f, f_1, f_2, \ldots : E \to \Bbbk$. We say that $f_1, f_2, ..:$ converges to f boundedly pointwise if $f_n \xrightarrow{n\to\infty} f$ pointwise and $\sup_n ||f_n|| < \infty$. We write $f_n \xrightarrow{n\to\infty}_{bp} f$*

In lean this does not exist yet, although dominated convergence exists.
How can one formulate this kind of convergence using filters?
Task: ??

lemma:bp⟩? mma:bp

Lemma 1.2. *Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability (or finite) measure space, and $X, X_1, X_2, \ldots : \Omega \to \Bbbk$ such that $X_n \xrightarrow{n\to\infty}_{bp} X$. Then, $\mathbf{E}[X_n] \xrightarrow{n\to\infty} \mathbf{E}[X]$.*

Proof. Note that the constant function $x \mapsto \sup_n ||f_n||$ is integrable (since $\mathbf{P}$ is finite), so the result follows from dominated convergence. □

2 Almost sure convergence and convergence in probability

Definition 2.1. *Let $X, X_1, X_2, \ldots,$ all E-valued random variables.*

1. *We say that $X_n \xrightarrow{n\to\infty} X$ almost everywhere if*
$$\mathbf{P}(\lim_{n\to\infty} X_n = X) = 1.$$
We also write $X_n \xrightarrow{n\to\infty}_{ae} X$.
2. *We say that $X_n \xrightarrow{n\to\infty} X$ in probability (or in measure) if, for all $\varepsilon > 0$,*
$$\lim_{n\to\infty} \mathbf{P}(r(X_n, X) > \varepsilon) = 0.$$

The two notions here are denoted `∀ᵐ (x : α) ∂P, Filter.Tendsto (fun n => X n x) Filter.atTop (nhds (X x))` and `MeasureTheory.TendstoInMeasure`, respectively.

?⟨l:aep⟩?

l:aep **Lemma 2.2.** *Let $X, X_1, X_2, ...$ be E-valued random variables with $X_n \xrightarrow{n\to\infty}_{ae} X$. Then, $X_n \xrightarrow{n\to\infty}_p X$.*

This is called MeasureTheory.tendstoInMeasure_of_tendsto_ae in mathlib.

?⟨l:puni⟩? l:puni **Lemma 2.3** (Uniqueness, limit in probability). *Let $X, Y, X_1, X_2, ...$ be E-valued random variables with $X_n \xrightarrow{n\to\infty}_p X$ and $X_n \xrightarrow{n\to\infty}_p Y$. Then, $X = Y$, almost surely.*

This does not exist in mathlib yet.

Proof. We write, using monotone convergence and Lemma 2.2

$$\begin{aligned}\mathbf{P}(X \neq Y) &= \lim_{\varepsilon\downarrow 0} \mathbf{P}(r(X,Y) > \varepsilon) \\ &\le \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \mathbf{P}(r(X,X_n) > \varepsilon/2) \\ &\quad + \mathbf{P}(r(Y,X_n) > \varepsilon/2) = 0.\end{aligned}$$

□

3 Separating algebras and characteristic functions

Definition 3.1 (Separating class of functions). *Let $\mathcal{M} \subseteq \mathcal{C}_b(E, \Bbbk)$.*

1. *If, for all $x, y \in E$ with $x \neq y$, there is $f \in \mathcal{M}$ with $f(x) \neq f(y)$, we say that $\mathcal{M}$ separates points.*
2. *If, for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(E)$,*

$$\mathbf{P} = \mathbf{Q} \quad \textit{iff} \quad \mathbf{P}[f] = \mathbf{Q}[f] \textit{ for all } f \in \mathcal{M},$$

   we say that $\mathcal{M}$ is separating in $\mathcal{P}(E)$.

3. *If (i) $1 \in \mathcal{M}$ and (ii) if $\mathcal{M}$ is closed under sums and products, then we call $\mathcal{M}$ a (sub-)algebra. If $\Bbbk = \mathbb{C}$, and (iii) if $\mathcal{M}$ is closed under complex conjugation, we call $\mathcal{M}$ a star-(sub-)algebra.*

In mathlib, 1. and 3. of the above definition are already implemented:

```
structure Subalgebra (R : Type u) (A : Type v)
  [CommSemiring R] [Semiring A] [Algebra R A]
  extends Subsemiring : Type v

abbrev Subalgebra.SeparatesPoints {α : Type u_1}
  [TopologicalSpace α] {R : Type u_2}
  [CommSemiring R] {A : Type u_3}
  [TopologicalSpace A] [Semiring A] [Algebra R A]
  [TopologicalSemiring A] (s : Subalgebra R C(α, A))
  : Prop
```

The latter is an extension of Set.SeparatesPoints, which works on any set of functions.

For the first result, we already need that (E, r) has a metric structure.

This is now implemented.

?⟨l:unique⟩? :unique **Lemma 3.2.** *$\mathcal{M} := \mathcal{C}_b(E, \Bbbk)$ is separating.*

Proof. We restrict ourselves to $\Bbbk = \mathbb{R}$, since the result for $\Bbbk = \mathbb{C}$ follows by only using functions with vanishing imaginary part. Let $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(E)$. We will prove that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all A closed. Since the set of closed sets is a π-system generating the Borel-σ-algebra, this suffices for $\mathbf{P} = \mathbf{Q}$. So, let A be closed and $g = 1_A$ be the indicator function. Let $g_n(x) := (1 - nr(A,x))^+$ (where $r(A,y) := \inf_{y\in A} r(y,x)$) and note that $g_n(x) \xrightarrow{n\to\infty} 1_A(x)$. Then, we have by dominated convergence

$$\mathbf{P}(A) = \lim_{n\to\infty} \mathbf{P}[g_n] = \lim_{n\to\infty} \mathbf{Q}[g_n] = \mathbf{Q}(A),$$

and we are done. $\square$

We will use the Stone-Weierstrass Theorem below. Note that this requires E to be compact.

```
theorem ContinuousMap.starSubalgebra_
↪   topologicalClosure_eq_top_of_separatesPoints
{𝕜 : Type u_2} {X : Type u_1} [IsROrC 𝕜]
↪   [TopologicalSpace X] [CompactSpace X]
(A : StarSubalgebra 𝕜 C(X, 𝕜)) (hA :
↪   Subalgebra.SeparatesPoints A.toSubalgebra) :
StarSubalgebra.topologicalClosure A = ⊤
```

We also need (as proved in the last project) that compact sets are measurable.

```
theorem innerRegular_isCompact_isClosed_
↪   measurableSet_of_complete_countable
[PseudoEMetricSpace α] [CompleteSpace α]
↪   [SecondCountableTopology α] [BorelSpace α]
(P : Measure α) [IsFiniteMeasure P] :
P.InnerRegular (fun s => IsCompact s ∧ IsClosed s)
↪   MeasurableSet
```

The proof of the following result follows [**?**, Theorem 3.4.5].

This does not exist in mathlib yet.

?⟨T:wc3⟩? T:wc3 **Theorem 1** (Algebras separating points and separating algebras)**.** *Let (E, r) be a complete and separable extended pseudo-metric space, and $\mathcal{M} \subseteq \mathcal{C}_b(E, \Bbbk)$ be a star-sub-algebra that separates points. Then, $\mathcal{M}$ is separating.*

Proof. Let $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(E)$, $\varepsilon > 0$ and K compact, such that $\mathbf{P}(K) > 1 - \varepsilon$, $\mathbf{Q}(K) > 1 - \varepsilon$, and $g \in \mathcal{C}_b(E, \Bbbk)$. According to the Stone-Weierstrass Theorem, there is $(g_n)_{n=1,2,\ldots}$ in $\mathcal{M}$ with

$$\sup_{x \in K} |g_n(x) - g(x)| \xrightarrow{n\to\infty} 0. \tag{1}$$

?eq:wc9? eq:wc9

So, (note that $C := \sup_{x \geq 0} x e^{-x^2} < \infty$)

$$\begin{aligned}
\big|\mathbf{P}[ge^{-\varepsilon g^2}] - \mathbf{Q}[ge^{-\varepsilon g^2}]\big| \leq \; & \big|\mathbf{P}[ge^{-\varepsilon g^2}] - \mathbf{P}[ge^{-\varepsilon g^2}; K]\big| \\
& + \big|\mathbf{P}[ge^{-\varepsilon g^2}; K] - \mathbf{P}[g_n e^{-\varepsilon g_n^2}; K]\big| \\
& + \big|\mathbf{P}[g_n e^{-\varepsilon g_n^2}; K] - \mathbf{P}[g_n e^{-\varepsilon g_n^2}]\big| \\
& + \big|\mathbf{P}[g_n e^{-\varepsilon g_n^2}] - \mathbf{Q}[g_n e^{-\varepsilon g_n^2}]\big| \\
& + \big|\mathbf{Q}[g_n e^{-\varepsilon g_n^2}] - \mathbf{Q}[g_n e^{-\varepsilon g_n^2}; K]\big| \\
& + \big|\mathbf{Q}[g_n e^{-\varepsilon g_n^2}] - \mathbf{Q}[ge^{-\varepsilon g^2}; K]\big| \\
& + \big|\mathbf{Q}[ge^{-\varepsilon g^2}; K] - \mathbf{Q}[ge^{-\varepsilon g^2}]\big|
\end{aligned}$$

We bound the first term by

$$\big|\mathbf{P}[ge^{-\varepsilon g^2}] - \mathbf{P}[ge^{-\varepsilon g^2}; K]\big| \leq \frac{C}{\sqrt{\varepsilon}} \mathbf{P}(K^c) \leq C\sqrt{\varepsilon},$$

and analogously for the third, fifth and last. The second and second to last vanish for $n \to \infty$ due to (1). Since

$\mathcal{M}$ is an algebra, we can approximate, using dominated convergence,

$$\begin{aligned}\mathbf{P}[g_n e^{-\varepsilon g_n^2}] &= \lim_{m\to\infty} \mathbf{P}[\underbrace{g_n\Big(1 - \frac{\varepsilon g_n^2}{m}\Big)^m}_{\in\mathcal{M}}] \\ &= \lim_{m\to\infty} \mathbf{Q}[\underbrace{g_n\Big(1 - \frac{\varepsilon g_n^2}{m}\Big)^m}_{\in\mathcal{M}}] = \mathbf{Q}[g_n e^{-\varepsilon g_n^2}],\end{aligned}$$

so the fourth term vanishes for $n \to \infty$ as well. Concluding,

$$\big|\mathbf{P}[g] - \mathbf{Q}[g]\big| = \lim_{\varepsilon\to 0} \big|\mathbf{P}[ge^{-\varepsilon g^2}] - \mathbf{Q}[ge^{-\varepsilon g^2}]\big| \le 4C \lim_{\varepsilon\to 0} \sqrt{\varepsilon} = 0.$$

Since g was arbitrary and $\mathcal{C}_b(E, \Bbbk)$ is separating by Lemma 3.2, we find $\mathbf{P} = \mathbf{Q}$. □

We now come to characteristic functions and Laplace transforms.

This does not exist in mathlib yet.

?⟨Pr:char1⟩?

Pr:char1

Proposition 3.3 (Characteristic function unique)**.** *A probability measure $\mathbf{P} \in \mathcal{P}(\mathbb{R}^d)$ is uniquely given by its characteristic function.*
In other words, if $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(\mathbb{R}^d)$ are such that $\int e^{itx}\mathbf{P}(dx) = \int e^{itx}\mathbf{Q}(dx)$ for all $t \in \mathbb{R}^d$. Then, $\mathbf{P} = \mathbf{Q}$.

Proof. The set

$$\mathcal{M} := \Big\{x \mapsto \sum_{k=1}^n a_k e^{it_k x}; n \in \mathbb{N}, a_1, ..., a_n \in \mathcal{C}, t_1, ..., 1_n \in \mathbb{R}^d\Big\}$$

separates points in $\mathbb{R}^d$. Since $\mathcal{M} \subseteq \mathcal{C}_b(\mathbb{R}^d, \Bbbk)$ contains 1, is closed under sums and products, and closed under complex conjugation, it is a star-subalgebra of $\mathcal{C}_b(E, \mathbb{C})$. So, the assertion directly follows from Theorem 1. □