Beating the Sum-Rate Capacity of the Binary Adder Channel with Non-Signaling Correlations

Omar Fawzi, **Paul Fermé**

arXiv:2206.10968

June 30, 2022

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2. Multiple-access channels:
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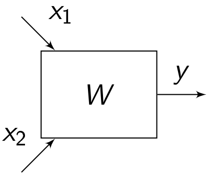
3. Our contribution:
   - Efficient algorithm computing lower bounds on non-signaling assisted capacity of MACs.
   - Increase capacity of the binary adder channel.

The Multiple-Access Channel coding problem

- Problem $\mathrm{S}(W, k_1, k_2)$: send k_1 (resp. k_2) messages through sender 1 (resp. 2) with encoding procedure $e_1 : [k_1] \to \mathcal{X}_1$ (resp. $e_2 : [k_2] \to \mathcal{X}_2$) and decoding $d : \mathcal{Y} \to [k_1] \times [k_2]$.
- Objective: maximize probability of success.



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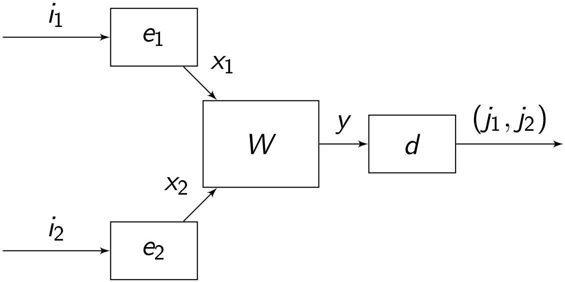

$\mathrm{S}(W, k_1, k_2)$ written as an optimization program

$$
\begin{aligned}
\underset{e_1,e_2,d}{\text{maximize}} \quad & \frac{1}{k_1 k_2} \sum_{x_1,x_2,y,i_1,i_2} W(y|x_1x_2)e_1(x_1|i_1)e_2(x_2|i_2)d(i_1i_2|y) \\
\text{subject to} \quad & \sum_{x_1 \in \mathcal{X}_1} e_1(x_1|i_1) = 1, \forall i_1 \in [k_1] \\
& \sum_{x_2 \in \mathcal{X}_2} e_2(x_2|i_2) = 1, \forall i_2 \in [k_2] \\
& \sum_{j_1 \in [k_1], j_2 \in [k_2]} d(j_1j_2|y) = 1, \forall y \in \mathcal{Y} \\
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\end{aligned}
$$

- [Barman and Fawzi, 2018]: $\mathrm{S}(W, k)$ is NP-hard to approximate within ratio $> 1 - e^{-1}$, so is $\mathrm{S}(W, k_1, k_2)$.

Capacity regions

Definition (Capacity region $\mathcal{C}(W)$ of a MAC W)

A rate pair (R_1, R_2) is achievable if:

$$\lim_{n \to +\infty} \mathrm{S}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 \ .$$

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$\mathcal{C}(W)$ is the set of all achievable rate pairs.

Theorem ([Liao, 1973, Ahlswede, 1973])

$\mathcal{C}(W)$ is the closure of all rate pairs (R_1, R_2) such that:

$$R_1 \leq I(X_1 : Y|X_2) \ , \ R_2 \leq I(X_2 : Y|X_1) \ , \ R_1 + R_2 \leq I((X_1 X_2) : Y) \ ,$$

for inputs $(X_1, X_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ following a product law $P_{X_1} \times P_{X_2}$, and $Y \in \mathcal{Y}$ the outcome of W.

Capacity regions

Definition (Capacity region $\mathcal{C}_0(W)$ of a MAC W)

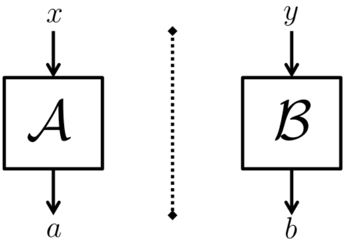
A rate pair (R_1, R_2) is achievable with zero-error if:

$$\exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, \mathrm{S}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 \ .$$

$\mathcal{C}_0(W)$ is the set of all achievable rate pairs with zero-error.

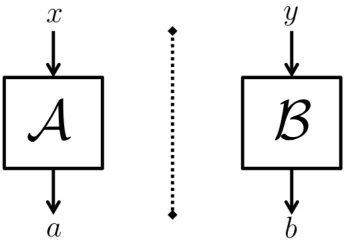
- No theorem characterizing $\mathcal{C}_0(W)$ so far.

Non-signaling correlations



What form can take $p(ab|xy)$?

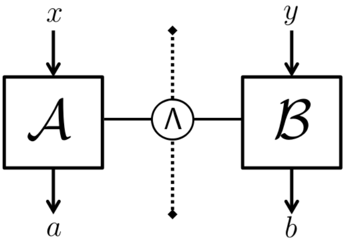
Non-signaling correlations



What form can take $p(ab|xy)$?

- Nothing more given : $p(ab|xy) = \mathcal{A}(a|x) \times \mathcal{B}(b|y)$

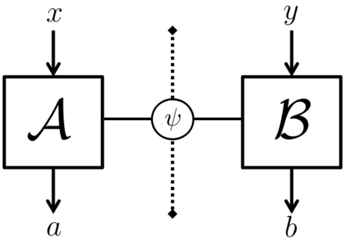
Non-signaling correlations



What form can take $p(ab|xy)$?

- Shared RV Λ : $p(ab|xy) = \sum_\lambda \Lambda(\lambda)\mathcal{A}_\lambda(a|x)\mathcal{B}_\lambda(b|y)$

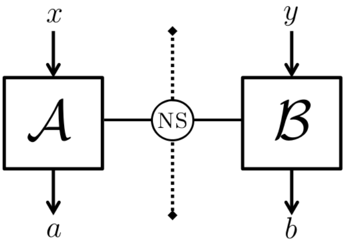
Non-signaling correlations



What form can take $p(ab|xy)$?

- Shared quantum entangled ψ : $p(ab|xy) = \langle\psi| \mathcal{A}_a^x \otimes \mathcal{B}_b^y |\psi\rangle$

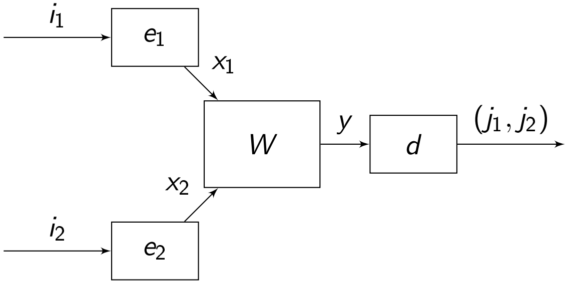
Non-signaling correlations



What form can take $p(ab|xy)$?

- **Any** $p(ab|xy)$ **s.t.** $p(a|xy) = p(a|xy')$ and $p(b|xy) = p(b|x'y)$

The non-signaling assisted MAC coding problem



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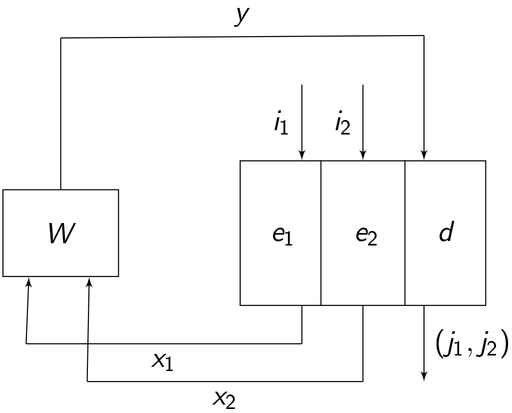

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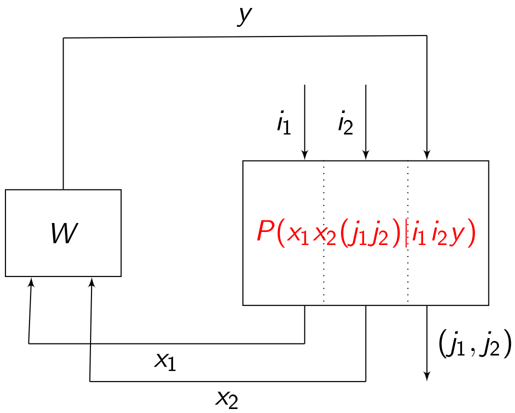

$\mathrm{S}^{\mathrm{NS}}(W, k_1, k_2)$ written as a *linear* program

$$\begin{aligned}
\underset{P}{\text{maximize}} \quad & \frac{1}{k_1 k_2} \sum_{x_1, x_2, y, i_1, i_2} W(y|x_1 x_2) {\color{red} P(x_1 x_2 (i_1 i_2)|i_1 i_2 y)} \\
\text{subject to} \quad & \sum_{x_1} P(x_1 x_2 (j_1 j_2)|i_1 i_2 y) = \sum_{x_1} P(x_1 x_2 (j_1 j_2)|i_1' i_2 y) \\
& \sum_{x_2} P(x_1 x_2 (j_1 j_2)|i_1 i_2 y) = \sum_{x_2} P(x_1 x_2 (j_1 j_2)|i_1 i_2' y) \\
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& \sum_{x_1, x_2, j_1, j_2} P(x_1 x_2 (j_1 j_2)|i_1 i_2 y) = 1 \\
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\end{aligned}$$

Non-signaling assisted capacity regions

Definition (Capacity region $\mathcal{C}^{\mathrm{NS}}(W)$ of a MAC W)

(R_1, R_2) achievable with non-signaling assistance if:

$$\lim_{n \to +\infty} \mathrm{S}^{\mathrm{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 \ .$$

$\mathcal{C}^{\mathrm{NS}}(W)$ is the set of all achievable rate pairs with non-signaling assistance.

Non-signaling assisted capacity regions

Definition (Capacity region $\mathcal{C}_0^{\mathrm{NS}}(W)$ of a MAC W)

(R_1, R_2) achievable with zero-error and non-signaling assistance if:

$$\exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, \mathrm{S}^{\mathrm{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 \ .$$

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Proposition (Lower bounds on $\mathcal{C}_0^{\mathrm{NS}}(W)$)

For any MAC W and integers k_1, k_2, n:

$$\mathrm{S}^{\mathrm{NS}}(W^{\otimes n}, k_1, k_2) = 1 \implies \left(\frac{\log_2(k_1)}{n}, \frac{\log_2(k_2)}{n} \right) \in \mathcal{C}_0^{\mathrm{NS}}(W) \subseteq \mathcal{C}^{\mathrm{NS}}(W) \ .$$

Symmetrization and main result

- $W^{\otimes n} := n$ independent copies of W.
- LP size $\Omega(|W|^n)$: hard to compute $\mathrm{S}^{\mathrm{NS}}(W^{\otimes n}, k_1, k_2)$ *a priori*.

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$\mathrm{S}^{\mathrm{NS}}(W^{\otimes n}, k_1, k_2)$ is the solution of a linear program of size bounded by $O\left(n^{|\mathcal{X}_1|\cdot|\mathcal{X}_2|\cdot|\mathcal{Y}|-1}\right)$: it can be computed in time poly(n).

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- Formally, for any MAC W invariant under the action of a group G, $\mathrm{S}^{\mathrm{NS}}(W, k_1, k_2)$ solution of a linear program of polynomial size in the number of orbits of that action.
- Apply to $W^{\otimes n}$ and $G = S_n$, the group of permutations on $[n]$.

The Binary Adder Channel

- For bits x_1, x_2, $W_{\mathrm{BAC}}(x_1 x_2) := x_1 + x_2$ (deterministic):

$$W_{\mathrm{BAC}}(00) = 0; W_{\mathrm{BAC}}(01) = W_{\mathrm{BAC}}(10) = 1; W_{\mathrm{BAC}}(11) = 2 \ .$$

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- Thanks to [Liao, 1973, Ahlswede, 1973] and their general single-letter formula, its classical capacity region is known:

$$\mathcal{C}(W_{\mathrm{BAC}}) = \{(R_1, R_2) : R_1 \leq 1, R_2 \leq 1, R_1 + R_2 \leq 3/2\} \ .$$

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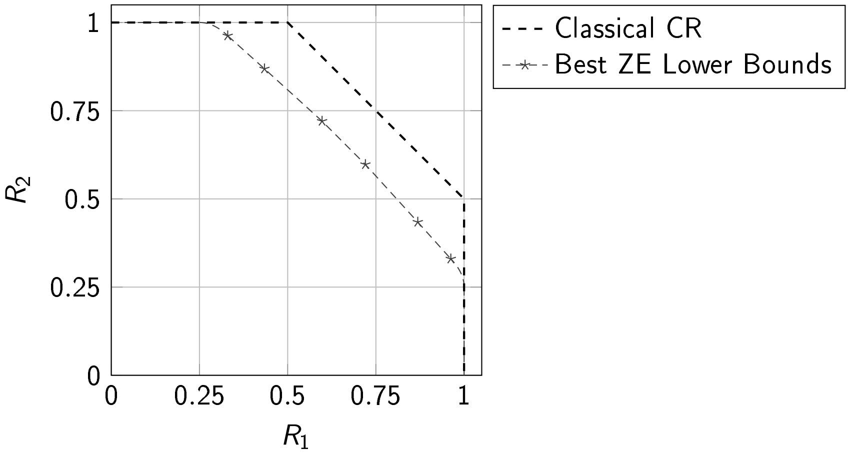
- Only lower bounds known on its zero-error capacity region.
- [Mattas and Östergård, 2005]: $\log_2(240/6) \simeq 1.3178$ best achievable sum-rate with zero-error so far.

Numerical results for W_{BAC}



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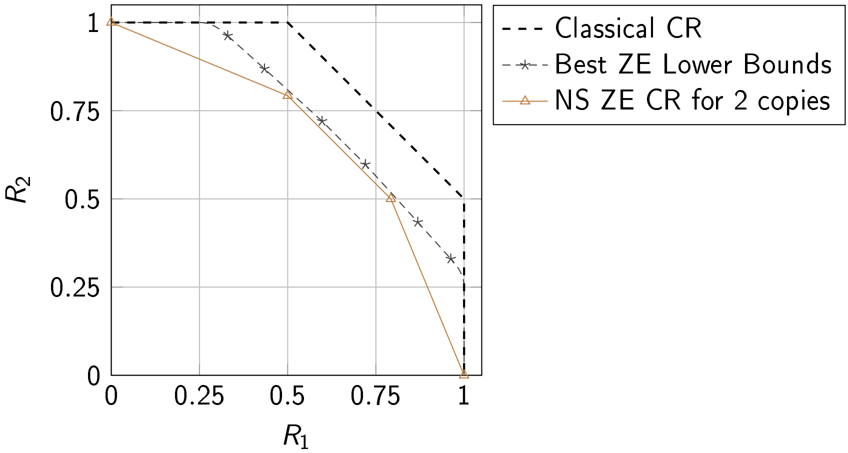

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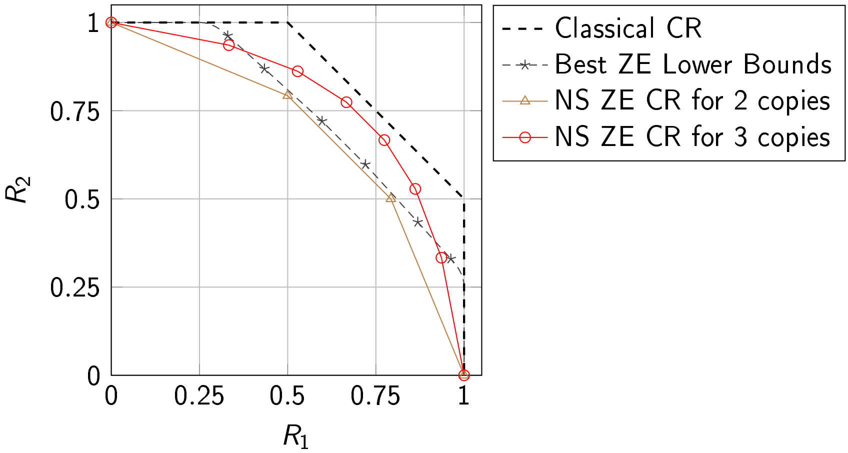

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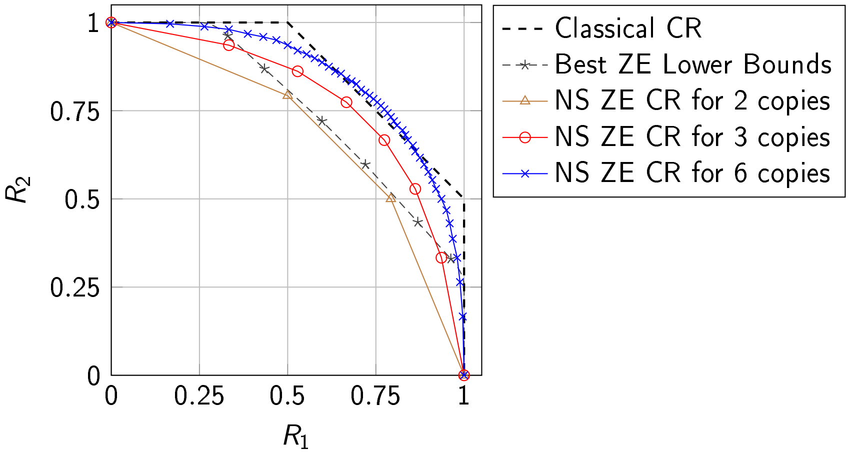
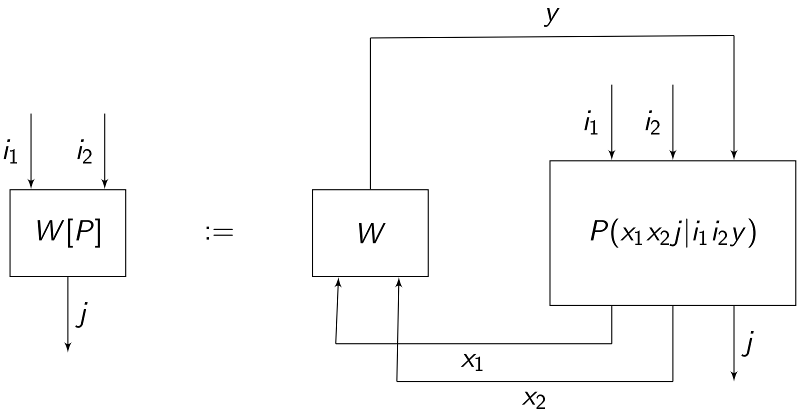


Handling errors with concatenated codes



Method to find lower bounds on $\mathcal{C}^{\mathrm{NS}}(W)$

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Proposition

The following (R_1, R_2) are in $\mathcal{C}^{\mathrm{NS}}(W)$:

$$R_1 \leq \frac{I(I_1 : J|I_2)}{n}\ , \ R_2 \leq \frac{I(I_2 : J|I_1)}{n}\ , \ R_1 + R_2 \leq \frac{I((I_1, I_2) : J)}{n}\ ,$$

for inputs $(I_1, I_2) \sim q_1 \times q_2$, and J outcome of $W^{\otimes n}[P]$.

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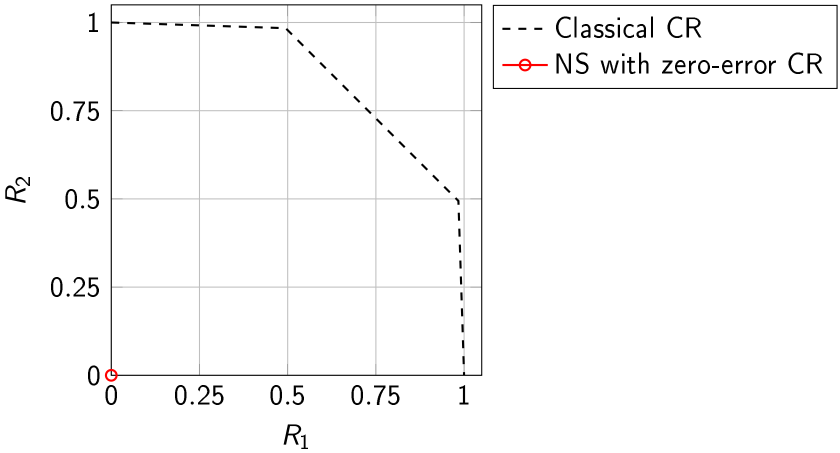
for inputs $(I_1, I_2) \sim q_1 \times q_2$, and J outcome of $W^{\otimes n}[P]$.

- Focus on noisy BAC $W_{\mathrm{BAC},\varepsilon,\varepsilon}$: bit flip error of ε on each input.

Numerical results for $W_{\mathrm{BAC},\varepsilon,\varepsilon}$ with $\varepsilon = 10^{-3}$



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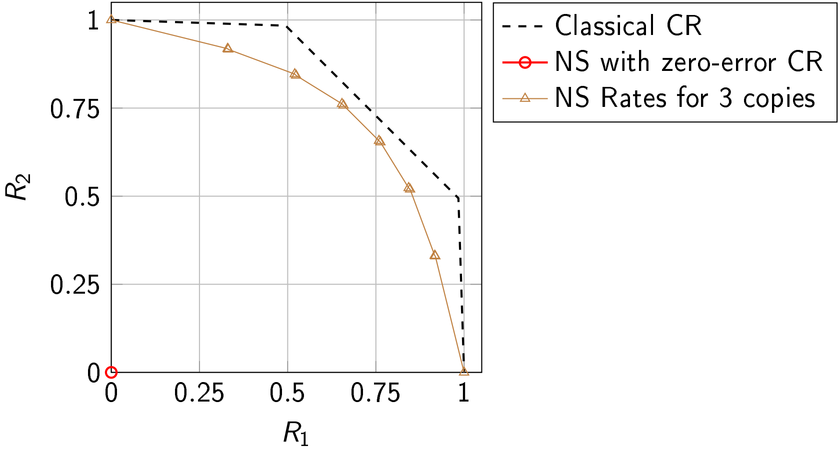

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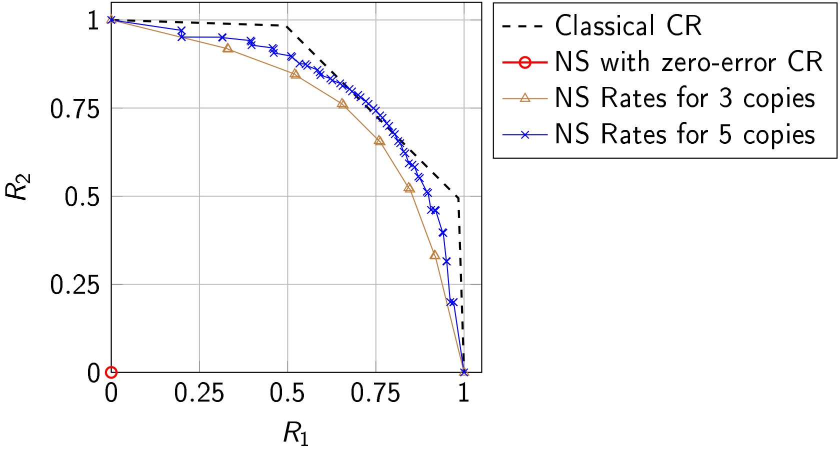

Conclusion and open problems

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   - Handle noisy channels through concatenated codes: Previous gap persists for noisy BAC.

2. Open problems:
   - Suggests that quantum entanglement may increase the capacity of such channels: still open for the BAC !
   - How are linked $\mathcal{C}^{\mathrm{NS}}(W)$ and the set of (R_1, R_2) such that:

     $$R_1 \leq I(X_1 : Y|X_2)\,,\ R_2 \leq I(X_2 : Y|X_1)\,,\ R_1+R_2 \leq I((X_1X_2) : Y)$$

     for inputs $(X_1, X_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ following any input law $P_{X_1X_2}$?

Thank you for listening!

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