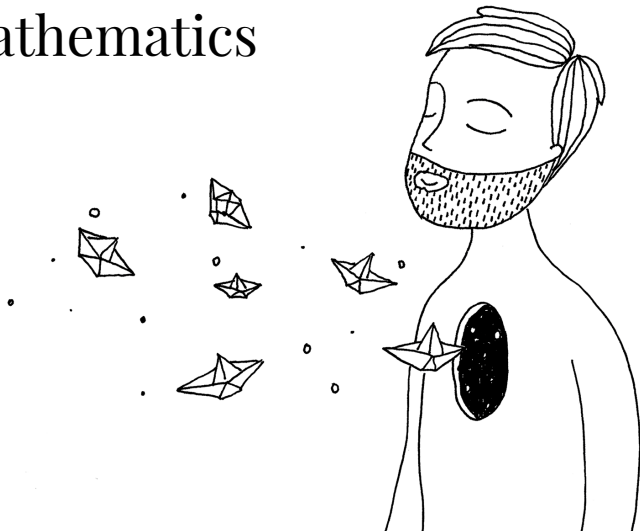


# 4509 – Bridging Mathematics

Differentiation

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## Definition

The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **differentiable** in  $\hat{x} \in \mathbb{R}$  if the limit

$$\lim_{h \rightarrow 0} \frac{f(\hat{x} + h) - f(\hat{x})}{h}$$

exists. If it does, we denote it as  $\frac{df(\hat{x})}{dx}$ , or  $f'(\hat{x})$ , or  $f_x(\hat{x})$ .

## Definition

The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is **partially differentiable** in  $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$  with respect to  $x_j$  if the limit

$$\lim_{h \rightarrow 0} \frac{f(\hat{x}_1, \dots, \hat{x}_j + h, \hat{x}_{j+1}, \dots, \hat{x}_n) - f(\hat{x}_1, \dots, \hat{x}_j, \hat{x}_{j+1}, \dots, \hat{x}_n)}{h}$$

exists. If it does, we denote it as  $\frac{\partial f(\hat{x})}{\partial x_j}$ , or  $f_{x_j}(\hat{x})$ .

## Definition

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is partially differentiable in each coordinate of  $\hat{x}$ , then its **gradient** is defined as:

$$\nabla f(\hat{x}) = \begin{pmatrix} \frac{\partial f(\hat{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\hat{x})}{\partial x_i} \\ \vdots \\ \frac{\partial f(\hat{x})}{\partial x_n} \end{pmatrix}$$

The gradient indicates the direction of steepest ascent.

## Definition

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is partially differentiable in each coordinate of  $\hat{x}$ , then the **Jacobian** matrix (also denoted  $\nabla f$ , and  $Df$ ) is defined as:

$$\nabla f(\hat{x}) = \begin{pmatrix} \frac{\partial f_1(\hat{x})}{\partial x_1} & \frac{\partial f_1(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\hat{x})}{\partial x_n} \\ \frac{\partial f_2(\hat{x})}{\partial x_1} & \frac{\partial f_2(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\hat{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\hat{x})}{\partial x_1} & \frac{\partial f_m(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_m(\hat{x})}{\partial x_n} \end{pmatrix}$$

where in this case  $f_i(\hat{x})$  represents the coordinate  $i$  of  $f(\hat{x})$ .

## Definition

The **higher order derivative** is defined recursively, consider the derivative of  $f$ , of order  $n$ , on  $\hat{x}$  as,

$$f^{(n)}(\hat{x}) = (f^{(n-1)}(\hat{x}))'$$

$$f^{(1)}(\hat{x}) = f'(\hat{x})$$

## Definition

The second partial derivative is defined as:

$$\frac{\partial^2 f(\hat{x})}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i} \left( \frac{\partial f(\hat{x})}{\partial x_j} \right)$$

It is well defined if both limits exist.

## Definition

Let  $f$  be a differentiable function for any  $x \in J$ , then we say  $f$  is **differentiable in  $J$** . If  $f'$  is well defined and continuous on  $J$ , we say that  $f$  is **continuously differentiable** on  $J$  or  $C^1$ . If  $f'$  is differentiable at every  $\hat{x} \in J$  we say  $f$  is **twice differentiable**, and if  $f''$  is continuous, we say that  $f$  is **twice continuously differentiable** or  $C^2$  in  $J$ .

In general, if  $f \in C^n$  then we say that  $f$  is  $n$ -times differentiable, and  $f^{(n)}$  is continuous.

## Definition

The **Hessian** of a function  $f$  is the matrix with the second partial derivatives of  $f$ .

$$H(f, \hat{x}) = \begin{pmatrix} \frac{\partial^2 f(\hat{x})}{(\partial x_1)^2} & \frac{\partial^2 f(\hat{x})}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f(\hat{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\hat{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\hat{x})}{(\partial x_2)^2} & \cdots & \frac{\partial^2 f(\hat{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\hat{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\hat{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\hat{x})}{(\partial x_n)^2} \end{pmatrix}$$



# Conjecture

Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  differentiable, it holds that:

1.  $(f + g)'(x) = f'(x) + g'(x)$
2.  $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
3.  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
4. If  $h(x) = f(g(x))$ , then  $h'(x) = f'(g(x)) \cdot g'(x)$ . The glorious **chain rule!**
5. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^k \rightarrow \mathbb{R}^n$ . Now let  $h : \mathbb{R}^k \rightarrow \mathbb{R}$  as  $h(x) = f(g(x))$ , with  $x = (x_1, \dots, x_j, \dots, x_k)$ .

$$\frac{\partial h}{\partial x_j} = \sum_{i=1}^n \frac{\partial f(x)}{\partial (g(x))_i} \frac{\partial (g(x))_i}{\partial x_j}$$

With  $(g(x))_i$  the coordinate  $i$  of the vector  $g(x) \in \mathbb{R}^n$ .

## Theorem (Implicit Function)

*Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  a function such that  $f(z) = 0$ , with  $z = (z_1, z_2, \dots, z_n)$ .*

*If  $f_{x_1}(z) \neq 0$ , then there exists a differentiable function  $g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  such that  $z_1 = g(z_2, z_3, \dots, z_n)$ , and  $f(g(x_2, \dots, x_n), x_2, \dots, x_n) = 0$  for any  $(x_2, \dots, x_n)$  near  $(z_2, \dots, z_n)$ .*

## Theorem (Mean Value)

*Consider a differentiable function  $f : [a, b] \rightarrow \mathbb{R}$ , then there is  $c \in [a, b]$  such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

## Conjecture

*Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable.  $f$  is strictly increasing if and only if  $f'(x) > 0$  for any  $x \in \mathbb{R}$ .*

## Definition

Given  $x_0 \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  differentiable (at least  $k$  times). The  $k$  – order Taylor series at  $x_0$  is:

$$T(x_0, f)(x) := \sum_{i=0}^k \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i$$

## Conjecture (l'Hôpital's rule)

*Let  $f$  and  $g$  be differentiable functions such that one of the following two conditions hold:*

- $\lim_{x \rightarrow x_0} f(x) = 0$  and  $\lim_{x \rightarrow x_0} g(x) = 0$ , or
- $\lim_{x \rightarrow x_0} |f(x)| = \infty$  and  $\lim_{x \rightarrow x_0} |g(x)| = \infty$ .

*Then  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ .*