Problem Set

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- The deadline to deliver the solution, if you want any feedback, is September 2nd at midnight.
- The solution must be delivered by e-mail to `paulo.fagandini@novasbe.pt` using LaTeX. The `pdf` as well as the `tex` file must be provided.
- You are allowed to work in groups to solve the exercises, however each `tex` (and therefore `pdf`) file must be individually produced, the idea is that you learn and practice LaTeX.
- Solutions must be detailed, carried out and explained step by step.

Questions

1. Consider the matrix $A \in \mathbb{R}^{n\times n}$. Show that if $AB = BA$ then a matrix B must also be in $\mathbb{R}^{n\times n}$.[1]

> **Solution:** Assume it is not. Assume $B \in \mathbb{R}^{n\times k}$ with $k \neq n$. If so, then the l.h.s. has dimesions $n \times k$. The r.h.s. however, presents a problem, because pre-multiplying A requires that B has n columns (for the r.h.s. the number of rows is irrelevant). Therefore k cannot be different from n. You can make a simmilar argument for $B \in \mathbb{R}^{k\times n}$ for $k \neq n$ showing that for $AB = BA$ it is necessary that $B \in \mathbb{R}^{n\times n}$.
>
> Moreover, it would have been enough to say that if AB and BA are both a matrix of some dimension, then B would need to be in $\mathbb{R}^{n\times n}$.

2. Let $A \subset B$, with A closed, and B compact. Show that A is compact.

> **Solution:** Because B is compact, then it is bounded, *i.e.* $\forall b_0 \in B,\ \exists \epsilon_{b_0} > 0$ such that $\forall b \in B, b \in B(b_0, \epsilon_{b_0})$ or $B \subseteq B(b_0, \epsilon_{b_0})$. Because $A \subset B$, then $A \subset B(b_0, \epsilon_{b_0}$ and therefore A is bounded. Because A is closed by hypothesis, then A is compact.

3. Maximize the function $f(x, y) = x^2 + x + 4y^2$ subject to the following constraints $2x + 2y \leq 1$, $x \geq 0$, and $y \geq 0$.

> **Solution:** Very simple, with the KKT conditions the result is $x^* = 0$, $y^* = \frac{1}{2}$. Clearly only constraints $2x + 2y \leq 1$ and $x \geq 0$ are binding, and then the Lagrangian can be solved with those constraints in equality.

4. Compute the first and second order Taylor polynomial of $f(x) = e^x$ around $x = 0$. Show which polynomial approaches better the true function using $x = 1$ as an example.

> **Solution:** The first order TP of f around 0 is $1 + x$, and the second order is $1 + x + \frac{x^2}{2}$. If we evaluate at 1, we obtain 2 for the first order approximation, and 2.5 for the second order approximation. Clearly the second order is a better approximation, as e^1 is $2.718...$

[1] Note: B cannot be a scalar.

5. Show that $f(x_1, x_2) = x_1^2 + x_2^2$ is convex.

Solution: Using the differential approach, letting $x, y \in \mathbb{R}^2$, we need to show that $f(y) - f(x) \geq Df(X)(x-y)$

$$Df(x)(x-y) = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix} (y-x) = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix} \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix}$$

We have then:

$$2x_1(y_1 - x_1) + 2x_2(y_2 - x_2) = 2x_1y_1 - 2x_1^2 + 2x_2y_2 - 2x_2^2$$

Going back to the initial problem we need to verify that:

$$y_1^2 + y_2^2 - (x_1^2 + x_2^2) \geq 2x_1y_1 - 2x_1^2 + 2x_2y_2 - 2x_2^2$$

or

$$y_1^2 - 2y_1x_1 + x_1^2 + y_2^2 - 2x_2y_2 + x_2^2 \geq 0 \;\Leftrightarrow\; (y_1 - x_1)^2 + (y_2 - x_2)^2 \geq 0$$

Which always holds.

6. Consider the expenditure function

$$e(p, u) = \min\{p_1x_1 + \ldots + p_nx_n : u(x) \geq u\}$$

show that $e(p, u)$ is concave in p.

Solution:

Take any two pairs of prices and quantities that satisfy the equation for some given utility level u, for example (p, x) and (p', x'). The convex combination between p and p'' can be writteng as $tp + (1-t)p''$ $\forall t \in [0, 1]$. Let x'' be the expenditure minimizing bundle, *i.e.*,

$$e(p'', u) = p'' \cdot x'' = tp \cdot x'' + (1-t)p' \cdot x''$$

Note that x'' is not the optimal choice of quantities be it for p or for p'', and therefore:

$$p \cdot x'' \geq e(p, u) \quad p' \cdot x'' \geq e(p', u)$$

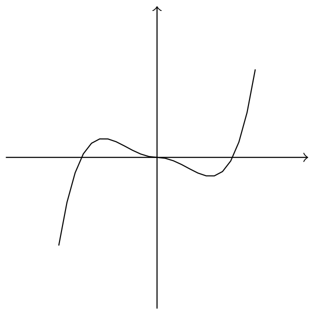
Replacing in the previous expression we obtain:

$$e(p'', u) \geq te(p, u)) + (1-t)e(p', u)$$

So e is concave in p.

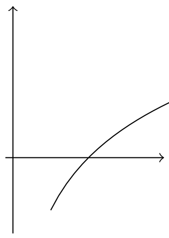
7. Find out which of the following functions is concave, quasiconcave or neither.
   (a) $x^4 - x^2$

   Solution: Neither.



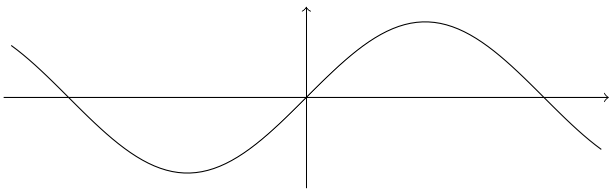
(b) $\ln(x)$

Solution: Concave and quasiconcave.



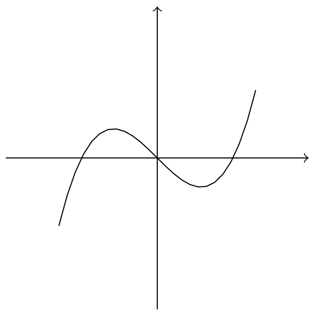
(c) $sin(x)$

Solution: Neither.



(d) $x^3 - x$

Solution: Neither.



8. Present an example for a convex, concave, quasiconcave (but not concave), quasiconvex (but non convex). Present a plot and a graphical description or reasoning.

Solution: Convex and concave, if lacking imagination any parabola or inverted parabola will do.

For a quasiconcave function, for example the normal density (Bell curve) is quasiconcave and not concave, while $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = log(|x| + 1)$ is quasiconvex and not convex (also $-f(x)$ with f being the normal density would work).

9. Explain what is a concave set.

Solution: Is the closest animal to the unicorn.