Problem Set 3
Topology and Differentiation

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Topology and Continuity

1. Show that the open ball around $x_0 \in \mathbb{R}$, $B(x_0, r)$ with $r > 0$ is an open set.
2. Let $A \subseteq \mathbb{R}^n$. Consider the collection of sets $A_i \subseteq \mathbb{R}^n$, $i \in I$. Show that
   (a) $int(A) \subseteq A$
   (b) A is open, if and only if $int(A) = A$.
   (c) If A_i is open, for any i, then $\bigcup_{i \in I} A_i$ is open.
   (d) If A_i is open, for any i and I is finite, then $\bigcap_{i \in I} A_i$ is open.
   (e) ϕ and $\mathbb{R}^n$ are open sets and closed at the same time.
   (f) If A_i is closed for any i, then $\cap_{i \in I} A_i$ is closed.
   (g) If A_i is closed for every i, and I is finite, then $\cup_{i=I} A_i$ is closed.
3. Let $A_i = \left(\frac{-1}{i}, \frac{1}{i}\right)$. show that $\cup_{i \in \mathbb{N}} A_i = \{0\}$, and show that $\{0\}$ is not open.
4. Show that the sequence $x_t = K$, with $K \in \mathbb{R}$ constant, then x_t converges, and its limit is K.
5. Show that if the sequence x_t converges, then $X = \{x_t | t \in \mathbb{N}\}$ is bounded.
6. Show that if $x_t \to x_0$ and $y_t \to x_0$, both sequences in $\mathbb{R}$, then $(x_t + y_t) \to (x_0 + y_0)$.
7. Show that if the sequence $x_t \to x^*$ is such that, for any t $x_t > 0$, then $x^* \geq 0$.
8. Show that if K is compact, and A is closed, then $K \cap A$ is compact.
9. Show that if $a_n \to a$, then $A = \{a_n, n \in \mathbb{N}\}$ is bounded. Discuss if A is closed or not.
10. Let x_t a sequence such that $\left|\frac{x_{t+1}}{x_t}\right| \to L$. Show that if $0 < L < 1$ then the sequence x_t converges.
11. Let x_t be a sequence such that $\sqrt[t]{|x_t|} \to L$. Show that if $0 < L < 1$ then x_t converges.
12. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $f(x_0) > 0$, then there is $r > 0$, such that for $x \in B(x_0, r)$ open, then $f(x) > 0$.
13. Let $f : \mathbb{R} \to \mathbb{R}$ continuous, let $B \subseteq \mathbb{R}$ open, show that $f^{-1}(B)$ is also open.

Differentiation

14. Show that $f(x) = x^2$ is differentiable. Find its derivative.

15. Show that $f(x) = |x|$ is not differentiable.

16. Find regions where the function $f(x) = x^2 - 3x^2 + x$ is increasing.

17. Find the Taylor series, of order k and around $x_0 = 0$, of the following functions.
    (a) $f(x) = e^x$
    (b) $f(x) = \sin(x)$
    (c) $f(x) = \cos(x)$

18. The Taylor series for a function $f : \mathbb{R} \to \mathbb{R}$ of order 2, and around x_0 corresponds to:

$$T_2(x_0, f)(x) = f(x_0) + \nabla^t f(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^t H(f, x_0)(x - x_0)$$

    Find $T_2(x_0, f)$ when $x_0 = (1, 1)$ and $f(x_1, x_2) = x_1^\alpha x_2^\beta$.

19. Using L'Hôpital's rule compute the following limits:
    (a) $\lim_{x\to 2} \frac{e^{x^2} - e^4}{x-2}$
    (b) $\lim_{x\to\infty} \frac{\sqrt[3]{x}}{\log(x)}$
    (c) $\lim_{x\to a} \frac{x-a}{\ln(x)-\ln(a)}$
    (d) $\lim_{x\to 0} \frac{e^x - 1}{x}$

20. Show that $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \frac{1}{2}x$ has a unique fixed point.

21. Does the function $f : [a, b] \to [a, b]$, with a and b finite, such that $f(x) = \frac{h(g(x)+1)}{\phi(x)}$, with h, g, and ϕ continuous, has a fixed point? If so, is this unique?