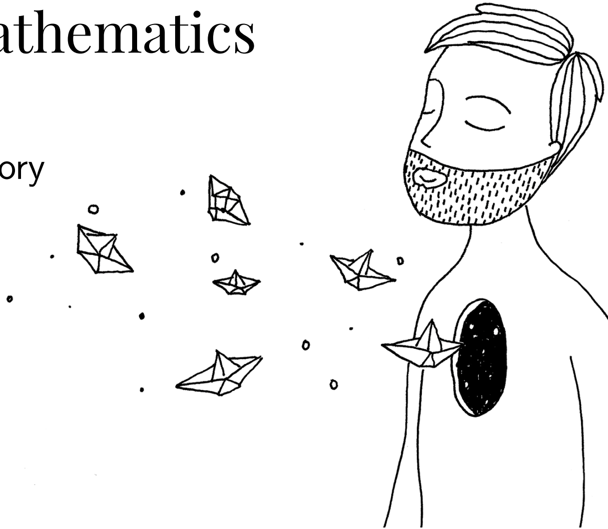
4509 - Bridging Mathematics

Introduction to Measure Theory and Integration

PAULO FAGANDINI



NOVA SCHOOL OF BUSINESS & ECONOMICS

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σ-Algebras

Definition

Let X be a set. An *algebra* is a collection $\mathcal{A}$ of subsets of X such that:

1. $\emptyset \in \mathcal{A}$
2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$
3. If $A_1, A_2, ..., A_n \in \mathcal{A}$ then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ and $\bigcap_{i=1}^{n} A_i \in \mathcal{A}$

If 3 holds for countable infinite sets A_i (*i.e.* you replace the n by ∞), then $\mathcal{A}$ is a $\sigma -$ *algebra*.

Example

These are examples for $\mathcal{A}$ being a $\sigma -$ *algebra*

1. Let $X = \mathbb{R}$, and $\mathcal{A}$ the set of all the subsets of $\mathbb{R}$.
2. Let $X = [0, 1]$ and let $\mathcal{A} = \left\{\emptyset, X, \left[0, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right]\right\}$

Definition

The pair $(X, \mathcal{A})$ is called a *measurable space*. A set A is *measurable* if $A \in \mathcal{A}$

Lemma

If $\mathcal{A}_\alpha$ is a $\sigma-$ algebra for each $\alpha \in I$, with I an index set, then $\cap_{\alpha \in I} \mathcal{A}_\alpha$ is a $\sigma-$ algebra

Proof.

1. If $\mathcal{A}_\alpha$ is a $\sigma-$ *algebra*, therefore $\emptyset \in \mathcal{A}_\alpha \ \forall \alpha \in I$, and then $\emptyset \in \cap_{\alpha \in I} \mathcal{A}_\alpha$
2. Let $S_i \in \cap_{\alpha \in I} \mathcal{A}_\alpha$. It follows that $S_i \in \mathcal{A}_\alpha \ \forall \alpha \in I$, and also $S_i^c \in \mathcal{A}_\alpha \ \forall \alpha \in I$, but then $S_i^c \in \cap_{\alpha \in I} \mathcal{A}_\alpha$.
3. Choose a collection $\{S_i\}_i^\infty \in \cap_{\alpha \in I} \mathcal{A}_\alpha$. Now given that S_i must also be in every $\mathcal{A}_\alpha$, their intersection is also in $\mathcal{A}_\alpha$, and therefore it must be in $\cap_{\alpha \in I} \mathcal{A}_\alpha$.

$\square$

Let $\mathcal{C}$ be a collection of subsets of X, define:

$$\sigma(\mathcal{C}) = \cap \{\mathcal{A}_\alpha | \mathcal{A}_\alpha \text{ is a } \sigma - \textit{algebra},\ \mathcal{C} \subset \mathcal{A}_\alpha\}$$

this is, the intersection of all $\sigma -$ *algebras* containing $\mathcal{C}$. Note that $\sigma(\mathcal{C})$ is non empty, as at least the $\sigma -$ *algebra* $\mathcal{P}(X)$ contains $\mathcal{C}$. Using the previous lemma, we have that $\sigma(\mathcal{C})$ is itself a $\sigma -$ *algebra*. We call this the $\sigma -$ *algebra generated by* $\mathcal{C}$, or that $\mathcal{C}$ generates the $\sigma -$ *algebra* $\sigma(\mathcal{C})$.

Fact

Continuing with the previous definition we can state that:

1. *If* $\mathcal{C}_1 \subset \mathcal{C}_2$, *then* $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$.
2. *Since* $\sigma(\mathcal{C})$ *is a* $\sigma -$ *algebra, then* $\sigma\left(\sigma(\mathcal{C})\right) = \sigma(\mathcal{C})$.

Definition

If X has some structure, for example if it is a metric space, then we can consider open sets in X. If $\mathcal{G}$ is the collection of open subsets of X, then $\sigma(\mathcal{G})$ is the **Borel** $\sigma -$ *algebra* on X, and it is denoted as $\mathcal{B}$. The elements of $\mathcal{B}$ are called *Borel sets*, and are said to be *Borel measurable*.

Proposition

If $X = \mathbb{R}$, then the Borel $\sigma -$ algebra $\mathcal{B}$ is generated by each of the following collection of sets:

1. $\mathcal{C}_1 = \{(a, b) | a, b \in \mathbb{R}\}$
2. $\mathcal{C}_2 = \{[a, b] | a, b \in \mathbb{R}\}$
3. $\mathcal{C}_3 = \{(a, b] | a, b \in \mathbb{R}\}$
4. $\mathcal{C}_4 = \{(a, \infty) | a, b \in \mathbb{R}\}$

Proof.

1. Let $\mathcal{G}$ be the collection of open sets. By definition $\sigma(\mathcal{G})$ is the Borel $\sigma -$ *algebra*. Since every element of $\mathcal{C}_1$ is open, then $\mathcal{C}_1 \subset \mathcal{G}$, and consequently $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{G}) = \mathcal{B}$.

$\square$

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Measures

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Lebesgue Integral