Functions

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A *function* takes one element from a set, and associates it with an element of another set.

Definition

f is a **function** from A to B, if it links each element from A to a single element from B. The set A is called the *domain of* f, and the set B is called the *codomain of* f.

The notation for a function is $f : A \to B$, and if $y = f(x)$ we say that $(x, y) \in f$.

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Consider a function $f:\mathbb{R}^n \to \mathbb{R}$, the **Graph** of f, $Gr(f)$ is defined as:

$$Gr(f) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | y = f(x)\}$$

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Note: More generally neither the domain needs to be $\mathbb{R}^n$ nor the codomain needs to be $\mathbb{R}$, the case given above is just the most common situation in economics.

Definition

Consider $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

1. Sum: $(f + g) : \mathbb{R} \to \mathbb{R}$, and $(f + g)(x) = f(x) + g(x)$.
2. Product: $(f \cdot g) : \mathbb{R} \to \mathbb{R}$, and $(f \cdot g)(x) = f(x)g(x)$
3. Division: $(f/g) : \mathbb{R} \to \mathbb{R}$, and $(f/g)(x) = \frac{f(x)}{g(x)}$. This is only well defined when $g(x) \neq 0$.
4. Scaling: If $\alpha \in \mathbb{R}$, $(\alpha f) : \mathbb{R} \to \mathbb{R}$, and $(\alpha f)(x) = \alpha f(x)$

Definition

Consider the functions $f : B \to C$, and $g : A \to B$, then the **composite** function $f \circ g : A \to C$ is defined as

$$(f \circ g)(x) = f(g(x))$$

Definition

Consider sets A and B, and the function $f : A \to B$.

1. f is **injective** if, for a and a' in A, such that $a \neq a'$, then $f(a) \neq f(a')$.
2. f is **surjective** if, for any $b \in B$, exists $a \in A$ such that $f(a) = b$.
3. f is **bijective** if it is both, injective and surjective at the same time.

Quick Quiz - 5 Minutes

Classify the following functions

Function	Classification
$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$	
$f : \mathbb{R} \to [-1, 1], f(x) = sin(x)$	
$f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$	

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$f : \mathbb{R} \to [-1, 1], f(x) = sin(x)$	Surjective
$f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$	Bijective

Conjecture

If $f : A \to B$ is a bijective function, then there exists a unique function $g : B \to A$, bijective, such that

$$g(f(x)) = x$$

g is called the inverse of f, also known as f^{-1}.

Conjecture

Let $f : B \to C$, and $g : A \to B$ be both invertible functions, then $f \circ g$ is invertible. Moreover,

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Proof.

Existence:

- Let $g = \{(b, a) | (a, b) \in f\}$.

Functions

Proof.

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- If $(b, a_1), (b, a_2) \in g$, then $(a_1, b), (a_2, b) \in f$, but f is injective, so $a_1 = a_2$. Then g is a function.

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- The domain of g is $\{b | (b, a) \in g\} = \{b | (a, b) \in f\} = f(X)$.

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- The domain of g is $\{b | (b, a) \in g\} = \{b | (a, b) \in f\} = f(X)$.
- Let $(b, a_2) \in g$ and $(a_1, b) \in f$. Then $(a_2, b) \in f$, and given f injective, we have $a_1 = a_2$. Then $g \circ f = \{(a, a) | a \in A\} = Id$.

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- Let $f^{-1} = g$

Homework, show that g is bijective. Hint: Go with contradiction. $\square$

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- So $g(b) \neq h(b)$, but $g(f(a)) \neq h(f(a))$.
- But $g \circ f$ and $h \circ f$ are both the identity so...
- $g(f(a)) = a \neq a = h(f(a))$, contradiction!

So the inverse must be unique.

Composite Invertible

Proof.

- $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ f \circ f^{-1} \circ g^{-1}$.

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- $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ f \circ f^{-1} \circ g^{-1}$.
- $g \circ f \circ f^{-1} \circ g^{-1} = g \circ Id \circ g^{-1}$.

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- $g \circ f \circ f^{-1} \circ g^{-1} = g \circ Id \circ g^{-1}$.
- $g \circ Id \circ g^{-1} = g \circ g^{-1} = Id$.

Trivial to show that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = Id$. as well, using the same steps.

$\square$

Definition

Consider $f : \mathbb{R}^n \to \mathbb{R}$ and $\hat{y}$ in the codomain.

1. The **level curve** of f at $\hat{y}$ is:

$$\mathcal{C}_{\hat{y}} = \{(x, \hat{y}) \in \mathbb{R}^{n+1} | f(x) = \hat{y}\}$$

2. The **isoquant** curve of f at $\hat{y}$ is:

$$I_{\hat{y}} = \{x \in \mathbb{R}^n | f(x) = \hat{y}\}$$

Definition

Consider the function $f : \mathbb{R} \to \mathbb{R}$, and any pair x and y in $\mathbb{R}$ such that $x < y$, we say that

1. f is **increasing** if $f(x) \leq f(y)$.
2. f is **decreasing** if $f(x) \geq f(y)$.

If the inequalities are strict, then you add the word *strictly* to increasing or decreasing. A non decreasing function is also known as monotonically increasing. Conversely, a non increasing function is also known as monotonically decreasing.

Definition

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$, and any pair x and y in $\mathbb{R}^n$ such that $y_i = x_i$ for every $i = 1, ., j-1, j+1, .., n$, and $y_i = x_i + \epsilon$, with $\epsilon > 0$ we say that

1. f is **increasing in the component** j if

$$f(x_1, \dots, x_j, \dots, x_n) \leq f(x_1, \dots, x_j + \epsilon, \dots, x_n)$$

If the inequalities are strict, then you add the word *strictly* to increasing or decreasing.

Definition

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$, x and y in $\mathbb{R}^n$, and $\lambda \in [0, 1]$, we say that:

1. f is **convex** if
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$
2. f is **concave** if
$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

If the inequalities are strict, then you add the word *strictly* to convex or concave.

Quick Quiz - 15 minutes

Consider you want to avoid a bad behavior from a certain agent. You catch the agent in such behavior with probability p. The agent has an initial wealth of W, and you can determine a fine F in case you catch him/her. The agent gets a benefit of M for missbehaving.

Consider for this exercise that a risk neutral/lover/averse agent has a linear(identity)/(str)convex/(str)concave utility function over money (monotonically increasing). Find the relationship between the *lowest* fine necessary for a risk neutral and a risk lover/averse agent to behave properly. **You need to show your conclusions mathematically.**

Solution

Start with the reference, the risk neutral. So you can find an F_0 such that

$$u(W) = pu(W + M - F_0) + (1 - p)u(W + M)$$

as $u()$ is monotonically increasing (identity) and linear, we know it is bijective, therefore exists u^{-1},

$$F_0 = W + M - u^{-1}\left(\frac{u(W) - (1 - p)u(W + M)}{p}\right)$$

furthermore, this is trivial, as the identity is equal to the inverse... the point is that exists and can be found (indeed $F_0 = M/p$).

Solution

Now that F_0 is well defined, let's compare it with our risk averse agent. We know that F_0 is such that:

$$W = p(W + M - F_0) + (1 - p)(W + M)$$

By the definition of concavity (for the risk averse) we know that

$$u(W) > pu(W + M - F_0) + (1 - p)u(W + M)$$

and therefore F_0 is too big!, note is the only thing we can adjust, and $u()$ is monotonically increasing in the argument, so we can decrease F_0 and still the Agent prefers to behave!
Same rationale can be applied to the risk lover agent.