

Topology and Continuity

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Definition

The **open ball** centered around $x_0 \in \mathbb{R}^n$ and radius $r > 0$ is defined as

$$B(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$$

while the **closed ball** centered around x_0 and radius $r > 0$ is

$$\overline{B}(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$$

Definition

- ▶ Let $A \subset \mathbb{R}^n$. $x_0 \in A$ is **interior**, if there is $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq A$.
- ▶ Let $A \subseteq \mathbb{R}^n$, the **interior** of A , denoted as $int(A)$, is the set of all its interior points, $int(A) = \{x \in A | \exists \epsilon > 0, B(x_0, \epsilon) \subseteq A\}$.
- ▶ The set $A \subseteq \mathbb{R}^n$ is **open** if $A \setminus int(A) = \emptyset$.
- ▶ The set A is **closed** if A^c is open.

Definition

- ▶ The **closure** of A , denoted as \overline{A} , is the smallest closed set that contains A .
- ▶ The **boundary** of A , denoted as ∂A , is defined as $\overline{A} \setminus \text{int}(A)$.

Definition

$A \subset \mathbb{R}^n$ is **bounded** if there is an open ball that contains A .

Definition

A set $A \subseteq \mathbb{R}^n$ is said to be **compact** if it is closed and bounded.

Definition

A **sequence** is any function $f : \mathbb{N} \rightarrow \mathbb{R}$.

Definition

The sequence x_t **converges** to x_0 if, for any open ball B containing x_0 , exists $t_\epsilon \in \mathbb{N}$ such that for $t \geq t_\epsilon$, $x_t \in B$. It is denoted as $x_t \rightarrow x_0$. x_0 is called the **limit** of x_t .

Conjecture

If a sequence converges, then its limit is unique.

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You know what is coming, ... Quiz! Think on a way to prove it... 10 min.

Proof.

Assume it is not unique, so:

1. $x_t \rightarrow x_0$ and also $x_t \rightarrow x_1$, and $x_1 \neq x_0$.

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2. $\exists t_\epsilon^0, t_\epsilon^1 \in \mathbb{N}$ such that for $t_\epsilon^* > \max\{t_\epsilon^0, t_\epsilon^1\}$ $x_t \in B(t_0, \epsilon)$ and $x_t \in B(t_1, \epsilon) \forall t > t^*$.

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3. Let $|x_0 - x_1| = \delta$. Choose $\epsilon = \delta/2$. So there is t^* such that $|x_t - x_0| < \delta/2$ and $|x_t - x_1| < \delta/2$.

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3. Let $|x_0 - x_1| = \delta$. Choose $\epsilon = \delta/2$. So there is t^* such that $|x_t - x_0| < \delta/2$ and $|x_t - x_1| < \delta/2$.
 $|x_0 - x_1| = |x_0 - x_t + x_t - x_1| = |(x_0 - x_t) + (-x_1 + x_t)| \leq |x_0 - x_t| + |x_1 - x_t| < 2\epsilon = \delta$,
contradiction!



Definition

- ▶ The sequence x_t is **increasing** if for any $t \in \mathbb{N}$, $x_t \leq x_{t+1} \in \mathbb{R}$.
- ▶ If x_t is increasing, it is called **bounded from above** if $x_t \leq c, \forall t \in \mathbb{N}$.

Conjecture

If the sequence x_t is increasing and bounded from above, then it converges.

Definition

Let x_t be a sequence. A **subsequence** of x_t is a sequence built by removing some of the elements of x_t without changing its order. Let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be increasing, then $y_t = x_{\phi(t)}$ is a subsequence of x_t .

Definition

Given a sequence x_t , x^* is a **cluster point** of x_t , if there is a subsequence of x_t that converges to x^* .

Conjecture

A bounded sequence converges if and only if it has only one cluster point.

Let $x_1^*, x_2^*, \dots, x_p^*$ be cluster points of x_t .

Definition

- ▶ The **upper bound** of x_t is defined as $\max\{x_1^*, x_2^*, \dots, x_p^*\}$.
- ▶ The **lower bound** of x_t is defined as $\min\{x_1^*, x_2^*, \dots, x_p^*\}$.

Conjecture

Let $A \subseteq \mathbb{R}^n$.

- ▶ A is closed if and only if any convergent sequence $x_t \subseteq A$ has its limit in A . If $x_t \subseteq A, x_t \rightarrow x_0 \Leftrightarrow x_0 \in A$.
- ▶ A is compact if and only if for any sequence $x_t \subseteq A$, there is a convergent subsequence.
- ▶ $\overline{A} = \{x^* | \exists x_t \in A, x_t \rightarrow x^*\}$

Definition

Let $A, C \subseteq \mathbb{R}^n$ such that $C \subseteq A$. We'll say that C is **dense** in A if and only if $\overline{C} = A$.

Definition

Consider $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$. $f(x)$ **converges** to $\alpha \in \mathbb{R}^n$ when $x \in \mathbb{R}^m$ goes to $x_0 \in \mathbb{R}^m$, if for any sequence $x_n \rightarrow x_0$, $f(x_n) \rightarrow \alpha$. This is written as $\lim_{x \rightarrow x_0} f(x) = a$.

Definition

$f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is **continuous** in $x_0 \in \mathbb{R}^m$ if, for any sequence $x_t \rightarrow x_0$ it holds that $f(x_t) \rightarrow f(x_0)$

Definition

If $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous for all $x_0 \in A \subseteq \mathbb{R}^m$, then it is continuous in A .

A more conventional definition of continuity is:

Definition

A function is said to be **continuous** on the set $S \subseteq \mathbb{R}^n$ if for every $a \in S$, and any $\epsilon > 0$ there exists δ such that for any $x \in S$ that satisfies $|x - a| \leq \delta$ implies $|f(x) - f(a)| \leq \epsilon$.

Conjecture

The sum, product, division or composition of continuous functions is continuous.

Conjecture

Let $A \subseteq \mathbb{R}^m$, and given $\mathcal{F} = \{f : A \rightarrow \mathbb{R}^m, f \text{ continuous in } A\}$, it holds that \mathcal{F} is a vector space.

Conjecture

Let $K \subseteq \mathbb{R}^n$ be compact and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a continuous function. Then $f(K)$ is compact.

Definition

Let $K \subseteq \mathbb{R}^n$ and $f : K \rightarrow \mathbb{R}$. The **maximum**(x_M) and the **minimum**(x_m) of f are defined as:

- ▶ $f(x_M) \geq f(x) \quad \forall x \in K$
- ▶ $f(x_m) \leq f(x) \quad \forall x \in K$

These are also known as *global maximum* and *global minimum*

Conjecture

Let $f : K \rightarrow \mathbb{R}$ be continuous and K compact, then x_M and x_m exist.

Definition

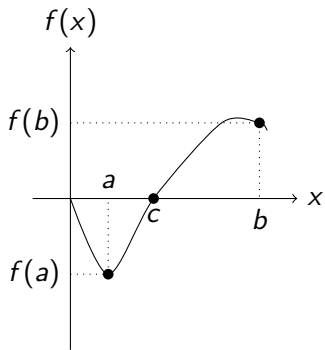
A set A is said to be connected if, for any $a, b \in A$, there is a continuous function $\phi : [0, 1] \rightarrow A$, such that $\phi(0) = a$ and $\phi(1) = b$.

Theorem Bolzano

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous. Let $a, b \in \mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$, then there is $c \in \mathbb{R}$ such that $f(c) = 0$.

Theorem Weierstrass

Let $[a, b] \subseteq \mathbb{R}$, let $f : [a, b] \rightarrow \mathbb{R}$ continuous. Then for any $u \in (a, b)$, there is at least one c such that $f(c) = u$.



Brouwer fixed point theorem in \mathbb{R}

Theorem

Let $f : K \rightarrow K$ continuous, with $K \subseteq \mathbb{R}$ compact and convex.¹ Then there is \bar{x} such that $f(\bar{x}) = \bar{x}$.

Proof.

- ▶ Let $f : [0, 1] \rightarrow [0, 1]$ continuous.
- ▶ Let $g(x) = f(x) - x$.
- ▶ $g(0) = f(0) - 0 = f(0)$, but $f(0) \geq 0$, so $g(0) \geq 0$
- ▶ $g(1) = f(1) - 1$, but $f(1) \leq 1$, so $f(1) - 1 \leq 0$, or $g(1) \leq 0$.
- ▶ Then, because of the proposition we just saw, there must be \bar{x} such that $g(\bar{x}) = 0$, or $f(\bar{x}) = \bar{x}$.

¹A.k.a. interval.

