Selected exercises from Susskind's General Relativity (TTM4)

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Here are my solutions to selected exercises from *L. Susskind & A. Cabannes, General Relativity: The Theoretical Minimum, Allen Lane 2023* (fourth volume of The Theoretical Minimum series). Solutions to some exercises (here marked with an asterisk *) were hinted at in the book, to which reference is made. Instead, some exercises left to the reader are solved here.

1. Equivalence Principle and Tensor Analysis

*Exercise 1

Page 18

[*If we are falling freely in a uniform gravitational field, prove that we feel no gravity and that things float around us like in the International Space Station.*]

Hint:

In the stationary frame $z = -\frac{1}{2}gt^2$. Passing to the primed coordinates (free-falling frame): $z' = z + \frac{1}{2}gt^2$. So, in the free-falling frame $z' = 0$.

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*Exercise 2

Page 30

[*Is it possible to find a curved surface and a lattice of rods arranged on it that cannot be flattened out, but can change shape?*]

Hint:

This exercise is answered in the text.

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2. Tensor mathematics

Exercise 1

Page 65

[*Prove that, in an orthonormal basis, equation (5) is equivalent to equation (6).*
Hint: Do it in two dimensions. Then - it is slightly more involved - we encourage you to try to do it in any dimension.]

Solution:

Equation (5) is the geometric definition of dot product:

$$\mathbf{V} \cdot \mathbf{W} = |\mathbf{V}||\mathbf{W}| \cos\theta$$

Equation (6) is the algebraic definition of dot product:

$$\mathbf{V} \cdot \mathbf{W} = V^1W^1 + V^2W^2 + \cdots + V^NW^N$$

In an orthonormal base:

$$\mathbf{V} = V^i\mathbf{e}_i$$

$$\mathbf{W} = W^j\mathbf{e}_j$$

$$\mathbf{V} \cdot \mathbf{W} = V^iW^j\mathbf{e}_i \cdot \mathbf{e}_j = V^iW^j\delta_ij = V^iW^i$$

In an orthonormal base contravariant and covariant components are exactly the same. In plane old components, by geometric definition:

$$\mathbf{V} \cdot \mathbf{e}_i = |\mathbf{V}||\mathbf{e}_i| \cos\theta_i = |\mathbf{V}| \cos\theta_i = V^i$$

where θ_i is the angle between $\mathbf{V}$ and $\mathbf{e}_i$, so

$$\mathbf{V} \cdot \mathbf{W} = \mathbf{V} \cdot (W^i \mathbf{e}_i = (\mathbf{V} \cdot \mathbf{e}_i) W^i = V^i W^i$$

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3. Flatness and Curvature

*Exercise 1

Page 109
[*Explain why the space can be flat and nevertheless the Christoffel symbols not zero.*]
Hint: see discussion in the text.

■

*Exercise 2

Page 109
[*Explain why the covariant derivative of the metric tensor is always zero.*]
Hint: see discussion in the text.

■

Exercise 3

Page 109
[*On Earth, with the polar coordinates θ for latitude and ϕ for longiture, find*
1. the metric tensor g_{mn}
2. its inverse g^{mn}
3. the Christoffel symbols at point (θ, ϕ).]
Note:
This exercise can be conjoined with Exercise 1 on page 133, chapter 4, items 1. and 2..
Solution:
The ordinary distance on a 2-sphere of unitary radius is $ds^2 = d\theta^2 + \cos^2\theta d\phi^2$. We get:

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{pmatrix}$$

$$g^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\cos^2\theta} \end{pmatrix}$$

$$\Gamma^{\theta}{}_{\phi\phi} = \sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = -\tan\theta$$

All other components are zero.

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4. Geodesics and Gravity

Exercises left to the reader

On page 132 it is required to proof that equation (16) produces a vector of lenght one.
Solution:
Equation (16) reads:

$$t^m = \frac{dX^m}{dS}$$

From equation (15) on page 132 ($dS^2 = g_{mn} dX^m dX^n$) and from the definition of vector length (equation (11) on page 67):

$$\mathbf{t} \cdot \mathbf{t} = t^m t^n g_{mn} = g_{mn} \frac{dX^m}{dS} \frac{dX^n}{dS} = 1$$

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On page 129 it is stated that "Parallel-transporting a vector ... preserves its length. It can be shown as a consequence of equation (14)."

Solution:

Equation (14) (on page 127) reads $dV^n + \Gamma^n{}_{mr} V^r dX^m = 0$. Its covariant counterpart is $dV_n - \Gamma^r{}_{mn} V_r\, dX^m = 0$. The ordinary change in length is

$$d\left(\mathbf{V}\cdot\mathbf{V}\right) = V_m\, dV^m + V^m dV_m$$

Plugging in it $dV^m = -\Gamma^m{}_{nr} V^r dX^n$ and $dV_m = \Gamma^r{}_{mn} V_r\, dX^n$ we get $d\left(\mathbf{V}\cdot\mathbf{V}\right) = 0$

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Exercise 1

Note:

Exercise 1 continues on pages 133–134 of chapter 1, items 3., 4. and 5..

[*3. Show that the tangent vector to a meridian has everywhere components $t^1 = 1$ and $t^2 = 0$.*

4. Show that the tensor that is the covariant derivative of this tangent vector is

$$\begin{pmatrix} 1 & 0 \\ 0 & -\tan\theta \end{pmatrix}$$

5. Show that if we follow a meridian, the covariant change of the tangent vector is always zero.]

Solution:

3. Remember that $t^\theta = \frac{d\theta}{dS}$, $t^\phi = \frac{d\phi}{dS}$. As t is tangent to a meridian $\phi = const$. Hence $dS = d\theta$, coming up with $t^\theta = 1$ and $t^\phi = 0$.

4. From equation (8) on page 124

$$D_r t^m = \partial_r t^m + \Gamma_{rn}{}^m t^n$$

and the solution of item 3. it follows that $\partial_r t^m = 0$, whence $D_r t^m = \Gamma_{r\theta}{}^m$.

5. From

$$Dt^n = dt^n + \Gamma_{mr}{}^n t^r dX^m$$

and the solution of item 3. it follows that $dt^n = 0$, whence $Dt^n = \Gamma_{m\theta}{}^n dX^m$.

We get:

$$Dt^\theta = \Gamma_{m\theta}{}^\theta dX^m = 0$$

as $\Gamma_{m\theta}{}^\theta = 0$;

$$Dt^\phi = \Gamma_{m\theta}{}^\phi dX^m = 0$$

as $\Gamma_{\theta\theta}{}^\phi = 0$; and $\Gamma_{\phi\theta}{}^\phi d\phi = -\tan\theta d\phi = 0$ as $d\phi = 0$ on the meridian.

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Exercise 2

Page 149

[*In figure 13, what is the speed, relative to the stationary frame, of the observer who sees R, S, and T as simultaneous events?*]

Solution:

From equation (31) on page 146 the coordinates of the events R, S, and T, are:

$$\begin{aligned} X_R &= \cosh\omega & T_R &= \sinh\omega \\ X_S &= 2\cosh\omega & T_S &= 2\sinh\omega \\ X_T &= 3\cosh\omega & T_T &= 3\sinh\omega \end{aligned}$$

Lorentz transforming the coordinates:

$$\begin{aligned} X' &= \gamma(X - vT) & T' &= \gamma(T - vX) \\ X'_R &= \gamma(\cosh\omega - v\sinh\omega) & T'_R &= \gamma(\sinh\omega - v\cosh\omega) \\ X'_S &= 2\gamma(\cosh\omega - v\sinh\omega) & T'_S &= 2\gamma(\sinh\omega - v\cosh\omega) \\ X'_T &= 3\gamma(\cosh\omega - v\sinh\omega) & T'_T &= 3\gamma(\sinh\omega - v\cosh\omega) \end{aligned}$$

where v is the speed of the sought-after observer. Imposing the simultaneity condition $T'_R - T'_S = T'_S - T'_T = \gamma(\sinh\omega - v\cosh\omega) = 0$ we get

$$v = \tanh\omega$$

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Exercises left to the reader

Let's revert to page 142 in order to prove that from the rest frame of the first observer (R), the distance to the second observer (S) grows.

Solution:

From the inertial frame the segment RS at some time T is, *by definition*, of fixed length. Lorentz inverse-transforming from the rest frame of R (primed coordinates) to the non-primed frame:

$$|RS| = X_S - X_R = \gamma(X'_S - X'_R) + \gamma v(T'_S - T'_R)$$

$$0 = T_S - T_R = \gamma(T'_S - T'_R) + \gamma v(X'_S - X'_R)$$

This leads to

$$|RS| = \gamma(X'_S - X'_R) - \gamma v^2(X'_S - X'_R) = \frac{1}{\gamma}(X'_S - X'_R)$$

and finally we get the sought-after distance from the observer R to the observer S: $\gamma|RS| > |RS|$. As a result, R sees S moving away, S sees R moving back.

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On page 147 the proper time is given as $\tau = r\omega$ without proof. Let's prove it.

Solution:

From

$$d\tau^2 = dt^2 - dx^2$$

Plugging the (r, ω) coordinates in the differentials:

$$d\tau^2 = (r\cosh\omega d\omega)^2 - (r\sinh\omega d\omega)^2 = r^2(\cosh\omega^2 - \sinh\omega^2)d\omega^2 = r^2 d\omega^2$$

that leads to $\tau = r\omega$ as expected.

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On page 148 it is required to show the equal spacing of observers R, S, T, ... to any value of ω, i.e. $|RS| = |ST| = \cdots$.

Solution:

Indeed:

$$|RS|^2 = (X_S - X_R)^2 - (T_S - T_R)^2 = (2\cosh\omega - \cosh\omega)^2 - (2\sinh\omega - \sinh\omega)^2 = \cosh^2\omega - \sinh^2\omega = 1$$

$$|ST|^2 = (X_T - X_S)^2 - (T_T - T_S)^2 = (3\cosh\omega - 2\cosh\omega)^2 - (3\sinh\omega - 2\sinh\omega)^2 = \cosh^2\omega - \sinh^2\omega = 1$$

$$\ldots \qquad \ldots \qquad \ldots$$

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On page 151 it is left as an exercise to prove that the magnitude of the acceleration is constant on each hyperbola of figure 14 (with a different acceleration for each hyperbola, depending on r).

Solution:

Remembering the definitions of the ordinary and relativistic velocities and accelerations on page 150, and introducing the proper time in the formulas of the coordinates:

$$X = r\cosh\tau/r$$

$$T = r\sinh\tau/r$$

we get:

$$a^1 = \frac{1}{r}\cosh\tau/r$$

$$a^0 = \frac{1}{r}\sinh\tau/r$$

$$|a|^2 = (a^1)^2 - (a^0)^2 = \frac{1}{r^2}(\cosh^2\tau/r - \sinh^2\tau/r) = \frac{1}{r^2}$$

$$|a| = \frac{1}{r}$$

The ordinary acceleration is

$$a^x = \frac{d^2x}{dt^2}$$

Passing to Cartesian coordinates the second derivative can be directly computed:

$$x = r\sqrt{1+(t/r)^2}$$

$$a^x = \frac{1}{r}\left(1+(t/r)^2\right)^{-3/2}$$

Note:
Notice that for $t=0$ the ordinary acceleration a^x coincides with the magnitude of the proper acceleration $|a|$. In the book it is remarked that the (proper) acceleration on a given trajectory in figure 14, for example the (proper) acceleration at point N on the hyperbola $R=2$, is the ordinary acceleration. And it is the constant (proper) acceleration we would experience all along the trajectory. In other words, the observer at point R will experience a constant acceleration $|a|$. At point N the inertial observer would determine an acceleration $a^x = |a|$ (that diminishes later on). Notice that $|a|$, the magnitude of the relativistic (four-)acceleration, coincides with $\gamma^3 a^x$ in the case that the (ordinary) (three-) acceleration a^x and the velocity have the same direction (this is of no concern for uni-dimensional motion); a is constant and Lorentz-invariant (only the magnitude, not the direction) in the particular case of an object accelerating parallel to its velocity; $\gamma^3\mathbf{a}$, where $\mathbf{a}$ is the ordinary (three-)acceleration, is the proper acceleration. (Do not confuse with the spatial part of the four-acceleration $a^i = \frac{d^2x^i}{d\tau^2}$). Let's recall a few definitions:
proper velocity: $v^i = \frac{dx^i}{d\tau}$
proper acceleration: $\alpha^i = \frac{dv^i}{dt}$
(relativistic) (four-)acceleration: $a^i = \frac{d^2x^i}{d\tau^2}$
(ordinary) (three-)acceleration: $a^x = \frac{d^2x}{dt^2}$
spatial part of the four-acceleration: $a^1 = \frac{d^2x}{d\tau^2} = \frac{1}{r}\cosh\frac{\tau}{r}$
The magnitudes of the relevant quantities are recapitulated in the following tables:

hyperbolic motion	
magnitudes	formulas
(four-)acceleration (1)	$\frac{1}{r}$
spatial part of (1)	$a^1 = \frac{\gamma}{r}$ (the time part is $a^0 = \frac{\sqrt{1-\gamma^2}}{r}$)
proper acceleration	$\frac{1}{r}$
ordinary acceleration	$\frac{\gamma^{-3}}{r}$
ordinary velocity	$v = \tanh\frac{\tau}{r}$
proper velocity	$\gamma v = \sinh\frac{\tau}{r}$
gamma	$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\frac{\tau}{r}$

parabolic motion	
magnitudes	formulas
ordinary acceleration	a
proper acceleration	$\gamma^3 a$
ordinary velocity	at
proper velocity	γat
space part of four-acceleration	$\gamma^4 a$
magnitude of four-acceleration	$\gamma^3 a$
gamma	$\gamma = \frac{1}{\sqrt{1-(at)^2}}$ $(t=\gamma\tau)$
constraints of parabolic motion	$a = const.$ for $0 \le t < \frac{c}{a}$, $a(t) = \frac{c}{t}$ for $t \ge \frac{c}{a}$.

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5. Metric for a Gravitational Field

Exercise 1

Page 179
[*Given the metric $g_{\mu\nu}(X)$ show that the Euler-Lagrange equation (16) (we drop the "s"), to minimize the proper time along a trajectory in space-time,*

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}^m} = \frac{\partial\mathcal{L}}{\partial X^m}$$

where the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}(X)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}$$

is equivalent to the definition of a geodesic given by equation (6), which says that the tangent vector to the trajectory in space-time stays constant:

$$\frac{d^2X^\mu}{d\tau^2} = -\Gamma^\mu{}_{\sigma\rho}\frac{dX^\sigma}{d\tau}\frac{dX^\rho}{d\tau}]$$

Solution:

$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}(X)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}} = -m\frac{d\tau}{dt}$$

$$\frac{\partial\mathcal{L}}{\partial X^m} = \frac{m}{2}\frac{\partial_m g_{\mu\nu}\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}{\frac{d\tau}{dt}} = \frac{m}{2}\frac{\partial_m g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{d\tau}{dt}\frac{dX^\nu}{d\tau}\frac{d\tau}{dt}}{\frac{d\tau}{dt}} = \frac{m}{2}\partial_m g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\frac{d\tau}{dt}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{X}^m} = \frac{m}{2}\frac{g_{\mu m}\frac{dX^\mu}{dt} + g_{m\nu}\frac{dX^\nu}{dt}}{\frac{d\tau}{dt}} = \frac{m}{2}\left(g_{\mu m}\frac{dX^\mu}{d\tau} + g_{m\nu}\frac{dX^\nu}{d\tau}\right)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}^m} = \frac{m}{2}\frac{d}{dt}\left(g_{\mu m}\frac{dX^\mu}{d\tau} + g_{m\nu}\frac{dX^\nu}{d\tau}\right) = \frac{m}{2}\left(\partial_\lambda g_{\mu m}\frac{dX^\lambda}{d\tau}\frac{dX^\mu}{d\tau} + \partial_\lambda g_{m\nu}\frac{dX^\lambda}{d\tau}\frac{dX^\nu}{d\tau} + g_{\mu m}\frac{d^2X^\mu}{d\tau^2} + g_{m\nu}\frac{d^2X^\nu}{d\tau^2}\right)\frac{d\tau}{dt}$$

$$\frac{1}{2}\partial_m g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = \frac{1}{2}\left(\partial_\lambda g_{\mu m}\frac{dX^\lambda}{d\tau}\frac{dX^\mu}{d\tau} + \partial_\lambda g_{m\nu}\frac{dX^\lambda}{d\tau}\frac{dX^\nu}{d\tau} + g_{\mu m}\frac{d^2X^\mu}{d\tau^2} + g_{m\nu}\frac{d^2X^\nu}{d\tau^2}\right)$$

$$-\frac{1}{2}\left(\partial_\lambda g_{\mu m}\frac{dX^\lambda}{d\tau}\frac{dX^\mu}{d\tau} + \partial_\lambda g_{m\nu}\frac{dX^\lambda}{d\tau}\frac{dX^\nu}{d\tau} - \partial_m g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\right) = g_{m\nu}\frac{d^2X^\nu}{d\tau^2}$$

$$-\frac{1}{2}\left(\partial_\nu g_{\mu m}\frac{dX^\nu}{d\tau}\frac{dX^\mu}{d\tau} + \partial_\mu g_{m\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} - \partial_m g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\right) = g_{m\nu}\frac{d^2X^\nu}{d\tau^2}$$

$$-\frac{1}{2}\left(\partial_\nu g_{\mu m} + \partial_\mu g_{m\nu} - \partial_m g_{\mu\nu}\right)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = g_{m\nu}\frac{d^2X^\nu}{d\tau^2}$$

$$-\frac{1}{2}\left(\partial_\rho g_{\sigma m} + \partial_\sigma g_{m\rho} - \partial_m g_{\sigma\rho}\right)\frac{dX^\sigma}{d\tau}\frac{dX^\rho}{d\tau} = g_{m\mu}\frac{d^2X^\mu}{d\tau^2}$$

$$-g^{m\mu}\cdot\frac{1}{2}\left(\partial_\rho g_{\sigma m} + \partial_\sigma g_{m\rho} - \partial_m g_{\sigma\rho}\right)\frac{dX^\sigma}{d\tau}\frac{dX^\rho}{d\tau} = g^{m\mu}g_{m\mu}\frac{d^2X^\mu}{d\tau^2}$$

$$-\Gamma^\mu{}_{\sigma\rho}\frac{dX^\sigma}{d\tau}\frac{dX^\rho}{d\tau} = \frac{d^2X^\mu}{d\tau^2}$$

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Exercise left to the reader

On page 192 it is left to the reader the calculation of the Hamiltonian (equation (36)).
Solution:
The Lagrangian is

$$\mathcal{L} = -m\sqrt{\left(1-\frac{2MG}{r}\right) - \left(\frac{1}{1-\frac{2MG}{r}}\right)\dot{r}^2}$$

The Hamiltonian is

$$H = p\dot{r} - \mathcal{L}$$

where

$$p = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\left(\frac{1}{1-\frac{2MG}{r}}\right)\dot{r}}{\sqrt{\left(1-\frac{2MG}{r}\right) - \left(\frac{1}{1-\frac{2MG}{r}}\right)\dot{r}^2}}$$

assuming a unitary test mass ($m = 1$). We get:

$$H = \frac{\left(\frac{1}{1-\frac{2MG}{r}}\right)\dot{r}^2}{\sqrt{\left(1-\frac{2MG}{r}\right) - \left(\frac{1}{1-\frac{2MG}{r}}\right)\dot{r}^2}} + \sqrt{\left(1-\frac{2MG}{r}\right) - \left(\frac{1}{1-\frac{2MG}{r}}\right)\dot{r}^2} = \frac{\left(1-\frac{2MG}{r}\right)}{\sqrt{\left(1-\frac{2MG}{r}\right) - \left(\frac{1}{1-\frac{2MG}{r}}\right)\dot{r}^2}}$$

i.e. the equation (36) (**Note:** in the book $\dot{r}^2$ in equation (36) is misprinted as $\dot{r}$). Hence, solving for $\dot{r}^2$:

$$\dot{r}^2 = \left(1 - \frac{2MG}{r}\right)^2 - \frac{\left(1 - \frac{2MG}{r}\right)^3}{E^2}$$

i.e. the equation (37).

■

Exercise 2

Page 193
[*Show that from equation (36) for the energy, and equation (37) for* $\dot{r}^2$, *it follows that*

$$\dot{r} \approx \sqrt{\frac{r - 2MG}{2MG}}$$

as $r \to 2MG$]
Note:
[In the book the above equation is misprinted. Read it $\dot{r} \approx \sqrt{\left(\frac{r-2MG}{2MG}\right)^2}$ or $\dot{r} \approx \frac{r-2MG}{2MG}$.]
Solution:
Assuming a radial falling of an unitary mass from infinity (assuming at infinity $\dot{r}_\infty = 0$), then from equation (36) $E = 1$, and from equation (37), taking $r = 2MG + \epsilon$, we get to the first order approximation in ϵ:

$$\dot{r} = \sqrt{\left(1 - \frac{2MG}{r}\right)^2 - \left(1 - \frac{2MG}{r}\right)^3} \approx \sqrt{\left(\frac{r - 2MG}{2MG + \epsilon}\right)^2 \left(\frac{2MG}{2MG + \epsilon}\right)} \approx \frac{r - 2MG}{2MG}$$

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Exercises left to the reader

On page 211 it is left to the reader to solve equation (16) for $\dot{\phi}$.
Solution:
Equation (16) (for the angular momentum) is

$$L = \frac{mr^2\dot{\phi}}{\sqrt{\mathcal{F}(r) - r^2\dot{\phi}^2}}$$

where $\mathcal{F} = 1 - \frac{2MG}{r}$. Resolving for $\dot{\phi}$:

$$\dot{\phi} = \frac{L\sqrt{\mathcal{F}(r)}}{\sqrt{r^2L^2 + m^2r^4}}$$

Plugging it back into equation (15)

$$E = \frac{\mathcal{F}(r)m}{\sqrt{\mathcal{F}(r) - r^2\dot{\phi}^2}}$$

we get equation (17) for E as a function of r and given angular momentum:

$$E = m\frac{\sqrt{\mathcal{F}(r)}\sqrt{r^2\left(\frac{L}{m}\right)^2 + r^4}}{r^2}$$

■

On page 213 it is left to the reader to show that the point where E is stationary is at $r = 3MG$.
Solution:
To this aim we differentiate the RHS of equation (19), that is

$$E = L\frac{\sqrt{\mathcal{F}}}{r}$$

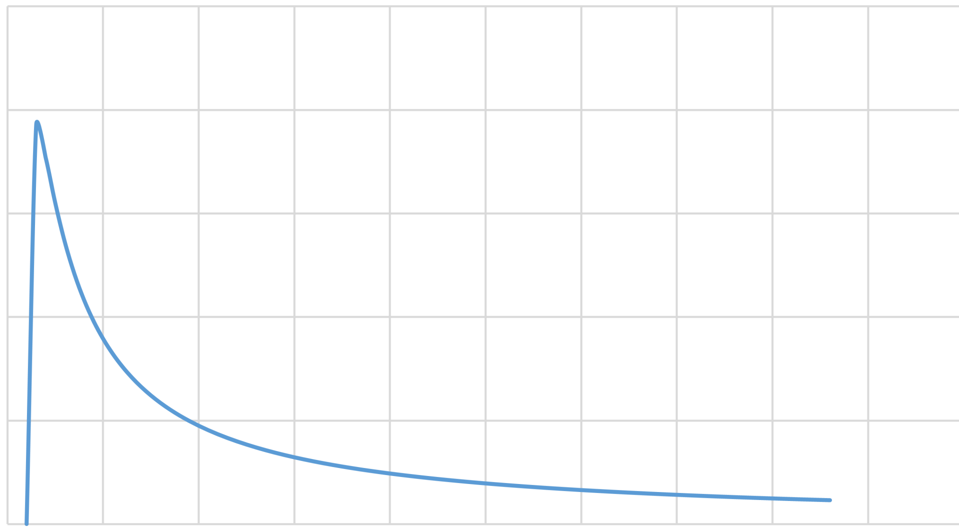

Figure 1: Energy of a test particle as a function of the radial distance (arbitrary units).

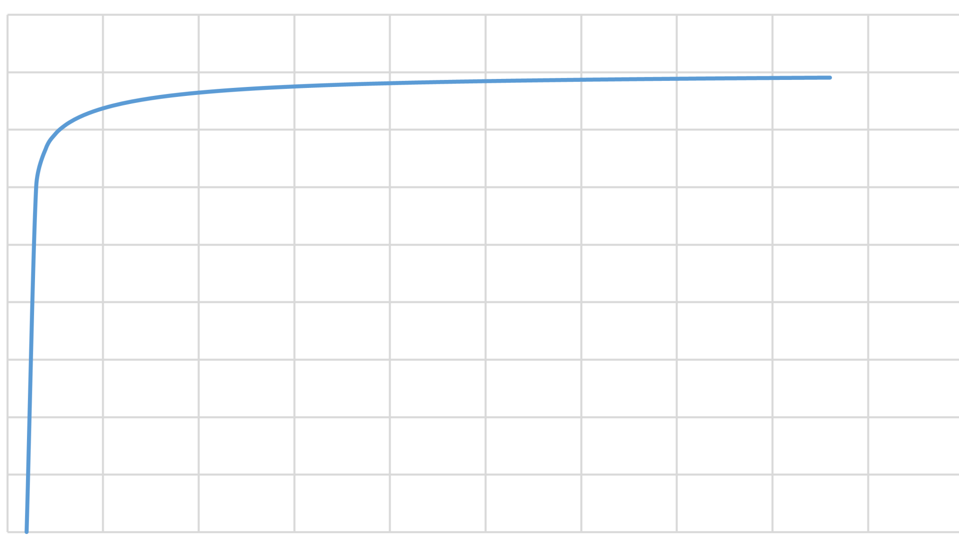

Figure 2: Energy of a massive body as a function of the radial distance (arbitrary units).

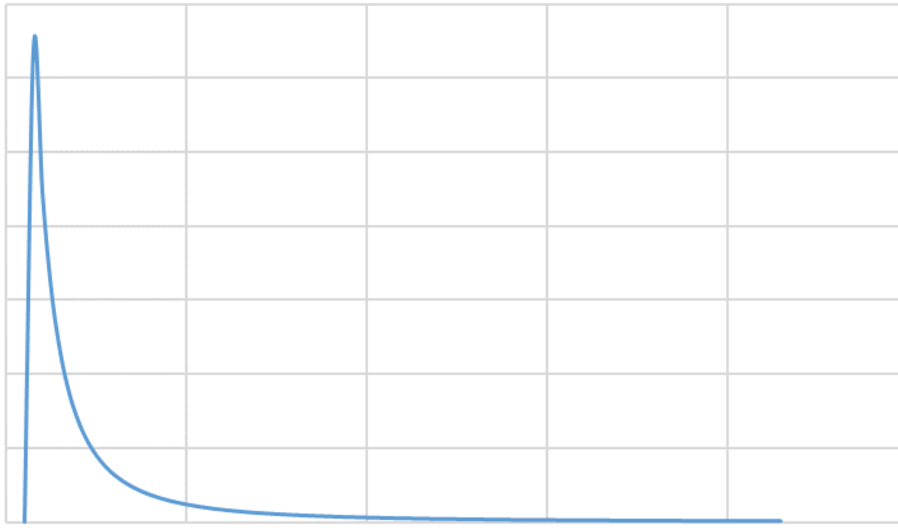

Figure 3: Energy of a photon as a function of the radial distance (arbitrary units).

w.r.t. r and set the derivative equal to zero:

$$\frac{d}{dr}\left(\frac{\sqrt{\mathcal{F}}}{r}\right) = 0$$

(leaving aside the multiplicative factor L). We get:

$$\frac{d}{dr}\left(\frac{\sqrt{\mathcal{F}}}{r}\right) = \sqrt{\mathcal{F}}\frac{dr^{-1}}{dr} + \frac{1}{r}\frac{d\sqrt{\mathcal{F}}}{dr} = -r^{-2}\left(\sqrt{\mathcal{F}} - \frac{\frac{MG}{r}}{\sqrt{\mathcal{F}}}\right) = 0$$

whence

$$\mathcal{F} = \frac{MG}{r}$$

i.e.

$$1 - \frac{2MG}{r} = \frac{MG}{r}$$

so $r = 3MG$, that does not depend on L. The calculation of the second derivative is pretty tedious but straightforward; the second derivative is negative at $r = 3MG$, i.e. E reaches a maximum at $r = 3MG$ (also cf. figure 7 on page 213).

■

On page 215 it is recommended to compare the calculations done for the photons with the corresponding Newtonian calculations.
Solution:
Let's start with the radial falling of a test mass. The newtonian Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - \left(-\frac{MGm}{r}\right)$$

and the linear momentum $p = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$. The Hamiltonian is

$$H = p\dot{r} - \mathcal{L} = \frac{1}{2}m\dot{r}^2 - \frac{MGm}{r}$$

Resolving for $\dot{r}$:

$$\dot{r} = \sqrt{\frac{2E}{m} + \frac{2MG}{r}}$$

For a radial light ray ($m \to 0$) beamed from point r the latter expression diverges. For an orbiting particle the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\Omega}^2\right) - \left(-\frac{MGm}{r}\right)$$

where $d\Omega^2 = d\theta^2 + \cos^2\theta d\phi^2$. In the plane $\theta = 0$ (the symmetry of the system allows this choice without loss of generality), $d\theta = 0$ and $d\Omega$ reduces to $d\phi$. The linear momentum is, as before, $p_r = m\dot{r}$ and the angular momentum $L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$. Resolving for $\dot{\phi}$:

$$\dot{\phi} = \frac{L}{mr^2}$$

The Hamiltonian is

$$H = p_r\dot{r} + L\dot{\phi} - \mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{MGm}{r}$$

Resolving for $\dot{r}$:

$$\dot{r} = \sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} + \frac{2MG}{r}}$$

Again, the above expression diverges for an orbiting light ray ($m \to 0$). As for a circular orbit $\dot{r} = 0$ light rays are not allowed to follow circular orbits. For a circular orbit ($\dot{r} = 0$) of a massive particle:

$$E = \frac{L^2}{2mr^2} - \frac{MGm}{r}$$

Given L, the (effective radial potential) energy is a function of r: $E = E(r)$. Differentiating the RHS of the above expression w.r.t. r and setting the derivative equal to zero we get the condition

$$\frac{dE}{dr} = -\frac{L^2}{mr^3} + \frac{MGm}{r^2} = 0$$

The equilibrium position is $r = \frac{L^2}{MGm^2}$, that depends on L, i.e. a massive particle is can follow circular orbits at any distance depending on L. It is easy to show that the second derivative is $\frac{L^2}{mr^4} > 0$ at the distance of the orbital radius r, i.e. the (effective radial potential) energy reaches a minimum (stable equilibrium position) at the distance of the circular orbit.
Note:
Alternatively the energy equation can be solved w.r.t. r:

$$E = \frac{L^2}{2mr^2} - \frac{MGm}{r}$$

$$2mr^2E + 2MGm^2r - L^2 = 0$$

$$r = \frac{-2MGm^2 \pm \sqrt{\Delta}}{4mE}$$

and imposing $\Delta = 4M^2G^2m^4 + 8mEL^2 = 0$, i.e. $E = -\frac{M^2G^2m^3}{2L^2}$, we get the radius of the circular orbit $r = \frac{L^2}{MGm^2}$.
Note:
For comparison with the Newtonian calculations in the general case $\dot{r} \neq 0$ we need the formulas in chapter 6. So let's skip to chapter 6.

■

6. Black Holes

*Exercise 1

Page 217
[*Explain why a light ray emitted from inside the photon sphere can escape, but a light ray cannot enter the photon sphere and come out again.*]
Hint: refer to figure 7 on page 213.
Note:
Let's continue the Newtonian calculations in chapter 5 in the general case $\dot{r} \neq 0$ starting from the equations (11) and (14) in this chapter, that is:

$$L = \frac{mr^2\dot{\phi}}{\sqrt{g}}$$

$$E = \frac{m\mathcal{F}}{\sqrt{g}}$$

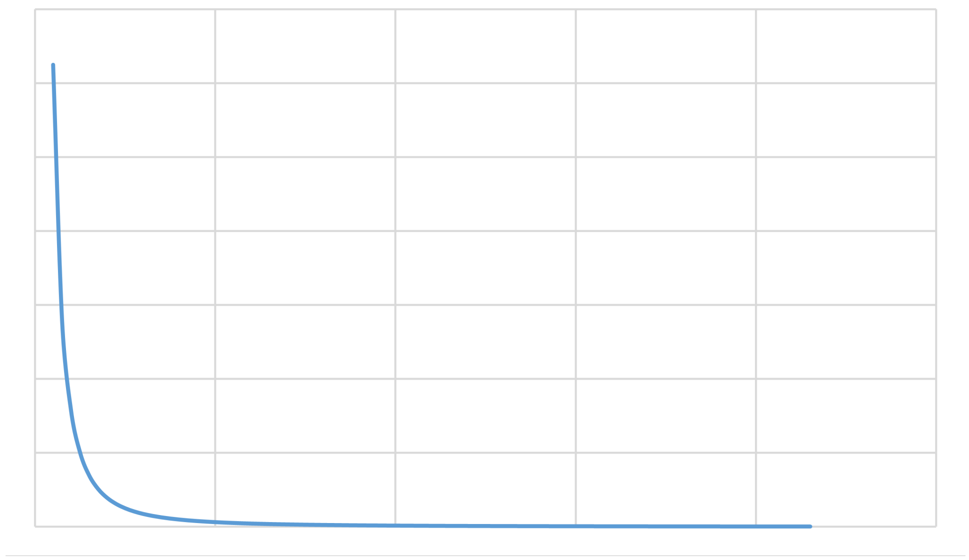

Figure 4: Effective radial potential of a test particle in the Newtonian approximation (arbitrary units).

where $g = \mathcal{F} - \dot{r}^2\mathcal{F}^{-1} - r^2\dot{\phi}^2$. Solving (11) for $r^2\dot{\phi}^2$:

$$r^2\dot{\phi}^2 = \frac{L^2}{m^2r^2}g$$

$$g = \mathcal{F} - \dot{r}^2\mathcal{F}^{-1} - \frac{L^2}{m^2r^2}g$$

and solving (14) for g:

$$g = \frac{m^2\mathcal{F}^2}{E^2}$$

$$\dot{r}^2 = \mathcal{F}^2 - \left(1 + \frac{L^2}{m^2r^2}\right)g\mathcal{F} = \mathcal{F}^2 - \left(1 + \frac{L^2}{m^2r^2}\right)\frac{m^2\mathcal{F}^3}{E^2}$$

$$\frac{1}{2}m\dot{r}^2 = \left(\frac{m}{2}\mathcal{F}^2 - \frac{m}{2}\frac{\mathcal{F}^3}{(E/m)^2}\right) - \frac{L^2}{2mr^2}\frac{\mathcal{F}^3}{(E/m)^2}$$

$$\frac{1}{2}m\dot{r}^2\left(\frac{E}{mc^2}\right)^2 = \left(\frac{mc^2}{2}\mathcal{F}^2\left(\frac{E}{mc^2}\right)^2 - \frac{mc^2}{2}\mathcal{F}^3\right) - \frac{L^2}{2mr^2}\mathcal{F}^3$$

where we have reintroduced for clarity c in SI units. With the approximations $\mathcal{F}^2 \approx 1 - \frac{2MG}{rc^2} + \cdots$, $\mathcal{F}^3 \approx 1 - \frac{3MG}{rc^2} + \cdots$, $E \approx mc^2 + \varepsilon$, $\left(\frac{E}{mc^2}\right)^2 \approx 1 + 2\frac{\varepsilon}{mc^2} + \cdots$:

$$\frac{1}{2}m\dot{r}^2 \approx \left(\frac{mc^2}{2}\left(1 + 2\frac{\varepsilon}{mc^2}\right) - 2\frac{mMG}{r} + \cdots\right) - \left(\frac{mc^2}{2} - 3\frac{mMG}{r} + \cdots\right) - \frac{L^2}{2mr^2}\mathcal{F}^3$$

In the limit $c \to \infty$ $\mathcal{F} = 1$ in the last term in the RHS. Rearranging we get the Newtonian approximation:

$$\varepsilon \approx \frac{1}{2}m\dot{r}^2 - \frac{mMG}{r} + \frac{L^2}{2mr^2}$$

Note:
In the case $\varepsilon = \frac{1}{2}mv_\infty^2 > 0$ the (hyperbolic) Newtonian trajectory does not make it to the singularity ($r = 0$) unless $L = 0$. If it were you would get $\lim_{r\to 0}\dot{r} = 0$ (as the particle would make it to the singularity) and $\lim_{r\to 0}\left(v_\infty^2 r^2\right) = \lim_{r\to 0}\left(-2MGr + \frac{L^2}{m^2}\right)$; but $0 \neq \lim_{r\to 0}\left(-2MGr + \frac{L^2}{m^2}\right)$ unless $L = 0$ (cf. figure 1 on page 199 of the book).

Note:
The non-approximated calculation can be done starting from equation (17) for $\dot{r} = 0$

$$E = m\frac{\sqrt{\mathcal{F}(r)}\sqrt{r^2\left(\frac{L}{m}\right)^2 + r^4}}{r^2}$$

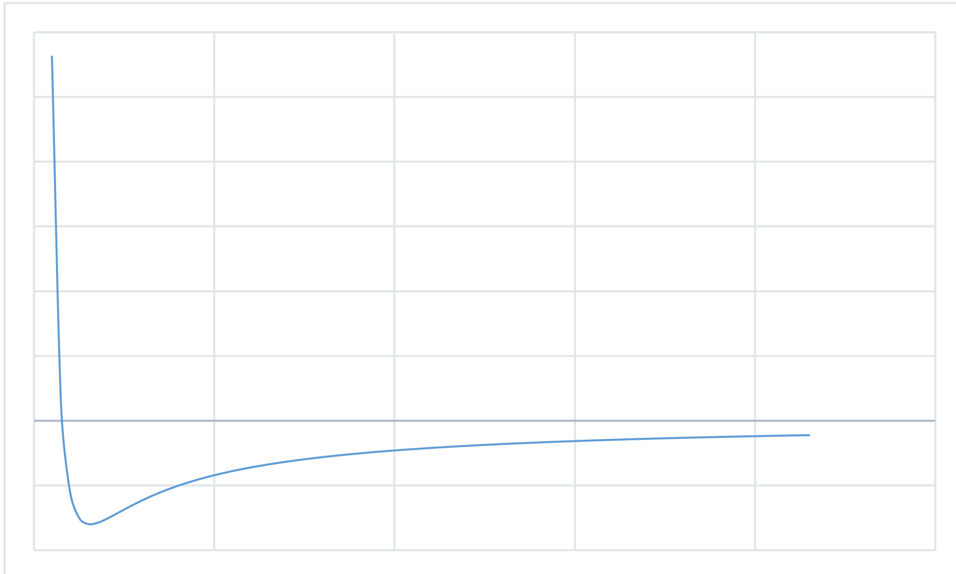

Figure 5: Effective radial potential of a massive body in the Newtonian approximation (arbitrary units).

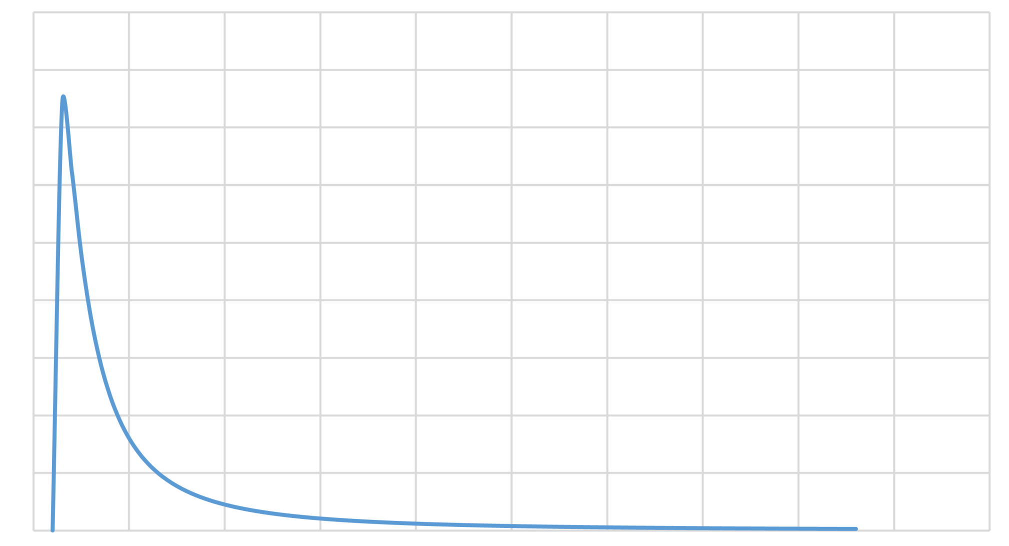

Figure 6: Effective radial potential of a test particle (arbitrary units).

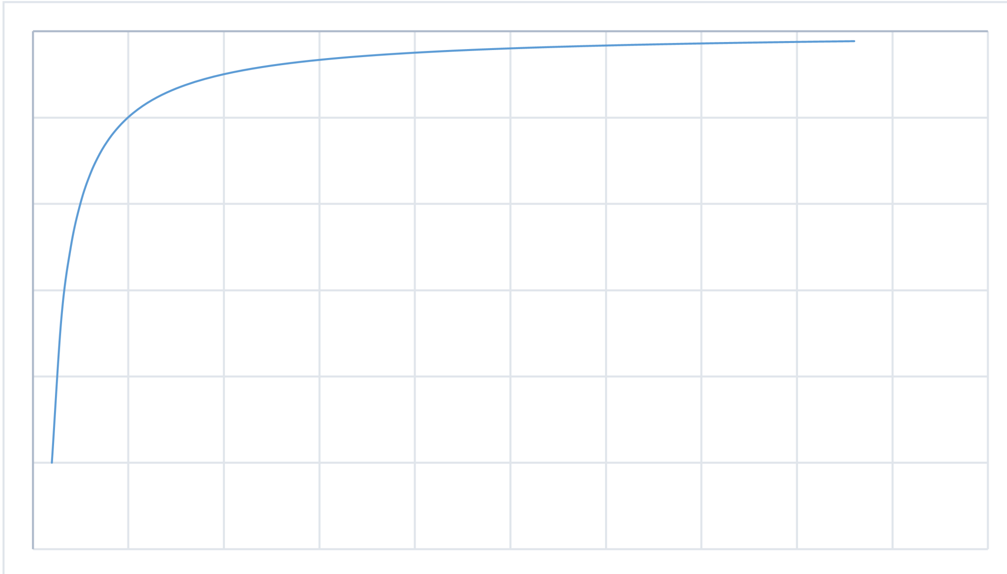

Figure 7: Effective radial potential of a massive body (arbitrary units).

that can be rearranged as

$$E = L\frac{\sqrt{\mathcal{F}}}{r}\sqrt{1+\left(\frac{mr}{L}\right)^2}$$

Observe that for $r \to 2MG$ $E = 0$ and for $r \to \infty$ the line $E = mc^2$ is an asymptote for the E curve. Squaring and rearranging:

$$\frac{E^2}{m^2} - 1 = -\frac{2MG}{r} + \frac{L^2}{m^2}\left(\frac{1}{r^2} - \frac{2MG}{r^3}\right)$$

The RHS behaves as an effective radial potential energy $V_{eff}(r)$. We can find the roots (positions of the maximum and minimum) and the minimum conditions under which there will be a minimum by deriving the LHS and setting the derivative of $V_{eff}(r)$ to zero. This is equivalent to setting the derivative of the energy to zero ($\frac{dE}{dr} = 0$):

$$\frac{d}{dr}\left(\frac{E^2}{m^2} - 1\right) = \frac{2E}{m^2}\cdot\frac{dE}{dr} = \frac{2MG}{r^2} - \frac{L^2}{m^2}\left(\frac{2}{r^3} - \frac{6MG}{r^4}\right) = 0$$

$$r^2 - \frac{L^2}{MGm^2}r + 3\frac{L^2}{m^2} = 0$$

$$r = \frac{L^2}{2MGm^2} \pm \sqrt{\left(\frac{L^2}{2MGm^2}\right)^2 - 3\frac{L^2}{m^2}}$$

Depending on $\Delta = \left(\frac{L^2}{2MGm^2}\right)^2 - 3\frac{L^2}{m^2}$ we can have two real roots (a maximum and a minimum) if $\Delta > 0$ and no real roots if $\Delta < 0$. For $\Delta = \left(\frac{L^2}{2MGm^2}\right)^2 - 3\frac{L^2}{m^2} = 0$ we have a minimum stable circular orbit for a given L: $r = \frac{L^2}{2MGm^2}$. Observing that $\Delta = \left(\frac{L^2}{2MGm^2}\right)^2 - 3\frac{L^2}{m^2} = r\frac{L^2}{2MGm^2} - 3\frac{L^2}{m^2} = 0$, we get $r = 6MG$, independent of L.

■

7. Falling into a Black Hole

*Exercise 1

Page 255
[*Use the various diagrams we drew to describe what a third person, say Charlie, following Alice sometime behind her would see of Alice and of Bob at different times.*]
Hint: see discussion in the text.

■

8. Formation of a Black Hole

*Exercise 1

Page 291
[*Suppose that there is a spherical mirror at some distance farther than the Schwarzschild radius, and that when the in-falling radiations hit the mirror, they are reflected back outward.*
1. In figure 23, draw the trajectory of the shell of radiations.
2. Discuss in which region people are doomed.]
Hint: refer to figure 23 on page 289. [The trajectory of the rays reflected off the mirror is slanted at $+45°$ and make it to scry$+$; and that's all, no people doomed.]

■

9. *Einstein Field Equations

Note: this chapter does not have any exercises.

■

10. Gravitational Waves

Exercise left to the reader

On page 342 it is left to the reader to compute the correction h in equation (17):

$$dX^2 + dY^2 = (dX')^2 + (dY')^2 + h_{mn} dX'^m dX'^n$$

where:

$$dX = dX' + \frac{\partial f}{\partial X'^m} dX'^m$$

$$dY = dY' + \frac{\partial g}{\partial X'^m} dX'^m$$

Solution:
In the appropriate approximations:

$$dX = dX' + \frac{\partial f}{\partial X'} dX' + \frac{\partial f}{\partial Y'} dY' = \left(1 + \frac{\partial f}{\partial X'}\right) dX' + \frac{\partial f}{\partial Y'} dY' \approx dX' + \frac{\partial f}{\partial Y'} dY'$$

$$dY = dY' + \frac{\partial g}{\partial Y'} dY' + \frac{\partial g}{\partial X'} dX' = \left(1 + \frac{\partial f}{\partial Y'}\right) dY' + \frac{\partial g}{\partial X'} dX' \approx dY' + \frac{\partial g}{\partial X'} dX'$$

$$dX^2 \approx \left(dX' + \frac{\partial f}{\partial Y'} dY'\right)^2 \approx (dX')^2 + \frac{\partial f}{\partial Y'} (dX' dY' + dY' dX')$$

$$dY^2 \approx \left(dY' + \frac{\partial g}{\partial X'} dX'\right)^2 \approx (dY')^2 + \frac{\partial g}{\partial X'} (dX' dY' + dY' dX')$$

$$dX^2 + dY^2 = (dX')^2 + (dY')^2 + \left(\frac{\partial f}{\partial Y'} + \frac{\partial g}{\partial X'}\right) dX'^m dX'^n$$

So we get

$$h_{mn} = \frac{\partial f}{\partial Y'} + \frac{\partial g}{\partial X'}$$

where the terms of a higher order of magnitude in smallness have been omitted.

■