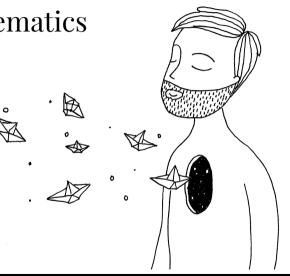
4509 - Bridging Mathematics

Markov Chains

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Discrete-time Markov chains

- 1. Time is indexed by an integer variable, say n.
- 2. At period n, the **state** of the chain is denoted by X_{n} .
- 3. S is a finite set of possible states, then $X_n \in S$.
- 4. We will allow for m different states, then $S = \{1, 2, ..., m\}$, for $m \in \mathbb{N}$.



Discrete-time Markov chains

Definition

Markov Chain The Markov chain is described in terms of its **transition probabilities** p_{ij} : whenever the state happens to be i, there is probability p_{ij} that the next state is equal to j:

$$p_{ij} = P(X_{n+1} = j | X_n = i), \quad i, j \in \mathcal{S}$$

with $p_{ij} \geq 0$ and $\sum_{j=1}^{m} p_{ij} = 1 \ \forall i$.



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with $p_{ij} \geq 0$ and $\sum_{j=1}^{m} p_{ij} = 1 \ \forall i$.

Note: the probability does not depend on time, nor anything else than the present state.



How to specify then a Markov Model?

- Identify:
 - 1. \mathcal{S} the set of states.
 - 2. the set of possible transitions, (i, j) where pij > 0
 - 3. the values for those p_{ij}
- The Markov chain specified by this model is a sequence of r.v.s $X_0, X_1, X_2, ...$, that can take values in S, and which satisfy:

$$P(X_{n+1} = j | X_n = i, \{X_{\nu} = i_{\nu}\}_{\nu=0}^{n-1}) = p_{ij}$$

for any n, and any $i, j \in \mathcal{S}$, and all possible sequences i_0, \ldots, i_{n-1} of earlier states.



It is convenient to sort all these probabilities in a two-dimensional array like this:

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

This is called the **Transition Probability Matrix**. This matrix is defined as having in each row i and column j the probability of transitioning from state i to state j.



Example, Bertsekas and Tsitsiklis (2008)

Alice is taking a probability class and in each week, she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively) . If she is behind in the given week, the probability that she will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively) . We assume that these probabilities do not depend on whether she was up-to-date or behind in previous weeks, so the problem has the typical Markov chain character (the future depends on the past only through the present)



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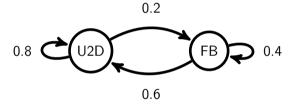
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The transition probability matrix:

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$



The transition probability graph:





We have said that the state today depends only on the state in the previous period. This is true, however, we can get around this constraint.

Consider the following example:

- 1. A working machine can be working the next day with probability p, and be broken with probability 1 p.
- 2. A broken machine can be working the next day with probability q, and remain broken with probability 1-q.

However, what happens if the machine cannot be fixed for, say, 4 straight days? Maybe we need to buy a new one. To model this we can introduce new states to our system.



- 1. A working machine can be working the next day with probability p, and be 1-day broken with probability 1 p, and zero for n-days broken for n > 1.
- 2. A 1-day broken machine can be working the next day with probability q, and become 2-day broken with probability 1-q, and zero for n-days broken for $n \neq 2$.
- 3. A 2-days broken machine can be working the next day with probability r, and become broken for 3 days with probability 1-r, and zero for n-days broken for $n \neq 3$.
- 4. A 3-days broken machine can be working the next day with probability s, and become broken for 4 days with probability 1-s, and zero for n-days broken for $n \neq 4$.
- 5. A 4-days broken machine can be working with probability 1, and zero for all the other broken states.



Definition (*n*-Step Transition Probabilities)

Let $r_{ij}(n)$ represent the probability that the state after n time periods will be j, given that the current state is i.

$$r_{ij}(n) = P(X_n = j | X_0 = i)$$

Proposition (Chapman-Kolmogorov)

The n-step transition probabilities can be generated by the recursive formula:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$
, for $n > 1$, and all i, j

starting with

$$r_{ij}(1) = p_{ij}$$



Note that this is an element of the following matrix:

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}^n$$



This "realization" allow us to be able to ask and answer some interesting questions:

- What can we say about limits? What happens as $n \to \infty$?
- The dependence of the state at *n* over the initial state becomes smaller as *n* increases.
- What can we say qualitatively about the behavior of this markov chain?



$$A = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$



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Definition (Accessible state)

A state j is accessible from a state i if $\exists n \in \mathbb{N}$ such that the n-step transition probability $r_{ij}(n)$ is positive, i.e., if there is positive probability of reaching j, starting from i, after some number of periods.



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Let A(i) be the set of states that are accessible from i. We say that i is **recurrent** if $\forall j \in A(i) \Rightarrow i \in A(j)$.



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Let A(i) be the set of states that are accessible from i. We say that i is **recurrent** if $\forall j \in A(i) \Rightarrow i \in A(j)$.

Definition (Transient state)

A state is called **transient** if it is not recurrent.



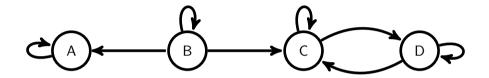
Corollary

A recurrent state will be visited an infinity amount of times.

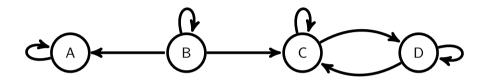
Corollary

A transient state will be visited a finite amount of times.



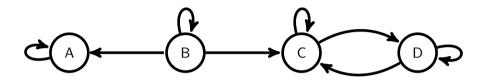






Recurrent

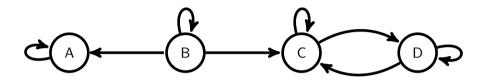




Recurrent 7

Transient



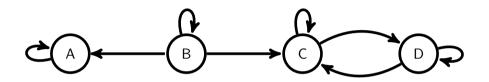


Recurrent

Transient

Recurrent





Recurrent

Transient

Recurrent

Recurrent



Definition (Recurrent class)

If i is a recurrent state, the set of sattes A(i) that are accessible from i form a **recurrent class** (or simply a class), meaning that states in A(i) are all accessible from each other, and no state outside A(i) is accessible from them.



Steady state behavior

When we talk about steady state in Markov Chains, it is not the "state" that is steady, but the probabilities of arriving to a certain state, remember the example we had before?

$$\pi_i = P(X_n = j)$$
, when *n* is large.



Theorem (Steady-State Convergence Theorem)

Consider a Markov chain with a single recurrent class, which is periodic. Then, the states j are associated with steady-state probabilities π_j that have the following properties:

1. For each j, we have

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j, \quad \forall i$$

2. The π_i are the unique solution to the system of equations below:

$$\pi_j = \sum_{k=1}^m \pi_k p_{kj}, \quad j = 1, \dots, m,$$

$$1 = \sum_{k=1}^m \pi_k$$

3. We have



Note that the steady-state probabilities add up to 1...



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Therefore these form a probability distribution on the state space, this is called the **stationary distribution** of the chain.



Definition (Balance Equations)

The equations

$$\pi_j = \sum_{k=1}^m \pi_k p_{kj}, \quad j = 1, \dots, m,$$

are called **balance equations**, and they are a direct consequence of the first part of the Steady-State Convergence Theorem, and the Chapman-Kolmogorov equation.

Definition (Normalization Equation)

The equation

$$sum_{k=1}^m \pi_k = 1$$

is known as the normalization equation.



Example

Consider our original example: $p_{11}=0.8$ $p_{12}=0.2$ $p_{21}=0.6$ $p_{22}=0.4$ Clearly, on the limit $r_{ij}\to\pi_j$ if this converges, then the balance equations say:

$$\pi_1 = \pi_1 \mathbf{p}_{11} + \pi_2 \mathbf{p}_{21}$$
 $\pi_2 = \pi_1 \mathbf{p}_{12} + \pi_2 \mathbf{p}_{22}$

Which, replacing, become:

$$\pi_1 = 0.8\pi_1 + 0.6\pi_2$$
 $\pi_2 = 0.2\pi_1 + 0.4\pi_2$

Solving, we obtain $\pi_1=3\pi_2$ in both equations, which together with the normalization equation $\pi_1+\pi_2=1$ lead us to:

$$\pi_1 = 0.75$$

$$\pi_2 = 0.25$$

