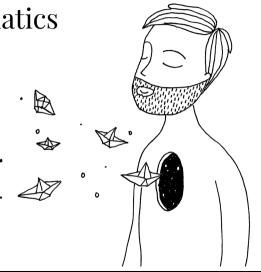
4509 - Bridging Mathematics

Concavity and Quasiconcavity

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Definition (Concave and Convex function)

A real-valued function $f: \mathcal{U} \subseteq \mathbb{R}^n$ is **concave** if $\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$:

$$f(tx + (1-t)y) \ge tf(x) + (1-t)f(y)$$

and it is convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

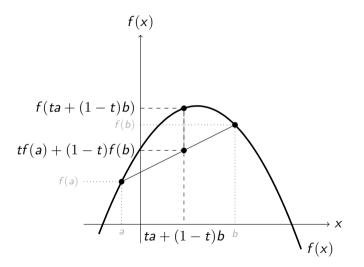


Concavity and convexity of a function have also a geometric interpretation.

Proposition

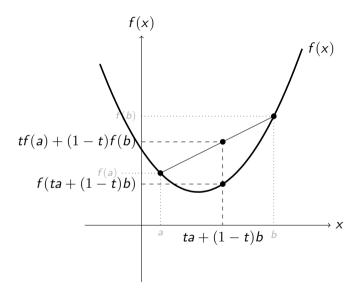
A function $f: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is concave (convex) $\Leftrightarrow \forall x, y \in \mathbb{R}^n$ the secant line connecting x and y lies below (above) the graph of f.







Convexity





Definition (Convex Set)

A set \mathcal{U} is said to be *convex* if $\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$ it holds that:

$$tx + (1 - t)y \in \mathcal{U}$$

Facts:

- 1. It is not uncommon to call tx + (1 t)y with $t \in [0, 1]$ a convex combination between x and y.
- 2. A concave set, contrary to functions, is not a thing.



Proposition

All convex or concave functions must have a convex domain.

Proposition

- 1. f is concave if and only if the set below its graph $\{(x,y):y\leq f(x)\}$ is convex.
- 2. f is convex if and only if the set above its graph $\{(x,y):y\geq f(x)\}$ is convex.



Theorem

Let $f: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Then, f is concave (convex) if and only if its restriction to every line segment in \mathcal{U} is a concave (convex) function of one variable.



Proof.

 \Leftarrow

Let $x, y \in \mathcal{U}$ and g(t) = f(tx + (1 - t)y). By hypothesis, g is concave.

$$f(tx + (1 - t)y) = g(t)$$

$$= g(t \times 1 + (1 - t) \times 0)$$

$$\ge tg(1) + (1 - t)g(0)$$

$$= tf(x) + (1 - t)f(y)$$

And then f is concave.



Proof.

 \Rightarrow

Let s_1 and s_2 in the domain of g. Let $t \in [0,1]$.

$$g(ts_1 + (1-t)s_2) = f((ts_1 + (1-t)s_2)x + (1-(ts_1 + (1-t)s_2))y)$$

$$= f(t(s_1x + (1-s_1)y) + (1-t)(s_2x + (1-s_2)y))$$

$$\geq tf(s_1x + (1-s_1)y) + (1-t)f(s_2x + (1-s_2)y)$$

$$= tg(s_1) + (1-t)g(s_2)$$

And then g is concave.



Theorem

Let f be a C^1 function on an interval $I \subset \mathbb{R}$. Then, f is concave on I if and only if:

$$f(y) - f(x) \le f'(x)(y - x), \ \forall x, y \in I$$

f is convex if:

$$f(y) - f(x) \ge f'(x)(y - x), \ \forall x, y \in I$$



Proof.

 \Rightarrow

Let f be a concave function on I, $x, y \in I$ with $x \neq y$, and $t \in (0,1]$. Then,

$$tf(y) + (1-t)f(x) \le f(ty + (1-t)x)$$

$$f(y) - f(x) \le \frac{f(ty + (1-t)x) - f(x)}{t}$$

$$= \frac{f(ty + (1-t)x) - f(x)}{t(y-x)}(y-x)$$

Letting $t \to 0$ we have:

$$tf(y) + (1-t)f(x) \le f'(x)(y-x)$$



Proof.

$$\Leftarrow$$

Let
$$f(y) - f(x) \le f'(x)(x - y) \ \forall x, y \in I$$
. Then,

$$f(x) - f((1-t)x + ty) \le f'((1-t)x + ty)(x - ((1-t)x + ty))$$

= $-tf'((1-t)x + ty)(y - x)$

Equivalently,

$$f(y) - f((1-t)x + ty) \le (1-t)f'((1-t)x + ty)(y-x)$$

Multiply the first by (1-t) and the second by t, and adding up we get:

$$(1-t)f(x) + tf(y) \le f((1-t)x + ty)$$
 i.e. f is concave.



Theorem

Let f be a C^1 function on convex $\mathcal{U} \subseteq \mathbb{R}^n$. Then f is concave on \mathcal{U} if and only if for all $x, y \in \mathcal{U}$:

$$f(y) - f(x) \le Df(x)(y - x)$$

Similarly, f is convex on \mathcal{U} if and only if for all $x, y \in \mathcal{U}$:

$$f(y) - f(x) \ge Df(x)(y - x)$$



Definition

Quasiconcave and Quasiconvex function A function f defined on a convex $\mathcal{U} \subseteq \mathbb{R}^n$ is **quasiconcave** if for any $a \in \mathbb{R}$

$$C_a^+ \equiv \{x \in \mathcal{U} : f(x) \ge a\}$$

is a convex set.

Similarly, f is said to be **quasiconvex** if for any $a \in \mathbb{R}$

$$C_a^- \equiv \{x \in \mathcal{U} : f(x) \le a\}$$

is a convex set.



Theorem

Let f be a function defined on a convex set $\mathcal{U} \subseteq \mathbb{R}^n$. Then, the following statements are equivalent to each other:

- 1. f is a quasiconcave function on \mathcal{U} .
- 2. $\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$,

$$f(x) \ge f(y) \quad \Rightarrow \quad f(tx + (1-t)y) \ge f(y)$$

3. $\forall x, y \in \mathcal{U}$ and $\forall t \in [0, 1]$,

$$f(tx + (1-t)y) \ge \min\{f(x), f(y)\}\$$



Suppose that f is a C^1 function on an open convex $\mathcal{U} \subseteq \mathbb{R}^n$. Then, f is quasiconcave on \mathcal{U} if and only if:

$$f(y) \ge f(x) \quad \Rightarrow \quad Df(x)(y-x) \ge 0$$

f is quasiconvex on $\mathcal U$ if and only if

$$f(y) \le f(x) \Rightarrow Df(x)(y-x) \le 0$$



Proof.

 \Rightarrow

Let f be quasiconcave on \mathcal{U} and that $f(y) \geq f(x)$ for some $x, y \in \mathcal{U}$. Then, $\forall t \in [0, 1]$

$$f(x + t(y - x)) \ge f(x)$$

Since

$$\frac{f(x+t(y-x))-f(x)}{t} \ge 0$$

and $\forall t \in (0,1)$, we multiply by $\frac{(y-x)}{(y-x)}$ and let $t \to 0$ to get

$$Df(x)(y-x) \ge 0$$

