Midterm

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Do not, in any circumstance, separate the sheets of this Exam. You can use whatever space not dedicated to answer for draft. In the last page you will find the *formulae*. You are allowed to use a calculator.
Each multiple choice is worth 1 point. Mistakes in these questions deduct 0.2 points from your grade. Each written answer is worth 1 point as well, in these ones, mistakes do not deduct from your grade.
The exam has a total of 20 marks, and must be completed in at most 90 minutes.

Name and student number: ______________________________

DRAFT

Multiple Choice

1. (1 point) The Laspeyres Price Index measures the change in the price of a basket of goods using which period's quantities as weights?

   A. The current period only (Q_t)

   B. The base period only (Q_0)

   C. The average of the base and current periods ($\frac{Q_0+Q_t}{2}$)

   D. The geometric mean of the base and current periods ($\sqrt{Q_0 \times Q_t}$)

2. (1 point) The Fisher Price Index is considered the "ideal" index because it is calculated as the geometric mean of which two indices?

   A. The Consumer Price Index (CPI) and the Producer Price Index (PPI)

   B. The Laspeyres Price Index and the Paasche Price Index

   C. The Quantity Index and the Value Index

   D. The Simple Aggregative Index and the Weighted Aggregative Index

3. (1 point) The expression of an individual's degree of belief about the chance that an event will occur is called ...... probability.

   A. objective

   B. classical

   C. relative frequency

   D. subjective

4. Suppose you roll a pair of dice. Let A be the event that you observe an even number. Let B be the event that you observe a number greater than seven.

   (a) (1 point) What is the intersection of events A and B?

      A. $[8, 10, 12]$

      B. $[7, 8, 9, 10, 11, 12]$

      C. $[2, 4, 6, 8, 10, 12]$

      D. $[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$

   (b) (1 point) What is $A^c \cap B$?

      A. $[9, 11]$

      B. $[2, 3, 4, 6]$

      C. $[5, 7, 8, 10, 12]$

      D. $[3, 5]$

5. (1 point) In a furniture manufacturing plant, a customer survey indicates that blemishes in the finish are a major concern. The table shown below displays a quality manager's probability assessment of the number of defects in the finish of new furniture.

Number of defects	0	1	2	3	4	5
Probability	0.34	0.25	0.19	0.11	0.07	0.04

Let A be the event that there is at least one defect and let event B be that there is at most three defects. Which of the following statements is true?

A. $P(A \cup B) = 0.96$
B. $P(A) = 0.34$
C. $P(B^C) = 0.89$
D. $P(A \cap B) = 0.55$

6. (1 point) Consider the following probability distribution. Which of the following is true?

x	0	1	2	3	4	5	6	7
$P(x)$	0.05	0.16	0.19	0.24	0.18	0.11	0.05	0.02

A. $P(2 < X < 5) = 0.42$
B. $P(X > 6) = 0.07$
C. $P(X \geq 3) = 0.64$
D. $P(X \leq 6) = 0.93$

True or False

7. (1 point) __F__ Consider the following index, $I_{t=0} = 100$, $I_{t=1} = 110$, $I_{t=2} = 120$, $I_{t=3} = 130$. The average growth rate is 10%.

8. (1 point) __F__ A key advantage of the Laspeyres Price Index is that it requires collecting new quantity data for the basket of goods in every period.

9. (1 point) __T__ If X and Y are two variables with $E(XY) = 10.56$, $E(X) = 4.22$, and $E(Y) = 5.34$, then $Cov(X, Y) = -11.97$.

10. (1 point) __T__ The expected value of the sum of two random variables is the sum of their expected values.

11. (1 point) __F__ Events A and B are said to be statistically independent if their union is the sample space Ω.

Open Questions

12. An investor bought shares in five companies in 2009. Over the five years to 2014 he increased his number of shares in some companies and decreased the number in others. The table shows the prices of the shares in 2009 and 2014 together with the numbers of shares held at those times. The share prices are in pence.

	2009		2014	
Company	Price	Shares	Price	Shares
Apple	105.2	1,400	69.3	700
Boeing	600.0	500	885.5	650
Citigroup	99.2	1,000	214.1	2,000
Dell	531.0	750	387.9	385
Enel	250.2	850	356.0	850

(a) (1 point) Compute, for each company, the simple price index in 2014, taking 2009 as the base year. Compute also, the index for the total value of the inverstor's portfolio in 2014 taking 2009 as the base year.

Solution: For the simple index:

Company	I_{2009}
Apple	$69.3/105.2 = 0.659$ or 65.9
Boeing	$885.5/600 = 1.476$ or 147.6
Citigroup	$214.1/99.2 = 2.158$ or 215.8
Dell	$387.9/531 = 0.731$ or 73.1
Enel	$356/250.2 = 1.423$ or 142.3

For the whole portfolio, we multiply the holdings in each stock, in each year (value, i.e. price times shares) and add them up:

$v_{2009} = 147,280 + 300,000 + 99,200 + 398,250 + 212,670 = 1,157,400$

$v_{2014} = 48,510 + 575,575 + 428,200 + 149,341.5 + 302,600 = 1,504,226.5$

For the portfolio, the value index is then : $1,504,226.5/1,157,400 = 1.299$ or 129.9.

(b) (1 point) Compute the Laspeyres index of share price relatives for these data.

Solution: The Laspeyre's Price Index is given, in this case by:

$$LPI_{2014|2009} = \frac{\sum p_i^{2014} q_i^{2009}}{\sum p_i^{2009} q_i^{2009}}$$

In this case:

$$\frac{69.3 \times 1400 + 885 \times 500 + 214.1 \times 1000 + 387.9 \times 750 + 356 \times 850}{V^{2009} = 1,157,400} = 1.16$$

13. From a pool of participants in a recent Marathon, you observe that 85% finished the race. From those who finished the race, you know that only 80% have some kind of training. From among those who did not finish, only 16% had some training.

If you pick randomly a participant of the race:

(a) (1 point) What is the likelihood this participant had some training?

Solution: Note first:

- $P(F) = 0.85$, $P(F^C) = 0.15$
- $P(T|F) = 0.80$
- $P(T|F^C) = 0.16$

To find the probability of having some training $P(T)$ we need to use the Law of Total Probability:

$$P(T) = P(T|F)P(F) + P(T|F^c)P(F^C)$$

or

$$P(T) = 0.8 \times 0.85 + 0.16 \times 0.15 = 0.704$$

(b) (1 point) If the participant trained, what is the probability she/he completed the race?

Solution: For thise we use the Bayes Theorem:

$$P(F|T) = \frac{P(T|F)P(F)}{P(T)} = \frac{0.80 \times 0.85}{0.704} \approx 0.9659$$

14. Let X be a continuous random variable with the following *pdf*:

$$f_X(x) = \begin{cases} \theta x^3 & x \in [0,2] \\ 0 & x \notin [0,2] \end{cases}$$

(a) (1 point) Show that $\theta = 1/4$. Conclude showing that for $x \in [0,2]$, $F_X(x) = \frac{x^4}{16}$, and describe what happens for $x \notin [0,2]$

Solution:

$$\int_0^2 \theta x^3 dx = \theta \left[\frac{x^4}{4}\right]_0^2 = \theta\,[4-0] = 4\theta$$

and this must equal 1, then $\theta = 1/4$.
Replacing back we obtain

$$F_X(x) = \int_0^x \frac{1}{4} t^3 dt = \frac{1}{4}\frac{x^4}{4} = \frac{x^2}{16}$$

for $x \in [0,2]$. $F_x(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 2$.

(b) (1 point) What is the probability that $x \in [1,2]$?

Solution:

$$P(1 \leq X \leq 2) = F(2) - F(1) = 1 - \frac{1}{16} = \frac{15}{16}$$

15. At the Death Star, stormtroopers rotate between attending the emperor and attending Darth Vador. At any moment during a day, let

$$X \in \{0,1,2\} \quad \text{be the number of stormtroopers attending the Emperor,}$$
$$Y \in \{0,1\} \quad \text{be the number of stormtroopers attending Darth Vader.}$$

Their joint probability is given by

$X\backslash Y$	0	1
0	0.10	0.05
1	0.20	0.30
2	0.05	0.30

(a) (1 point) Find the marginal $f_X(x)$ for $x = 0, 1, 2$ and compute $P(X \geq 1 \mid Y = 1)$.

Solution: Marginals:

$$f_X(0) = 0.10 + 0.05 = 0.15$$
$$f_X(1) = 0.20 + 0.30 = 0.50$$
$$f_X(2) = 0.05 + 0.30 = 0.35.$$

Also $P(Y = 1) = 0.05 + 0.30 + 0.30 = 0.65$. Hence

$$P(X \geq 1 \mid Y = 1) = \frac{P(X = 1, Y = 1) + P(X = 2, Y = 1)}{P(Y = 1)}$$
$$= \frac{0.30 + 0.30}{0.65} = \frac{0.60}{0.65} = \frac{12}{13} \approx 0.9231.$$

(b) (1 point) Compute $\text{Cov}(X, Y)$.

Solution: Now expectations:

$$E[X] = 0 \cdot 0.15 + 1 \cdot 0.50 + 2 \cdot 0.35 = 0.50 + 0.70 = 1.20$$
$$E[Y] = 0 \cdot 0.35 + 1 \cdot 0.65 = 0.65.$$

Compute $E[XY]$: only cells with $y = 1$ contribute:

$$E[XY] = 1 \cdot 1 \cdot 0.30 + 2 \cdot 1 \cdot 0.30 = 0.30 + 0.60 = 0.90.$$

Thus

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0.90 - (1.20)(0.65) = 0.90 - 0.78 = 0.12.$$

Formulae

Index Numbers

- $i_{t|0} = \frac{x_t}{x_0} \times 100$ with $x_t, x_0 > 0$
- $\delta_{t,0} = \frac{x_t - x_0}{x_0}$
- $\delta_{t,0} = i_{t|0} - 1$
- $r_{t|0} = \left(i_{t|0}\right)^{1/k} - 1$
- $LPI_{t|0} = \frac{\sum p_t^k q_0^k}{\sum p_0^k q_0^k}$
- $LQI_{t|0} = \frac{\sum p_0^k q_t^k}{\sum p_0^k q_0^k}$
- $PPI_{t|0} = \frac{\sum p_t^k q_t^k}{\sum p_0^k q_t^k}$
- $PQI_{t|0} = \frac{\sum p_t^k q_t^k}{\sum p_t^k q_0^k}$

$$FPI_{t|0} = \sqrt{LPI_{t|0} PPI_{t|0}}$$
$$FQI_{t|0} = \sqrt{LQI_{t|0} PQI_{t|0}}$$

Probability Theory

Conditional probability

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ with $P(B) > 0$

Total Probability Theorem

$P(B) = \sum_{i=1}^{n} P(A_i) P(B|A_i)$

Bayes Theorem

$P(A_i|B) = \frac{P(A_i) P(B|A_i)}{\sum_{i=1}^{n} P(A_i) P(B|A_i)}$

Random variables

Expected Value

- $\mu_X \equiv E[X]$
- $\mu_X = \sum x_i f(x_i)$
- $\mu_X = \int x f(x) dx$

Variance

- $\sigma_X^2 = Var(X) = V[X]$
- $\sigma_X^2 = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$

Discrete Joint Random Variables

Joint density function

$f_{XY}(x, y) = P(X = x_i, Y = y_i) = p_{ij}$ with $p_i = \sum_j p_{ij}$ and $p_j = \sum_i p_{ij}$

Expected Value

$E[g(X, Y)] = \sum_i \sum_j g(x_i, y_j)P(X = x_i, Y = y_j)$

Variance, Covariance, Correlation

- $cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X]E[Y]$
- $V[X \pm Y] = V[X] + V[Y] \pm 2cov(X, Y)$
- $cov(a + bX, c + dY) = bdcov(X, Y)$ with $a, b, c, d \in \mathbb{R}$
- $\rho = \frac{cov(X,Y)}{\sqrt{V[X]V[Y]}}$ with $\sigma_X, \sigma_Y > 0$