Vectors

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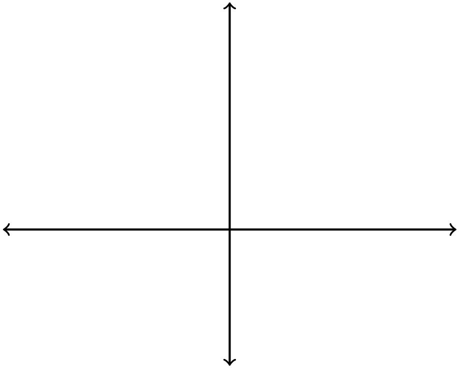
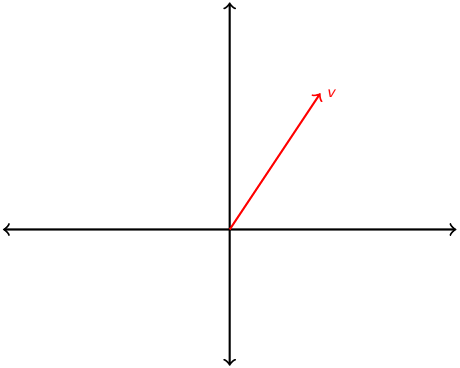
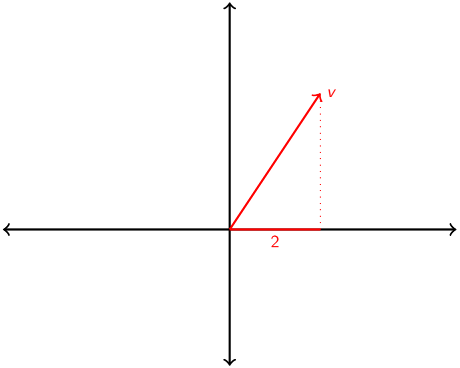
Notation

Notation is important. For this set of slides consider:

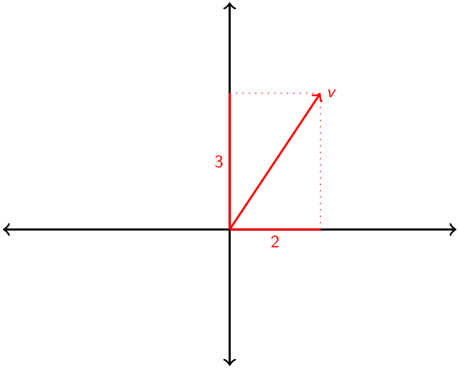
1. Lowercase for elements of a *vector*, v_i.
2. Uppercase for vectors/matrices, V.
3. Calligraphic uppercase for sets, e.g., set $\mathcal{S}$.

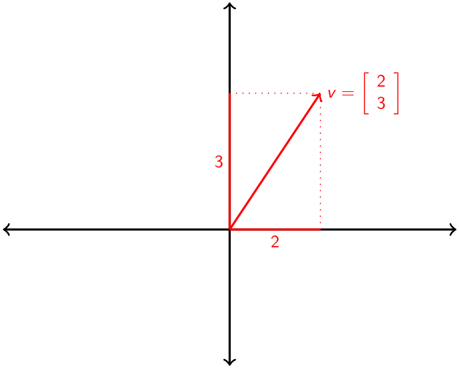


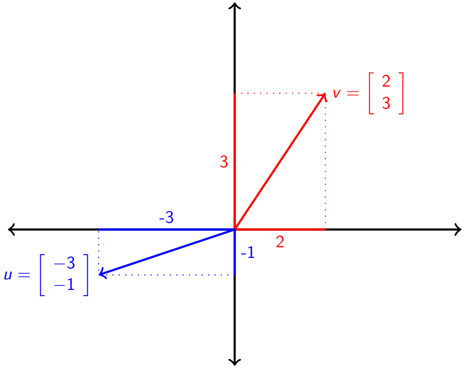

v

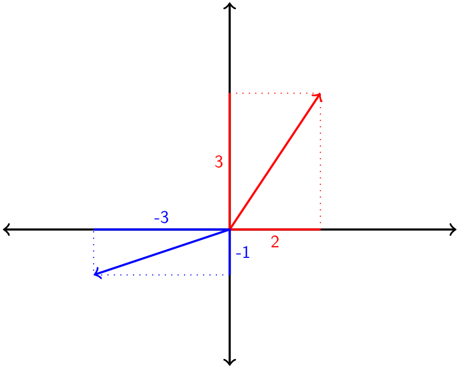


v
2

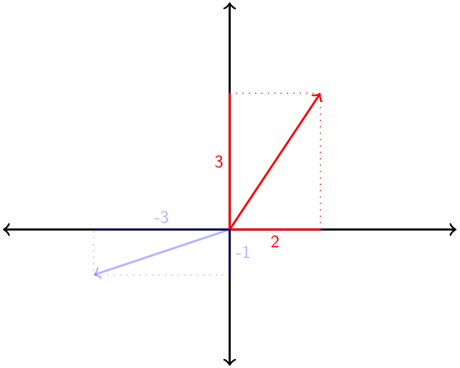

v
3
2

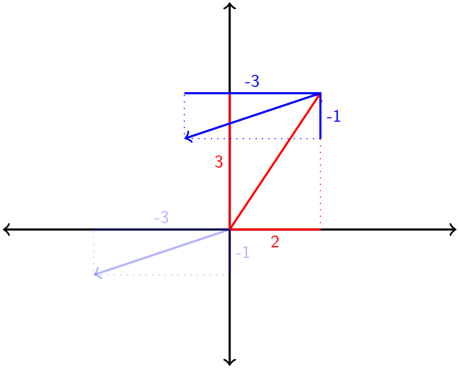

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
3
2

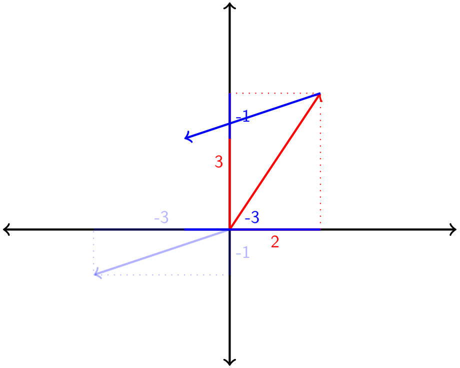

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
3
-3
2
-1
$u = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

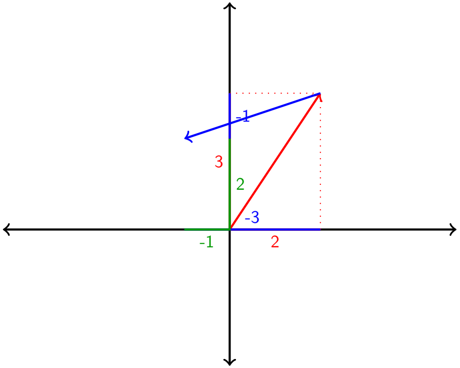


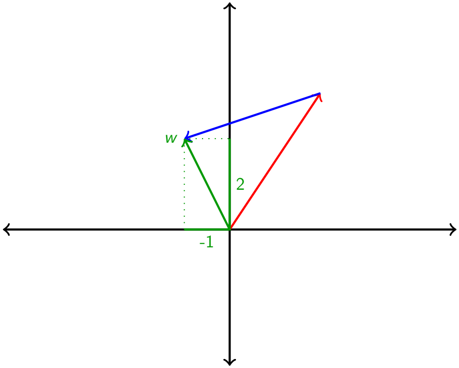
$$u + v?$$

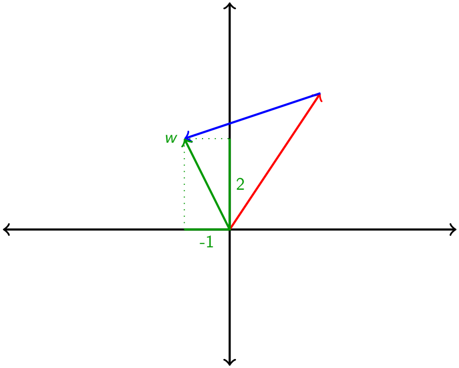

3
-3
2
-1


-3
-1
3
-3
2
-1

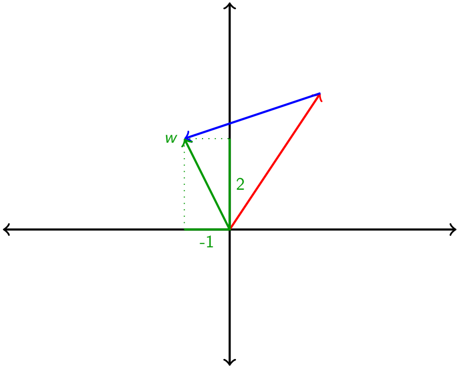

-1
3
-3
-3
2
-1


-1
3
2
-3
-1
2


w
2
-1

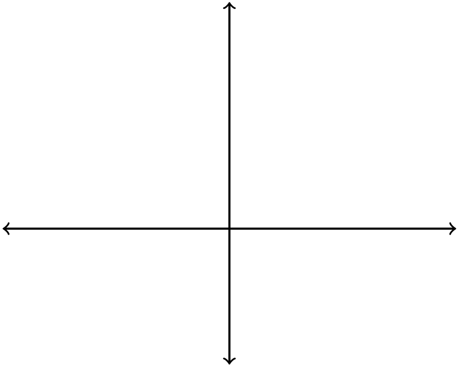


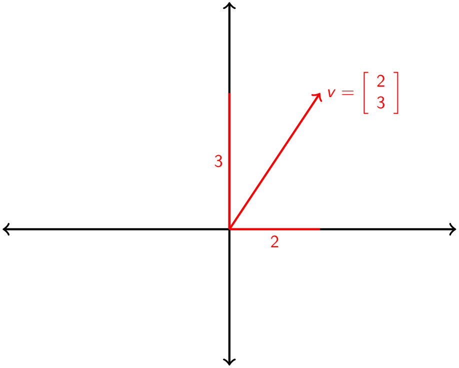
$$u + v = w$$

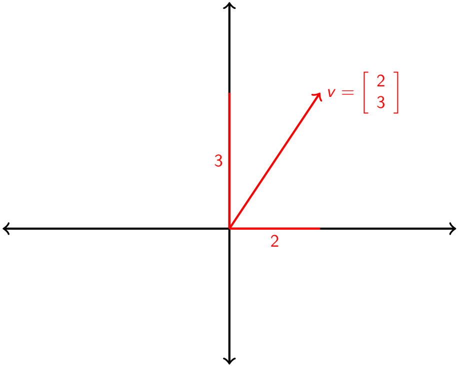


$$u + v = w$$

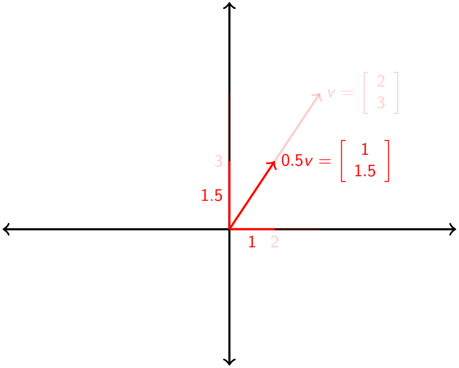
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

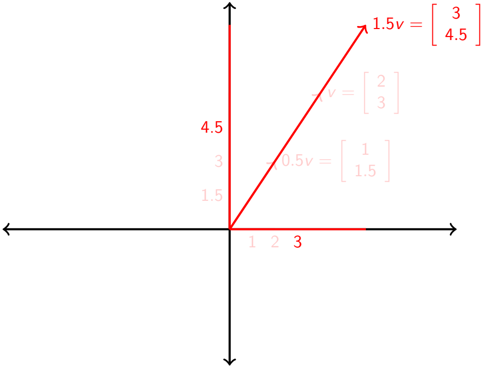


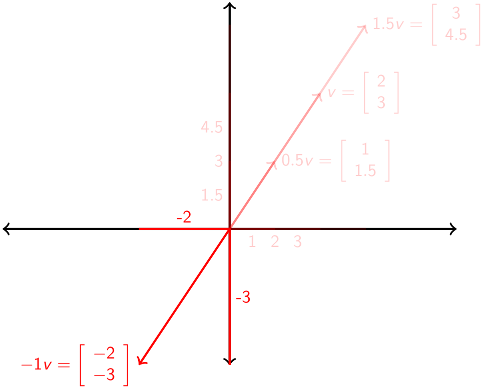

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
3
2

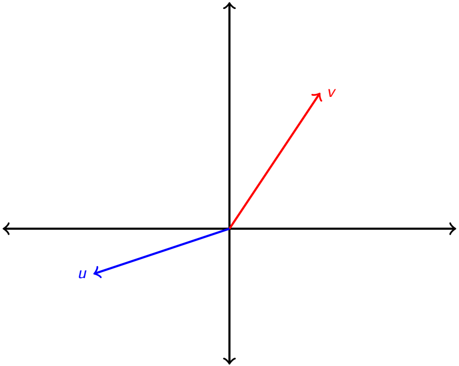


$$0.5v?$$

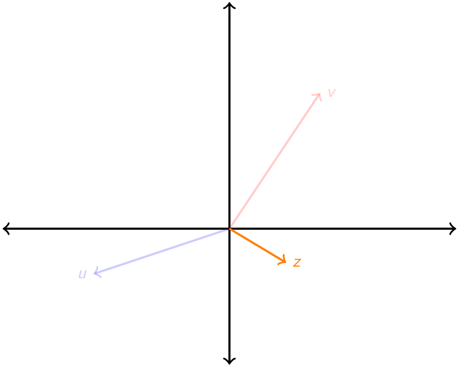

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$0.5v = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$
3
1.5
1
2

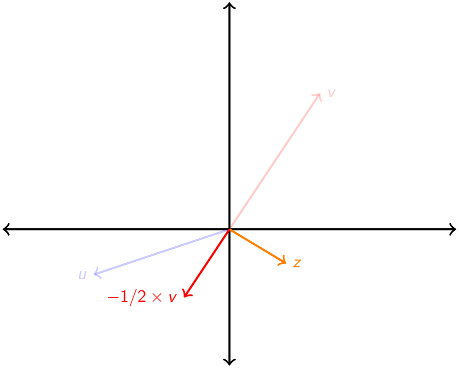

$1.5v = \begin{bmatrix} 3 \\ 4.5 \end{bmatrix}$
$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$0.5v = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$
4.5
3
1.5
1
2
3


$1.5v = \begin{bmatrix} 3 \\ 4.5 \end{bmatrix}$
$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$0.5v = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$
4.5
3
1.5
-2
1 2 3
-3
$-1v = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

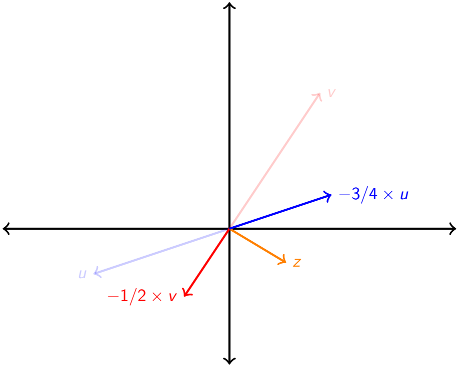


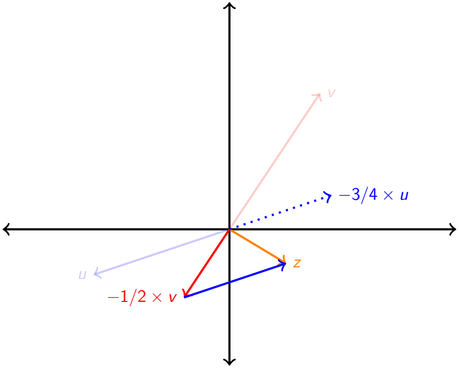
v
u

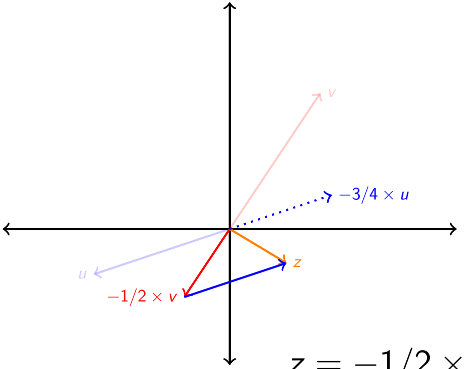

v
z
u



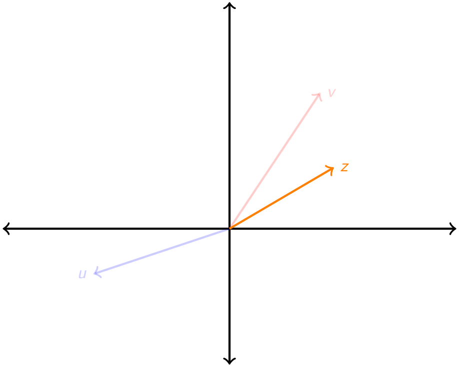
v
z
u
$-1/2 \times v$


v
$-3/4 \times u$
z
u
$-1/2 \times v$

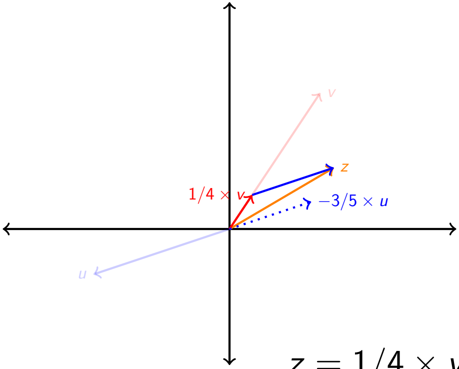

v
$-3/4 \times u$
z
u
$-1/2 \times v$



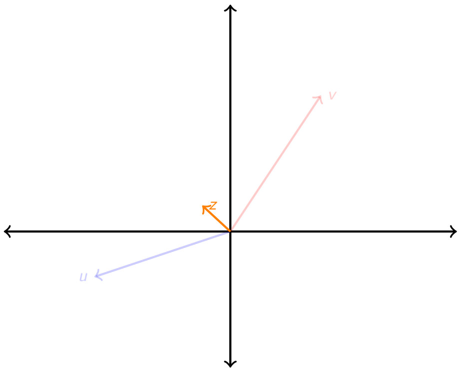
$$z = -1/2 \times v + -3/4 \times u$$

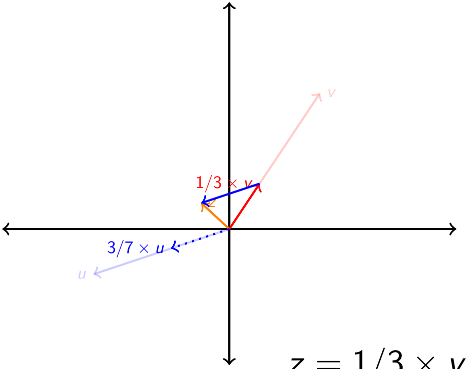


v
z
u



$$z = 1/4 \times v + -3/5 \times u$$

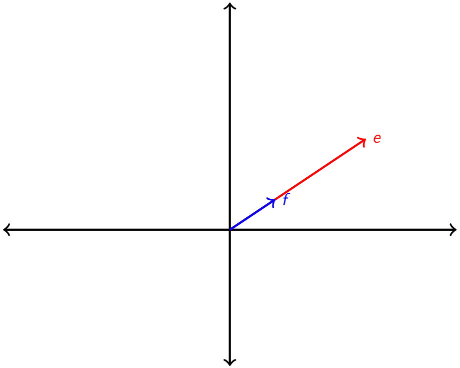

v
z
u



$$z = 1/3 \times v + 3/7 \times u$$

We can write any vector in the plane as the result of the product and sum of u and v (a.k.a. a *linear combination*). These vectors, are not special, except for 1 thing... they are linearly independent.

Consider now these two vectors e and f...



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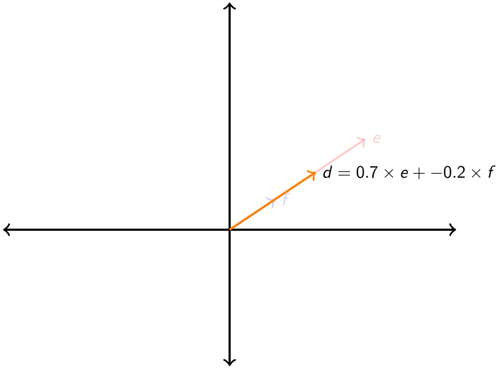

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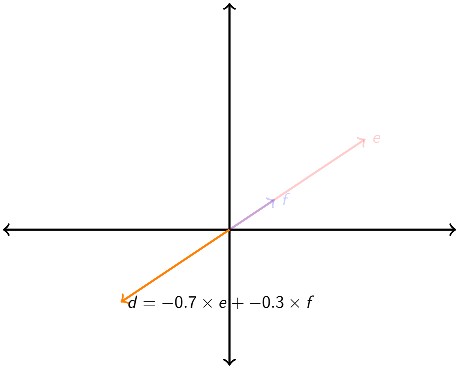

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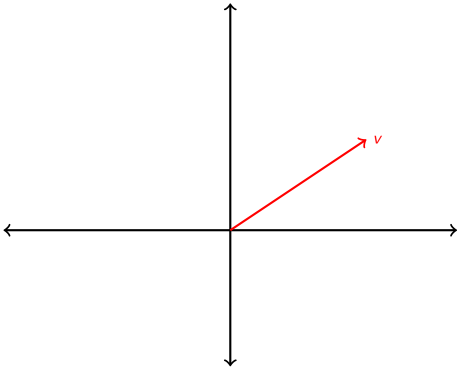
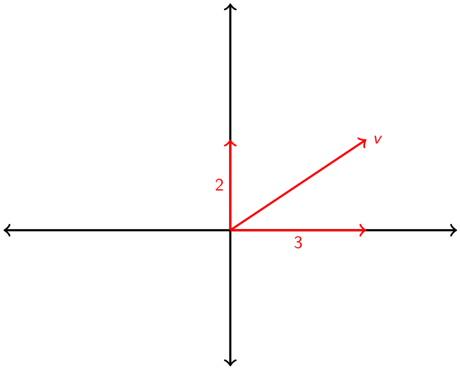
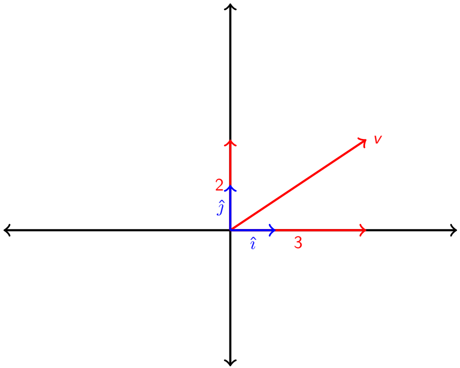
We can only “create” vectors along the same line, the line that goes in the direction of vectors e and f. These vectors are linearly dependent.

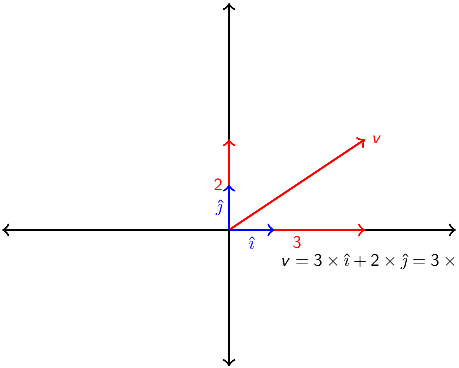
Actually, we only needed one of them to create all the others that we could draw!


v


v
2
3



v
2
$\hat{\jmath}$
$\hat{\imath}$
3



$$v = 3\times\hat{\imath} + 2\times\hat{\jmath} = 3\times\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\hat{\imath}} + 2\times\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\hat{\jmath}} = \begin{bmatrix}3\\2\end{bmatrix}$$

Definition

A **vector** is an element V of $\mathbb{R}^n$, for $n \geq 2$. A scalar is an element of $\mathbb{R}$.

Vectors are to be written as columns, example:

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_{n-1} \\ v_n \end{pmatrix} \quad \in \quad \mathbb{R}^n$$

Let $X, Y \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, then

1. The sum,

$$X + Y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

2. Scalar multiplication,

$$\alpha X = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$0 \in \mathbb{R}^n$ is,

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

that is a vector of dimension $n \times 1$ filled with zeroes.

Definition

A **vector space** $\mathcal{S}$, satisfies that, for any $A, B \in \mathcal{S}$, and $\alpha \in \mathbb{R}$,

- $(A + B) \in \mathcal{S}$
- $\alpha A \in \mathcal{S}$

It is trivial to show that $\mathbb{R}^n$ is a vector space.

Definition

A nonempty set $\mathcal{S} \subseteq \mathbb{R}^n$ is a **vector subspace** of $\mathbb{R}^n$ if, with the vector addition and the scalar multiplication it is a vector space by itself.

Conjecture

Let $\mathcal{V} \subseteq \mathbb{R}^n$, nonempty. $\mathcal{V}$ is a vector subspace of $\mathbb{R}^n$ if and only if,

1. $0 \in \mathcal{V}$,
2. $a, b \in \mathcal{V}$, $\alpha \in \mathbb{R}$, then $a + \alpha b \in \mathcal{V}$

Conjecture

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1. $0 \in \mathcal{V}$,
2. $a, b \in \mathcal{V}$, $\alpha \in \mathbb{R}$, then $a + \alpha b \in \mathcal{V}$

Quick quiz! 15 min to prove it!

Proof.

- $\Rightarrow$... If $\mathcal{V}$ is v.s. of $\mathbb{R}^n$, we know that the scalar multiplication and the sum is in the space. Because scalar mult. we know that $\alpha b \in \mathcal{V}$, so the sum must be in $\mathcal{V}$ too.

Proof.

- $\Rightarrow$... If $\mathcal{V}$ is v.s. of $\mathbb{R}^n$, we know that the scalar multiplication and the sum is in the space. Because scalar mult. we know that $\alpha b \in \mathcal{V}$, so the sum must be in $\mathcal{V}$ too.
- $\Leftarrow$... If $a + \alpha b \in \mathcal{V}$, then it holds in particular for $\alpha = 1$, so the sum is *closed* in the space. Also, let $a = 0$, and you have the scalar multiplication. Then $\mathcal{V}$ must be a v.s.

$\square$

Definition

Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a set of k vectors, then, $Z \in \mathbb{R}^n$ is a **linear combination** of the vectors $\{V_i\}_{i=1}^k$in $\mathcal{V}$ if there are scalars α_j $j = 1, ..., k$ such that,

$$Z = \sum_{j=1}^{k} \alpha_j V_j$$

Definition

A **linear subspace** generated by the vectors in $\mathcal{V}$, represented $L(\mathcal{V})$, is the set of all the linear combinations of those vectors.

Conjecture

1. Let $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, such that $\mathcal{V} \subseteq \mathcal{W}$, then $L(\mathcal{V}) \subseteq L(\mathcal{W})$
2. If $Y \in L(\mathcal{V})$, then $L(\{Y\} \cup \mathcal{V}) = L(\mathcal{V})$
3. Given a nonempty $\mathcal{V} \subseteq \mathbb{R}^n$, then $L(\mathcal{V})$ is a vector subspace of $\mathbb{R}^n$.

Conjecture

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3. Given a nonempty $\mathcal{V} \subseteq \mathbb{R}^n$, then $L(\mathcal{V})$ is a vector subspace of $\mathbb{R}^n$.

Quick quiz! Prove it $\rightarrow$ 15 min.

Proof.

1. Trivial. If $X \in L(\mathcal{V}) \Rightarrow X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i$, and because $\mathcal{V} \subseteq \mathcal{W}$ those vectors are also part of $\mathcal{W}$, so $X \in L(\mathcal{W})$, so $L(\mathcal{V}) \subseteq L(\mathcal{W})$.

Proof.

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2. - As $\mathcal{V} \subseteq \mathcal{V} \cup \{Y\}$, we have that $L(\mathcal{V}) \subseteq L(\mathcal{V} \cup \{Y\})$.

Proof.

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   - We need then $L(\mathcal{V} \cup \{Y\}) \subseteq L(\mathcal{V})$.

Proof.

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   - We need then $L(\mathcal{V} \cup \{Y\}) \subseteq L(\mathcal{V})$.
   - Let $X \in L(\mathcal{V} \cup \{Y\})$, then there are scalars α_i such that

$$X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta Y$$

Proof.

1. Trivial. If $X \in L(\mathcal{V}) \Rightarrow X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i$, and because $\mathcal{V} \subseteq \mathcal{W}$ those vectors are also part of $\mathcal{W}$, so $X \in L(\mathcal{W})$, so $L(\mathcal{V}) \subseteq L(\mathcal{W})$.
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   - Let $X \in L(\mathcal{V} \cup \{Y\})$, then there are scalars α_i such that

   $$X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta Y$$

   - As $Y \in L(\mathcal{V})$ there are scalars γ_i such that $Y = \sum_{v_i \in \mathcal{V}} \gamma_i v_i$

Proof.

1. Trivial. If $X \in L(\mathcal{V}) \Rightarrow X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i$, and because $\mathcal{V} \subseteq \mathcal{W}$ those vectors are also part of $\mathcal{W}$, so $X \in L(\mathcal{W})$, so $L(\mathcal{V}) \subseteq L(\mathcal{W})$.
2.
   - As $\mathcal{V} \subseteq \mathcal{V} \cup \{Y\}$, we have that $L(\mathcal{V}) \subseteq L(\mathcal{V} \cup \{Y\})$.
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   - Let $X \in L(\mathcal{V} \cup \{Y\})$, then there are scalars α_i such that

$$X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta Y$$

   - As $Y \in L(\mathcal{V})$ there are scalars γ_i such that $Y = \sum_{v_i \in \mathcal{V}} \gamma_i v_i$
   - $X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta \left(\sum_{v_i \in \mathcal{V}} \gamma_i v_i \right) = \sum_{v_i \in \mathcal{V}} (\alpha_i + \beta \gamma_i) v_i$. But $\alpha + \beta\gamma$ is a scalar, so

Proof.

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2. - As $\mathcal{V} \subseteq \mathcal{V} \cup \{Y\}$, we have that $L(\mathcal{V}) \subseteq L(\mathcal{V} \cup \{Y\})$.
   - We need then $L(\mathcal{V} \cup \{Y\}) \subseteq L(\mathcal{V})$.
   - Let $X \in L(\mathcal{V} \cup \{Y\})$, then there are scalars α_i such that

   $$X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta Y$$

   - As $Y \in L(\mathcal{V})$ there are scalars γ_i such that $Y = \sum_{v_i \in \mathcal{V}} \gamma_i v_i$
   - $X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta \left(\sum_{v_i \in \mathcal{V}} \gamma_i v_i\right) = \sum_{v_i \in \mathcal{V}} (\alpha_i + \beta\gamma_i) v_i$. But $\alpha + \beta\gamma$ is a scalar, so
   - $X \in L(\mathcal{V})$, proof is complete.

Proof.

1. Trivial. If $X \in L(\mathcal{V}) \Rightarrow X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i$, and because $\mathcal{V} \subseteq \mathcal{W}$ those vectors are also part of $\mathcal{W}$, so $X \in L(\mathcal{W})$, so $L(\mathcal{V}) \subseteq L(\mathcal{W})$.
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   - We need then $L(\mathcal{V} \cup \{Y\}) \subseteq L(\mathcal{V})$.
   - Let $X \in L(\mathcal{V} \cup \{Y\})$, then there are scalars α_i such that

   $$X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta Y$$

   - As $Y \in L(\mathcal{V})$ there are scalars γ_i such that $Y = \sum_{v_i \in \mathcal{V}} \gamma_i v_i$
   - $X = \sum_{v_i \in \mathcal{V}} \alpha_i v_i + \beta \left(\sum_{v_i \in \mathcal{V}} \gamma_i v_i\right) = \sum_{v_i \in \mathcal{V}} (\alpha_i + \beta\gamma_i) v_i$. But $\alpha + \beta\gamma$ is a scalar, so
   - $X \in L(\mathcal{V})$, proof is complete.
3. 0 belongs to any $L()$, as it is the case with scalars $= 0$. Now, let $X, Y \in L(\mathcal{V})$ and $\gamma \in \mathbb{R}$; $X + \gamma Y = \sum_{v_i \in \mathcal{V}} (\alpha_i + \gamma\beta_i) v_i$ if we write each vector as a linear comb. For the same argument used before, we complete the proof.

$\square$

Definition

A set of k vectors $\mathcal{V} \subseteq \mathbb{R}^n$ is **linearly independent** if, $\forall \alpha_j \in \mathbb{R}$

$$\sum_{j=1}^{k} \alpha_k V_j = 0 \quad \Leftrightarrow \quad \alpha_j = 0$$

Definition

Conversely, if there are $\{\alpha_i\}_{i=1}^{k}$, with at least one $\alpha_k \neq 0$, then they are **linearly dependent.**

Definition

The set of vectors $\mathcal{X} \subseteq \mathcal{V}$ **generates** the vector subspace $\mathcal{V} \subseteq \mathbb{R}^n$ if any $V \in \mathcal{V}$ can be written as a linear combination of the vectors in $\mathcal{X}$.

Moreover, if the vectors in $\mathcal{X}$ are linearly independent, then $\mathcal{X}$ is called a **basis** of $\mathcal{V}$.

Conjecture

Let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ a basis of the vector subspace $\mathcal{V} \subseteq \mathbb{R}^n$. Then, for any $V \in \mathcal{V}$, there are **unique** scalars $\{\alpha_i\}_{i=1}^k$ such that

$$V = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_k X_k$$

Conjecture

Any set of n linearly independent vectors $\mathcal{X} \subseteq \mathbb{R}^n$, generates $\mathbb{R}^n$

Definition

The **dimension** of a vector space $\mathcal{V}$ is the maximum number of *l.i.* vectors that generates it. This number coincides with the number of vectors in any basis of the space. It is denoted $dim(\mathcal{V})$.

Definition

Given $X, Y \in \mathbb{R}^n$, the **inner product** corresponds to:

$$X \cdot Y = \sum_{j=1}^{n} x_j y_j \quad \in \mathbb{R}$$

Definition

The **Euclidean norm** of a vector $X \in \mathbb{R}^n$ is:

$$||X|| = \sqrt{X \cdot X} = \sqrt{\sum_{j=1}^{n} x_j^2} \quad \in \mathbb{R}$$

Definition

For $X, Y \in \mathbb{R}^n$, the **Euclidean distance** between them is defined as:

$$d(X, Y) = ||X - Y||$$

Definition

For $X, Y \in \mathbb{R}^n$, both different from zero, the **angle** between them, denoted as $\angle(X, Y)$, is defined as the value that satisfies,

$$\cos(\angle(X, Y)) = \frac{X \cdot Y}{||X|| \cdot ||Y||} \quad \in \quad [-1, 1]$$

Definition

Two vectors X, Y are **orthogonal**, if $\angle(X, Y) = 90^\circ$, or equivalently, $X \cdot Y = 0$. It is denoted as $X \perp Y$.

Definition

Let $X \in \mathbb{R}^n$. If $||X|| = 1$, X is a **unit vector**.

Definition

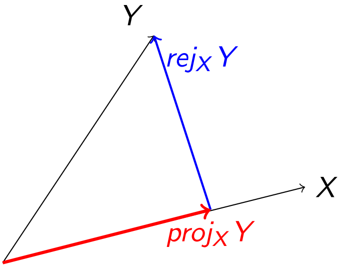
Consider two vectors $X, Y \in \mathbb{R}^n$, both different from zero. The **projection** of Y over X is defined as:

$$proj_X Y = Y \cdot \frac{X}{||X||}$$

The **rejection**, is defined as:

$$rej_X Y = Y - proj_X Y$$

The rejection is orthogonal to X.

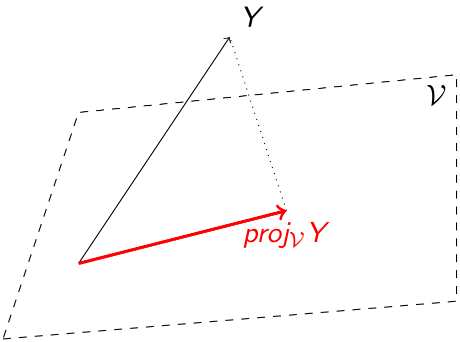


In econometrics, the endogenous variable would be Y. We try to explain it with the exogenous variable X, so we "project" Y over X. Of course, what is not explained, the error, is $rej_X Y$.

Definition

The **projection** of a vector Y over a subspace $\mathcal{V}$ defined by a basis $\{X_1, X_2, \ldots, X_k\}$ is the vector $proj_{\mathcal{V}} Y$, and it must satisfy that

$$[proj_{\mathcal{V}} Y - Y] \perp X_i \quad \forall i = 1, ..., k$$



Here we could be projecting the exogenous variable over two explanatory variables...