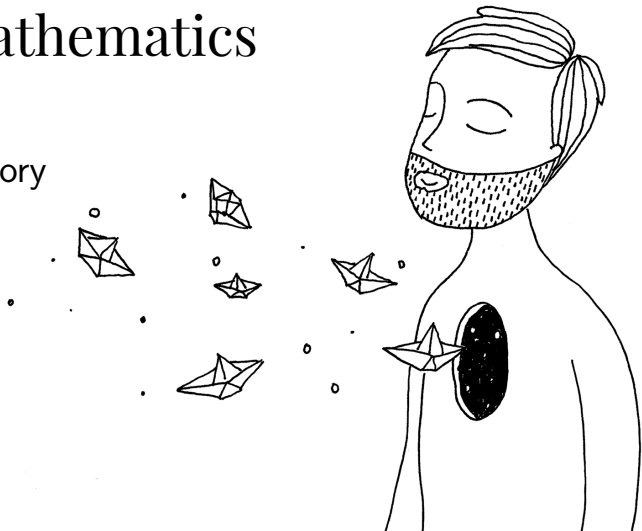


4509 – Bridging Mathematics

Introduction to Measure Theory
and Integration

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σ -Algebras

Definition

Let X be a set. An *algebra* is a collection \mathcal{A} of subsets of X such that:

1. $\emptyset \in \mathcal{A}$
2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$
3. If $A_1, A_2, \dots, A_n \in \mathcal{A}$ then $\bigcup_{i=1}^n A_i \in \mathcal{A}$ and $\bigcap_{i=1}^n A_i \in \mathcal{A}$

If 3 holds for countable infinite sets A_i (i.e. you replace the n by ∞), then \mathcal{A} is a σ – *algebra*.

Example

These are examples for \mathcal{A} being a σ – *algebra*

1. Let $X = \mathbb{R}$, and \mathcal{A} the set of all the subsets of \mathbb{R} .
2. Let $X = [0, 1]$ and let $\mathcal{A} = \{ \emptyset, X, [0, \frac{1}{2}], (\frac{1}{2}, 1] \}$

Definition

The pair (X, \mathcal{A}) is called a *measurable space*. A set A is *measurable* if $A \in \mathcal{A}$

Lemma

If \mathcal{A}_α is a σ – algebra for each $\alpha \in I$, with I an index set, then $\cap_{\alpha \in I} \mathcal{A}_\alpha$ is a σ – algebra

Proof.

1. If \mathcal{A}_α is a σ – algebra, therefore $\emptyset \in \mathcal{A}_\alpha \ \forall \alpha \in I$, and then $\emptyset \in \cap_{\alpha \in I} \mathcal{A}_\alpha$
2. Let $S_i \in \cap_{\alpha \in I} \mathcal{A}_\alpha$. It follows that $S_i \in \mathcal{A}_\alpha \ \forall \alpha \in I$, and also $S_i^c \in \mathcal{A}_\alpha \ \forall \alpha \in I$, but then $S_i^c \in \cap_{\alpha \in I} \mathcal{A}_\alpha$.
3. Choose a collection $\{S_i\}_i^\infty \in \cap_{\alpha \in I} \mathcal{A}_\alpha$. Now given that S_i must also be in every \mathcal{A}_α , their intersection is also in \mathcal{A}_α , and therefore it must be in $\cap_{\alpha \in I} \mathcal{A}_\alpha$.



Let \mathcal{C} be a collection of subsets of X , define:

$$\sigma(\mathcal{C}) = \cap \{ \mathcal{A}_\alpha \mid \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra, } \mathcal{C} \subset \mathcal{A}_\alpha \}$$

this is, the intersection of all σ -algebras containing \mathcal{C} . Note that $\sigma(\mathcal{C})$ is non empty, as at least the σ -algebra $\mathcal{P}(X)$ contains \mathcal{C} . Using the previous lemma, we have that $\sigma(\mathcal{C})$ is itself a σ -algebra. We call this the σ -algebra generated by \mathcal{C} , or that \mathcal{C} generates the σ -algebra $\sigma(\mathcal{C})$.

Fact

Continuing with the previous definition we can state that:

1. *If $\mathcal{C}_1 \subset \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$.*
2. *Since $\sigma(\mathcal{C})$ is a σ – algebra, then $\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$.*

Definition

If X has some structure, for example if it is a metric space, then we can consider open sets in X . If \mathcal{G} is the collection of open subsets of X , then $\sigma(\mathcal{G})$ is the **Borel** σ – *algebra* on X , and it is denoted as \mathcal{B} . The elements of \mathcal{B} are called *Borel sets*, and are said to be *Borel measurable*.

Proposition

If $X = \mathbb{R}$, then the Borel σ – algebra \mathcal{B} is generated by each of the following collection of sets:

1. $\mathcal{C}_1 = \{(a, b) | a, b \in \mathbb{R}\}$
2. $\mathcal{C}_2 = \{[a, b] | a, b \in \mathbb{R}\}$
3. $\mathcal{C}_3 = \{(a, b] | a, b \in \mathbb{R}\}$
4. $\mathcal{C}_4 = \{(a, \infty) | a \in \mathbb{R}\}$

Proof.

1. Let \mathcal{G} be the collection of open sets. By definition $\sigma(\mathcal{G})$ is the Borel σ – *algebra*. Since every element of \mathcal{C}_1 is open, then $\mathcal{C}_1 \subset \mathcal{G}$, and consequently $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{G}) = \mathcal{B}$.



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Measures



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Lebesgue Integral