Problem Set 1
Sets and Functions

Paulo Fagandini

Sets

1. Consider sets A, B, and C. Using the definition of "subset" show that:
   (a) $A \subseteq A$.

   Solution: Note that $\forall x \in A$, we have that $x \in A$, and therefore $A \subseteq A$.

   (b) If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

   Solution: If $A \subseteq B$, then $\forall x \in A$ we have that $x \in B$. Now, if $B \subseteq C$, then $\forall z \in B$, we have that $z \in C$, and as $x \in B$, then $x \in C$. Therefore $\forall x \in A$, $x \in C$, which means that $A \subseteq C$.

   (c) If $A \subseteq B$, and $B \subseteq A$, then $A = B$.

   Solution: First part, if $A \subseteq B$, then $\forall x \in A \Rightarrow x \in B$. Now as $B \subseteq A$ as well, then $\forall x \in B \Rightarrow x \in A$. Then $\forall x$, $x \in A \Leftrightarrow x \in B$, or $A = B$.

   (d) $\emptyset \subseteq A$, and $A \subseteq \mathcal{U}$.

   Solution: $\emptyset \subseteq A$ if $\forall z \in \emptyset \Rightarrow z \in A$. But there is no z in $\emptyset$, so $\forall z, z \in \emptyset$ is false, and by logic, false implies anything, in particular $z \in A$, and therefore $\emptyset \subseteq A$. Now, $A \subseteq \mathcal{U}$ implies that $\forall x \in A \Rightarrow x \in \mathcal{U}$. By definition $\mathcal{U}$ contains all the elements, in particular x, so $A \subseteq \mathcal{U}$.

2. Consider the sets A, B, and C. Show that:
   (a) $A \times \emptyset = \emptyset \times A = \emptyset$
   (b) $A \times B = B \times A \Leftrightarrow A = B$
   (c) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   (d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

3. Consider sets A, B, and C. Show that:
   (a) $A \cap B \subseteq A$
   (b) $(A \cap B)^c = A^c \cup B^c$
   (c) Is it true that if $A \cap B = \emptyset$, and $B \cap C = \emptyset$, then $A \cap C = \emptyset$?
   (d) Find set A such that $A \subseteq A \times A$.

4. Given sets A and B, such that $\#A = m$ and $\#B = n$,
   (a) Find $\#(A \times B)$.

(b) Find $\#(A^k)$.

(c) Find $\#\mathcal{P}(A \times B)$.

(d) Show that $\#(A \cup B) = \#A + \#B - \#(A \cap B)$.

Real Numbers

5. Let $x, y \in \mathbb{R}$. Show that:
   (a) $|x| \geq 0$
   (b) $|x + y| \leq |x| + |y|$
   (c) $|x| - |y| \leq |x - y|$
   (d) $|xy| = |x||y|$

6. Solve the following inequalities,
   (a) $|x - 3| < 9$

   Solution: $x \in (-6, 12)$

   (b) $|x - 1| + |x - 2| \geq 1$

   Solution: Given that $|\cdot| \geq 0$, we have that the left hand side $x \in (-\infty, 0] \cup [2, \infty)$, and the right hand side $x \in (-\infty, 1] \cup [3, \infty)$ So obviously for $x \in (-\infty, 1] \cup [2, \infty)$ satisfies the inequality. Note that for $x \in (1, 2)$ both inequalitie

   (c) $|x - 1| + |x + 1| < 2$

7. Let $x, y \in \mathbb{R}$. Prove that:
   (a) $max(x, y) = \frac{x+y+|y-x|}{2}$
   (b) $min(x, y) = \frac{x+y-|y-x|}{2}$

8. Let $S = [0, 1]$, interval, and $A_s = \left[0, \frac{1}{1+s}\right]$.
   (a) Find $\bigcap_{s \in S} A_s$.
   (b) Find $\bigcup_{s \in S} A_s$

9. Consider $x_0 \in \mathbb{R}$ and $\delta \in \mathbb{R}_{++}$. Let $A_\delta = \{x \in \mathbb{R} | \quad |x - x_0| \leq \delta\}$. Show that $\bigcap_{\delta > 0} A_\delta = \{x_0\}$

10. Show that $A \subseteq \mathbb{R}$ is bounded if and only if there is $c \in \mathbb{R}$ such that for any $x \in A$, $|a| \leq c$.

11. Find $max(), min(), sup(), inf()$ if applicable for the following sets:
    (a) $A = \{x \in \mathbb{R} | \quad |x + a| - 2x \leq 4\}$
    (b) $A = \{x \in \mathbb{R} | \quad x^2 - 3x + 2 \geq 0\}$
    (c) $A = \{x \in \mathbb{R} | \quad |x - 2a| - |x + a| > 12\}$

Functions

12. Consider $f, g : \mathbb{R} \to \mathbb{R}$, with $f(x) = x^2 - 4x + 3$ and $g(x) = e^{2x^2}$, find:
    (a) $f + 2g$
    (b) $f \circ g$
    (c) $g \circ g$
    (d) $\frac{f}{f \circ f}$
    (e) $f \cdot g$
13. Show that, in general, $f \circ g \neq g \circ f$.
14. Let $f(x) = ax + b$, such that $f \circ f(x) = 4x + 3$. Find $f(5)$.
15. Let $f(x) = 7x + 2$, find $g(x)$ such that $f \circ g(x) = x$.
16. Show that $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = e^x$ is injective but not bijective.
17. Given $\alpha > 0$, find if the function $f(x) = x^\alpha$ is injective, surjective or bijective.
18. Show that $f(x) = ax^2 + bx + c$ is not injective in $\mathbb{R}$.
19. Given $\alpha > 0$, show that $f(x) = x^\alpha$ is invertible in $\mathbb{R}_{++}$.
20. Find the isoquants at $y_0 > 0$ for the following functions:
    (a) $f(x_1, x_2) = x_1^2 + x_2^2$.
    (b) $f(x_1, x_2) = x_1^2 \cdot x_2^2$.
    (c) $f(x_1, x_2) = \max\{x_1, x_2\}$.
21. Show that if f is strictly increasing or decreasing, it must be injective.
22. Provide a function that while being injective, is not strictly increasing or decreasing.
23. Show that if f and g are strictly increasing functions, then $f \circ g$ is also strictly increasing.
24. If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and invertible, what can you say about the increasingness of f^{-1}?
25. Show that if the real valued function $f(x)$ is convex, then $g = -f(x)$ is concave.
26. Show that if $g > 0$ and f is increasing then

$$\frac{f(x+h) - f(x)}{h} > 0$$

27. Given $f : \mathbb{R} \to \mathbb{R}$, and $A \subseteq \mathbb{R}$, define $f(A) := \{f(a) | a \in A\}$. Let $S = sup(A)$. If f is strictly increasing, is it true that $f(S) = sup(f(A))$?