

# Supplement to *The Catalan Light Cone*

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## Abstract

This companion document collects additional appendices and proof sketches that are omitted from the lean arXiv v1 of *The Catalan Light Cone*. It is interpretation-neutral and intended to preserve auxiliary results and extended technical development without expanding the main paper.

## A A Conditional Uniqueness Theorem for the Dyck-Prefix Substrate

This appendix isolates a minimal set of structural axioms under which the Dyck-prefix substrate is not merely an example but is determined uniquely up to isomorphism. The result is interpretation-neutral: it assumes no particular computational semantics (evaluation order, combinators, rewriting), only a ranked causal-growth structure with a one-sided admissibility boundary.

### A.1 Ranked growth posets

**Definition A.1** (Ranked growth poset). *A ranked growth poset is a triple  $(S, \preceq, \text{rk})$  where  $S$  is a countable set,  $\preceq$  is a partial order on  $S$ , and  $\text{rk} : S \rightarrow \mathbb{N}$  is a rank function such that:*

- (i) *for every  $t \in S$  the set  $\{s \in S : s \preceq t\}$  is finite, and*
- (ii)  *$\text{rk}$  is strictly order-increasing along cover relations.*

Write  $s \prec t$  when  $s \preceq t$  and  $s \neq t$ , and write  $s \triangleleft t$  if  $s \prec t$  and there is no  $u$  with  $s \prec u \prec t$  (i.e.  $t$  covers  $s$ ).

**Definition A.2** (Two-move height structure). *Let  $(S, \preceq, \text{rk})$  be a ranked growth poset with a distinguished root  $s_\emptyset \in S$  satisfying  $\text{rk}(s_\emptyset) = 0$ . A two-move height structure on  $S$  consists of a function  $h : S \rightarrow \mathbb{Z}_{\geq 0}$  (height) with  $h(s_\emptyset) = 0$  such that every cover relation  $s \triangleleft t$  satisfies*

$$\text{rk}(t) = \text{rk}(s) + 1 \quad \text{and} \quad h(t) = h(s) \pm 1.$$

We call a cover with  $h(t) = h(s) + 1$  an up-step and a cover with  $h(t) = h(s) - 1$  a down-step.

### A.2 Axioms for the Catalan core

The uniqueness theorem below follows from four axioms capturing: (i) discrete single-step growth, (ii) exactly two local move types (up/down), (iii) a one-sided boundary in which down-steps are disabled at height 0, and (iv) an *unfolded* (collision-free) history space.

**Definition A.3** (Catalan core axioms). *A ranked growth poset  $(S, \preceq, \text{rk})$  with root  $s_\emptyset$  and two-move height structure  $h$  satisfies the Catalan core axioms if:*

- (A1) (**Ranked single-step growth**) For every cover  $s \triangleleft t$ , one has  $\text{rk}(t) = \text{rk}(s) + 1$  and  $h(t) = h(s) \pm 1$ .
- (A2) (**Local determinism by move type**) For each  $s \in S$ , there exists at most one up-step successor  $t_\uparrow$  with  $s \triangleleft t_\uparrow$  and  $h(t_\uparrow) = h(s) + 1$ , and at most one down-step successor  $t_\downarrow$  with  $s \triangleleft t_\downarrow$  and  $h(t_\downarrow) = h(s) - 1$ .
- (A3) (**One-sided boundary admissibility**) For each  $s \in S$ , an up-step successor exists, while a down-step successor exists iff  $h(s) > 0$ .
- (A4) (**No collisions / unique history**) Each  $t \in S$  admits a unique saturated chain (cover chain) from the root:

$$s_\emptyset = s_0 \triangleleft s_1 \triangleleft \cdots \triangleleft s_{\text{rk}(t)} = t.$$

### A.3 Dyck-prefix poset

Let  $\Sigma = \{+, -\}$ . For a word  $w = w_1 \cdots w_n \in \Sigma^n$ , define the partial-sum height

$$H_w(k) = \sum_{j=1}^k \xi(w_j), \quad \text{where } \xi(+) = +1, \xi(-) = -1.$$

Call  $w$  *admissible* if  $H_w(k) \geq 0$  for all  $k$ . Let  $\mathcal{C}$  be the set of all admissible words (Dyck prefixes), partially ordered by prefix:  $u \preceq v$  iff  $u$  is a prefix of  $v$ . Let  $\ell(w) = n$  be length and  $H(w) = H_w(n)$  be final height.

**Definition A.4** (Dyck-prefix poset). *The Dyck-prefix poset is  $(\mathcal{C}, \preceq, \ell)$  with root  $\varepsilon$  (the empty word).*

### A.4 Uniqueness theorem

**Theorem A.1** (Dyck-prefix uniqueness). *Let  $(S, \preceq, \text{rk})$  be a ranked growth poset with root  $s_\emptyset$  and a two-move height structure  $h$ . If  $S$  satisfies the Catalan core axioms of Definition A.3, then there exists a unique bijection*

$$\pi : S \rightarrow \mathcal{C}$$

*such that for every  $s \in S$ ,*

$$\ell(\pi(s)) = \text{rk}(s), \quad H(\pi(s)) = h(s),$$

*and  $\pi$  is an order isomorphism:*

$$s \preceq t \iff \pi(s) \preceq \pi(t) \quad (\text{prefix order}).$$

*In particular,  $(S, \preceq, \text{rk})$  is isomorphic to the Dyck-prefix poset.*

*Proof.* By (A4), each  $s \in S$  admits a unique cover chain  $s_\emptyset = s_0 \triangleleft \cdots \triangleleft s_n = s$  where  $n = \text{rk}(s)$ . Define a word  $\pi(s) \in \Sigma^n$  by setting

$$\pi(s)_k = \begin{cases} + & \text{if } h(s_k) = h(s_{k-1}) + 1, \\ - & \text{if } h(s_k) = h(s_{k-1}) - 1. \end{cases}$$

This is well-defined and has length  $\ell(\pi(s)) = n$ . By construction, the partial sums of  $\pi(s)$  coincide with the height along the chain,

$$H_{\pi(s)}(k) = h(s_k) \geq 0 \quad \text{for all } k,$$

so  $\pi(s)$  is admissible and  $\pi(s) \in \mathcal{C}$ . Moreover  $H(\pi(s)) = H_{\pi(s)}(n) = h(s)$ .

We now show that  $\pi$  is bijective. Given any admissible word  $w = w_1 \cdots w_n \in \mathcal{C}$ , construct a chain in  $S$  by starting at  $s_0 = s_\emptyset$  and for each  $k \geq 1$  taking the successor specified by  $w_k$ :

$$s_k = \begin{cases} \text{the (unique) up-step successor of } s_{k-1} & \text{if } w_k = +, \\ \text{the (unique) down-step successor of } s_{k-1} & \text{if } w_k = -. \end{cases}$$

Existence and uniqueness follow from (A2)–(A3); admissibility of  $w$  ensures that a down-step is only requested when the current height is positive. Let  $s := s_n$  be the resulting node at rank  $n$ . Then  $\pi(s) = w$ , proving surjectivity. Injectivity follows from (A4): distinct nodes have distinct cover chains and hence distinct step sequences.

Finally,  $\pi$  preserves order. If  $s \preceq t$ , then the unique chain to  $t$  contains the chain to  $s$  as an initial segment, so  $\pi(s)$  is a prefix of  $\pi(t)$ . Conversely, if  $\pi(s)$  is a prefix of  $\pi(t)$ , then the constructed chain for  $\pi(t)$  passes through the constructed chain for  $\pi(s)$ , giving  $s \preceq t$ . Uniqueness of  $\pi$  is immediate from (A4), since  $\pi(s)$  is forced by the unique cover chain to  $s$ .  $\square$

## A.5 Catalan enumeration at completion

To recover the usual Catalan counting, impose a completion boundary condition: completion occurs when the height returns to zero at even rank.

**Definition A.5** (Completed histories). *For  $n \in \mathbb{N}$ , define the set of completed histories at rank  $2n$  by*

$$S_{2n}^{\text{comp}} := \{s \in S : \text{rk}(s) = 2n \text{ and } h(s) = 0\}.$$

**Corollary A.1** (Catalan counting). *Under the hypotheses of Theorem A.1, the number of completed histories satisfies*

$$\#(S_{2n}^{\text{comp}}) = C_n = \frac{1}{n+1} \binom{2n}{n},$$

*the  $n$ th Catalan number.*

*Proof.* By Theorem A.1,  $S_{2n}^{\text{comp}}$  is in bijection with admissible words of length  $2n$  with final height 0, i.e. Dyck words of semilength  $n$ . These are counted by the Catalan number  $C_n$ .  $\square$

## A.6 Quotients and structural sharing

The “no collisions” axiom (A4) asserts that  $S$  is an *unfolded* history space: each node encodes a distinct growth history. This does not preclude gauge equivalences or multi-way computation graphs; rather, those arise by quotienting  $S$  by an equivalence relation.

**Remark A.1** (Quotients and gauge identification). *Let  $\sim$  be an equivalence relation on  $S$  representing gauge or rewrite identifications. The quotient set  $S/\sim$  may exhibit merges (a DAG-like state graph), even if  $S$  is collision-free. Theorem A.1 characterizes the unfolded core; quotient structure is encoded by the choice of  $\sim$  and acts as an identification of histories rather than a primitive feature of the substrate.*

**Remark A.2** (Structural sharing vs. semantic collisions). *Axiom (A4) concerns semantic identity in  $S$  (distinct histories are distinct nodes). It is compatible with immutability and structural sharing in implementations (e.g. hash-consing), which may share representation of common substructures without identifying distinct history nodes.*

## A.7 Scope of the result

Theorem A.1 establishes a core uniqueness statement: given two local move types changing an integer height by  $\pm 1$ , a one-sided boundary disabling down-steps at height 0, and an unfolded history space, the substrate is forced to be Dyck-prefix order. This statement is independent of additional semantic layers (program interpretations, rewriting dynamics, or amplitude assignments), which can be imposed on top of the substrate without affecting the isomorphism.

## B Unfoldings and Covers of Growth Posets

Many natural representations of computation or gauge-identified state spaces exhibit merges: distinct growth histories may lead to the same state, producing a DAG-like multi-way graph rather than a tree. The core uniqueness theorem in Appendix A characterizes the *unfolded* history space. This appendix makes that relationship explicit by defining a canonical unfolding (history cover) for a ranked growth poset and showing that the Catalan uniqueness statement applies upstairs even when merges exist downstairs.

### B.1 Growth graphs

**Definition B.1** (Rooted ranked growth poset). *A rooted ranked growth poset is a ranked growth poset  $(S, \preceq, \text{rk})$  (Definition A.1) equipped with a distinguished root  $s_\emptyset \in S$  such that for every  $s \in S$  one has  $s_\emptyset \preceq s$  (i.e. every node is reachable from the root).*

**Definition B.2** (Directed growth graph). *Let  $(S, \preceq, \text{rk})$  be a rooted ranked growth poset. Its directed growth graph is the rooted directed graph  $G(S) = (V, E, s_\emptyset)$  where  $V = S$  and  $(s, t) \in E$  iff  $s \prec t$  (i.e.  $t$  covers  $s$ ). We regard each edge as a single-step growth move.*

**Definition B.3** (Move type induced by height). *If  $(S, \preceq, \text{rk})$  carries a two-move height structure  $h$  (Definition A.2), then each directed edge  $s \rightarrow t$  in  $G(S)$  has a move type*

$$\text{type}(s \rightarrow t) \in \{+, -\} \quad \text{defined by} \quad \text{type}(s \rightarrow t) = \begin{cases} + & \text{if } h(t) = h(s) + 1, \\ - & \text{if } h(t) = h(s) - 1. \end{cases}$$

### B.2 Rooted covers (local isomorphism in one step)

To model “the same local growth rule everywhere” we use a minimal notion of cover: a map that is locally bijective on outgoing edges from each node. This avoids introducing general poset coverings and suffices for the unfolding argument below.

**Definition B.4** (Rooted directed cover). *Let  $G = (V, E, r)$  and  $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{r})$  be rooted directed graphs. A function  $\varphi : \tilde{V} \rightarrow V$  is a rooted directed cover if:*

$$(i) \quad \varphi(\tilde{r}) = r;$$

(ii) *for every  $\tilde{v} \in \tilde{V}$ , the map*

$$\varphi_* : \text{Out}(\tilde{v}) \rightarrow \text{Out}(\varphi(\tilde{v})), \quad (\tilde{v} \rightarrow \tilde{w}) \mapsto (\varphi(\tilde{v}) \rightarrow \varphi(\tilde{w}))$$

*is a bijection, where  $\text{Out}(x) = \{x \rightarrow y \in E\}$  denotes the set of outgoing edges from  $x$ .*

*If edges carry move types  $\{+, -\}$ , we additionally require that  $\varphi_*$  preserves move type (i.e. the bijection matches  $+$ -edges to  $+$ -edges and  $-$ -edges to  $-$ -edges).*

### B.3 The canonical unfolding (history cover)

**Definition B.5** (Unfolding as rooted path space). *Let  $(S, \preceq, \text{rk})$  be a rooted ranked growth poset and let  $G(S)$  be its directed growth graph. Define the unfolding  $\tilde{S}$  to be the set of all finite directed paths in  $G(S)$  starting at the root:*

$$\tilde{S} := \{(s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n) : s_0 = s_\emptyset, (s_{k-1}, s_k) \in E\}.$$

Write  $\tilde{s} = (s_0 \rightarrow \cdots \rightarrow s_n)$  and define:

- (i) the endpoint map  $\varphi : \tilde{S} \rightarrow S$  by  $\varphi(\tilde{s}) = s_n$ ;
- (ii) the rank  $\tilde{\text{rk}}(\tilde{s}) = n$  (path length);
- (iii) the prefix order on  $\tilde{S}$ :  $\tilde{u} \preceq \tilde{v}$  iff  $\tilde{u}$  is an initial segment (prefix) of  $\tilde{v}$ .

If  $S$  has a height function  $h$ , define the lifted height  $\tilde{h}(\tilde{s}) := h(\varphi(\tilde{s}))$ .

**Lemma B.1** (The unfolding is collision-free). *Every element  $\tilde{s} \in \tilde{S}$  has a unique predecessor chain from the root in the prefix order, i.e.  $\tilde{S}$  satisfies the “no collisions / unique history” property (A4) of Definition A.3.*

*Proof.* By construction, each  $\tilde{s}$  is itself a rooted path  $(s_0 \rightarrow \cdots \rightarrow s_n)$ . Its strict prefixes are exactly the initial segments  $(s_0 \rightarrow \cdots \rightarrow s_k)$  for  $0 \leq k < n$ , and these form the unique saturated chain from the root to  $\tilde{s}$  under prefix inclusion.  $\square$

**Lemma B.2** (The endpoint map is a rooted directed cover). *Let  $G(\tilde{S})$  be the directed growth graph of the unfolding, whose edges append one growth step:*

$$(s_0 \rightarrow \cdots \rightarrow s_n) \rightarrow (s_0 \rightarrow \cdots \rightarrow s_n \rightarrow s_{n+1}) \quad \text{whenever } s_n \rightarrow s_{n+1} \text{ is an edge in } G(S).$$

*Then the endpoint map  $\varphi : \tilde{S} \rightarrow S$  from Definition B.5 is a rooted directed cover in the sense of Definition B.4. If  $S$  carries a two-move height structure, then  $\varphi$  preserves move type.*

*Proof.* Clearly  $\varphi$  maps the length-0 path  $(s_\emptyset)$  to  $s_\emptyset$ . Fix a path  $\tilde{s} = (s_0 \rightarrow \cdots \rightarrow s_n)$ . Outgoing edges from  $\tilde{s}$  in  $G(\tilde{S})$  are in bijection with outgoing edges from  $s_n$  in  $G(S)$  by appending the corresponding last step  $s_n \rightarrow s_{n+1}$ . Under  $\varphi$ , each appended edge maps to exactly that edge  $s_n \rightarrow s_{n+1}$ , giving a bijection on outgoing edges. If move types are present, appending an up-step or down-step in  $\tilde{S}$  maps to an up-step or down-step in  $S$  by Definition B.3, so type is preserved.  $\square$

### B.4 Dyck-prefix structure upstairs

**Proposition B.1** (Lift of the Catalan core axioms). *Suppose  $(S, \preceq, \text{rk})$  is a rooted ranked growth poset with a two-move height structure  $h$  satisfying axioms (A1)–(A3) of Definition A.3 (ranked single-step growth, local determinism by move type, and one-sided boundary admissibility), but not necessarily (A4). Then the unfolding  $(\tilde{S}, \preceq, \tilde{\text{rk}})$  with lifted height  $\tilde{h}$  satisfies (A1)–(A4).*

*Proof.* Axiom (A4) holds by Lemma B.1. For (A1)–(A3), each cover in  $\tilde{S}$  appends exactly one edge of  $G(S)$ , so ranks increase by one and height changes by  $\pm 1$  exactly as in  $S$ . Moreover, by local determinism in  $S$ , from any endpoint there is at most one  $+$ -successor and at most one  $-$ -successor; by Lemma B.2 the same holds in the unfolding at every path. Finally, one-sided boundary admissibility is preserved: a  $-$ -extension exists in  $\tilde{S}$  precisely when the endpoint height is positive.  $\square$

**Theorem B.1** (Dyck-prefix characterization of the unfolding). *Under the hypotheses of Proposition B.1, the unfolding  $\tilde{S}$  is (canonically) order-isomorphic to the Dyck-prefix poset (Definition A.4). In particular, completed unfolded histories are counted by Catalan numbers as in Corollary A.1.*

*Proof.* By Proposition B.1,  $\tilde{S}$  satisfies the full Catalan core axioms (A1)–(A4). The conclusion follows by applying Theorem A.1 to  $\tilde{S}$ .  $\square$

**Remark B.1** (Canonical quotient and “collisions”). *The unfolding comes with a canonical surjection  $\varphi : \tilde{S} \rightarrow S$  (the endpoint map). Collisions/merges in  $S$  correspond exactly to identifications of distinct unfolded histories:*

$$\tilde{u} \sim_{\varphi} \tilde{v} \iff \varphi(\tilde{u}) = \varphi(\tilde{v}).$$

*Thus, as a set of states reachable from the root,  $S$  is naturally identified with the quotient  $\tilde{S}/\sim_{\varphi}$ . For the distinction between semantic identification and structural sharing in implementations, see Remark A.2.*

## B.5 A Reduction-Theoretic “Full Uniqueness” Statement

The core uniqueness theorem (Theorem A.1) shows that a collision-free two-move growth system with a one-sided boundary is forced to be Dyck-prefix order. This subsection records a complementary reduction statement: if a canonical notion of *intrinsic state* (defined by future-cone isomorphism) is already a minimal one-counter system, and if the projection to intrinsic state is locally bijective on labeled moves, then the Catalan core is forced as the unfolded normal form.

### B.5.1 Future cones and intrinsic-state equivalence

**Definition B.6** (Future cone as a rooted directed graph). *Let  $(S, \preceq, \text{rk})$  be a rooted ranked growth poset with directed growth graph  $G(S) = (S, E, s_0)$  (Definition B.2). For  $s \in S$ , define the future cone at  $s$  to be the rooted directed subgraph*

$$\text{Cone}(s) := (S_s, E_s, s),$$

*where  $S_s = \{t \in S : s \preceq t\}$  and  $E_s = \{(u, v) \in E : u, v \in S_s\}$ . If edges in  $G(S)$  are labeled by a finite move alphabet (in particular  $\{+, -\}$ ), we regard  $\text{Cone}(s)$  as an edge-labeled rooted directed graph.*

**Definition B.7** (Cone isomorphism and intrinsic-state equivalence). *Assume edges are labeled by move type in  $\{+, -\}$ . For  $s, t \in S$ , write  $s \equiv t$  if there exists a rooted directed graph isomorphism*

$$\psi : \text{Cone}(s) \cong \text{Cone}(t)$$

*sending root to root and preserving move labels. This defines an equivalence relation  $\equiv$  on  $S$ , called intrinsic-state equivalence. Let  $M := S/\equiv$  be the set of equivalence classes, with root  $m(s_0) \in M$ , and write*

$$m : S \rightarrow M, \quad m(s) = [s]_{\equiv}$$

*for the canonical projection.*

**Remark B.2** (Relocatable futures). *The equivalence relation  $\equiv$  captures a precise form of relocatability: states are indistinguishable if and only if their reachable futures are isomorphic as rooted labeled growth graphs. The quotient  $M = S/\equiv$  is a canonical “coarsest” state descriptor that preserves the full future growth structure.*

### B.5.2 Intrinsic dynamics and the cover hypothesis

**Definition B.8** (Intrinsic transition graph). *Let  $G(S)$  be labeled by  $\{+, -\}$ . Define a rooted labeled directed graph  $G(M)$  on  $M$  by declaring a labeled edge  $x \rightarrow_{\pm} y$  to exist if there are  $s, t \in S$  such that  $s \leq t$  is a  $\pm$ -labeled edge of  $G(S)$  and  $m(s) = x$ ,  $m(t) = y$ .*

**Definition B.9** (One-counter intrinsic dynamics). *We say  $(S, \preceq, \text{rk})$  has one-counter intrinsic dynamics if there exists a bijection*

$$\iota : M \rightarrow \mathbb{Z}_{\geq 0}$$

*with  $\iota(m(s_0)) = 0$  such that, for every intrinsic state  $x \in M$  with  $\iota(x) = k$ :*

- (i) (**Two primitive moves**) *there exists exactly one  $+$ -successor  $y_+$  of  $x$  in  $G(M)$  with  $\iota(y_+) = k + 1$ ;*
- (ii) (**One-sided boundary**) *there exists a  $-$ -successor  $y_-$  of  $x$  in  $G(M)$  iff  $k > 0$ , and in that case  $\iota(y_-) = k - 1$ ;*
- (iii) (**No other intrinsic moves**)  *$x$  has no other outgoing edges in  $G(M)$  besides  $x \rightarrow_+ y_+$  and (when  $k > 0$ )  $x \rightarrow_- y_-$ .*

*Equivalently,  $G(M)$  is isomorphic (as a rooted labeled graph) to the standard one-counter graph on  $\mathbb{Z}_{\geq 0}$  with edges  $k \rightarrow_+ k + 1$  for all  $k$  and edges  $k \rightarrow_- k - 1$  for  $k > 0$ .*

**Remark B.3** (Minimality content). *Definition B.9 should be read as a minimality hypothesis: the intrinsic descriptor  $M = S/\equiv$  is already a single nonnegative integer, and it admits exactly two primitive move types with a one-sided boundary at 0.*

**Definition B.10** (Cover-consistency of the intrinsic projection). *Assume edges are labeled by  $\{+, -\}$ . We say the intrinsic projection  $m : S \rightarrow M$  is cover-consistent if it is a rooted directed cover (Definition B.4) from the labeled growth graph  $G(S)$  to the intrinsic graph  $G(M)$  (Definition B.8), i.e. for every  $s \in S$  the induced map on outgoing edges  $\text{Out}(s) \rightarrow \text{Out}(m(s))$  is a label-preserving bijection.*

**Remark B.4.** *Under the one-counter identification  $\iota : M \rightarrow \mathbb{Z}_{\geq 0}$  of Definition B.9, cover-consistency of  $m$  is equivalent to requiring that the composite  $h := \iota \circ m : S \rightarrow \mathbb{Z}_{\geq 0}$  is a rooted directed cover from  $G(S)$  to the standard one-counter graph on  $\mathbb{Z}_{\geq 0}$ .*

### B.5.3 Reduction to the Dyck-prefix normal form

**Theorem B.2** (Reduction to a Dyck-prefix cover). *Let  $(S, \preceq, \text{rk})$  be a rooted ranked growth poset whose directed growth graph  $G(S)$  is labeled by move type in  $\{+, -\}$ . Assume:*

- (H1) (**Intrinsic-state quotient**) *intrinsic-state equivalence  $\equiv$  is defined by label-preserving cone isomorphism as in Definition B.7, with quotient  $m : S \rightarrow M$ ;*
- (H2) (**One-counter intrinsic dynamics**)  *$S$  satisfies Definition B.9;*
- (H3) (**Cover consistency**) *the intrinsic projection  $m$  is a rooted directed cover as in Definition B.10;*
- (H4) (**Tiered single-step growth**) *for every cover  $s \leq t$ ,  $\text{rk}(t) = \text{rk}(s) + 1$ .*

Let  $\tilde{S}$  be the unfolding (history cover) of  $S$  (Definition B.5) with endpoint map  $\varphi : \tilde{S} \rightarrow S$ . Then  $\tilde{S}$  is (canonically) order-isomorphic to the Dyck-prefix poset (Definition A.4), and hence  $S$  is a quotient (folding) of a Dyck-prefix cover via  $\varphi$  (Remark B.1).

*Proof.* Define a height function on  $S$  by composing the intrinsic-state projection with the one-counter identification,

$$h(s) := \iota(m(s)) \in \mathbb{Z}_{\geq 0}.$$

By cover consistency (Definition B.10) and one-counter intrinsic dynamics (Definition B.9), each  $s \in S$  has exactly one outgoing  $+$ -edge, and it has an outgoing  $-$ -edge iff  $h(s) > 0$ . Moreover, because  $m$  is label-preserving and  $\iota$  identifies  $G(M)$  with the standard one-counter graph, along any cover edge  $s < t$  one has  $h(t) = h(s) + 1$  for a  $+$ -edge and  $h(t) = h(s) - 1$  for a  $-$ -edge. Together with the tier assumption  $\text{rk}(t) = \text{rk}(s) + 1$ , this shows that  $S$  admits a two-move height structure satisfying axioms (A1)–(A3) of Definition A.3.

Applying Theorem B.1 now yields that the unfolding  $\tilde{S}$  is Dyck-prefix order-isomorphic. The quotient statement for  $S$  is then exactly Remark B.1.  $\square$

**Remark B.5** (What is and is not proved). *Theorem B.2 is a “full uniqueness” statement in the reduction sense: once relocatable futures are formalized via cone isomorphism and the intrinsic state space is assumed to be a one-counter with two primitive moves, together with the cover-consistency hypothesis (no hidden multiplicity in the projection  $m$ ), the Catalan substrate is forced as the unfolded normal form. What is not claimed is that all reasonable substrates satisfy these minimality hypotheses; establishing that requires separate domain-specific arguments.*

## B.6 Interpretive takeaway

Theorems A.1 and B.1 together support the following robust core statement: whenever a ranked two-move growth system with a one-sided boundary is present, the unfolded history space is forced to be Dyck-prefix order, and any merged multi-way representation is a quotient of that Catalan cover.

# C Additional Technical Notes

This appendix collects optional bookkeeping and auxiliary constructions. It is separate from the main formal development and may be skipped on first reading.

## C.1 Entropy of coarse-graining and information rate

Fix a tier  $n$  and an observable (deterministic coarse-graining)  $f : \mathcal{D}_n \rightarrow \mathcal{X}$ . For  $x \in \mathcal{X}$  write

$$N(x) := \#\{w \in \mathcal{D}_n : f(w) = x\},$$

so that  $f^{-1}(x)$  is the equivalence class of histories identified as the same outcome. If  $f$  is prefix-local in the sense of Remark 4.1, then the multiplicities  $N(x)$  admit transfer recursions on the induced state space.

**Multiplicity entropy.** Define the (microcanonical) entropy of the full ensemble at tier  $n$  by

$$S_n := \log \#(\mathcal{D}_n),$$

and the conditional entropy of an outcome  $x$  by

$$S(x) := \log N(x).$$

The information eliminated by selecting outcome  $x$  is the entropy drop

$$\Delta S(x) := S_n - S(x) = \log\left(\frac{\#(\mathcal{D}_n)}{N(x)}\right).$$

(Any logarithm base may be used; base 2 yields units of bits.)

**Remark C.1** (Retrospective vs. prospective counts). *The multiplicity entropy  $S(x) = \log N(x)$  measures how many fine-grained histories are identified as the same outcome at tier  $n$ . A complementary forward-looking quantity is the number of admissible continuations of a realized prefix into higher tiers (the size of its local cone; see Section 2.9), which may be studied by counting completions as a function of current height (Lemma 2.1).*

**Completion multiplicity and future-cone entropy.** Fix a Dyck prefix  $u$  of length  $k$  and height  $h$ . For a target tier  $n$  with  $2n \geq k$ , define the completion multiplicity

$$M_n(u) := \#\{w \in \mathcal{D}_n : u \preceq w\},$$

the number of completed histories at tier  $n$  consistent with the partial history  $u$ . This is the finite-tier size of the local cone rooted at  $u$ . It depends only on the remaining step budget  $s := 2n - k$  and the current height  $h$ , and admits the explicit ballot-number formula of Lemma 2.1. The associated future-cone entropy is

$$S_n^{\text{cone}}(u) := \log M_n(u).$$

**Information rate as rate of possibility reduction.** Let  $m$  denote the number of selection events (local contractions) along a history, and let computational proper time be  $\tau = \tau_0 m$  for a fixed scale  $\tau_0 > 0$ . We define the information rate associated with outcome  $x$  to be the information loss per unit computational proper time,

$$R(x) := \frac{\Delta S(x)}{\tau}.$$

In the simplest case of a single event ( $m = 1$ ), this reduces to  $R(x) = \Delta S(x)/\tau_0$ . One may also adopt a coarser tier-wise selection model in which  $m$  is identified with the tier index  $n$  (one selection per tier boundary), but we keep these notions separate in general.

**Gauge-invariant counting.** When histories admit a redundancy under commuting spacelike-separated updates, one may quotient the fine-grained history set at tier  $n$  by the induced gauge equivalence relation  $\sim_g$  of Definition 5.1. When  $\mathcal{D}_n$  is taken to parametrize such fine-grained histories at tier  $n$ , define  $\bar{\mathcal{D}}_n := \mathcal{D}_n / \sim_g$ . If  $f$  is gauge-invariant (constant on  $\sim_g$ -orbits), define

$$\bar{N}(x) := \#\{[w] \in \bar{\mathcal{D}}_n : f(w) = x\}, \quad \bar{\Delta S}(x) := \log\left(\frac{\#(\bar{\mathcal{D}}_n)}{\bar{N}(x)}\right),$$

and use  $\bar{\Delta S}$  in place of  $\Delta S$ . This removes overcounting due solely to reordering of independent collapses.

## C.2 Size bookkeeping for subtree-selection collapses

In addition to ensemble-level multiplicities, some collapse rules admit a simple size bookkeeping at the level of individual trees. Let  $T$  be a finite full binary tree. Define its *internal size*  $U(T)$  (number of internal nodes) recursively by

$$U(()) := 0, \quad U(\bullet(L, R)) := 1 + U(L) + U(R),$$

where  $()$  denotes a leaf and  $\bullet(L, R)$  denotes a binary pair.

**Lemma C.1** (Size drop under subtree selection). *Consider the local collapse that replaces a pair  $\bullet(L, R)$  by one of its children:*

$$\bullet(L, R) \rightsquigarrow L \quad \text{or} \quad \bullet(L, R) \rightsquigarrow R.$$

*Then the size drop is*

$$U(\bullet(L, R)) - U(L) = 1 + U(R), \quad U(\bullet(L, R)) - U(R) = 1 + U(L).$$

*In particular, if the collapse rule keeps the larger child in  $U$  (i.e. keeps  $\arg \max\{U(L), U(R)\}$ ), then the drop is  $1 + \min\{U(L), U(R)\}$ .*

*Proof.* Immediate from the defining recursion  $U(\bullet(L, R)) = 1 + U(L) + U(R)$ .  $\square$

## C.3 Fields on words, prefixes, and nodes (optional)

We use the word “field” as shorthand for a complex-valued function on one of the Catalan objects already in play. Several closely related state spaces are useful in different contexts.

**Fields on completed histories (fixed tier).** Fix  $n$  and consider a function  $\Phi_n : \mathcal{D}_n \rightarrow \mathbb{C}$  assigning an amplitude (or observable value) to each completed history  $w \in \mathcal{D}_n$ . The associated Hilbert space is  $\ell^2(\mathcal{D}_n)$  with inner product

$$\langle \psi, \phi \rangle := \sum_{w \in \mathcal{D}_n} \overline{\psi(w)} \phi(w).$$

**Fields on prefixes (the full cone).** Let  $\mathcal{C}$  denote the set of Dyck prefixes (admissible partial histories). A prefix field is a function  $\Phi : \mathcal{C} \rightarrow \mathbb{C}$ , which may be restricted to a fixed length slice  $\mathcal{C}^{(k)} := \{p \in \mathcal{C} : |p| = k\}$  when needed.

**Fields on nodes of a fixed tree.** Given  $w \in \mathcal{D}_n$ , let  $T(w)$  be its associated full binary tree. A node field is a function  $\phi_w : \text{Int}(T(w)) \rightarrow \mathbb{C}$  on the internal nodes of that tree.

**Remark C.2.** *These notions live on different objects (tiers, the prefix poset, or a single tree) and are independent of any within-tier ordering convention on  $\mathcal{D}_n$ .*

## C.4 Subtree indicators as a multiscale spanning family (optional)

Let  $T$  be a finite rooted tree and write  $\text{Int}(T)$  for its internal nodes. Each  $v \in \text{Int}(T)$  determines a rooted subtree  $T_v$ , and hence a subset  $\text{Int}(T_v) \subseteq \text{Int}(T)$ . Define the subtree indicator

$$\chi_v : \text{Int}(T) \rightarrow \{0, 1\}, \quad \chi_v(u) := \mathbf{1}\{u \in \text{Int}(T_v)\}.$$

**Lemma C.2** (Subtree indicators form a basis). *The family  $\{\chi_v : v \in \text{Int}(T)\}$  is a basis of the vector space of complex-valued functions on  $\text{Int}(T)$ .*

*Proof.* Order the internal nodes by nonincreasing depth (deepest first), and let  $M$  be the square matrix with entries  $M_{uv} := \chi_v(u)$ . Then  $M_{vv} = 1$  for all  $v$ , while  $M_{uv} = 0$  whenever  $u$  precedes  $v$  in this order (a node cannot be a descendant of a deeper node). Thus  $M$  is triangular with ones on the diagonal, hence invertible. Therefore the indicators are linearly independent and, since their number equals  $\#\text{Int}(T)$ , they form a basis.  $\square$

**Corollary C.1** (Explicit inversion). *Let  $f : \text{Int}(T) \rightarrow \mathbb{C}$  be any function. There is a unique family of coefficients  $\{a_v\}_{v \in \text{Int}(T)}$  such that*

$$f = \sum_{v \in \text{Int}(T)} a_v \chi_v.$$

Writing  $\text{par}(v)$  for the parent of  $v$  (for  $v \neq \text{root}(T)$ ), these coefficients are given by

$$a_{\text{root}(T)} = f(\text{root}(T)), \quad a_v = f(v) - f(\text{par}(v)) \quad (v \neq \text{root}(T)).$$

*Proof.* For each  $u \in \text{Int}(T)$ ,  $(\sum_v a_v \chi_v)(u) = \sum_{v: u \in \text{Int}(T_v)} a_v = \sum_{v \preceq u} a_v$ , where  $v \preceq u$  means that  $v$  is an ancestor of  $u$ . With the stated choice of coefficients, this ancestor sum telescopes along the unique root-to- $u$  chain to yield  $f(u)$ . Uniqueness follows from Lemma C.2.  $\square$

**Remark C.3.** *This basis is “multiscale”: indicators of deep subtrees localize to fine regions of  $T$ , while indicators near the root encode coarse structure. Any choice of orthonormalization yields an orthonormal basis adapted to the rooted tree geometry.*

## C.5 Operators on a fixed history tree (optional)

In addition to tier-wise state spaces (fields on  $\mathcal{D}_n$ ), one may also consider dynamics *within* a fixed realized history by placing operators on the internal nodes of its tree.

**Node Hilbert space.** Fix  $w \in \mathcal{D}_n$  and let  $T(w)$  be its associated full binary tree. Write  $V_w := \text{Int}(T(w))$  and consider  $\ell^2(V_w)$  with inner product  $\langle \psi, \phi \rangle := \sum_{v \in V_w} \overline{\psi(v)} \phi(v)$ .

**Adjacency and Laplacian.** Let  $G_w = (V_w, E_w)$  be any finite undirected graph on  $V_w$  (for example, connect each internal node to its internal children). Define  $A_{G_w}$ ,  $D_{G_w}$ , and the graph Laplacian and generator by

$$\Delta_{G_w} := D_{G_w} - A_{G_w}, \quad L_{G_w} := -\Delta_{G_w}.$$

**Heat and Schrödinger evolutions.** The corresponding “internal-time” heat equation is

$$\partial_\tau u = L_{G_w} u,$$

and the corresponding unitary Schrödinger evolution is

$$i \partial_t \psi = -L_{G_w} \psi = \Delta_{G_w} \psi.$$

**Remark C.4.** *This within-history operator framework is independent of the tier-growth Markov dynamics and of coherent summation over histories: it simply records that, once a graph structure is specified on the internal nodes of a fixed Catalan tree, standard graph-Laplacian constructions yield discrete diffusion and Schrödinger-type evolutions on that fixed combinatorial background.*

## C.6 Operators on tier slices (optional)

The main text emphasizes two dynamics on the Catalan substrate: tier growth (prefix extension) and coherent summation over histories. Independently, one may also consider *slice dynamics* on a fixed tier by endowing the finite set  $\mathcal{D}_n$  with an auxiliary adjacency graph. This subsection records the standard operator framework for such constructions.

**Tier Hilbert space.** We take the tier state space to be  $\ell^2(\mathcal{D}_n)$  as in Section C.3.

**Adjacency graphs.** Let  $G_n = (\mathcal{D}_n, E_n)$  be any finite undirected graph on  $\mathcal{D}_n$ . The choice of  $G_n$  is additional structure: different graphs induce different notions of locality on the tier. A canonical example is the rotation graph (the associahedron adjacency) on full binary trees, where edges correspond to single associativity rotations [17, 21].

**Associahedra and planar tree amplitudes (scattering-amplitude tie-in).** The associahedron adjacency on  $\mathcal{D}_n$  is also natural from the perspective of scattering amplitudes. For the planar tree-level sector of bi-adjoint cubic scalar theory ( $\phi^3$ ), Arkani-Hamed, Bai, He, and Yan identify an associahedron in planar kinematic space and show that the tree amplitude is the corresponding canonical form of this positive geometry [19]. From this viewpoint, the Catalan enumeration of planar cubic tree diagrams is not merely counting: the associahedron organizes factorization channels geometrically, and different triangulations correspond to different diagrammatic expansions of the same canonical form (see, e.g., the review [22]). For quartic interactions, an analogous positive-geometry description involves Stokes polytopes rather than associahedra [20].

**Tamari/Dyck/alt-Tamari choices on the same tier.** The point of introducing an auxiliary graph  $G_n$  is that the underlying state set  $\mathcal{D}_n$  supports multiple natural notions of tier-locality coming from classical Catalan posets. The rotation graph is the undirected adjacency underlying the Tamari order; one may likewise equip  $\mathcal{D}_n$  with adjacency induced by the Dyck (“Stanley”) lattice on Dyck paths, or more generally by the family of  $\delta$ -Tamari (alt-Tamari) posets interpolating between these extremes. These alternatives use different covering relations on the same Catalan tier and therefore induce different graph Laplacians  $\Delta_{G_n}$  and different “free” tier Hamiltonians, but they live on a common configuration space  $\mathcal{D}_n$  [17, 21].

**Linear intervals as “1D corridors” in Catalan posets.** A useful robustness fact is that certain one-dimensional substructures are invariant across these Catalan posets: Chenevière proves that, for each fixed  $n$  and each height parameter  $k$  (in the sense of [21]), the Tamari lattice and the Dyck lattice have the same number of *linear intervals* (intervals whose Hasse diagram is a chain), and moreover all alt-Tamari posets share this same count at each height [21]. In the present language, this says that the number of “diamond-free corridors” (regions with a unique maximal chain) is stable under a wide class of tier-local adjacency choices, reinforcing the theme that many distinct dynamics can be layered on a single Catalan substrate without changing its most rigid combinatorial invariants.

**Adjacency and Laplacian.** Define the adjacency operator  $A_{G_n}$  and the degree operator  $D_{G_n}$  on  $\ell^2(\mathcal{D}_n)$  by

$$(A_{G_n}\psi)(w) := \sum_{w' \sim w} \psi(w'), \quad (D_{G_n}\psi)(w) := \deg(w) \psi(w),$$

where  $w' \sim w$  denotes adjacency in  $G_n$ . The (combinatorial) graph Laplacian is

$$\Delta_{G_n} := D_{G_n} - A_{G_n},$$

and the associated diffusion generator is

$$L_{G_n} := -\Delta_{G_n}.$$

Then  $\Delta_{G_n}$  is self-adjoint and positive semidefinite, while  $L_{G_n}$  is self-adjoint and negative semidefinite.

**Discrete heat and Schrödinger equations.** The heat equation on the tier graph is the linear ODE

$$\partial_\tau u = L_{G_n} u,$$

with solution  $u(\tau) = e^{\tau L_{G_n}} u(0)$ . Because  $-\Delta_{G_n}$  is self-adjoint and nonpositive,  $e^{\tau L_{G_n}}$  is a contraction semigroup. The corresponding unitary “free” Schrödinger evolution is

$$i \partial_t \psi = -L_{G_n} \psi = \Delta_{G_n} \psi,$$

with solution  $\psi(t) = e^{-it\Delta_{G_n}} \psi(0)$ .

**Remark C.5.** *This optional tier-graph framework does not fix a preferred choice of adjacency  $G_n$  and is not used elsewhere in the paper. Its purpose is to make explicit that, once a tier-local notion of neighbourhood is specified, discrete diffusion and Schrödinger-type evolutions on the Catalan state space follow by standard graph-Laplacian constructions (compare Remark A.5 and Remark A.6 for the tier-growth Markov structure induced by Dyck conditioning).*

## D Computational Foundations

This appendix records proof sketches for the computational claims used in the main text, including the Catalan universality statement for program structure and the disjoint-commutation property of Lemma D.2. For a lean v1, extended examples and additional formal development are omitted.

**Lemma D.1** (Internalizing Symbols as Motifs). *Let  $\Sigma$  be a finite (or countable) alphabet of atomic symbols, and consider finite applicative terms over  $\Sigma$  (binary application with atoms). There is an injective encoding of such labeled application trees into the Catalan family  $\mathcal{T}$  of unlabeled full binary trees. One explicit construction fixes two distinct tag trees  $A, B \in \mathcal{T}$ , assigns each  $\sigma \in \Sigma$  a distinct motif  $S_\sigma \in \mathcal{T} \setminus \{A, B\}$ , and defines*

$$E(\sigma) := \bullet(A, S_\sigma), \quad E(t \cdot u) := \bullet(B, \bullet(E(t), E(u))),$$

where  $t \cdot u$  denotes application and  $\bullet(\cdot, \cdot)$  denotes binary pairing of subtrees. The left tag distinguishes atoms from applications, so  $E$  is recursively decodable and therefore injective.

**Catalan universality for program structure.** The following is a proof sketch of Proposition 3.1 in the main paper.

*Proof sketch.* As described in classical treatments of combinatory logic and the  $\lambda$ -calculus [8, 4], application is a binary operation, and every term therefore possesses a unique representation as a full binary tree: internal nodes encode application, and leaves encode variables, constants, or combinators. This establishes a canonical embedding of programs into  $\mathcal{T}$ .

Conversely, any full binary tree with labeled leaves may be interpreted as a well-formed program term by reading internal nodes as applications and leaves as atomic symbols, yielding a unique term up to  $\alpha$ -equivalence.

Operational semantics are defined via local tree rewrites. A  $\beta$ -redex  $(\lambda x.M) N$  contracts by replacing the parent application with  $M[x := N]$ ; SKI reductions replace specific subtrees according to fixed patterns. In each case, the output remains a full binary tree, so evaluation never leaves  $\mathcal{T}$ . Because nondeterministic redex choices correspond to branching in the space of trees, each complete reduction sequence is a path through the Catalan possibility space, completing the correspondence.  $\square$

**Lemma D.2** (Commutation of disjoint reductions). *Let  $T$  be a full binary tree and consider any local rewrite system whose single-step reductions replace a rooted subtree matching a finite pattern by a new subtree, leaving the rest of  $T$  unchanged. Suppose two single-step reductions are applicable at positions  $p$  and  $q$  whose rooted subtrees are disjoint (neither position lies on the root-to-node path of the other). Let  $T_p$  denote the result of applying the reduction at  $p$ , and similarly  $T_q$ . Then both reductions remain applicable after the other, and they commute:*

$$(T_p)_q \equiv (T_q)_p.$$

*In particular, disjoint reductions form a commuting diamond as in the usual commuting-diamond picture.*

*Proof.* Since  $p$  and  $q$  lie in disjoint subtrees, contracting at  $p$  rewrites only the subtree rooted at  $p$  and leaves the subtree rooted at  $q$  unchanged. Symmetrically, contracting at  $q$  leaves the subtree at  $p$  unchanged. Because the two rewrite steps act on disjoint parts of the tree, performing both contractions yields the same result regardless of order.  $\square$

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