# The Geometry of Possibility: From Binary Roots to Complex Phase

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**Abstract**

At the smallest conceivable scale, where space and time lose their classical smoothness, physical reality may reduce to combinatorial logic: binary distinctions arranged in self-consistent causal order. This paper develops a discrete, geometric interpretation of quantum phase and causal propagation from first principles, beginning with Dyck words and Catalan structures. It unifies logic, probability, and geometry through the causal cone: a lattice of binary histories whose statistical and algebraic properties reproduce core features of quantum mechanics, relativity, and thermodynamic expansion. Depth and breadth within the cone correspond to conjugate observables, bound by a discrete uncertainty principle. Phase evolution appears as a rotation around the cone's rim, and the continuum limit yields the Schrödinger equation. The result is a framework where spacetime, probability, and causality emerge from the oscillatory geometry of binary information itself.

## 1 Introduction

The continuum has long been the language of physics, yet at the Planck scale it becomes an assumption without experimental foundation. If the universe is ultimately digital, its substrate must consist of the simplest possible units of distinction: binary choices. Every such choice bifurcates reality into two futures—yes/no, open/close, something/nothing—and together these form a tree of causal possibilities. The structure of that tree is governed by the *Catalan numbers*:

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1}$$

which enumerate every balanced sequence of $n$ pairs of parentheses, or equivalently, every way to maintain logical consistency through nested causes and effects. Each balanced sequence (a *Dyck word*) represents a self-contained history: a consistent pattern of openings and closures, actions and completions.

When plotted in a space whose axes measure *depth* (nesting) and *breadth* (branching), the set of all Dyck words of length $2n$ fills a triangular region—the discrete analogue of a light cone. This *causal cone* encodes what can influence what when propagation is limited to one binary step per tick.

## 2 The Discrete Causal Cone

A Dyck path is defined recursively as a sequence of steps $\pm 1$ obeying

$$h_{t+1} = h_t \pm 1, \qquad h_t \geq 0, \quad h_0 = h_{2n} = 0. \tag{2}$$

The height $h_t$ counts the number of active causal contexts at step $t$. Because each step changes the depth by one unit, the path is bounded by

$$0 \leq h_t \leq \min\{t,\, 2n - t\}. \tag{3}$$

This region is the discrete equivalent of a light cone: no causal influence can propagate faster than one level per tick.

Two special paths mark its boundaries:

- the **chain** $((\ldots))$, which advances along the cone's rim at maximum speed (pure temporal depth),
- the **star** $()()()\ldots()$, which remains flat (pure spatial breadth, simultaneous events).

Every intermediate Dyck path lies within the cone, tracing a distinct compromise between these extremes. At fixed $n$, all $C_n$ histories share the same causal depth and may be regarded as simultaneous layers of possibility.

## 3 Breadth Sectors and Narayana Numbers

Histories of equal depth differ in how their causal energy divides between concentration and expansion. This variation is described by the *Narayana numbers*:

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \qquad \sum_{k=1}^{n} N(n,k) = C_n. \tag{4}$$

Here $k$ counts the number of peaks or leaves (terminal events) in the corresponding binary tree. $k = 1$ gives the chain, $k = n$ the star.

Plotting $N(n,k)$ over $k$ yields an approximately Gaussian shape centered near $k \approx n/2$. This distribution expresses an *entropic bias* toward balanced configurations, providing a combinatorial analogue of the universe's observed expansion: breadth dominates over extreme depth, but both persist in tension.

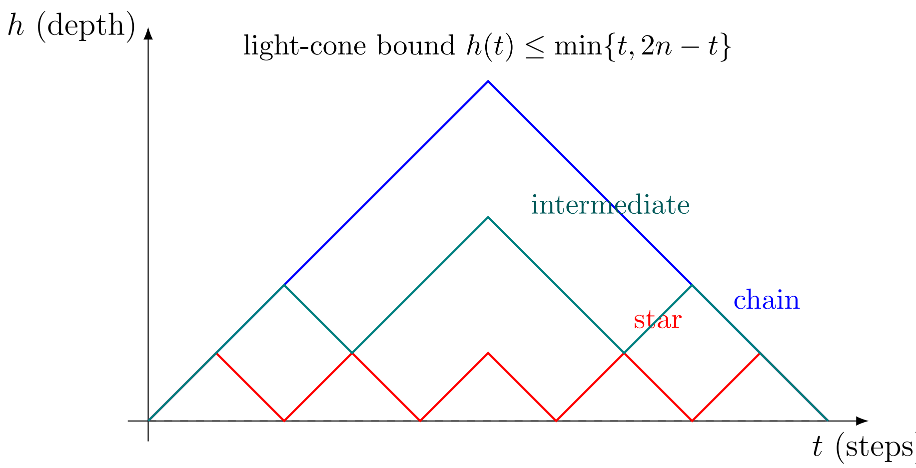


Figure 1: The discrete causal cone for $n = 5$ (path length $2n = 10$). The chain (blue) rides the rim; the star (red) hugs the base; an intermediate history (teal) weaves through the interior.

# 4 Depth–Breadth Duality and a Discrete Uncertainty Principle

For any full binary tree with leaves at depths $d_i$, the Kraft equality holds:

$$\sum_{i=1}^{k} 2^{-d_i} = 1. \tag{5}$$

From this follows the inequality

$$\overline{d} \geq \log_2 k. \tag{6}$$

The average depth $\overline{d}$ and the number of leaves $k$ thus satisfy an inverse relation: many shallow branches require less depth, and vice versa. Multiplying through,

$$k\, 2^{-\overline{d}} \leq 1, \tag{7}$$

we obtain a structural analogue of the Heisenberg relation $\Delta x\, \Delta p \geq \hbar/2$. Depth and breadth behave as conjugate variables; neither can be specified independently with arbitrary precision.

# 5 Phase and Interference of Histories

In quantum mechanics, the uncertainty between conjugate observables arises from the interference of complex phases. Each possible history carries an amplitude

$$\Psi(w) = A(w) e^{iS(w)/\hbar}, \tag{8}$$

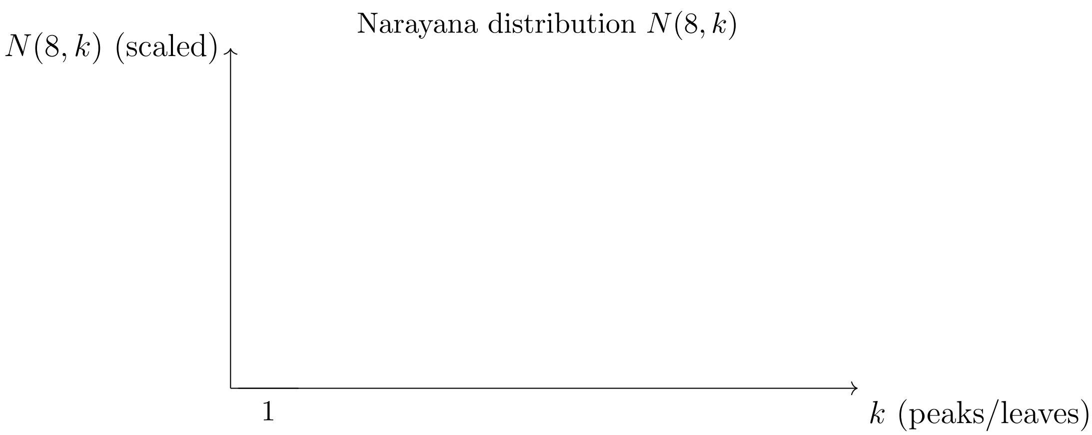


Figure 2: Narayana sectors for $n = 8$. Most histories live near the middle sectors ($k \approx 4, 5$), reflecting an entropic bias toward balanced depth/breadth.

where $S(w)$ is an *action* associated with the structure of the Dyck path (its total area or cumulative depth). Summing over all histories gives

$$\Psi_{\text{total}} = \sum_{w} A(w) e^{iS(w)/\hbar}. \tag{9}$$

Where $S(w)$ varies rapidly between nearby configurations, phases cancel; where it is stationary, they reinforce. The stable, phase-coherent bundles correspond to the *classical trajectories* of the cone.

This discrete summation is the combinatorial analogue of the Feynman path integral (Feynman 1948), with Dyck words serving as fundamental histories.

# 6 Imaginary Rotation Along the Cone's Rim

Ordering all $C_n$ Dyck words cyclically defines a shift operator

$$U|w_j\rangle = |w_{j+1}\rangle, \qquad U^{C_n} = I. \tag{10}$$

The eigenvectors of $U$ form a discrete Fourier basis,

$$|\tilde{k}\rangle = \frac{1}{\sqrt{C_n}} \sum_{j} e^{2\pi i jk/C_n} |w_j\rangle, \tag{11}$$

with eigenvalues $e^{-i2\pi k/C_n}$. Each step along the rim advances phase by $\Delta\theta = 2\pi/C_n$, a discrete *imaginary rotation*. At small $n$, these appear as quantized pulses; as $n$ grows, the motion approximates continuous precession on the Bloch sphere.

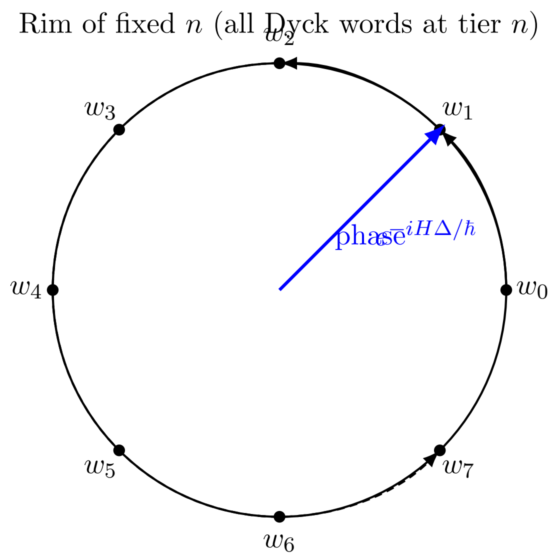


Figure 3: A cyclic ordering of Dyck words at fixed $n$ realizes a unitary shift $U$, i.e., a discrete imaginary rotation in Hilbert space. Fourier eigenmodes diagonalize $U$.

# 7 Temporal Interpretation and Simultaneity

Two complementary notions of time emerge:

1. **Causal time.** All states at depth $n$ are simultaneous, forming one layer of the expanding cone. From the lightlike rim, they appear coexistent.
2. **Phase time.** Within that layer, each history carries an internal phase depending on its structural proportion:

$$\tau(w) = \frac{h_{\max}(w)}{n}, \qquad \theta(w) = \omega n + \beta\tau(w) \pmod{2\pi}. \tag{12}$$

   Histories sharing equal $\theta$ define helices winding from the apex to the rim—slices of constant phase through the combinatorial lattice.

When $\omega$ is large, corresponding to the speed-of-light limit, these helices collapse into single simultaneity surfaces; time appears continuous.

# 8 Expansion Dynamics and Entropic Pressure

Each new layer $n \to n+1$ arises from two fundamental operations:

$$(W) : S \mapsto (S), \quad \text{(wrap/deepen)} \tag{13}$$
$$(B) : S \mapsto SS, \quad \text{(branch/widen)}. \tag{14}$$

These are the primitive causal moves of the universe. Wraps preserve peak number $k$; branches increase it by one. Since branching configurations outnumber wrapping ones, breadth statistically dominates over depth, producing a natural drive toward expansion. This entropic asymmetry functions as a combinatorial analogue of dark energy.

## 9 Continuum Limit and Emergent Wave Dynamics

As $n \to \infty$, discrete Dyck paths converge to continuous *Brownian excursions*. Rescaling time and height,

$$t \to nx, \quad h \to n^{1/2}y, \tag{15}$$

yields the diffusion equation

$$\partial_t \psi = \tfrac{1}{2}\partial_{yy}\psi, \tag{16}$$

with reflective boundary at $y = 0$. Introducing complex phase rotation $e^{iS/\hbar}$ converts this diffusion into the Schrödinger equation:

$$i\hbar\,\partial_t \Psi = -\frac{\hbar^2}{2m}\partial_{yy}\Psi. \tag{17}$$

Thus, the familiar wave mechanics of quantum theory emerge as the continuum limit of discrete binary causality.

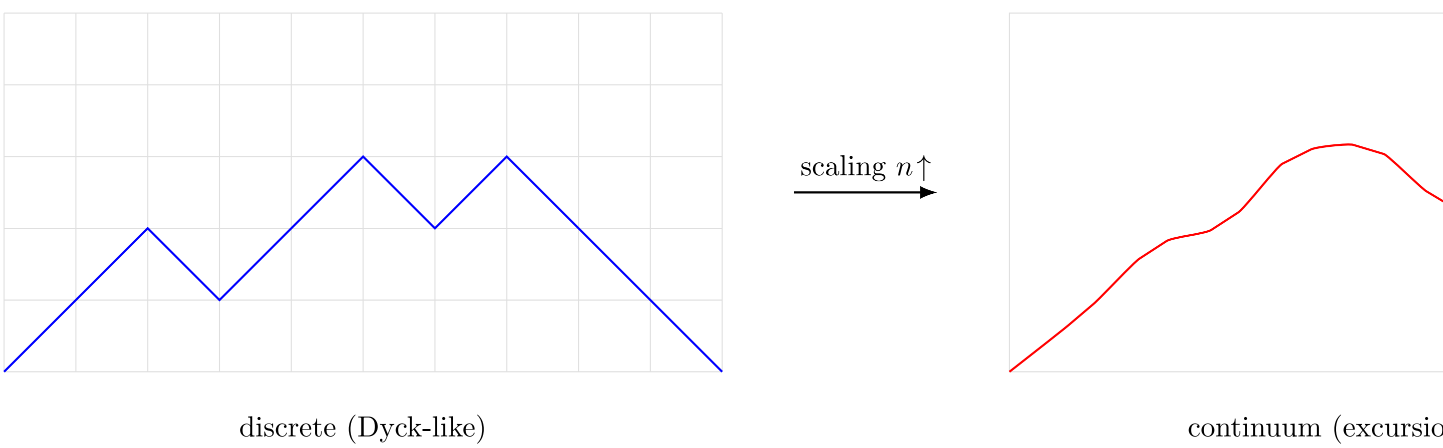


Figure 4: Scaling limit from discrete Dyck paths to a smooth random excursion; with complex phase, diffusion deforms into Schrödinger evolution.

## 10 Discreteness and the Planck Boundary

Information-theoretic arguments, such as the Bekenstein bound, imply finite information per area—roughly one bit per Planck square. The Catalan substrate satisfies this constraint:

each binary event contributes one unit of causal information. At macroscopic scales, the spacing between events is too fine to resolve, giving the illusion of continuity; at the Planck scale, discreteness reemerges as the fundamental texture of spacetime.

# 11 Extremal States: Chain and Star

Two limiting geometries bracket all possible histories:

- **Chain** $((\ldots))$**:** maximal depth, minimal breadth; pure temporal propagation, analogous to a null ray moving at $c$.
- **Star** $()()()$**:** minimal depth, maximal breadth; simultaneous, non-propagating structure.

Every real process is a compromise between these poles, oscillating between concentration and expansion—between time and space. This rhythmic exchange constitutes the *heartbeat of the causal cone.*

# 12 Discussion and Outlook

The causal cone framework unites logical, geometric, and probabilistic domains:

- Binary logic gives rise to structure (Dyck words).
- Geometry emerges from causal ordering (light-cone shape).
- Probability and entropy arise from combinatorial multiplicity (Narayana distribution).
- Quantum phase emerges as rotation within this space (unitary shift).
- Thermodynamic expansion results from entropic imbalance (breadth dominance).

Several research directions follow naturally:

1. Define structural actions $S(w)$ encoding curvature or energy.
2. Analyze the spectral properties of the shift operator $U$ on the associahedron graph.
3. Derive Lorentz-invariant field equations through coarse-graining of discrete dynamics.
4. Compare Catalan entropy with black-hole information scaling.

The core principle may be summarized as follows:

> *The universe is a self-rotating cone of possibilities. Each layer represents a discrete moment of causal depth; each rotation, a renewal of phase; each interference, a decision about what becomes real.*

Depth and breadth—time and space—are dual projections of a single complex oscillation: the imaginary rotation that turns binary information into the continuous world of experience.

# Appendix: Generating Functions and Asymptotics

The Catalan numbers satisfy the generating function

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}, \tag{18}$$

with asymptotic form

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}. \tag{19}$$

The Narayana numbers share the generating function

$$N(x, y) = \sum_{n,k \geq 1} N(n, k) x^n y^k = \frac{1 - x(1 + y) - \sqrt{1 - 2x(1 + y) + x^2(1 - y)^2}}{2x}. \tag{20}$$

The normalized distribution

$$P_n(k) = \frac{N(n, k)}{C_n} \tag{21}$$

tends toward a Gaussian with mean $\mu = (n + 1)/2$ and variance $\sigma^2 = n/8$ as $n \to \infty$, describing the entropic preference for intermediate breadth.

# Figure Suggestions (for future rendering)

1. **Figure 1:** Dyck paths for $n = 0$ to 4, showing the discrete causal cone.
2. **Figure 2:** Narayana distribution $N(n, k)$ visualized as a probability density.
3. **Figure 3:** Mapping between Dyck paths and complex phases along the cone's rim (imaginary rotation).
4. **Figure 4:** Continuum limit illustrating transition from discrete cone to continuous wavefront.

# References

[1] Feynman, R. P. (1948). Space-Time Approach to Non-Relativistic Quantum Mechanics. *Reviews of Modern Physics*, 20(2), 367–387.

[2] Noether, E. (1918). Invariant Variation Problems. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen.*

[3] Penrose, R. (2004). *The Road to Reality.* Vintage.

[4] Stanley, R. P. (1999). *Enumerative Combinatorics, Vol. 2*. Cambridge University Press.

[5] Zeilberger, D. (1985). A combinatorial proof of the Narayana number formula. *Discrete Mathematics*, 44(3), 325–326.