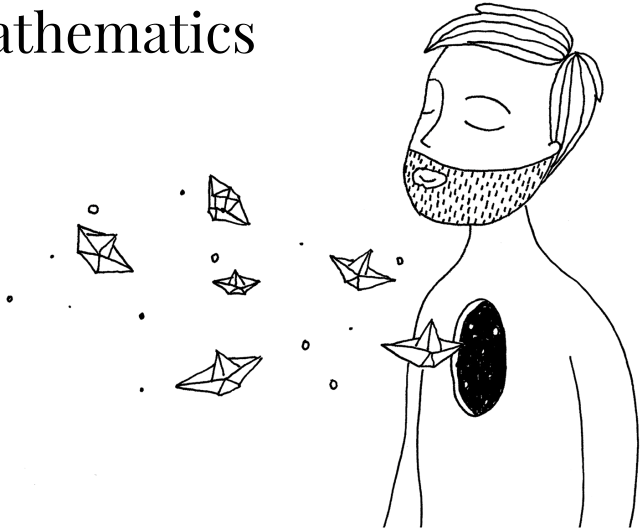
4509 - Bridging Mathematics

Correspondences



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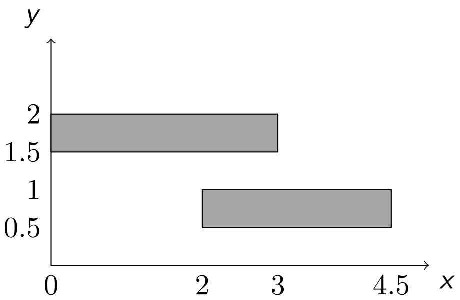
Definition

A **correspondence** or **set-valued function** is a function that maps elements from one set, the domain, to subsets of another set.

$$\Gamma : A \twoheadrightarrow B$$
$$a \in A \rightarrow \Gamma(a) \subseteq B$$

Any idea on how to turn Γ into a function?

Example



$$\Gamma(x) = \begin{cases} [1.5, 2] & \text{if } x \in [0, 2] \\ [1.5, 2] \cup [0.5, 1] & \text{if } x \in [2, 3] \\ [0.5, 1] & \text{if } x \in [3, 4.5] \end{cases}$$

Example

The budget set is another often used example in economics. For a given W, we have:

$$B(p) = \{x \in \mathbb{R}^n | p'x \leq W\}$$

Then $B(p)$ is a correspondence $B : \mathbb{R}^n_{++} \twoheadrightarrow \mathbb{R}^n_{+}$.

Definition

Let $\Gamma : X \twoheadrightarrow Y$. A **selection** is a function $g : X \to Y$ such that $g(x) \in \Gamma(x)\ \forall x \in X$.

Definition

A correspondence Γ is said to be **upper-hemicontinuous (uhc)** at $x_0 \in X$ if, for any open set such that V

$$f(x_0) \subseteq V \Rightarrow \exists \text{ open } U \subseteq X : [x_0 \in U \text{ and } x \in U \Rightarrow f(x) \subseteq V]$$

Definition

A correspondence Γ is said to be **uhc** if Γ is uhc $\forall x \in X$.

Definition

A correspondence Γ is said to be **lower-hemicontinuous (lhc)** at $x \in X$ if for any open set V such that

$$f(x_0) \cap V \neq \emptyset \Rightarrow \exists \text{ open } U \subseteq X : [x_0 \in U \text{ and } x \in U \Rightarrow f(x) \cap V \neq \emptyset]$$

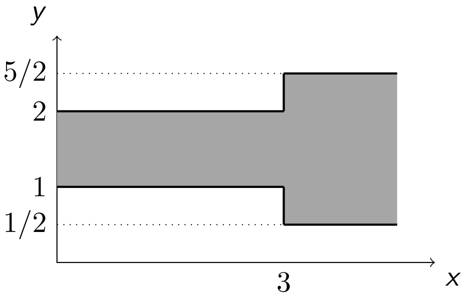
Definition

A correspondence Γ is said to be **lhc** if Γ is lhc $\forall x \in X$.

Example

Consider the correspondence Γ, defined as:

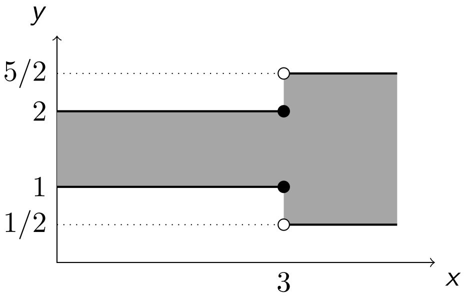
$$\Gamma(x) = \begin{cases} [1,2] & , 0 \leq x \leq 3 \\ \left[\frac{1}{2}, \frac{5}{2}\right] & , x \geq 3 \end{cases}$$



Example

Consider the correspondence Γ, defined as:

$$\Gamma(x) = \begin{cases} [1, 2] & , 0 \leq x \leq 3 \\ \left[\frac{1}{2}, \frac{5}{2}\right] & , x > 3 \end{cases}$$



Definition

A correspondence is said to be **continuous** if it is upper and lower hemicontinuous.

The case of the Budget Set

Consider the correspondence $B : \mathbb{R}^n_{++} \twoheadrightarrow \mathbb{R}^n_+$:

$$B(p) = \{x \in \mathbb{R}^n_+ | p'x \leq p'x_0\}$$

where $x_0 \in \mathbb{R}^n_+$ is the initial endowment of the consumer.
Trivially $x_0 \in B(p)\ \forall p \in \mathbb{R}^n_{++}$.
Well established is also, the fact that changing p by λp , $\lambda \in \mathbb{R}_{++}$ does not affect $B(p)$, *i.e.* $B(p) = B(\lambda p)$ with $\lambda \in \mathbb{R}_{++}$.

The case of the Budget Set

Recall that p is a vector in $\mathbb{R}^n_{++}$, and therefore multiplying by a scalar just changes the length of the vector. We obtain then that any vector in the same *line* arrives, under B to the same budget set.

The case of the Budget Set

Definition

A **simplex** is a n dimensional subspace of $\mathbb{R}^{n+1}$ given by

$$\Delta^n = \left\{ z \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} z_i = 1,\ z_i \geq 0\ \forall i = 1, .., n \right\}$$

Now, we can chose $p \in \Delta^n$, *i.e.* we can normalize the prices. Why is this useful? We have “compacted” our domain! This will be useful in your near future. Of course you would need to restrict the set replacing $z_i \geq 0$ by $z_i > 0$

The case of the Budget Set

Theorem

B is a closed correspondence.

Note that B closed implies that $Gr(B)$ is closed in $\Delta^n \times \mathbb{R}^n_{++}$. A convenient definition of closed set is that any convergent sequence in a closed sets, converges to the set. So the steps to prove this would be:

1. Take the sequences $(p_n, x_n) \in Gr(B)$.
2. Assume that $p_n \to p^*$ and $x_n \to x^*$.
3. Then, prove that $(p^*, x^*) \in Gr(B)$.

The case of the Budget Set

Proof.

1. $(p_n, x_n) \in Gr(B) \;\Rightarrow\; p_n x_n \leq p_n x_0 \Rightarrow p_n(x_n - x_0) \leq 0$
2. $p_n \to p^*$, $x_n \to x^* \Rightarrow p_n x_n \to p^* x^*$ and $p_n x_0 \to p^* x_0$.
3. $p_n(x_n - x_0) \to p^*(x^* - x_0) \Rightarrow p^*(x^* - x_0) \leq 0$
4. $p^* x^* \leq p^* x_0$

And therefore $(p^*, x^*) \in Gr(B)$, which means $Gr(B)$ is closed. Not only, but we can also say that $x^* \in B(p^*)$. $\square$

The case of the Budget Set

Theorem

B is uhc.

If $B(p_0)$ is uhc, we need to show that for any open V such that $B(p_0) \subset V$, there is open U where if $p_0 \in U$ and $p \in U$ then $B(p) \subset V$.

In order to complete this proof we need one definition and one theorem.

Definition

The distance between two sets, A and B is defined as:

$$d(A, B) := inf\{||a - b|| \mid a \in A, b \in B\}$$

Theorem

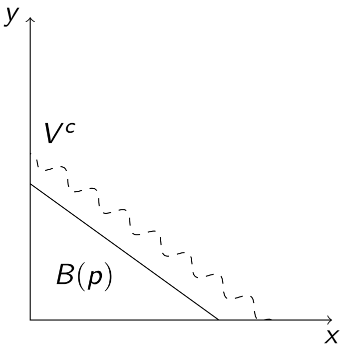
Let A and B be nonempty disjoint sets in $\mathbb{R}^n$. If A is compact, and B is closed, then $d(A, B) > 0$:

$$\exists \epsilon : [a \in A \ \wedge b \in B] \Rightarrow \ ||a - b|| \geq \epsilon$$

Proof.

1. We know that $B(p)$ is compact.
2. If $B(p) \in V$, then $B(p)$ and V^c are disjoint. Moreover, given that V is open, V^c is closed.
3. Use the theorem we just saw, and you obtain that $\exists \epsilon > 0$ such that the budget constraint is "separated" from V.
4. Take $\epsilon/2$ to rotate the budget constraint up to $x_i \epsilon/2$. This rotation implies a new price vector, say $\hat{p}$ (recall, rotates on the endowment bundle)
5. Now define U as the set of all the prices larger than the new prices in each corner. That ensures that the new budget constraints will be below all the possible rotations.
$$U := \left\{ p \in S | p_i > p_i^0 + \frac{\epsilon}{2} \right\}$$
6. But those rotations were by a small enough amount as to be contained in V, and therefore all the prices in U will also generate $B(p) \subset V$.

□


y
V^c
$B(p)$
x

Theorem

B is lhc.