Solutions to problems in Leighton and Vogt's textbook *"Exercises in Introductory Physics" (Addison-Wesley, 1969)*

Pier F. Nali

(Revised April 10, 2016)

CHAPTER 1

Atoms in Motion

B-3

B-3.

Ordinary air has a density of about 0.001 g cm^{-3}, while liquid air has a density of about 1.0 g cm^{-3}.

a) Estimate the number of air molecules per cm^3 in ordinary air and in liquid air.

b) Estimate the mass of an air molecule.

c) Estimate the average distance an air molecule should travel between collisions at normal temperatures and pressures (NTP). This distance is called the mean free path.

d) Estimate at what pressure, in normal atmospheres, a vacuum system should be operated in order that the mean free path be about one meter.

a) $n_G \sim 10^{19}\ cm^{-3}$

$n_L \sim 10^{22}\ cm^{-3}$

b) $m \sim 10^{-23} g$

c) $\ell \sim 10^{-5} cm$

d) $P \sim 10^{-7}$ atm.

Avogadro's number $N_0 = 6.025 \times 10^{23}$ molecules/mole

a) $n_G \sim \dfrac{N_0}{Molar\_vol.\_at\_STP} \sim \dfrac{6.025\times10^{23}\,\text{molecules / mole}}{22.41\times10^3\,\text{cm}^3\text{ / mole}} \sim 10^{19}\,\text{molecules/cm}^3$.

$$\frac{n_L}{n_G} \sim \frac{liquid\_air\_density}{air\_density} \sim \frac{1.0\text{ g cm}^{-3}}{0.001\text{ g cm}^{-3}} \sim 10^3 \Rightarrow n_L \sim 10^{22}\ \text{molecules/cm}^3.$$

b) $m \sim \dfrac{air\_density}{n_G} \sim \dfrac{0.001\text{ g cm}^{-3}}{2.7\times10^{19}\,\text{molecules / cm}^3} \sim 10^{-23}\ \text{g/molecule}$.

c) Said σ the (approx.) diameter of an air molecule the inverse of the mean free path is $\dfrac{1}{\ell} \sim \sigma^2 \cdot n_G$, where $\sigma \sim \sqrt[3]{\dfrac{1}{n_L}} \sim 10^{-7}\,\text{cm}$, assuming for a rough estimate that the molecules are condensed in liquid air in such a way that each molecule occupies a cubic cell of side σ. Thus $\dfrac{1}{\ell} \sim \left(10^{-7}\,\text{cm}\right)^2 \times 10^{19}\,\text{molecules/cm}^3 \sim 10^5\,\text{cm}^{-1}$ or, calculating the reciprocal, the mean free path $\ell \sim 10^{-5}\,\text{cm}$.

d) From $P : P^{(STP)} \sim n_G : n_G^{(STP)} \sim \ell^{(STP)} : \ell$,

$P^{(STP)} \sim 1$ atm,

$\ell^{(STP)} \sim 10^{-5}$ cm,

$\ell \sim 1$ m,

one gets $P \sim 10^{-7}$ atm.

B-4

B-4.

The intensity of a collimated, parallel beam of K atoms is reduced 3.0% by a layer of A gas 1.0 mm thick at a pressure of 6.0 x 10^{-4} mm Hg. Calculate the effective target area per argon atom.

B-4.

A ≈ 1.4 x 10^{-14} cm^2

Avogadro's number N_0 = 6.025 x 10^{23} molecules/mole

Said P the pressure of argon (A) gas spread in a layer of volume V (at constant temperature), we have

$$\frac{P}{P^{(STP)}} = \frac{Molar\_vol.\_at\_STP}{V} \Rightarrow V = \frac{P^{(STP)} \times Molar\_vol.\_at\_STP}{P} = \frac{760\text{ mm Hg} \times 22.41\times10^3\text{cm}^3\text{/mole}}{6.0\times10^{-4}\,\text{mm Hg}} =$$

$= 2.839\times10^{10}\,\text{cm}^3\text{/mole}$

and

$$number\_density\left(\mathrm{A}\_atoms\_per\_cm^3\right)=\frac{N_0}{V}=\frac{6.025\times10^{23}\,\mathrm{atoms/mole}}{2.839\times10^{10}\,\mathrm{cm^3/mole}}=2.123\times10^{13}\,\mathrm{atoms\,/\,cm^3}\,.$$

Said A a constant of proportionality representing the effective target area per A atom and expressing (on experimental basis) the percent intensity reduction of the atomic beam of K atoms as $A\times number\_density\times thickness=3.0\ \%$ one gets

$$A=\frac{3.0\ \%}{number\_density\times thickness}=\frac{3.0\times10^{-2}}{2.123\times10^{13}\,\mathrm{atoms/cm^3}\times10^{-1}\,\mathrm{cm}}\approx1.4\times10^{-14}\,\mathrm{cm^2/atom}\,.$$

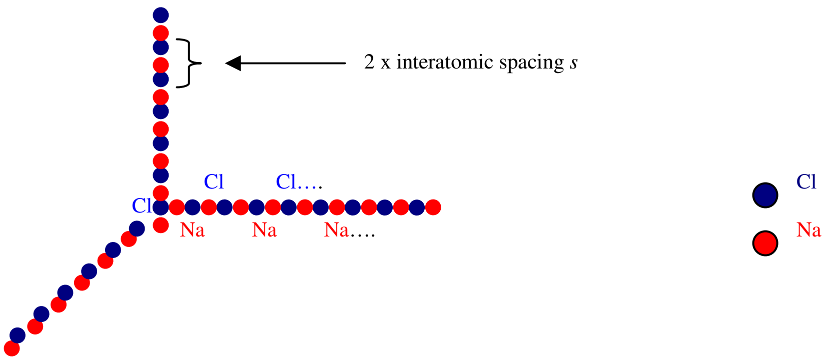
B-5

B-5.

X-ray diffraction studies show that NaCl crystals have a cubic lattice, with a spacing of 2.820 Å between nearest neighbours. Look up the density and molecular weight of NaCl and calculate Avogadro's number. (This is one of the most precise experimental methods for N_0.)

B-5.

6.02×10^{23} molecules/mole



The cubic lattice has $N\times N\times N$ points and $(N-1)\times(N-1)\times(N-1)$ spacings between nearest neighbours (Cl and Na ions in alternance). Then $lattice\_vol.=(N-1)^3\cdot s^3\approx N^3s^3$ (for large N) and

$$number\_density\left(atoms/\mathrm{\mathring{A}}^3\right)\approx\frac{N^3}{N^3s^3}=\frac{1}{s^3}=\frac{1}{\left(2.820\,\mathrm{\mathring{A}}\right)^3}=\frac{1}{22.4258\times10^{-24}}\,\mathrm{cm^{-3}}\,.$$

Since the NaCl molecule is diatomic

$$density\_of\_\mathrm{NaCl}=\frac{\tfrac{1}{2}\,number\_density\times molecular\_weight\_of\_\mathrm{NaCl}}{N_0}\,,$$

where:

$$\boxed{\begin{array}{l} density\_of\_\mathrm{NaCl} = 2.165\ \mathrm{g/cm^3}, \\ molecular\_weight\_of\_\mathrm{NaCl} = 58.454\ \mathrm{g/mole}, \end{array}}$$

and hence

$$N_0 = \frac{\frac{1}{2} number\_density \times molecular\_weight\_of\_\mathrm{NaCl}}{density\_of\_\mathrm{NaCl}} = 6.02\times10^{23}\,\mathrm{molecules/mole}\,.$$

B-6

B-6.

Boltwood and Rutherford found that radium in equilibrium with its disintegration products produced 13.6×10^{10} Helium atoms per second per gram of radium. They also measured that the disintegration of 192 mg of radium produced 0.0824 $\mathrm{mm^3}$ of Helium per day at STP. Use these data to calculate:

a) The number of helium atoms per $\mathrm{cm^3}$ of gas at STP.

b) Avogadro's number.

Reference: Boltwood and Rutherford, Phil. Mag. 22, 586, 1911.

B-6.

a) $\approx 2.7 \times 10^{19}$ atoms $\mathrm{cm^{-3}}$

b) $\approx 6.1 \times 10^{23}$ molecules/mole

The number of helium atoms expelled per second by one gram of radium in equilibrium with its disintegration products is $13.6\times10^{10}\,\mathrm{atoms\ g^{-1}\ s^{-1}} = 117.504\times10^{14}\,\mathrm{atoms\ g^{-1}\ day^{-1}}$. The corresponding volume of helium produced per day by one gram of radium is $\frac{0.0824\times10^{-3}\,\mathrm{cm^3\ day^{-1}}}{192\times10^{-3}\,\mathrm{g}} = 4.291\times10^{-4}\,\mathrm{cm^3\ g^{-1} day^{-1}}$ and thus the number density of helium atoms at STP is

a) $$n_{He}^{(STP)} = \frac{117.504\times10^{14}\,\mathrm{atoms\ g^{-1}\ day^{-1}}}{4.291\times10^{-4}\,\mathrm{cm^3\ g^{-1}\ day^{-1}}} \approx 2.7\times10^{19}\,\mathrm{atoms/cm^3}$$

and the calculated Avogadro's number

b)
$$N_0 = n_{He}^{(STP)} \times Molar\_vol.\_at\_STP \approx 2.7\times10^{19}\,\mathrm{atoms\ cm^{-3}} \times 22.41\times10^3\,\mathrm{cm^3/mole} \approx 6.1\times10^{23}\,\mathrm{atoms/mole}\,.$$

C-1

C-1.

Rayleigh found that 0.81 mg of olive oil on a water surface produced a mono-molecular layer 84 cm in diameter. What value of Avogadro's number results?

Note: Approximate composition $H(CH_2)_{18}COOH$, in a linear chain. Density 0.8 g cm^{-3}.

Reference: Rayleigh, Proc. Roy. Soc., 47, 364 (1890).

C-1.

$\sim 10^{24}$ molecules/mole

The molecular weight of $H(CH_2)_{18}COOH$ is approximately 298 g/mole and the molar volume is $\dfrac{molecular\_weight}{density} \approx \dfrac{298 \text{ g mole}^{-1}}{0.8 \text{ g cm}^{-3}} \approx 372.5 \text{ cm}^3\text{/mole}$.

The volume of oil is deduced by dividing 0.81 mg by its density, i.e. $\dfrac{0.81\times10^{-3}\text{g}}{0.8 \text{ g cm}^{-3}} \approx 1.0\times10^{-3}\text{cm}^3$.

The surface area over which this volume of oil is spread is $\frac{1}{4}\pi\times84^2\text{cm}^2 \approx 5.542\times10^3\text{cm}^2$, so the thickness of the oil film (calculated as if its density were the same as in non spread state) is $l \approx \dfrac{0.81\times10^{-3}\text{g}/0.8 \text{ g cm}^{-3}}{\frac{1}{4}\pi\times84^2\text{cm}^2} \approx \dfrac{1.0\times10^{-3}\text{cm}^3}{5.542\times10^3\text{cm}^2} \approx 0.2\times10^{-6}\text{cm}$.

If the oil molecules were arranged in linear chains 20 atoms high ($\approx$ thickness l of the oil film) $\times$ 4 atoms wide ($\approx \frac{1}{5}\times l$) so that a molecule occupies a volume roughly estimated (assuming in liquid state a rotational degree of motion around the main axis of the chain) as $l\times\frac{1}{5}l\times\frac{1}{5}l = \dfrac{l^3}{25} \approx \dfrac{\left(0.2\times10^{-6}\text{cm}\right)^3}{25} \approx 3.2\times10^{-22}\text{cm}^3$.

The reciprocal of the latter (0.3×10^{22} molecules/cm^3) is the number density $\dfrac{N_0}{Molar\_vol.}$. So

$N_0 \approx 0.3\times10^{22}\text{molecules/cm}^3\times Molar\_vol. \approx 0.3\times10^{22}\text{mol./cm}^3\times372.5 \text{ cm}^3\text{/mole} \sim 10^{24}\text{molecules/mole}$.

C-2

C-2.

About 1860, Maxwell showed that the viscosity of a gas is given by

$$\eta = 1/3\ \rho v\ \ell$$

where ρ is the density, v the mean speed, and ℓ, the mean free path. The latter quantity he had earlier shown to be $\ell = 1/(\sqrt{2}\ \pi N_g\ \sigma^2)$, where σ is the diameter of the molecule. Loschmidt (1865) used the measured value of η, ρ(gas), and ρ(solid) together with Joule's calculated v to determine N_g, the number of molecules per cm^3 in a gas at STP. He assumed the molecules to be hard spheres, tightly packed in a solid. Given $\eta = 2.0 \times 10^{-4}$ g $cm^{-1}s^{-1}$ for air at STP, ρ(liquid) ~ 1 g cm^{-3}, ρ(gas) ~ 1×10^{-3} g cm^{-3}, and v ~ 500 m s^{-1}, calculate N_g.

C-2.

$N_g \sim 0.7 \times 10^{19}$ molecules/cm^3.

Said m the mass of a gas molecule the gas density is $\rho(\text{gas}) = m \cdot N_g$. If one assume for sake of simplicity that in a solid the molecules are tightly packed in cells of side σ then $\rho(\text{solid}) \approx m/\sigma^3$ (1). Thus

$$\frac{\rho(\text{gas})}{\rho(\text{solid})} \approx N_g \cdot \sigma^3 \Rightarrow \sigma \approx \sqrt[3]{\frac{1}{N_g} \cdot \frac{\rho(\text{gas})}{\rho(\text{solid})}} \Rightarrow N_g \sigma^2 \approx \sqrt[3]{N_g} \cdot \left[\frac{\rho(\text{gas})}{\rho(\text{solid})}\right]^{2/3}.$$

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(1) For our purposes here we may ignore that in closed packaging configuration the proportion of space occupied by spherical particles is $\Delta = \frac{nr\_of\_particles\_per\_cell \times V_{(particle)}}{V_{(unitcell)}} = \frac{\pi}{3\sqrt{2}} \approx 74,048\%$ and therefore the number of particles per unit cell is $\Delta \cdot \frac{V_{(unitcell)}}{V_{(particle)}} = \frac{\pi}{3\sqrt{2}} \cdot \frac{\sigma^3}{\frac{\pi}{6}\sigma^3} = \sqrt{2}$ particles/cell rather than 1 particle/cell, $\rho(\text{solid})$ is $\sqrt{2} \cdot m/\sigma^3$ rather than m/σ^3 and so on, so the subsequent results should contain $\frac{N_g}{\sqrt{2}}$ in locus of N_g. This is the densest packing of equal spheres in three dimensions, as proved by Gauss in 1831. See Gauss, C. F. "Besprechung des Buchs von L. A. Seeber: Intersuchungen über die Eigenschaften der positiven ternären quadratischen Formen usw." Göttingsche Gelehrte Anzeigen (1831, July 9) 2, 188-196, 1876 = Werke, II (1876), 188-196. For a proof of this result see, e.g., J. W. S. Cassels, —An Introduction to the Geometry of Numbers, Springer-Verlag, 1971 [Chap. II, Th. III]

Comparing $\ell = 1\big/\left(\sqrt{2}\pi N_g \sigma^2\right)$ and $\eta = \frac{1}{3}\rho(\text{gas}) \cdot v \cdot \ell$ one also gets

$$N_g \sigma^2 = 1\big/\left(\sqrt{2}\pi \ell\right) = \rho(\text{gas}) \cdot v \big/ \left(\sqrt{2}\pi \cdot 3\eta\right).$$

Comparing again the right-hand sides of latter two expressions for $N_g \sigma^2$ and cubing the second members one gets

$$N_g \cdot \left[\frac{\rho(\text{gas})}{\rho(\text{solid})}\right]^2 \approx \left[\rho(\text{gas})\right]^3 \cdot v^3 \big/ \left(54\sqrt{2}\pi^3 \cdot \eta^3\right)$$

and finally – assuming on experimental basis(2) $\rho(\text{solid}) \approx \rho(\text{liquid})$ -

$$N_g \approx \frac{\rho(\text{gas})\left[\rho(\text{liquid})\right]^2 \cdot v^3}{54\sqrt{2} \cdot \pi^3 \cdot \eta^3} \sim 0.7 \times 10^{19}\,\text{molecules/cm}^3.$$

(2) This assumption is valid where we are dealing only whit orders of magnitude and is based on the observed condensation behaviour of many substances. Unfortunately, at the times of Loschimdt's work (1865) was not yet known how to liquefy the air, so he could not directly observe condensation behaviour for air. However, the density of a substance in the condensed state can be estimated by its chemical composition, as Loschmidt did. See Porterfield, William W., and Walter Kruse. "Loschmidt and the Discovery of the Small.", *J. Chem. Educ.* , 1995, 72 (10), p 870

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CHAPTER 2

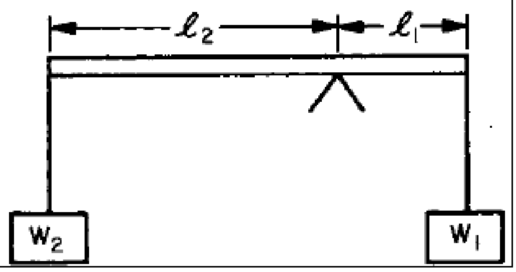
Conservation of Energy, Statics

Refer to The Feynman Lectures on Physics, Vol. I, Ch. 4.

1

1.

Use the principle of virtual work to establish the formula for an unequal-arm balance: $W_1 \ell_1 = W_2 \ell_2$. (Neglect the weight of the cross-beam.)



According to the principle of virtual work (PVW) "the sum of the heights times the weights does not change", said h_1 and h_2 the heights of the weights W_1 and W_2 respectively, then $\Delta(W_1 h_1 + W_2 h_2) = W_1 \Delta h_1 + W_2 \Delta h_2 = 0$.

If we assume that the weight W_1 goes down Δh_1 as the weight W_2 rises Δh_2, then $\Delta h_1 = -\frac{\ell_1}{\ell_2} \Delta h_2 \Rightarrow W_1 \ell_1 = W_2 \ell_2$.

2

2. Extend the formula obtained in the previous exercise to include a number of weights hung at various distances from the pivot point:

$$\sum_i W_i \ell_i = 0$$

(Distances on one side of the fulcrum are considered positive and on the other side, negative.)

The "sum of the heights times the weights" for an arbitrary number of weights becomes $\sum_i W_i \Delta h_i = 0$ and considering the distances positive or negative depending on the side of the fulcrum, so that $\Delta h_i = \frac{\ell_i}{\ell_j}\Delta h_j$ (or $\Delta h_i \propto \ell_i$ for any *i*, being $\frac{\Delta h_j}{\ell_j}$ independent of *j*), it becomes thus $\sum_i W_i \ell_i = 0$.

3

3. A body is acted upon by n forces and is in static equilibrium. Use the principle of virtual work to prove that:

a) If n = 1, the magnitude of the force must be zero. (A trivial case.)

b) If n = 2, the two forces must be equal in magnitude, opposite in direction, and collinear.

c) If n = 3, the forces must be coplanar and their lines of action must pass through a single point.

d) For any n, the sum of the products of the magnitude of a force F_i times the cosine of the angle Δ_i between the force and any fixed line, is zero:

$$\sum_{i=1} F_i \cos \Delta_i = 0.$$

The PVW can be expressed in a more general form as $\sum_i^N \left(\sum_j^n \mathbf{F}_{ij} \right) \bullet \Delta \mathbf{r}_i = 0$, where the sum over *j* is extended to all the *n* forces acting on the mass *i* and the sum over *i* is extended to all the *N* masses.

In the present case (a single body) $N = 1$, so the above formula reduces to $\left(\sum_i^n \mathbf{F}_i \right) \bullet \Delta \mathbf{r} = 0$.

a) For $n = 1$ is $\mathbf{F} \bullet \Delta \mathbf{r} = 0$; so, for the arbitrariness of the choice of the vector $\Delta \mathbf{r}$, $\mathbf{F} = 0$.

b) For $n=2$ is $(\mathbf{F}_1+\mathbf{F}_2)\bullet\Delta\mathbf{r}=0$ and, again for the arbitrariness of the choice of the vector $\Delta\mathbf{r}$, is $\mathbf{F}_1+\mathbf{F}_2=0$, whence $\mathbf{F}_1=-\mathbf{F}_2$; so, the two forces are equal in magnitude, opposite in direction and collinear.

c) For $n=3$ is, again, $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3=0$. If we consider the plane on which lie the two vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ (apart from the trivial case in which the two vectors are parallel, which degenerates to the case of $n=2$), $\mathbf{F}_3$ also lies in the same pane, being expressible as a linear combination of the other two vectors. Let $\mathbf{X_0}$ the vector joining the origin to the point of intersection of the lines of action of the two vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ and $\mathbf{X}$ the vector joining the origin to an arbitrary point of one or the other line of action. Then the two lines of action are defined by the system of equations $\begin{cases}(\mathbf{X}-\mathbf{X}_0)\wedge\mathbf{F}_1=0\\(\mathbf{X}-\mathbf{X}_0)\wedge\mathbf{F}_2=0\end{cases}$.

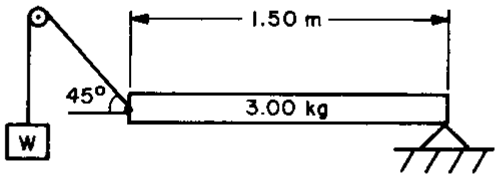
By adding the latter two equation, immediately follows that $\mathbf{X_0}$ also lies on the line of action of $\mathbf{F}_3$. $\mathbf{X_0}$ defines a single point (through which the three lines pass) as each line of action cannot intersect each other in more than one point (provided the lines are not parallel).

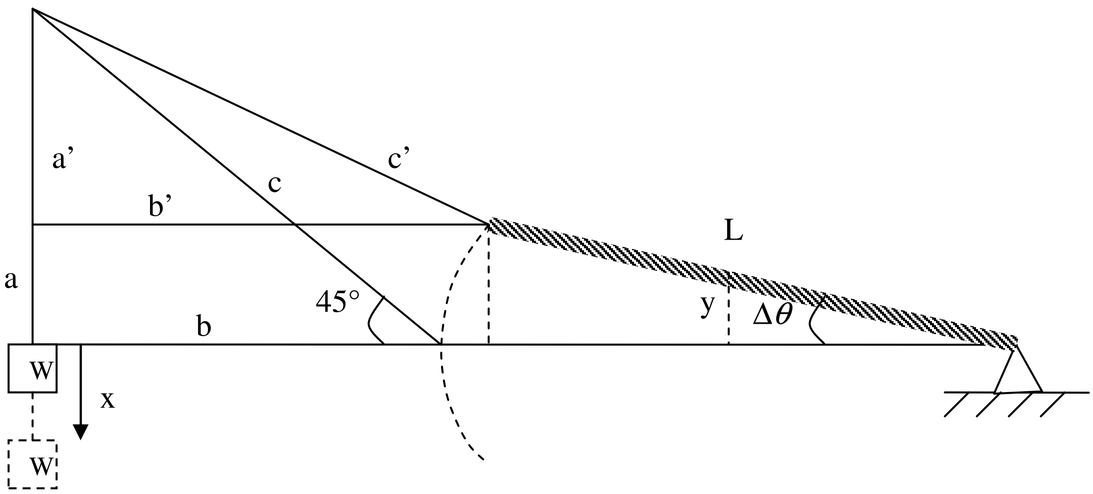
d) From $\mathbf{F}_i\bullet\Delta\mathbf{r}=F_i\Delta\mathrm{r}\cdot\cos\Delta_i$ it follows that $\left(\sum_{i=1}^{n}\mathbf{F}_i\right)\bullet\Delta\mathbf{r}=\left(\sum_{i=1}^{n}F_i\cos\Delta_i\right)\cdot\Delta r=0$, and – again for the arbitrariness of the choice of Δr - $\sum_{i=1}^{n}F_i\cos\Delta_i=0$.

A-1

A-1.

A uniform plank 1.5 m long and weighing 3.00 kg is pivoted at one end. The plank is held in equilibrium in a horizontal position by a weight and pulley arrangement, as shown. Find the weight W needed to balance the plank. Neglect friction.





A-1.

$$W = \frac{3}{\sqrt{2}} \text{ kg-wts}$$

Suppose the weight W goes down *x* as the centre of mass of the plank rises *y*. Therefore (refer to the above figure) $y = \frac{L}{2}\sin\Delta\theta$, $x = -(c - c')$ and

$a' = a - L\sin\Delta\theta,$

$b' = b + L(1-\cos\Delta\theta),$

$c' = \sqrt{a'^2 + b'^2}.$

So – assuming $\Delta\theta$ small - $c' = c\sqrt{1 - 2\frac{aL}{c^2}\sin\Delta\theta + 4\frac{L(L+b)}{c^2}\sin^2\frac{\Delta\theta}{2}} \cong c\left(1 - \frac{aL}{c^2}\sin\Delta\theta\right)$ and $x \cong -\frac{aL}{c}\sin\Delta\theta$,

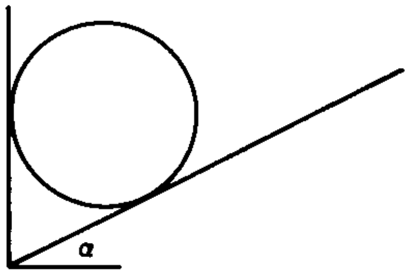
where we have used the trigonometric double-angle formula $1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$ and the approx. $\sqrt{1+x} \cong 1 + \frac{x}{2}$ for *x* small.

Being also $c=\sqrt{2}\cdot a$ it follows $x \cong -\frac{L}{\sqrt{2}}\sin\Delta\theta = -\sqrt{2}\cdot y$. From PVW then we have $W\cdot(-x) = W_{(plank)}\cdot y$, whence $W = -\frac{y}{x}W_{(plank)} = \frac{W_{(plank)}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$ kg-wts .
(note: the length of the plank doesn't affect this derivation).

A-2

A-2.

A ball of radius 3.0 cm and weight 1.00 kg rests on a plane tilted at an angle α with the horizontal and also touches a vertical wall. Both surfaces have negligible friction. Find the force with which the ball presses on each plane.



A-2.

$F_P = \dfrac{1}{\cos\alpha}$ kg-wt

$F_W = \tan\alpha$ kg-wt

Let W the weight of the ball, F_P and F_W the forces exerted on the ball by the plane and by the wall respectively and made use of PVW in the form $\sum_{i=1}^{n} F_i \cos\Delta_i = 0$.
Thus, with respect to an horizontal line joining the point in which the ball touches the wall with the centre of mass of the ball, $F_w\cos(0) + F_P\cos\left(\frac{\pi}{2}+\alpha\right) = 0$, that is $F_w - F_P\sin\alpha = 0$; and, with respect to a vertical line lying in the plane of the sheet and passing through the centre of mass of the ball, $F_W\cos\frac{\pi}{2} + W\cos\pi + F_P\cos\alpha = 0$, that is $-W + F_P\cos\alpha = 0$.
So

$$F_P = \frac{W}{\cos\alpha},$$

$$F_W = W\tan\alpha,$$

and the forces on the planes are

$$F_P = \frac{1}{\cos\alpha}\ \text{kg-wt},$$

$$F_W = \tan\alpha\ \text{kg-wt}.$$

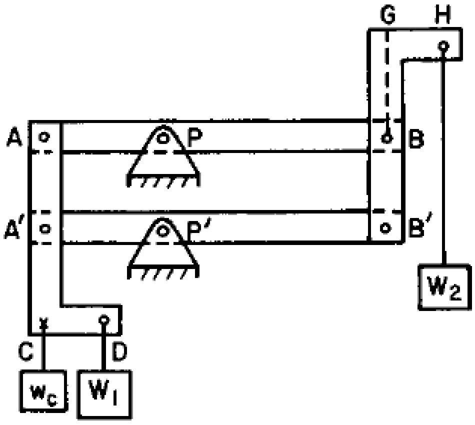
A-3

A-3.

The jointed parallelogram frame AA'BB' is pivoted (in a vertical plane) on the pivots P and P'. There is negligible friction in the pins at A, A', B, B', P, and P'. The members AA'CD and B'BGH are rigid and identical in size.

$$AP = A'P' = \tfrac{1}{2}\, PB = \tfrac{1}{2}\, P'B'$$

Because of the counterweight w_c, the frame is in balance without the loads W_1 and W_2. If a 0.50 kg weight W_1 is hung from D, what weight W_2, hung from H, is needed to produce equilibrium?



A-3.

W_2 = 0:25 kg-wts

Again make use of PVW, this time in the form $\sum_{i=1}^{n} W_i h_i = 0$.

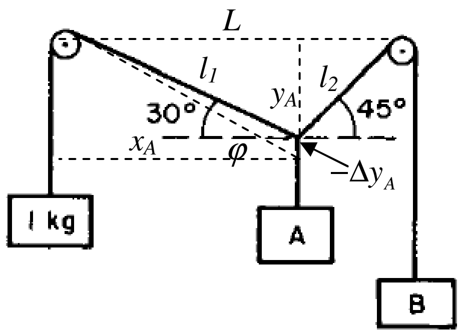
- Without the loads: $\left(W_C + W_{AA'CD}\right)\cdot AP = W_{BB'GH}\cdot PB$, that is $W_C + W_{AA'CD} = 2W_{BB'GH}$.
- With the loads, assume the weight W_1 goes down *x* as the weight W_2 rises *y*, being $x = -\dfrac{1}{2}y$.

Thus $\left(W_C + W_{AA'CD} + W_1\right)\cdot(-x) = \left(W_{BB'GH} + W_2\right)\cdot y$, that is $W_C + W_{AA'CD} + W_1 = 2\left(W_{BB'GH} + W_2\right)$, and subtracting to this equation the one without the loads above one gets $W_1 = 2W_2$, so $W_2 = \frac{1}{2}W_1 = 0.25$ kg-wts.

A-4

A-4.

The system shown is in static equilibrium. Use the principle of virtual work to find the weights A and B. Neglect the weight of the strings and the friction in the pulleys.



<u>A-4.</u>

$$A = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \text{ kg-wts}$$

$$B = \sqrt{\frac{3}{2}} \text{ kg-wts}$$

The weight A can move both vertically and horizontally.

- Vertical displacement of A (assumed downwards): $W_A \cdot \left(-\Delta y_A\right) = W_{1\text{ kg}} \cdot \Delta y_{_{1\text{ kg}}} + W_B \cdot \Delta y_B$.
- Horizontal displacement of A (assuming B rises Δy_B as A moves to the left): $W_{1\text{ kg}} \cdot \left(-\Delta y_{_{1\text{ kg}}}\right) = W_B \cdot \Delta y_B$

<u>Case 1 – vertical displacement</u>

For the flattened triangle $\triangle$ (l_1, Δy_A, $l_1 + \Delta l_1$), using Carnot's theorem and assuming Δy_A small, $l_1 + \Delta l_1 = \sqrt{l_1^{\,2} + 2l_1 \Delta y_A \cos\left(\frac{\pi}{2} + \varphi\right) + \Delta y_A^2} = \sqrt{l_1^{\,2} + 2l_1 \Delta y_A \left(-\sin\varphi\right) + \Delta y_A^2} = l_1\sqrt{1 - 2\left(\frac{\Delta y_A}{l_1}\right)\sin\varphi + \frac{\Delta y_A^2}{l_1^2}}$.

Neglecting the term of order Δy_A^2 and using the approx. $\sqrt{1+x} \cong 1 + \frac{1}{2}x$, then $l_1 + \Delta l_1 \cong l_1\sqrt{1 - 2\left(\frac{\Delta y_A}{l_1}\right)\sin\varphi} \cong l_1\left(1 - \frac{\Delta y_A}{l_1}\sin\varphi\right)$ and $\Delta l_1 \cong l_1\left(1 - \frac{\Delta y_A}{l_1}\sin\varphi\right) - l_1 \cong -\Delta y_A \sin\varphi$, where $\sin\varphi = \dfrac{y_A}{l_1 + \Delta l_1} \cong \dfrac{y_A}{l_1}$. So $\Delta l_1 \cong -\Delta y_A \dfrac{y_A}{l_1}$.

Similarly, for the triangle (not shown) $\triangle$ (l_2, Δy_A, $l_2 + \Delta l_2$), $\Delta l_2 \cong -\Delta y_A \dfrac{y_A}{l_2}$. But $\Delta l_1 = \Delta y_{_{1\text{ kg}}}$ and $\Delta l_2 = \Delta y_{_B}$, whence

$$W_A \cdot \left(-\Delta y_A\right) \cong W_{1\text{ kg}} \cdot \left(-\Delta y_A\right)\frac{y_A}{l_1} + W_B \cdot \left(-\Delta y_A\right)\frac{y_A}{l_2} \to W_A \cong W_{1\text{ kg}} \cdot \frac{y_A}{l_1} + W_B \cdot \frac{y_A}{l_2} =$$
$$= W_{1\text{ kg}} \cdot \sin 30° + W_B \cdot \sin 45° = \tfrac{1}{2}W_{1\text{ kg}} + \tfrac{1}{\sqrt{2}}W_B = \tfrac{1}{2} + \tfrac{1}{\sqrt{2}}W_B.$$

<u>Case 2 – horizontal displacement</u>

In the same way, for the triangle (not even shown as well) $\triangle$ ($l_1 - \Delta l_1$, Δx_A, l_1) with Δx_A small, $l_1 - \Delta l_1 = \sqrt{l_1^{\,2} + 2l_1 \Delta x_A \cos 30° + \Delta x_A^2} = l_1\sqrt{1 + 2\left(\frac{\Delta x_A}{l_1}\right)\cos 30° + \frac{\Delta x_A^2}{l_1^2}}$, and, with the above approximations, $l_1 - \Delta l_1 \cong l_1\sqrt{1 + 2\left(\frac{\Delta x_A}{l_1}\right)\cos 30°} \cong l_1\left(1 + \frac{\Delta x_A}{l_1}\cos 30°\right) \to \Delta l_1 \cong l_1 - l_1\left(1 + \frac{\Delta x_A}{l_1}\cos 30°\right) \cong -\Delta x_A \cos 30°$.

Substituting $\cos 30° = \dfrac{x_A}{l_1}$ in the above expressions one gets $\Delta l_1 \cong -\Delta x_A \dfrac{x_A}{l_1}$, and similarly, for triangle $\triangle$ ($l_2 + \Delta l_2$, Δx_A, l_2): $\Delta l_2 \cong \Delta x_A \cos 45° = \Delta x_A \dfrac{L - x_A}{l_2}$.

So from PVW

$$W_{1\text{ kg}} \cdot \left(-\Delta y_{_{1\text{ kg}}}\right) = W_B \cdot \Delta y_B \to W_{1\text{ kg}} \cdot \left(-\Delta l_1\right) = W_B \cdot \Delta l_2 \to W_{1\text{ kg}} \cdot \Delta x_A \frac{x_A}{l_1} = W_B \cdot \Delta x_A \frac{L - x_A}{l_2}$$

$$\to W_{1\text{ kg}} \cdot \frac{x_A}{l_1} = W_B \cdot \frac{L - x_A}{l_2} \to W_{1\text{ kg}} \cdot \cos 30° = W_B \cdot \cos 45° \to W_{1\text{ kg}} \cdot \tfrac{\sqrt{3}}{2} = W_B \cdot \tfrac{\sqrt{2}}{2}$$

$$\to W_B = \sqrt{\tfrac{3}{2}} \cdot W_{1\text{ kg}}.$$

Whence finally $W_A \cong \frac{1}{2} + \frac{1}{\sqrt{2}} W_B = \frac{1}{2} + \frac{\sqrt{3}}{2} W_{1\text{ kg}}$. In summary:

$W_A = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$ kg-wts,

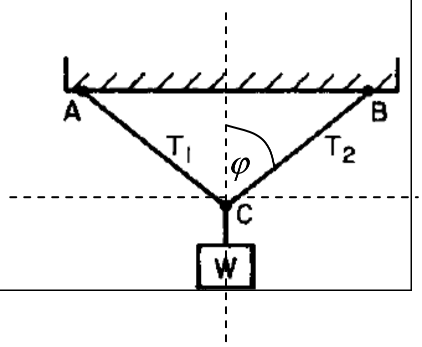
$W_B = \sqrt{\frac{3}{2}}$ kg-wts.

A-5

A-5.

A weight W = 50 lb. is suspended from the mid-point of a wire ACB as shown.

AC = CB = 5 ft. AB = 5 $\sqrt{2}$ ft. Find the tension in the wire.



A-5.

$T_1 = T_2 = \dfrac{50}{\sqrt{2}}$ lb-wts

From the PVW (in the form $\sum_{i=1}^{n} F_i \cos \Delta_i = 0$):

- Case 1 – vertical fixed line:

$$T_1 \cos\varphi + T_2 \cos\varphi + W \cos\pi = 0 \to \left(T_1 + T_2\right)\cos\varphi = W \to \left(T_1 + T_2\right)\sqrt{1 - \sin^2\varphi} = W \to$$

$$\left(T_1 + T_2\right)\sqrt{1 - \left(\frac{\text{AB}/2}{\text{AC}}\right)^2} = W \to \tfrac{1}{\sqrt{2}}\left(T_1 + T_2\right) = W.$$

- Case 2 – horizontal fixed line:

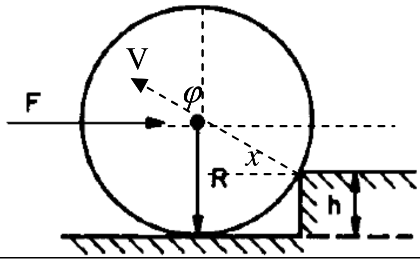
$$-T_1 \cos\left(\tfrac{\pi}{2} - \varphi\right) + T_2 \cos\left(\tfrac{\pi}{2} - \varphi\right) + W \cos\tfrac{\pi}{2} = 0 \to T_1 = T_2.$$

Thus $T_1 = T_2 = \dfrac{W}{\sqrt{2}} = \dfrac{50}{\sqrt{2}}$ lb-wts

A-7

A-7.

What horizontal force F (applied at the axle) is required to push a wheel of weight W and radius R over a block of height h?



A-7.

$$F = W\frac{\sqrt{h(2R - h)}}{R - h}$$

Let V the reaction force of the edge in contact with the wheel and assume null the reaction force of the wall at the ground soon as the force F is applied to the axle. Then from PVW $\sum_{i=1}^{n} F_i \cos\Delta_i = 0$:

- Case 1 – vertical fixed line: $V\cos\varphi + W\cos\pi = 0 \to V\cos\varphi = W \to V\frac{R-h}{R} = W$.
- Case 2 – horizontal fixed line: $F\cos(0) + V\cos\left(\frac{\pi}{2}+\varphi\right) + W\cos\frac{\pi}{2} = 0 \to F = V\sin\varphi = V\frac{x}{R}$,

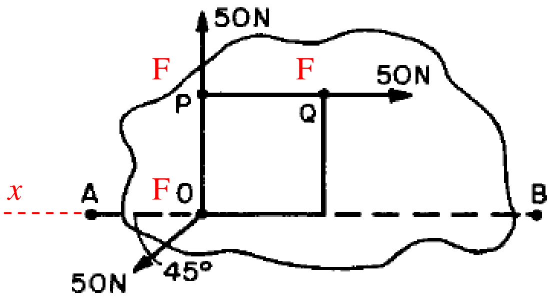
where $x = \sqrt{R^2 - (R-h)^2} = \sqrt{h(2R-h)}$.

Thus $F = W\frac{x}{R-h} = W\frac{\sqrt{h(2R-h))}}{R-h}$.

A-9

A-9.

A flat steel plate floating on mercury is acted upon by three forces at three corners of a square of side 0.100 m, as shown. Find a single fourth force which will hold the plate in equilibrium. Give the magnitude, direction, and point of application along the line AB.



A-9.

F = 20.7 N, 45^o

.34 m L of 0

Using $\sum_{i=1}^{n} F_i \cos\Delta_i = 0$ we have:

- horizontal line $\overline{AB}$:

$F_O\cos(\pi+45°)+F_P\cos\frac{\pi}{2}+F_Q\cos 0°+F_x\cos(\pi+\varphi)=0 \Rightarrow -F\cos 45°+F-F_x\cos\varphi=0$,

- vertical line $\overline{OP}$:

$F_O\cos\left(\frac{\pi}{2}+45°\right)+F_P\cos 0°+F_Q\cos\frac{\pi}{2}+F_x\cos\left(\frac{\pi}{2}+\varphi\right)=0 \Rightarrow -F\sin 45°+F-F_x\sin\varphi=0$;

and confronting:

$\cos\varphi=\sin\varphi$, whence $\varphi=45°$ or $\varphi=225°$.

F_x is positive when $\varphi=45°$ therefore it is parallel to, and has the same direction of, F_O. Its magnitude is $F_x=F\left(\sqrt{2}-1\right)=20.7\text{ N}$, 45° down from $\overline{AB}$.

When the plate is in equilibrium the torque, with respect to the corner *O*, is

$F_x\cdot x\cdot\sin(\pi-\varphi)+\overline{OP}\cdot F_Q+0\cdot F_P+0\cdot F_O=0 \Rightarrow F_x\cdot x\cdot\sin\varphi+F\cdot\overline{OP}=0 \Rightarrow$

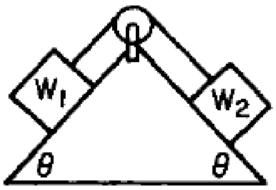
$$x=-\frac{F\cdot\overline{OP}}{F_x\sin 45°}=-\frac{50\text{ N}\times 0{,}100\text{ m}}{20.7\text{ N}\times 0.7071}=-0{,}34\text{ m},$$

i.e. .34 m left of O.

A-10

A-10.

In the absence of friction, how fast will the weights W_1 and W_2 be going when they travel a distance D, starting from rest? $(W_1 > W_2)$



A-10.

$$v = \sqrt{2gD \frac{W_1 - W_2}{W_1 + W_2} \sin\theta}$$

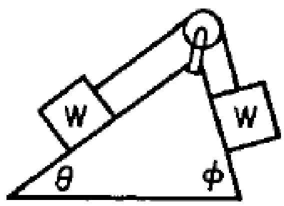
Assume $W_1 > W_2$. Then as W_1 goes down $-D\sin\theta$ the weight W_2 rises $+D\sin\theta$. From the conservation of energy $\frac{1}{2}\left[\frac{W_1+W_2}{g}\right]\cdot v^2 + W_1\cdot(-D\sin\theta) + W_2\cdot D\sin\theta = 0$.

Thus $v = \sqrt{2gD\frac{W_1-W_2}{W_1+W_2}\sin\theta}$.

A-11

A-11.

In the figure, the weights are equal, and there is negligible friction. If the system is released from rest, how fast are the weights moving when they have gone a distance D?



A-11.

$$v = \sqrt{gD\,(\sin\varphi - \sin\theta)}$$

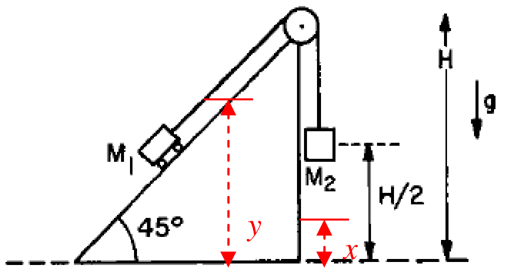
See the previous problem. In the present case,

$$\tfrac{1}{2}\cdot\frac{2W}{g}\cdot v^2 + W\cdot D\sin\theta + W\cdot(-D\sin\phi) = 0 \Rightarrow v = \sqrt{gD(\sin\phi - \sin\vartheta)}\ .$$

B-1

B-1.

A mass M_1 slides on a 45^o inclined plane of height H as shown. It is connected by a flexible cord of negligible mass over a small pulley (neglect its mass) to an equal mass M_2 hanging vertically as shown. The length of the cord is such that the masses can be held at rest both at height H/2. The dimensions of the masses and the pulley are negligible compared to H. At time t = 0 the two masses are released.



a) For t > 0 calculate the vertical acceleration of M_2.

b) Which mass will move downward? At what time t_1 will it strike the ground?

c) If the mass in (b) stops when it hits the ground, but the other mass keeps moving, show whether or not it will strike the pulley.

B-1.

a) $a = -\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) g$

b) M_2

$t_1 = \sqrt{\dfrac{2H}{g\left(1 - \frac{1}{\sqrt{2}}\right)}}$

c) No

At $t=0$ $\left(K.E.\right)_1 = \left(K.E.\right)_2 = 0, \left(P.E.\right)_1 = \left(P.E.\right)_2 = Mg\, {}^{H}\!/_{2}$, being $M_1 = M_2 = M$

At $t>0$ $\left(K.E.\right)_1 = \left(K.E.\right)_2 = \tfrac{1}{2}Mv^2, \left(P.E.\right)_1 = Mgy, \left(P.E.\right)_2 = Mgx$

Length of the cord: $\frac{H}{2}\left(1+\sqrt{2}\right)$

Length of the cord on the side of M_2: $H-x$

Length of the cord on the side of M_1: $\frac{H}{2}\left(\sqrt{2}-1\right)+x$

Height of M_2: x

Height of M_1: $y=\frac{H}{2}\left(1-\frac{1}{\sqrt{2}}\right)-\frac{x}{\sqrt{2}}$

Then at $t>0$ $\left(P.E.\right)_1 = Mgy = Mg\left[\frac{H}{2}\left(1-\frac{1}{\sqrt{2}}\right)-\frac{x}{\sqrt{2}}\right]$ and from the conservation of energy

$$\left(K.E.\right)_{1,t=0}+\left(K.E.\right)_{2,t=0}+\left(P.E.\right)_{1,t=0}+\left(P.E.\right)_{2,t=0}=\left(K.E.\right)_{1,t>0}+\left(K.E.\right)_{2,t>0}+\left(P.E.\right)_{1,t>0}+\left(P.E.\right)_{2,t>0}\Rightarrow$$

$$0+0+Mg\frac{H}{2}+Mg\frac{H}{2}=\frac{1}{2}Mv^2+\frac{1}{2}Mv^2+Mg\left[\frac{H}{2}\left(1-{}^{1}\!/\!{}_{\sqrt{2}}\right)-\frac{x}{\sqrt{2}}\right]+Mgx\Rightarrow$$

$$MgH=Mv^2+Mgx\left(1-{}^{1}\!/\!{}_{\sqrt{2}}\right)+Mg\frac{H}{2}\left(1-{}^{1}\!/\!{}_{\sqrt{2}}\right).$$

Taking the derivative with respect to t of each member of the above expression

$0=2Mva+Mgv\left(1-{}^{1}\!/\!{}_{\sqrt{2}}\right)+0$, where we used $v=\frac{dx}{dt}$ and $\frac{dv^2}{dt}=2v\frac{dv}{dt}=2va$.

a) So the vertical acceleration of M_2 at $t>0$ is $a=-\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)g$.

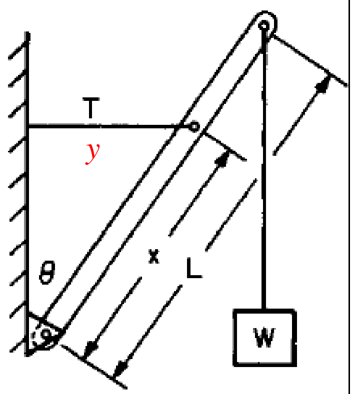
b) Being $a<0$ the mass M_2 will move downward. The eight of M_2 is $x=\frac{H}{2}$ and its velocity $v(0)=0$ at $t=0$ whereas $x=\frac{1}{2}at^2+\frac{H}{2}$ at $t>0$; then it will strike the ground ($x=0$) at time $t_1=\sqrt{-H/a}=\sqrt{\frac{2H}{g\left(1-\frac{1}{\sqrt{2}}\right)}}$.

c) Being $y=\frac{H}{2}\left(1-\frac{1}{\sqrt{2}}\right)-\frac{x}{\sqrt{2}}$ the height of M_1, then for $x=0$ is $y=\frac{H}{2}\left(1-\frac{1}{\sqrt{2}}\right)<H$; so the other mass ($M_1$) will no strike the pulley.

B-2

B-2.

A derrick is made of a uniform boom of length L and weight w, pivoted at its lower end. It is supported at an angle θ with the vertical by a horizontal cable attached at a point a distance x from the pivot, and a weight W is slung from its upper end. Find the tension in the horizontal cable.



B-2.

$$T = \frac{L}{x}\left(W + \frac{w}{2}\right) \tan \theta$$

When the boom rotates an angle $\Delta\theta$ (assumed small, i.e. neglecting terms of the order if $(\Delta\theta)^2$ or higher) around the pivot the length y of the horizontal cable will change of $\Delta y \simeq x\cos\theta\Delta\theta$. This can be seen using Carnot's theorem and the approx. $\sqrt{1+x} \cong 1+\frac{1}{2}x$ to find the length of the side opposite to the angle θ:

$$\Delta y = y' - y = \sqrt{x^2 + x^2\cos^2\theta - 2x^2\cos\theta\cos(\theta+\Delta\theta)} - x\sin\theta \cong$$
$$x\sin\theta\left(\sqrt{1+2\,\mathrm{ctg}\,\theta\Delta\theta} - 1\right) \simeq x\sin\theta\cdot\mathrm{ctg}\,\theta\Delta\theta = x\cos\theta\Delta\theta.$$

Or, in a more straightforward way,

$$y' \cong x\sin(\theta+\Delta\theta) = x\sin\theta\cos\Delta\theta + x\cos\theta\sin\Delta\theta \cong x\sin\theta + x\cos\theta\Delta\theta \Rightarrow$$
$$\Delta y = y' - y \cong x\cos\theta.$$

The height of the weight W changes of $\Delta Z = L\cos(\theta+\Delta\theta) - L\cos\theta \cong -L\sin\theta\Delta\theta$ and the weight w of the boom (assumed applied to the centre of mass of the boom) moves of

$\Delta z = \frac{L}{2}\cos(\theta+\Delta\theta) - \frac{L}{2}\cos\theta \cong -\frac{L}{2}\sin\theta\Delta\theta$.

Then from PVW:

$$T\Delta y + W\Delta Z + w\Delta z = 0 \Rightarrow T\cdot x\cos\theta\Delta\theta - W\cdot L\sin\theta\Delta\theta - w\cdot\tfrac{L}{2}\sin\theta\Delta\theta \cong 0 \Rightarrow$$
$$T = \frac{L}{x}\left(W + \frac{w}{2}\right)\tan\theta.$$

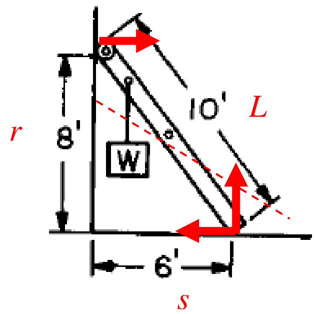
B-3

B-3.

A uniform ladder 10 ft. long with rollers at the top end leans against a smooth vertical wall. The ladder weighs 30 lb. A weight W = 60 lb is hung from a rung 2.5 feet from the top end. Find

a) The force with which the rollers push on the wall.

b) The horizontal and vertical forces with which the ladder pushes on the ground.



B-3.

a) 45 lb-wts

b) 45 lb-wts, 90 lb-wts

Let $L = 10$ ft the length of the ladder, $r =$ 8 ft the initial height of the rollers at the top end and suppose that, slightly tilting the ladder leaning against the vertical wall, the rollers would go down Δr.

Then (assuming Δr small)

$$L^2 = (r+\Delta r)^2 + (s+\Delta s)^2 \Rightarrow L^2 \cong r^2 + 2r\Delta r + s^2 + 2s\Delta s = (r^2+s^2) + 2(r\Delta r + s\Delta s) = L^2 + 2(r\Delta r + s\Delta s) \Rightarrow$$
$$\Rightarrow r\Delta r + s\Delta s = 0.$$

Thus $\Delta s = -\frac{r}{s}\Delta r$.

Furthermore, said $h_{W,w}$ the heights of the weights W and w, hangings from rungs at $L_{W,w}$ feet from the bottom end, then $h_{W,w} = \frac{L_{W,w}}{L} r$ and $\Delta h_{W,w} = \frac{L_{W,w}}{L}\Delta r$.

W is hung $l = 2.5$ feet from the top end and therefore $L_W = L - l$ whereas $L_w = \frac{L}{2}$.

Let R, H and V, respectively, the force of the roller on the wall, the horizontal and vertical forces of the ladder on the ground, then, from PVW,

$$H\Delta s + W\Delta h_W + w\Delta h_w = 0 \Rightarrow H\left(-\frac{r}{s}\Delta r\right) + W\frac{L_W}{L}\Delta r + w\frac{L_w}{L}\Delta r = 0 \Rightarrow$$

$$-H\frac{r}{s} + W\frac{L_W}{L} + w\frac{L_w}{L} = 0 \Rightarrow H = \frac{s}{Lr}(WL_W + wL_w) = \frac{s}{Lr}\left[W(L-l) + w\frac{L}{2}\right].$$

Now, from PVW in the form $\sum_{i=1}^{n} F_i \cos\Delta_i = 0$ one gets $R = H$ and $V = W + w$, whence

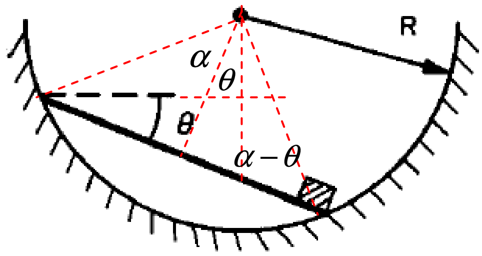
a) $R = 45$ lb-wts

b) $H = 45$ lb-wts, $V = 90$ lb-wts

B-4

B-4.

A plank of weight W and length $\sqrt{3}$ R lies in a smooth circular trough of radius R. At one end of the plank is a weight W/2. Calculate the angle θ at which the plank lies when it is in equilibrium.



B-4.

$\theta = 30^\circ$

Let $L = \sqrt{3}R$ the length of the plank. Then we have $\sin\alpha = \dfrac{L}{2R} = \dfrac{\sqrt{3}}{2} \Rightarrow \alpha = 60°$.

For the heights of the weights W and w one gets that $h_W = R - \sqrt{R^2 - \left(\frac{L}{2}\right)^2}\cos\theta$,

$h_w = R - R\cos(\alpha-\theta)$ and (for a small tilt $\Delta\theta$)

$$\Delta h_W \cong \sqrt{R^2 - \left(\tfrac{L}{2}\right)^2}\sin\theta\Delta\theta = R\sqrt{1-\left(\tfrac{L}{2R}\right)^2}\sin\theta\Delta\theta = R\cos\alpha\sin\theta\Delta\theta = \frac{R}{2}\sin\theta\Delta\theta,$$

being $\cos\alpha = \frac{1}{2}$, $\Delta h_w \cong R\sin(\alpha-\theta)\Delta\theta$.

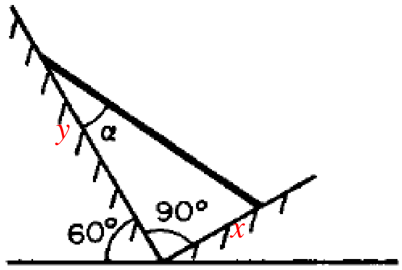
From PVW, remembering that Δh_W and Δh_w lie in opposite directions, we have $W\Delta h_W = w\Delta h_w$, whence

$$W\frac{R}{2}\sin\theta\Delta\theta = wR\sin(\alpha-\theta)\Delta\theta = \frac{W}{2}R\sin(\alpha-\theta)\Delta\theta \Rightarrow \sin\theta = \sin(\alpha-\theta) \Rightarrow \theta = \alpha-\theta \Rightarrow \theta = \frac{\alpha}{2} = 30°.$$

B-5

B-5.

A uniform bar of length ℓ and weight W is supported at its ends by two inclined planes as shown. (Neglect friction.) From the PVW find the angle α at which the bar is in equilibrium.



B-5.

$\alpha = 30^\circ$

From PVW we have $W\Delta x\sin 30^\circ + W\Delta y\sin 60^\circ = 0$.

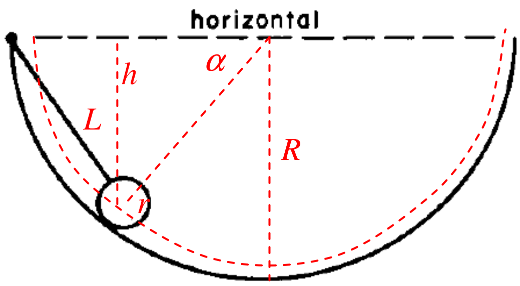
From the rectangular triangle shown (see problem B-3) $x\Delta x + y\Delta y = 0 \Rightarrow \Delta y = -\frac{x}{y}\Delta x = -\tan\alpha\Delta x$.

Then

$W\Delta x\sin 30^\circ + W\Delta y\sin 60^\circ = 0 \Rightarrow \Delta x\sin 30^\circ + \Delta y\sin 60^\circ = 0 \Rightarrow \Delta x\sin 30^\circ - \tan\alpha\Delta x\sin 60^\circ = 0 \Rightarrow$
$\sin 30^\circ - \tan\alpha\sin 60^\circ = 0 \Rightarrow \sin 30^\circ - \tan\alpha\cos 30^\circ = 0 \Rightarrow \tan\alpha = \frac{\sin 30^\circ}{\cos 30^\circ} = \tan 30^\circ$,

and therefore $\alpha = 30^\circ$.

B-6



B-6.

A solid small sphere of radius 4.5 cm and weight W, is to be suspended by a string from the ends of a smooth hemispherical bowl of radius 49 cm. It is found that if the string is any shorter than 40 cm, it breaks. By PVW what is the breaking strength of the string?

B-6.

F = 0.6 W

Being L the length of the string and r the radius of the sphere, then, from Carnot's theorem, $(L+r)+\Delta L = \sqrt{R^2 + (R-r)^2 - 2R(R-r)\cos(\alpha+\Delta\alpha)}$.

If $\Delta\alpha$ is small $(L+r)+\Delta L \cong \sqrt{R^2+(R-r)^2-2R(R-r)\cos\alpha+2R(R-r)\sin\alpha\Delta\alpha}$.

But $R^2+(R-r)^2-2R(R-r)\cos\alpha=(L+r)^2$, thus

$$(L+r)+\Delta L \cong \sqrt{(L+r)^2+2R(R-r)\sin\alpha\Delta\alpha} \cong (L+r)\sqrt{1+\frac{2R(R-r)}{(L+r)^2}\sin\alpha\Delta\alpha} \simeq$$

$$(L+r)+\frac{R(R-r)}{(L+r)}\sin\alpha\Delta\alpha \Rightarrow \Delta L \simeq \frac{R(R-r)}{(L+r)}\sin\alpha\Delta\alpha.$$

Furthermore, being h the height of the centre of the sphere, $\Delta h=(R-r)\sin(\alpha+\Delta\alpha)-(R-r)\sin\alpha \cong (R-r)\cos\alpha\Delta\alpha$.

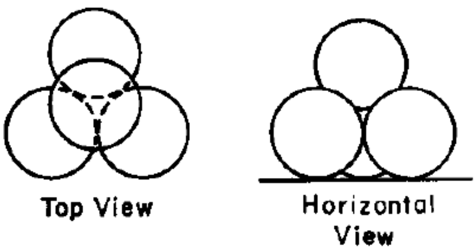
Let F the strength of the string, from PVW: $F\Delta L=W\Delta h \Rightarrow F=W\dfrac{\Delta h}{\Delta L}\cong W\dfrac{L+r}{R}\cot\alpha$. One also gets

$R-r=49-4.5$ cm=44.5cm

$L+r=40+4.5$ cm=44.5cm

So the triangle is isosceles so that $\cot\alpha={}^1\!/_2 \Rightarrow F\cong W\dfrac{L+r}{2h}$. Let h the height of the isosceles, then $h=\sqrt{(L+r)^2-\left({}^R\!/_2\right)^2}=\sqrt{44.5^2-24.5^2}=\sqrt{1380}$ and $F\cong W\dfrac{L+r}{2h}=W\dfrac{44.5}{2\times\sqrt{1380}}=\dfrac{22.25}{\sqrt{1380}}W\simeq 0.6W$.

B-7

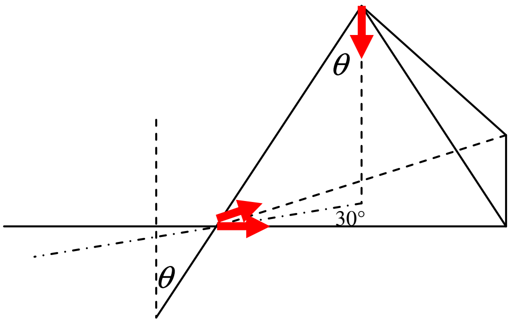


B-7.

An ornament for a courtyard at a World's Fair is to be made up of four identical, frictionless metal spheres, each weighing $2\sqrt{6}$ tonwts. The spheres are to be arranged as shown, with three resting on a horizontal surface and touching each other; the fourth is to rest freely on the other three. The bottom three are kept from separating by spot welds at the points of contact with each other. Allowing for a factor of safety of 3, how much tension must the spot welds withstand?

B-7.

2 ton-wts



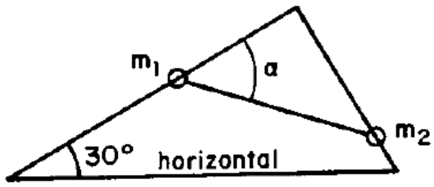
The centres of the spheres lie on the vertices of a regular tetrahedron of edge a and height $\frac{\sqrt{6}a}{3}$, then $\cos\theta = \frac{\sqrt{6}}{3}$.

Let w_a the component of the weight w along each edge a. From symmetry considerations $3\times w_a \cos\theta = w$, whence $w_a = \frac{w}{3\cos\theta} = \frac{w}{\sqrt{6}} = 2$ ton-wts.

Along the horizontal surface of the tetrahedron w_a has a component $w_a \sin\theta = w_a \cdot \frac{\sqrt{3}}{3}$ and the two welds at the contact points of each sphere with the two other exert along the same direction a force $2T\cos 30° = T\sqrt{3}$, being T the tension of each weld.

Then from PVW (in the form $\sum_{i=1}^{n} F_i \cos\Delta_i = 0$) and fixing the horizontal direction, one gets $T\sqrt{3} = w_a \cdot \frac{\sqrt{3}}{3} \Rightarrow T = \frac{w_a}{3}$, and thus, for a factor of safety of 3, $T_{(3)} = 3\times T = w_a = 2$ ton-wts.

B-8



B-8.

A rigid wire frame is formed in a right triangle, and set in a vertical plane as shown. Two beads of masses m_1 = 100 gram, m_2 = 300 gram slide without friction on the wires, and are connected by a cord. When the system is in static equilibrium, what is the tension in the cord, and what angle α does it make with the first wire?

B-8.

$T = 265$ g-wts

$\alpha = 79^{\circ}.1$

Let l the length of the cord, $h_1,\quad h_2$ the heights of the two masses when they lie at distances $c_1,\quad c_2$ from the top in downwards direction along the two legs of the right triangle.

Then from PVW $T\Delta l = m_1\Delta h_1 + m_2\Delta h_2$ where:

$\Delta h_1 = \Delta c_1 \sin 30° = \frac{1}{2}\Delta c_1$

$\Delta h_2 = \Delta c_2 \sin 60° = \frac{\sqrt{3}}{2}\Delta c_2$

Being

$c_1 = l\cos\alpha$

$c_2 = l\sin\alpha$

one gets $\Delta c_1 = -l\sin\alpha\Delta\alpha + \cos\alpha\Delta l \Rightarrow \Delta h_1 = -\frac{l}{2}\sin\alpha\Delta\alpha + \frac{1}{2}\cos\alpha\Delta l$,

$\Delta c_2 = l\cos\alpha\Delta\alpha + \sin\alpha\Delta l \Rightarrow \Delta h_2 = \frac{\sqrt{3}}{2} l\cos\alpha\Delta\alpha + \frac{\sqrt{3}}{2}\sin\alpha\Delta 1$

Thus

$T\Delta l = m_1\Delta h_1 + m_2\Delta h_2 = \left(\frac{1}{2}m_1\cos\alpha + \frac{\sqrt{3}}{2}m_2\sin\alpha\right)\Delta l - \left(\frac{1}{2}m_1\sin\alpha - \frac{\sqrt{3}}{2}m_2\cos\alpha\right)l\Delta\alpha$.

$\Delta l,\quad \Delta\alpha$ being independent of one another, tacking $\Delta l = 0$ one finds that the system is in static equilibrium when $m_1\sin\alpha - \sqrt{3}m_2\cos\alpha = 0 \Rightarrow \tan\alpha = 3\sqrt{3}$.

Reading the tables one gets $\alpha = 79°.1$ and, taking $\Delta\alpha = 0$, $T = \frac{1}{2}m_1\cos 79°.1 + \frac{\sqrt{3}}{2}m_2\sin 79°.1 = 265$ g-wts.

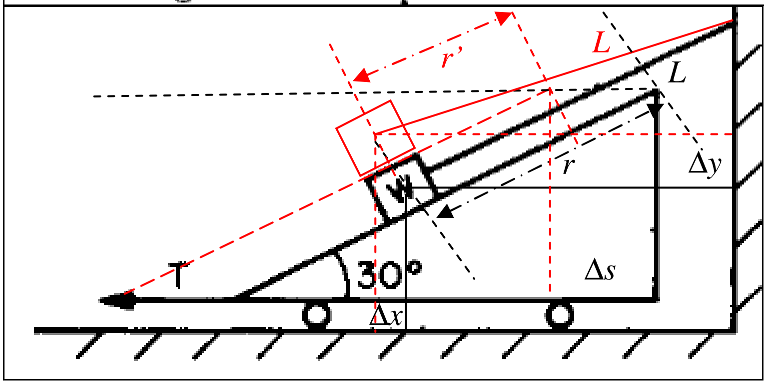
B-9

B-9.

Find the tension T needed to hold the cart in equilibrium, if there is no friction.

a) Using the principle of virtual work.

b) Using force components.



B-9.

$T = \frac{\sqrt{3}}{4} W$

Being $\Delta L = 0$ we have $x\Delta x + y\Delta y = 0 \Rightarrow \Delta y = -\frac{x}{y}\Delta x = -\cot 30°\Delta x$.

But $\Delta x = r'\cos 30° + \Delta s - r\cos 30° = \Delta s - (r - r')\cos 30°$.

Assuming Δs small we have $r - r' \cong \Delta s \cos 30°$.

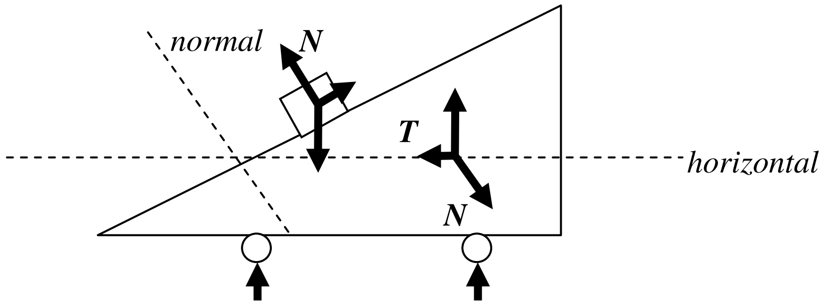
Thus $\Delta x \cong \Delta s\left(1-\cos^2 30°\right) = \Delta s \sin^2 30°$.

a) from PVW

$$T\Delta s + W\Delta y = 0 \Rightarrow T = -W\frac{\Delta y}{\Delta s} = -W\left(\frac{-\cot 30°\Delta x}{\Delta s}\right) = W\left(\frac{\cot 30°\cdot\sin^2 30°\Delta s}{\Delta s}\right) \Rightarrow$$

$$T = W\cos 30°\sin 30° = \frac{\sqrt{3}}{4}W.$$

b)



Using the method of the force components (FC) on cart we have:

$T\cos 180° + N\cos 300° = 0 \Rightarrow T = \dfrac{\sqrt{3}}{2}N$ (horizontal).

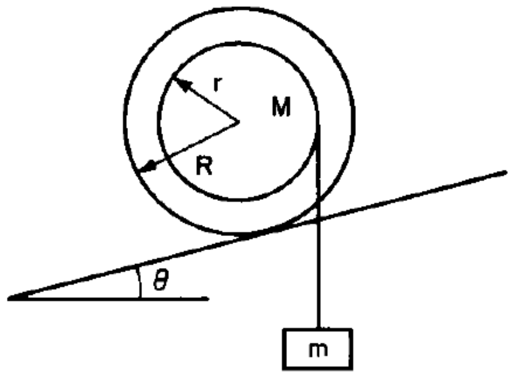
Using FC on W: $N\cos 0° + W\cos 240° = 0 \Rightarrow N = \dfrac{1}{2}W$.

So $T = \dfrac{\sqrt{3}}{4}W$.

B-10

B-10.

A bobbin of mass M = 3 kg consists of a central cylinder of radius r = 5 cm and two end plates of radius R = 6 cm. It is placed on a slotted incline on which it will roll but not slip, and a mass m = 4.5 kg is suspended from a cord wound around the bobbin. It is observed that the system is in static equilibrium. What is the angle of tilt θ of the incline?



B-10.

$\theta = 30°$

Assume the centre of the bobbin goes down ΔH and m rises Δh as the bobbin rolls downwards. Then from PVW: $M\Delta H = m\Delta h$.

For a rolling of the bobbin of $\Delta L = R\Delta\alpha$ then M goes down $\Delta H = \Delta L \sin\theta$, whereas m rises (assuming $\Delta\alpha$ small) $\Delta l \cong r\Delta\alpha = \frac{r}{R}\Delta L$ and goes down ΔH.

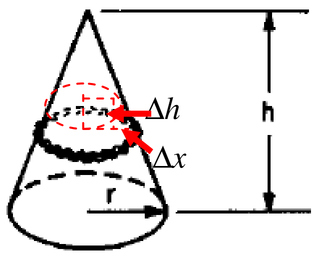
Thus

$$\Delta h = \Delta l - \Delta H = \Delta L\left(\frac{r}{R} - \sin\theta\right) \Rightarrow M\,\Delta L \sin\theta = m\Delta L\left(\frac{r}{R} - \sin\theta\right) \Rightarrow$$
$$M \sin\theta = m\frac{r}{R} - m\sin\theta \Rightarrow (M+m)\sin\theta = m\frac{r}{R} \Rightarrow \sin\theta = \frac{m}{M+m}\frac{r}{R} = 0{,}5 \Rightarrow$$
$$\theta = 30°.$$

B-11

B-11.

A loop of flexible chain, of total weight W, rests on a smooth right circular cone of base radius r and height h. The chain rests in a horizontal circle on the cone, whose axis is vertical. Find the tension in the chain. Neglect friction.



B-11.

$$T = \frac{Wh}{2\pi r}$$

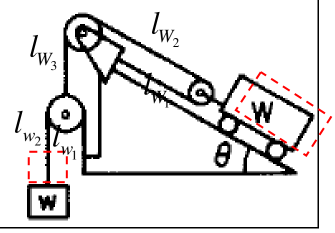
If the chain rises Δh resting in a horizontal circle, then its radius varies of $\Delta x = \frac{r}{h}\Delta h$ and its length of $\Delta l = 2\pi\Delta x$.

From PVW: $W\Delta h = T\Delta l \Rightarrow W\Delta h = T\cdot 2\pi\Delta x = T\cdot 2\pi\frac{r}{h}\Delta h \Rightarrow T = \frac{Wh}{2\pi r}$.

B-12

B-12.

A cart on an inclined plane is balanced by the weight w. All parts have negligible frcition. Find the weight W of the cart.



B-12.

$$W = \frac{4w}{\sin\theta}$$

Assuming w rises Δh_w whereas W goes down Δh_W, being:

$$\Delta\left(l_{W_1}+l_{W_2}+l_{W_3}\right)=0\,, \Delta l_{W_1}=\Delta l_{W_2} \Rightarrow \Delta l_{W_1}=-\tfrac{1}{2}\Delta l_{W_3}\,,\ \Delta\left(l_{w_1}+l_{w_2}\right)=0\,,\ \Delta l_{W_3}=-\Delta l_{w_1}=\Delta l_{w_2}\,,$$

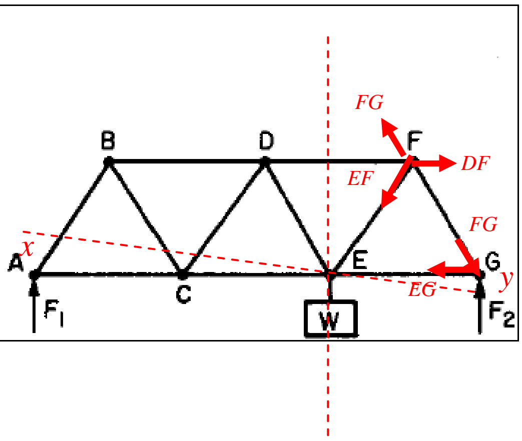
then $\Delta h_w = \Delta l_{w_2} + \Delta l_{W_3} = 2\Delta l_{w_2}$, $\Delta h_W = \Delta l_{W_1}\sin\theta = -\tfrac{1}{2}\Delta l_{W_3}\sin\theta = -\tfrac{1}{2}\Delta l_{w_2}\sin\theta = -\tfrac{1}{4}\Delta h_w \sin\theta$.

From PVW: $w\Delta h_w + W\Delta h_W = 0$. So $w\Delta h_w - \tfrac{1}{4}\sin\theta\cdot W\Delta h_w = 0 \Rightarrow W = \dfrac{4w}{\sin\theta}$.

B-13

B-13.

A bridge truss is constructed as shown. All joints may be considered frictionless pivots and all members rigid, weightless, and of equal length. Find the reaction forces F_1 and F_2 and the force in the member DF.



B-13

$F_1 = \frac{W}{3}$

$F_2 = \frac{2W}{3}$

$F = \frac{4W}{3\sqrt{3}}$

From PVW: $F_1 x = F_2 y$, but $x:2 = y:1 \Rightarrow F_2 = 2F_1$.

Using $\sum_i F_i \cos\Delta_i = 0$ then, with respect to the vertical direction, one gets

$$F_1 - W + F_2 = 0 \Rightarrow F_1 + F_2 = W \Rightarrow 3F_1 = W \Rightarrow F_1 = \frac{W}{3},\ F_2 = \frac{2W}{3}\ .$$

Using FC at the joint G:

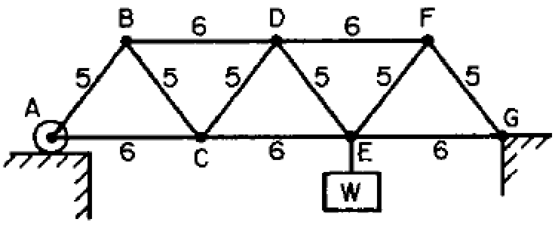
. vertical components: $F_2 \cos 0° + T_{FG} \cos 150° = 0 \Rightarrow F_2 - T_{FG}\frac{\sqrt{3}}{2} = 0 \Rightarrow T_{FG} = \frac{2F_2}{\sqrt{3}}$.

Using FC at the joint F:

- horizontal components: $T_{FG} \cos 120° + T_{EF} \cos 240° + F \cos 0° = 0 \Rightarrow F = \frac{1}{2}\left(T_{FG} + T_{EF}\right)$
- vertical components: $T_{FG} \cos 30° + T_{EF} \cos 150° = 0 \Rightarrow \frac{\sqrt{3}}{2}\left(T_{FG} - T_{EF}\right)00 \Rightarrow T_{EF} = T_{FG}$

whence $F = T_{FG} = \frac{2F_2}{\sqrt{3}} = \frac{4W}{3\sqrt{3}}$.

B-14



B-14.

In the truss shown, all diagonal struts are of length 5 units and all horizontal ones are of length 6 units. All joints are freely hinged, and the weight of the truss is negligible.

a) Which of the members could be replaced with flexible cables, for the load position shown?

b) Find the forces in struts BD, and DE.

B-14.

a) AC, CE, EG, BC, EF, ED

b) $BD = \frac{W}{2}$

$DE = \frac{5}{12} W$

a) members AB, BD, DF, FG should be rigid in order that A remains in fixed position; CD also rigid to fix C and D. The elements that could be flexible are therefore AC, CE, EG, BC, EF, ED.

b) meanwhile note that $\cos\hat{A} = \frac{3}{5}$, $\sin\hat{A} = \frac{4}{5}$. From the result of the preceding problem is then

$$F_1 = \frac{W}{3} \ , \ F_2 = \frac{2W}{3} \ .$$

Using FC at the joint A:

- horizontal: $\text{AB}\cos\hat{A} = \text{AC}$
- vertical: $\text{AB}\sin\hat{A} = F_1 = \frac{W}{3} \Rightarrow \text{AB} = \frac{5W}{12}$

Using FC at the joint B:

- horizontal: $(\text{AB}+\text{BC})\cos\hat{A} = \text{BD}$

- vertical: $AB=BC \Rightarrow BD=2AB\cos\hat{A}=\dfrac{W}{2}$

Using FC at the joint C:

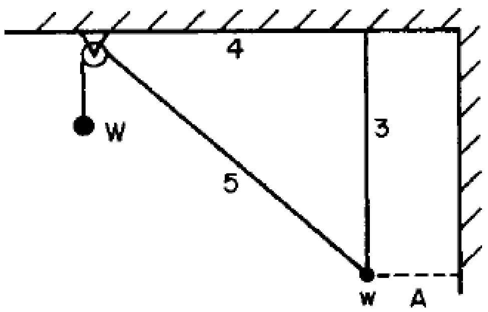
- vertical: $BC = CD$

Using FC at the joint D:

- vertical: $DE = CD \Rightarrow DE=CD=BC=AB=\dfrac{5W}{12}$

So $BD=\dfrac{W}{2}$, $DE=\dfrac{5W}{12}$.

B-15



B-15.

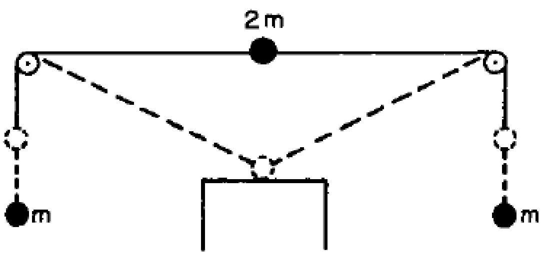
In the system shown, a pendulum bob of weight w is initially held in the vertical position by a thread A. When this thread is burned, releasing the pendulum, it swings to the left and barely reaches the ceiling at its maximum swing. Find the weight W. (Neglect friction, the radius of the pulley, and the finite sizes of the weights.)

B-15.

$W = \frac{3}{4}$ w

At its maximum swing the pendulum has zero velocity. Then from the conservation of energy $W\Delta h_W + w\Delta h_w = 0 \Rightarrow W = -w\dfrac{\Delta h_w}{\Delta h_W}$. At the maximum swing (pendulum at the ceiling) $\Delta h_w = 3$, $\Delta h_W = -4$. Thus $W = \frac{3}{4}w$.

B-16



B-16.

Two equal masses m are attached to a third mass 2m by equal lengths of fine thread and the thread is passed over two small pulleys with negligible friction situated 100 cm apart. The mass 2m is initially held level with the pulleys midway between them, and is then released from rest. When it has descended a distance of 50 cm it strikes a table top. How fast is it then moving?

B-16.

$V = 196 \text{ cm s}^{-1}$

As the third mass reaches the table top, the two equal masses rise $50\left(\sqrt{2}-1\right)\text{cm}$; when the third mass has descended a distance Δh the two equal masses have risen by $50\left(\sqrt{1+\left(\frac{\Delta h}{50}\right)^2}-1\right)$.

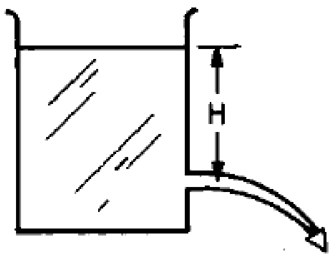
Then, said V and v the final velocities (in module) of the third mass and of the two equal masses respectively, $v = \frac{V}{\sqrt{2}}$.

The kinetic energy varies of $\frac{1}{2}(2m)V^2 + 2\cdot\frac{1}{2}mv^2 = m\left(V^2+v^2\right) = \frac{3}{2}mV^2$ and the potential energy of $2(mg)\cdot 50\left(\sqrt{2}-1\right) - (2mg)\cdot 50 = -2mg\cdot 50\left(2-\sqrt{2}\right)$.

From the conservation of energy: $\frac{3}{2}mV^2 = 2mg\cdot 50\left(2-\sqrt{2}\right) \Rightarrow V = \sqrt{\frac{200g\left(2-\sqrt{2}\right)}{3}}$.

With $g = 981 \text{ cm s}^{-2}$ thus one obtains $V = 196 \text{ cm s}^{-1}$.

B-17



B-17.

A tank of cross-sectional area A contains a liquid having density ρ. The liquid squirts freely from a small hole of area A a distance H below the free surface of the liquid. If the liquid has no internal friction (viscosity), with what speed does it emerge?

B-17.

$v = \sqrt{2gH}$

Let's consider a small layer of liquid of thickness Δz a distance h up from the hole.

Its potential energy of position is $\rho g \mathrm{A} \Delta z \cdot h$.

The internal tension force due to the weight of the overlying liquid, $\rho g \mathrm{A}(H-h)$, corresponds to a pressure $p = \rho g (H-h)$ and an internal energy $p \mathrm{A} \Delta z = \rho g \mathrm{A}(H-h)\Delta z$.

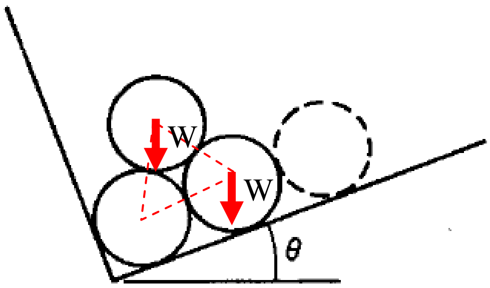
The total potential energy is the sum of the position energy and of the internal energy: $\rho g \mathrm{A} \Delta z \cdot h + \rho g \mathrm{A}(H-h)\Delta z = \rho g \mathrm{A} H \Delta z$ (independent of h, and hence also valid for $h = 0$ corresponding to the hole position).

Soon as the hole is open the above energy is converted to kinetic energy, so $\rho g \mathrm{A} H \Delta z = \frac{1}{2}(\rho \mathrm{A} \Delta \mathrm{z}) v^2 \Rightarrow v = \sqrt{2gH}$.

C-1

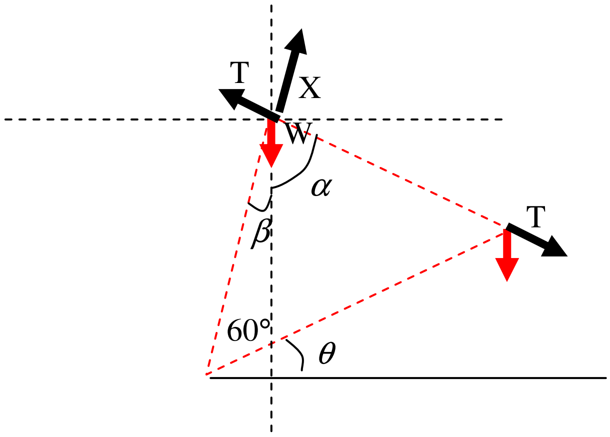
C-1.

Smooth, identical logs are piled in a stake truck. The truck is forced off the highway and comes to rest on an even keel lengthwise but with the bed at an angle θ with the horizontal. As the truck is unloaded, the removal of the log shown dotted leaves the remaining three in a condition where they are just ready to slide, that is, if θ were any smaller, the logs would fall down. Find θ.



C-1.

$$\theta = \tan^{-1}\frac{1}{3\sqrt{3}} = 10.^\circ 9$$



$$\beta + 60° + \theta = 90° \Rightarrow \beta = 30° - \theta$$
$$\alpha = 60° - \beta = 30° + \theta$$

From FC:

- vert. $T\cos\alpha + X\cos\beta = W$

- horiz. $T\sin\alpha = X\sin\beta$

- obliq. $W\sin\theta = T\cos 60° = \dfrac{T}{2}$

(Note that in the shown arrangement $T_{(\text{limit})} = 2W\sin\theta$).

In terms of $\cot\beta$:

$$X\cos\beta = X\sin\beta\cot\beta = T\sin\alpha\cot\beta \Rightarrow T\cos\alpha + T\sin\alpha\cot\beta = W\ .$$

But

$$\begin{aligned}
\cos\alpha &= \cos\left(30°+\theta\right) = \cos 30°\cos\theta - \sin 30°\sin\theta = \tfrac{\sqrt{3}}{2}\cos\theta - \tfrac{1}{2}\sin\theta \\
\sin\alpha &= \sin\left(30°+\theta\right) = \sin 30°\cos\theta + \cos 30°\sin\theta = \tfrac{1}{2}\cos\theta + \tfrac{\sqrt{3}}{2}\sin\theta \\
\cot\beta &= \left(\cot 30°\cot\theta + 1\right)\big/\left(\cot\theta - \cot 30°\right) = \left(\sqrt{3}\cot\theta+1\right)\big/\left(\cot\theta-\sqrt{3}\right)
\end{aligned}$$

Thus we have

$$T\left(\cos\alpha+\sin\alpha\cot\beta\right)=W \Rightarrow T\left[\tfrac{\sqrt{3}}{2}\cos\theta-\tfrac{1}{2}\sin\theta+\left(\tfrac{1}{2}\cos\theta+\tfrac{\sqrt{3}}{2}\sin\theta\right)\cdot\frac{\sqrt{3}\cot\theta+1}{\cot\theta-\sqrt{3}}\right]=W=\frac{T}{2\sin\theta}$$

$$\Rightarrow \sqrt{3}\cos\theta\sin\theta-\sin^2\theta+\left(\cos\theta\sin\theta+\sqrt{3}\sin^2\theta\right)\cdot\frac{\sqrt{3}\cot\theta+1}{\cot\theta-\sqrt{3}}=1.$$

In terms of $\cot\theta$:

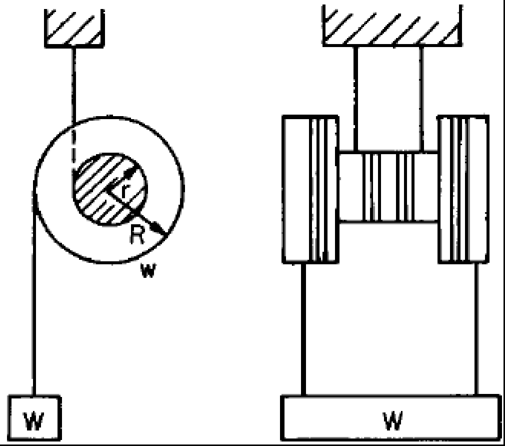
$$\begin{aligned}
\cos\theta\sin\theta &= \frac{\cot\theta}{1+\cot^2\theta} \\
\sin^2\theta &= \frac{1}{1+\cot^2\theta}
\end{aligned}$$

Thus $\dfrac{\sqrt{3}\cot\theta-1}{1+\cot^2\theta}+\dfrac{\cot\theta+\sqrt{3}}{1+\cot^2\theta}\cdot\dfrac{\sqrt{3}\cot\theta+1}{\cot\theta-\sqrt{3}}=1\Rightarrow\dfrac{2\sqrt{3}}{\cot\theta-\sqrt{3}}=1\Rightarrow\cot\theta=3\sqrt{3}\Rightarrow\tan\theta=\dfrac{1}{3\sqrt{3}}$ and finally $\theta=\tan^{-1}\dfrac{1}{3\sqrt{3}}=10°{,}9$.

C-2

C-2.

A spool of weight w has radii r and R. It is wound with cord, and suspended from a fixed support by two cords wound on the smaller radius. A weight W is then suspended from two cords wound on the larger radius, as shown. W is chosen so that the spool is just balanced. Find W.



C-2.

$$W = \frac{wr}{R - r}$$

As w moves of a (small) distance Δh_w the spool rotates an angle $\Delta\alpha \cong \dfrac{\Delta h_w}{r}$ and W moves in the same direction by the same distance Δh_w due to the vertical displacement of the spool, and in the opposite direction of a distance $\Delta h_W \cong R\Delta\alpha$ due to the rotation of the spool.

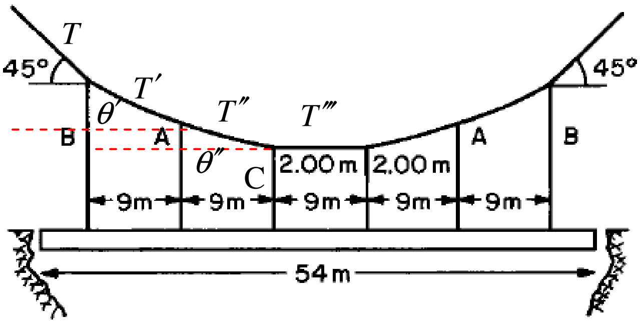
Then from PVW:

$w\Delta h_w = W\left(\Delta h_W - \Delta h_w\right) = W\left(R\Delta\alpha - r\Delta\alpha\right) = W\left(R-r\right)\Delta\alpha \Rightarrow wr\Delta\alpha = W\left(R-r\right)\Delta\alpha$, whence $W = \dfrac{wr}{R-r}$.

C-3

C-3.

A suspension bridge is to span a deep gorge 54 m wide. The roadway will consist of a steel truss supported by six pairs of vertical cables spaced 9.0 m apart, each cable to carry an equal share of the 48.0×10^3 kg weight. The two pairs of cables nearest the center are to be 2.00 m long. Find the proper lengths for the remaining vertical cables, and the maximum tension in the two longitudinal cables, if the latter are to be at a 45° angle with the horizontal at their ends.



C-3.

A = 5 m

B = 11 m

$T_{max} \approx 34 \times 10^3$ kg-wts

Define $t' = \tan\theta' = \dfrac{\text{B-A}}{9\text{ m}}$, $t'' = \tan\theta'' = \dfrac{\text{A-C}}{9\text{ m}}$ and $w = \dfrac{4.8\times10^3\text{ kg-wts}}{6}$.

From FC at the ends of B, A, C:

B)

Horiz. 1. $$\frac{T}{\sqrt{2}} = T'\frac{1}{\sqrt{1+t'^2}}$$

Vert. 2. $$\frac{T}{\sqrt{2}} = T'\frac{t'}{\sqrt{1+t'^2}} + w$$

A)

Horiz. 3. $T'\frac{1}{\sqrt{1+t'^2}}=T''\frac{1}{\sqrt{1+t''^2}}$

Vert. 4. $T'\frac{t'}{\sqrt{1+t'^2}}=T''\frac{t''}{\sqrt{1+t''^2}}+w$

C)

Horiz. 5. $T''\frac{1}{\sqrt{1+t''^2}}=T'''$

Vert. 6. $T''\frac{t''}{\sqrt{1+t''^2}}=w$

From 3. and 4.: $T''\frac{t'}{\sqrt{1+t''^2}}=T''\frac{t''}{\sqrt{1+t''^2}}+w\Rightarrow T''\frac{(t'-t'')}{\sqrt{1+t''^2}}=w$.

From 6. and the latter: $T''\frac{t''}{\sqrt{1+t''^2}}=T''\frac{(t'-t'')}{\sqrt{1+t''^2}}$.

Thus $t''=t'-t''\Rightarrow t''=\frac{t'}{2}$.

From 6. and 4.: $T'\frac{t'}{\sqrt{1+t'^2}}=2w$, whence from 2. $\frac{T}{\sqrt{2}}=3w$.

Finally, from 1., $3wt'=2w\Rightarrow t'=\frac{2}{3}\Rightarrow t''=\frac{1}{3}$. Thus $\frac{\text{B-A}}{9\text{ m}}=\frac{2}{3}$, $\frac{\text{A-C}}{9\text{ m}}=\frac{1}{3}$.

But C = 2.00 m, so A=5 m, B=11 m.

To determine T_{max} let's return back to 1. – 6.:

1'. $\frac{T}{\sqrt{2}}=\frac{3T'}{\sqrt{13}}$, whence $T>T'$

2'. $\frac{T}{\sqrt{2}}=\frac{2T'}{\sqrt{13}}+w$

3'. $\frac{3T'}{\sqrt{13}}=\frac{3T''}{\sqrt{10}}$, whence $T'>T''$

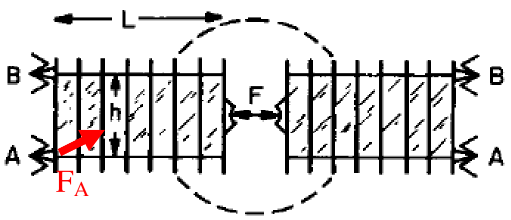
4'. $\dfrac{2T'}{\sqrt{13}} = \dfrac{T''}{\sqrt{10}} + w = 2w$ using 6'. below, from which $T' = \sqrt{13}w$

5'. $\dfrac{3T''}{\sqrt{10}} = T'''$, whence $T'' > T'''$

6'. $\dfrac{T''}{\sqrt{10}} = w$

Finally, therefore, $T_{\max} = T = \dfrac{3\sqrt{2}}{\sqrt{13}} T' = 3\sqrt{2}w = \dfrac{4.8\times10^3 \text{ kg-wts}}{\sqrt{2}} \approx 34\times10^3$ kg-wts .

C-4



C-4.

The insulating support structure of a Tandem Van de Graaff may be represented as two blocks of about uniform density, length L, height h and weight W, supported from vertical bulkheads by pivot joints (A and B) and forced apart by a screw jack (F) at the center. Since the material of the blocks cannot support tension, the jack must be adjusted to give zero force on the upper pivot.

a) What force F is required?

b) What is the total force on one of the lower pivots A?

C-4.

a) $F = W\dfrac{L}{h}$

b) $|\vec{F}_A| = W\sqrt{1 + \left(\dfrac{L}{h}\right)^2}$

$\theta = \tan^{-1}\left(\dfrac{h}{L}\right)$

a) Given the condition $\overrightarrow{F_B} = 0$, from PVW (in the form $\sum_i W_i \ell_i = 0$), with respect to A, one gets $Fh = WL \Rightarrow F = W\dfrac{L}{h}$.

b) From FC:

h. $F = \left|\overrightarrow{F_A}\right| \cos\theta$

v. $W = \left|\overrightarrow{F_A}\right| \sin\theta$

whence $\tan\theta = \dfrac{W}{F} = \dfrac{h}{L} \Rightarrow \theta = \tan^{-1}\left(\dfrac{h}{L}\right)$ and

$$\left|\overrightarrow{F_A}\right| = W\frac{L}{h}\cdot\frac{1}{\cos\theta} = W\frac{L}{h}\sqrt{1+\tan^2\theta} = W\frac{L}{h}\sqrt{1+\left(\frac{h}{L}\right)^2} = W\sqrt{1+\left(\frac{L}{h}\right)^2}\ .$$

Solutions to problems in Leighton and Vogt's textbook
"Exercises in Introductory Physics" (Addison-Wesley, 1969)

Pier F. Nali

(Revised April 10, 2016)

CHAPTER 3

Kepler's Laws and Gravitation

Refer to The Feynman Lectures on Physics, Vol. I, Ch. 7.

1. Some properties of the ellipse.

The size and shape of an ellipse are determined by specifying the values of any two of the following quantities:

a : the semi major axis

b : the semi minor axis

c : the distance from the center to one focus

e : the eccentricity

r_p: the perihelion (or perigee) distance (the closest distance from a focus to the ellipse)

r_a: the aphelion (or apogee) distance (the farthest distance from a focus to the ellipse.

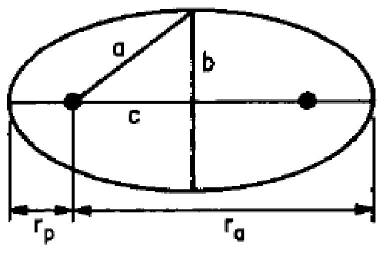
The relationships of these various quantities are as follows:

$a^2 = b^2 + c^2$

$e = c/a$ (definition of e)

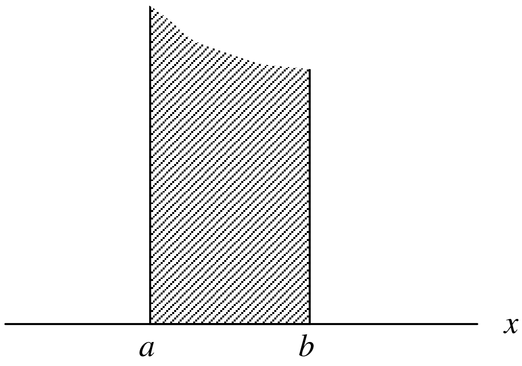
$r_p = a - c = a(1 - e)$

$r_a = a + c = a(1 + e)$



Show that the area of an ellipse is given by $A = \pi ab$.

In general, for a curve having parametric equations $x=\varphi(t),\quad y=\psi(t)$, the curvilinear trapezoid area in the figure below

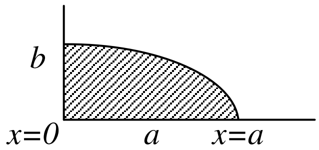


where $a=\varphi(t_1),\quad b=\varphi(t_2)$ and $\psi(t)\geq 0$ in the interval (t_1,t_2), is $\mathrm{A}=\int_{t_1}^{t_2}\psi(t)\cdot\varphi'(t)\,dt$.

We can use this formula to calculate the area of an ellipse having parametric equations $x=a\cos\theta,\quad y=b\sin\theta$.

From the symmetry of the ellipse we can calculate the area of a quarter ellipse between the limits $x=0\quad\left(\vartheta=\frac{\pi}{2}\right),\ x=a\quad(\vartheta=0)$.

Hence $\dfrac{\mathrm{A}}{4}=\int_{\frac{\pi}{2}}^{0}b\sin\theta\cdot(-a\sin\theta)\,d\theta=ab\int_{0}^{\frac{\pi}{2}}\sin^2\theta d\theta=\frac{\pi}{4}ab\Rightarrow \mathrm{A}=\pi\mathrm{ab}$.



A-1

A-1.

The distance of the moon from the center of the earth varies from 363,300 km at perigee to 405,500 km at apogee, and its period is $27^{\mathrm{d}}.322$. A certain artificial earth satellite is orbiting so that its perigee height from the surface of the Earth is 225 km, and its apogee height is 710 km. The mean diameter of the Earth is 12,756 km. What is the sidereal period of this satellite?

A-1.

$t \approx 1.6$ hr

For the moon $a_{☽} = \frac{1}{2}\left(r_{a☽} + r_{p☽}\right) = \dfrac{405{,}500 \text{ km} + 363{,}300 \text{ km}}{2} = 384{,}500 \text{ km}$

For the Earth sat.

$$a = \tfrac{1}{2}\left(r_a + r_p\right) = \tfrac{1}{2}\left(r_\oplus + 710 \text{ km} + r_\oplus + 225 \text{ km}\right) = \tfrac{1}{2}\left(d_\oplus + 935 \text{ km}\right) = \frac{12{,}756 \text{ km} + 935 \text{ km}}{2} = 6{,}845.5 \text{ km}$$

Thus $t = \left(\dfrac{a}{a_{☽}}\right)^{\frac{3}{2}} t_{☽} = \left(\dfrac{6{,}845.5 \text{ km}}{384{,}500 \text{ km}}\right)^{\frac{3}{2}} \times 27^d.322 \approx 0^d.065 \approx 1.6 \text{ hr}$

A-2

A-2.

The eccentricity of the Earth's orbit is 0.0167. Find the ratio of its maximum speed in its orbit to its minimum speed.

A-2.

1.033

From the Kepler's second law $r_a v_a = r_p v_p \Rightarrow \dfrac{v_p}{v_a} = \dfrac{r_a}{r_p} = \dfrac{a+c}{a-c} = \dfrac{1+e_\oplus}{1-e_\oplus} = \dfrac{1+0.0167}{1-0.0167} = \dfrac{1.0167}{0.9833} = 1.033$.

A-3

A-3.

The radii of the Earth and the moon are 6378 km and 1738 km, respectively, and their masses are in the ratio 81.3 to 1.000. Calculate the acceleration of gravity at the surface of the moon. $g_\oplus = 9.80 \text{ m s}^{-2}$.

A-3.

$1.6 \text{ m s}^{-2} \simeq \dfrac{g_\oplus}{6}$

$$g_{\oplus} = G\frac{M_{\oplus}}{R_{\oplus}^2},\quad g_{\leftmoon} = G\frac{M_{\leftmoon}}{R_{\leftmoon}^2} \Rightarrow \frac{g_{\leftmoon}}{g_{\oplus}} = \frac{M_{\leftmoon}}{M_{\oplus}}\left(\frac{R_{\oplus}}{R_{\leftmoon}}\right)^2 \Rightarrow g_{\leftmoon} = \frac{M_{\leftmoon}}{M_{\oplus}}\left(\frac{R_{\oplus}}{R_{\leftmoon}}\right)^2 g_{\oplus} = \frac{1.000}{81.3}\times\left(\frac{6,378\text{ km}}{1,738\text{ km}}\right)^2\times 9.80\text{ m s}^{-2} =$$

$$1.6\text{ m s}^{-2} \simeq \frac{g_{\oplus}}{6}$$

A-4

A-4.

In 1986, Halley's comet is expected to return on its seventh trip around the sun since the days in 1456 when people were so frightened that they offered prayers in the churches "to be saved from the Devil, the Turk, and the comet". In its most recent perihelion on April 19, 1910, it was observed to pass near the sun at a distance 0.60 A.U.

a) How far does it go from the sun at the outer extreme of its orbit?

b) What is the ratio of its maximum orbital speed to its minimum speed?

A-4.

a) ~ 35.2 A.U.

b) 59

a) From the Kepler's third law $\left(\frac{T_{\text{comet}}}{T_{\oplus}}\right)^2 = \left(\frac{a_{\text{comet}}}{a_{\oplus}}\right)^3 \Rightarrow a_{\text{comet}} = \left(\frac{T_{\text{comet}}}{T_{\oplus}}\right)^{\frac{2}{3}} \cdot a_{\oplus}$. Thus

$$a_{\text{comet}} = \left(\frac{(1986-1456)/7}{1\text{ yr}}\right)^{\frac{2}{3}} \times 1\text{ A.U.} \approx (75.7)^{\frac{2}{3}}\text{ A.U.} \approx 17.9\text{ A.U. and}$$

$$r_{a_{\text{comet}}} = 2a_{\text{comet}} - r_{p_{\text{comet}}} \approx 2\times 17.9\text{ A.U. - }0.60\text{ A.U.} \approx 35.8\text{ A.U. - }0.60\text{ A.U.} \sim 35.2\text{ A.U.}$$

b) From the Kepler's second law $v_{\max} r_{p_{\text{comet}}} = v_{\min} r_{a_{\text{comet}}} \Rightarrow \frac{v_{\max}}{v_{\min}} = \frac{r_{a_{\text{comet}}}}{r_{p_{\text{comet}}}} = \frac{35.2\text{ A.U.}}{0.60\text{ A.U.}} \sim 59$.

A-5

A-5.

A satellite in a circular orbit near the earth's surface has a typical period of about 100 minutes. What should be the radius of its orbit (in earth radii) for a period of 24 hours.

A-5.

$\sim 5.9\ R_E$

From the Kepler's third law

$$\left(\frac{T_{24\text{ h}}}{T_{100\text{ min}}}\right)^2 = \left(\frac{R_{24\text{ h}}}{R_E}\right)^3 \Rightarrow R_{24\text{ h}} = \left(\frac{T_{24\text{ h}}}{T_{100\text{ min}}}\right)^{\frac{2}{3}} \cdot R_E = \left(\frac{24\times 60\text{ min}}{100\text{ min}}\right)^{\frac{2}{3}} \cdot R_E = (14.4)^{\frac{2}{3}} \cdot R_E \sim 5.9 R_E\,.$$

B-1

B-1.

A true "Syncom" satellite rotates synchronously with the earth. It always remains in a fixed position with respect to a point P on the earth's surface.

a) Consider the straight line connecting the center of the earth with the satellite. If P lies on the intersection of this line with the earth's surface, can P have any geographic latitude or what restrictions do exist? Explain.

b) What is the distance r_s from the earth's center of a Syncom satellite of mass m? Express r_s in units of the earth-moon distance $r_{⊕☽}$

Note: Consider the earth a uniform sphere. You may use $T_☽ = 27$ days for the moon's period.

B-1.

a) $\lambda = 0$

b) $r_s = \frac{1}{9}\, r_{⊕☽}$

a) The geographic latitude of P is restricted to $\lambda = 0$ (equator). Although a synchronous orbit doesn't need to be equatorial, a body in a synchronous non equatorial orbit could not remain in a fixed position with respect to P: it would appear to oscillate between the north and south of the Earth's equator. Moreover, a body in a synchronous non circular (elliptical) orbit would appear to oscillate East-and-West. The combination of the two latter (synchronous non-circular non-equatorial) would appear as a path in the shape of eight.

b) From the Kepler's third law $t_s(=1\text{ day})=\left(\frac{r_s}{r_{\oplus\rightmoon}}\right)^{\frac{3}{2}} T_{\rightmoon}(=27\text{ days}) \Rightarrow r_s = \left(\frac{1}{27}\right)^{\frac{2}{3}} r_{\oplus\rightmoon} = \frac{1}{9} r_{\oplus\rightmoon}$.

B-2

B-2.

a) Comparing data describing the Earth's orbital motion about the sun with data for the moon's orbital motion about the Earth, determine the mass of the sun relative to the mass of the Earth.

b) Io, a moon of Jupiter, has an orbital period of revolution of $1^d.769$ and an orbital radius of 421,800 km. Determine the mass of Jupiter in terms of the mass of the Earth.

B-2.

a) $\frac{M_\odot}{M_\oplus} = 3.33 \times 10^5$

b) $\frac{M_4}{M_\oplus} = 318$

a) From the Kepler's third law $\frac{T_{\rightmoon}^2}{a_{\oplus\rightmoon}^3} = \frac{4\pi^2}{GM_\oplus}$, $\frac{T_\oplus^{\ 2}}{a_{\odot\oplus}^{\ 3}} = \frac{4\pi^2}{GM_\odot} \Rightarrow \left(\frac{T_{\rightmoon}}{T_\oplus}\right)^2 \cdot \left(\frac{a_{\odot\oplus}}{a_{\oplus\rightmoon}}\right)_3 = \frac{M_\odot}{M_\oplus}$.

From Earth and Moon data:

$r_{P_\oplus} = 1.47\times10^{11}$ m, $r_{A_\oplus} = 1.52\times10^{11}$ m, $T_\oplus = 365^d.242$
$r_{P\rightmoon} = 3.63\times10^{8}$ m, $r_{A\rightmoon} = 4.06\times10^{11}$ m, $T_{\rightmoon} = 27^d.322$

$$a_{\oplus} = \tfrac{1}{2}\left(r_{P_{\oplus}} + r_{A_{\oplus}}\right) = 1.50\times10^{11}\ \text{m}$$

$$a_{☽} = \tfrac{1}{2}\left(r_{P☽} + r_{A☽}\right) = 3.84\times10^{8}\ \text{m}$$

Thus $\dfrac{M_{\odot}}{M_{\oplus}} = \left(\dfrac{27^{d}.322}{365^{d}.242}\right)^2 \cdot \left(\dfrac{1.50\times10^{11}\ \text{m}}{3.84\times10^{8}\ \text{m}}\right)^3 = 3.33\times10^{5}$

b) $\dfrac{T_{♃}^2}{a_{♃}^3} = \dfrac{4\pi^2}{GM_{♃}}, \quad \dfrac{T_{☽}^2}{a_{\oplus☽}^3} = \dfrac{4\pi^2}{GM_{\oplus}} \Rightarrow \left(\dfrac{T_{☽}}{T_{♃}}\right)^2 \cdot \left(\dfrac{a_{♃}}{a_{\oplus☽}}\right)^3 = \dfrac{M_{♃}}{M_{\oplus}}$

$a_{♃} = 421{,}800$ km, $\quad a_{\oplus☽} = 384{,}000$km, $\quad T_{♃} = 1^{d}.769$

Thus $\dfrac{M_{♃}}{M_{\oplus}} = \left(\dfrac{27^{d}.322}{1^{d}.769}\right)^2 \cdot \left(\dfrac{421{,}800\ \text{km}}{384{,}000\ \text{km}}\right)^3 = 318$.

B-3

<u>B-3.</u>

Two stars, a and b, move around one another under the influence of their mutual gravitational attraction. If the semi major axis of their relative orbit is observed to be R, measured in astronomical units (AU) and their period of revolution is T years, find an expression for the sum of the mass, $m_a + m_b$, in terms of the mass of the sun.

<u>B-3.</u>

$$m_a + m_b = \frac{R^3}{T^2}\ m_{\odot}$$

For elliptic orbits $m_a + m_b = \dfrac{4\pi^2 R^3}{GT^2}$.

For the Earth-Sun system $m_{\odot} + m_{\oplus}\left(\cong m_{\odot}\right) = \dfrac{4\pi^2\left(1\ \text{A.U.}\right)^3}{G\left(1\ \text{yr}\right)}$.

Thus $m_a + m_b = \frac{R^3}{T^2} m_{\odot}$ with R expressed in A.U. and T in years.

B-4

B-4.

If the attractive gravitational force between a very large central sphere $\underline{M}$ and a satellite $\underline{m}$ in orbit about it were actually $\vec{F} = -\frac{GMm}{R^{(3+a)}}\vec{R}$, (where $\vec{R}$ is the vector between them) how would Kepler's second and third law be modified? (In discussing the third law, you should assume a circular orbit.)

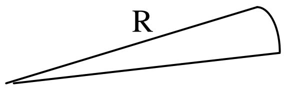
B-4.

II : unchanged

III : $T^2 = \frac{4\pi^2}{GM} R^{(3+a)}$

II: unchanged.

The Kepler's second low says that the areal velocity is constant, i.e., said $d\ell$ the arc length of the trajectory of $\underline{m}$ and R the distance from $\underline{M}$:



$\frac{dA}{dt} = \frac{1}{2}R\frac{d\ell}{dt} = \frac{1}{2}Rv = \frac{1}{2m}R(mv) = \frac{L}{2m} = \text{const.}$, being L the orbital angular momentum of $\underline{m}$.

But L remains constant for central forces, so the areal velocity also doesn't change. Indeed,

$$\frac{d\vec{L}}{dt} = \vec{R}\times\vec{F} = \vec{R}\times\left[-\frac{GMm}{R^{(3+a)}}\vec{R}\right] = -\frac{GMm}{R^{(3+a)}}\left(\vec{R}\times\vec{R}\right) = 0, \text{ being } \vec{R}\times\vec{R}\equiv 0.$$

Thus also L cost. and $\frac{dA}{dt}$ so well.

III: For a circular orbit $\vec{F} = -m\omega^2\vec{R}$, where $\omega = \frac{2\pi}{T}$. Hence

$$-\frac{GMm}{R^{(3+a)}}\vec{R} = -m\omega^2\vec{R} \Rightarrow \omega^2 = \frac{GM}{R^{(3+a)}} \Rightarrow \frac{4\pi^2}{T^2} = \frac{GM}{R^{(3+a)}} \Rightarrow T^2 = \frac{4\pi^2}{GM}R^{(3+a)}.$$

C-1

C-1.

In making laboratory measurements of g, how precise does one have to be to detect diurnal variations in g due to the moon's gravitation? For simplicity, assume that your laboratory is so located that the moon passes through zenith and nadir. Also, neglect earth-tide effects.

C-1.

$$\frac{\Delta g}{g} = 7 \times 10^{-6}$$

Ignoring the moon's gravitation one can assume $g = \dfrac{GM_\oplus}{R_\oplus^2}$.

Moon at zenith: $g_z = g - \dfrac{GM_{\leftmoon}}{\left(R_{\oplus\leftmoon} - R_\oplus\right)^2}$, at nadir: $g_n = g + \dfrac{GM_{\leftmoon}}{\left(R_{\oplus\leftmoon} + R_\oplus\right)^2}$.

Thus $\dfrac{\Delta g}{g} = \dfrac{g_n - g_z}{g} = \dfrac{M_{\leftmoon}}{M_\oplus}\left[\dfrac{1}{\left(\dfrac{R_{\oplus\leftmoon}}{R_\oplus}+1\right)^2} + \dfrac{1}{\left(\dfrac{R_{\oplus\leftmoon}}{R_\oplus}-1\right)^2}\right]$.

But $\dfrac{R_{\oplus\leftmoon}}{R_\oplus} = \dfrac{3.84\times10^8 \text{ m}}{6.38\times10^6 \text{ m}}$, $\dfrac{R_{\oplus\leftmoon}}{R_\oplus}+1 = \dfrac{3.90\times10^8 \text{ m}}{6.38\times10^6 \text{ m}}$, $\dfrac{R_{\oplus\leftmoon}}{R_\oplus}-1 = \dfrac{3.78\times10^8 \text{ m}}{6.38\times10^6 \text{ m}}$, $\dfrac{M_{\leftmoon}}{M_\oplus} = \dfrac{7.34\times10^{22} \text{ kg}}{5.98\times10^{24} \text{ kg}}$, so

$$\frac{\Delta g}{g} = \frac{7.34}{5.98}\times\left[\left(\frac{6.38}{3.90}\right)^2 + \left(\frac{6.38}{3.78}\right)^2\right]\times10^{-6} = 1.23\times\left(2.68+2.85\right)\times10^{-6} = 1.23\times5.53\times10^{-6} =$$

$=$ (approx.) 7×10^{-6}.

C-2

C-2.

An eclipsing binary star system is one whose orbital plane nearly contains the line of sight, so that one star eclipses the other periodically. The relative orbital velocity of the two components can be measured from the Doppler shift of their spectral lines. Let T and V be the observed period in days and orbital velocity in km s^{-1}. Find the total mass of the binary system in solar masses. Note: The mean distance from the earth to the sun is 1.50×10^{8} km.

C-2.

$M = 1.02 \times 10^{-7}\ TV^{3}\ M_{\odot}$

$$M = \frac{4\pi^2 a^3}{GT^2}, \quad M_{\odot} = \frac{4\pi^2 a_{\oplus\odot}^3}{GT_{\oplus}^2} \Rightarrow M = \left(\frac{a}{a_{\oplus\odot}}\right)^3 \cdot \left(\frac{T_{\oplus}}{T}\right)^2 .$$

Being T in days and V in km s^{-1} ,

$$V = \frac{2\pi a[\mathrm{km}]}{\left\{T[\mathrm{s}] = T\left[\mathrm{days}\times\frac{\mathrm{seconds}}{\mathrm{day}}\right]\right\}} = \frac{2\pi a[\mathrm{km}]}{T\times 86,400[\mathrm{s}]} \Rightarrow a = \frac{86,400}{2\pi}TV .$$

Thus $M = \left(86,400\ \mathrm{km} \big/ 2\pi\times 1.50\times 10^{8}\ \mathrm{km}\right)^3 \times (365\ \mathrm{days})^2 \times TV^3 \cdot M_{\odot} = 1.02\times 10^{-7} TV^3 M_{\oplus}$.

C-3

C-3.

A comet rounds the sun at a perihelion distance of $R_p = 1.00 \times 10^6$ km. At this point its velocity is 500.0 km s^{-1}.

a) What is the radius of curvature of the orbit at perihelion (in km)?

b) For an ellipse with semi-major axis a and semi-minor axis b, the radius of curvature at perihelion is $R_c = b^2/a$. If you know R_c and R_p you should be able to write a relation involving a and only these quantities. Do so, and find a.

c) If you were able to solve for a from the above information, you should be able to calculate the period of the comet. Write the relation, defining all symbols.

C-3.

a) $R_c = 1.88 \times 10^6$ km

b) $a = \dfrac{R_p^2}{2R_p - R_c} = 8.33 \times 10^6$ km

c) $T_c = \dfrac{2\pi a\sqrt{aR_c}}{vR_p} \approx 4.8$ days

For the ellipse the radius of curvature is $R_c = \dfrac{\left(R_p R_a\right)^{\frac{3}{2}}}{ab}$,

where $R_a = a(1+e) \;\; > \;\; R_p = a(1-e), \;\; b = a\sqrt{1-e^2}$.

Thus $R_c = R_p(1+e) = R_a(1-e)$ and $\dfrac{1}{r} = \dfrac{1}{R_c}(1-e\cos\varphi)$.

a) From the conservation of energy: $\frac{1}{2}v^2 - \frac{GM_\odot}{R_p} = \frac{1}{2}v_a^2 - \frac{GM_\odot}{R_a}$ (simplifying the comet mass), where v is the velocity at perihelion, and from the Kepler's second law $vR_p = v_a R_a$.

Thus $\frac{1}{2}\left(v^2 - v_a^2\right) = GM_\odot\left(\frac{1}{R_p} - \frac{1}{R_a}\right) = 2e\frac{GM_\odot}{R_c}$,

$$v_a = v\frac{R_p}{R_a} = v\frac{1-e}{1+e} \Rightarrow v - v_a = 2v\frac{e}{1+e}, \quad v + v_a = 2v\frac{1}{1+e} \;,$$

so $\frac{1}{2}\left(v^2 - v_a^2\right) = 2v^2\frac{e}{(1+e)^2} = 2e\frac{GM_\odot}{R_c}$, whence

$$R_c = \frac{GM_\odot}{v^2}\cdot\left(\frac{R_c}{R_p}\right)^2 \Rightarrow R_c = \frac{\left(vR_p\right)^2}{GM_\odot} = \frac{\left(500.0\text{ km s}^{-1}\times 1.00\times 10^6\text{ km}\right)^2}{1.33\times 10^{11}\text{ km}^3\text{ s}^{-2}} = 1.88\times 10^6\text{ km} \;.$$

b) $$1+e = \frac{R_c}{R_p} \Rightarrow e = \frac{R_c}{R_p} - 1, \quad a = \frac{R_p}{1-e} = \frac{R_p}{2-\frac{R_c}{R_p}} = \frac{1.00\times 10^6\text{ km}}{2-\frac{1.88}{1.00}} = \frac{10^6\text{ km}}{0.12} = 8.33\times 10^6\quad\text{km} \;.$$

c) From the Kepler's third law $T_c^2 = \frac{4\pi^2 a^3}{GM_\odot} \Rightarrow T_c = \frac{2\pi a\sqrt{a}}{\sqrt{GM_\odot}}$.

From a)

$$\sqrt{GM_\odot} = \frac{vR_p}{\sqrt{R_c}} \Rightarrow T_c = \frac{2\pi a\sqrt{aR_c}}{vR_p} = \frac{2\pi\times 8.33\times 10^6\text{ km}\sqrt{8.33\times 10^6\text{ km}\times 1.88\times 10^6\text{ km}}}{500.0\text{ Km s}^{-1}\times 1.00\times 10^6\text{ km}} \simeq 414{,}000\text{ s}$$

$$T_c\left[\text{days}\right] = T_c\left[\text{s}\right]\times\left[\frac{\text{days}}{\text{s}}\right] = T_c\left[\text{s}\right]\times\left[\frac{\text{seconds}}{\text{day}}\right]^{-1} = \frac{(\simeq)414{,}000\text{ s}}{86{,}400\text{ s day}^{-1}} \approx 4.8\text{ days} \;.$$