Supplement to *The Catalan Light Cone*

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Abstract

This companion document collects additional appendices and proof sketches that are omitted from the lean arXiv v1 of *The Catalan Light Cone*. It is interpretation-neutral and intended to preserve auxiliary results and extended technical development without expanding the main paper.

A A Conditional Uniqueness Theorem for the Dyck-Prefix Substrate

This appendix isolates a minimal set of structural axioms under which the Dyck-prefix substrate is not merely an example but is determined uniquely up to isomorphism. The result is interpretation-neutral: it assumes no particular computational semantics (evaluation order, combinators, rewriting), only a ranked causal-growth structure with a one-sided admissibility boundary.

A.1 Ranked growth posets

Definition A.1 (Ranked growth poset). *A* ranked growth poset *is a triple* $(S, \preceq, \mathrm{rk})$ *where* S *is a countable set,* $\preceq$ *is a partial order on* S*, and* $\mathrm{rk} : S \to \mathbb{N}$ *is a rank function such that:*

(i) for every $t \in S$ *the set* $\{s \in S : s \preceq t\}$ *is finite, and*

(ii) rk *is strictly order-increasing along cover relations.*

Write $s \prec t$ *when* $s \preceq t$ *and* $s \neq t$*, and write* $s \lessdot t$ *if* $s \prec t$ *and there is no* u *with* $s \prec u \prec t$ *(i.e.* t covers s*).*

Definition A.2 (Two-move height structure). *Let* $(S, \preceq, \mathrm{rk})$ *be a ranked growth poset with a distinguished root* $s_\emptyset \in S$ *satisfying* $\mathrm{rk}(s_\emptyset) = 0$*. A* two-move height structure *on* S *consists of a function* $h : S \to \mathbb{Z}_{\geq 0}$ *(height) with* $h(s_\emptyset) = 0$ *such that every cover relation* $s \lessdot t$ *satisfies*

$$\mathrm{rk}(t) = \mathrm{rk}(s) + 1 \qquad \textit{and} \qquad h(t) = h(s) \pm 1.$$

We call a cover with $h(t) = h(s) + 1$ *an* up-step *and a cover with* $h(t) = h(s) - 1$ *a* down-step.

A.2 Axioms for the Catalan core

The uniqueness theorem below follows from four axioms capturing: (i) discrete single-step growth, (ii) exactly two local move types (up/down), (iii) a one-sided boundary in which down-steps are disabled at height 0, and (iv) an *unfolded* (collision-free) history space.

Definition A.3 (Catalan core axioms). *A ranked growth poset* $(S, \preceq, \mathrm{rk})$ *with root* $s_\emptyset$ *and two-move height structure* h *satisfies the* Catalan core axioms *if:*

(A1) ***(Ranked single-step growth)*** *For every cover $s \lessdot t$, one has* $\mathrm{rk}(t) = \mathrm{rk}(s) + 1$ *and* $h(t) = h(s) \pm 1$.

(A2) ***(Local determinism by move type)*** *For each $s \in S$, there exists* at most one *up-step successor $t_\uparrow$ with $s \lessdot t_\uparrow$ and $h(t_\uparrow) = h(s) + 1$, and* at most one *down-step successor $t_\downarrow$ with $s \lessdot t_\downarrow$ and $h(t_\downarrow) = h(s) - 1$.*

(A3) ***(One-sided boundary admissibility)*** *For each $s \in S$, an up-step successor exists, while a down-step successor exists* iff $h(s) > 0$.

(A4) ***(No collisions / unique history)*** *Each $t \in S$ admits a unique saturated chain (cover chain) from the root:*

$$s_\emptyset = s_0 \lessdot s_1 \lessdot \cdots \lessdot s_{\mathrm{rk}(t)} = t.$$

A.3 Dyck-prefix poset

Let $\Sigma = \{+, -\}$. For a word $w = w_1 \cdots w_n \in \Sigma^n$, define the partial-sum height

$$H_w(k) = \sum_{j=1}^{k} \xi(w_j), \qquad \text{where } \xi(+) = +1,\ \xi(-) = -1.$$

Call w *admissible* if $H_w(k) \geq 0$ for all k. Let $\mathcal{C}$ be the set of all admissible words (Dyck prefixes), partially ordered by prefix: $u \preceq v$ iff u is a prefix of v. Let $\ell(w) = n$ be length and $H(w) = H_w(n)$ be final height.

Definition A.4 (Dyck-prefix poset)**.** *The* Dyck-prefix poset *is $(\mathcal{C}, \preceq, \ell)$ with root ε (the empty word).*

A.4 Uniqueness theorem

Theorem A.1 (Dyck-prefix uniqueness)**.** *Let $(S, \preceq, \mathrm{rk})$ be a ranked growth poset with root $s_\emptyset$ and a two-move height structure h. If S satisfies the Catalan core axioms of Definition A.3, then there exists a unique bijection*

$$\pi : S \to \mathcal{C}$$

such that for every $s \in S$,

$$\ell(\pi(s)) = \mathrm{rk}(s), \qquad H(\pi(s)) = h(s),$$

and π is an order isomorphism:

$$s \preceq t \iff \pi(s) \preceq \pi(t) \quad \textit{(prefix order).}$$

In particular, $(S, \preceq, \mathrm{rk})$ is isomorphic to the Dyck-prefix poset.

Proof. By (A4), each $s \in S$ admits a unique cover chain $s_\emptyset = s_0 \lessdot \cdots \lessdot s_n = s$ where $n = \mathrm{rk}(s)$. Define a word $\pi(s) \in \Sigma^n$ by setting

$$\pi(s)_k = \begin{cases} + & \text{if } h(s_k) = h(s_{k-1}) + 1, \\ - & \text{if } h(s_k) = h(s_{k-1}) - 1. \end{cases}$$

This is well-defined and has length $\ell(\pi(s)) = n$. By construction, the partial sums of $\pi(s)$ coincide with the height along the chain,

$$H_{\pi(s)}(k) = h(s_k) \geq 0 \quad \text{for all } k,$$

so $\pi(s)$ is admissible and $\pi(s) \in \mathcal{C}$. Moreover $H(\pi(s)) = H_{\pi(s)}(n) = h(s)$.

We now show that π is bijective. Given any admissible word $w = w_1 \cdots w_n \in \mathcal{C}$, construct a chain in S by starting at $s_0 = s_\emptyset$ and for each $k \geq 1$ taking the successor specified by w_k:

$$s_k = \begin{cases} \text{the (unique) up-step successor of } s_{k-1} & \text{if } w_k = +, \\ \text{the (unique) down-step successor of } s_{k-1} & \text{if } w_k = -. \end{cases}$$

Existence and uniqueness follow from (A2)–(A3); admissibility of w ensures that a down-step is only requested when the current height is positive. Let $s := s_n$ be the resulting node at rank n. Then $\pi(s) = w$, proving surjectivity. Injectivity follows from (A4): distinct nodes have distinct cover chains and hence distinct step sequences.

Finally, π preserves order. If $s \preceq t$, then the unique chain to t contains the chain to s as an initial segment, so $\pi(s)$ is a prefix of $\pi(t)$. Conversely, if $\pi(s)$ is a prefix of $\pi(t)$, then the constructed chain for $\pi(t)$ passes through the constructed chain for $\pi(s)$, giving $s \preceq t$. Uniqueness of π is immediate from (A4), since $\pi(s)$ is forced by the unique cover chain to s. □

A.5 Catalan enumeration at completion

To recover the usual Catalan counting, impose a completion boundary condition: completion occurs when the height returns to zero at even rank.

Definition A.5 (Completed histories). *For $n \in \mathbb{N}$, define the set of* completed histories *at rank $2n$ by*

$$S_{2n}^{\text{comp}} := \{s \in S : \text{rk}(s) = 2n \text{ and } h(s) = 0\}.$$

Corollary A.1 (Catalan counting). *Under the hypotheses of Theorem A.1, the number of completed histories satisfies*

$$\#\big(S_{2n}^{\text{comp}}\big) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

the nth Catalan number.

Proof. By Theorem A.1, S_{2n}^{comp} is in bijection with admissible words of length $2n$ with final height 0, i.e. Dyck words of semilength n. These are counted by the Catalan number C_n. □

A.6 Quotients and structural sharing

The "no collisions" axiom (A4) asserts that S is an *unfolded* history space: each node encodes a distinct growth history. This does not preclude gauge equivalences or multi-way computation graphs; rather, those arise by quotienting S by an equivalence relation.

Remark A.1 (Quotients and gauge identification). *Let $\sim$ be an equivalence relation on S representing gauge or rewrite identifications. The quotient set $S/\!\sim$ may exhibit merges (a DAG-like state graph), even if S is collision-free. Theorem A.1 characterizes the unfolded core; quotient structure is encoded by the choice of $\sim$ and acts as an identification of histories rather than a primitive feature of the substrate.*

Remark A.2 (Structural sharing vs. semantic collisions). *Axiom (A4) concerns semantic identity in S (distinct histories are distinct nodes). It is compatible with immutability and structural sharing in implementations (e.g. hash-consing), which may share representation of common substructures without identifying distinct history nodes.*

A.7 Scope of the result

Theorem A.1 establishes a core uniqueness statement: given two local move types changing an integer height by ± 1, a one-sided boundary disabling down-steps at height 0, and an unfolded history space, the substrate is forced to be Dyck-prefix order. This statement is independent of additional semantic layers (program interpretations, rewriting dynamics, or amplitude assignments), which can be imposed on top of the substrate without affecting the isomorphism.

B Unfoldings and Covers of Growth Posets

Many natural representations of computation or gauge-identified state spaces exhibit merges: distinct growth histories may lead to the same state, producing a DAG-like multi-way graph rather than a tree. The core uniqueness theorem in Appendix A characterizes the *unfolded* history space. This appendix makes that relationship explicit by defining a canonical unfolding (history cover) for a ranked growth poset and showing that the Catalan uniqueness statement applies upstairs even when merges exist downstairs.

B.1 Growth graphs

Definition B.1 (Rooted ranked growth poset)**.** *A* rooted ranked growth poset *is a ranked growth poset* $(S, \preceq, \mathrm{rk})$ *(Definition A.1) equipped with a distinguished root* $s_\emptyset \in S$ *such that for every* $s \in S$ *one has* $s_\emptyset \preceq s$ *(i.e. every node is reachable from the root).*

Definition B.2 (Directed growth graph)**.** *Let* $(S, \preceq, \mathrm{rk})$ *be a rooted ranked growth poset. Its* directed growth graph *is the rooted directed graph* $G(S) = (V, E, s_\emptyset)$ *where* $V = S$ *and* $(s,t) \in E$ *iff* $s \lessdot t$ *(i.e.* t *covers* s*). We regard each edge as a single-step growth move.*

Definition B.3 (Move type induced by height)**.** *If* $(S, \preceq, \mathrm{rk})$ *carries a two-move height structure* h *(Definition A.2), then each directed edge* $s \to t$ *in* $G(S)$ *has a* move type

$$\mathrm{type}(s \to t) \in \{+, -\} \quad \textit{defined by} \quad \mathrm{type}(s \to t) = \begin{cases} + & \textit{if } h(t) = h(s) + 1, \\ - & \textit{if } h(t) = h(s) - 1. \end{cases}$$

B.2 Rooted covers (local isomorphism in one step)

To model "the same local growth rule everywhere" we use a minimal notion of cover: a map that is locally bijective on outgoing edges from each node. This avoids introducing general poset coverings and suffices for the unfolding argument below.

Definition B.4 (Rooted directed cover)**.** *Let* $G = (V, E, r)$ *and* $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{r})$ *be rooted directed graphs. A function* $\varphi : \widetilde{V} \to V$ *is a* rooted directed cover *if:*

(i) $\varphi(\widetilde{r}) = r$*;*

(ii) for every $\widetilde{v} \in \widetilde{V}$*, the map*

$$\varphi_* : \mathrm{Out}(\widetilde{v}) \to \mathrm{Out}(\varphi(\widetilde{v})), \qquad (\widetilde{v} \to \widetilde{w}) \mapsto (\varphi(\widetilde{v}) \to \varphi(\widetilde{w}))$$

is a bijection, where $\mathrm{Out}(x) = \{x \to y \in E\}$ *denotes the set of outgoing edges from* x*.*

If edges carry move types $\{+, -\}$*, we additionally require that* φ_* *preserves move type (i.e. the bijection matches* $+$*-edges to* $+$*-edges and* $-$*-edges to* $-$*-edges).*

B.3 The canonical unfolding (history cover)

Definition B.5 (Unfolding as rooted path space). *Let $(S, \preceq, \mathrm{rk})$ be a rooted ranked growth poset and let $G(S)$ be its directed growth graph. Define the* unfolding *$\widetilde{S}$ to be the set of all finite directed paths in $G(S)$ starting at the root:*

$$\widetilde{S} := \{(s_0 \to s_1 \to \cdots \to s_n) \ : \ s_0 = s_\emptyset, \ (s_{k-1}, s_k) \in E\}.$$

Write $\widetilde{s} = (s_0 \to \cdots \to s_n)$ and define:

(i) the endpoint *map $\varphi : \widetilde{S} \to S$ by $\varphi(\widetilde{s}) = s_n$;*

(ii) the rank *$\widetilde{\mathrm{rk}}(\widetilde{s}) = n$ (path length);*

(iii) the prefix order *on $\widetilde{S}$: $\widetilde{u} \preceq \widetilde{v}$ iff $\widetilde{u}$ is an initial segment (prefix) of $\widetilde{v}$.*

If S has a height function h, define the lifted height $\widetilde{h}(\widetilde{s}) := h(\varphi(\widetilde{s}))$.

Lemma B.1 (The unfolding is collision-free). *Every element $\widetilde{s} \in \widetilde{S}$ has a unique predecessor chain from the root in the prefix order, i.e. $\widetilde{S}$ satisfies the "no collisions / unique history" property (A4) of Definition A.3.*

Proof. By construction, each $\widetilde{s}$ is itself a rooted path $(s_0 \to \cdots \to s_n)$. Its strict prefixes are exactly the initial segments $(s_0 \to \cdots \to s_k)$ for $0 \leq k < n$, and these form the unique saturated chain from the root to $\widetilde{s}$ under prefix inclusion. □

Lemma B.2 (The endpoint map is a rooted directed cover). *Let $G(\widetilde{S})$ be the directed growth graph of the unfolding, whose edges append one growth step:*

$$(s_0 \to \cdots \to s_n) \ \to \ (s_0 \to \cdots \to s_n \to s_{n+1}) \quad \textit{whenever } s_n \to s_{n+1} \textit{ is an edge in } G(S).$$

Then the endpoint map $\varphi : \widetilde{S} \to S$ from Definition B.5 is a rooted directed cover in the sense of Definition B.4. If S carries a two-move height structure, then φ preserves move type.

Proof. Clearly φ maps the length-0 path $(s_\emptyset)$ to $s_\emptyset$. Fix a path $\widetilde{s} = (s_0 \to \cdots \to s_n)$. Outgoing edges from $\widetilde{s}$ in $G(\widetilde{S})$ are in bijection with outgoing edges from s_n in $G(S)$ by appending the corresponding last step $s_n \to s_{n+1}$. Under φ, each appended edge maps to exactly that edge $s_n \to s_{n+1}$, giving a bijection on outgoing edges. If move types are present, appending an up-step or down-step in $\widetilde{S}$ maps to an up-step or down-step in S by Definition B.3, so type is preserved. □

B.4 Dyck-prefix structure upstairs

Proposition B.1 (Lift of the Catalan core axioms). *Suppose $(S, \preceq, \mathrm{rk})$ is a rooted ranked growth poset with a two-move height structure h satisfying axioms (A1)–(A3) of Definition A.3 (ranked single-step growth, local determinism by move type, and one-sided boundary admissibility), but not necessarily (A4). Then the unfolding $(\widetilde{S}, \preceq, \widetilde{\mathrm{rk}})$ with lifted height $\widetilde{h}$ satisfies (A1)–(A4).*

Proof. Axiom (A4) holds by Lemma B.1. For (A1)–(A3), each cover in $\widetilde{S}$ appends exactly one edge of $G(S)$, so ranks increase by one and height changes by ± 1 exactly as in S. Moreover, by local determinism in S, from any endpoint there is at most one $+$-successor and at most one $-$-successor; by Lemma B.2 the same holds in the unfolding at every path. Finally, one-sided boundary admissibility is preserved: a $-$-extension exists in $\widetilde{S}$ precisely when the endpoint height is positive. □

Theorem B.1 (Dyck-prefix characterization of the unfolding)**.** *Under the hypotheses of Proposition B.1, the unfolding $\widetilde{S}$ is (canonically) order-isomorphic to the Dyck-prefix poset (Definition A.4). In particular, completed unfolded histories are counted by Catalan numbers as in Corollary A.1.*

Proof. By Proposition B.1, $\widetilde{S}$ satisfies the full Catalan core axioms (A1)–(A4). The conclusion follows by applying Theorem A.1 to $\widetilde{S}$. $\square$

Remark B.1 (Canonical quotient and "collisions")**.** *The unfolding comes with a canonical surjection $\varphi : \widetilde{S} \to S$ (the endpoint map). Collisions/merges in S correspond exactly to identifications of distinct unfolded histories:*

$$\widetilde{u} \sim_{\varphi} \widetilde{v} \quad \iff \quad \varphi(\widetilde{u}) = \varphi(\widetilde{v}).$$

Thus, as a set of states reachable from the root, S is naturally identified with the quotient $\widetilde{S}/\sim_{\varphi}$. For the distinction between semantic identification and structural sharing in implementations, see Remark A.2.

B.5 A Reduction-Theoretic "Full Uniqueness" Statement

The core uniqueness theorem (Theorem A.1) shows that a collision-free two-move growth system with a one-sided boundary is forced to be Dyck-prefix order. This subsection records a complementary reduction statement: if a canonical notion of *intrinsic state* (defined by future-cone isomorphism) is already a minimal one-counter system, and if the projection to intrinsic state is locally bijective on labeled moves, then the Catalan core is forced as the unfolded normal form.

B.5.1 Future cones and intrinsic-state equivalence

Definition B.6 (Future cone as a rooted directed graph)**.** *Let $(S, \preceq, \mathrm{rk})$ be a rooted ranked growth poset with directed growth graph $G(S) = (S, E, s_\emptyset)$ (Definition B.2). For $s \in S$, define the* future cone *at s to be the rooted directed subgraph*

$$\mathrm{Cone}(s) \ := \ \big(S_s, E_s, s\big),$$

where $S_s = \{t \in S : \ s \preceq t\}$ and $E_s = \{(u, v) \in E : \ u, v \in S_s\}$. If edges in $G(S)$ are labeled by a finite move alphabet (in particular $\{+, -\}$), we regard $\mathrm{Cone}(s)$ as an edge-labeled rooted directed graph.

Definition B.7 (Cone isomorphism and intrinsic-state equivalence)**.** *Assume edges are labeled by move type in $\{+, -\}$. For $s, t \in S$, write $s \equiv t$ if there exists a rooted directed graph isomorphism*

$$\psi : \mathrm{Cone}(s) \ \cong \ \mathrm{Cone}(t)$$

sending root to root and preserving move labels. This defines an equivalence relation $\equiv$ on S, called intrinsic-state equivalence*. Let $M := S/\equiv$ be the set of equivalence classes, with root $m(s_\emptyset) \in M$, and write*

$$m : S \to M, \qquad m(s) = [s]_\equiv$$

for the canonical projection.

Remark B.2 (Relocatable futures)**.** *The equivalence relation $\equiv$ captures a precise form of relocatability: states are indistinguishable if and only if their reachable futures are isomorphic as rooted labeled growth graphs. The quotient $M = S/\equiv$ is a canonical "coarsest" state descriptor that preserves the full future growth structure.*

B.5.2 Intrinsic dynamics and the cover hypothesis

Definition B.8 (Intrinsic transition graph)**.** *Let $G(S)$ be labeled by $\{+,-\}$. Define a rooted labeled directed graph $G(M)$ on M by declaring a labeled edge $x \to_{\pm} y$ to exist if there are $s, t \in S$ such that $s \lessdot t$ is a $\pm$-labeled edge of $G(S)$ and $m(s) = x$, $m(t) = y$.*

Definition B.9 (One-counter intrinsic dynamics)**.** *We say $(S, \preceq, \mathrm{rk})$ has* one-counter intrinsic dynamics *if there exists a bijection*

$$\iota : M \;\to\; \mathbb{Z}_{\geq 0}$$

with $\iota(m(s_\emptyset)) = 0$ such that, for every intrinsic state $x \in M$ with $\iota(x) = k$:

(i) ***(Two primitive moves)*** *there exists exactly one $+$-successor y_+ of x in $G(M)$ with $\iota(y_+) = k + 1$;*

(ii) ***(One-sided boundary)*** *there exists a $-$-successor y_- of x in $G(M)$ iff $k > 0$, and in that case $\iota(y_-) = k - 1$;*

(iii) ***(No other intrinsic moves)*** *x has no other outgoing edges in $G(M)$ besides $x \to_+ y_+$ and (when $k > 0$) $x \to_- y_-$.*

Equivalently, $G(M)$ is isomorphic (as a rooted labeled graph) to the standard one-counter graph on $\mathbb{Z}_{\geq 0}$ with edges $k \to_+ k + 1$ for all k and edges $k \to_- k - 1$ for $k > 0$.

Remark B.3 (Minimality content)**.** *Definition B.9 should be read as a minimality hypothesis: the intrinsic descriptor $M = S/\equiv$ is already a single nonnegative integer, and it admits exactly two primitive move types with a one-sided boundary at 0.*

Definition B.10 (Cover-consistency of the intrinsic projection)**.** *Assume edges are labeled by $\{+,-\}$. We say the intrinsic projection $m : S \to M$ is* cover-consistent *if it is a rooted directed cover (Definition B.4) from the labeled growth graph $G(S)$ to the intrinsic graph $G(M)$ (Definition B.8), i.e. for every $s \in S$ the induced map on outgoing edges $\mathrm{Out}(s) \to \mathrm{Out}(m(s))$ is a label-preserving bijection.*

Remark B.4. *Under the one-counter identification $\iota : M \to \mathbb{Z}_{\geq 0}$ of Definition B.9, cover-consistency of m is equivalent to requiring that the composite $h := \iota \circ m : S \to \mathbb{Z}_{\geq 0}$ is a rooted directed cover from $G(S)$ to the standard one-counter graph on $\mathbb{Z}_{\geq 0}$.*

B.5.3 Reduction to the Dyck-prefix normal form

Theorem B.2 (Reduction to a Dyck-prefix cover)**.** *Let $(S, \preceq, \mathrm{rk})$ be a rooted ranked growth poset whose directed growth graph $G(S)$ is labeled by move type in $\{+,-\}$. Assume:*

(H1) ***(Intrinsic-state quotient)*** *intrinsic-state equivalence $\equiv$ is defined by label-preserving cone isomorphism as in Definition B.7, with quotient $m : S \to M$;*

(H2) ***(One-counter intrinsic dynamics)*** *S satisfies Definition B.9;*

(H3) ***(Cover consistency)*** *the intrinsic projection m is a rooted directed cover as in Definition B.10;*

(H4) ***(Tiered single-step growth)*** *for every cover $s \lessdot t$, $\mathrm{rk}(t) = \mathrm{rk}(s) + 1$.*

Let $\widetilde{S}$ be the unfolding (history cover) of S (Definition B.5) with endpoint map $\varphi : \widetilde{S} \to S$. Then $\widetilde{S}$ is (canonically) order-isomorphic to the Dyck-prefix poset (Definition A.4), and hence S is a quotient (folding) of a Dyck-prefix cover via φ (Remark B.1).

Proof. Define a height function on S by composing the intrinsic-state projection with the one-counter identification,

$$h(s) := \iota(m(s)) \in \mathbb{Z}_{\geq 0}.$$

By cover consistency (Definition B.10) and one-counter intrinsic dynamics (Definition B.9), each $s \in S$ has exactly one outgoing $+$-edge, and it has an outgoing $-$-edge iff $h(s) > 0$. Moreover, because m is label-preserving and ι identifies $G(M)$ with the standard one-counter graph, along any cover edge $s \lessdot t$ one has $h(t) = h(s)+1$ for a $+$-edge and $h(t) = h(s)-1$ for a $-$-edge. Together with the tier assumption $\mathrm{rk}(t) = \mathrm{rk}(s) + 1$, this shows that S admits a two-move height structure satisfying axioms (A1)–(A3) of Definition A.3.

Applying Theorem B.1 now yields that the unfolding $\widetilde{S}$ is Dyck-prefix order-isomorphic. The quotient statement for S is then exactly Remark B.1. □

Remark B.5 (What is and is not proved)**.** *Theorem B.2 is a "full uniqueness" statement in the reduction sense: once relocatable futures are formalized via cone isomorphism and the intrinsic state space is assumed to be a one-counter with two primitive moves, together with the cover-consistency hypothesis (no hidden multiplicity in the projection m), the Catalan substrate is forced as the unfolded normal form. What is not claimed is that all reasonable substrates satisfy these minimality hypotheses; establishing that requires separate domain-specific arguments.*

B.6 Interpretive takeaway

Theorems A.1 and B.1 together support the following robust core statement: whenever a ranked two-move growth system with a one-sided boundary is present, the unfolded history space is forced to be Dyck-prefix order, and any merged multi-way representation is a quotient of that Catalan cover.

C Additional Technical Notes

This appendix collects optional bookkeeping and auxiliary constructions. It is separate from the main formal development and may be skipped on first reading.

C.1 Entropy of coarse-graining and information rate

Fix a tier n and an observable (deterministic coarse-graining) $f : \mathcal{D}_n \to \mathcal{X}$. For $x \in \mathcal{X}$ write

$$N(x) := \#\{w \in \mathcal{D}_n : f(w) = x\},$$

so that $f^{-1}(x)$ is the equivalence class of histories identified as the same outcome. If f is prefix-local in the sense of Remark 4.1, then the multiplicities $N(x)$ admit transfer recursions on the induced state space.

Multiplicity entropy. Define the (microcanonical) entropy of the full ensemble at tier n by

$$S_n \;:=\; \log \#(\mathcal{D}_n),$$

and the conditional entropy of an outcome x by

$$S(x) \;:=\; \log N(x).$$

The information eliminated by selecting outcome x is the entropy drop

$$\Delta S(x) \;:=\; S_n - S(x) \;=\; \log\Big(\frac{\#(\mathcal{D}_n)}{N(x)}\Big).$$

(Any logarithm base may be used; base 2 yields units of bits.)

Remark C.1 (Retrospective vs. prospective counts)**.** *The multiplicity entropy $S(x) = \log N(x)$ measures how many fine-grained histories are identified as the same outcome at tier n. A complementary forward-looking quantity is the number of admissible continuations of a realized prefix into higher tiers (the size of its local cone; see Section 2.9), which may be studied by counting completions as a function of current height (Lemma 2.1).*

Completion multiplicity and future-cone entropy. Fix a Dyck prefix u of length k and height h. For a target tier n with $2n \geq k$, define the completion multiplicity

$$M_n(u) \;:=\; \#\{w \in \mathcal{D}_n : u \preceq w\},$$

the number of completed histories at tier n consistent with the partial history u. This is the finite-tier size of the local cone rooted at u. It depends only on the remaining step budget $s := 2n-k$ and the current height h, and admits the explicit ballot-number formula of Lemma 2.1. The associated future-cone entropy is

$$S_n^{\mathrm{cone}}(u) \;:=\; \log M_n(u).$$

Information rate as rate of possibility reduction. Let m denote the number of selection events (local contractions) along a history, and let computational proper time be $\tau = \tau_0 m$ for a fixed scale $\tau_0 > 0$. We define the information rate associated with outcome x to be the information loss per unit computational proper time,

$$R(x) \;:=\; \frac{\Delta S(x)}{\tau}.$$

In the simplest case of a single event ($m = 1$), this reduces to $R(x) = \Delta S(x)/\tau_0$. One may also adopt a coarser tier-wise selection model in which m is identified with the tier index n (one selection per tier boundary), but we keep these notions separate in general.

Gauge-invariant counting. When histories admit a redundancy under commuting spacelike-separated updates, one may quotient the fine-grained history set at tier n by the induced gauge equivalence relation $\sim_g$ of Definition 5.1. When $\mathcal{D}_n$ is taken to parametrize such fine-grained histories at tier n, define $\bar{\mathcal{D}}_n := \mathcal{D}_n/\sim_g$. If f is gauge-invariant (constant on $\sim_g$-orbits), define

$$\bar{N}(x) := \#\{[w] \in \bar{\mathcal{D}}_n : f(w) = x\}, \qquad \bar{\Delta S}(x) := \log\Big(\frac{\#(\bar{\mathcal{D}}_n)}{\bar{N}(x)}\Big),$$

and use $\bar{\Delta S}$ in place of ΔS. This removes overcounting due solely to reordering of independent collapses.

C.2 Size bookkeeping for subtree-selection collapses

In addition to ensemble-level multiplicities, some collapse rules admit a simple size bookkeeping at the level of individual trees. Let T be a finite full binary tree. Define its *internal size* $U(T)$ (number of internal nodes) recursively by

$$U(\texttt{()}) := 0, \qquad U(\bullet(L,R)) := 1 + U(L) + U(R),$$

where `()` denotes a leaf and $\bullet(L,R)$ denotes a binary pair.

Lemma C.1 (Size drop under subtree selection)**.** *Consider the local collapse that replaces a pair $\bullet(L,R)$ by one of its children:*

$$\bullet(L,R) \rightsquigarrow L \qquad \text{or} \qquad \bullet(L,R) \rightsquigarrow R.$$

Then the size drop is

$$U(\bullet(L,R)) - U(L) = 1 + U(R), \qquad U(\bullet(L,R)) - U(R) = 1 + U(L).$$

In particular, if the collapse rule keeps the larger child in U (i.e. keeps $\arg\max\{U(L), U(R)\}$), then the drop is $1 + \min\{U(L), U(R)\}$.

Proof. Immediate from the defining recursion $U(\bullet(L,R)) = 1 + U(L) + U(R)$. □

C.3 Fields on words, prefixes, and nodes (optional)

We use the word "field" as shorthand for a complex-valued function on one of the Catalan objects already in play. Several closely related state spaces are useful in different contexts.

Fields on completed histories (fixed tier). Fix n and consider a function $\Phi_n : \mathcal{D}_n \to \mathbb{C}$ assigning an amplitude (or observable value) to each completed history $w \in \mathcal{D}_n$. The associated Hilbert space is $\ell^2(\mathcal{D}_n)$ with inner product

$$\langle \psi, \phi \rangle := \sum_{w \in \mathcal{D}_n} \overline{\psi(w)}\, \phi(w).$$

Fields on prefixes (the full cone). Let $\mathcal{C}$ denote the set of Dyck prefixes (admissible partial histories). A prefix field is a function $\Phi : \mathcal{C} \to \mathbb{C}$, which may be restricted to a fixed length slice $\mathcal{C}^{(k)} := \{p \in \mathcal{C} : |p| = k\}$ when needed.

Fields on nodes of a fixed tree. Given $w \in \mathcal{D}_n$, let $T(w)$ be its associated full binary tree. A node field is a function $\phi_w : \mathrm{Int}(T(w)) \to \mathbb{C}$ on the internal nodes of that tree.

Remark C.2. *These notions live on different objects (tiers, the prefix poset, or a single tree) and are independent of any within-tier ordering convention on $\mathcal{D}_n$.*

C.4 Subtree indicators as a multiscale spanning family (optional)

Let T be a finite rooted tree and write $\mathrm{Int}(T)$ for its internal nodes. Each $v \in \mathrm{Int}(T)$ determines a rooted subtree T_v, and hence a subset $\mathrm{Int}(T_v) \subseteq \mathrm{Int}(T)$. Define the subtree indicator

$$\chi_v : \mathrm{Int}(T) \to \{0,1\}, \qquad \chi_v(u) := \mathbf{1}\{u \in \mathrm{Int}(T_v)\}.$$

Lemma C.2 (Subtree indicators form a basis)**.** *The family $\{\chi_v : v \in \mathrm{Int}(T)\}$ is a basis of the vector space of complex-valued functions on $\mathrm{Int}(T)$.*

Proof. Order the internal nodes by nonincreasing depth (deepest first), and let M be the square matrix with entries $M_{uv} := \chi_v(u)$. Then $M_{vv} = 1$ for all v, while $M_{uv} = 0$ whenever u precedes v in this order (a node cannot be a descendant of a deeper node). Thus M is triangular with ones on the diagonal, hence invertible. Therefore the indicators are linearly independent and, since their number equals $\#\mathrm{Int}(T)$, they form a basis. □

Corollary C.1 (Explicit inversion)**.** *Let $f : \mathrm{Int}(T) \to \mathbb{C}$ be any function. There is a unique family of coefficients $\{a_v\}_{v\in\mathrm{Int}(T)}$ such that*

$$f \;=\; \sum_{v\in\mathrm{Int}(T)} a_v\,\chi_v.$$

Writing $\mathrm{par}(v)$ for the parent of v (for $v \neq \mathrm{root}(T)$), these coefficients are given by

$$a_{\mathrm{root}(T)} = f(\mathrm{root}(T)), \qquad a_v = f(v) - f(\mathrm{par}(v)) \quad (v \neq \mathrm{root}(T)).$$

Proof. For each $u \in \mathrm{Int}(T)$, $(\sum_v a_v\chi_v)(u) = \sum_{v:\,u\in\mathrm{Int}(T_v)} a_v = \sum_{v\preceq u} a_v$, where $v \preceq u$ means that v is an ancestor of u. With the stated choice of coefficients, this ancestor sum telescopes along the unique root-to-u chain to yield $f(u)$. Uniqueness follows from Lemma C.2. □

Remark C.3. *This basis is "multiscale": indicators of deep subtrees localize to fine regions of T, while indicators near the root encode coarse structure. Any choice of orthonormalization yields an orthonormal basis adapted to the rooted tree geometry.*

C.5 Operators on a fixed history tree (optional)

In addition to tier-wise state spaces (fields on $\mathcal{D}_n$), one may also consider dynamics *within* a fixed realized history by placing operators on the internal nodes of its tree.

Node Hilbert space. Fix $w \in \mathcal{D}_n$ and let $T(w)$ be its associated full binary tree. Write $V_w := \mathrm{Int}(T(w))$ and consider $\ell^2(V_w)$ with inner product $\langle \psi, \phi\rangle := \sum_{v\in V_w} \overline{\psi(v)}\,\phi(v)$.

Adjacency and Laplacian. Let $G_w = (V_w, E_w)$ be any finite undirected graph on V_w (for example, connect each internal node to its internal children). Define A_{G_w}, D_{G_w}, and the graph Laplacian and generator by

$$\Delta_{G_w} := D_{G_w} - A_{G_w}, \qquad L_{G_w} := -\Delta_{G_w}.$$

Heat and Schrödinger evolutions. The corresponding "internal-time" heat equation is

$$\partial_\tau u = L_{G_w} u,$$

and the corresponding unitary Schrödinger evolution is

$$i\,\partial_t \psi = -L_{G_w}\psi = \Delta_{G_w}\psi.$$

Remark C.4. *This within-history operator framework is independent of the tier-growth Markov dynamics and of coherent summation over histories: it simply records that, once a graph structure is specified on the internal nodes of a fixed Catalan tree, standard graph-Laplacian constructions yield discrete diffusion and Schrödinger-type evolutions on that fixed combinatorial background.*

C.6 Operators on tier slices (optional)

The main text emphasizes two dynamics on the Catalan substrate: tier growth (prefix extension) and coherent summation over histories. Independently, one may also consider *slice dynamics* on a fixed tier by endowing the finite set $\mathcal{D}_n$ with an auxiliary adjacency graph. This subsection records the standard operator framework for such constructions.

Tier Hilbert space. We take the tier state space to be $\ell^2(\mathcal{D}_n)$ as in Section C.3.

Adjacency graphs. Let $G_n = (\mathcal{D}_n, E_n)$ be any finite undirected graph on $\mathcal{D}_n$. The choice of G_n is additional structure: different graphs induce different notions of locality on the tier. A canonical example is the rotation graph (the associahedron adjacency) on full binary trees, where edges correspond to single associativity rotations [17, 21].

Associahedra and planar tree amplitudes (scattering-amplitude tie-in). The associahedron adjacency on $\mathcal{D}_n$ is also natural from the perspective of scattering amplitudes. For the planar tree-level sector of bi-adjoint cubic scalar theory (ϕ^3), Arkani-Hamed, Bai, He, and Yan identify an associahedron in planar kinematic space and show that the tree amplitude is the corresponding canonical form of this positive geometry [19]. From this viewpoint, the Catalan enumeration of planar cubic tree diagrams is not merely counting: the associahedron organizes factorization channels geometrically, and different triangulations correspond to different diagrammatic expansions of the same canonical form (see, e.g., the review [22]). For quartic interactions, an analogous positive-geometry description involves Stokes polytopes rather than associahedra [20].

Tamari/Dyck/alt-Tamari choices on the same tier. The point of introducing an auxiliary graph G_n is that the underlying state set $\mathcal{D}_n$ supports multiple natural notions of tier-locality coming from classical Catalan posets. The rotation graph is the undirected adjacency underlying the Tamari order; one may likewise equip $\mathcal{D}_n$ with adjacency induced by the Dyck ("Stanley") lattice on Dyck paths, or more generally by the family of δ-Tamari (alt-Tamari) posets interpolating between these extremes. These alternatives use different covering relations on the same Catalan tier and therefore induce different graph Laplacians Δ_{G_n} and different "free" tier Hamiltonians, but they live on a common configuration space $\mathcal{D}_n$ [17, 21].

Linear intervals as "1D corridors" in Catalan posets. A useful robustness fact is that certain one-dimensional substructures are invariant across these Catalan posets: Chenevière proves that, for each fixed n and each height parameter k (in the sense of [21]), the Tamari lattice and the Dyck lattice have the same number of *linear intervals* (intervals whose Hasse diagram is a chain), and moreover all alt-Tamari posets share this same count at each height [21]. In the present language, this says that the number of "diamond-free corridors" (regions with a unique maximal chain) is stable under a wide class of tier-local adjacency choices, reinforcing the theme that many distinct dynamics can be layered on a single Catalan substrate without changing its most rigid combinatorial invariants.

Adjacency and Laplacian. Define the adjacency operator A_{G_n} and the degree operator D_{G_n} on $\ell^2(\mathcal{D}_n)$ by

$$(A_{G_n}\psi)(w) := \sum_{w' \sim w} \psi(w'), \qquad (D_{G_n}\psi)(w) := \deg(w)\, \psi(w),$$

where $w' \sim w$ denotes adjacency in G_n. The (combinatorial) graph Laplacian is

$$\Delta_{G_n} := D_{G_n} - A_{G_n},$$

and the associated diffusion generator is

$$L_{G_n} := -\Delta_{G_n}.$$

Then Δ_{G_n} is self-adjoint and positive semidefinite, while L_{G_n} is self-adjoint and negative semidefinite.

Discrete heat and Schrödinger equations. The heat equation on the tier graph is the linear ODE

$$\partial_\tau u = L_{G_n} u,$$

with solution $u(\tau) = e^{\tau L_{G_n}} u(0)$. Because $-\Delta_{G_n}$ is self-adjoint and nonpositive, $e^{\tau L_{G_n}}$ is a contraction semigroup. The corresponding unitary "free" Schrödinger evolution is

$$i\, \partial_t \psi = -L_{G_n} \psi = \Delta_{G_n} \psi,$$

with solution $\psi(t) = e^{-it\Delta_{G_n}} \psi(0)$.

Remark C.5. *This optional tier-graph framework does not fix a preferred choice of adjacency G_n and is not used elsewhere in the paper. Its purpose is to make explicit that, once a tier-local notion of neighbourhood is specified, discrete diffusion and Schrödinger-type evolutions on the Catalan state space follow by standard graph-Laplacian constructions (compare Remark A.5 and Remark A.6 for the tier-growth Markov structure induced by Dyck conditioning).*

D Computational Foundations

This appendix records proof sketches for the computational claims used in the main text, including the Catalan universality statement for program structure and the disjoint-commutation property of Lemma D.2. For a lean v1, extended examples and additional formal development are omitted.

Lemma D.1 (Internalizing Symbols as Motifs)**.** *Let Σ be a finite (or countable) alphabet of atomic symbols, and consider finite applicative terms over Σ (binary application with atoms). There is an injective encoding of such labeled application trees into the Catalan family $\mathcal{T}$ of unlabeled full binary trees. One explicit construction fixes two distinct tag trees $A, B \in \mathcal{T}$, assigns each $\sigma \in \Sigma$ a distinct motif $S_\sigma \in \mathcal{T} \setminus \{A, B\}$, and defines*

$$E(\sigma) \;:=\; \bullet(A, S_\sigma), \qquad E(t \cdot u) \;:=\; \bullet\big(B, \bullet(E(t), E(u))\big),$$

where $t \cdot u$ denotes application and $\bullet(\,\cdot\,,\,\cdot\,)$ denotes binary pairing of subtrees. The left tag distinguishes atoms from applications, so E is recursively decodable and therefore injective.

Catalan universality for program structure. The following is a proof sketch of Proposition 3.1 in the main paper.

Proof sketch. As described in classical treatments of combinatory logic and the λ-calculus [8, 4], application is a binary operation, and every term therefore possesses a unique representation as a full binary tree: internal nodes encode application, and leaves encode variables, constants, or combinators. This establishes a canonical embedding of programs into $\mathcal{T}$.

Conversely, any full binary tree with labeled leaves may be interpreted as a well-formed program term by reading internal nodes as applications and leaves as atomic symbols, yielding a unique term up to α-equivalence.

Operational semantics are defined via local tree rewrites. A β-redex $(\lambda x.M)\,N$ contracts by replacing the parent application with $M[x := N]$; SKI reductions replace specific subtrees according to fixed patterns. In each case, the output remains a full binary tree, so evaluation never leaves $\mathcal{T}$. Because nondeterministic redex choices correspond to branching in the space of trees, each complete reduction sequence is a path through the Catalan possibility space, completing the correspondence. □

Lemma D.2 (Commutation of disjoint reductions)**.** *Let T be a full binary tree and consider any local rewrite system whose single-step reductions replace a rooted subtree matching a finite pattern by a new subtree, leaving the rest of T unchanged. Suppose two single-step reductions are applicable at positions p and q whose rooted subtrees are disjoint (neither position lies on the root-to-node path of the other). Let T_p denote the result of applying the reduction at p, and similarly T_q. Then both reductions remain applicable after the other, and they commute:*

$$(T_p)_q \;\equiv\; (T_q)_p.$$

In particular, disjoint reductions form a commuting diamond as in the usual commuting-diamond picture.

Proof. Since p and q lie in disjoint subtrees, contracting at p rewrites only the subtree rooted at p and leaves the subtree rooted at q unchanged. Symmetrically, contracting at q leaves the subtree at p unchanged. Because the two rewrite steps act on disjoint parts of the tree, performing both contractions yields the same result regardless of order. □

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