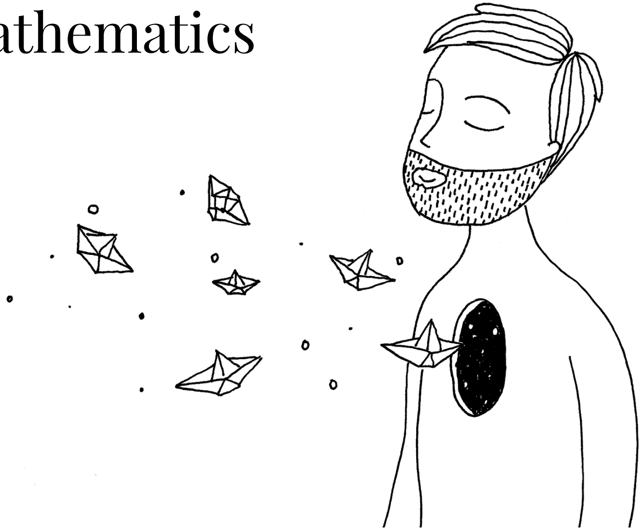
4509 - Bridging Mathematics

Optimization



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Let $f : \mathbb{R}^n \to \mathbb{R}$, and let $S \subseteq \mathbb{R}^n$, the problem is:

$$\min_{x \in S} \quad f(x)$$

Or finding $x^* \in S$ such that

$$f(x^*) \leq f(x) \quad \forall x \in S$$

Definition

If $x \in S$, x is said to be **feasible** for the optimization problem $\min_{x \in S} f(x)$.

Note that to minimize $f(x)$ is equivalent to maximize $-f(x)$.

Definition

x_0 is a **local minimum** of f in S if, there is an open ball $B(x_0, r)$ such that for any $x \in B(x_0, r) \cap S$ it holds that $f(x_0) \leq f(x)$.

x^* is a **global minimum** of f in S if for any $x \in S$ it holds that $f(x^*) \leq f(x)$.

Definition

Let the $\arg\min_{x \in S} f(x)$ represent the set of all global solutions to the problem of minimizing f with $x \in S$.

Note: For a maximization problem you have equivalently local maximum, global maximum, and $\arg\max_{x \in S} f(x)$.

Conjecture

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Let $S \in \mathbb{R}^n$ be compact. Then the problem

$$\max_{x \in S} f(x)$$

has at least one solution, or equivalently $\arg\min_{x \in S} f(x) \neq \emptyset$.

Conjecture

If $S_1 \subseteq S_2$, then $min(f, S_1) \geq min(f, S_2)$.

Prove the first one, 10 min.

Proof.

- f is continuous in S, and S is compact, so $f(S)$ is compact.

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- $\exists s \in S$ such that $f(s) = \sup f(S)$.

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Maybe it can be shown with contradiction, choosing an x, then see if it is the maximizer, done, if it is not, it is because you know of another that gives you a higher value in $f(x)$, so you move to that one. Then you can build a sequence, which you now it has a subsequence that converges in S (S is compact), and therefore the limit must be the maximizer, and therefore $f(x_n)$ converges to the maximum of the function as well.

□

Now the second one 5 min.

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Proof.

Trivial. By contradiction. Let $s_1 \in S \leq s_2 \in S$. As $s_i \in S_1 \Rightarrow s_1 \in S_2$, because $S_1 \subseteq S_2$, then s_2 cannot be a minimizer.

$\square$

These are the necessary conditions for optimality for the unconstrained problem $(S = \mathbb{R}^n)$, when f is at least twice differentiable:

Conjecture

x_0 is a local optimum of $f : \mathbb{R}^n \to \mathbb{R}$ if:

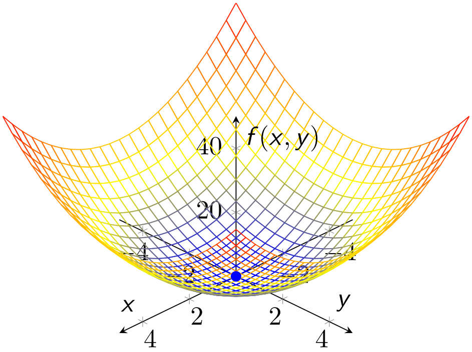
1. *First Order Condition: $\nabla f(x_0) = 0$.*
2. *Second Order Conditions:*
    2.1 *$H(f, x_0) \geq 0$ (Hessian positive semi definite) then x_0 is a local minimum.*
    2.2 *$H(f, x_0) \leq 0$ (Hessian negative semi definite) then x_0 is a local maximum.*

Example

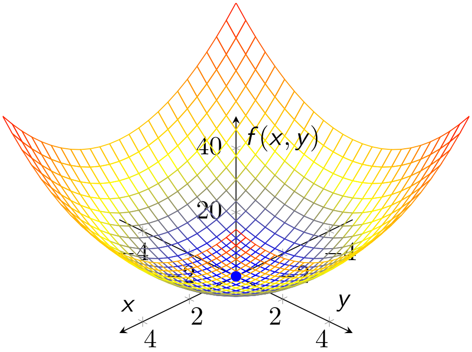
Let $f(x,y) = x^2 + y^2$,

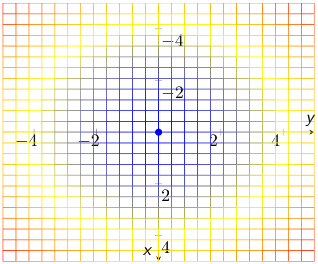
1. FOC: $\frac{\partial f}{\partial x}(x^*) = 0$ and $\frac{\partial f}{\partial y}(x^*) = 0$
   1.1 $f_x = 2x$, so $f_x = 0 \Rightarrow x^* = 0$
   1.2 $f_y = 2y$, so $f_y = 0 \Rightarrow y^* = 0$
2. SOC:
   2.1 Hessian:
   $$\mathcal{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$
   2.2 $\det(\mathcal{H} - \lambda I) = (2-\lambda)^2$, so $\det(\mathcal{H} - \lambda I) = 0 \Rightarrow \lambda = 2$
   2.3 All *e.v.* are strictly positive, so $\mathcal{H}$ is positive definite.
   2.4 Finally $(0,0)$ is a minimum.

Example



Example



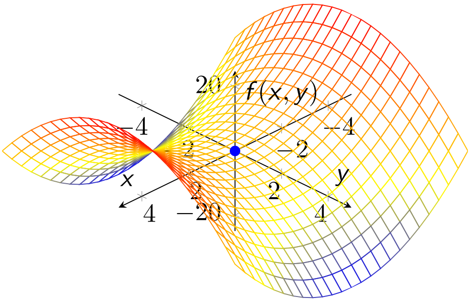


Example

Let $f(x, y) = x^2 - y^2$,

1. FOC: $\frac{\partial f}{\partial x}(x^*) = 0$ and $\frac{\partial f}{\partial y}(x^*) = 0$
    1.1 $f_x = 2x$, so $f_x = 0 \Rightarrow x^* = 0$
    1.2 $f_y = -2y$, so $f_y = 0 \Rightarrow y^* = 0$
2. SOC:
    2.1 Hessian:
    $$\mathcal{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$
    2.2 $\det(\mathcal{H} - \lambda I) = -(4 - \lambda^2)$, so $\det(\mathcal{H} - \lambda I) = 0 \Rightarrow \lambda = \pm 2$
    2.3 The *e.v.*s are not strictly positive or negative, so we cannot say anything about the positiveness of $\mathcal{H}$.
    2.4 Finally we cannot say that $(0, 0)$ is a minimum or a maximum.

Example



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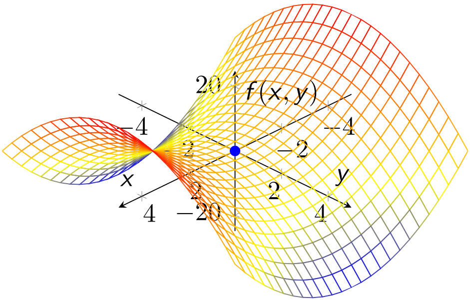
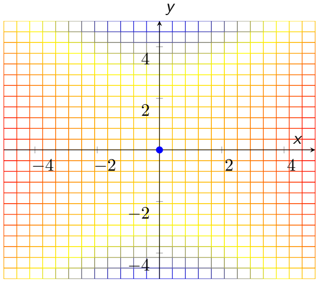




Equality Constraints

Consider now the constrained problem, and assume that $S \subseteq \mathbb{R}^n$ can be described as a set of equations that $x \in \mathbb{R}^n$ must satisfy, say $h_i(x) = 0$.

$$S = \{x \in \mathbb{R}^n | h_i(x) = 0, \quad i = 1, ..., m\}$$

The problem is now

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ s.t. \quad & h_i(x) = 0, \quad i = 1, ..., m \end{aligned}$$

From now on, to simplify notation, we will use $\min_x$ instead of $\min_{x \in \mathbb{R}^n}$.

Definition (Mangasarian-Fromowitz constraint qualification)

The feasible point $x^* \in \mathbb{R}^n$ is said to be **regular** if the set of gradients $\nabla h_i(x^*)$ for $i = 1, ..., m$ is l.i.

If this is not satisfied, the solution you find might not be an optimum.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$, continuous and differentiable.
Consider the following optimization problem:

$$\begin{aligned} \min_x \quad & f(x) \\ s.t. \quad & h_i(x) = 0, \quad i = 1, ..., n \end{aligned}$$

Define the function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that:

$$\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$$

$\mathcal{L}$ is called the **Lagrangian**, and λ_i for $i = 1, ..., m$ are called the **Lagrange multipliers**.

Theorem

Let x^* to be a local minimum of f, such that $h_i(x^*) = 0$ for $i = 1, ..., m$. Also let x^* be regular. Then, there is a vector $\lambda = (\lambda_1, \ldots, \lambda_m)^t \in \mathbb{R}^m$ such that:

$$\frac{\partial f(x^*)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial h_j(x^*)}{\partial x_i} = 0, \quad i = 1, \ldots, n$$

$$\frac{\partial f(x^*)}{\partial \lambda_j} = 0, \quad j = 1, ..., m$$

Theorem (Second Order Conditions)

Let x^* be a local minimum for f, satisfying $h_j(x^*) = 0$ for every $j = 1, ..., m$. Assume further that x^* is regular. Consider $\lambda \in \mathbb{R}^m$ the vector of Lagrange multipliers of the problem, then the matrix

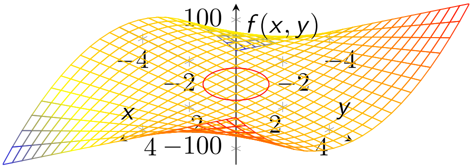
$$\mathcal{H} = H(f, x^*) + \sum_{j=1}^{m} \lambda_j H(h_j, x^*)$$

is positive semi definite in the set $M := \{y \in \mathbb{R}^n | \nabla h_j(x^*) \cdot y = 0, \forall j = 1, ..., m\}$

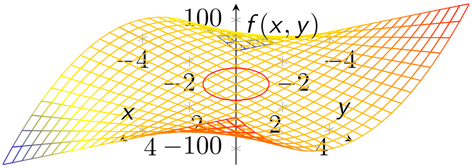
Example

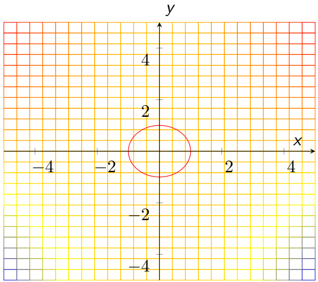
Let $f(x, y) = x^2 y$. Maximize $f(x, y)$ such that $x^2 + y^2 = 1$.

Example



Example





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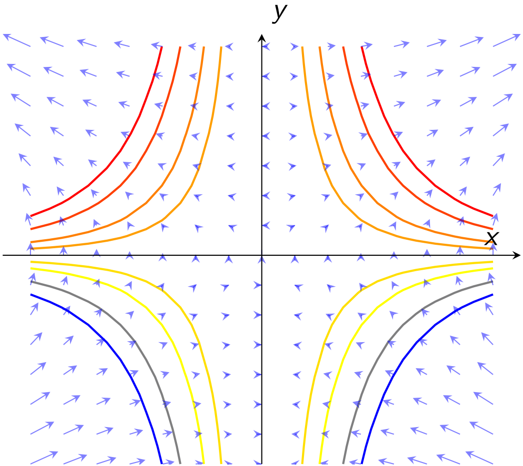

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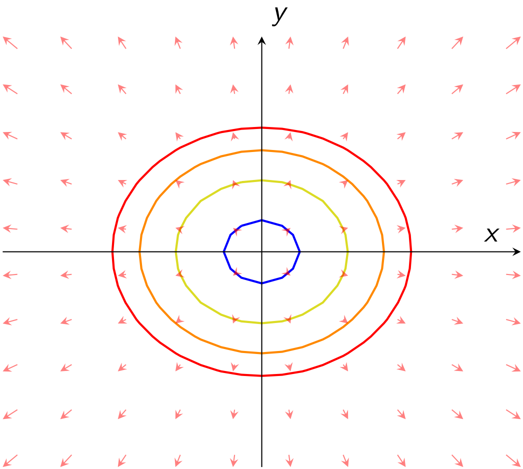

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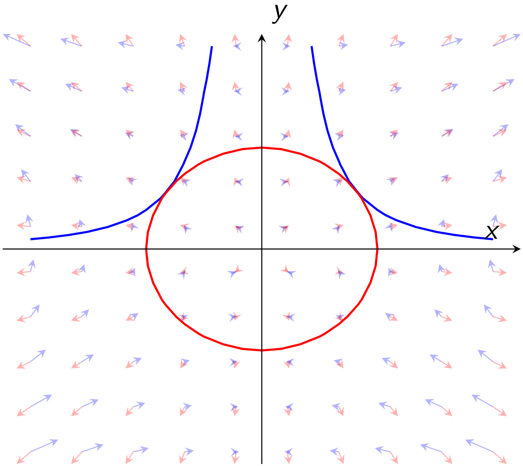

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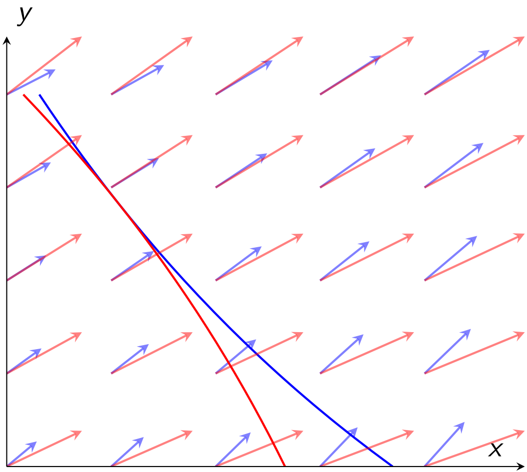

Example

The Lagrangian is:

$$\mathcal{L} = x^2 y + \lambda(x^2 + y^2 - 1)$$

Inequality Constraints

Assume now that $S \subseteq \mathbb{R}^n$ can be described as a set of equations and inequalities that $x \in \mathbb{R}^n$ must satisfy, say $h_i(x) = 0$ and $g_j(x) \leq 0$.

$$S = \{x \in \mathbb{R}^n | h_i(x) = 0, i = 1, ..., m\} \cap \{x \in \mathbb{R}^n | g_j(x) \leq 0, j = 1, ..., p\}$$

The problem is now

$$\begin{aligned} \min_x \quad & f(x) \\ s.t. \quad & h_i(x) = 0, \quad i = 1, ..., m \\ & g_j(x) \leq 0, \quad j = 1, ..., p \end{aligned}$$

Definition

Let $x^* \in \mathbb{R}^n$ be such that $h_i(x^*) = 0, i = 1, ..., m$ and $g_j(x^*) \leq 0, j = 1, ..., p$. x^* is called **regular** for the constraints if the set of gradients

$$\{\nabla h_i(x^*), \nabla g_j(x^*), i = 1, ..., m, \quad j = \in J_A\}$$

is *l.i.*, where $J_A \subseteq \{1, ..., p\}$ represents the active constraints in x^*.

Definition

Let x^* be a solution to the problem in the previous slide. The inequality constraint $g_k(x^*)$ is called **active**, if $g_k(x^*) = 0$. Otherwise it is considered **slack**.

If you knew ex-ante which constraints are active, you can use the Lagrange method to find the solution.

Theorem (Karush-Kuhn-Tucker)

Let x^* be a local minimum for the problem:

$$\begin{aligned} \min_x \quad & f(x) \\ s.t. \quad & h_i(x) = 0, i = 1, \ldots, m \\ & g_j(x) \leq 0, j = 1, \ldots, p \end{aligned}$$

Such that x^* is regular for the constraints, then there are multipliers $\lambda_i, i = 1, ..., m$ and $\mu_j, j = 1, ..., p$ such that:

1. $\mu_j \geq 0$ *for* $j = 1, ..., p$.
2. $\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j=1}^p \mu_j \nabla g_j(x^*) = 0$.
3. $\sum_{j=1}^p \mu_j g_j(x^*) = 0$

Theorem (Second Order Conditions)

Let x^* be a local minimum of f that satisfies $h_i(x^*) = 0, i = 1, \ldots, m$, $g_j(x^*) \leq 0, j = 1, \ldots, p$. Assume further than x^* is regular for the constraints. Then the matrix

$$\mathcal{H} = H(f, x^*) + \sum_{i=1}^{m} \lambda_i H(h_i, x^*) + \sum_{j=1}^{p} \mu_j H(g_j, x^*)$$

Is positive semi definite in the set that is orthogonal to the active constraints:
$M := \{y \in \mathbb{R}^n | \nabla h_j(x^*) \cdot y = 0, \forall j = 1, ..., m\} \cap \{y \in \mathbb{R}^n | \nabla g_k(x^*) \cdot y = 0, k \in J_A\}$

Theorem (Envelope's Theorem)

Consider the following optimization problem,

$$\begin{aligned} \min_{x} \quad & f(x,\alpha) \\ s.t. \quad & h_j(x,\alpha) = 0, j = 1, ..., m \end{aligned}$$

Where $\alpha = (\alpha_1, ..., \alpha_l) \in \mathbb{R}^l$ *are parameters of the problem. Consider further that all the functions (*f*, hs) are continuously differentiable. Let* $x(\alpha)$ *a solution and* $min(f)(\alpha) = f(x(\alpha), \alpha)$ *the minimum value taken by* f*. Then,* $\forall k = 1, ..., l$ *holds that:*

$$\frac{d \min(f)(\alpha)}{d\alpha_k} = \frac{\partial f(x(\alpha),\alpha)}{\partial \alpha_k} + \sum_{j=1}^{m} \lambda_j \frac{\partial h_j(x(\alpha),\alpha)}{\partial \alpha_k}$$

Where λ_j *is the multiplier of the optimality conditions.*

Theorem (Berge's Maximum)

Consider the following optimization problem

$$\begin{aligned} \max_{x} \quad & f(x, \alpha) \\ s.t. \quad & g_j(x, \alpha) \leq 0, j = 1, ..., p \end{aligned}$$

Assume that for α^*, the solution is $x^* = x(\alpha^*)$. Then, if f and the gs are continuous in (x^*, α^*), and the set defined by the inequality constraints is compact, then the function $\max(f)(\alpha) = f(x^*(\alpha), \alpha)$ is continuous in α^*. Furthermore, if the solution $x^*(\alpha)$ is unique, then, $x^*(\alpha)$ is also continuous.