Differentiability

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Definition

The function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** in $\hat{x} \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(\hat{x} + h) - f(\hat{x})}{h}$$

exists. If it does, we denote it as $\frac{df(\hat{x})}{dx}$, or $f'(\hat{x})$, or $f_x(\hat{x})$.

Definition

The function $f : \mathbb{R}^n \to \mathbb{R}$ is **partially differentiable** in $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ with respect to x_j if the limit

$$\lim_{h \to 0} \frac{f(\hat{x}_1, \ldots, \hat{x}_j + h, \hat{x}_{j+1}, \ldots, \hat{x}_n) - f(\hat{x}_1, \ldots, \hat{x}_j, \hat{x}_{j+1}, \ldots, \hat{x}_n)}{h}$$

exists. If it does, we denote it as $\frac{\partial f(\hat{x})}{\partial x_j}$, or $f_{x_j}(\hat{x})$.

Definition

If $f : \mathbb{R}^n \to \mathbb{R}$ is partially differentiable in each coordinate of $\hat{x}$, then its **gradient** is defined as:

$$\nabla f(\hat{x}) = \begin{pmatrix} \frac{\partial f(\hat{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\hat{x})}{\partial x_i} \\ \vdots \\ \frac{\partial f(\hat{x})}{\partial x_n} \end{pmatrix}$$

Definition

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is partially differentiable in each coordinate of $\hat{x}$, then the **Jacobian** matrix (also denoted ∇f, and Df) is defined as:

$$\nabla f(\hat{x}) = \begin{pmatrix} \frac{\partial f_1(\hat{x})}{\partial x_1} & \frac{\partial f_1(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\hat{x})}{\partial x_n} \\ \frac{\partial f_2(\hat{x})}{\partial x_1} & \frac{\partial f_2(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\hat{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\hat{x})}{\partial x_1} & \frac{\partial f_m(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_m(\hat{x})}{\partial x_n} \end{pmatrix}$$

where in this case $f_i(\hat{x})$ represents the coordinate i of $f(\hat{x})$.

Definition

The **higher order derivative** is defined recursively, consider the derivative of f, of order n, on $\hat{x}$ as,

$$f^{(n)}(\hat{x}) = (f^{(n-1)}(\hat{x}))'$$
$$f^{(1)}(\hat{x}) = f'(\hat{x})$$

Definition

The second partial derivative is defined as:

$$\frac{\partial^2 f(\hat{x})}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i}\left(\frac{\partial f(\hat{x})}{\partial x_j}\right)$$

It is well defined if both limits exist.

Definition

The **Hessian** of a function f is the matrix with the second partial derivatives of f.

$$H(f,\hat{x}) = \begin{pmatrix} \frac{\partial^2 f(\hat{x})}{(\partial x_1)^2} & \frac{\partial^2 f(\hat{x})}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f(\hat{x})}{\partial x_n \partial x_1} \\ \frac{\partial^2 f(\hat{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\hat{x})}{(\partial x_2)^2} & \cdots & \frac{\partial^2 f(\hat{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\hat{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\hat{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\hat{x})}{(\partial x_n)^2} \end{pmatrix}$$

Conjecture

Let $f, g : \mathbb{R} \to \mathbb{R}$ differentiable, it holds that:

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
3. $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
4. If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$. The glorious **chain rule!**.
5. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^k \to \mathbb{R}^n$. Now let $h : \mathbb{R}^k \to \mathbb{R}$ as $h(x) = f(g(x))$, with $x = (x_1, \ldots, x_j, \ldots, x_k)$.

$$\frac{\partial h}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial (g(x))_i} \frac{\partial (g(x))_i}{\partial x_j}$$

With $(g(x))_i$ the coordinate i of the vector $g(x) \in \mathbb{R}^n$.

Theorem Implicit Function

Let $f : \mathbb{R}^n \to \mathbb{R}$ a function such that $f(z) = 0$, with $z = (z_1, z_2, ..., z_n)$.
If $f_{x_1}(z) \neq 0$, then there exists a differentiable function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $z_1 = g(z_2, z_3, ..., z_n)$, and $f(g(z_2, ..., z_n), z_2, ..., z_n) = 0$ for any $(x_2, ..., x_n)$ near $(z_2, ..., z_n)$.

Theorem Intermediate Value

Consider a differentiable function $f : [a, b] \to \mathbb{R}$, then there is $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Definition

$f : \mathbb{R}^n \to \mathbb{R}^n$ is called **locally Lipschitz continuous** if for any $x_0 \in \mathbb{R}^n$, there is a neighborhood V_{x_0} and a constant $L > 0$ such that for any $x, y \in V_{x_0}$ it holds that

$$||f(x) - f(y)|| \leq L||x - y||$$

L is called the **Lipschitz constant**.
If L does not depend on x_0, it is called simply a **Lipschitz continuous**, and furthermore, if $L < 1$ it is called a **contraction**.

Theorem Banach fixed point

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction, then there is a single $x^* \in \mathbb{R}^n$ such that $f(x^*) = x^*$.

Theorem Brower fixed point in $\mathbb{R}^n$

Consider $B_n \subseteq \mathbb{R}^n$ the unit open ball (an open ball of radius 1). Let $f : B_n \to B_n$ continuous. Then f has a fixed point in B_n, that is, there is $x^* \in B_n$ such that $f(x^*) = x^*$.

Conjecture

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. f is strictly increasing if and only if $f'(x) > 0$ for any $x \in \mathbb{R}$.

Definition

Given $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ differentiable (at least k times). The $k-$*order* Taylor series at x_0 is:

$$T(x_0, f)(x) := \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

Conjecture l'Hôpital's rule

Let f and g be differentiable functions such that one of the following two conditions hold:

- $\lim_{x \to x_0} f(x) = 0$ and $\lim_{x \to x_0} g(x) = 0$, or
- $\lim_{x \to x_0} |f(x)| = \infty$ and $\lim_{x \to x_0} |g(x)| = \infty$.

Then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.