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Simultaneous Conjoint Measurement: A New Type of Fundamental Measurement

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The essential character of what is classically considered, e.g., by N. R. Campbell, the fundamental measurement of extensive quantities is described by an axiomatization for the comparision of effects of (or responses to) arbitrary *combinations* of "quantities" of a *single specified kind.* For example, the effect of placing one arbitrary combination of masses on a pan of a beam balance is compared with another arbitrary combination on the other pan. Measurement on a ratio scale follows from such axioms. In this paper, the essential character of simultaneous *conjoint* measurement is described by an axiomatization for the comparision of effects of (or responses to) *pairs* formed from *two specified kinds* of "quantities." The axioms apply when, for example, the effect of a pair consisting of one mass and one difference in gravitational potential on a device that responds to momentum is compared with the effect of another such pair. Measurement on interval scales which have a common unit follows from these axioms; usually these scales can be converted in a natural way into ratio scales.

A close relation exists between conjoint measurement and the establishment of response measures in a two-way table, or other analysis-of-variance situations, for which the "effects of columns" and the "effects of rows" are additive. Indeed, the discovery of such measures, which are well known to have important practical advantages, may be viewed as the discovery, via conjoint measurement, of fundamental measures of the row and column variables. From this point of view it is natural to regard conjoint measurement as factorial measurement.

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INTRODUCTION

Norman Robert Campbell (1920, 1928) emphasized the conceptual importance, to physics and, for him at least, to any science worthy of the name, of a particular form of additive measurement. For four decades this emphasis has had a profound influence. In many respects this has been healthy, but in others it may have been unduly restrictive. It seems to us important to reduce these restrictions in several ways, among them by finding:

(1) a more reasonable treatment of experimental errors (which Campbell assumed were bounded),

(2) a formulation covering measurements and scales that possess the conventionally desirable properties only approximately, and

(3) treatments of kinds of fundamental measurement based on operations other than unrestricted, side-by-side combination (which we shall call concatenation) of the things bieng measured.

This paper contributes to the third of these tasks.

In the axiomatization to follow, we begin with an ordering of pairs of objects that represents their effects or a subject's responses to them. But, unlike the classic theories, these objects are not treated as elementary; rather, each is assumed to be adequately described by an ordered pair of components. For example, they might be physical objects with components of mass and gravitational potential or pure tones with components of energy and frequency. It is not assumed that the objects themselves or either of their components can be concatenated in any natural way. The role usually played by the concatenation operation is replaced by the fact that the objects are ordered pairs. From the axioms we give, simultaneous measurement on interval scales is obtained for each kind of quantity separately and for their joint effects. In many situations, it is possible, by added information or by an added postulate, to convert these into ratio scales; often an exponential transformation is first introduced to convert from an additive to a multiplicative representation.

Extension of these results to the simultaneous conjoint measurement of three or more quantities is direct, and will be discussed elsewhere.

I. REPRESENTATION THEOREMS

If $\circ$ represents an operation of concatenation, it is well known that fundamental extensive measurement, as described by Campbell, corresponds to, but is not axiomatized in terms of, the existence of a real-valued function X that "measures" the several quantities and their concatenations in such a way that $X(A \circ B) = X(A) + X(B)$ and that $X(A) \leqslant X(B)$ is equivalent to $A \leqslant B$. This is to say that the axioms, e.g., Hölder (1901) and Suppes (1951), describe situations in which the "effects of" or "responses to " A, B, and $A \circ B$ can be given numerical expression (in a suitably invariant way) so that the effect of $A \circ B$ is the sum of the effects of A and of B, and so that the numerical measure of A is less than or equal to that of B if and only if the effect of A is less than or equal to the effect of B.

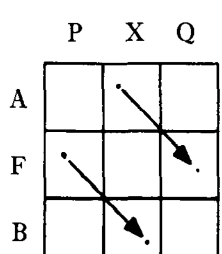
Conjoint measurement corresponds to, but is not axiomatized in terms of, the existence of two real-valued functions φ and ψ that "measure" the effects of the two classes of variables in such a way that the over-all effect of (A, P) is the sum of the "effect" of A as measured by φ and of the "effect" of B as measured by ψ. Thus, the numerical proposition $\varphi(A) + \psi(P) \leqslant \varphi(B) + \psi(Q)$ is equivalent to the statement "the effect of (A, P) is less than or equal to the effect of (B, Q)." The parallel to measurement by concatenation is clear.

A simple graphical way to display the simultaneous effects of two factors is in a two-way table, where the rows correspond to the values of one factor, the columns to the values of the other factor, and the entries correspond to the effects of, or responses to, their joint presence. The conclusion from the axioms of conjoint measurement is that we can "measure" the effect, or response, quantitatively in such a way that (a) the observed ordering of the cells is preserved by the natural ordering of the numbers assigned, (b) the measure for any cell is the sum of a function of its row component and another function of its column compoment, and (c) each of these functions is unique up to the positive linear transformations of interval measurement. Thus, we are able, at a single stroke, to measure both the factors and the responses.

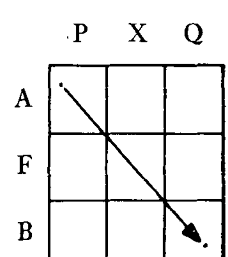
In these terms, the axioms may be roughly described as follows. The first states that the given ordering of cells satisfies mild consistency conditions, including transitivity. The second axiom requires that each of the two factors be sufficiently extensive and finely graded so that, given any cell and any row (or column), we can find a column (or row) such that the new cell thus defined has the same effect or produces the same response as the originally given cell. The third axiom we turn to in a moment. The fourth is an Archimedean condition that states that no nonzero change is "infinitely small" when compared with any other change.

In terms of rows A, F, and B, and columns P, X, and Q, the third axiom can be represented graphically by:

if



then



where the cell at the tail of an arrow dominates (has at least as large an effect as) the cell at the head. The verbal interpretation is that if a change from A to F overbalances a change from X to Q, and if a change from F to B overbalances one from P to X, then the over-all combined change from A to F to B overbalances the combined change from P to X to Q.

II. ADDITIVITY IN A NEW LIGHT

Seeking response measures which make the effects of columns and the effects of rows additive in an analysis-of-variance situation has become increasingly popular as the advantages of such parsimonious descriptions, whether exact or approximate, have become more appreciated. In spite of the practical advantages of such response measures, objections have been raised to their quest, the primary ones being (a) that such "tampering" with data is somehow unethical, and (b) that one should be interested in fundamental results, not merely empirical ones.

For those who grant, in the situations where it is natural, the fundamental character of measurement axiomatized in terms of concatenation, the present measurement theory overcomes both of these objections since its existence shows that qualitatively described "additivity" over pairs of factors of responses or effects is just as axiomatizable as concatenation. Indeed, the additivity is axiomatizable in terms of axioms that lead to scales of the highest repute: interval and ratio scales.

Moreover, as we will illustrate by a simple example, the axioms of conjoint measurement apply naturally to problems of classical physics and permit the measurement of conventional physical quantities on ratio scales.

In the various fields, including the behavioral and biological sciences, where factors producing orderable effects and responses deserve both more useful and more fundamental measurement, the moral seems clear: when no natural concatenation operation exists, one should try to discover a way to measure factors and responses such that the "effects" of different factors are additive.

III. A MECHANICAL EXAMPLE

Consider a simple mechanical example of conjoint measurement involving the joint effects of mass and gravitational potential difference in producing momentum. Let a pendulum hanging *in vacuo* be fitted with auxiliary horizontal arms that end in sticky pans, and arrange it so that pairs of spherical pebbles of the same material can be dropped on the pans *simultaneously* from repeatable points of release. We record, qualitatively, the altitude of release and identity of each pebble and the direction of the first swing of the pendulum. Such a device is, in essence, a two-pan ballistic pendulum that permits us to compare momentum transfer. If A and B represent altitudes of release, or, more precisely, differences in gravitational potential between the release points and the pans, and P and Q represent masses of the pebbles, then the device allows us to compare directly the effect of (A, P) with the effect of (B, Q) when the two pebbles are dropped simultaneously. (The usual precautions to have lever arms of equal magnitude are assumed.)

If our axioms apply to this device, as they would according to classic mechanics, then it follows that suitable functions of both mass and difference in gravitational potential can be measured on interval scales. Moreover, by assuming that the pendulum compares momentum and that momentum is the *product* of a function of the differences in gravitational potential with a function of mass, then it can be shown that our interval scales are the logarithms, to some base, of these two functions.

If we take antilogarithms (= exponentials), to any base, of the interval scale measurements, the results will be ratio scale measurements of positive powers of mass and difference in gravitational potential. As will be shown, the interval scales have a common unit; accordingly, if the antilogarithms are to the same base, these powers will be the same. Thus, with these physical assumptions, our axioms provide a method for the simultaneous fundamental measurement of momentum, mass, and gravitational potential; measurement of the same general nature as that usually given for mass using an ordinary two-pan balance.

All three of these physical quantities can, of course, be measured in conventional manners susceptible to concatenation. For example, differences in gravitational potential can be compared by dropping *in vacuo* pebbles of matched mass into two liquid-filled vessels in a differential calorimeter. (Such a comparison is surely not less clumsy than that of the ballistic pendulum.) We do not claim that conjoint measurement supersedes classic measurement by concatenation, but only that neither is more fundamental than the other.

Other standard physical quantities can be equally easily measured by conjoint techniques. A beam balance with many pans permits conjoint measurement of weight and distance from a pivot to the pan support. A differential calorimeter permits conjoint measurement of resistance and quantity of charge, or of displacement and force exerted. Any direct comparison of photon numbers or light intensity permits conjoint measurement of effective slit aperture and source brightness. In all of these cases, the axiom system holds, and conjoint measurement provides measures, on interval scales, that are logarithmically related to the conventional physical measures.

IV. BEHAVIORAL AND ECONOMIC EXAMPLES

That we can devise alternative ways to measure familiar physical quantities is philosophically interesting, but is of little practical significance to physics as long as conventional measurement based on concatenation is possible. In the behavioral and biological sciences, however, these new methods may be of considerable importance. Many of the quantities that one would like to measure, and that many scientists have felt it should be possible to measure, do not come within the scope of the classical axiomatization because no one has been able to devise a natural concatenation operation (e.g., cf. Cattell, 1962). Usually, however, the stimuli being scaled have at least two distinguishable aspects, each of which affects the judgment being made. For example, it appears that subjects can order pure tones according to their loudness, and considerable evidence shows that loudness, so determined, depends upon both intensity and frequency. But it is far from clear how to define a concatenation of the pure tone (I, F) of intensity I and frequency F with another pure tone (I', F'). However, if a subject's ordering of tones according to loudness should satisfy our axioms, then our results show that one can then assign measures to intensity, frequency, and tones such that the "loudness" of a pure tone is determined by the sum of an intensity contribution and a frequency contribution. Under these circumstances, by taking exponentials, "loudness" of tones could be equally well viewed as the *product* of intensity and frequency components. Whether the subject-determined orderings in this or any other case actually satisfy our axiom system is an empirical problem about which little is now known.

More generally, a question raised throughout the social and behavioral sciences is whether two independent variables contribute independently to an over-all effect or response. The usual approach is to attach to each pair of values of the variables a numerical measure of effect that preserves the order of effects and then to test for independence using an additive statistical model, probably one of the conventional analysis of variance models. When dependence (interaction) is shown to exist, one is uncertain whether the dependence is real or whether another measure would have shown a different result. Certain familiar transformations are often applied in an effort to reduce the danger of the second possibility, but they are unlikely to approach exhausting the infinite family of monotonic transformations, so that one cannot be too sure of the reality of an apparent interaction. Our results show that additive independence exists provided that our axioms are satisfied; of these, the most essential one from a substantive point of view is the cancellation axiom, which is also a necessary condition for an additive representation to exist. Thus, one could test the cancellation axiom by examining a sufficiently voluminous body of ordinal data directly, without introducing any numerical measures and, thereby, test the primary ingredient in additive independence. In some applications this should be more convincing than present techniques.

It should be noted that the adjective "additive" is important throughout the above paragraph. There can be a form of multiplicative independence in which, because φ and ψ assume both positive and negative values, the representation $\varphi(A)\,\psi(P)$ cannot be converted into an additive representation by a logarithmic transformation. In these cases, as is easy to see, the cancellation axiom does not hold. An appropriate modification of our cancellation axiom has been developed by Roskies (1963) which, when coupled with our results, leads to an essentially multiplicative representation.

Within economics, the problem of establishing a "cardinal" (interval or ratio) measure of utility over commodity bundles is classical. When the bundles have only two components, the ordering and indifference relations can be displayed by the familiar graphical device of indifference curves. The cancellation axiom, which has been used by Adams and Fagot (1959), Debreu (1959, 1960), and Suppes and Winet (1955) in this connection, can be expressed as follows: If we fix any two indifference curves, we may form inscribed flights of stairs from alternating horizontal and vertical seats cut off by the two given indifference curves. If now we take any two such flights of

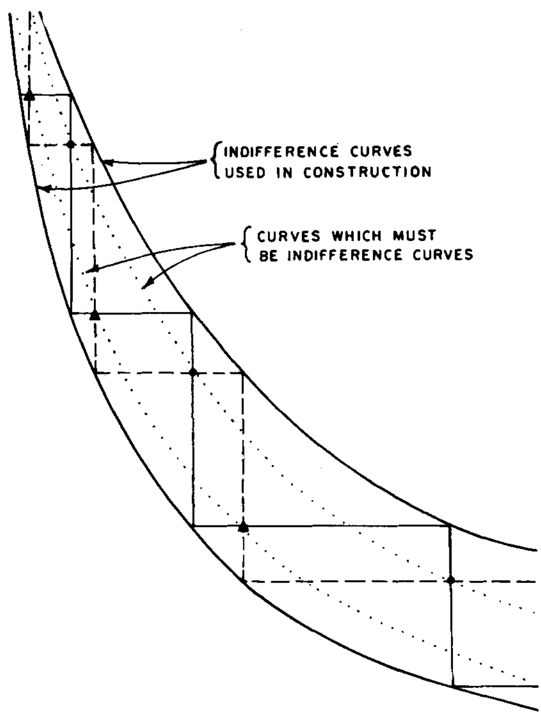


FIG. 1. The cancellation axiom illustrated for indifference curves. If two "flights of stairs" are inscribed between two indifference curves, as shown, then alternate intersections lie on the same indifference curve when the cancellation axiom is true.

stairs, which will intersect one another repeatedly, the cancellation axiom states that *alternate* intersections lie on the same indifference curve. Figure 1 illustrates this for two particular indifference curves and two flights of stairs.

V. THE FINITE AXIOMS: ORDERING, SOLUTION, AND CANCELLATION

The first steps in our formal development rest upon only three axioms; the fourth or Archimedean axiom is needed only at a later stage. In this section we state the first three and draw some of their more direct consequences.

Let $\mathscr{A}$ be a set with typical elements, A, B, C, $\cdots$, F, G, H, $\cdots$ and $\mathscr{P}$ a set with typical elements P, Q, R, $\cdots$, X, Y, Z, $\cdots$; then $\mathscr{A} \times \mathscr{P}$ consists of pairs (A, P), (A, Q), (B, Q), etc. Let $\geqslant$ be a binary relation on such pairs. [Thus $\geqslant$ is equivalent to a subset of $(\mathscr{A} \times \mathscr{P}) \times (\mathscr{A} \times \mathscr{P})$.]

(VA) Ordering Axiom (Axiom 1). *$\geqslant$ is a weak ordering, i.e.:*

(VB, Reflexivity), *$(A, P) \geqslant (A, P)$ holds for all A in $\mathscr{A}$ and P in $\mathscr{P}$,*

(VC, Transitivity), *$(A, P) \geqslant (B, Q)$ and $(B, Q) \geqslant (C, R)$ imply $(A, P) \geqslant (C, R)$,*

(VD, Connectedness), *Either $(A, P) \geqslant (B, Q)$, or $(B, Q) \geqslant (A, P)$, or both.*

(VE) Definition. *For A, B in $\mathscr{A}$ and P, Q in $\mathscr{P}$,*

$$(A, P) = (B, Q) \quad \textit{if and only if} \quad (A, P) \geqslant (B, Q) \quad \textit{and} \quad (B, Q) \geqslant (A, P)$$
$$(A, P) > (B, Q) \quad \textit{if and only if} \quad \text{not } [(B, Q) \geqslant (A, P)].$$

(VF) Solution (of Equations) Axiom (Axiom 2). *For each A in $\mathscr{A}$ and P, Q in $\mathscr{P}$, the equation $(F, P) = (A, Q)$ has a solution F in $\mathscr{A}$, and for each A, B in $\mathscr{A}$ and P in $\mathscr{P}$ the equation $(A, X) = (B, P)$ has a solution X in $\mathscr{P}$.*

(VG) Cancellation Axiom (Axiom 3). *For A, F, B in $\mathscr{A}$ and P, X, Q in $\mathscr{P}$, $(A, X) \geqslant (F, Q)$ and $(F, P) \geqslant (B, X)$ imply $(A, P) \geqslant (B, Q)$.*

We now begin with:

(VH) Theorem. *If axioms 1 to 3 hold, then $(A, P) \geqslant (B, P)$ for any P in $\mathscr{P}$ implies $(A, X) \geqslant (B, X)$ for all X in $\mathscr{P}$, and $(A, P) \geqslant (A, Q)$ for any A in $\mathscr{A}$ implies $(F, P) \geqslant (F, Q)$ for all F in $\mathscr{A}$.*

Proof. By the first of VF, there exists an F in $\mathscr{A}$ such that $(F, X) = (A, P)$. But $(A, P) \geqslant (B, P)$, and, by transitivity, $(F, X) \geqslant (B, P)$. Together with $(A, P) = (F, X)$ this implies $(A, X) \geqslant (B, X)$ by VG. The second statement is proved similarly.

Together, these results justify:

(VJ) Definition. *If axioms 1 to 3 hold, define $A \geqslant B$ if, for some P in $\mathscr{P}$, and therefore for all P in $\mathscr{P}$, $(A, P) \geqslant (B, P)$. Similarly, define $P \geqslant Q$ if, for some A in $\mathscr{A}$, and therefore for all A in $\mathscr{A}$, $(A, P) \geqslant (A, Q)$. Define $A = B$ if and only if $A \geqslant B$ and $B \geqslant A$. Define $A > B$, $P = Q$, and $P > Q$ accordingly.*

(VK) Theorem. *If axioms 1 to 3 hold, then both the relation $\geqslant$ over $\mathscr{A}$ and the relation $\geqslant$ over $\mathscr{P}$ are weak orderings.*

(VL) Theorem. *If axioms 1 to 3 hold, $A \geqslant B$ and $P \geqslant Q$ imply $(A, P) \geqslant (B, Q)$.*

PROOF. If $A \geqslant B$ and $P \geqslant Q$,

$$
\begin{aligned}
(A, Q) &\geqslant (B, Q) && \text{(by VH)}\\
(B, P) &\geqslant (B, Q) && \text{(by VH)}\\
(A, P) &\geqslant (B, Q) && \text{(by VG).}
\end{aligned}
$$

REMARK. We often refer to uses of VJ and VL as "by transfer."

REMARK. Since $\geqslant$ on $\mathscr{A} \times \mathscr{P}$, $\geqslant$ on $\mathscr{A}$, and $\geqslant$ on $\mathscr{P}$ are all weak orderings, the corresponding $=$ are equivalence relations, and it is an immediate consequence of VL that $A = A'$ and $P = P'$ imply $(A, P) = (A', P')$, although the converse need not hold. There are thus two equivalent interpretations for all that follows: (a) A, P, and (A, P) refer to equivalence classes, and the solutions of VF are unique; (b) A, P, and (A, P) are elements some of which may be equivalent, and the solutions of VF or of chains of such equations are unique up to equivalence. We shall leave both alternatives open by saying, where appropriate, "unique up to $=$."

The axioms we have introduced are not too different from those used in theories of measurement based on a concatenation operation. The ordering axiom is about as weak as might reasonably be expected. Requiring "negative" as well as "positive" solutions to equations is a stronger assumption than is made in the concatenation case, as we shall comment upon later. The cancellation axiom is our substitute for a commutativity axiom about concatenation. It is not easy to judge its strength heuristically, but in terms of the discussion at the end of Section I, it does not appear to be unusually strong.

VI. DSS'S AND THE ARCHIMEDEAN AXIOM: CONCLUSIONS

The central tool in classical measurement by concatenation is the notion, and free constructability, of standard sequences (ss) of magnitudes corresponding to an arithmetic progression. The corresponding notion here is the following:

(VIA) CHARACTERIZATION. *A doubly infinite series of pairs* $\{A_i, P_i\}$, $i = 0, \pm 1, \pm 2, \cdots$, *with* A_i *in* $\mathscr{A}$ *and* P_i *in* $\mathscr{P}$, *is a dual standard sequence (dss) provided that*

(VIB) $(A_m, P_n) = (A_p, P_q)$ *whenever* $m + n = p + q$ *for positive, zero, or negative integers,* m, n, p, *and* q.
A dss is trivial *if for all* i *either* $A_i = A_0$ *or* $P_i = P_0$, *in which case both hold by transfer.*

Under axioms 1, 2, and 3, dss's are as freely constructable as ss's are under Campbell's axioms. (This is proved in Section IX.)

The necessary Archimedean axiom can now be stated as:

(VIC) ARCHIMEDEAN AXIOM (AXIOM 4). *If* $\{A_i, P_i\}$ *is a non-trivial dss,* B *is in* $\mathscr{A}$, *and* Q *is in* $\mathscr{P}$, *then there exist (positive or negative) integers* n *and* m *such that*

$$(A_n, P_n) \geqslant (B, Q) \geqslant (A_m, P_m).$$

We now indicate our main results, leaving more general statements of the results and their proofs to later sections.

(VID) ARCHIMEDEAN EXISTENCE THEOREM. *If axioms 1 to 4 hold, there exist real-valued functions* φ *and* ψ *defined over* $\mathscr{A}$ *and* $\mathscr{P}$, *respectively, such that*

(VIE) $\varphi(F) + \psi(X) \geqslant \varphi(G) + \psi(Y)$ *if and only if* $(F, X) \geqslant (G, Y)$,

(VIF) $\varphi(F) \geqslant \varphi(G)$ *if and only if* $F \geqslant G$, *and*

(VIG) $\psi(X) \geqslant \psi(Y)$ *if and only if* $X \geqslant Y$.

COROLLARY. *Under the hypothesis of* VID,

(VIH) *if* $\{F_i, X_i\}$ *is any dss and n is any integer, the following are all equal:*

$$\varphi(F_n) - \varphi(F_0), \qquad \psi(X_n) - \psi(X_0),$$
$$n \cdot [\varphi(F_1) - \varphi(F_0)], \qquad n \cdot [\psi(X_1) - \psi(X_0)].$$

NOTE. This theorem and its corollary are the special case of XIIB for which $p = 1 = q$, $r = 0$.

(VIJ) ARCHIMEDEAN UNIQUENESS THEOREM. *If* φ, ψ *and* φ^*, ψ^* *are two pairs of functions satisfying* VIE-VIG, *there are constants* $a > 0$, b *and* c, *such that*

$$\varphi^*(F) = a \cdot \varphi(F) + b$$
$$\psi^*(X) = a \cdot \psi(X) + c$$

Consequently, φ *measures the elements of* $\mathscr{A}$ *and* ψ *measures the elements of* $\mathscr{P}$ *on interval scales with a common unit.*

NOTE. This theorem is the special case of XIIG, for which $p = 1 = q$ and $r = 0$.

If either (a) we have a distinguished pair of elements F_0 in $\mathscr{A}$ and X_0 in $\mathscr{P}$ for which it is reasonable to require $\varphi(F_0) = 0 = \psi(X_0)$, or (b) we find it reasonable to exponentiate $\varphi(F)$ and $\psi(X)$, using $\Phi(F) = e^{\varphi(F)}$ and $\Psi(X) = e^{\psi(X)}$ as measures of F and X, we will have measured the elements of $\mathscr{A}$ and $\mathscr{P}$ on ratio scales. [In case (a), these ratio scales even retain a common unit!]

VII. RELATIONS TO MEASUREMENT BY CONCATENATION

To the best of our knowledge, this is the first axiomatization of simultaneous conjoint measurement, and it is the only entirely algebraic theory based on an ordering of ordered pairs of elements from distinct sets. Thus no direct comparisons can be made with other measurement axiom systems. There are, however, various ways of interpreting within our system axiomatizations of a concatenation operation. For the axiomatizer, rather than for the user of measurement, there is some interest in what follows from our results, and in discovering which of our axioms fail to hold under such interpretations.

In the first interpretation, we let both $\mathscr{A}$ and $\mathscr{P}$ consist of the elements to be concatenated and we let $(A, P) \leftrightarrow A \circ P$. If we now consider a standard (e.g., cf. Suppes, 1951) axiomatization of mass or length measurement, it is not difficult to see that our axioms 1 and 4 follow from the usually assumed weak ordering and Archimedean axioms. Our cancellation axiom 3 can be established as follows:

Suppose that $(A, X) \geqslant (F, Q)$ and $(F, P) \geqslant (B, X)$; then from commutativity and associativity we have

$$\begin{aligned}(A \circ P) \circ (F \circ X) &= (A \circ X) \circ (F \circ P)\\ &\geqslant (F \circ Q) \circ (B \circ X)\\ &= (B \circ Q) \circ (F \circ X).\end{aligned}$$

Applying the usual right cancellation axiom, $A \circ P \geqslant B \circ Q$, i.e., $(A, P) \geqslant (B, Q)$, thus proving axiom 3.

The existence of solutions to equations, axiom 2, is somewhat more subtle. The problem arises from the fact that in the usual axiomatizations of concatenation the equation $A \circ X = B$ has a solution X if and only if $A < B$, whereas there is no such restriction in our system.

This difficulty is avoided in the second interpretation, by taking both $\mathscr{A}$ and $\mathscr{P}$ to consist of the concatenatable elements, their formal negatives, and a formal zero, 0, and by extending the definition of concatenation, defining $A \circ (-B) \geqslant C$ to mean $A \geqslant C \circ B$ and taking $A \circ 0$ to be A. This extension does not lead to inconsistencies, and axiom 2 holds in this augmented system, which, of course, does not itself satisfy the axioms for nonnegative measurement.

Applied directly, our results yield two functions φ and ψ which are only constrained to be *interval* scales, which seems weaker than the classical results. As we now show, this is not really so. Because $A \circ B = B \circ A$, we have $\varphi(A) + \psi(B) = \varphi(B) + \psi(A)$ for all A, B, so $\varphi(A) - \psi(A)$ is constant. And $A \circ 0 = A$ for all A uniquely defines 0, so that the added condition, which we are free to impose, $\varphi(0) = 0 = \psi(0)$, requires $\varphi = \psi$ and restricts their common freedom to that of ratio scales. Thus, this interpretation of concatenation in terms of conjoint measurement does lead to the full strength of the classical results.

A third interpretation is also illuminating. Suppose $\mathscr{A}$ consists of the concatenable elements and $\mathscr{P}$ consists of all real numbers. For positive integers p, q, choose

$$(A, p) \leftrightarrow A^{(1)} \circ A^{(2)} \circ \cdots \circ A^{(p)}, \qquad \text{where all} \qquad A^{(i)} = A,$$

and

$$(B, q) \leftrightarrow B^{(1)} \circ B^{(2)} \circ \cdots \circ B^{(q)}, \qquad \text{where all} \qquad B^{(j)} = B.$$

For other positive numbers P, Q, the relation between formal symbols $(A, P) \geqslant (B, Q)$ is defined to mean $(A, p) \geqslant (B, q)$ for all p, q with $pQ \geqslant qP$. The extension to nonpositive P, Q is made as usual. Once the existence of $\chi(A)$ has been established by the usual proof, it is relatively easy to establish axioms 1 to 4, and $(A, 0)$ can again be used to convert the interval scale into a ratio scale.

Finally, it is of considerable interest to try to interpret conjoint measurement in terms of concatenation. The simplest approach is to treat conjoint measurement as a scheme of very limited concatenation.

Let us suppose that A consists of labels for certain "rows," and that P consists of labels for certain "columns." Consider all "shifts" from one cell to another, e.g., from $\langle A, P\rangle$ to $\langle B, Q\rangle$. Certain pairs of shifts can be combined in a natural way, e.g., "$\langle A, P\rangle$ to $\langle B, Q\rangle$" and "$\langle B, Q\rangle$ to $\langle C, R\rangle$" clearly ought to produce "$\langle A, P\rangle$ to $\langle C, R\rangle$." If we think of $[A, B]$ as the class of shifts $\{\langle A, P\rangle \text{ to } \langle B, P\rangle \mid \text{some } P\}$, and $[P, Q]$ as the class of shifts $\{\langle A, P\rangle \text{ to } \langle A, Q\rangle \mid \text{some } A\}$, we are led to a similarly limited concatenation of such classes. If one now attemps to define equivalences and extend the definition of concatenation, one is eventually led to the general approach of the present paper.

Krantz (1964) has succeeded in defining operations on the equivalence classes of $\mathscr{A} \times \mathscr{P}$, one for each choice of an A_0 in $\mathscr{A}$ and P_0 in $\mathscr{P}$, which can play the role of concatenation operations. He defines $(A, P) \circ (A', P')$ to be (B, Q) where B is a solution to $(A, P) = (B, P_0)$ and Q is a solution to $(A', P') = (A_0, Q)$. This leads to an alternative proof of our result.

VIII. RELATIONS WITH BEHAVIORAL SCIENCE AXIOM SYSTEMS

Four groups of studies, largely motivated by measurement problems in economics and the behavioral sciences, are related to our formulation of conjoint measurement. The most closely related from the point of view of generality is the study by Adams and Fagot (1959) in which they derived some necessary conditions from the assumption of a representation of the type given in our existence theorem (VID) including, among other things, the ordering and cancellation axioms (1 and 3). In contrast to our work, and to the other studies we shall mention, they did little toward finding a sufficient set of axioms for (VID) to hold.

Debreu (1960) established a result that is closely related to ours in that it leads to an additive, order-preserving, real-valued representation; however, it differs in that it depends upon topological assumptions which, as this present paper demonstrates, can be replaced by simple algebraic assumptions. His assumptions, when specialized to two components, are:

(a) there is a weak ordering $\geqslant$ on $\mathscr{A} \times \mathscr{P}$,

(b) $\mathscr{A}$ and $\mathscr{P}$ are both connected and separable topological spaces,

(c) the sets

$$\{(A, X) \text{ in } \mathscr{A} \times \mathscr{P} \mid (A, X) \geqslant (B, Y)\}$$

and

$$\{(A, X) \text{ in } \mathscr{A} \times \mathscr{P} \mid (A, X) \leqslant (B, Y)\}$$

are closed for every (B, Y) in $\mathscr{A} \times \mathscr{P}$,

(d) the independence property asserted in Theorem VH holds, and

(e) the relations induced on $\mathscr{A}$ and on $\mathscr{P}$ by Definition VJ each have at least two equivalence classes.

From these assumptions Debreu showed that there exist continuous real-valued functions ϕ and ψ defined on $\mathscr{A}$ and $\mathscr{P}$, respectively, such that for (A, P) and (B, Q) in $\mathscr{A} \times \mathscr{P}$,

$$\phi(A) + \psi(P) \geqslant \phi(B) + \psi(Q) \qquad \text{if and only if} \qquad (A, P) \geqslant (B, Q),$$

and $\phi + \psi$ is unique up to positive linear transformations.

Earlier, Suppes and Winet (1955) (see also Suppes and Zinnes, 1963) stated an axiom system for what they called infinite difference systems. The axioms were formulated in terms of a binary relation Q on $\mathscr{A}$ and a quaternary relation R on $\mathscr{A} \times \mathscr{A}$, where $(A, X)\, R(B, Y)$ is interpreted intuitively as meaning that the difference from A to X is not less than the difference from B to Y. Thus, in terms of our notation,

$$(A, X)\, R(B, Y) \qquad \text{if and only if} \qquad (A, Y) \geqslant (B, X).$$

It is not difficult to see that $\geqslant$ is transitive (or satisfies the cancellation axiom) if and only if R satisfies the cancellation axiom (or is transitive). When these translations are made, the Suppes and Winet axiom system is seen to include explicitly the ordering and cancellation axioms (their axioms A3, A4, and A9). In addition, it includes a version of the solution axioms (A6 and A10) and a version of the Archimedean axiom (A11). Their axioms A1, A2, A7, and A8 establish connections between the binary relation Q and the quaternary relation R; these are much the same connections as we used to justify definition VJ. Their axiom A5 postulates that R is commutative in the sense that $(A, X)\, R(X, A)$. From these axioms they show that there exists a real-valued function χ on $\mathscr{A}$ such that

$$AQB \qquad \text{if and only if} \qquad \chi(A) \geqslant \chi(B),$$

and

$$(A, X)\, R(B, Y) \qquad \text{if and only if} \qquad | \chi(A) - \chi(X) | \geqslant | \chi(B) - \chi(Y) |;$$

this function is unique up to positive linear transformations. Note the introduction of absolute values in the representation of the quaternary relation.

Davidson and Suppes (1956) (see also Davidson, Suppes, and Siegel, 1957) modified this system to apply to a finite $\mathscr{A}$ in which the elements are "equally spaced in utility"; such finite difference systems are closely related to our notion of a dss.

Pfanzagl (1959a, b) has recently presented an axiomatization of concatenation that differs somewhat from usual theories of extensive measurement. It involves the usual weak ordering axiom; a strong form of monotonicity (or cancellation), namely, that if $A \gtreqless B$, then for all C in $\mathscr{A}$, $A \circ C \gtreqless B \circ C$; continuity of the concatenation operation; and the following bisymmetry condition: for all A, B, C, and D in $\mathscr{A}$,

$$(A \circ B) \circ (C \circ D) = (A \circ C) \circ (B \circ D).$$

Making the identification $(A, B) = A \circ B$, our ordering axiom (VA) follows from his and our solution axiom (VF) follows from his continuity axiom. To show our cancellation axiom (VG), suppose that $(A, X) \geqslant (F, Q)$ and $(F, P) \geqslant (B, X)$. Applying bisymmetry three times and monotonicity twice, we see that

$$\begin{aligned}(A \circ P) \circ (X \circ F) &= (A \circ X) \circ (P \circ F)\\ &\geqslant (F \circ Q) \circ (P \circ F)\\ &= (F \circ P) \circ (Q \circ F)\\ &\geqslant (B \circ X) \circ (Q \circ F)\\ &= (B \circ Q) \circ (X \circ F).\end{aligned}$$

Thus, by monotonicity, $(A, P) = A \circ P \geqslant B \circ Q = (B, Q)$.

Pfanzagl's system does not explicitly include an Archimedean axiom. Presumably the role it fills in our scheme is filled by his bisymmetry requirement for the concatenation of more than two elements.

The representation that he establishes is that there exists an order preserving function χ on $\mathscr{A}$ and numbers p, q, and r, with $p, q > 0$, such that

$$\chi(A \circ B) = p\chi(A) + q\chi(B) + r.$$

The function χ is invariant up to positive linear transformations, i.e., measurement is on an interval scale. This representation is the special case of XIIC for which $\mathscr{A} = \mathscr{P}$ and $\varphi = \psi$.

IX. DUAL STANDARD SEQUENCES

We now define dual standard sequences and show that the definition implies both the characterization VIA and that dss exist under weak assumptions.

(IXA) Definition. *A doubly infinite sequence of pairs* $\{A_i, P_i\}$, $i = 0, \pm 1, \pm 2, \cdots$ *where, for each* i, A_i *is in* $\mathscr{A}$ *and* P_i *is in* $\mathscr{P}$, *is a* dual standard sequence *if, for each* i,

(IXB) $(A_i, P_{i+1}) = (A_{i+1}, P_i)$,

(IXC) $(A_{i+1}, P_{i-1}) = (A_i, P_i)$.

The dss $\{A_i, P_i\}$ *is said to be on* (A_0, P_0) through A_1, *or* through P_1, *or* through (A_1, P_1) *as may be appropriate.*

NOTE. In IXJ it is shown that either A_0, P_0, A_1 or A_0, P_0, P_1 are sufficient to determine a dss up to $=$.

(IXD) THEOREM. *If axioms 1 to 3 hold, if* $\{A_i, P_i\}$ *is a dss, and if* m, n, p, q *are integers, then*

(IXE) $(A_m, P_n) = (A_p, P_q)$ *whenever* $m + n = p + q$.

NOTE. IXE is equivalent to VIB; its use will be indicated "by dss."

PROOF. First, we establish that $(A_m, P_n) = (A_n, P_m)$. With no loss of generality, we may assume $m \leqslant n$ and so we can write $n = m + k$, $k \geqslant 0$. For $k = 0$, the result clearly holds. For $k = 1$, it is IXB. For $k > 1$, we proceed inductively.

$$
\begin{aligned}
(A_{m+k-1}, P_{m+k}) &= (A_{m+k}, P_{m+k-1}) && \text{(by IXB)}\\
(A_m, P_{m+k-1}) &= (A_{m+k-1}, P_m) && \text{(induction hypothesis)}\\
\therefore\ (A_m, P_{m+k}) &= (A_{m+k}, P_m) && \text{(by cancellation, VG)}
\end{aligned}
$$

Next, letting $k = |m - n| \geqslant 0$, we show that $(A_m, P_n) = (A_{m+1}, P_{n-1})$. For $k = 0$, this is IXC. We proceed inductively. By the first part of the proof, there is no loss of generality in assuming $m < n$ so $n = m + k$, and

$$
\begin{aligned}
& (A_{m+k-1}, P_m) = (A_{m+k}, P_{m-1}) && \text{(because } |m + k - 1 - m| = k - 1 \text{ and the induction hypothesis)}\\
& (A_{m+k-1}, P_m) = (A_m, P_{m+k-1}) && \text{(first part of proof)}\\
\therefore\ & (A_{m+k}, P_{m-1}) = (A_m, P_{m+k-1}) && \text{(by transitivity)}\\
& (A_m, P_m) = (A_{m+1}, P_{m-1}) && \text{(by IXC)}\\
\therefore\ & (A_{m+1}, P_{m+k-1}) = (A_{m+k}, P_m) && \text{(by cancellation, VG)}\\
\therefore\ & (A_m, P_{m+k}) = (A_{m+1}, P_{m+k-1}) && \text{(first part of proof).}
\end{aligned}
$$

A simple induction upon this result completes the proof.

(IXF) THEOREM. *If* $\{A_i, P_i\}$ *and* $\{B_j, Q_j\}$ *are dss's, and if* $(A_f, Q_y) \geqslant (B_g, P_x)$, *then*

(IXG) $(A_{f+m}, Q_{y+n}) \geqslant (B_{g+n}, P_{x+m})$ *for all integers* m *and* n.

PROOF.

$$
\begin{aligned}
& (A_f, Q_y) \geqslant (B_g, P_x) && \text{(by hypothesis)}\\
& (B_g, Q_{y+n}) = (B_{g+n}, Q_y) && \text{(by dss, IXE)}\\
\therefore\ & (A_f, Q_{y+n}) \geqslant (B_{g+n}, P_x) && \text{(by cancellation, VG)}\\
& (A_{f+m}, P_x) = (A_f, P_{x+m}) && \text{(by dss)}\\
\therefore\ & (A_{f+m}, Q_{y+n}) \geqslant (B_{g+n}, P_{x+m}) && \text{(by cancellation).}
\end{aligned}
$$

(IXH) Corollary. *If* $\{A_i, P_i\}$ *and* $\{B_i, Q_i\}$ *are dss's for which* $(A_f, Q_y) = (B_g, P_x)$, *then* $(A_{f+m}, Q_{y+n}) = (B_{g+n}, P_{x+m})$ *for all integers m and n.*

Proof. IXH follows from two applications of IXF: to $(A_f, Q_y) \geqslant (B_g, P_x)$ and to $(B_g, P_x) \geqslant (A_f, Q_y)$.

(IXJ) Existence Theorem. *If axioms 1 to 3 hold, there is a dss on* (A_0, P_0) *through* A_1, *or through* P_1, *or through* (A_1, P_1) *provided that* $(A_0, P_1) = (A_1, P_0)$. *This dss is unique up to* $=$.

Proof. If either A_1 or P_1 is given, but not both, take the missing element to be the solution of $(A_0, P_1) = (A_1, P_0)$. The dss is now constructed inductively. For $i = 0$ and 1 the elements are given. For $i > 1$, if A_i, A_{i+1}, and P_i have been constructed, P_{i+1} is the solution to IXB whose existence is guaranteed by axiom 2. If A_i, P_{i-1}, and P_i have been constructed, A_{i+1} is the solution to IXC. For $i < 0$, if A_i, A_{i+1}, and P_i have been constructed, P_{i-1} is the solution to IXC. If A_i, P_i, and P_{i-1} have been constructed, A_{i-1} is the solution to IXB. If $\{A_i, P_i\}$ and $\{A_i', P_i'\}$ are two dss's so constructed, then $A_i = A_i'$ and $P_i = P_i'$ follow inductively using the second remark following VL.

(IXK) Definition. *A dss is* increasing *if* $A_i < A_{i+1}$ *and* $P_i < P_{i+1}$ *for all integers i, positive or negative.*

(IXL) Lemma. *A dss with* $A_1 > A_0$ *or with* $P_1 > P_0$ *is increasing.*

Proof. First, if $A_1 > A_0$, then $(A_0, P_1) = (A_1, P_0) > (A_0, P_0)$, and so, by transitivity and transfer, $P_1 > P_0$. Similarly, if $P_1 > P_0$, then $A_1 > A_0$. Using this, we now show the dss is increasing.

$$
\begin{array}{rll}
& (A_{i+1}, P_1) > (A_{i+1}, P_0) & \text{(by transfer)} \\
& (A_{i+1}, P_0) = (A_i, P_1) & \text{(by dss)} \\
\therefore & (A_{i+1}, P_1) > (A_i, P_1) & \text{(by transitivity)} \\
\therefore & A_{i+1} > A_i & \text{(by transfer).}
\end{array}
$$

The proof that $P_{i+1} > P_i$ is similar.

(IXM) Corollary. *Either a dss* $\{A_i, P_i\}$ *is increasing, or it is trivial (i.e.,* $A_i = A_j$ *and* $P_i = P_j$ *for all* i, j*), or the reverse dss* $\{A_{-i}, P_{-i}\}$ *is increasing.*

X. LEMMAS ABOUT DSS'S

In the next five lemmas we shall be concerned with dss's on (A, P) for some fixed (A, P). We shall use * to indicate related elements of the opposite kind. Thus we may write either $\{F_i, F_i^*\}$ or $\{X_i^*, X_i\}$ for a dss $\{F_i, X_i\}$.

(XA) MULTIPLICATION LEMMA. *Suppose axioms 1 to 3 hold. Let* $\{F_i, F_i^*\}$ *be a dss on* (A, P) *and for each integer* $n \neq 0$ *let* $\{(F_n)_i, (F_n)_i^*\}$ *be a dss on* (A, P) *through* F_n. *Then*

(XB) $(F_n)_i = F_{ni}$ *and* $(F_n)_i^* = F_{ni}^*$.

PROOF. For any n, $(F_n)_1 = F_n$. Hence

$$\begin{aligned} (A, F_n^*) &= (F_n, P) && \text{(by dss for } F) \\ (F_{n1}, P) &= ((F_n)_1, P) && \text{(by transfer)} \\ ((F_n)_1, P) &= (A, (F_n)_1^*) && \text{(by dss for } F_n) \\ (A, F_n^*) &= (A, (F_n)_1^*) && \text{(by transitivity)} \\ F_n^* &= (F_n)_1^* && \text{(by transfer),} \end{aligned}$$

and XB is established for $i = 1$. Proceeding by induction from i to $i + 1$

$$\begin{aligned} ((F_n)_{i+1}, P) &= ((F_n)_i, (F_n)_1^*) && \text{(by dss)} \\ &= (F_{ni}, F_n^*) && \text{(by induction)} \\ &= (F_{(i+1)n}, P) && \text{(by dss),} \end{aligned}$$

whence $(F_n)_{i+1} = F_{(i+1)n}$ by transfer. The inductive step for $(F_n)_i^*$ can be taken similarly, and XB is established for $i \geqslant 0$.

If $i < 0$, then

$$\begin{aligned} (F_{ni}, F_{-ni}^*) &= (A, P) && \text{(by dss for } F) \\ (A, P) &= ((F_n)_i, (F_n)_{-i}^*) && \text{(by dss for } F^*) \\ &= ((F_n)_i, F_{-ni}^*) && \text{(by result for } -i > 0) \\ \therefore (F_{ni}, F_{-ni}^*) &= ((F_n)_i, F_{-ni}^*) && \text{(by transitivity)} \\ \therefore F_{ni} &= (F_n)_i && \text{(by transfer).} \end{aligned}$$

The extension to $(F_n)_i^*$ is immediate, and XB is completely established.

(XC) TWO-*dss* LEMMA. *If axioms 1 to 3 hold, if* $\{F_i, F_i^*\}$ *and* $\{G_i, G_i^*\}$ *are dss's on* (A, P) *through* F *and* G, *respectively, and if* $F < G$, *then* $F_i < G_i$ *for all* $i > 0$.

PROOF. For $i = 1$, $F_1 = F < G = G_1$. Since by dss, $(F_1, F_{-1}^*) = (A, P) = (G_1, G_{-1}^*)$ and $F_1 < G_1$, we must have $F_{-1}^* > G_{-1}^*$. Proceeding by induction,

$$\begin{aligned} (F_i, P) &= (F_{i+1}, F_{-1}^*) && \text{(by dss)} \\ (G_i, P) &= (G_{i+1}, G_{-1}^*) && \text{(by dss)} \\ (F_i, P) &< (G_i, P) && \text{(by induction)} \\ \therefore (F_{i+1}, F_{-1}^*) &< (G_{i+1}, G_{-1}^*) && \text{(by transitivity).} \end{aligned}$$

If $F_{i+1} \geqslant G_{i+1}$, then since $F^*_{-1} > G^*_{-1}$, we would have $(F_{i+1}, F^*_{-1}) > (G_{i+1}, G^*_{-1})$, which is a contradiction; hence, $F_{i+1} < G_{i+1}$, and the induction is complete.

(XD) FOUR-*dss* LEMMA. *Suppose axioms 1 to 3 hold and that* $\{F_i, F^*_i\}$, $\{X^*_i, X_i\}$, $\{G_i, G^*_i\}$, *and* $\{Y^*_i, Y_i\}$ *are dss's on* (A, P) *through* F, X, G, *and* Y, *respectively. If* $(F, X) \geqslant (G, Y)$, *then for* $k > 0$, $(F_k, X_k) \geqslant (G_k, Y_k)$.

PROOF. Let H satisfy $(H, X) = (G, Y)$ and let $\{H_i, H^*_i\}$ be the dss on (A, P) through H, which exists by IXJ. Thus, by the choice of H, $(H_1, X_1) = (G_1, Y_1)$ and so, by IXH with $m = n = k - 1$, $(H_k, X_k) = (G_k, Y_k)$ for all k. Since $H_1 \leqslant F_1$, it follows from XC that $H_k \leqslant F_k$ for $k > 0$, and so $(F_k, X_k) \geqslant (G_k, Y_k)$ by transfer.

(XE) COMPARABILITY LEMMA. *Suppose axioms 1 to 3 hold and that* $\{F_i, F^*_i\}$ *is a dss and* $\{B_i, B^*_i\}$ *an increasing dss on* (A, P). *If* $m, n > 0$, j, *and* k *are integers such that* $B_j \geqslant F_m$ *and* $B_k \leqslant F_n$, *then* $j/m \geqslant k/n$.

PROOF. By XA, $\{B_{jn}, B^*_{jn}\}$ and $\{F_{mn}, F^*_{mn}\}$ are dss's on (A, P) through B_j and F_m, respectively. If $B_j \geqslant F_m$ and $n > 0$, then by XC, $B_{jn} \geqslant F_{mn}$. Similarly, if $B_k \leqslant F_n$ and $m > 0$, then $B_{km} \leqslant F_{mn}$. By transitivity, $B_{jn} \geqslant B_{km}$, so $jn \geqslant km$, which was to be proved.

(XF) LEMMA. *Suppose that axioms 1 to 3 hold. If* $(F, X) \geqslant (G, Y)$ *and* $\{B_i, B^*_i\}$ *is any dss for which there are integers* f, g, x, *and* y *such that*

$$B_{f+1} > F \geqslant B_f \qquad B_g > G \geqslant B_{g-1}$$

$$B^*_x > X \geqslant B^*_{x-1} \qquad B^*_{y+1} > Y \geqslant B^*_y,$$

then $f + x + 1 > y + g - 1$.

Note that $\{B_i, B^*_i\}$ must be increasing.

PROOF. Observe that

$$\begin{aligned}
(B_g, B^*_{f-g+x+1}) &= (B_{f+1}, B^*_x) && \text{(by dss)}\\
&> (F, X) && \text{(by transfer)}\\
&\geqslant (G, Y) && \text{(by hypothesis)}\\
&\geqslant (B_{g-1}, B^*_y) && \text{(by transfer)}\\
&= (B_g, B^*_{y-1}) && \text{(by dss).}
\end{aligned}$$

Therefore, by transfer, $y - 1 < f - g + x + 1$, and so $f + x + 1 > y + g - 1$.

XI. THE EXISTENCE OF REAL-VALUED FUNCTIONS

(XIA) DEFINITION. *If* $\{B_i\}$ *is a set of elements of* $\mathscr{A}$, *its* convex cover $\mathscr{C}\{B_i\}$ *consists of all A in* $\mathscr{A}$ *such that* $B_j \leqslant A \leqslant B_k$ *for some j and k. If* $\{B_i^*\}$ *is a set of elements of* $\mathscr{P}$, *its* convex cover $\mathscr{C}\{B_i^*\}$ *consists of all P in* $\mathscr{P}$ *such that* $B_g^* \leqslant P \leqslant B_h^*$ *for some g and h.*

(XIB) LEMMA. *If axioms 1 to 3 hold and* $\{B_i, B_i^*\}$ *is an increasing dss, then for any A in* $\mathscr{C}\{B_i\}$, *there exists an integer* ν *such that* $B_\nu > A \geqslant B_{\nu-1}$, *and for any P in* $\mathscr{C}\{B_i^*\}$, *there exists an integer* μ *such that* $B_\mu^* > P \geqslant B_{\mu-1}$.

PROOF. By the definitions of $\mathscr{C}\{B_i\}$ and $\mathscr{C}\{B_i^*\}$, there exist integers j, k, g, and h such that $B_k \geqslant A \geqslant B_j$ and $B_h^* \geqslant P \geqslant B_g^*$. The result follows by a simple finite induction.

(XIC) LEMMA. *If axioms 1 to 3 hold for* $\geqslant$ *over* $\mathscr{A} \times \mathscr{P}$, *and if* $\{B_i, B_i^*\}$ *is a dss, then axioms 1 to 3 hold for* $\geqslant$ *over* $\mathscr{C}\{B_i\} \times \mathscr{C}\{B_i^*\}$.

PROOF. That axioms 1 (= VA) and 3 (= VG) hold for the Cartesian product of any pair of subsets of $\mathscr{A}$ and $\mathscr{P}$ is trivial. To prove axiom 2 (= VF), suppose that A is in $\mathscr{C}\{B_i\}$ and that P, Q are in $\mathscr{C}\{B_i^*\}$. Then there are integers j, k, g, h, e, and f such that $B_j \leqslant A \leqslant B_k$, $B_g^* \leqslant P \leqslant B_h^*$, $B_e^* \leqslant Q \leqslant B_f^*$. Let F be the solution of $(F, P) = (A, Q)$. Then, by transfer and dss, $(F, B_g^*) \leqslant (F, P) = (A, Q) \leqslant (B_k, B_f^*) = (B_{k+f-g}, B_g^*)$ so that $F \leqslant B_{k+f-g}$. A similar argument shows that $B_{j+e-h} \leqslant F$, so that F is in $\mathscr{C}\{B_i\}$. The remainder of axiom 2 follows similarly.

(XID) COROLLARY (TO XIC AND IXJ). *If axioms 1 to 3 hold, if* $\{B_i, B_i^*\}$ *is a dss, and if* F_1 *is in* $\mathscr{C}\{B_i\}$ *or* F_1^* *is in* $\mathscr{C}\{B_i^*\}$, *then there is a dss* $\{F_i, F_i^*\}$ *on* (B_0, B_0^*) *through* F_1 *or* F_1^* *with* F_j *in* $\mathscr{C}\{B_i\}$ *and* F_j^* *in* $\mathscr{C}\{B_i^*\}$ *for all j.*

REMARK. The proof of XIC shows that the conclusion holds for any $\{F_i, F_i^*\}$ on (B_0, B_0^*) through F_1 or F_1^*.

If $\{B_i, B_i^*\}$ is a given increasing dss, F_1 is in $\mathscr{C}\{B_i\}$ or, as is equivalent, F_1^* is in $\mathscr{C}\{B_i^*\}$, and if $n > 0$, XID shows that F_n is in $\mathscr{C}\{B_i\}$, and XIB shows that there exist j and k so that both $B_j \geqslant F_n$ and $B_k \leqslant F_n$. But XE shows that these inequalities divide the rational numbers into two sets, all those in the one being less than or equal to all those in the other, thus defining a Dedekind cut. So there is a single real number which simultaneously is the sup (least upper bound) of the lower set and the inf (greatest lower bound) of the upper set. Consequently, the following definition is meaningful.

(XIE) DEFINITION OF φ_B AND ψ_B. *Suppose axioms 1 to 3 hold. Let an increasing dss*

$\{B_i, B_i^*\}$ be fixed, let F be in $\mathscr{C}\{B_i\}$, let X be in $\mathscr{C}\{B_i^*\}$, and let $\{F_i, F_i^*\}$ and $\{X_i, X_i^*\}$ be dss's on (B_0, B_0^*) through $F = F_1$ and through $X = X_1$, respectively. Define

$$\varphi_B(F) = \inf \left\{ \frac{j}{m} \,\middle|\, B_j \geqslant F_m, m > 0 \right\} = \sup \left\{ \frac{k}{n} \,\middle|\, B_k \leqslant F_n, n > 0 \right\}$$

$$\psi_B(X) = \inf \left\{ \frac{j}{m} \,\middle|\, B_j^* \geqslant X_m, m > 0 \right\} = \sup \left\{ \frac{k}{n} \,\middle|\, B_k^* \leqslant X_n, n > 0 \right\}.$$

We now have:

(XIF) Existence Theorem. *Suppose that axioms 1 to 3 hold, that $p > 0$, $q > 0$, and r are given real numbers, and that $\{B_i, B_i^*\}$ is an increasing dss. There exist real-valued functions φ, ψ, and θ, defined on $\mathscr{C}\{B_i\}$, $\mathscr{C}\{B_i^*\}$, and $\mathscr{C}\{B_i\} \times \mathscr{C}\{B_i^*\}$, respectively, such that*

(XIG) $\theta(F, X) = p\varphi(F) + q\psi(X) + r$,

(XIH) $(F, X) \geqslant (G, Y)$ *implies* $\theta(F, X) \geqslant \theta(G, Y)$,

(XIJ) $F \geqslant G$ *implies* $\varphi(F) \geqslant \varphi(G)$,

(XIK) $X \geqslant Y$ *implies* $\psi(X) \geqslant \psi(Y)$,

(XIL) $\varphi(B_1) > \varphi(B_0)$ *and* $\psi(B_1^*) > \psi(B_0^*)$,

Proof. With no loss of generality we may suppose that $p = q = 1$ and $r = 0$, for if we have $\theta'(F, X) = \varphi'(F) + \psi'(X)$ satisfying XIH through XIK, then it is clear that $\theta = \theta' - r$, $\varphi = (\varphi' - r)/p$, and $\psi = (\psi' - r)/q$ satisfy XIG through XIK.

If $F \geqslant B_0$, define $\varphi(F) = \varphi_B(F)$ where φ_B is as in XIE. If $F < B_0$, then define P_{-1} as the solution in $\mathscr{C}\{B_i^*\}$, by XIC, of $(F, B_0^*) = (B_0, P_{-1})$ and define F_{-1} as the solution in $\mathscr{C}\{B_i\}$ of $(F_{-1}, P_{-1}) = (B_0, B_0^*)$. Since $F_{-1} > B_0$, we may let $\varphi(F) = -\varphi_B(F_{-1})$. Proceed in a similar manner for ψ, and set $\theta(F, X) = \varphi(F) + \psi(X)$. In what follows we work through the details only for positive φ and ψ, i.e., only for $F, G > B_0$ and $X, Y > B_0^*$.

Suppose that $(F, X) \geqslant (G, Y)$. By IXJ and IXL, since $F, G > B_0$ and $X, Y > B_0^*$, there exist increasing dss's $\{F_i, F_i^*\}$, $\{X_n^*, X_n\}$, $\{G_n, G_n^*\}$, and $\{Y_m^*, Y_m\}$ on (B_0, B_0^*) through F, X, G, and Y, respectively. By XD, $(F_n, X_n) \geqslant (G_n, Y_n)$ for $n > 0$. By XIB and XIC, for any $n > 0$, there exist integers f, x, g, and y such that

$$B_{f+1} > F_n \geqslant B_f, \qquad B_g > G_n \geqslant B_{g-1}$$

$$B_x^* > X_n \geqslant B_{x-1}^*, \qquad B_{y+1}^* > Y_n \geqslant B_y^*.$$

Thus, by XF, $f + x + 1 > g + y - 1$. But from the definitions of φ_B and ψ_B

$$\begin{aligned}\varphi_B(F) + \psi_B(X) &\geqslant \frac{f+x-1}{n} \\ &= \frac{f+x+1}{n} - \frac{2}{n} \\ &> \frac{g+y-1}{n} - \frac{2}{n} \\ &= \frac{g+y+1}{n} - \frac{4}{n} \\ &\geqslant \varphi_B(G) + \psi_B(Y) - \frac{4}{n}.\end{aligned}$$

Since n is arbitrary, we have, since $F, G > B_0$, $X, Y > B_0^*$,

$$\theta(F, X) = \varphi_B(F) + \psi_B(X) \geqslant \varphi_B(G) + \psi_B(Y) = \theta(G, Y),$$

which proves XIH.

Suppose $F \geqslant G$, then by transfer $(F, B_0^*) \geqslant (G, B_0^*)$, and by the choice of θ and by what we have just seen,

$$\theta(F, B_0^*) = \varphi_B(F) + \psi_B(B_0^*) \geqslant \theta(G, B_0^*) = \varphi_B(G) + \psi_B(B_0^*).$$

Thus $\varphi(F) = \varphi_B(F) \geqslant \varphi_B(G) = \varphi(G)$, which proves XIJ. The proof of XIK is similar.

Finally, XIL follows from definition XIE since

$$\varphi(B_i) = \varphi_B(B_i) = i \qquad \text{and} \qquad \psi(B_i) = \psi_B(B_i^*) = i \qquad \text{for} \qquad i \geqslant 0.$$

(XIM) Corollary. *If axioms 1 to 3 hold, if* $\{B_i, B_i^*\}$ *is an increasing dss, and if* $\{G_j, G_j^*\}$ *is any dss with* G_j *in* $\mathscr{C}\{B_i\}$ *and* G_j^* *in* $\mathscr{C}\{B_i^*\}$ *for all* j, *then for any functions* φ *and* ψ *satisfying* XIJ *and* XIK *over* $\mathscr{C}\{B_i\}$ *and* $\mathscr{C}\{B_i^*\}$, *respectively*

$$\varphi(G_n) - \varphi(G_0) = n[\varphi(G_1) - \varphi(G_0)]$$

$$\psi(G_n^*) - \psi(G_0^*) = n[\psi(G_1^*) - \psi(G_0^*)].$$

Proof. For $n = 0$ and $n = 1$, XIM clearly holds. For $n > 1$ we proceed by induction. By dss, $(G_{n+1}, G_0^*) = (G_n, G_1^*)$ and $(G_0, G_1^*) = (G_1, G_0^*)$, so by XIG, XIJ, and XIK

$$\varphi(G_{n+1}) + \psi(G_0^*) = \varphi(G_n) + \psi(G_1^*)$$

$$\varphi(G_0) + \psi(G_1^*) = \varphi(G_1) + \psi(G_0^*).$$

Adding and cancelling,

$$\varphi(G_{n+1}) + \varphi(G_0) = \varphi(G_n) + \psi(G_1).$$

Using this and the induction hypothesis,

$$\begin{aligned}\varphi(G_{n+1}) - \varphi(G_0) &= \varphi(G_n) - \varphi(G_0) + \varphi(G_1) - \varphi(G_0)\\ &= n[\varphi(G_1) - \varphi(G_0)] + \varphi(G_1) - \varphi(G_0)\\ &= (n+1)\,[\varphi(G_1) - \varphi(G_0)].\end{aligned}$$

If $n < 0$, then by dss $(G_{-n}, G_0^*) = (G_0, G_{-n}^*)$ and $(G_n, G_{-n}^*) = (G_0, G_0^*)$. Using XIJ and XIK, adding the resulting equations, and using the result for positive integers yields

$$\begin{aligned}\varphi(G_n) - \varphi(G_0) &= -\,[\varphi(G_{-n}) - \varphi(G_0)]\\ &= -\,(-\,n)\,[\varphi(G_1) - \varphi(G_0)]\\ &= n[\varphi(G_1) - \varphi(G_0)].\end{aligned}$$

A parallel proof holds for ψ.

(XIN) UNIQUENESS THEOREM. *Suppose that axioms 1 to 3 hold, real numbers $p > 0$, $q > 0$, and r are given, and an increasing dss $\{B_i, B_i^*\}$ is fixed. If θ, φ, ψ and θ', φ', ψ' are two sets of functions satisfying* XIF *with the same p, q, r, and $\{B_i, B_i^*\}$, then there are real constants $a > 0$, b, and c such that*

$$\begin{aligned}\theta' &= a\theta + b + c,\\ \varphi' &= a\varphi + b,\\ \psi' &= a\psi + c,\end{aligned}$$

i.e., the three scales are interval scales with consistent units.

PROOF. With no loss of generality we may assume that $p = q = 1$ and $r = 0$, that $\varphi' = \varphi_B$ and $\psi' = \psi_B$ are the functions defined in XIE, and so $\theta_B = \varphi_B + \psi_B$ and $\varphi_B(B_0) = \psi_B(B_0^*) = 0$. We show that $[\varphi(F) - \varphi(B_0)]/\varphi_B(F)$ is a constant for $F \neq B_0$, where F is in $\mathscr{C}\{B_i\}$. Suppose then, with no loss of generality, there exists a $G \neq B_0$ such that

$$\frac{\varphi(F) - \varphi(B_0)}{\varphi_B(F)} > \frac{\varphi(G) - \varphi(B_0)}{\varphi_B(G)}.$$

Furthermore, we can assume $F, G > A$, for suppose $F < A$, then define Q and $H > A$ by

$$(F, B_0^*) = (B_0, Q)$$

$$(B_0, B_0^*) = (H, Q).$$

By the properties of φ, ψ, φ_B, and ψ_B,

$$\begin{aligned}
\varphi(F) + \psi(B_0^*) &= \varphi(B_0) + \psi(Q) \\
\varphi_B(F) &= \psi_B(Q) \\
\varphi(B_0) + \psi(B_0) &= \varphi(H) + \psi(Q) \\
0 &= \varphi_B(H) + \psi_B(Q).
\end{aligned}$$

Thus,

$$\begin{aligned}
\varphi(H) - \varphi(B_0) &= -[\varphi(F) - \varphi(B_0)] \\
\varphi_B(H) &= -\varphi_B(F),
\end{aligned}$$

and so

$$\frac{\varphi(H) - \varphi(B_0)}{\varphi_B(H)} = \frac{\varphi(F) - \varphi(B_0)}{\varphi_B(F)}.$$

A similar argument shows that we may assume $G > B_0$. Since all of the terms of the inequality are positive, it can be rewritten as

$$\frac{\varphi_B(G)}{\varphi_B(F)} - \frac{\varphi(G) - \varphi(B_0)}{\varphi(F) - \varphi(B_0)} > \frac{1}{n}$$

for some integer $n > 0$. Let $\{F_i, F_i^*\}$ and $\{G_i, G_i^*\}$ be dss's on (B_0, B_0^*) through F and G, respectively. By IXL they are both increasing, by XID they can be taken in $\mathscr{C}\{B_i\}$ and $\mathscr{C}\{B_i^*\}$, and so by XIB there exists an integer m such that $F_m > G_n \geqslant F_{m-1}$; hence by XIM

$$\begin{aligned}
m\varphi_B(F) &= m[\varphi_B(F) - \varphi_B(B_0)] \\
&= \varphi_B(F_m) \\
&> \varphi_B(G_n) \\
&= \varphi_B(G_n) - \varphi_B(B_0) \\
&= n[\varphi_B(G) - \varphi_B(B_0)] \\
&= n\varphi_B(G),
\end{aligned}$$

and so

$$\frac{\varphi_B(G)}{\varphi_B(F)} < \frac{m}{n}.$$

Similarly,

$$\begin{aligned}
n[\varphi(G) - \varphi(B_0)] &= \varphi(G_n) - \varphi(B_0) \\
&\geqslant \varphi(F_{m-1}) - \varphi(B_0) \\
&= (m-1)[\varphi(F) - \varphi(B_0)],
\end{aligned}$$

and so,

$$\frac{\varphi(G) - \varphi(B_0)}{\varphi(F) - \varphi(B_0)} \geqslant \frac{m-1}{n}.$$

Thus,

$$\frac{1}{n} < \frac{\varphi_B(G)}{\varphi_B(F)} - \frac{\varphi(G) - \varphi(B_0)}{\varphi(F) - \varphi(B_0)} < \frac{m}{n} - \frac{m-1}{n} = \frac{1}{n},$$

which is impossible. So φ and φ_B are linearly related.

Similar proofs show that ψ and ψ_B are linearly related and that θ and θ_B are linearly related. Suppose that the equations are:

$$\varphi_B = a\varphi + b, \qquad \psi_B = a'\psi + c, \qquad \theta_B = a''\theta + d.$$

We show that $a = a' = a''$ and $d = b + c$. Keeping in mind that

$$\varphi_B(B_0) = \psi_B(B_0^*) = \theta_B(B_0, B_0^*) = 0,$$

$$a\varphi(F) + b = \varphi_B(F) = \theta_B(F, B_0^*) = a''\theta(F, B_0^*) + d = a''\varphi(F) + a''\psi(B_0^*) + d.$$

Since this must hold for all F, $a = a''$. A similar argument with ψ shows that $a' = a''$.

Finally,

$$\begin{aligned} \theta_B(B_0, B_0^*) &= 0 \\ &= a\theta(B_0, B_0^*) + d \\ &= a\varphi(B_0) + a\psi(B_0^*) + d \\ &= \varphi_B(B_0) - b + \psi_B(B_0^*) - c + d \\ &= d - b - c. \end{aligned}$$

The proof is concluded.

XII. THE ARCHIMEDEAN AXIOM AND ITS CONSEQUENCES

We now add the Archimedean axiom (VIC = axiom 4).

(XIIA) LEMMA. *If axioms 1 to 3 hold, axiom 4 is equivalent to:*

if $\{B_i, B_i^*\}$ *is an increasing dss, then* $\mathscr{C}\{B_i\}$ *is* $\mathscr{A}$ *and* $\mathscr{C}\{B_i^*\}$ *is* $\mathscr{P}$.

PROOF. Suppose axiom 4 holds and that $\{B_i, B_i^*\}$ is an increasing dss, then for any A in $\mathscr{A}$, there are integers m and n such that

$$(B_n, B_n^*) \geqslant (A, B_0) \geqslant (B_m, B_m^*),$$

whence, by dss,

$$(B_{2n}, B_0) \geqslant (A, B_0) \geqslant (B_{2m}, B_0).$$

So, by transfer, $B_{2n} \geqslant A \geqslant B_{2m}$; hence A is in $\mathscr{C}\{B_i\}$. The proof for the other component is similar.

The converse is obvious.

We can now strengthen the existence theorem XIF by simplifying the hypotheses and replacing implication in XIF to XIK by equivalence.

(XIIB) ARCHIMEDEAN EXISTENCE THEOREM. *Suppose that axioms 1 to 4 hold and that $p > 0$, $q > 0$, and r are given real numbers. There exist real-valued functions φ, ψ, and θ defined on $\mathscr{A}$, $\mathscr{P}$, and $\mathscr{A} \times \mathscr{P}$, respectively, such that*

(XIIC) $\theta(F, X) = p\varphi(F) + q\psi(X) + r$,

(XIID) $(F, X) \geqslant (G, Y)$ *is equivalent to* $\theta(F, X) \geqslant \theta(G, Y)$,

(XIIE) $F \geqslant G$ *is equivalent to* $\varphi(F) \geqslant \varphi(G)$, and

(XIIF) $X \geqslant Y$ *is equivalent to* $\psi(X) \geqslant \psi(Y)$.

PROOF. If all A in $\mathscr{A}$ are equal or, equivalently, if all P in $\mathscr{P}$ are equal, then any constant functions satisfy XIIC through XIIF. Otherwise, there is an increasing dss in $\mathscr{A} \times \mathscr{P}$ to which XIF applies. By XIIA, this establishes XIIC and the implications from left to right of XIID to XIIF. We thus need only consider the opposite implications.

Let some increasing $\{B_i, B_i^*\}$ be given, let $\varphi_B(F), \psi_B(X)$ be defined as in XIE, and let $\varphi(F)$ and $\psi(X)$ be defined as in the proof of XIF. We continue to restrict ourselves to $F, G \geqslant B_0$. Consider first $\varphi(F) \geqslant \varphi(G)$. If $\varphi(F) > \varphi(G)$, then $F > G$; for if $F \leqslant G$, then by XIJ $\varphi(F) \leqslant \varphi(G)$, contrary to assumption. It suffices, therefore, to show that $\varphi(F) = \varphi_B(F) = \varphi_B(G) = \varphi(G)$ implies $F = G$. Suppose not, then with no loss of generality we may assume $F > G$. By IXJ and IXL there exists an increasing dss $\{C_i, C_i^*\}$ on (G, B_0^*) through F, and since $F = C_1$ and $G = C_0$, $\varphi(C_1) = \varphi(C_0)$. By definition of a dss, $(C_1, C_0^*) = (C_0, C_1^*)$, so by what has just been shown

$$\varphi(C_1) + \psi(C_0^*) = \theta(C_1, C_0^*) = \theta(C_0, C_1^*) = \varphi(C_0) + \psi(C_1^*),$$

hence $\psi(C_0^*) = \psi(C_1^*)$. Also, by definition of a dss, $(C_2, C_0^*) = (C_1, C_1^*)$, and so $\varphi(C_2) + \psi(C_0^*) = \varphi(C_1) + \psi(C_1^*)$, and so $\varphi(C_2) = \varphi(C_1) = \varphi(C_0)$. Proceeding inductively, $\varphi(C_i) = \varphi(C_0)$ and $\psi(C_1^*) = \psi(C_0^*)$ for all i. For any B_ν, XIB and XIIA imply there exists some i such that $C_i > B_\nu \geqslant C_{i-1}$, whence

$$\varphi(C_0) = \varphi(C_i) \geqslant \varphi(B_\nu) \geqslant \varphi(C_{i-1}) = \varphi(C_0);$$

hence $\varphi(B_\nu) = \varphi(C_0)$. However, by XIE, $\varphi(B_\nu) = \varphi_B(B_\nu) = \nu$. Since ν is an arbitrary integer, we have a contradiction. Thus, $F = G$ which completes the proof of XIIE.

A parallel proof shows that $\psi(X) \geqslant \psi(Y)$ implies $X \geqslant Y$, which completes the proof of XIIF.

Finally, suppose $\theta(F, X) \geqslant \theta(G, Y)$. If $\theta(F, X) > \theta(G, Y)$, then as above $(F, X) > (G, Y)$. If $\theta(F, X) = \theta(G, Y)$, then let Z be the solution of $(F, X) = (G, Z)$. Now, $\theta(F, X) = \theta(G, Y)$ implies

$$\varphi(F) + \psi(X) = \varphi(G) + \psi(Y),$$

and $(F, X) = (G, Z)$ implies

$$\varphi(F) + \psi(X) = \varphi(G) + \psi(Z).$$

Thus, $\psi(Y) = \psi(Z)$, whence by XIK, $Y \geqslant Z$ and $Z \geqslant Y$ so that $(G, Y) = (G, Z)$, and by transitivity $(F, X) = (G, Y)$, which completes the proof of XIID.

(XIIG) ARCHIMEDEAN UNIQUENESS THEOREM. *Suppose that axioms 1 to 4 hold and real numbers* $p > 0$, $q > 0$, *and* r *are given. If* θ, φ, ψ *and* θ', φ', ψ' *are two sets of functions satisfying* XIIB, *there are real constants* $a > 0$, b, *and* c *such that*

$$\begin{aligned} \theta' &= a\theta + b + c, \\ \varphi' &= a\varphi + b, \\ \psi' &= a\psi + c, \end{aligned}$$

i.e., the three scales (defined on $\mathscr{A} \times \mathscr{P}$, $\mathscr{A}$, *and* $\mathscr{P}$, *respectively) are interval scales with consistent units.*

PROOF. If all A are equal, or if all P are equal, only constant functions can satisfy XIIB. Otherwise there will be an increasing dss $\{B_i, B_i^*\}$ and XIN and XIIA imply XIIG.

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