Quadratic form of particle- or quasiparticle-number conserving fermion operators

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Here, we prove the following result: if $\hat{A}$ is a fermionic operator that conserves total particle or quasiparticle number in a model space of finite dimension N_s, then, in any particle or quasiparticle basis, $\hat{A}$ can be written in quadratic form

$$\hat{A} = \frac{1}{2}\eta^\dagger \mathcal{A} \eta + A_0 \tag{1}$$

where $\eta = \left(a_1, ..., a_{N_s}, a_1^\dagger, ..., a_{N_s}^\dagger\right)^T$, $\{a_k, a_k^\dagger\}$ are the fermion particle or quasiparticle annihilation and creating operators for the chosen model space basis, and A_0 is a constant. Furthermore, $\mathcal{A}$ always has the property that $\sigma\mathcal{A}$ is skew-symmetric, i.e.

$$(\sigma\mathcal{A})^T = -\sigma\mathcal{A} \tag{2}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

In any (quasi)particle basis, an operator that conserves (quasi)particle number can be written

$$\hat{A} = \sum_{i,j=1}^{N_s} \mathbf{A}_{ij} a_i^\dagger a_j \tag{4}$$

We can rewrite $\hat{A}$ by using the anti-commutation properties of the fermion creation and annihilation operators

$$\begin{aligned} \hat{A} &= \frac{1}{2}\sum_{i,j=1}^{N_s} \mathbf{A}_{ij}(a_i^\dagger a_j + a_i^\dagger a_j) \\ &= \frac{1}{2}\sum_{i,j=1}^{N_s} \mathbf{A}_{ij}\left(a_i^\dagger a_j + \delta_{ij}\right) - \mathbf{A}_{ji}^T a_j a_i^\dagger \\ &= \frac{1}{2}\eta^\dagger \mathcal{A}\eta + \frac{1}{2}\mathrm{tr}\left(\mathbf{A}\right) \end{aligned} \tag{5}$$

where δ_{ij} is the Kronecker delta function and

$$\mathcal{A} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & -\mathbf{A}^T \end{pmatrix} \tag{6}$$

Thus, we have written $\hat{A}$ in the form (1). $\sigma\mathcal{A}$ is given by

$$\sigma\mathcal{A} = \begin{pmatrix} 0 & -\mathbf{A}^T \\ \mathbf{A} & 0 \end{pmatrix} = -(\sigma\mathcal{A})^T \tag{7}$$

Thus, $\sigma\mathcal{A}$ is skew-symmetric.

Any particle or quasiparticle basis can be reached from any other such basis by a general Bogoliubov transformation. All that remains is to show that a general Bogoliubov transformation does not change the properties in Eqs. (1) or (2). The general Bogoliubov transformation from the basis $\{a, a^\dagger\}$ to the new basis $\{\alpha, \alpha^\dagger\}$ can be written [1]

$$\alpha_i^\dagger = \sum_k \left(U_{ki} a_k^\dagger + V_{ki} a_k\right) \tag{8}$$

A unitary change of basis corresponds to a Bogoliubov transformation in which V is zero. We can write transformation in more compact notation as

$$\xi = W^\dagger \eta \tag{9}$$

where $\xi = \left(\alpha_1, ..., \alpha_{N_s}, \alpha_1^\dagger, ..., \alpha_{N_s}^\dagger\right)^T$ and W is a unitary matrix of dimension $2N_s \times 2N_s$ with the general form

$$W = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix} \tag{10}$$

Consequently,

$$\hat{A} = \frac{1}{2}\xi^\dagger \mathcal{A}' \xi + A_0 \tag{11}$$

where $\mathcal{A}' = W^\dagger \mathcal{A} W$. Elementary matrix multiplication shows that

$$\sigma W^\dagger = W^T \sigma = \begin{pmatrix} V^T & U^T \\ U^\dagger & V^\dagger \end{pmatrix} \tag{12}$$

Therefore, because $\sigma\mathcal{A}$ is skew-symmetric, $\sigma\mathcal{A}'$ is also skew-symmetric

$$\sigma\mathcal{A}' = W^T \sigma \mathcal{A} W = -W^T (\sigma\mathcal{A})^T W = -(W^T \sigma \mathcal{A} W)^T \ . \tag{13}$$

Thus, under a general Bogoliubov transformation, $\hat{A}$ retains the form of Eq. (1), where $\sigma\mathcal{A}$ is skew-symmetric in keeping with property (2). Our proof is complete.

[1] P. Ring and P. Schuck, *The Nuclear Many-Body Problem.* (Springer-Verlag, New York, 1980).