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PHYS 550A
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Quantum Mechanics Homework IV

Problem 2

2. Prove the relations

$$|j, \pm m\rangle = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}}(J_{\mp})^{j-m}\,|j, \pm j\rangle\,, \qquad |j, \pm j\rangle = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}}(J_{\pm})^{j-m}\,|j, \pm m\rangle$$

Let us begin by defining the J_+ and J_- operators as functions, where

$\hat{J}_{\pm}|j, m\rangle = \sqrt{j\ (j+1) - m\ (m \pm 1)}\,|j, m \pm 1\rangle$.

In[∘]:=
```
J₊[{C_, j_, m_}] := {C √(j (j + 1) - m (m + 1)), j, m + 1} // FullSimplify
J₋[{C_, j_, m_}] := {C √(j (j + 1) - m (m - 1)), j, m - 1} // FullSimplify
```

We can check to make sure that we defined them properly.

In[∘]:=
```
J₊[{1, j, m}]
J₋[{1, j, m}]
```

Out[∘]= $\left\{\sqrt{(j-m)\ (1+j+m)},\ j,\ 1+m\right\}$

Out[∘]= $\left\{\sqrt{(1+j-m)\ (j+m)},\ j,\ -1+m\right\}$

We need to apply these operators iteratively to a set of eigenstates so we can check that the Nest function accomplishes what we need.

In[∘]:=
```
Nest[J₊, {1, j, m}, 2]
Nest[J₋, {1, j, m}, 2]
```

Out[∘]= $\left\{\sqrt{(j-m)\ (1+j+m)}\ \sqrt{(-1+j-m)\ (2+j+m)},\ j,\ 2+m\right\}$

Out[∘]= $\left\{\sqrt{(2+j-m)\ (-1+j+m)}\ \sqrt{(1+j-m)\ (j+m)},\ j,\ -2+m\right\}$

Now we can define both cases (±) of the right-hand side of the first equation in the problem as a function.

In[∘]:=
```
Prob2APlus[j_, m_] := Module[
  {vec},
  vec = {Sqrt[(j + m)! / ((2 j)! (j - m)!)], 1, 1} Nest[J₋, {1, j, j}, j - m];
  If[vec[[1]] == 1, vec[[1]] = ""];
  Return[StringForm["``|``,``⟩", vec[[1]], vec[[2]], vec[[3]]]]
 ]
Prob2AMinus[j_, m_] := Module[
  {vec},
  vec = {Sqrt[(j + m)! / ((2 j)! (j - m)!)], 1, 1} Nest[J₊, {1, j, -j}, j - m];
  If[vec[[1]] == 1, vec[[1]] = ""];
  Return[StringForm["``|``,``⟩", vec[[1]], vec[[2]], vec[[3]]]]
 ]
```

We can check that we defined the function properly.

In[∘]:=
```
Prob2APlus[0, 0]
Prob2AMinus[0, 0]
```

Out[∘]= |0,0⟩

Out[∘]= |0,0⟩

We can create a table that compares the left and right-hand sides of both cases of the first equation of the problem for $j \in \{0, 1, 2\}$ where $-j \le m \le j$.

In[∘]:=
```
headers = {"j", "m", "Sqrt[(j + m)! / ((2 j)! (j - m)!)] (J₋)^(j-m)|j,j⟩",
   "|j,m⟩", "Sqrt[(j + m)! / ((2 j)! (j - m)!)] (J₊)^(j-m)|j,-j⟩", "|j,-m⟩"};
mat = ConstantArray[0, {10, 6}];
mat[[1]] = headers;
```

In[∘]:=
```
ind = 2;
For[i = 0, i ≤ 2, i++,
 For[k = -i, k ≤ i, k++,
  mat[[ind]] = {i, k, Prob2APlus[i, k], StringForm["|``,``⟩", i, k],
    Prob2AMinus[i, k], StringForm["|``,``⟩", i, -k]};
  ind = ind + 1
 ]
]
Grid[mat, Frame → All]
```

Out[∘]=

j	m	$\sqrt{\frac{(j+m)!}{(2j)!\,(j-m)!}}\,(J_-)^{j-m}\lvert j,j\rangle$	$\lvert j,m\rangle$	$\sqrt{\frac{(j+m)!}{(2j)!\,(j-m)!}}\,(J_+)^{j-m}\lvert j,-j\rangle$	$\lvert j,-m\rangle$
0	0	$\lvert 0,0\rangle$	$\lvert 0,0\rangle$	$\lvert 0,0\rangle$	$\lvert 0,0\rangle$
1	-1	$\lvert 1,-1\rangle$	$\lvert 1,-1\rangle$	$\lvert 1,1\rangle$	$\lvert 1,1\rangle$
1	0	$\lvert 1,0\rangle$	$\lvert 1,0\rangle$	$\lvert 1,0\rangle$	$\lvert 1,0\rangle$
1	1	$\lvert 1,1\rangle$	$\lvert 1,1\rangle$	$\lvert 1,-1\rangle$	$\lvert 1,-1\rangle$
2	-2	$\lvert 2,-2\rangle$	$\lvert 2,-2\rangle$	$\lvert 2,2\rangle$	$\lvert 2,2\rangle$
2	-1	$\lvert 2,-1\rangle$	$\lvert 2,-1\rangle$	$\lvert 2,1\rangle$	$\lvert 2,1\rangle$
2	0	$\lvert 2,0\rangle$	$\lvert 2,0\rangle$	$\lvert 2,0\rangle$	$\lvert 2,0\rangle$
2	1	$\lvert 2,1\rangle$	$\lvert 2,1\rangle$	$\lvert 2,-1\rangle$	$\lvert 2,-1\rangle$
2	2	$\lvert 2,2\rangle$	$\lvert 2,2\rangle$	$\lvert 2,-2\rangle$	$\lvert 2,-2\rangle$

We can repeat this process for the second equation in the problem.

In[∘]:=
```
Prob2BPlus[j_, m_] := Module[
  {vec},
  vec = {Sqrt[(j + m)! / ((2 j)! (j - m)!)], 1, 1} Nest[J₊, {1, j, m}, j - m];
  If[vec[[1]] == 1, vec[[1]] = ""];
  Return[StringForm["``|``,``⟩", vec[[1]], vec[[2]], vec[[3]]]]
 ]
Prob2BMinus[j_, m_] := Module[
  {vec},
  vec = {Sqrt[(j + m)! / ((2 j)! (j - m)!)], 1, 1} Nest[J₋, {1, j, -m}, j - m];
  If[vec[[1]] == 1, vec[[1]] = ""];
  Return[StringForm["``|``,``⟩", vec[[1]], vec[[2]], vec[[3]]]]
 ]
```

In[∘]:=
```
Prob2BPlus[0, 0]
Prob2BMinus[0, 0]
```

Out[∘]= $|0,0\rangle$

Out[∘]= $|0,0\rangle$

In[∘]:=
```
headers = {"j", "m", "√((j + m)! / ((2 j)! (j - m)!)) (J₊)^(j-m) |j,m⟩",
   "|j,j⟩", "√((j + m)! / ((2 j)! (j - m)!)) (J₋)^(j-m) |j,-m⟩", "|j,-j⟩"};
mat = ConstantArray[0, {10, 6}];
mat[[1]] = headers;
```

In[∘]:=
```
ind = 2;
For[i = 0, i ≤ 2, i++,
 For[k = -i, k ≤ i, k++,
  mat[[ind]] = {i, k, Prob2BPlus[i, k], StringForm["|``,``⟩", i, i],
    Prob2BMinus[i, k], StringForm["|``,``⟩", i, -i]};
  ind = ind + 1
 ]
]
Grid[mat, Frame → All]
```

Out[∘]=

j	m	$\sqrt{\frac{(j+m)!}{(2j)!\,(j-m)!}}\,(J_+)^{j-m}\lvert j,m\rangle$	$\lvert j,j\rangle$	$\sqrt{\frac{(j+m)!}{(2j)!\,(j-m)!}}\,(J_-)^{j-m}\lvert j,-m\rangle$	$\lvert j,-j\rangle$
0	0	$\lvert 0,0\rangle$	$\lvert 0,0\rangle$	$\lvert 0,0\rangle$	$\lvert 0,0\rangle$
1	-1	$\lvert 1,1\rangle$	$\lvert 1,1\rangle$	$\lvert 1,-1\rangle$	$\lvert 1,-1\rangle$
1	0	$\lvert 1,1\rangle$	$\lvert 1,1\rangle$	$\lvert 1,-1\rangle$	$\lvert 1,-1\rangle$
1	1	$\lvert 1,1\rangle$	$\lvert 1,1\rangle$	$\lvert 1,-1\rangle$	$\lvert 1,-1\rangle$
2	-2	$\lvert 2,2\rangle$	$\lvert 2,2\rangle$	$\lvert 2,-2\rangle$	$\lvert 2,-2\rangle$
2	-1	$\lvert 2,2\rangle$	$\lvert 2,2\rangle$	$\lvert 2,-2\rangle$	$\lvert 2,-2\rangle$
2	0	$\lvert 2,2\rangle$	$\lvert 2,2\rangle$	$\lvert 2,-2\rangle$	$\lvert 2,-2\rangle$
2	1	$\lvert 2,2\rangle$	$\lvert 2,2\rangle$	$\lvert 2,-2\rangle$	$\lvert 2,-2\rangle$
2	2	$\lvert 2,2\rangle$	$\lvert 2,2\rangle$	$\lvert 2,-2\rangle$	$\lvert 2,-2\rangle$