The Catalan Light Cone: A Recursive Substrate for Causal Geometry, Quantum Amplitudes, and Computation

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Abstract

We investigate the Catalan family of combinatorial structures—Dyck paths, full binary trees, and balanced parenthesis expressions—as a unified discrete substrate from which causal geometry, quantum amplitude propagation, and universal computation jointly emerge. A central observation is that the Dyck constraint induces a natural causal order. When organized by growth tier and lateral spread, the Catalan lattice forms a discrete cone whose extremal configurations reproduce a light-cone–like causal envelope. Classical invariance-principle results show that constrained Dyck walks converge in the scaling limit to Brownian excursions; the induced probability flow satisfies the heat equation and, under Wick rotation, the free Schrödinger equation. Via a uniform structural mapping—the pairs expansion—Dyck trees are placed in bijection with SKI and λ-calculus term graphs. Under this identification, causal extension corresponds to functional application, while local collapse corresponds to computational reduction. Structural equivalences of the substrate induce gauge-like redundancies, and disjoint subtrees commute analogously to spacelike-separated operators. The paper distinguishes rigorously established correspondences—scaling limits, diffusion dynamics, prefix-causal structure, and computational universality in the sense of standard SKI/λ encodings—from conjectural extensions concerning measurement, actualization, and interaction structure. Taken together, these correspondences show that a single recursive constraint can reproduce, at a structural and kinematical level, large portions of the operational framework of relativistic quantum theory and universal computation, without introducing additional primitives. We do not attempt to derive interactions or physical constants; the aim is to isolate the minimal recursive structure common to these domains.

1 Introduction

Discrete approaches to fundamental physics have long suggested that continuum spacetime and quantum dynamics may emerge from deeper combinatorial structure. Examples include causal sets [4], discrete random surfaces and Causal Dynamical Triangulations (CDT) [1, 2], spin networks and loop quantum gravity [14], tensor networks [13], and rewriting systems inspired by λ-calculus and combinatory logic. Typically, however, these models require multiple independent ingredients: a relation or graph for causal structure, an algebra for computation, and additional rules for quantum propagation. This work explores a more economical possibility: that a *single* recursive structure simultaneously supports all three.

The focus is the *Catalan substrate*, the family of structures counted by the Catalan numbers [15], including Dyck paths, full binary trees, and balanced parenthesis expressions. These objects are usually studied in enumerative combinatorics, probability theory, and theoretical computer science. Here they are treated instead as a space of *admissible histories* generated by a minimal growth constraint.

The central claim is that the Catalan substrate admits three tightly coupled interpretations:

(i) a discrete causal geometry with a light-cone–like envelope,

(ii) a natural path-integral dynamics with diffusive and wave-like continuum limits,

(iii) a universal computational calculus via λ- and SKI-term graphs.

These interpretations do not require distinct primitives; they arise from different readings of the same recursive object.

The geometric aspect follows from prefix order and growth bounds intrinsic to Dyck paths. The dynamical aspect follows from classical results on conditioned random walks and Brownian excursions [12, 10]. The computational aspect follows from the well-known equivalence between binary trees, cons-pair structures, and λ-calculus or SKI combinators [5, 7, 3]. Taken together, these results show that spacetime-like causal structure, quantum wave dynamics, and computation can be viewed as complementary manifestations of a single recursive substrate.

The exposition proceeds as follows. Section 2 establishes the discrete causal geometry of the Catalan lattice and its interpretation as a light cone. Subsequent sections place amplitude propagation and computation on this structure, analyze continuum limits, and discuss collapse, locality, and interaction. Interpretive considerations concerning origin, vacuum structure, and time are collected in a clearly labeled appendix, separate from the formal claims.

2 The Catalan Light Cone as a Discrete Causal Geometry

2.1 Dyck paths and growth tiers

A Dyck path of semilength n is a walk on the integers satisfying

$$H_{k+1} = H_k \pm 1, \qquad H_k \geq 0, \qquad H_0 = H_{2n} = 0.$$

Equivalently, Dyck paths are balanced parenthesis strings or full binary trees with n internal nodes. The number of such paths is the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Each up–down pair () represents a minimal unit of growth. The integer

$$t = n$$

will be called the *tier* and will be interpreted as a discrete proper time.

2.2 Prefix order and causality

Dyck paths carry a natural partial order by prefix inclusion. If a Dyck word u is a prefix of v, then u represents a causal ancestor of v. Conversely, prefixes that diverge represent causally incompatible futures. This prefix order defines a discrete causal structure:

- every node has a unique causal past,
- multiple incompatible futures may branch from the same prefix,
- cycles are prohibited by construction.

No additional causal axiom is required; causality is enforced combinatorially by the Dyck constraint.

2.3 Extremal configurations: chain and star

At fixed tier t there are many Dyck paths. Two extremal configurations play a distinguished role:

- the *chain* (or spine)

$$(((\cdots))),$$

  fully nested, with maximal depth and minimal spread;

- the *star*

$$()()()\cdots(),$$

  fully separated, with minimal depth and maximal spread.

All other configurations interpolate between these extremes. Together, the set of Dyck paths at tier t forms a discrete envelope bounded by the chain and the star.

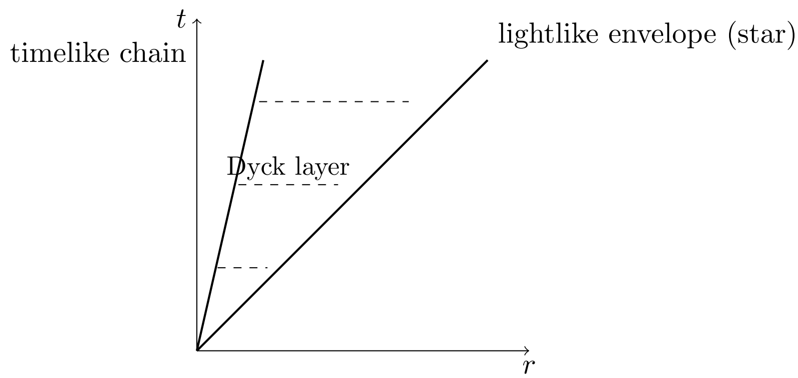

Figure 1: The Catalan light cone. Tier t (the number of Dyck units) plays the role of proper time, while breadth r measures spatial radius. All Dyck configurations at fixed tier lie between the fully nested chain (timelike extreme) and the fully separated star (lightlike envelope). Discrete Dyck layers approximate constant-time hypersurfaces, and the bound $r \leq t$ is enforced combinatorially.

2.4 Breadth as spatial extent

Define the *breadth* $r(w)$ of a Dyck path w to be the size of a largest level set in the associated full binary tree:

$$r(w) := \max_{\ell}\{\text{number of nodes at depth } \ell\}.$$

Equivalently, $r(w)$ is the maximal number of non-overlapping pairs at a common nesting depth. For a Dyck word of tier t there are t matched pairs in total, so

$$1 \leq r(w) \leq t,$$

with $r = 1$ for the fully nested chain and $r = t$ for the fully separated star. The inequality

$$r \leq t$$

is enforced purely by the recursive constraint. It is the discrete analogue of the relativistic light-cone bound $|\Delta x| \leq \Delta t$ (in units with $c = 1$).

2.5 Depth–breadth tradeoff

Let $h(w)$ denote the maximum height of a Dyck path, i.e. its maximal nesting depth. Depth and breadth are not independent. Interpreting the Dyck tree as a full binary prefix code, assign to each leaf i its depth d_i (root at depth 0). Then the Kraft equality holds:

$$\sum_i 2^{-d_i} = 1.$$

In particular, at any fixed depth ℓ there can be at most 2^ℓ leaves at that depth. Since $r(w)$ is the size of a largest level set, there is some depth ℓ_* at which $r(w)$ nodes appear, and hence

$$r(w) \le 2^{\ell_*} \le 2^{h(w)} \quad \Rightarrow \quad h(w) \ge \log_2 r(w).$$

Thus configurations with large breadth necessarily have logarithmically large depth, while very deep trees must concentrate most of their leaves in narrow antichains. This structural constraint is a combinatorial analogue of the tension between spatial spread and temporal commitment and is the standard Kraft bound for prefix codes [6].

$$\begin{array}{ll} ((())) & \quad (h=3,\ r=1) \\ (()()) & \quad (h=2,\ r=2) \\ (())() & \quad (h=2,\ r=2) \\ ()(()) & \quad (h=2,\ r=2) \\ ()()() & \quad (h=1,\ r=3) \end{array}$$

Figure 2: All Dyck words of tier $n = 3$, ordered from maximal nesting (chain) to maximal separation (star). These five configurations exhaust the discrete causal possibilities at fixed proper time. Depth h and breadth r interpolate between the two extremes, illustrating the intrinsic tradeoff enforced by the Dyck constraint. Higher tiers replicate this structure at larger scale.

2.6 Cone structure

Organizing Dyck paths by tier t and breadth r yields a discrete cone:

- each tier is a "constant-time" slice,
- the chain defines the timelike axis,
- the star defines the lightlike boundary,
- admissible configurations fill the interior.

This structure will be referred to as the *Catalan light cone*.

2.7 Scaling behavior

Classical results on conditioned random walks show that typical Dyck paths at tier t have height and breadth of order $\sqrt{t}$ [12, 10, 8]. Extremal configurations saturate the linear bound $r \le t$, while typical configurations lie deep within the cone. This separation between extremal and typical behavior mirrors the role of null, timelike, and spacelike trajectories in relativistic geometry.

Theorem 2.1 (Discrete light-cone bound and scaling). *Let w be a Dyck word of semilength t and breadth $r(w)$ as above. Then*

$$1 \leq r(w) \leq t.$$

Moreover, for a uniformly random Dyck word of semilength t, the typical height and breadth are of order $\sqrt{t}$.

The first statement follows from the definition of $r(w)$ and the fact that there are t internal nodes, while the scaling behavior is a consequence of invariance-principle results for conditioned random walks [12, 10, 8].

2.8 Summary

The Catalan substrate supports a discrete causal geometry determined entirely by recursive constraint. Without introducing a manifold, metric, or causal relation by fiat, it yields:

- a partial order interpretable as causality,
- a cone-shaped causal envelope,
- intrinsic bounds on spatial extension,
- well-defined constant-time layers.

Subsequent sections place dynamical rules—quantum amplitudes and computational reduction—on this geometry.

3 Recursive Pairing and Universal Computation

3.1 Pairs expansion

Every Dyck path admits a unique decomposition into nested pairs. Writing parentheses explicitly, the simplest nontrivial closure of the empty expression () is

$$(()),$$

which contains a single internal pairing. Iterating this rule produces the entire Catalan family.

This recursive pairing induces a uniform transformation, referred to here as the *pairs expansion*, which places Dyck trees in bijection with binary application trees. Under this correspondence:

- each matched pair corresponds to a cons cell,
- containment corresponds to functional application,
- sibling subtrees correspond to arguments at equal precedence.

The transformation is purely structural and preserves prefix order.

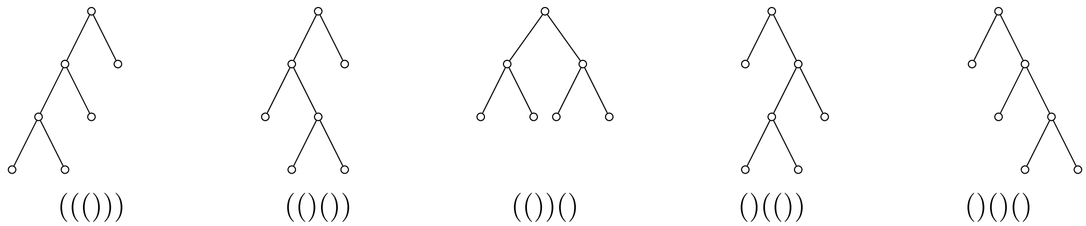

Figure 3: Binary-tree representations of the five Dyck words of semilength 3, corresponding to Fig. 2. From left to right: maximal nesting (chain) through mixed cases to maximal separation (star).

3.2 Connection to λ-calculus and SKI

Binary trees are a standard representation of λ-terms and SKI combinators [5, 7, 3]. Variables may be represented by leaf positions, abstraction by structural capture, and application by tree composition. Under the pairs expansion, each Dyck tree canonically determines an unlabeled application graph. When variables are suppressed, the resulting graphs coincide with the structure graphs used in combinatory logic. No additional primitives beyond recursive pairing are required to obtain this representation.

Choosing a finite set of tree patterns to represent the SKI combinators and interpreting local tree rewrites as SKI reduction therefore equips the Catalan substrate with a standard universal calculus: every partial recursive function can be encoded as an SKI term, and hence by a finite Dyck tree, and every computation corresponds to a sequence of local tree transformations. In this sense, the Catalan substrate is *computationally universal*. What is new here is that the same underlying objects simultaneously carry a causal and geometric interpretation.

3.3 Reduction and local collapse

In the computational interpretation, reduction corresponds to local pattern replacement. A redex occupies a finite region of a tree and may be reduced without reference to distant subtrees. This locality mirrors the causal structure established in Section 2. From the perspective of the Catalan lattice, reduction may be viewed as *collapse*: a locally ambiguous structure is replaced by a simpler one consistent with the global constraint. Importantly, collapse does not alter causal ancestry; it refines an already-admissible history. Confluence of reduction in the λ-calculus ensures that independent local reductions commute. This computational fact will later support an interpretation of spacelike commutativity.

3.4 Summary

Recursive pairing suffices to encode universal computation. Via the pairs expansion, Dyck trees and application graphs are two views of the same structure. Local computational reduction aligns naturally with causal locality on the Catalan light cone.

4 Quantum Amplitudes on the Catalan Lattice

4.1 Histories as paths

Interpreting Dyck paths as admissible histories suggests assigning weights to each history. Let $\mathcal{D}_t$ denote the set of Dyck paths of tier t. A state at tier t may be represented as a formal superposition

$$\Psi_t = \sum_{w \in \mathcal{D}_t} \psi(w)\,|w\rangle.$$

Local extensions of a Dyck path correspond to admissible future steps. Thus, time evolution is governed by transitions that respect the Dyck constraint.

4.2 Path integrals and conditioned walks

Dyck paths are random walks conditioned to remain nonnegative and return to zero. Classical results show that, when rescaled appropriately, ensembles of such paths converge to Brownian excursions [12, 10]. Assigning equal weight to all admissible paths yields a discrete path integral. More general amplitude assignments may depend on local features such as height or curvature, provided the Dyck constraint is preserved.

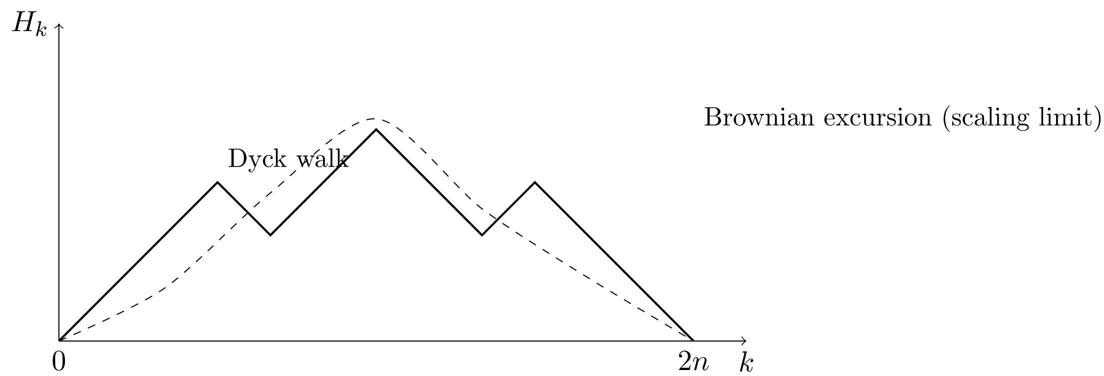

Figure 4: A Dyck path as a nearest-neighbour walk (H_k) constrained to stay nonnegative and return to zero at time $2n$. Under diffusive rescaling of k and H_k, ensembles of such paths converge to Brownian excursions, providing the bridge to the heat and Schrödinger equations discussed in the text.

4.3 Diffusion limit

Let $n \to \infty$ and rescale time and height by

$$t \mapsto n\tau, \qquad h \mapsto \sqrt{n}\,x.$$

Under this scaling, the probability density $\rho(\tau, x)$ for conditioned walks converges to the density of a Brownian excursion on $x \geq 0$. It follows from the invariance principle that ρ satisfies the heat equation

$$\partial_\tau \rho = \frac{1}{2}\,\partial_x^2 \rho, \tag{1}$$

with boundary conditions enforcing reflection or absorption at $x = 0$. Full derivations may be found in [12, 10].

4.4 Schrödinger equation

More formally, if $\rho(\tau, x)$ denotes the real heat kernel on $x \geq 0$, analytic continuation in the diffusion parameter, $\tau \mapsto it$, produces a complex-valued kernel $\psi(t, x)$ satisfying the free Schrödinger equation

$$i\partial_t \psi = -\frac{1}{2}\,\partial_x^2 \psi. \tag{2}$$

Boundary conditions at $x = 0$ are carried over from the diffusive regime (e.g. reflecting or absorbing), and the choice of boundary does not affect the existence of the continuum limit itself. Thus, quantum wave dynamics arises here as the analytic continuation of diffusive propagation on the Catalan lattice, in line with the classical connection between diffusion and Schrödinger evolution [9, 11]. No separate quantization procedure is required; the wave equation is inherited from the scaling limit of constrained combinatorial growth.

4.5 Relation to discrete quantum gravity

Similar scaling behavior appears in two-dimensional quantum gravity and random surface models. In particular, Causal Dynamical Triangulations (CDT) enforce a preferred foliation and causal constraint that parallels the prefix order of Dyck paths [1, 2]. In CDT, the continuum limit is taken after summing over causally admissible triangulations. Here the admissible structures are Dyck paths rather than triangulations, but the organizing principle—restriction to histories that respect a causal growth rule—is the same.

5 Locality, Commutation, and Interaction

5.1 Disjoint subtrees

Two subtrees of a Dyck tree that share no common ancestor beyond a given prefix are causally independent. Operations localized to one subtree do not affect the other. In the computational interpretation, this corresponds to independent reductions. In the amplitude interpretation, it corresponds to commuting operators acting on spacelike-separated regions.

Multiple Dyck paths may represent the same abstract computation or the same coarse-grained geometry. Such redundancies may be quotiented out without changing observable predictions, yielding equivalence classes of histories. This redundancy plays a role analogous to gauge symmetry: distinct internal descriptions correspond to the same external behavior.

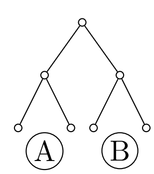

Figure 5: Schematic of disjoint subtrees A and B hanging from a common ancestor. Local operations (reductions, amplitude updates, or collapse rules) supported on A commute with those supported on B as long as neither subtree lies in the causal past of the other. This combinatorial notion of disjoint subtrees provides the substrate analogue of spacelike separation and microcausality.

5.2 Collapse and selection

Both computation and amplitude propagation require selection:

- computational reduction chooses a redex,
- measurement-like selection chooses an outcome.

In the Catalan substrate, selection operates locally, refining rather than destroying structure. The global constraint ensures consistency after selection. The formal development of collapse probabilities lies beyond the scope of this paper and is treated here only structurally.

5.3 Summary

Locality, commutation, and interaction emerge directly from the causal and recursive structure of the Catalan lattice. The same principles underlie both computational reduction and quantum amplitude propagation.

6 Discussion and Limitations

The results presented here establish a shared structural basis for causal geometry, quantum dynamics, and computation. Several limitations should be emphasized:

- Physical constants and interactions are not derived.
- Only free (noninteracting) wave dynamics appears explicitly.
- Collapse probabilities are not fixed uniquely by the structure.

These limitations reflect a deliberate restriction of scope. The goal has been to isolate the minimal recursive structure common to multiple domains, not to provide a complete physical theory.

A Interpretive Appendix: Void, Potential, and Recursive Actualization

This appendix records an interpretive perspective that motivates the formal development of the paper. The material here is not required for the results of the main text. It is included to clarify the conceptual picture suggested by the recursive structures analyzed above.

A.1 Void and first differentiation

Let $()$ denote the null structure: a state with no internal distinction and no recursive content. Formally, it is the base case of the Catalan construction. The minimal nontrivial closure of $()$ under recursive pairing is

$$(()).$$

This object introduces an internal relation without introducing multiplicity. It is the smallest structure capable of supporting further recursive growth while remaining globally consistent with the Dyck constraint. In this sense, $(())$ represents the first differentiation of the void: not an object placed *in* an existing space, but the emergence of relational structure itself.

A.2 Possibility space from recursive expansion

Iterating the pairing rule generates the full Catalan family. Each construction step introduces new admissible extensions while preserving all previous structural commitments. The resulting set of Dyck paths may be read as a space of mutually compatible but not jointly realizable futures. From this perspective, the Catalan lattice represents a structured *possibility space*. The Dyck constraint does not merely limit growth; it organizes it, enforcing consistency across all scales.

A.3 Temporal interpretation

The recursive construction index naturally induces an ordering. Each step corresponds to the introduction of new structure relative to what has already been fixed. This ordering provides a discrete notion of time internal to the construction, without presupposing an external temporal parameter. At each stage, two complementary processes are present:

- expansion, in which new admissible configurations are introduced;
- restriction, in which incompatible possibilities are locally excluded.

Temporal progression may be read as the alternation between these processes: the opening of new potential followed by selection consistent with prior structure. The formal theory requires only that both processes respect the global recursive constraint.

A.4 Emergent spacetime properties

The causal and geometric features described in the main text arise directly from this alternation. Recursive depth tracks accumulated structural commitment, while breadth tracks contemporaneous branching. Bounds on breadth as a function of depth yield a cone-shaped causal envelope. Under scaling limits, this discrete structure supports diffusion and wave propagation. These continuum behaviors do not require additional geometric axioms; they emerge from the organization of possibility imposed by the recursive constraint.

A.5 Interpretive status

Nothing in this appendix asserts a physical identity between recursive pairing and any particular physical process. The interpretations offered here are meant to guide intuition rather than to extend the formal claims of the paper. The central formal result remains unchanged: a single recursive constraint suffices to generate a structured possibility space exhibiting causal order, wave dynamics, and computational universality. The interpretive perspective suggests how these features may be interpreted as aspects of a unified underlying process, but the validity of the formal results does not depend on that reading.

B Conclusion

This paper has examined the Catalan family of recursive structures as a common substrate for causal geometry, quantum amplitude propagation, and computation. The Dyck constraint induces a natural causal order and a discrete cone-shaped geometry exhibiting light-cone–like bounds. Classical invariance principles show that ensembles of admissible histories converge in the continuum limit to Brownian excursions, yielding the heat equation and, under analytic continuation, the free Schrödinger equation. Through the pairs expansion, the same structures encode universal computation via λ-calculus and SKI combinators. These correspondences require no additional primitives beyond recursive pairing and constraint.

Spacetime-like geometry, wave dynamics, and computation emerge as complementary aspects of a single recursive system. Several open problems remain, including the incorporation of interaction terms, the determination of collapse probabilities, and the connection to physical constants. The results presented here establish a minimal and structurally unified foundation upon which such extensions may be explored.

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