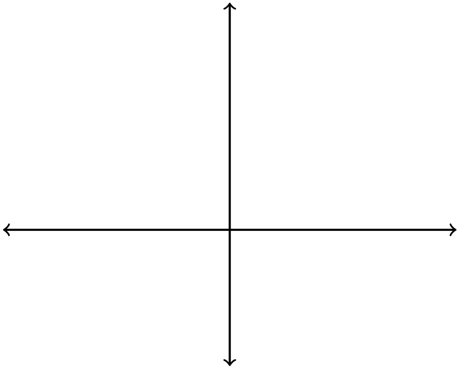
Matrices

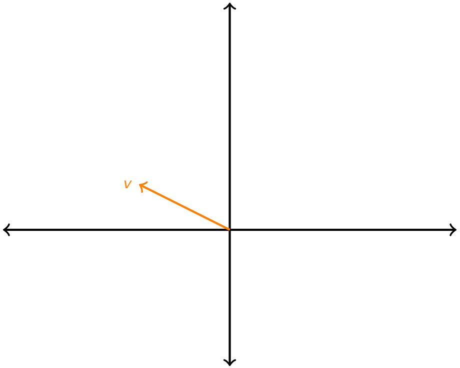
Paulo Fagandini

NOVA SCHOOL OF
BUSINESS & ECONOMICS

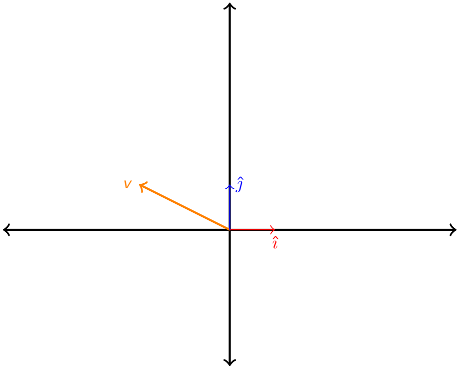
Introduction



Introduction



$$v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



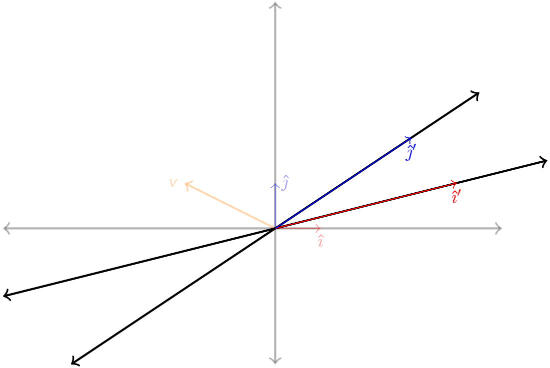
$$v = -2 \times \hat{\imath} + 1 \times \hat{\jmath} = -2 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Being not very rigorous, we can define a linear transformation as a transformation on every vector on the plane that must satisfy 2 things:

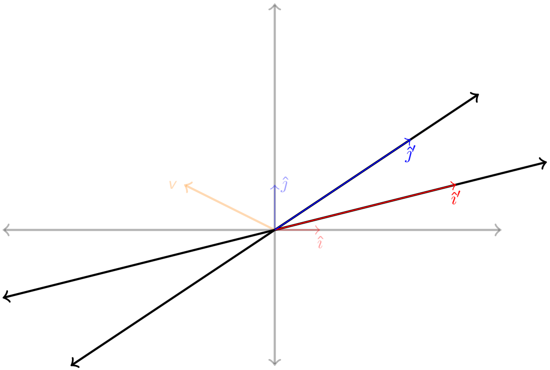
1. Lines must be transformed into lines
2. The origin must remain in the same place

We will deal with the formal definition and rigor later...

Introduction

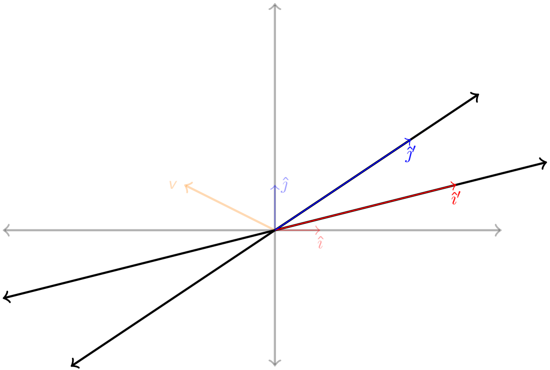


Introduction



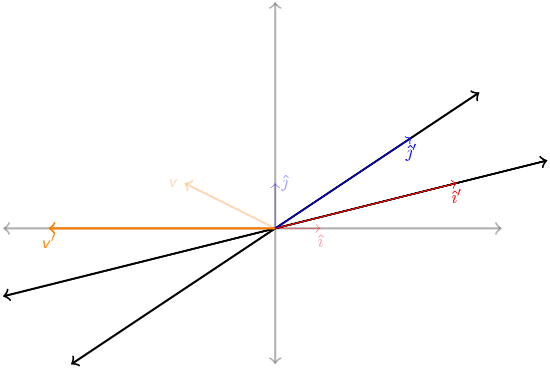
$$\hat{\imath}' = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \hat{\jmath}' = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Introduction

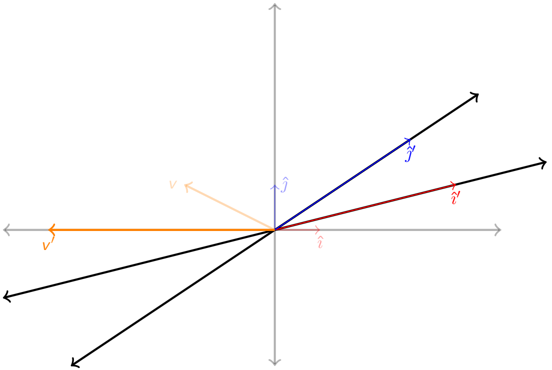


$$v = -2 \times \hat{\imath}' + 1 \times \hat{\jmath}' = -2 \times \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

Introduction

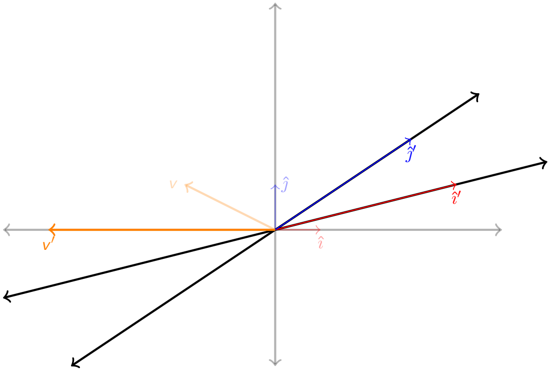


Introduction



$$w = \begin{bmatrix} x \\ y \end{bmatrix} \text{ lands on } x \times \begin{bmatrix} 4 \\ 1 \end{bmatrix} + y \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4x + 3y \\ 1x + 2y \end{bmatrix}$$

Introduction



$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}=x\times\hat{\imath}+y\times\hat{\jmath}$$

$$\begin{bmatrix}4 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4x+3y\\1x+2y\end{bmatrix}=x\times\hat{\imath}'+y\times\hat{\jmath}'$$

What about another vector in the same "direction" than v? say

$$z = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

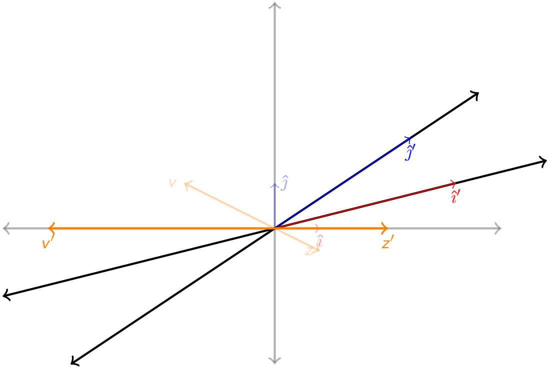
...

What about another vector in the same "direction" than v? say

$$z = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

...

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$$



v
$\hat{j}$
$\hat{j}'$
$\hat{i}'$
v'
$\hat{i}$
z
z'

So transforming two vectors in the same line, they both end up also in the same line... keep this in mind.

Could we take back $\hat{\imath}'$ to $\hat{\imath}$ and $\hat{\jmath}'$ to $\hat{\jmath}$?

Could we take back $\hat{\imath}'$ to $\hat{\imath}$ and $\hat{\jmath}'$ to $\hat{\jmath}$?
Sure, if we apply the transform

$$\begin{bmatrix} 0.4 & -0.6 \\ -0.2 & 0.8 \end{bmatrix}$$

Could we take back $\hat{\imath}'$ to $\hat{\imath}$ and $\hat{\jmath}'$ to $\hat{\jmath}$?
Sure, if we apply the transform

$$\begin{bmatrix} 0.4 & -0.6 \\ -0.2 & 0.8 \end{bmatrix}$$

$$\begin{bmatrix} 0.4 & -0.6 \\ -0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying the inverse!

The important thing, is that if the vectors are linearly dependent, then we cannot invert the matrix, we just saw that two vectors that reside on the same line, end up in the same (although probably a different one) line.

Introduction

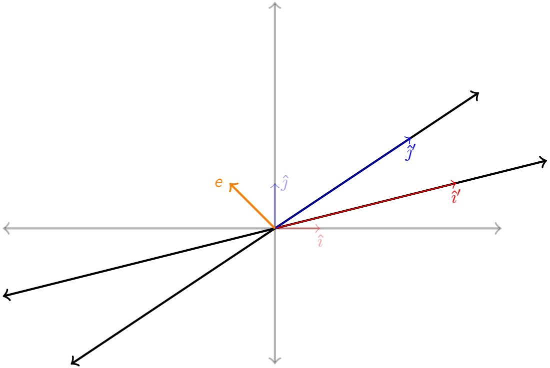
There are a couple of interesting vectors on the whole space when we apply this linear transformation...

Take for example the following vector: $e = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

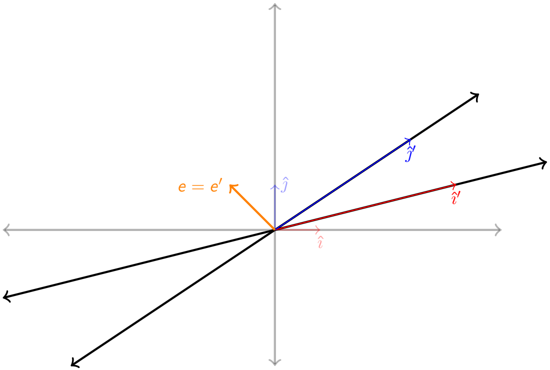
There are a couple of interesting vectors on the whole space when we apply this linear transformation...

Take for example the following vector:$e = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

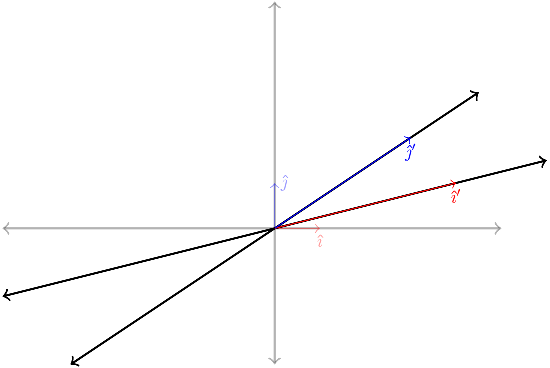

e
$\hat{j}$
$\hat{j}'$
$\hat{i}'$
$\hat{i}$

Introduction



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Introduction



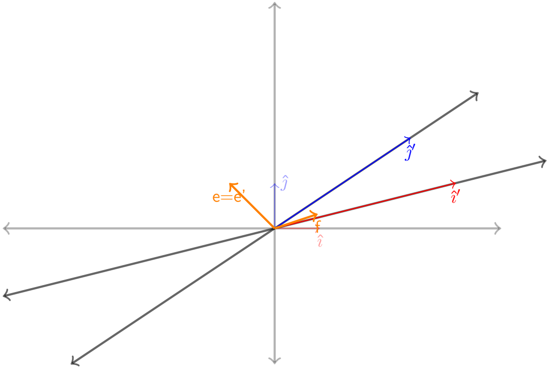
$$e.v._1 = \left[-1, 1\right], \quad \lambda_1 = 1$$

Or the vector: $f = \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix}$

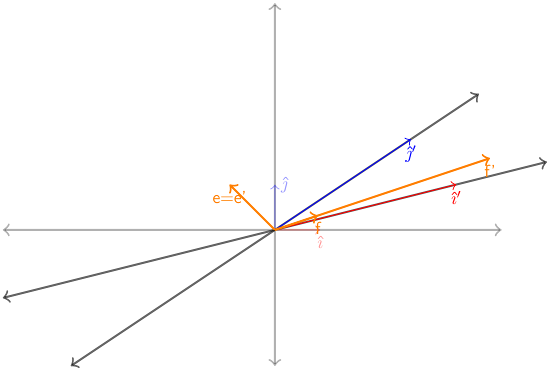
Or the vector: $f = \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix} = \begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix}$$

Introduction

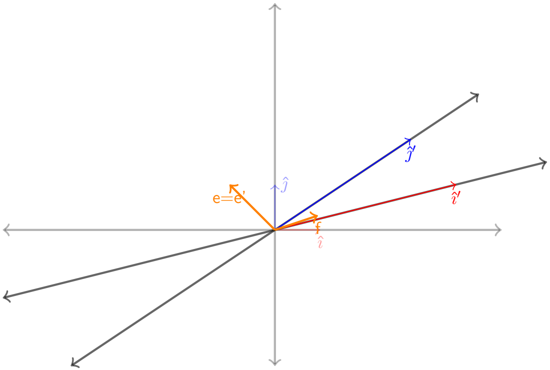


Introduction



$$5 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix} = \begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix}$$

Introduction



$$e.v._2 = \begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix}, \quad \lambda_2 = 5$$

What happens with these vectors?

	ev_1	ev_2
$A\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix}$

What happens with these vectors?

	ev_1	ev_2
$A\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix}$
$A^2\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 23.7171 \\ 7.9057 \end{bmatrix}$

What happens with these vectors?

	ev_1	ev_2
$A\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix}$
$A^2\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 23.7171 \\ 7.9057 \end{bmatrix}$
$A^3\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 118.585 \\ 39.528 \end{bmatrix}$

Introduction

What happens with these vectors?

	ev_1	ev_2
$A\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4.7434 \\ 1.5811 \end{bmatrix}$
$A^2\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 23.7171 \\ 7.9057 \end{bmatrix}$
$A^3\times$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 118.585 \\ 39.528 \end{bmatrix}$
λ	1	5

Definition

A real **matrix** is a rectangular array of real numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{nm} \end{pmatrix}$$

Where $a_{ij} \in \mathbb{R}$. A is said to be an element of $\mathbb{R}^{m \times n}$

A vector would be then a matrix with only 1 column!

Let $A, B \in \mathbb{R}^{m \times n}$. Let $C \in \mathbb{R}^{n \times l}$. Finally, let $\alpha \in \mathbb{R}$.

1. $[A + B]_{ij} = a_{ij} + b_{ij}$
2. $[A \cdot C]_{ik} = \sum_{j=1}^{n} a_{ij} \cdot c_{jk}$, and it has a dimension $m \times l$
3. $[\alpha A]_{ij} = \alpha a_{ij}$

Definition

Let $A \in \mathbb{R}^{m\times n}$, A's **transpose**, denoted $A^t \in \mathbb{R}^{n\times m}$ is such that its elements are:

$$a^t_{ij} = a_{ji}$$

Definition

Matrix $A \in \mathbb{R}^{m\times n}$ is said to be **squared** if $n = m$

Definition

Matrix A is said to be **symmetric** if $A^t = A$

Definition

Matrix A is said to be **antisymmetric** if $A^t = -A$

Definition

The **Identity** is a squared matrix $I_n \in \mathbb{R}^{n \times n}$ that has $I_{ij} = 0$ if $i \neq j$, and $I_{ij} = 1$ if $i = j$.
The identity has a nice property: $AI_n = I_m A = A$ for any $A \in \mathbb{R}^{m \times n}$.

Definition

Matrix A is **invertible**, if there is another matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = I$

Conjecture

Given $A, B, C \in \mathbb{R}^{n\times n}$

1. $A + B = B + A$
2. $A(BC) = (AB)C$
3. $A(B + C) = AB + AC$
4. $(A + B)^t = A^t + B^t$
5. $(AB)^t = B^t A^t$
6. $(A^t)^t = A$
7. If A and B are invertible, then AB and BA are invertible as well. Furthermore $(AB)^{-1} = B^{-1}A^{-1}$
8. If A is invertible, then $(A^t)^{-1} = (A^{-1})^t$

Quick quiz, 15 min, prove points 7 and 8. You can use points 1-6 as true and given.

Solution

7 Start with AB, multiply by A^{-1} from the left, you are left with $A^{-1}AB = IdB = B$. Now multiply by B^{-1}, so you get $B^{-1}A^{-1}AB = B^{-1}IdB = B^{-1}B = Id$. Then $(B^{-1}A^{-1})(AB) = Id$ so it must be that $B^{-1}A^{-1} = (AB)^{-1}$. To complete the proof, you need to show that you can do the same from the "right".

8 Start with $(A^{-1}A)^t = Id^t = Id$, use property 5 and you get $(A^{-1}A)^t = A^t(A^{-1})^t = Id$, then $(A^{-1})^t$ must be the inverse (again the only thing that is missing is to show that it works if you start with $(AA^{-1})^t$ as well which is trivial.

Conjecture

The set of the matrices in $\mathbb{R}^{m \times n}$, together with the sum and scalar multiplication is a vector space.

Conjecture

A squared matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if all of its columns are linearly independent.

Definition

Matrix $A \in \mathbb{R}^{m\times n}$ is **upper triangular** if it has the following shape:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2m} & \dots & a_{2n} \\ 0 & 0 & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{mm} & \dots & a_{mn} \end{bmatrix}$$

That is, it has zeroes below its main diagonal.

Conjecture

The set of the upper triangular matrices in $\mathbb{R}^{n\times m}$, with the sum and scalar multiplication is a vector subspace of $\mathbb{R}^{n\times m}$.

Definition

Matrix A is **lower triangular** if A^t is upper triangular.

Definition

Matrix A is **diagonal** if it is upper and lower triangular at the same time.

Definition

The **rank** of a matrix A, denoted by $rank(A)$ is the maximum number of linearly independent rows or columns of A.

A convenient way to write down a system of equations:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{1n}x_n & = & b_2 \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

It would be $AX = B$, where,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Definition

Given a system of equations $AX = B$,

- $\hat{X} \in \mathbb{R}^n$ is a **particular solution** of the system if $A\hat{X} = B$.
- X_0 is an **homogeneous solution** if $AX_0 = 0$.

Note that for $\lambda \in \mathbb{R}$, $A(\hat{X} + \lambda X_0) = A\hat{X} + \lambda AX_0 = B + 0 = B$.

NOVA SCHOOL OF BUSINESS & ECONOMICS

Definition

The **kernel** of $A \in \mathbb{R}^{m \times n}$ is defined as:

$$Ker(A) := \{X \in \mathbb{R}^n | AX = 0\}$$

Conjecture

$Ker(A) \subseteq \mathbb{R}^n$ is a vector subspace of $\mathbb{R}^n$

Definition

The dimension of $Ker(A)$ is called the **nullity** — or nullspace —, and it is denoted by $Null(A)$. If $Ker(A) = \{0\}$, then $Null(A) = 0$.

Note that a system of equations as the one shown before, has unique solution only if $Null(A) = 0$.

Conjecture

Matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $Null(A) = 0$.

Conjecture

$A \in \mathbb{R}^{n \times n}$ is invertible if and only if the system $AX = B$ as a unique solution, for any $B \in \mathbb{R}^n$.

Definition

The **image** of $A \in \mathbb{R}^{n \times n}$ is defined as:

$$Im(A) := \{Y \in \mathbb{R}^n | \exists X \in \mathbb{R}^n, Y = AX\} \equiv \{AX | X \in \mathbb{R}^n\}$$

Conjecture

Let $A \in \mathbb{R}^{n \times n}$. $Im(A)$ is a vector subspace of $\mathbb{R}^n$.

Definition

The dimension of $Im(A)$ is called the **range** of A. Let's denote it as $R(A)$.

Conjecture

Let $A \in \mathbb{R}^{n \times n}$. $R(A)$ is the number of l.i. columns of A.

Conjecture

Consider $A \in \mathbb{R}^{n \times n}$. It holds that $Null(A) + R(A) = n$.

Definition

The quadratic form associated to A is a function $Q_A : \mathbb{R}^n \to \mathbb{R}$ such that for any $X \in \mathbb{R}^n$,

$$Q_A(X) = X^t AX \in \mathbb{R}$$

Conjecture

For any matrix $A \in \mathbb{R}^n$, there are always symmetric and antisymmetric matrices S and T such that

$$A = S + T$$

Note: Let $S = \frac{A+A^t}{2}$ and $T = \frac{A-A^t}{2}$. While S is symmetric, T is antisymmetric,

Corollary

A quadratic form can be represented as

$$Q_A(X) = X^t S X$$

with S symmetric.

Quick quiz! 15 min to prove the corollary.

Solution

- Let $A = (S + T)$
- Then $Q_A(X) = X^t(S + T)X = X^t SX + X^t TX$
- But $X^t TX \in \mathbb{R}$, so $(X^t TX)^t = X^t TX$ (a number trasposed is the same number).
- So you end up that $X^t TX = (X^t TX)^t = X^t T^t X$
- But T is anytsymmetric so $T^t = -T$...
- Then $X^t TX = -X^t TX$, so if $X^t TX$ is the number z, you have $z = -z$, that only is true for $z = 0$.
- Then $Q_A(X) = X^t SX$

Definition

Let $A \in \mathbb{R}^{n \times n}$, symmetric. Consider the quadratic form $Q_A(X) = X^t A X$. If for any $X \in \mathbb{R}^n \setminus \{0\}$,

1. $Q_A(X) > 0$, A is **positive definite**,
2. $Q_A(X) \geq 0$, A is **positive semi-definite**,
3. $Q_A(X) < 0$, A is **negative definite**,
4. $Q_A(X) \leq 0$, A is **negative semi-definite**.

Definition

$\lambda \in \mathbb{C}$ is an **eigenvalue** (or characteristic value) of matrix $A \in \mathbb{R}^{n\times n}$ if there is a vector, called **eigenvector**, $X_\lambda \in \mathbb{R}^n \setminus \{0\}$ such that

$$AX_\lambda = \lambda X_\lambda$$

Conjecture

Let $A \in \mathbb{R}^{n \times n}$, and λ_1 and λ_2 two eigenvalues of A, with $\lambda_1 \neq \lambda_2$. If V_1 in the vector subspace associated to λ_1, and V_2 in the vector subspace of λ_2 then V_1 and V_2 are linearly independent.

Conjecture

Given $A \in \mathbb{R}^{n \times n}$ symmetric, then its eigenvalues are real valued.

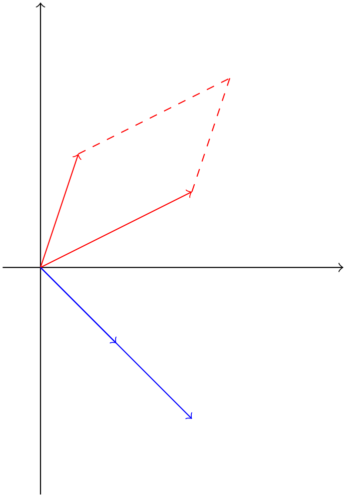
Definition

The **determinant** of a squared matrix A is the hyper-volume of the figure formed by the column vectors of the matrix.

Example

Consider the matrices,

$$A = \begin{pmatrix} 1/2 & 2 \\ 3/2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$



It is easy to see that, given our definition, $det(B) = 0$. It is also easy to show that $det(A) = |1/2 \times 1 - 3/2 \times 2| = 5/2$.

How to calculate the determinant of a big matrix? Recursively. Let $A \in \mathbb{R}^{n \times n}$. Define A_{ij} as:

$$A_{ij} = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$$

That is, what is left of A after removing row i and column j.

Then,

$$det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} det(A_{ik})$$

You can choose any i that you prefer.

Conjecture

- A squared matrix is invertible if and only if its determinant is different from zero.
- Take a finite set of matrices $\mathbb{A} \subseteq \mathbb{R}^{n \times n}$, with A_i being the *ith* element of $\mathbb{A}$ then,

$$det(A_1 A_2 \dots A_k) = det(A_1) det(A_2) \dots det(A_k)$$

- If A is invertible, then
$$det(A^{-1}) = \frac{1}{det(A)}$$
- For any squared A it holds that $det(A^t) = det(A)$.

Note that λ is an eigenvalue if

$$AX_\lambda = \lambda X_\lambda \qquad \text{with} \quad X_\lambda \neq 0$$

so λ is an eigenvalue of A if

$$(A - \lambda I)X_\lambda = 0$$

or $X_\lambda \in ker(A - \lambda I)$, which implies that $ker(A - \lambda I) \neq \{0\}$, and therefore $(A - \lambda I)$ must be not invertible! But if $(A - \lambda I)$ is not invertible, then $det(A - \lambda I) = 0$.

Corollary

λ is an eigenvalue of A if and only if $det(A - \lambda I) = 0$.

Definition

Given $A \in \mathbb{R}^{n \times n}$, the **characteristic polynomial** of A is defined as the function $p_A : \mathbb{R} \to \mathbb{R}$ such that

$$p_A(\lambda) = det[A - \lambda I]$$

So, λ is an eigenvalue of A if $p_A(\lambda) = 0$

Conjecture

If A is symmetric, then the eigenvectors of different eigenvalues are orthogonal.

For practical reasons, consider the matrix V as the matrix that has in its columns the eigenvectors of A, and $D(\lambda)$ the diagonal matrix that contains in the column i, the eigenvalue that corresponds to the eigenvector in the column i in V.
Note that:

$$AV = VD(\lambda) \quad \Leftrightarrow \quad A = VD(\lambda)V^{-1}$$

It can be shown that, given that the eigen vectors are l.i., $V^t = V^{-1}$ and therefore

$$A = VD(\lambda)V^{-1}$$

which is one of the fundamental properties of the symmetric matrices.

Conjecture

Given $A \in \mathbb{R}^{n \times n}$, symmetric. It holds that,

$$A = VDV^t$$

With D the diagonal with the eigenvalues of A and V the unit eigenvectors of A.

Conjecture

Let $A \in \mathbb{R}^{n \times n}$, symmetric.

1. A is positive definite if all the eigenvalues of A are strictly positive.
2. A is positive semi definite if all the eigenvalues are nonnegative.
3. A is negative definite if all the eigenvalues are strictly negative.
4. A is negative semidefinite if all the eigenvalues are non positive.

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if, for any $X, Y \in \mathbb{R}^n$, and for any $\alpha \in \mathbb{R}$

$$f(X+Y) = f(X) + f(Y), \quad f(\alpha X) = \alpha f(X)$$

Definition

The **trace** of a square matrix A ($tr(A)$) is the sum of the elements of its diagonal.

Theorem

Let $A \in \mathbb{R}^{n\times n}$, then:

- the product of the eigenvalues of A is equal to its determinant, that is,

$$det(A) = \prod_{i=1}^{n} \lambda_i$$

- the sum of the eigenvalues of A is equal to its trace, that is,

$$\sum_{i=1}^{n} a_{i,i} = \sum_{i=1}^{n} \lambda_i$$

- if A is a triangular matrix, then its eigenvalues are the coefficients in the principal diagonal of the matrix, *i.e.*,

$$\lambda_i = a_{i,i}$$