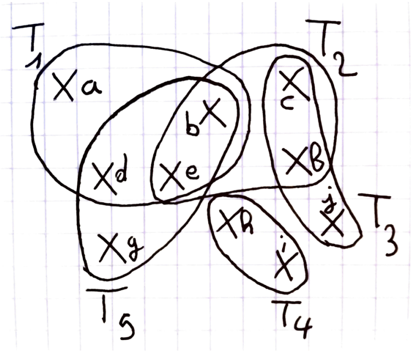
Tight Approximation Guarantees for Concave Coverage Problems

Siddharth Barman, Omar Fawzi, **Paul Fermé**

arXiv:2010.00970

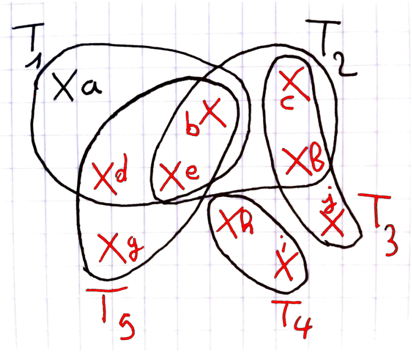
18/03/2021

The MaxCoverage problem



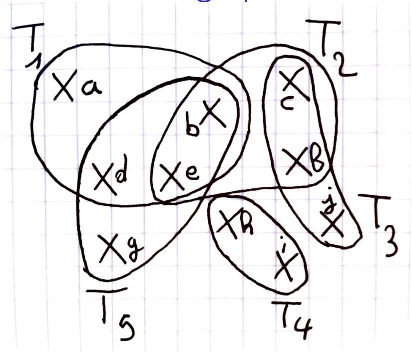
$$
\begin{aligned}
\text{maximize} \quad & C(S) := \left| \bigcup_{i \in S} T_i \right| \\
\text{subject to} \quad & |S| = k
\end{aligned}
$$

The MaxCoverage problem



$$\begin{aligned} \text{maximize} \quad & C(S) := \left| \bigcup_{i \in S} T_i \right| \\ \text{subject to} \quad & |S| = 3 \end{aligned}$$

The MaxCoverage problem



$$\begin{aligned} \text{maximize} \quad & C(S) := \left| \bigcup_{i \in S} T_i \right| \\ \text{subject to} \quad & |S| = k \end{aligned}$$

- NP-hard to approximate within a ratio $1 - e^{-1} + \varepsilon$ [Feige, 1998].
- As C is *submodular*, the natural *greedy* algorithm achieves the approximation ratio $1 - e^{-1}$ [Hochbaum, 1997].

A generalization

- What happens if we take into account elements covered several times?

A generalization

- What happens if we take into account elements covered several times?

- Network coverage:
  1. T_i: agents covered by antenna placement i.
  2. Goal: find k antenna locations maximizing coverage and bandwidth.

A generalization

- What happens if we take into account elements covered several times?

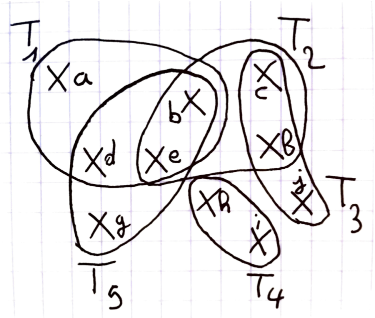
- Network coverage:
  1. T_i: agents covered by antenna placement i.
  2. Goal: find k antenna locations maximizing coverage and bandwidth.
- Multiwinner elections:
  1. T_i: electors approving candidate i.
  2. Goal: find k candidates maximizing the representative utility.

A generalization

- What happens if we take into account elements covered several times?

- Network coverage:
  1. T_i: agents covered by antenna placement i.
  2. Goal: find k antenna locations maximizing coverage and bandwidth.
- Multiwinner elections:
  1. T_i: electors approving candidate i.
  2. Goal: find k candidates maximizing the representative utility.

- The utility of an element covered c times will be given by $\varphi(c)$.
- We suppose φ nondecreasing concave and normalized, ie. $\varphi(0) = 0$, $\varphi(1) = 1$.

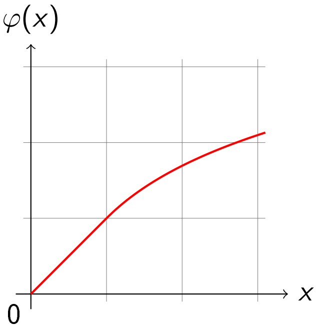


$$C^{\varphi}(S) = \dots$$

$$\text{maximize} \quad C^{\varphi}(S) := \sum_{a \in [n]} \varphi(|S|_a)$$

$$\text{subject to} \quad |S| = k$$

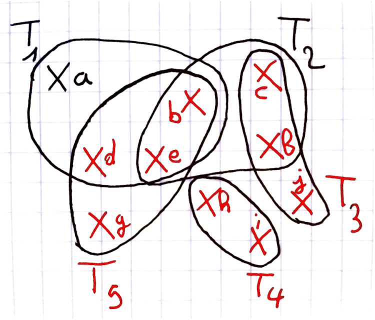
$$\text{with} \quad |S|_a := |\{i \in S : a \in T_i\}|$$

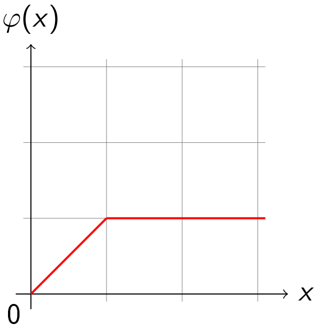


$$\text{maximize} \quad C^{\varphi}(S) := \sum_{a\in[n]} \min\{|S|_a, 1\}$$

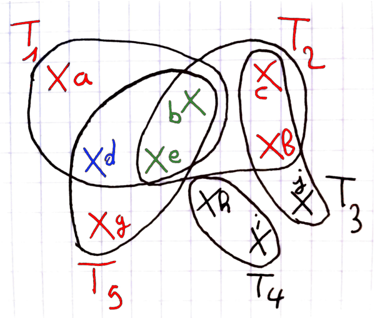
$$\text{subject to} \quad |S| = 3$$

$$\text{with} \quad |S|_a := |\{i \in S : a \in T_i\}|$$





$$C^{\varphi}(\{3,4,5\}) = 9$$

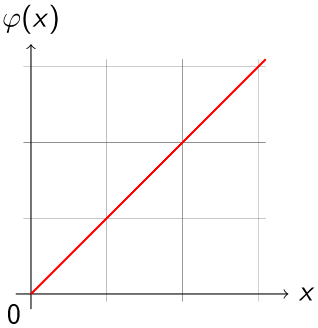


$$\text{maximize} \quad C^{\varphi}(S) := \sum_{a \in [n]} |S|_a$$

$$\text{subject to} \quad |S| = 3$$

$$\text{with} \quad |S|_a := |\{i \in S : a \in T_i\}|$$

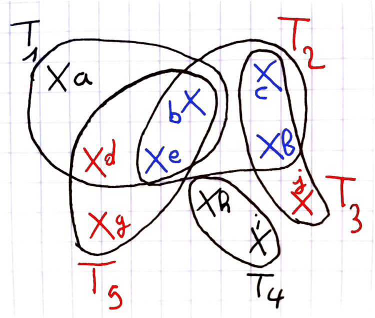
$$C^{\varphi}(\{1,2,5\}) = 12$$



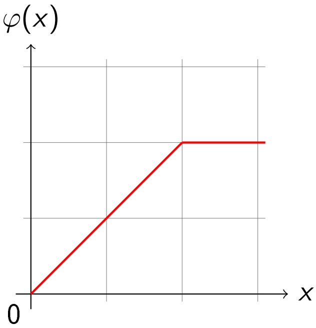
$$\text{maximize} \quad C^{\varphi}(S) := \sum_{a \in [n]} \min\{|S|_a, 2\}$$

$$\text{subject to} \quad |S| = 3$$

$$\text{with} \quad |S|_a := |\{i \in S : a \in T_i\}|$$



$$C^{\varphi}(\{2,3,5\}) = 11$$



Our results

φ-MaxCoverage problem:

$$\begin{aligned} &\text{maximize} && C^{\varphi}(S) := \sum_{a \in [n]} w_a \varphi(|S|_a) \\ &\text{subject to} && |S| = k \end{aligned}$$

Our results

φ-MaxCoverage problem:

$$\begin{aligned} \text{maximize} \quad & C^{\varphi}(S) := \sum_{a \in [n]} w_a \varphi(|S|_a) \\ \text{subject to} \quad & |S| = k \end{aligned}$$

Theorem (Main Result)

There exists a polynomial-time approximation algorithm achieving the *Poisson concavity ratio* of φ, defined by:

$$\alpha_{\varphi} := \min_{x \in \mathbb{N}^*} \alpha_{\varphi}(x), \text{ with } \alpha_{\varphi}(x) := \frac{\mathbb{E}[\varphi(\mathrm{Poi}(x))]}{\varphi(\mathbb{E}[\mathrm{Poi}(x)])} .$$

Furthermore for $\varphi(n) = o(n)$, it is NP-hard to approximate within a better ratio than $\alpha_{\varphi} + \varepsilon$.

Previous work

- [Sviridenko et al., 2017]: Generic algorithm for *submodular* maximization using the *curvature* c, ratio $1 - ce^{-1}$.
  $\Rightarrow$ We have shown that $\alpha_\varphi \geq 1 - ce^{-1}$, c curvature of C^φ.

Previous work

- [Sviridenko et al., 2017]: Generic algorithm for *submodular* maximization using the *curvature* c, ratio $1 - ce^{-1}$.
  $\Rightarrow$ We have shown that $\alpha_\varphi \geq 1 - ce^{-1}$, c curvature of C^φ.
- [Barman et al., 2020]: ℓ-MultiCoverage problem:
  $\varphi(j) := \min\{j, \ell\}$, ratio $1 - \frac{\ell^\ell e^{-\ell}}{\ell!}$, tight under UGC.
  $\Rightarrow$ One can compute $\alpha_\varphi = \alpha_\varphi(\ell) = 1 - \frac{\ell^\ell e^{-\ell}}{\ell!}$, tight if $\mathrm{P} \neq \mathrm{NP}$.

Previous work

- [Sviridenko et al., 2017]: Generic algorithm for *submodular* maximization using the *curvature* c, ratio $1 - ce^{-1}$.
  $\Rightarrow$ We have shown that $\alpha_\varphi \geq 1 - ce^{-1}$, c curvature of C^φ.
- [Barman et al., 2020]: ℓ-MultiCoverage problem:
  $\varphi(j) := \min\{j, \ell\}$, ratio $1 - \frac{\ell^\ell e^{-\ell}}{\ell!}$, tight under UGC.
  $\Rightarrow$ One can compute $\alpha_\varphi = \alpha_\varphi(\ell) = 1 - \frac{\ell^\ell e^{-\ell}}{\ell!}$, tight if $\mathrm{P} \neq \mathrm{NP}$.
- [Dudycz et al., 2020]: For *geometrically dominant* φ, ratio $\mathbb{E}[\varphi(\mathrm{Poi}(1))]$, tight if $\varphi(n) = o(n)$ and $\mathrm{P} \neq \mathrm{NP}$.
  $\Rightarrow$ We have shown that if φ is geometrically dominant, then $\alpha_\varphi = \alpha_\varphi(1) = \mathbb{E}[\varphi(\mathrm{Poi}(1))]$.

Some particular cases

φ-MaxCoverage	$\varphi(j)$	α_φ
MaxCoverage	$\min\{j, 1\}$	$1 - e^{-1}$
ℓ-MultiCoverage	$\min\{j, \ell\}$	$1 - \frac{\ell^\ell e^{-\ell}}{\ell!}$
PAV	$\sum_{i=1}^{j} \frac{1}{i}$	$\alpha_\varphi(1) \simeq 0.7965\ldots$
PAV capped at 3	$\sum_{i=1}^{\min\{j,3\}} \frac{1}{i}$	$\alpha_\varphi(1) \simeq 0.7910\ldots$
p-VTA	$\frac{1-(1-p)^j}{p}$	$\frac{1-e^{-p}}{p}$
0.1-VTA	$\frac{1-(1-0.1)^j}{0.1}$	$\frac{1-e^{-0.1}}{0.1} \simeq 0.9516\ldots$
0.1-VTA capped at 5	$\frac{1-(1-0.1)^{\min\{j,5\}}}{0.1}$	$\alpha_\varphi(5) \simeq 0.8470\ldots$

Table: Tight approximation ratios for particular choices of φ in the φ-MaxCoverage problem.

Linear Relaxation of φ-MaxCoverage

- *Relax and Round* strategy to achieve the ratio α_φ.

Linear Relaxation of φ-MaxCoverage

- *Relax and Round* strategy to achieve the ratio α_φ.
- We consider φ on $\mathbb{R}^+$, by extending it piecewise linearly:

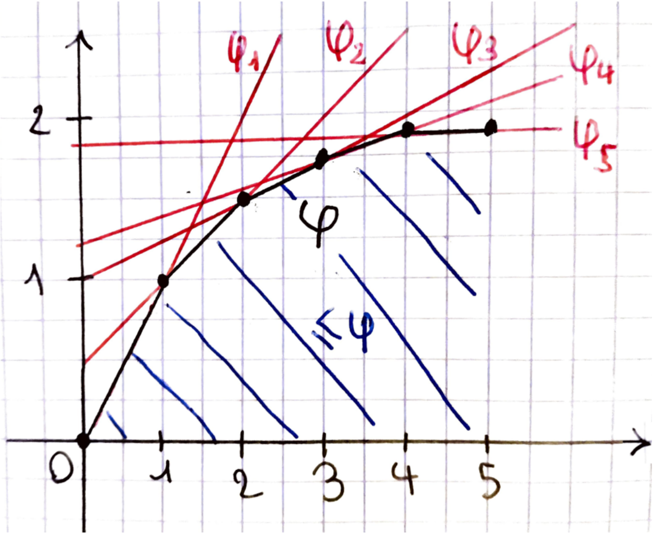
$$\begin{aligned}
\text{maximize} \quad & \sum_{a \in [n]} w_a c_a \\
\text{subject to} \quad & c_a \le \varphi(|x|_a), \forall a \in [n], \ \text{with } |x|_a := \sum_{i \in [m] : a \in T_i} x_i \\
& 0 \le x_i \le 1, \forall i \in [m] \\
& \sum_{i=1}^{m} x_i = k \ .
\end{aligned}$$

Linear Relaxation of φ-MaxCoverage

- *Relax and Round* strategy to achieve the ratio α_φ.
- We consider φ on $\mathbb{R}^+$, by extending it piecewise linearly:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{a \in [n]} w_a c_a \\
\text{subject to} \quad & c_a \leq \varphi(|x|_a), \forall a \in [n], \text{ with } |x|_a := \sum_{i \in [m] : a \in T_i} x_i \\
& 0 \leq x_i \leq 1, \forall i \in [m] \\
& \sum_{i=1}^{m} x_i = k \ .
\end{aligned}
$$

- Note that $c_a \leq \varphi(|x|_a)$ equivalent to $c_a \leq \varphi_j(|x|_a)$ for all $j \in [m]$, with φ_j linear interpolation of $\varphi(j-1)$ and $\varphi(j)$.


φ_1
φ_2
φ_3
φ_4
φ_5
φ
$\leq \varphi$
2
1
0
1
2
3
4
5

Pipage Rounding

C^{φ} *submodular*: use *pipage rounding* to get integral solution from fractional one.

Pipage Rounding

C^{φ} *submodular*: use *pipage rounding* to get integral solution from fractional one.

Definition (Multilinear extension)

The multilinear extension $F : [0,1]^m \to \mathbb{R}$ of f is defined by: $F(x_1, \ldots, x_m) := \mathbb{E}[f(X_1, \ldots, X_m)]$, $X_i \sim \mathrm{Ber}(x_i)$ independent.

Theorem ([Ageev and Sviridenko, 2004, Vondrák, 2007])

For f submodular and F computable in polynomial time, the *pipage rounding* procedure applied on a fractional solution x gives in polynomial time an integer solution x^{int} with $F(x^{\text{int}}) \geq F(x)$.

Pipage Rounding

C^φ *submodular*: use *pipage rounding* to get integral solution from fractional one.

Definition (Multilinear extension)

The multilinear extension $F : [0,1]^m \to \mathbb{R}$ of f is defined by: $F(x_1, \ldots, x_m) := \mathbb{E}[f(X_1, \ldots, X_m)]$, $X_i \sim \mathrm{Ber}(x_i)$ independent.

Theorem ([Ageev and Sviridenko, 2004, Vondrák, 2007])

For f submodular and F computable in polynomial time, the *pipage rounding* procedure applied on a fractional solution x gives in polynomial time an integer solution x^{int} with $F(x^{\mathrm{int}}) \geq F(x)$.

Applied to our setting, with x^* an optimal fractional solution:

$$C^\varphi(x^{\mathrm{int}}) \geq \mathbb{E}_{X\sim \mathrm{Ber}(x^*)}[C^\varphi(X)] \, .$$

Approximation Guarantee Theorem

Theorem

Let x, c be a feasible solution of our linear relaxation and $X \sim \mathrm{Ber}(x)$. We have:

$$\mathbb{E}_{X \sim \mathrm{Ber}(x)}[C^{\varphi}(X)] \geq \left(\min_{j \in [m]} \alpha_{\varphi}(j) \right) \sum_{a \in [n]} w_a c_a \ .$$

In particular, this implies that the described polynomial time algorithm has an approximation ratio of α_{φ}:

$$C^{\varphi}(x^{\mathrm{int}}) \underset{\text{Rounding}}{\overset{\text{Pipage}}{\geq}} \mathbb{E}_{X \sim \mathrm{Ber}(x^*)}[C^{\varphi}(X)] \overset{\text{AGT}}{\geq} \alpha_{\varphi} \sum_{a \in [n]} w_a c_a^*$$

$$\underset{\text{Relax}}{\geq} \alpha_{\varphi} \max_{S \subseteq [m] : |S| = k} C^{\varphi}(S) \ .$$

Key lemma

Lemma

For φ concave, and $p \in [0,1]^m$, we have
$\mathbb{E}\Big[\varphi\Big(\sum_{i=1}^m \mathrm{Ber}(p_i)\Big)\Big] \geq \mathbb{E}\Big[\varphi\Big(\mathrm{Poi}\Big(\sum_{i=1}^m p_i\Big)\Big)\Big]$.

Key lemma

Lemma

For φ concave, and $p \in [0,1]^m$, we have
$\mathbb{E}\left[\varphi\left(\sum_{i=1}^m \mathrm{Ber}(p_i)\right)\right] \geq \mathbb{E}\left[\varphi\left(\mathrm{Poi}\left(\sum_{i=1}^m p_i\right)\right)\right].$

- Notion of *convex order* used to prove this result:
  $X \leq_{\mathrm{cx}} Y \iff \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for any convex f.

Key lemma

Lemma

For φ concave, and $p \in [0,1]^m$, we have
$\mathbb{E}\left[\varphi\left(\sum_{i=1}^m \mathrm{Ber}(p_i)\right)\right] \geq \mathbb{E}\left[\varphi\left(\mathrm{Poi}\left(\sum_{i=1}^m p_i\right)\right)\right]$.

- Notion of *convex order* used to prove this result: $X \leq_{\mathrm{cx}} Y \iff \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for any convex f.
- We have $\forall p_i \in [0,1], \mathrm{Ber}(p_i) \leq_{\mathrm{cx}} \mathrm{Poi}(p_i)$.
- Convex order is preserved through convolution.
- $\sum_{i=1}^m \mathrm{Poi}(p_i) \sim \mathrm{Poi}\left(\sum_{i=1}^m p_i\right)$. So:

$$\sum_{i=1}^m \mathrm{Ber}(p_i) \leq_{\mathrm{cx}} \mathrm{Poi}\left(\sum_{i=1}^m p_i\right).$$

Key lemma

Lemma

For φ concave, and $p \in [0,1]^m$, we have
$\mathbb{E}\Big[\varphi\Big(\sum_{i=1}^m \mathrm{Ber}(p_i)\Big)\Big] \geq \mathbb{E}\Big[\varphi\Big(\mathrm{Poi}\left(\sum_{i=1}^m p_i\right)\Big)\Big]$.

- Notion of *convex order* used to prove this result:
  $X \leq_{\mathrm{cx}} Y \iff \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for any convex f.
- We have $\forall p_i \in [0,1], \mathrm{Ber}(p_i) \leq_{\mathrm{cx}} \mathrm{Poi}(p_i)$.
- Convex order is preserved through convolution.
- $\sum_{i=1}^m \mathrm{Poi}(p_i) \sim \mathrm{Poi}\left(\sum_{i=1}^m p_i\right)$. So:

$$\sum_{i=1}^m \mathrm{Ber}(p_i) \leq_{\mathrm{cx}} \mathrm{Poi}\left(\sum_{i=1}^m p_i\right).$$

- Apply this to convex $-\varphi$.

Resource Allocation in Multiagent Systems

- Algorithmic game theory: maximizing welfare among multiple agents [Paccagnan and Marden, 2018].
- φ-Resource Allocation problem:
  1. n resources,
  2. k agents,
  3. agent i has resources $\mathcal{A}_i = \{T_1^i, \dots, T_{m_i}^i\}$, with $T_j^i \subseteq [n]$,
  4. a counting function $\varphi : \mathbb{N} \to \mathbb{R}_+$,

  Goal: maximize $W^\varphi(A_1, A_2, \dots, A_k) := \sum_{a\in[n]} w_a \, \varphi(|A|_a)$ with $A_i \in \mathcal{A}_i$ and $|A|_a := |\{i \in [k] : a \in A_i\}|$.

Resource Allocation in Multiagent Systems

- Algorithmic game theory: maximizing welfare among multiple agents [Paccagnan and Marden, 2018].
- φ-Resource Allocation problem:
  1. n resources,
  2. k agents,
  3. agent i has resources $\mathcal{A}_i = \{T^i_1, \ldots, T^i_{m_i}\}$, with $T^i_j \subseteq [n]$,
  4. a counting function $\varphi : \mathbb{N} \to \mathbb{R}_+$,

  Goal: maximize $W^\varphi(A_1, A_2, \ldots, A_k) := \sum_{a\in[n]} w_a \, \varphi(|A|_a)$ with $A_i \in \mathcal{A}_i$ and $|A|_a := |\{i \in [k] : a \in A_i\}|$.

- φ-Resource Allocation is a φ-MaxCoverage instance subject to a *partition matroid* constraint.
- With adapted hardness proof, we get the same theorem as for φ-MaxCoverage.

An example: Vehicle-Target Assignment [Murphey, 2000]

- Particular case of φ-Resource Allocation:
    1. resources correspond to targets $[n]$,
    2. agents correspond to vehicles $[k]$,
    3. vehicle i has several possible assignments, described as target covering sets $\mathcal{A}_i \subseteq 2^{[n]}$,
    4. $\varphi^p(j) = \frac{1-(1-p)^j}{p}$, for some $p \in (0,1)$.

An example: Vehicle-Target Assignment [Murphey, 2000]

- Particular case of φ-Resource Allocation:
  1. resources correspond to targets $[n]$,
  2. agents correspond to vehicles $[k]$,
  3. vehicle i has several possible assignments, described as target covering sets $\mathcal{A}_i \subseteq 2^{[n]}$,
  4. $\varphi^p(j) = \frac{1-(1-p)^j}{p}$, for some $p \in (0,1)$.

- φ^p is nondecreasing concave and $\varphi(n) = o(n)$: we get the tight ratio $\alpha_{\varphi^p} = \frac{1-e^{-p}}{p}$.

- Capped version of this problem: $\varphi^p_\ell(j) := \varphi^p(\min\{j,\ell\})$. In particular, we recover the ℓ-MultiCoverage function when $p = 0$ and MaxCoverage for $p = 1$.

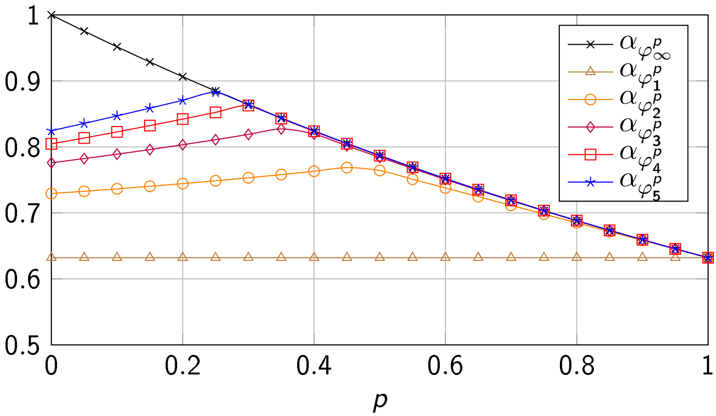


Figure: Tight approximation ratios $\alpha_{\varphi^p_\ell}$, where ℓ is the rank of the capped version of the p-Vehicle-Target Assignment problem. When $p = 0$, we recover the ℓ-MultiCoverage problem.

Links to *Price of Anarchy*

- [Paccagnan and Marden, 2018]: game-theoretic aspects of φ-Resource Allocation problem, and in particular VTA.
- Goal: bound welfare loss due to self-interested choice of $A_i \in \mathcal{A}_i$ by each agent i, defined as the *Price of Anarchy*.
- In that setting, the PoA can be computed via linear programs.

Links to *Price of Anarchy*

- [Paccagnan and Marden, 2018]: game-theoretic aspects of φ-Resource Allocation problem, and in particular VTA.
- Goal: bound welfare loss due to self-interested choice of $A_i \in \mathcal{A}_i$ by each agent i, defined as the *Price of Anarchy*.
- In that setting, the PoA can be computed via linear programs.

- Our hardness result $\Rightarrow$ upper bounds on those PoA.
- [Chandan et al., 2019]: in the particular case of the ℓ-MultiCoverage problem, PoA $= \alpha_\varphi$.
- However, numerically comparing α_φ for VTA with the optimal PoA bound, α_φ can in fact be strictly greater than the PoA guarantee.

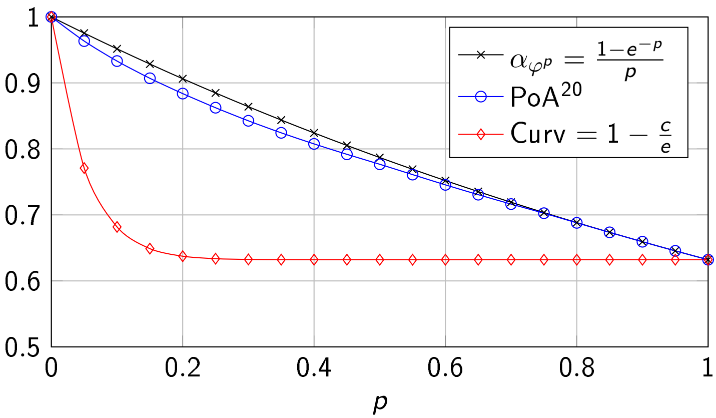


Figure: Comparison between the PoA and α_φ for the Vehicle-Target Assignment problem. Since the PoA only decreases when the number of players grows, this means that PoA $< \alpha_\varphi$ in that case.

Conclusion and Open Questions

- φ-MaxCoverage: generalization of MaxCoverage where having c copies of element a gives a value $\varphi(c)$.
- For nondecreasing concave φ, approximation guarantee given by the *Poisson concavity ratio* $\alpha_\varphi := \min_{x\in\mathbb{N}^*} \frac{\mathbb{E}[\varphi(\mathsf{Poi}(x))]}{\varphi(\mathbb{E}[\mathsf{Poi}(x)])}$.
- Tight for sublinear functions $\varphi(n) = o(n)$.
- Beats the curvature bound as soon as $\varphi(t) \neq 1 + (1-c)t$, where they are equal.

Conclusion and Open Questions

- φ-MaxCoverage: generalization of MaxCoverage where having c copies of element a gives a value $\varphi(c)$.
- For nondecreasing concave φ, approximation guarantee given by the *Poisson concavity ratio* $\alpha_\varphi := \min_{x\in\mathbb{N}^*} \frac{\mathbb{E}[\varphi(\mathrm{Poi}(x))]}{\varphi(\mathbb{E}[\mathrm{Poi}(x)])}$.
- Tight for sublinear functions $\varphi(n) = o(n)$.
- Beats the curvature bound as soon as $\varphi(t) \neq 1 + (1-c)t$, where they are equal.
- Open questions:
  1. Does there exist combinatorial algorithms that achieve α_φ?
  2. Does the hardness result remain true when $\varphi(n) \neq o(n)$?

Conclusion and Open Questions

- φ-MaxCoverage: generalization of MaxCoverage where having c copies of element a gives a value $\varphi(c)$.
- For nondecreasing concave φ, approximation guarantee given by the *Poisson concavity ratio* $\alpha_\varphi := \min_{x \in \mathbb{N}^*} \frac{\mathbb{E}[\varphi(\mathrm{Poi}(x))]}{\varphi(\mathbb{E}[\mathrm{Poi}(x)])}$.
- Tight for sublinear functions $\varphi(n) = o(n)$.
- Beats the curvature bound as soon as $\varphi(t) \neq 1 + (1 - c)t$, where they are equal.
- Open questions:
  1. Does there exist combinatorial algorithms that achieve α_φ?
  2. Does the hardness result remain true when $\varphi(n) \neq o(n)$?
- Thank you for listening!

Bibliography I

Ageev, A. A. and Sviridenko, M. (2004).
Pipage rounding: A new method of constructing algorithms with proven performance guarantee.
J. Comb. Optim., 8(3):307–328.

Barman, S., Fawzi, O., Ghoshal, S., and Gürpinar, E. (2020).
Tight approximation bounds for maximum multi-coverage.
In Bienstock, D. and Zambelli, G., editors, *Integer Programming and Combinatorial Optimization - 21st International Conference, IPCO 2020, London, UK, June 8-10, 2020, Proceedings*, volume 12125 of *Lecture Notes in Computer Science*, pages 66–77. Springer.

Bibliography II

Chandan, R., Paccagnan, D., and Marden, J. R. (2019).
Optimal mechanisms for distributed resource-allocation.
CoRR, abs/1911.07823.

Dudycz, S., Manurangsi, P., Marcinkowski, J., and Sornat, K. (2020).
Tight approximation for proportional approval voting.
In Bessiere, C., editor, *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020*, pages 276–282. ijcai.org.

Feige, U. (1998).
A threshold of ln n for approximating set cover.
J. ACM, 45(4):634–652.

Bibliography III

Hochbaum, D. S. (1997).
Approximation algorithms for NP-hard problems.
SIGACT News, 28(2):40–52.

Murphey, R. A. (2000).
Target-based weapon target assignment problems.
In *Nonlinear assignment problems*, pages 39–53. Springer.

Paccagnan, D. and Marden, J. R. (2018).
Utility design for distributed resource allocation–part II: Applications to submodular, covering, and supermodular problems.
CoRR, abs/1807.01343.

Bibliography IV

Sviridenko, M., Vondrák, J., and Ward, J. (2017).
Optimal approximation for submodular and supermodular optimization with bounded curvature.
Math. Oper. Res., 42(4):1197–1218.

Vondrák, J. (2007).
Submodularity in Combinatorial Optimization.
Univerzita Karlova, Matematicko-Fyzikální Fakulta.