4509 - Bridging Mathematics

Dynamic Optimization: Euler and how to optimally eat cake

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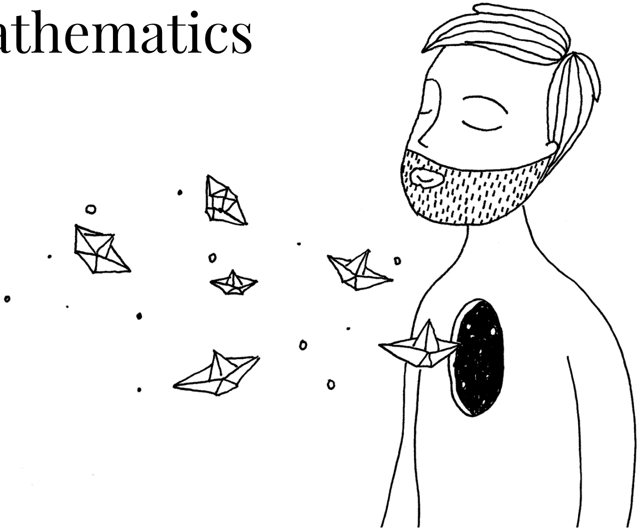

Dynamic Optimization: Basic Problem

In this section we review the basics/intuition of dynamic optimization. We are going to solve how to properly eat a cake!

Figure: How would you eat this cake... if it was the only food you would ever get!

Cake Eating Problem

1. Utility function $u(c) = \ln(c)$, where c is the slice of the cake you are eating.
2. You have a cake of size x.
3. Your discount factor is $\beta \in [0, 1)$.
4. You live forever.

Cake Eating Problem

So at each point in time:

1. you have x_t of cake
2. you get $\ln(c_t)$ of utility, and
3. you leave $x_{t+1} = x_t - c_t$ for the future.

Your only decision variable is *how much* cake eat at each period, which impacts on your utility today, but also on how much utility you will be able to get in the future.

Cake Eating Problem

So your problem is:

$$\sup_{\{c_t\}_t^\infty} \quad \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$
$$s.t. \quad x_{t+1} = x_t - c_t$$
$$c_t \geq 0 \quad \forall t$$
$$x_t \geq 0 \quad \forall t$$
$$x_0 > 0 \quad given$$

Keep this in mind...

Gen. Seq. Opt. Problem

Given $\beta \in [0, 1)$

$$\sup_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$
$$s.t. \quad x_{t+1} \in \Gamma(x_t)$$
$$x_0 \in X \subseteq \mathbb{R}^n$$

With $\Gamma(x_t) \neq \emptyset$ and $\Gamma(x_t) \subseteq X$, that is, only allow for feasible values for x_t. X is known as the state space, and x_t is then known as the...

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We can write what is the problem, using the notation we got from the set part...

$$
\begin{aligned}
&A = \{(x, y) : x \in X, y \in \Gamma(x)\} \\
&F : A \to \mathbb{R}
\end{aligned}
$$

And we actually can choose from

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty}, x_t \in \Gamma(x_{t-1}), t \in \mathbb{N}\}$$

So $\Pi(x_o)$ represents the set of *admissible paths* starting at x_0, and therefore the generic problem is equivalent to writing:

$$\sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

with x_0 given.

So how we solve this. It would help to get an idea of what we could expect of a solution.
Of course we cannot find x_t for every t explicitly, as there are an infinite number of those, however, we can find a function to generate them, this function is called *policy function*.

$$x_t = g(x_{t-1})$$

Then the solution would look like...

$$\{x_0, g(x_0), g(g(x_0)), ...\}$$

Cauchy's Criterion

A real sequence $\{r_t\}$ converges in $\mathbb{R}$ if and only if $\forall \epsilon > 0 \; \exists \mathbb{T}$ such that $\forall t, s > \mathbb{T}$ $|r_t - r_s| < \epsilon$

So we want that for $\mathbb{T} < T < S$,

$$\left|\sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) - \sum_{t=0}^{S} \beta^t F(x_t, x_{t+1})\right| = \left|\sum_{t=T+1}^{S} \beta^t F(x_t, x_{t+1})\right|$$

And

$$\left|\sum_{t=T+1}^{S} \beta^t F(x_t, x_{t+1})\right| \leq \sum_{t=T+1}^{S} \beta^t |F(x_t, x_{t+1})|$$

Assumption

To find the solution we need an extra assumption, that $F(x_t, x_{t+1})$ is bounded!...
$\exists M > 0$ such that $\forall (x, y) \in A \; |F(x, y)| \leq M$

$$\sum_{t=T+1}^{S} \beta^t |F(x_t, x_{t+1})| \leq \sum_{t=T+1}^{S} \beta^t M = M \sum_{t=T+1}^{S} \beta^t$$

As $\beta \in [0, 1)$

$$M \sum_{t=T+1}^{S} \beta^t \leq M \sum_{t=T+1}^{\infty} \beta^t = M\beta^{T+1} \sum_{t=0}^{\infty} \beta^t = M\beta^{T+1} \frac{1}{1-\beta}$$

Now we want, from Cauchy, that

$$M\beta^{T+1} \frac{1}{1-\beta} < \epsilon$$

Which can be achieved by choosing a sufficiently large T.

What did just happen??

We saw that $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ converges, so the objective function is well defined, if F is bounded on the feasible domain, so

$$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \in \mathbb{R}$$

The other assumption that would ensure a well defined objective function is $F \geq 0$. Note that as $\beta \geq 0$ this would ensure that the sequence is strictly increasing in T, and therefore or it would reach a limit, or it could diverge to $+\infty$. What we cannot have is the sequence having more than one accumulation points, because we wouldn't know what happens at the end.

Approaches

1. Dynamic Programming
2. Variational Approach: Euler's Equations

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1. Dynamic Programming
2. Variational Approach: Euler's Equations

We'll deal with Euler's equations here. Dynamic Programming although very useful is complex enough to be too much for a couple of hours lecture.

Variational Approach

Say $x^* \in \mathbb{R}^n$ is a maximizer of $f : \mathbb{R}^n \to \mathbb{R}$, then

$$f(x_1^*, x_2^*, x_3^*, ..., x_n^*) \geq f(x_1, x_2, x_3, ..., x_n) \quad \forall x \in \mathbb{R}^n$$

Which in turns implies that

$$f(x_1^*, x_2^*, x_3^*, ..., x_n^*) \geq f(x_1, x_2^*, x_3^*, ..., x_n^*) \quad \forall x_1 \in \mathbb{R}$$

If f is differentiable in x_1, then we would have

$$f_{x_1}(x_1^*, x_2^*, x_3^*, ..., x_n^*) = 0$$

as the first order condition. Moreover, we could generalize for each variable (assuming differentiability) to have

$$f_{x_i}(x_i^*, x_{-i}^*) = 0 \quad \forall i = 1, ..., n$$

Euler's Equations

Let $\beta \in (0,1)$ (note that if $\beta = 0$ then the problem is not dynamic).

Let $\{x_t^*\}_{t=0}^{\infty}$ be such that

$$\sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) = \sup_{\Pi(x_0^*)} \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) = \max_{\Pi(x_0^*)} \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*)$$

And let $\tau \in \mathbb{N}$ fixed (but arbitrary).

The contribution of x_τ^* to the objective function lies within the following terms

$$\beta^{\tau-1} F(x_{\tau-1}^*, x_\tau^*) + \beta^\tau F(x_\tau^*, x_{\tau+1}^*)$$

with

$$x_\tau^* \in \Gamma(x_{\tau-1}^*), \quad x_{\tau+1}^* \in \Gamma(x_\tau^*)$$

All the other terms, do not have x_τ^* in them.

Quick Quiz - 10 minutes

Note:

$$\beta^{\tau-1}F(x^*_{\tau-1},x^*_{\tau})+\beta^{\tau}F(x^*_{\tau},x^*_{\tau+1})= \max_{x\in\Gamma(x^*_{\tau-1}),x^*_{\tau+1}\in\Gamma(x)}\beta^{\tau-1}F(x^*_{\tau-1},x)+\beta^{\tau}F(x,x^*_{\tau+1})$$

Why? Prove it. Hint: Go by contradiction.

If $x^*_\tau \in int\Gamma(x^*_{\tau-1})$ and $x^*_{\tau+1} \in int\Gamma(x^*_\tau)$, then x^*_τ is a local maximizer of

$$\beta^{\tau-1}F(x^*_{\tau-1}, x) + \beta^\tau F(x, x^*_{\tau+1})$$

, and if $F()$ is differentiable, then

$$F'_2(x^*_{\tau-1}, x^*_\tau) + \beta F'_1(x^*_\tau, x^*_{\tau+1}) = 0$$

Which is the Euler's Equation. F'_i represents the derivative of F with respect to the i^{th} coordinate.

Euler's Equation

Conjecture

If $\{x_t^*\}_{t=0}^*$ *is optimal for the initial value* x_0^**, and* $F()$ *is differentiable, and if* $x_t^* \in int\Gamma(x_{t-1}^*) \forall t \in \mathbb{N}$ *then*

$$F_2'(x_{t-1}^*, x_t^*) + \beta F_1'(x_t^*, x_{t+1}^*) = 0 \quad \forall t \in \mathbb{N}$$

is a necessary condition for an interior optimizer.

Note, **necessary** is not the same as *sufficient*.
Now let's go back to our...

We had

$$\sup_{\{x_t\}} \sum_{t=0}^{\infty} \beta^t \ln(x_t - x_{t+1})$$

$$s.t. \quad x_{t+1} \in (0, x_t)$$

$$x_0 \quad given$$

Is the objective function well defined?

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By definition $x_0 > x_1 > x_2 > ... > 0$, so $\exists x_\infty = \lim_{t\to\infty} x_t$, and therefore $\lim_{t\to\infty} x_t - x_{t+1} = x_\infty - x_\infty = 0$...

Why does x_t converge? Quick Quiz $\rightarrow$ 5 minutes.

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Why does x_t converge? Quick Quiz $\rightarrow$ 5 minutes.

Monotonic and bounded! we can use the monotone convergence theorem.

If x_t is convergent, then $\exists T \in \mathbb{N}$ such that for $t > T$ $x_t - x_{t+1} < 1$ or $\ln(x_t - x_{t+1}) < 0$.
Let $S > T$,

$$\sum_{t=0}^{S} \beta^t \ln(x_t - x_{t+1}) = \underbrace{\sum_{t=0}^{T} \beta^t \ln(x_t - x_{t+1})}_{\in \mathbb{R}} + \underbrace{\sum_{t=T+1}^{S} \beta^t \ln(x_t - x_{t+1})}_{\text{decreasing in S}}$$

So there exists a limit in $\mathbb{R} \cup \{-\infty\}$.

Now

$$F(x_t, x_{t+1}) = ln(x_t - x_{t+1})$$

Leads to:

$$\beta^{t-1}[\ln(x_{t-1} - x_t) + \beta \ln(x_t - x_{t+1})]$$

And therefore the Euler equation is:

$$-\frac{1}{x_{t-1} - x_t} + \beta \frac{1}{x_t - x_{t+1}} = 0$$

Note that $x_{t-1} - x_t = c_{t-1}$ so

$$-\frac{1}{c_{t-1}} + \beta \frac{1}{c_t} = 0 \quad \Rightarrow \quad c_t = \beta c_{t-1} \quad \Rightarrow \quad c_t = \beta^t c_0$$

Now note, $c_0 = x_0 - x_1$, and have no clue about x_1 just yet, so we need an extra condition for x_1.

Use the fact that $\sum_{t=0}^{\infty} c_t \leq x_0$... you cannot eat more than the cake!
And $c_0 = x_0 - x_1$, $c_1 = x_1 - x_2$, $c_2 = x_2 - x_3$ ... $c_T = x_T - x_{T+1}$, so $\sum_{t=0}^{T} c_t = x_0 - x_{T+1}$, let $T \to \infty$, then so $\sum_{t=0}^{\infty} c_t = x_0 - x_\infty \leq x_0$, and consider that $x_\infty \geq 0$.

Note now that if $\sum_{t=0}^{\infty} c_t < x_0$, then c_t cannot be optimal, as there is cake left to be eaten!, so necessarily optimality implies $x_\infty = 0$, so the extra constraint is the **transversality condition**.

$$\lim_{T \to \infty} x_T = 0$$

$$
\begin{aligned}
c_t &= \beta^t c_0 \\
x_t - x_{t+1} &= \beta^t c_0 \\
x_{t+1} &= x_t - \beta^t c_0 \\
x_{t+1} &= (x_{t-1} - \beta^{t-1} c_0) - \beta^t c_0 \\
&\vdots \\
x_{t+1} &= x_0 - c_0 - \ldots - \beta^t c_0 \\
x_{t+1} &= x_0 - c_0 \frac{1 - \beta^{t+1}}{1 - \beta} \\
x_t &= x_0 - c_0 \frac{1 - \beta^t}{1 - \beta} \\
&\vdots \quad t \to \infty \\
x_\infty &= x_0 - c_0 \frac{1}{1 - \beta} = 0
\end{aligned}
$$

And replacing for x, we have

$$x_0 - \frac{c_0}{1-\beta} = 0$$
$$x_0 - \frac{x_0 - x_1}{1-\beta} = 0$$
$$x_1 = \beta x_0$$

And as $c_0 = x_0 - x_1 = x_0 - \beta x_0 = (1-\beta)x_0$, then if an optimal exists, then it is

$$c_t = \beta^t (1-\beta) x_0$$