Algorithmic aspects of Optimal Channel Coding: A geometric point of view

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Introduction

- How much classical information can we send through quantum channels?
- Shannon and Holevo's *channel capacity*: average number of bits per channel we can faithfully transmit in the limit of taking n channel copies.

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- Our setting: what is the *best* strategy (maximizing the probability of success) of sending k equiprobable messages through *one* channel.
- How efficiently can we compute a *good* strategy?

Contents

Channel Coding Function

Definition (Channel Coding Function)

$W = \{W_x\}_{x \in X}$ a classical-quantum channel, ie. a finite set of quantum states. $S \subseteq X$ a code for $|S|$ messages, best POVM gives

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\begin{aligned}
g_W(S) := \quad \underset{\Lambda}{\text{maximize}} \quad & \frac{1}{|S|} \sum_{x \in S} \mathrm{Tr}(\Lambda_x W_x) \\
\text{subject to} \quad & \sum_{x \in S} \Lambda_x = \mathbb{1} \\
& \Lambda_x \succcurlyeq 0, \forall x \in S
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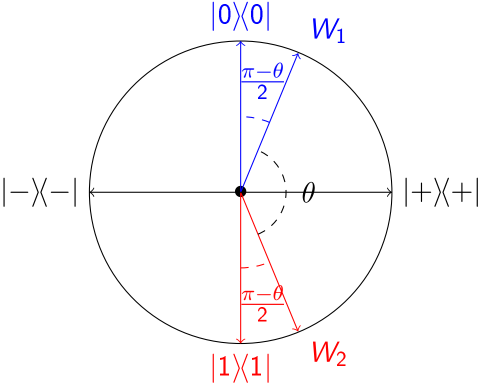
The probability of success of the best strategy is given by:

$$S(W, k) := \frac{1}{k} \max_{S \subseteq X, |S| \leq k} f_W(S)$$

First examples on the Bloch sphere



Any S such that $|S| = 1$

Best POVM = $\mathbb{1}$

$f(S) = 1, S(W,1) = 1$

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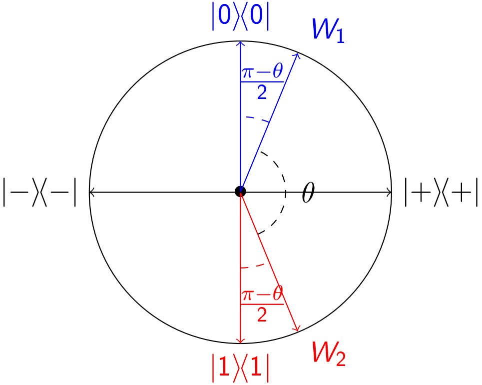

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Best POVM $= \mathbb{1}$
$f(S) = 1, S(W, 1) = 1$

$W = \{W_1, W_2\}$ pure states.
We take $S = \{1, 2\}$.
Best POVM is $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$:

$$f_W(S) = 2\cos^2\left(\frac{\pi-\theta}{4}\right)$$
$$= 1 + \sin\frac{\theta}{2}$$
$$S(W, 2) = \frac{1 + \sin\frac{\theta}{2}}{2}$$

Dual point of view

Finding a measurement is hard, we look at the (SDP) dual POV:

Property (Strong Duality)

$$
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Property ([Bae, 2013])

The optimal matrix ρ is unique, whereas the optimal POVM Λ isn't (proved via KKT conditions: primal and dual constraints mixed).

Remark

The problem of computing one image of f_W is known as *minimum-error quantum state discrimination*.

Geometric point of view

Definition (Penumbra (inspired by [Burgeth et al., 2007]))

$P(M) := \{N \in \mathcal{H}_d \text{ s.t. } N \preccurlyeq M\}$. It is a proper affine cone.

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Property

$Q_d := \mathcal{D}_d - \mathbb{1}_d$ centered set of states.

Then $\mathcal{D}(m, r) := m - rQ_d$ with $m \in T_1 := \{\mathrm{tr}(M) = 1\} \cap \mathcal{H}_d$ is the base of $P(m + r\mathbb{1}_d)$ included in T_1.

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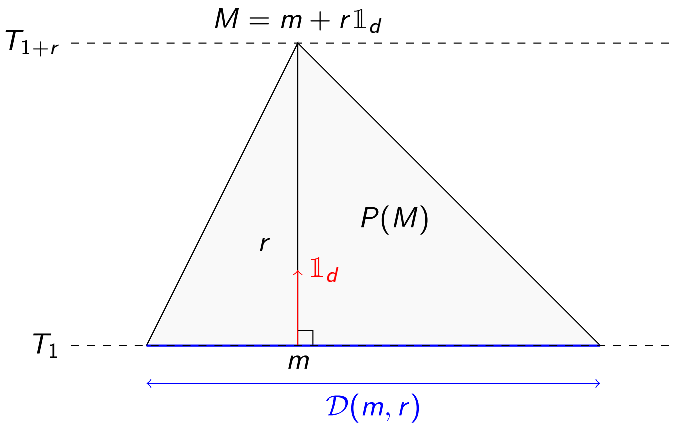
In particular for $W \in \mathcal{D}_d$ and any M with $\mathrm{tr}(M) \geq 1$:

$$W \in \mathcal{D}(m, r) \iff m + r\mathbb{1}_d \succcurlyeq W$$

$$M \succcurlyeq W \iff W \in \mathcal{D}(M - r_M\mathbb{1}_d, r_M) \text{ where } r_M := \mathrm{tr}(M) - 1$$

Drawing



Equivalent geometric formulation

Let W a finite set of points in $\mathcal{D}_d$, and S a subset of its indices:

MinTr: Find the minimum trace matrix ρ such that $\forall x \in S, \rho \succcurlyeq W_x$.

MEQP: Find the smallest $r \geq 0$ and $m \in T_1$ s.t. $\{W_x\}_{x \in S} \subseteq \mathcal{D}(m, r)$.

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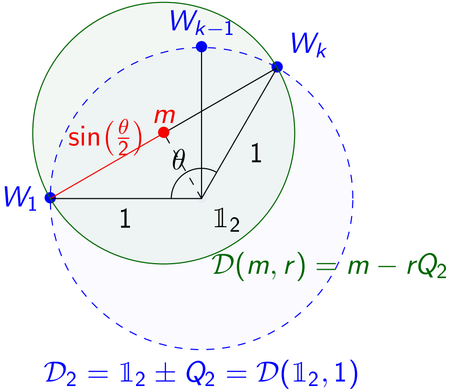
Theorem

ρ the solution of MinTr and m, r a solution of MEQP for W, S.
Then $\rho = m + r\mathbb{1}_d$. In particular, $f_W(S) = \mathrm{tr}(\rho) = 1 + r$.

Application in the Bloch sphere



For a set of k pure states:

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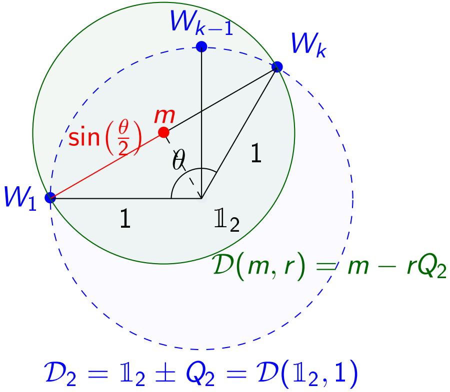

For a set of k pure states:

- All of them in one hemisphere: $r = \sin\left(\frac{\theta}{2}\right)$, so $f_W(S) = 1 + \sin\left(\frac{\theta}{2}\right)$.

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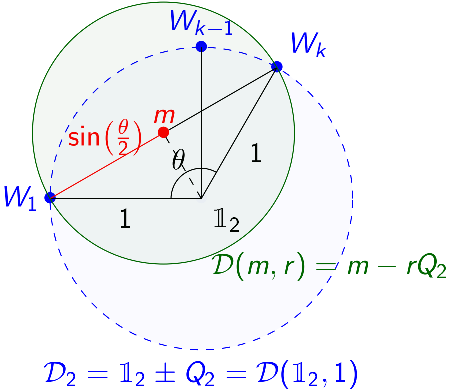

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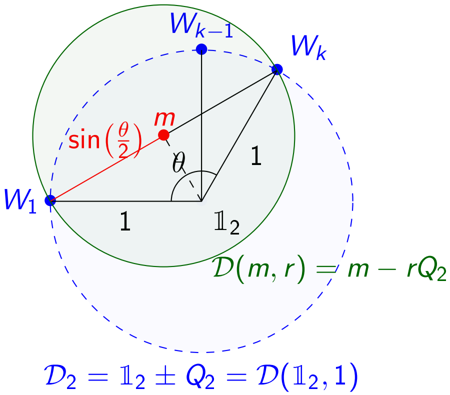

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We recover in particular the example of the beginning.

General Convex Problem

For $d > 2$, $\mathcal{D}_d$ not a ball, but still a *convex body*: a full-dimensional compact convex set in $\mathbb{R}^{d^2-1}$

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MECP: [Brandenberg and König, 2013] $P \subseteq \mathbb{R}^n$ compact (finite for us), $C \subseteq \mathbb{R}^n$ convex body, find the least dilatation factor $r \geq 0$, such that a translate of rC contains P:

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\begin{aligned}
R(P, C) := \quad & \text{minimize} \quad && r \\
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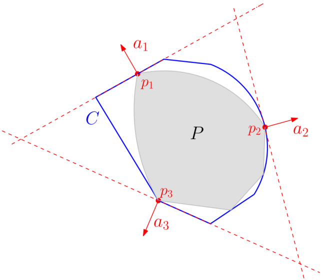
In particular, MEQP is the case where we take points $P \subseteq \mathcal{D}_d \subseteq \mathbb{R}^{d^2-1}$ and we take $C = -Q_d = \mathbb{1}_d - \mathcal{D}_d$.
In particular $f_W(S) = 1 + R(\{W_x\}_{x \in S}, -Q_d)$.

Optimality conditions

Theorem (Optimality conditions)

P is optimally contained in C iff:

1. $P \subseteq C$
2. For some $2 \leq k \leq n+1$, there exist $p_1, \ldots, p_k \in P$ and hyperplanes $H(a_i, 1)$ supporting P and C in p_i s.t. $0 \in \text{conv}\{a_1, \ldots, a_k\}$.



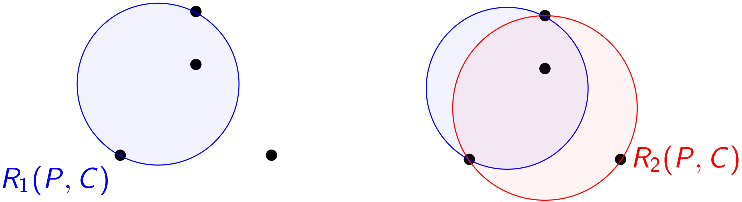
Core-radii and $S(W, k)$

Definition (Core-radii)

The *k-th core-radius* of P with respect to C is defined by:

$$R_k(P, C) := \max_{S \subseteq P, |S| \leq k+1} R(S, C)$$

In particular: $S(W, k) = \frac{1}{k}(1 + R_{k-1}(W, -Q_d))$



Properties on core-radii

Theorem (Helly's theorem)

If $\dim(P) \leq k$, then $R_k(P, C) = R(P, C)$. For us: d^2 measurement outputs used at most.

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Theorem (R_k/k decreases)

$\left(\dfrac{R_k(P, C)}{k}\right)_{k \in [n]}$ decreases. For us: $\left(\dfrac{S(W, k) - \frac{1}{k}}{1 - \frac{1}{k}}\right)_{k \in [n]}$ decreases.

R_k/k decreases I

- Let us show that $\left(\frac{R_k(P,C)}{k}\right) \leq \left(\frac{R_{k-1}(P,C)}{k-1}\right)$.

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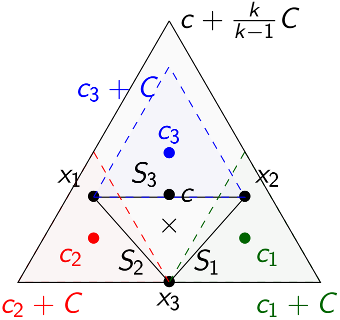
- $S = \{x_1, \dots x_{k+1}\} \subseteq P$ s.t. $R(S, C) = R_k(P, C)$: if it does not exist, then $R_k(P, C) = R_{k-1}(P, C)$ and we are done.
- We suppose $\sum_{i=1}^{k+1} x_i = 0$ and $R_{k-1}(P, C) = 1$.

- We will show that $R(S, C) \leq \frac{k}{k-1}$.

R_k/k decreases II



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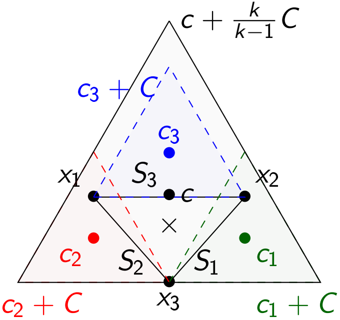

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- $-\frac{1}{k}x_j = \frac{1}{k}\sum_{i \neq j} x_j \in S_j$
  $x_j \in S_i$ for $i \neq j$

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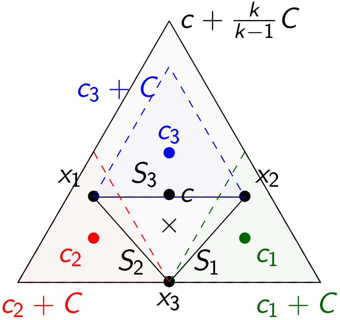

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- $S_i \subseteq c_i + C$ so $(k - \frac{1}{k})x_j \in \sum_{i=1}^{k+1} S_i \subseteq \sum_{i=1}^{k+1} c_i + (k+1)C$ by convexity of C.

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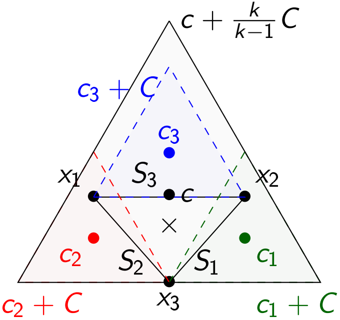

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- Implies $R(S, C) \leq \frac{k+1}{k-\frac{1}{k}} = \frac{k}{k-1}$
  with $c = \frac{k}{k-1}\sum_{i=1}^{k+1}\frac{1}{k+1}c_i$ QED.

Greedy algorithm

Algorithm 1: Greedy algorithm

Input: $k \in \{1, \dots, n\}$

Output: An *approximation* of $R_{k-1}(P, C)$: a set S_k of size k such that $R_{k-1}(P, C) \leq (1 + c)R(S_k, C)$

$S_0 = \emptyset$

for $i \in \{1, \dots, k\}$ **do**

  $p_i = \underset{p \in P - S_{i-1}}{\operatorname{argmax}} \; R(S_{i-1} \cup \{p\}, C)$

  $S_i = S_{i-1} \cup \{p_i\}$

return S_k

We assume we have an *oracle* that computes $R(S, C)$ for cost 1. Thus the greedy algorithm is polynomial. Does it give a good approximation ratio $1 + c$?

Regular simplex

- Take points in T^n and $C = -T^n$ where T^n is the n-regular simplex centered in 0. Previous drawing was T^2.
- Corresponds to the classical case of our channel problem: T^n is the space of probability distributions of dimension $n+1$.

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- In [Barman and Fawzi, 2017], it was shown that the greedy algrithm has an approximation ratio $1 + c = \frac{1}{1-e^{-1}}$ and that it is optimal if $P \neq NP$.
- This was done through the *submodularity* of f_W, which is the discrete equivalent of concavity.
- However, f_W is not submodular even for one qubit.

Ball

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Definition (Core-set)

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Theorem ([Badoiu and Clarkson, 2003])

An ϵ-core-set S_ϵ of $P \subseteq \mathbb{R}^n$ of size $\lceil \frac{2}{\epsilon} \rceil$ can be efficiently computed.
In particular, this gives a $(1 + \frac{2}{k})$-approximation of $R_k(P, C)$.

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Proof of Corollary.

Let ϵ such that $k + 1 = \lceil \frac{2}{\epsilon} \rceil = |S_\epsilon|$. Then $\epsilon \leq \frac{2}{k}$, so
$R_k(P, C) \leq R(P, C) \leq (1 + \epsilon)R(S_\epsilon, C) \leq (1 + \frac{2}{k})R(S_\epsilon, C)$. $\square$

Algorithm

Algorithm 2: [Badoiu and Clarkson, 2003] algorithm

Input: $k \in \{1, \dots, n\}$

Output: An $\frac{2}{k}$-core-set S_k of $P \subseteq \mathbb{R}^n$ of size $k+1$

$S_0 = \{p\}$ with p any point of P

$c_0 = p, r_0 = 0$ `// Center and radius of MEB of` S_0

for $i \in \{1, \dots, k\}$ **do**

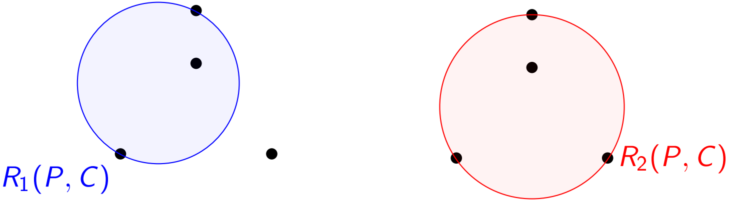
  $q_i =$ furthest point from c_{i-1} in P

  $S_i = S_{i-1} \cup \{q_i\}$

  c_i, r_i such that $\mathrm{MEB}(S_i) = \mathcal{B}(c_i, r_i)$

return S_k

Example



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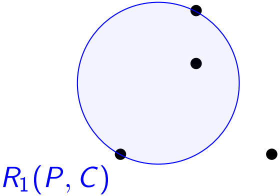

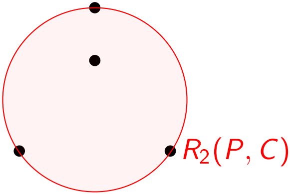

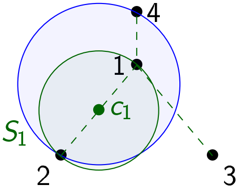

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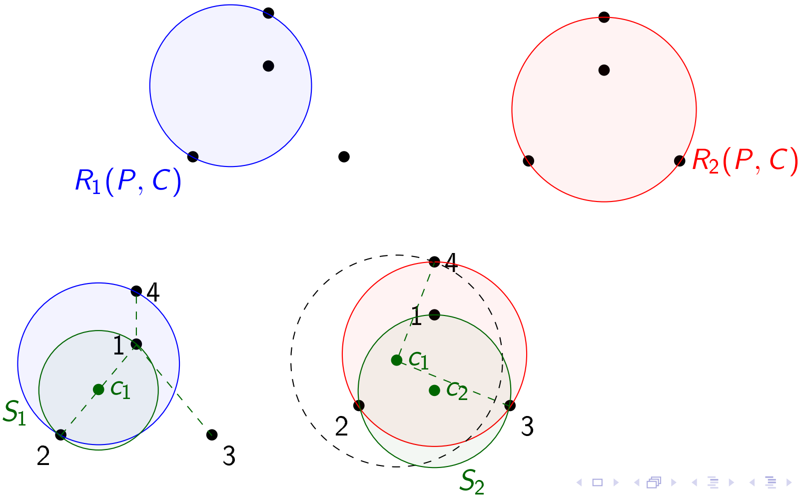

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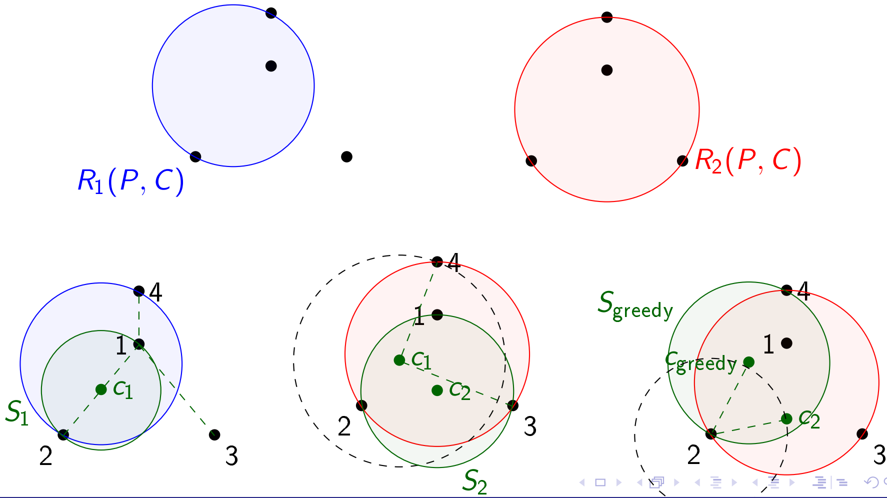

Key lemmas of the proof

Theorem on MECP optimality conditions restated for MEB gives:

Lemma (Half-space lemma)

$\mathcal{B}(P)$ is the minimum enclosing ball of $P \subseteq \mathbb{R}^n$ iff any closed half-space that contains the center $c_{\mathcal{B}(P)}$ also contains a point of P on the boundary of $\mathcal{B}(P)$.

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- Else, take $H \ni c_i$ with $H \perp \overrightarrow{c_i c_{i+1}}$, and $p \in \partial\mathcal{B}(S_i) \cap H^+ \cap S_i$
- Get two inequalities:
    1. $r_{i+1} \geq \|c_{i+1} - p\|_2 \geq \sqrt{r_i^2 + \|c_{i+1} - c_i\|_2^2}$
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  2. $r_{i+1} \geq \|c_{i+1} - q_{i+1}\|_2 \geq R - \|c_{i+1} - c_i\|_2$
- Implies that after $k = \lceil \frac{2}{\epsilon} \rceil$ steps, we get $R \leq (1+\epsilon) r_k$.

Optimality of this algorithm

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- Indeed, if $\epsilon > \frac{2}{k}$, then $k < \frac{2}{\epsilon}$ is of bounded size: exhaustive search of all sets of size k: $\mathcal{O}(|P|^{\frac{2}{\epsilon}})$ efficient for us.
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- FPTAS (ie. polynomial in $\frac{1}{\epsilon}$) exists?
- No! Reduction form *Exact cover by 3-sets* shows NP-hardness and that no FPTAS exists if $P \neq NP$.

Core-sets in general

Definition (Core-set)

$S \subseteq P$ is a ϵ-*core-set* of P if $R(S, C) \leq R(P, C) \leq (1 + \epsilon)R(S, C)$
Valid definition in general problem MECP.

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Not efficient in general:

Theorem ([Brandenberg and König, 2013])

For all $P, C \subseteq \mathbb{R}^n, \epsilon \geq 0$, there exists an ϵ-core-set of size at most $\lceil \frac{n}{1+\epsilon} \rceil + 1$.
Moreover, for any $\epsilon < 1$ there exist $P \subseteq \mathbb{R}^n$ and a 0-symmetric convex body C (ie. $C = -C$) such that no smaller subset of P suffices.
Also, this bound is optimal for the n-regular simplex.

Quantum states

- We look at $Q_d = \mathcal{D}_d - \mathbb{1}_d \subseteq \mathbb{R}^n$ with $n = d^2 - 1$.
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  if $k < \lceil \frac{d-1}{1+\epsilon} \rceil$, then for $|S| \leq k+1$: $R(\mathcal{D}_d, -Q_d) =$
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- ...?

Open questions

- Simplex worst convex shape and ball best convex shape?
- Greedy works well in general (constant approximation ratio)?
- If not in general, what properties do quantum states have that make greedy works?
  (experimentally we were unable to find counter-examples)

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