Temperature and energy measurements in ultra-low temperature Helium-3

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Specific heat capacity C_V as a function of the temperature T (in units of [J/K/m^3]) has been calculated from the entropy of a system of independent fermions, quasiparticles [1], as

$$C_V(T) = 2(2\pi)^{1/2} k_B N_0 \Delta \left(\frac{\Delta}{k_B T}\right) \exp\left(-\frac{\Delta}{k_B T}\right) \tag{1}$$

where k_B is the Boltzmann constant, N_0 is the density of quasiparticle states in the normal phase at Fermi energy for one spin component, Δ ($\approx 1.76 k_B T_c$) is the average gap energy at low temperature near the critical temperature for superfluidity T_c ($\simeq 930\,\mu$K). They all depend on pressure P.

$C_V(T)$ has a strong dependency with temperature (fig. 1)

To obtain the heat variation ΔQ between two temperatures T_1 and T_2 for a a given volume V of helium-3 is required the integration

$$\begin{aligned}
\Delta Q(T_1, T_2) = \Delta C_V V &= V \int_{T_1}^{T_2} C_V(T)\,dT && (2)\\
&= V \left(\int_0^{T_2} C_V(T)\,dT - \int_0^{T_1} C_V(T)\,dT\right) && (3)\\
&= V\left(I(T_2) - I(T_1)\right) && (4)
\end{aligned}$$

where

$$\begin{aligned}
I(T) = \int_0^T C_V(T)\,dT &= \sqrt{2}\pi N_0 \Delta^2 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\frac{\Delta}{k_B T}}} e^{-t^2}\,dt\right) && (5)\\
&= \sqrt{2}\pi N_0 \Delta^2 \mathrm{erfc}\sqrt{\frac{\Delta}{k_B T}} && (6)
\end{aligned}$$

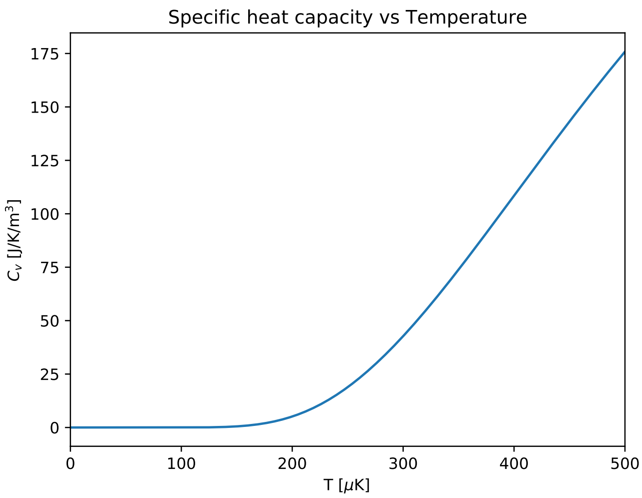

Figure 1: C_V vs T, below T_c. He3 has an extremely low heat capacity at low temperatures.

therefore

$$\Delta Q(T_1, T_2) = \sqrt{2}\pi N_0 \Delta^2 \left(\text{erfc}\sqrt{\frac{\Delta}{k_B T_2}} - \text{erfc}\sqrt{\frac{\Delta}{k_B T_1}} \right) V \tag{7}$$

We consider the 1 cm^3 volume of helium-3 at a certain base temperature T_0 so we can calculate the temperature variation ΔT for an energy deposition $\Delta Q(T_0, T_0 + \Delta T)$; since both N_0 and Δ are function of the pressure we calculate the energy deposition for 4 different pressures in the range (0–30) bar (fig. 2)

Higher pressures for higher system temperatures have equivalent effects in terms of temperature variations.

For a vibrating wire resonator (VWR), either from an amplitude sweep or a resonance tracking is possible to calculate the *resonance width* Δf which can be used to infer the temperature [2]. Since the resonance width is

$$\Delta f = \frac{2F}{\pi \rho d} \tag{8}$$

where d and ρ are respectively the diameter and the mass density of the wire, and F is the damping(?) force (in units of length, velocity and diameter), defined as

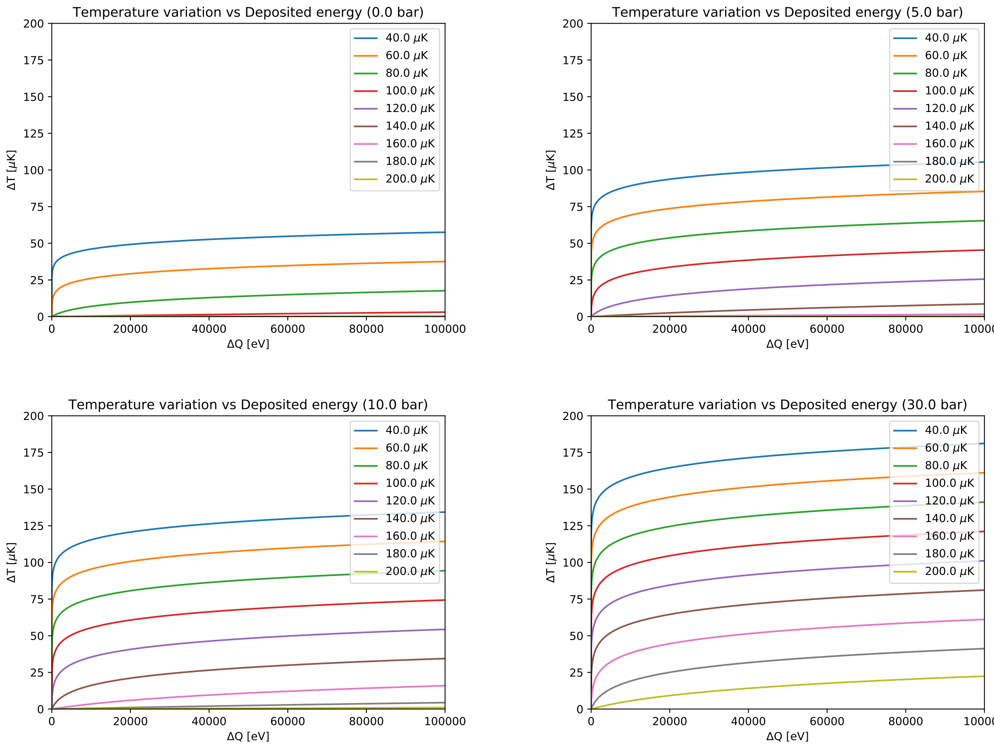

Figure 2: ΔT vs ΔQ for several T_0 in the range 40–200 μK for pressures of 0 bar (top-left), 5 bar (top-right), 10 bar (bottom-left), 30 bar (bottom-right).

$$F = \frac{\pi}{4} p_F^2 v_F N_0 \exp\left(-\frac{\Delta}{k_B T}\right) \tag{9}$$

The resonance width expressed in terms of temperature is

$$\Delta f = \frac{p_F^2 v_F N_0}{2\pi \rho d} \exp\left(-\frac{\Delta}{k_B T}\right) \tag{10}$$

e.g. in fig. 3 for a Niobium-Titanium wire of 150 nm diameter in a volume at 1 bar; therefore the temperature can be derived from the resonance width

$$T = -\frac{\Delta}{k_B \ln\left(\Delta f \frac{2\pi\rho d}{p_F^2 v_F N_0}\right)} \tag{11}$$

The increase of resonance width $\Delta(\Delta f)$ in response of an energy deposition ΔQ is linearly proportional, as shown in figures 4 and 5. The constant

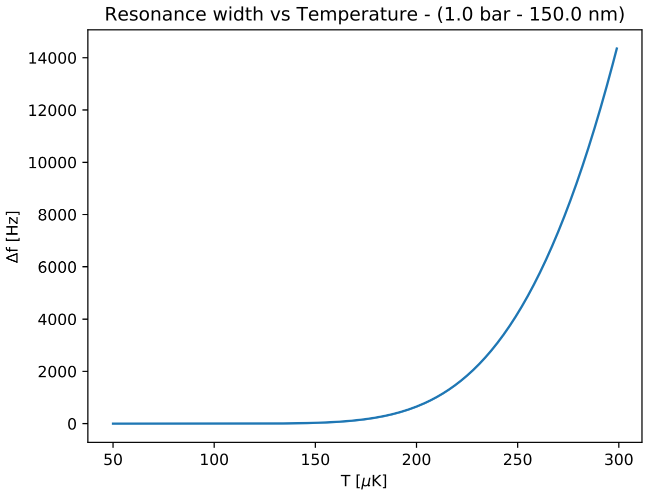

Figure 3: Resonance frequency width dependence on temperature for a nanowire.

of proportionality is inversely proportional to the base temperature (and pressure), so the other way around, in terms of energy deposition

$$\Delta Q = \alpha(T_0, P)\Delta(\Delta f), \alpha \propto T_0 \tag{12}$$

The increase of resonance width $\Delta(\Delta f)$ can be measured from fitting each bolometric recorded event, e.g. in figure 6 from [3], using the function

$$\Delta f(t) = \Delta f_{\text{base}} + \Delta(\Delta f)\left(\frac{\tau_b}{\tau_w}\right)^{\tau_w/(\tau_b - \tau_w)} \frac{\tau_b}{\tau_b - \tau_w}\left(e^{-t/\tau_b} - e^{-t/\tau_w}\right) \tag{13}$$

in order to extract the maximum variation $\Delta(\Delta f)$, where Δf_{base} is the base width at the base temperature of the helium, τ_w is the response time of the oscillating wire

$$\tau_w \simeq \frac{1}{\pi \Delta f} = \text{const} \tag{14}$$

and τ_b is the decay constant, proportional to the Kapitza resistance (the thermal boundary resistance limiting the heat conduction between the solid metal and the liquid helium)

$$\tau_b = R_K(T)C_V \tag{15}$$

Considering that there are three main variables, base temperature, pressure of the helium, wire diameter (considering fixed the wire material and

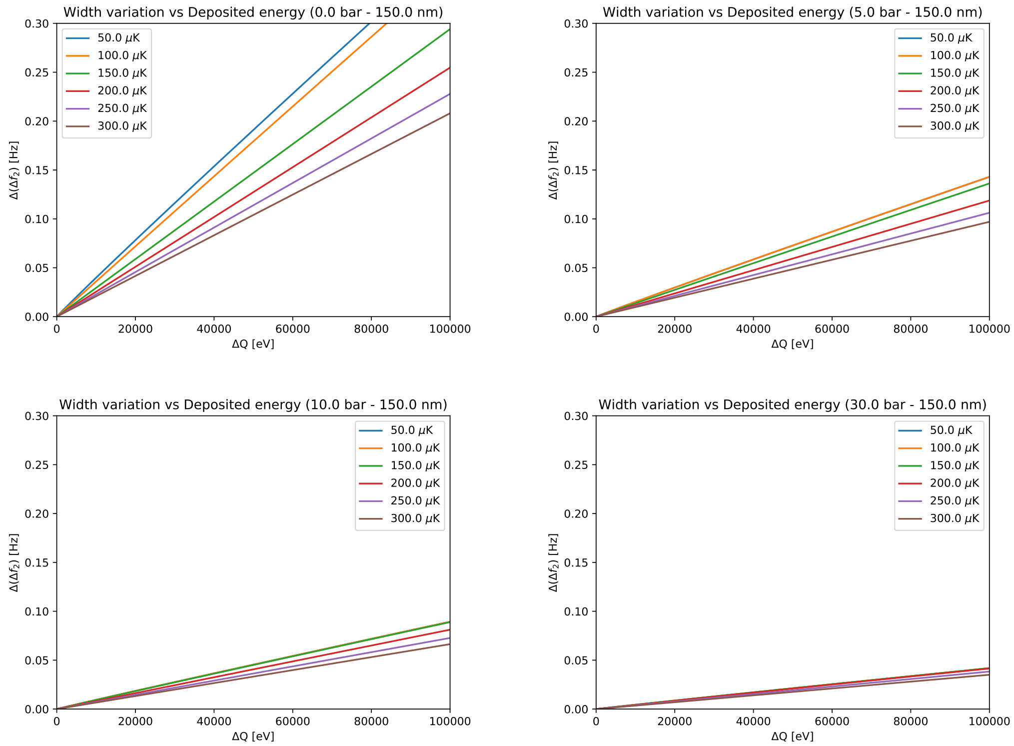

Figure 4: $\Delta(\Delta f)$ vs ΔQ for several T_0 in the range 50–300 μK for pressures of 0 bar (top-left), 5 bar (top-right), 10 bar (bottom-left), 30 bar (bottom-right).

the helium volume), all with an interplay between each other, we can at least extract few conclusions:

- For small wires there is an higher amplitude of the width response $\Delta(\Delta f)$
- For low temperatures there is an higher amplitude of the width response $\Delta(\Delta f)$
- For high pressures there is a lower width response for a certain energy deposition
- The response time of the oscillating wire is faster for high temperatures
- The decay time of the oscillating wire is faster for high temperatures (low Kapitza resistance), so low temperatures might cause some pile-up of events

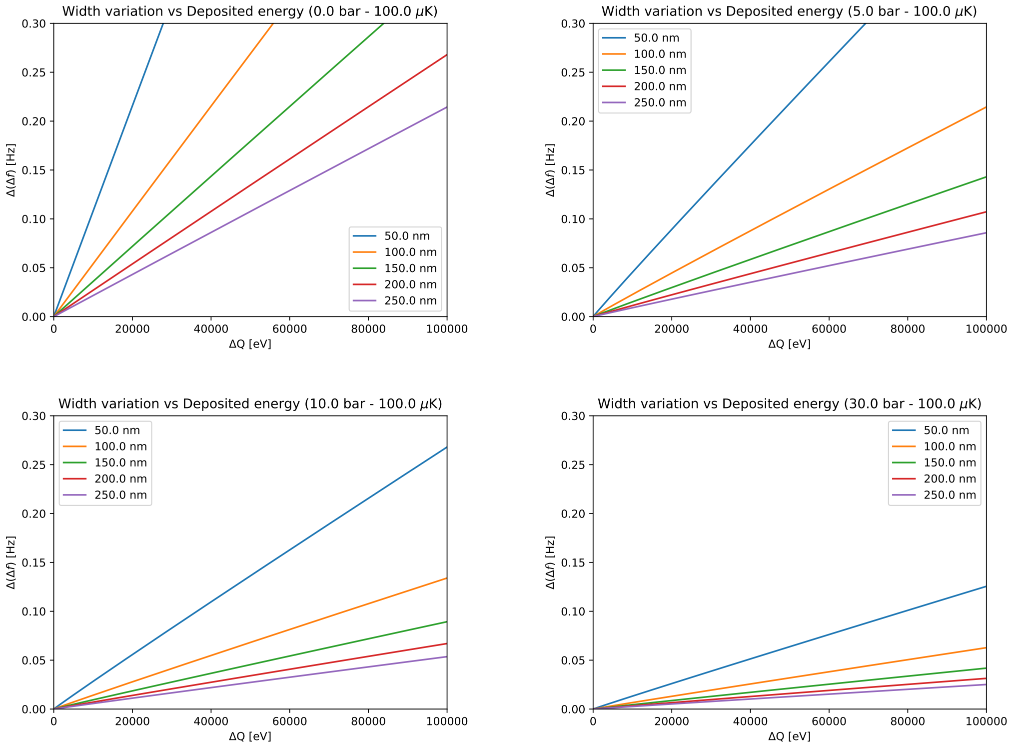

Figure 5: $\Delta(\Delta f)$ vs ΔQ for several wire diameters in the range 50–250 nm for pressures of 0 bar (top-left), 5 bar (top-right), 10 bar (bottom-left), 30 bar (bottom-right).

Ultimately the measurement done in the lab is done in terms of voltage readout of the oscillating wire, in two main steps:

1. *Sweep measurement* to extract the resonance frequency of the wire for a certain base temperature and the base width Δf_{base}. The quantities are extracted from the Lorentzian fits of the in-phase and out-of-phase signal (Voltage vs Frequency). This should provide the error $\sigma_{\Delta f_{\text{base}}}$.

2. Data acquisition with the setup sitting at the resonance frequency (*resonance tracking*) for a fixed voltage drive value V_D (amplitude of the injected voltage). The measurement is voltage height V_H. From the Lorentzian function we know that

$$\frac{V_H \Delta f}{V_D} = \text{const} = K \tag{16}$$

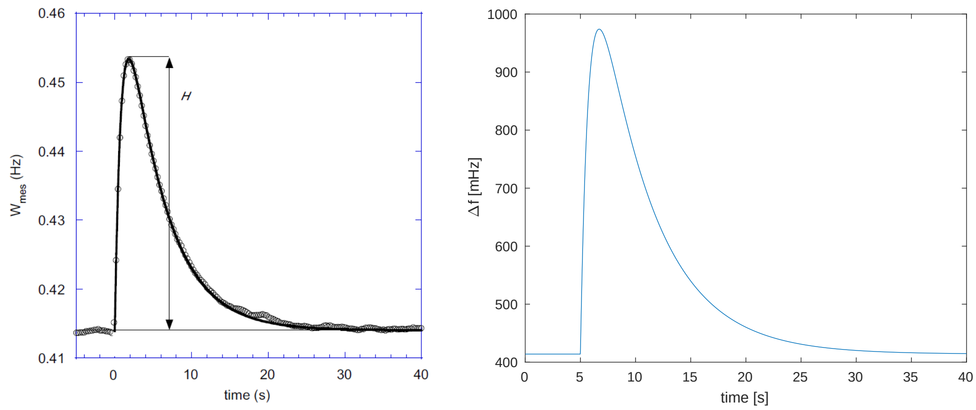

Figure 6: Example of an energy deposition event for a 4.5 μm wire in a volume of 0.14 cm^3 of He-3. H in the paper's notation is proportional to $\Delta(\Delta f)$ with respect to a base width [3] (*left*); fit function as in eq. 13 (*right*) with τ_b=5 s, τ_w=0.77 s and Δf_{base}=414 mHz.

in particular for the base values at the steady state (before the heat pulse)

$$\frac{V_{H_{\mathrm{base}}} \Delta f_{\mathrm{base}}}{V_D} = \mathrm{const} = K \tag{17}$$

so we are going to measure $\Delta f(t)$, as in figure 6, as

$$\Delta f(t) = \frac{V_D}{V_H(t)} K = \frac{V_{H_{\mathrm{base}}} \Delta f_{\mathrm{base}}}{V_H(t)} \tag{18}$$

1 Errors

In order to answer to the following questions, we need to find which is the measurement uncertainty coming from the laboratory setup:

- What is the minimum voltage variation we can measure, hence the threshold of the energy deposition?
- Which is the resolution in the resonance width variation (aka temperature variation), hence the resolution of the energy deposition?

We assume that the main uncertainty comes from the voltage measurement σ_{V_H} which propagates into the determination of Δf_{base} and into the quality of the fit to extract $\Delta(\Delta f)$

The error on the energy deposition should be

$$\sigma_{\Delta Q} \simeq \left| \frac{\partial \Delta Q}{\partial(\Delta(\Delta f))} \right| \sigma_{\Delta(\Delta f)} \overset{\text{eq.12}}{=} \alpha(T_0, P)\sigma_{\Delta(\Delta f)} \tag{19}$$

The analytical error propagation might prove to be ineffective; is worth assigning an error coming from the voltage measurement noise from the lock-in amplifier and do a systematic error study starting from a toy distribution generated $\Delta f(t)$ (where the noise is present both for the baseline and for the signal distribution); find an error on the fit done to extract $\Delta(\Delta f)$; propagate this error to ΔQ.

Since the data acquisition will fit Δf we need to add the uncertainty $\sigma_{\Delta f}$ on the toy generated function from eq. 13, where

$$\sigma_{\Delta f}(t) = \left| \frac{\partial \Delta f(t)}{\partial V_H(t)} \right| \sigma_{V_H} = \frac{(\Delta f(t))^2}{V_{H_{\text{base}}} \Delta f_{\text{base}}} \sigma_{V_H} \tag{20}$$

producing a distribution like in figure 7. To summarize the independent parameters present in the simulation are

- volume and pressure of the helium-3 cell (V, P)
- wire density (ρ)
- diameter of the oscillating wire (d)
- base temperature of the helium (T_0)
- decay constant and response time (τ_b, τ_w)
- base voltage height ($V_{H_{\text{base}}}$)
- error on the voltage measurement (σ_{V_H})

The randomization and fit of the pseudo-experiments is repeated N times, ultimately extracting the distribution of the deposited energy from the fitted $\Delta(\Delta f)$ using eq. 12, as in figure 8.

The error for the energy measurement can be related to σ/μ, so for a range of energies [0–100] KeV is possible to obtain distributions of the expected error for a certain configuration (figure 9)

In order to consider the error on the wire diameter we should take into account also the error on the base width, so

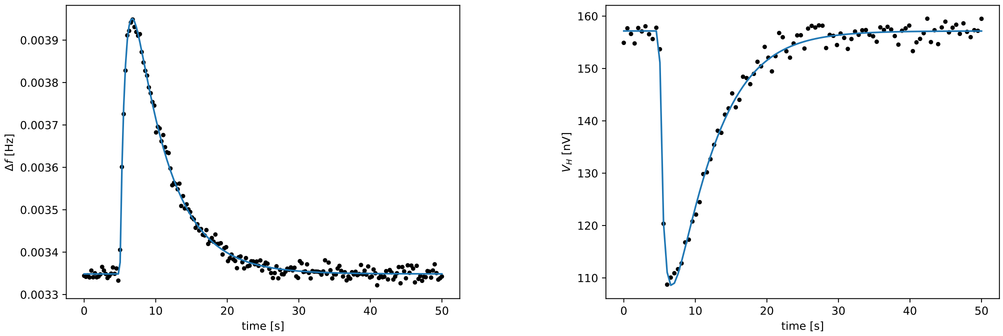

Figure 7: Example of a pseudo-experiment distribution of $\Delta f(t)$ (*left*) and $V_H(t)$ (*right*).

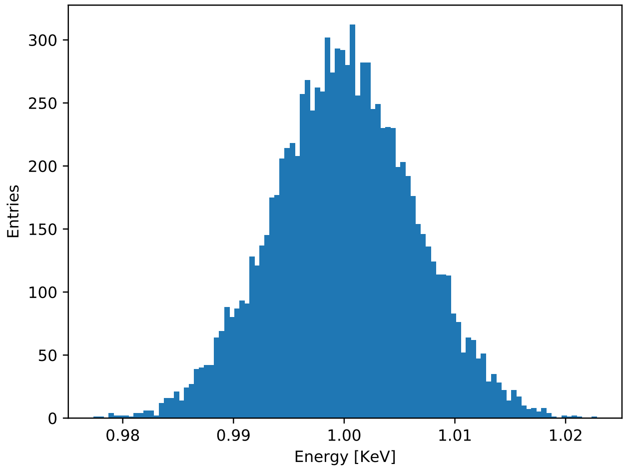

Figure 8: Energy distribution for 10000 toys.

$$\sigma^2_{\Delta Q} \simeq \left(\frac{\partial \Delta Q}{\partial \Delta f_{\text{base}}}\right)^2 \sigma^2_{\Delta f_{\text{base}}} + \left(\frac{\partial \Delta Q}{\partial(\Delta(\Delta f))}\right)^2 \sigma^2_{\Delta(\Delta f)} \tag{21}$$

$$= \left(\left.\frac{\partial \alpha}{\partial \Delta f_{\text{base}}}\right|_{(T_0,P)} \Delta(\Delta f)\right)^2 \sigma^2_{\Delta f_{\text{base}}} + \alpha^2(T_0,P)\sigma^2_{\Delta(\Delta f)} \tag{22}$$

We can define the base voltage $V_{H_{\text{base}}}$ fixing the wire velocity to v=1 mm/s

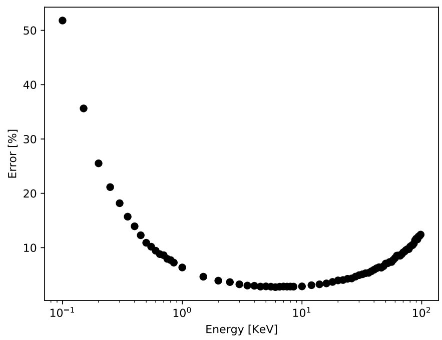

Figure 9: Error vs energy (in a certain configuration), for a 200 nm wire. The relative error increases with high energies because is proportional to the square of the width increase.

and assuming a field B=100 mT and a leg spacing D=2 mm, according to this describing the induced voltage of the wire moving in a magnetic field

$$V_H(t) = \frac{\pi}{4} B D v(t) \tag{23}$$

(where the geometrical constant $\pi/4$ corresponds to a semi-loop), $V_{H_{\text{base}}} \simeq 157\, nV$

References

[1] Vollhardt, D., Wolfle, P., The Superfluid Phases of Helium 3 (1990)

[2] Lawson, C.R., A Novel Measurement Device for use in Multiphase Helium-3 and 4 at Ultra-Low Temperatures, MPHys thesis (2014)

[3] Winkelmann et al, Bolometric calibration of a superfluid 3He detector for Dark Matter search: Direct measurement of the scintillated energy fraction for neutron, electron and muon events (2007)