Lecture Notes

The Feynman Lectures on Physics

Volume II – Maxwell's Equations

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$$\boxed{\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \frac{\rho}{\epsilon_0} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ c^2 \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\boldsymbol{j}}{\epsilon_0} \end{aligned}}$$

Ten thousand years from now . . . the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.

A	\mathrm{A}	α	\alpha	'al fə
B	\mathrm{B}	β	\beta	'bā tə
Γ	\Gamma	γ	\gamma	'ga mə
Δ	\Delta	δ	\delta	'del tə
E	\mathrm{E}	ϵ	\epsilon	'ep si län
Z	\mathrm{Z}	ζ	\zeta	'zā tə
H	\mathrm{H}	η	\eta	'ā tə
Θ	\Theta	θ	\theta	'thā tə
I	\mathrm{I}	ι	\iota	ī 'ō tə
K	\mathrm{K}	κ	\kappa	'ka pə
Λ	\Lambda	λ	\lambda	'lam də
M	\mathrm{M}	μ	\mu	m(y)ū
N	\mathrm{N}	ν	\nu	n(y)ū
Ξ	\Xi	ξ	\xi	zī
O	\mathrm{O}	o	\mathrm{o}	'ō mi krän
Π	\Pi	π	\pi	pī
P	\mathrm{P}	ρ	\rho	rō
Σ	\Sigma	σ	\sigma	'sig mə
T	\mathrm{T}	τ	\tau	taū
Υ	\Upsilon	υ	\upsilon	'(y)ūp sə län
Φ	\Phi	ϕ	\phi	fī
X	\mathrm{X}	χ	\chi	kī
Ψ	\Psi	ψ	\psi	(p)sī
Ω	\Omega	ω	\omega	ō 'mā gə

$$
\begin{aligned}
q_e &\approx 1.60 \times 10^{-19} \quad \text{Coulomb} \\
\epsilon_0 &\approx 8.854187817 \times 10^{-12} \quad \text{Farads/meter} \\
\mu_0 &= 4\pi \times 10^{-7} \quad \text{Henrys/meter} \\
c &\approx 2.99792458 \times 10^{8} \quad \text{meters/second}
\end{aligned}
$$

Chapter 1

Electromagnetism

1.1 Electrical forces

Consider a force like gravitation which varies predominantly inversely as the square of the distance, but which is about a *billion-billion-billion-billion* times stronger. And with another difference. There are two kinds of *charges*, which we call positive and negative, where like charges repel and unlike charges attract. There is such a force: the electrical force. Like the gravitational force, the electrical force F decreases inversely as the square of the distance r between two charges q_1 and q_2. This relationship is called Coulomb's law.

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

But Coulomb's law is not precisely true when charges are moving. The electrical forces depend also on the motions of the charges in a complicated way. One part of the force between moving charges we call the *magnetic* force, which is really one aspect of an electrical effect. This is why we call the subject "electromagnetism."

We can write the electromagnetic force on a charge q in a simple way. We use two vectors, $\boldsymbol{E}$ and $\boldsymbol{B}$, where $\boldsymbol{E}$ is the *electric field* at the location of q, and $\boldsymbol{B}$ is the *magnetic field* at the location of q. The electrical forces from all the other charges in the universe can be summarized by giving just these two vectors. The Lorentz force $\boldsymbol{F}$ on a charge q moving with a velocity $\boldsymbol{v}$ is

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{1.1}$$

From mechanics, we know how to calculate the motion of a particle, i.e., $\boldsymbol{F} = m\boldsymbol{a}$. The equation for the Lorentz force can be combined with the equation of motion so that if $\boldsymbol{E}$ and $\boldsymbol{B}$ are given, we can find the motions.

$$\frac{d}{dt}\left[\frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}}\right] = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{1.2}$$

An important simplifying principle is the *principle of superposition* of fields. If one set of charges produces field $\boldsymbol{E}_1$ and another set produces field $\boldsymbol{E}_2$ then the superposition $\boldsymbol{E}$ of these two fields is simply their sum. The principle of superposition holds for both electric fields and magnetic fields.

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_1 + \boldsymbol{E}_2 \\ \boldsymbol{B} &= \boldsymbol{B}_1 + \boldsymbol{B}_2 \end{aligned} \tag{1.3}$$

1.2 Electric and magnetic fields

We must extend our ideas of the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$. We have defined them in terms of the forces that are felt by a single charge q. Now we eliminate the charge and associate the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ with *the point* in space (x, y, z) that the charge occupies at time t.

Next, we associate with *every* point (x, y, z) in space two vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ which may be changing with time. Therefore we can view the electric and magnetic fields as *vector functions* of x, y, z, and t. The electric field is denoted as $\boldsymbol{E}(x, y, z, t)$ and the magnetic field is denoted as $\boldsymbol{B}(x, y, z, t)$. Each field represents three mathematical functions of x, y, z, and t, since a vector is specified by its three orthogonal components.

A *field* is any physical quantity which takes on different values at different points in space. It is precisely because $\boldsymbol{E}$ and $\boldsymbol{B}$ can be specified at every point in space that they are called fields.

There are simple relationships between field values at *one point* and field values at a *nearby point*. This fact allows us to completely describe the electric and magnetic fields with only a few such relationships in the form of differential equations. It is in terms of differential equations that the laws of electrodynamics are most simply written.

1.3 Characteristics of vector fields

There are two mathematically important properties of a vector field which we will use in our description of the laws of electricity from the field point of view. These two properties are *flux* and *circulation*.

Imagine a balloon made of porous rubber and inflated with air. We wish to quantify the outflow of air through its surface. The “flux of velocity” through the surface is the net amount of air going out through the surface per unit time. The “flux” through a surface element is equal to the component of the velocity perpendicular to the element times the area of the element. For any vector field, the *flux* through an arbitrary closed surface is defined as the average outward normal component of the vector multiplied by the area of the surface.

$$\text{Flux} = (\text{avg normal component}) \cdot (\text{surface area}) \tag{1.4}$$

Now imagine a turbulent body of cold water. The water is instantaneously frozen, all but the water inside a flexible hollow hoop which materializes at the same moment. The water inside the hoop continues to circulate due to its momentum. The “circulation” is the speed of this water times the circumference of the hoop. For any vector field, the *circulation* around an arbitrary closed curve is defined as the average tangential component of the vector multiplied by the circumference of the curve.

$$\text{Circ} = (\text{avg tangential component}) \cdot (\text{circumference}) \tag{1.5}$$

We extend these two ideas to electric and magnetic fields. With just these two ideas — flux and circulation — we can describe all the laws of electricity and magnetism at once.

1.4 The laws of electromagnetism

Here are four laws of electromagnetism. They are expressed using the two ideas, flux and circulation, in conjunction with the two fields, $\boldsymbol{E}$ and $\boldsymbol{B}$.

The first law describes the flux of $\boldsymbol{E}$. Given some arbitrary closed surface S' (e.g. a spherical membrane) and some charges in the space bounded by that surface, the flux of electric field $\boldsymbol{E}$ through closed surface S' is the net charge inside divided by ϵ_0 (a convenient constant).

$$\text{Flux of } \boldsymbol{E} \text{ thru } S' = \frac{\text{net charge inside}}{\epsilon_0} \tag{1.6}$$

The second law describes the circulation of $\boldsymbol{E}$. Given some arbitrary open surface S (e.g. a thin disc) whose edge is the curve C, the circulation of electric field $\boldsymbol{E}$ around curve C is equal to the negative of the time derivative of the flux of magnetic field $\boldsymbol{B}$ through open surface S.

$$\text{Circ of } \boldsymbol{E} \text{ around } C = -\frac{d}{dt}\Big(\text{flux of } \boldsymbol{B} \text{ thru } S\Big) \tag{1.7}$$

The third law describes the flux of $\boldsymbol{B}$. Given some arbitrary closed surface S' the flux of magnetic field $\boldsymbol{B}$ through closed surface S' is zero. There are no magnetic "charges" inside the surface.

$$\text{Flux of } \boldsymbol{B} \text{ thru } S' = 0 \tag{1.8}$$

The fourth law describes the circulation of $\boldsymbol{B}$. Given some arbitrary open surface S whose edge is the curve C, the circulation of magnetic field $\boldsymbol{B}$ around curve C is proportional to the time derivative of the flux of electric field $\boldsymbol{E}$ through open surface S, plus the flux of the electric current I through open surface S, divided by ϵ_0. The constant c^2 is the square of the velocity of light. It appears because magnetism is in reality a relativistic effect of electricity.

$$\begin{aligned} c^2(\text{Circ of } \boldsymbol{B} \text{ around } C) \quad = \quad & \frac{d}{dt}\Big(\text{flux of } \boldsymbol{E} \text{ thru } S\Big) \\ & + \frac{\text{flux of } I \text{ thru } S}{\epsilon_0} \end{aligned} \tag{1.9}$$

The most remarkable consequence of our equations is that the combination of Eq. (1.7) and Eq. (1.9) contains the explanation of the radiation of electromagnetic effects over large distances. Consider an antenna element: a current changes and causes the magnetic field around the element to change. By Eq. (1.7) the changing magnetic field causes a changing electric field. And by Eq. (1.9) the changing electric field causes a changing magnetic field. This interchange continues indefinitely. The electric and magnetic fields work their way through space without the need of charges or currents.

1.5 What are the fields?

All this business of fluxes and circulations is pretty abstract. What is *actually* happening? Field lines are helpful, but they do not contain the deepest principle of electrodynamics, which is the superposition principle. So what is the *most convenient* way to look at electrical effects? The best way is to use the abstract field idea. That it is abstract is unfortunate but necessary.

What happens when two charges in space move parallel to each other at the same speed? A stationary observer will observe a magnetic field surrounding the two charges. But an observer riding along with the charges will observe no magnetic field. So magnetism is really a relativistic effect.

Consider the apparatus in Figure 1-8. Two parallel wires each carry a current. When the currents are in the same direction, the two wires attract. The average speed of the electrons in the wires is about 0.01 centimeters per second, so v^2/c^2 is about 10^{-25}. In the Lorentz force equation, the term for the electrical force disappears because of the almost perfect balance — the wires have the same number of protons as electrons. The small relativistic term which we call the magnetic force is the only term left. It becomes the dominant term.

1.6 Electromagnetism in science and technology

It is amazing to consider that the amber and lodestone effects were the only phenomena known to the Greeks in which the effects of electricity or magnetism were apparent. But when the understanding of electromagnetism developed, technologies appeared that defied imagination: telegraphy, telephony, hydroelectric power, television.

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Chapter 2

Differential Calculus of Vector Fields

2.1 Understanding physics

The physicist needs a facility in looking at problems from several points of view. Ideas such as the field lines, capacitance, resistance, and inductance are, for such purposes, very useful. On the other hand, none of the heuristic models, such as field lines, is really adequate and accurate for all situations. There is only one precise way of presenting the laws, and that is by means of differential equations.

It will take you some time to understand what should happen in different situations. You will have to solve the equations. Each time you solve the equations, you will learn something about the character of the solutions. To keep these solutions in mind, it will be useful also to study their meaning in terms of field lines and of other concepts. This is the way you will really "understand" the equations.

What it means really to understand an equation — that is, in more than a strictly mathematical sense — was described by Dirac. He said: "I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it." A physical understanding is a completely unmathematical, imprecise, and inexact thing, but absolutely necessary for a physicist.

2.2 Scalar and vector fields — T and $\boldsymbol{h}$

Our ultimate goal is to explain the meaning of the laws given in Chapter 1. But to do this we must understand the mathematics of vector fields. We first review vector algebra: scalars, vectors, scalar products, and vector products.

A *scalar* m is a value that consists of a single component. The component is an element of the set of real numbers.

$$m \in \mathcal{R}$$

A *vector* $\boldsymbol{A}$ is a value that has three components — A_x, A_y, and A_z — from the rectangular coordinate system. Each component is an element of the set of real numbers.

$$\boldsymbol{A} = \Big(\; A_x \; , \; A_y \; , \; A_z \; \Big)$$

The *magnitude* of vector $\boldsymbol{A}$ is the scalar $|\boldsymbol{A}|$, sometimes written simply as A. We use the Pythagorean theorem, extended for three dimensions, to calculate the magnitude $|\boldsymbol{A}|$.

$$|\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

To add two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, we simply add their components.

$$\boldsymbol{A} + \boldsymbol{B} = \Big((A_x + B_x), (A_y + B_y), (A_z + B_z) \Big)$$

The product of a scalar m and a vector $\boldsymbol{A}$ is a vector. Each of the three components of $\boldsymbol{A}$ is multiplied by m.

$$m\boldsymbol{A} = \Big((mA_x), (mA_y), (mA_z) \Big)$$

The *scalar product* of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is a scalar. The scalar product is sometimes called the *dot product*. It has two forms. In the first form, each of the three components of vector $\boldsymbol{A}$ are multiplied by the three corresponding components of vector $\boldsymbol{B}$. In the second form, the magnitudes of $\boldsymbol{A}$ and $\boldsymbol{B}$ are multiplied by the cosine of the angle θ that separates $\boldsymbol{A}$ and $\boldsymbol{B}$.

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= \text{scalar} \\ &= A_x B_x + A_y B_y + A_z B_z \\ &= AB\cos\theta \end{aligned} \tag{2.1}$$

The *vector product* of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is a vector. The vector product is sometimes called the *cross product*. Each component of the result is a difference of products.

$$\begin{aligned} \boldsymbol{A} \times \boldsymbol{B} &= \text{vector} \\ &= \Big((A_y B_z - A_z B_y), (A_z B_x - A_x B_z), (A_x B_y - A_y B_x) \Big) \end{aligned} \tag{2.2}$$

The vector product of $\boldsymbol{A}$ with itself is simply the zero vector $(0,0,0)$ which is represented by the symbol $\mathbf{0}$. The proof is quick.

$$\boldsymbol{A} \times \boldsymbol{A} = \mathbf{0} \tag{2.3}$$

Here are some identities. The proofs are simple but somewhat tedious.

$$\boldsymbol{A} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = 0 \tag{2.4}$$

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C} \tag{2.5}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{2.6}$$

The *total differential* for three dimensions looks like the scalar product of two vectors.

$$\Delta f(x,y,z) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z \tag{2.7}$$

When we have a second derivative with two coefficients in the denominator, the coefficients are commutative.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{2.8}$$

A *scalar field* is a set of scalars, where each scalar is associated with a unique position (x,y,z) in space. For example, consider a concrete slab, one meter square and ten centimeters thick, where the top of the slab is heated by the sun and the bottom contacts some frozen ground. Heat flows through the slab from top to bottom. At any instant in time, each point inside the slab has a specific temperature T (degrees Celsius), and we have the scalar function $T(x,y,z)$. This example demonstrates that temperature is a scalar field.

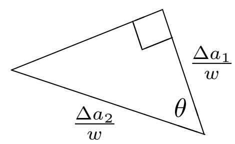
A *vector field* is a set of vectors, where each vector is associated with a unique position (x,y,z) in space. For example, consider the concrete slab again. At any instant in time, each point inside the slab has a specific heat flow $\boldsymbol{h}$ with a specific rate (Joules per square meter) and a specific direction. We therefore have the vector function $\boldsymbol{h}(x,y,z)$ which specifies a heat flow vector for each point inside the slab at some instant. This example demonstrates that heat flow is a vector field.

A *unit vector* is a vector with a magnitude of 1. We can use a unit vector to specify the heat flow vector $\boldsymbol{h}$. Let ΔJ be the amount of thermal energy that passes through a small area Δa. Let $\boldsymbol{e}_f$ be a unit vector ($f \equiv$ flow) that points in the same direction as $\boldsymbol{h}$. Then the magnitude of $\boldsymbol{h}$ is $\Delta J/\Delta a$ and

$$\boldsymbol{h} = \frac{\Delta J}{\Delta a}\boldsymbol{e}_f \tag{2.9}$$

We wish to specify how much heat flows through a small surface at *any* angle with respect to the flow. For example, we wish to specify how much heat flows through the surface Δa_2 in Figure 2-4.

The unit vector $\boldsymbol{n}$ is normal to surface Δa_2. The heat flow vector $\boldsymbol{h}$ is normal to surface Δa_1 and points in a different direction than unit vector $\boldsymbol{n}$. The two vectors are separated by the angle θ. The same amount of heat flows through Δa_1 and Δa_2. These two surface areas are related such that $\Delta a_1 = \Delta a_2 \cos\theta$. For proof, consider the prism implied in Figure 2-4.



The prism has width w. One face of the prism is the surface Δa_1. Another face is the surface Δa_2. A third face is a right triangle with angle θ, a hypotenuse $\Delta a_2/w$, and adjacent side $\Delta a_1/w$. The cosine of θ is defined as the length of the adjacent side divided by the length of the hypotenuse.

$$\begin{aligned} \cos\theta &= \frac{\Delta a_1/w}{\Delta a_2/w} = \frac{\Delta a_1}{\Delta a_2} \\ \Delta a_1 &= \Delta a_2 \cos\theta \end{aligned}$$

Rearrange to get the surface areas in the denominators.

$$\frac{1}{\Delta a_2} = \frac{1}{\Delta a_1}\cos\theta$$

Again we ask, what is the heat flow through Δa_2? It is $\Delta J/\Delta a_2$. And the magnitude of $\boldsymbol{h}$ is $\Delta J/\Delta a_1$. Now we multiply both sides of the equation above by ΔJ. This is the same as the scalar product of $\boldsymbol{h}$ and $\boldsymbol{n}$.

$$\frac{\Delta J}{\Delta a_2} = \frac{\Delta J}{\Delta a_1}\cos\theta = \boldsymbol{h}\cdot\boldsymbol{n} \tag{2.10}$$

The heat flow through Δa_2 is the scalar product of the heat flow vector $\boldsymbol{h}$ and the normal unit vector $\boldsymbol{n}$.

2.3 Derivatives of fields — the gradient

This section introduces the concept of the *gradient*. It defines $\boldsymbol{\nabla} T$ (pronounced "grad tee"), the gradient of the temperature function, as a vector whose three components in any Cartesian coordinate system are partial derivatives. It also demonstrates that vectors are the same regardless of the orgin and rotation of the coordinate system.

We start by asking, what is the derivative of a scalar field? Consider the three partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial z$. Perhaps these are the components of a vector.

$$\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial y}\right) \stackrel{?}{=} \text{a vector} \tag{2.11}$$

We wish to demonstrate that this is indeed a vector. We will show that when the coordinate system is rotated, the three derivatives transform among themselves in the correct way. Consider some vector $\boldsymbol{A}$ with components (A_x, A_y, A_z) and three numbers B_1, B_2, and B_3, which are combined with the components of $\boldsymbol{A}$ in the following way to compute some scalar S. Then by Eq. (2.1) it must be true that the three numbers B_1, B_2, and B_3 are the components (B_x, B_y, B_z) of some vector $\boldsymbol{B}$.

$$A_x B_1 + A_y B_2 + A_z B_3 = S \tag{2.12}$$

Now consider a solid block of material where the temperature at any point is $T(x, y, z)$ and the change in temperature between any two nearby points is $\Delta T(x, y, z)$. We use the total differential from Eq. (2.7) to specify ΔT.

$$\Delta T = \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z \tag{2.13}$$

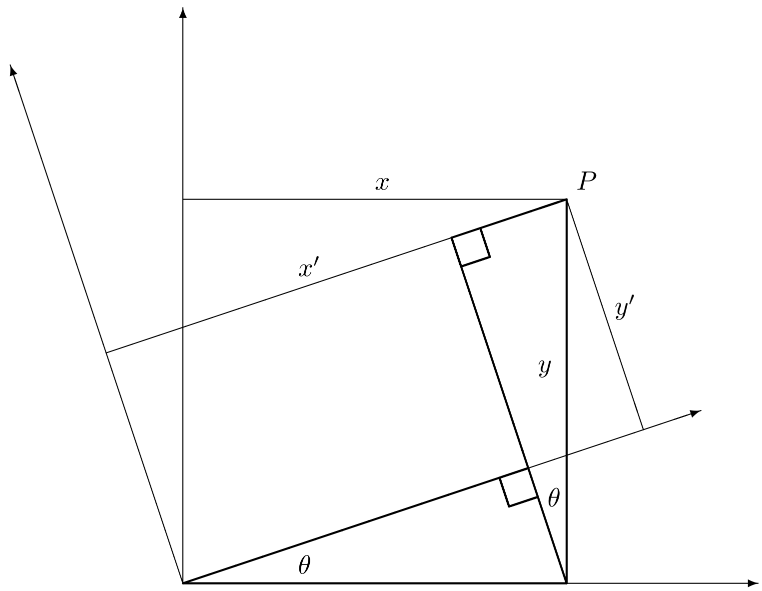
Note that the right side is in the form of a scalar product of two vectors. The second vector has the three components $\Delta x, \Delta y, \Delta z$. The first vector has the three components $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial z$. This first vector has a special notation: $\boldsymbol{\nabla} T$.

$$\text{grad } T = \boldsymbol{\nabla} T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial y}\right) \tag{2.14}$$

Let $\Delta \boldsymbol{R} = (\Delta x, \Delta y, \Delta z)$. Now we can write Eq. (2.13) in a more compact form that shows $\boldsymbol{\nabla} T$ is indeed a vector.

$$\Delta T = \boldsymbol{\nabla} T \cdot \Delta \boldsymbol{R} \tag{2.15}$$

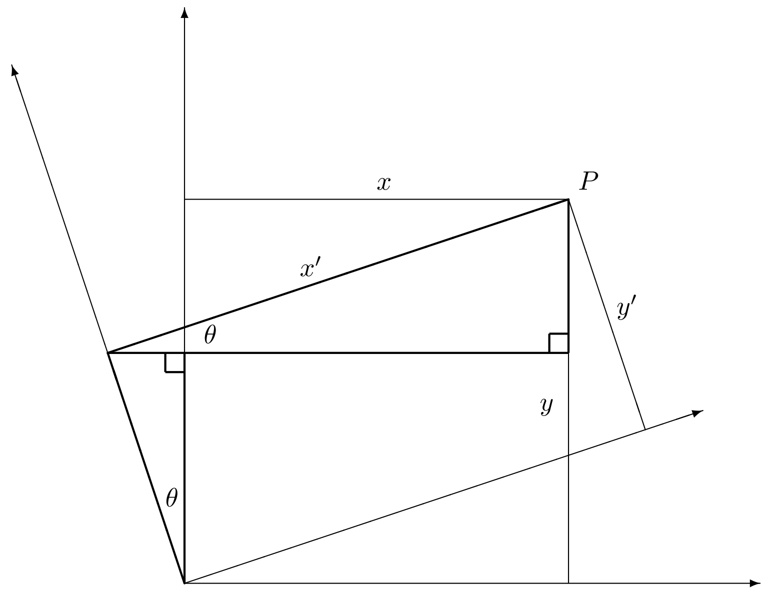
But what if we use a different coordinate system? We need to show that the components of the vector $\boldsymbol{\nabla} T$ transform in the same way as the components of the vector $\Delta \boldsymbol{R}$.



This figure shows the point P in two different coordinate systems. In one system P has the coordinates (x, y). In the other system P has the coordinates (x', y'). We use simple trigonometry to derive equations for x' and y' in terms of x and y. Consider the two right triangles, both drawn with thick lines. One has hypotenuse x and the other has hypotenuse y. The figure shows x' as a sum of two lengths and y' as a difference of two lengths.

$$x' = x\cos\theta + y\sin\theta \tag{2.16}$$

$$y' = y\cos\theta - x\sin\theta \tag{2.17}$$



This figure shows the same point P in the two different coordinate systems. However, we now derive equations for x and y in terms of x' and y'. Consider the two different right triangles. One has hypotenuse x' and the other has hypotenuse y'. The figure shows x as a diffence of two lengths and y as a sum of two lengths.

$$x = x' \cos\theta - y' \sin\theta \tag{2.18}$$

$$y = y' \cos\theta + x' \sin\theta \tag{2.19}$$

Figure 2-6(b) shows P_1 and P_2 in two different coordinate systems (x, y) and (x', y'). There is no third dimension z, so $\Delta z = 0$. P_1 and P_2 are chosen such that $\Delta y = 0$ also. From Eq. (2.13) we now have

$$\Delta T = \frac{\partial T}{\partial x} \Delta x \tag{2.20}$$

We can also write ΔT in terms of the second coordinate system.

$$\Delta T = \frac{\partial T}{\partial x'} \Delta x' + \frac{\partial T}{\partial y'} \Delta y' \tag{2.21}$$

Looking at Figure 2-6(b) we can write values for $\Delta x'$ and $\Delta y'$.

$$\Delta x' = \Delta x \cos\theta \tag{2.22}$$

$$\Delta y' = -\Delta x \sin\theta \tag{2.23}$$

We substitute these two values into Eq. (2.21). The result looks a lot like Eq. (2.20)

$$\begin{aligned} \Delta T &= \frac{\partial T}{\partial x'}\Delta x \cos\theta - \frac{\partial T}{\partial y'}\Delta x \sin\theta && (2.24) \\ &= \left(\frac{\partial T}{\partial x'}\cos\theta - \frac{\partial T}{\partial y'}\sin\theta\right)\Delta x && (2.25) \end{aligned}$$

except that...

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial x'}\cos\theta - \frac{\partial T}{\partial y'}\sin\theta \tag{2.26}$$

So $\boldsymbol{\nabla} T$ is definitely a vector.

2.4 The operator $\boldsymbol{\nabla}$

Now we can do something that is extremely amusing and ingenious — and characteristic of the things that make mathematics beautiful. We can abstract the gradient $\boldsymbol{\nabla}$ away from the T. We take the T out of equation (2.26) and leave the operators "hungry for something to differentiate."

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\cos\theta - \frac{\partial}{\partial y'}\sin\theta \tag{2.27}$$

We make $\boldsymbol{\nabla}$ a *vector operator*. Just as the vector $\boldsymbol{\nabla} T$ in equation (2.14) has three components, so does the vector operator $\boldsymbol{\nabla}$.

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{2.28}$$

Another way of writing the vector operator $\boldsymbol{\nabla}$ is $(\nabla_x, \nabla_y, \nabla_z)$ where

$$\nabla_x = \frac{\partial}{\partial x} \qquad \nabla_y = \frac{\partial}{\partial y} \qquad \nabla_z = \frac{\partial}{\partial z} \tag{2.29}$$

We need to remember that the operator $\boldsymbol{\nabla}$ always precedes a scalar variable. (For example, $\boldsymbol{\nabla} T$). An expression with $\boldsymbol{\nabla}$ on the right side of the scalar variable is meaningless. For example, $T\boldsymbol{\nabla}$ has an x-component that is not a number.

$$T\frac{\partial}{\partial x} \tag{2.30}$$

What is to be differentiated must be placed on the right of the $\boldsymbol{\nabla}$. The commutative law for multiplication does not apply for $\boldsymbol{\nabla}$. However, if we have a scalar T and a vector $\boldsymbol{A}$ we can represent their product either way.

$$T\boldsymbol{A} = \boldsymbol{A}T \tag{2.31}$$

What is the direction of the vector $\boldsymbol{\nabla} T$? On a three-dimensional graph, it's the direction of the steepest uphill slope of $T(x, y, z)$.

2.5 Operations with $\boldsymbol{\nabla}$

We can compute the dot product of $\boldsymbol{\nabla}$ with the vector field $\boldsymbol{h}$. We use equation (2.7), the sum of three products. Remember that $\boldsymbol{\nabla} = (\nabla_x, \nabla_y, \nabla_z)$ and $\boldsymbol{h} = (h_x, h_y, h_z)$.

$$\boldsymbol{\nabla} \cdot \boldsymbol{h} = \nabla_x h_x + \nabla_y h_y + \nabla_z h_z \tag{2.32}$$

If we replace the grad terms $(\nabla_x, \nabla_y, \nabla_z)$ with the hungry partials in equation (2.29) we get

$$\boldsymbol{\nabla} \cdot \boldsymbol{h} = \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} \tag{2.33}$$

We can use a different coordinate system. Instead of $\boldsymbol{\nabla}$ we use $\boldsymbol{\nabla}'$. This gives us

$$\boldsymbol{\nabla}' \cdot \boldsymbol{h} = \frac{\partial h_{x'}}{\partial x'} + \frac{\partial h_{y'}}{\partial y'} + \frac{\partial h_{z'}}{\partial z'} \tag{2.34}$$

Even though we have two different equations for two different coordinate systems, the two products are equal, because the scalar product of two vectors is invariant under a coordinate transformation. The vector field $\boldsymbol{h}$ is the same in both systems. The vector operators $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}'$ are also the same — they just have different variable names, depending on the coordinate system.

$$\boldsymbol{\nabla}' \cdot \boldsymbol{h} = \boldsymbol{\nabla} \cdot \boldsymbol{h} \tag{2.35}$$

The dot product of $\boldsymbol{\nabla}$ with any vector is called the *divergence*.

$$\boldsymbol{\nabla} \cdot \boldsymbol{h} = \text{div } \boldsymbol{h} \tag{2.36}$$

The cross product of $\boldsymbol{\nabla}$ with any vector is called the *curl*.

$$\boldsymbol{\nabla} \times \boldsymbol{h} = \text{curl } \boldsymbol{h} \tag{2.37}$$

The curl is a vector with three components. The calculation of each component is specified in equation (2.2) and given again here, but with the partial terms.

$$\left(\boldsymbol{\nabla} \times \boldsymbol{h}\right)_z = \nabla_x h_y - \nabla_y h_x = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} \tag{2.38}$$

$$\left(\boldsymbol{\nabla} \times \boldsymbol{h}\right)_x = \nabla_y h_z - \nabla_z h_y = \frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z} \tag{2.39}$$

$$\left(\boldsymbol{\nabla} \times \boldsymbol{h}\right)_y = \nabla_z h_x - \nabla_x h_z = \frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x} \tag{2.40}$$

The vector operator $\boldsymbol{\nabla}$ allows us to write Maxwell's equations in a succinct form. In the first equation, ρ (rho) represents the *electric charge density*, the amount of charge per unit volume. In the fourth equation, $\boldsymbol{j}$ represents the *electric current density*, the rate at which charge flows through a unit area per second. Of course, c represents the speed of light, and ϵ_0 is a convenient constant.

Maxwell's Equations

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t} \\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\
c^2\boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{\partial\boldsymbol{E}}{\partial t}+\frac{\boldsymbol{j}}{\epsilon_0}
\end{aligned} \tag{2.41}$$

2.6 The differential equation of heat flow

We can use vector notation to describe all sorts of elementary physics relations. In this section we describe heat flow using vector notation. We are given the slab of material in Figure 2-7(a). A is the area of the large face. d is the thickness of the slab. T_2 is the temperature on the hot face. T_1 is the area on the cold face. J is the thermal energy that passes through the slab per unit time. κ is the *thermal conductivity* constant.

$$J=\kappa\big(T_2-T_1\big)\frac{A}{d} \tag{2.42}$$

Now we complicate things. The slab becomes odd-shaped and the temperature within the slab varies in peculiar ways. Our strategy is to describe infinitesimal differences. We take an infinitesimal piece of the slab, as shown in Figure 2-7(b). The two parallel curves are isothermals. ΔA is the area of the small face (not shown in the figure). Δs is the distance between the isothermals. ΔT is the difference in temperature between the isothermals. κ is the same thermal conductivity constant as before.

$$\Delta J=\kappa\Delta T\frac{\Delta A}{\Delta s} \tag{2.43}$$

Now we transform the scalar equation (2.43) to a vector equation (2.44). This is explained using magnitudes. The first step is to divide both sides by ΔA and rearrange. This gets rid of the ΔA on the right side.

$$\frac{\Delta J}{\Delta A} = \kappa \frac{\Delta T}{\Delta s}$$

The first transformation is from scalar $\Delta J / \Delta A$ to vector $\boldsymbol{h}$. In equation (2.9), $\Delta J / \Delta A$ was defined as the magnitude of $\boldsymbol{h}$. The direction of $\boldsymbol{h}$ is perpendicular to the isothermals.

$$\frac{\Delta J}{\Delta A} = |\boldsymbol{h}|$$

The other transformation is from scalar $\Delta T / \Delta s$ to vector $\boldsymbol{\nabla} T$. $\Delta T / \Delta s$ is the rate of change of T with position. It's the maximum rate of change, and therefore the magnitude of $\boldsymbol{\nabla} T$.

$$\frac{\Delta T}{\Delta s} = |\boldsymbol{\nabla} T|$$

We substitute these two transformations into Equation (2.43) and then drop the magnitude bars. The direction of $\boldsymbol{\nabla} T$ is opposite that of $\boldsymbol{h}$, so there's a sign change.

$$\boldsymbol{h} = -\kappa \boldsymbol{\nabla} T \tag{2.44}$$

2.7 Second derivatives of vector fields

In this section we explore the various possible second derivatives. We are given three expressions: (1) the gradient $\boldsymbol{\nabla} T$, (2) the dot product $\boldsymbol{\nabla} \cdot \boldsymbol{h}$, and (3) the cross product $\boldsymbol{\nabla} \times \boldsymbol{h}$. We then take each one of these and try to compute the gradient, the dot product, and the vector product with $\boldsymbol{\nabla}$. Here are all nine possible combinations. Two result in a scalar, three in a vector, and four don't make any sense.

$$\begin{aligned}
&\boldsymbol{\nabla}(\boldsymbol{\nabla} T) \rightarrow ??\\
&\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) \rightarrow \text{scalar}\\
&\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) \rightarrow \text{vector}\\
&\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{h}) \rightarrow \text{vector}\\
&\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{h}) \rightarrow ??\\
&\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \cdot \boldsymbol{h}) \rightarrow ??\\
&\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \boldsymbol{h}) \rightarrow ??\\
&\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{h}) \rightarrow \text{scalar}\\
&\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{h}) \rightarrow \text{vector}
\end{aligned}$$

We dispense with the four expressions that don't make sense. The expression $\boldsymbol{\nabla}(\boldsymbol{\nabla}T)$ says to compute the gradient of gradient $\boldsymbol{\nabla}T$. This doesn't make sense because $\boldsymbol{\nabla}T$ is a vector and we can only compute the gradient of a scalar. The expression $\boldsymbol{\nabla}(\boldsymbol{\nabla}\times\boldsymbol{h})$ doesn't make sense for the same reason, because $(\boldsymbol{\nabla}\times\boldsymbol{h})$ is a vector. The expression $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{h})$ says to compute the dot product of vector $\boldsymbol{\nabla}$ and scalar $(\boldsymbol{\nabla}\cdot\boldsymbol{h})$. This doesn't make sense because we can only compute the dot product of two vectors. The expression $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\cdot\boldsymbol{h})$ says to compute the cross product of vector $\boldsymbol{\nabla}$ and scalar $(\boldsymbol{\nabla}\cdot\boldsymbol{h})$. This doesn't make sense because we can only compute the cross product of two vectors.

Out of the nine possible combinations, only five make sense. These are labeled (a) through (e) and summarized in the equation below. The implications of each expression is discussed in the remainder of this section.

$$\begin{array}{ll} \text{(a)} & \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}T\big) \\ \text{(b)} & \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}T\big) \\ \text{(c)} & \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\boldsymbol{h}\big) \\ \text{(d)} & \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{h}\big) \\ \text{(e)} & \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\boldsymbol{h}\big) \end{array} \tag{2.45}$$

Expression (b), $\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}T\big)$, is the curl of the gradient of T. The curl of any gradient is the zero vector $\mathbf{0}=(0,0,0)$.

$$\text{curl }\big(\text{grad }T\big)=\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}T\big)=\mathbf{0} \tag{2.46}$$

The reason is easily demonstrated. Consider the z component of the cross product. The two partial derivatives will always be identical to each other because of the identity in equation (2.8), so the subtraction will always result in 0.

$$\begin{aligned} \big[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}T)\big]_z &= \nabla_x(\boldsymbol{\nabla}T)_y-\nabla_y(\boldsymbol{\nabla}T)_x \\ &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right) \end{aligned} \tag{2.47}$$

Expression (d), $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{h})$, has the same form as equation (2.4). The proof is quick.

$$\boldsymbol{A}\cdot(\boldsymbol{A}\times\boldsymbol{B})=0 \tag{2.48}$$

We simply replace $\boldsymbol{A}$ with $\boldsymbol{\nabla}$ and $\boldsymbol{B}$ with $\boldsymbol{h}$. The divergence of the curl of any vector $\boldsymbol{h}$ is zero.

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{h})=\text{div }(\text{curl }\boldsymbol{h})=0 \tag{2.49}$$

Expressions (b) and (d) allow us to make two theorems. The first theorem is derived from equation (2.46) except that it uses the scalar ψ instead of the scalar T. The theorem turns things around a little. It says that if the curl of some vector A is zero, then A must be the gradient of some scalar ψ.

$$\begin{array}{ll} \textbf{Theorem}: & \\ \text{If} & \boldsymbol{\nabla} \times \boldsymbol{A} = 0 \\ \text{there is a} & \psi \\ \text{such that} & \boldsymbol{A} = \boldsymbol{\nabla}\psi \end{array} \tag{2.50}$$

The second theorem is derived from equation (2.49) except that it uses the vector $\boldsymbol{C}$ instead of the vector $\boldsymbol{h}$. Again, things are turned around a little. The theorem says that if the divergence of some vector D is zero, then D must be the curl of some vector $\boldsymbol{C}$.

$$\begin{array}{ll} \textbf{Theorem}: & \\ \text{If} & \boldsymbol{\nabla} \cdot \boldsymbol{D} = 0 \\ \text{there is a} & \boldsymbol{C} \\ \text{such that} & \boldsymbol{D} = \boldsymbol{\nabla} \times \boldsymbol{C} \end{array} \tag{2.51}$$

Expression (a), $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T)$, doesn't always result in 0, so we expand it. The three components of $\boldsymbol{\nabla}$ are ∇_x, ∇_y, and ∇_z. The three components of $\boldsymbol{\nabla} T$ are $(\nabla_x T)$, $(\nabla_y T)$, and $(\nabla_z T)$.

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) &= \nabla_x(\nabla_x T) + \nabla_y(\nabla_y T) + \nabla_z(\nabla_z T) \\ &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \end{aligned} \tag{2.52}$$

Because multiplication is associative, we can move the parentheses around. We then introduce a new symbol, ∇^2.

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) T = \nabla^2 T \tag{2.53}$$

The symbol ∇^2 is called the *Laplacian* and it's a scalar operator. Just like $\boldsymbol{\nabla}$, it's hungry for something to differentiate.

$$\text{Laplacian } = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.54}$$

Expression (e), $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{h})$, has the same form as equation (2.6).

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{2.55}$$

The second term on the right side, $\boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$, is the arithmetic product of vector $\boldsymbol{C}$ and scalar $\boldsymbol{A} \cdot \boldsymbol{B}$. Since multiplication is commutative, we can replace this term with $(\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$.

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C} \tag{2.56}$$

Now we replace both $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\boldsymbol{\nabla}$. We also replace $\boldsymbol{C}$ with $\boldsymbol{h}$.

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{h}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{h}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{h} \tag{2.57}$$

We can use the Laplacian in the second term on the right side.

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{h}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{h}) - \nabla^2 \boldsymbol{h} \tag{2.58}$$

Expression (c), $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot h)$, is just some vector. There isn't anything remarkable about it. We now summarize things. We also add expression (f), the Laplacian of vector $\boldsymbol{h}$. The vector $\nabla^2 \boldsymbol{h}$ has three components: $(\nabla^2 \boldsymbol{h}_x)$, $(\nabla^2 \boldsymbol{h}_y)$, and $(\nabla^2 \boldsymbol{h}_z)$.

$$\begin{array}{ll}
\text{(a)} & \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) = \nabla^2 T = \text{a scalar field} \\
\text{(b)} & \boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = (0, 0, 0) \\
\text{(c)} & \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{h}) = \text{a vector field} \\
\text{(d)} & \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{h}) = 0 \\
\text{(e)} & \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{h}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{h}) - \nabla^2 \boldsymbol{h} \\
\text{(f)} & (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{h} = \nabla^2 \boldsymbol{h} = \text{a vector field}
\end{array} \tag{2.59}$$

The dot product $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})$ makes sense and defines the Laplacian. But the cross product $(\boldsymbol{\nabla} \times \boldsymbol{\nabla})$ doesn't make sense, because of the identity in equation (2.3), $\boldsymbol{A} \times \boldsymbol{A} = 0$.

2.8 Pitfalls

There are two pitfalls associated with the use of $\boldsymbol{\nabla}$. The first concerns the cross-product of gradients. Consider the expression $(\boldsymbol{\nabla}\psi) \times (\boldsymbol{\nabla}\phi)$. We might be tempted to mentally replace each $\boldsymbol{\nabla}$ with an $\boldsymbol{A}$ so that we would have $(\boldsymbol{A}\psi) \times (\boldsymbol{A}\phi)$. This expression reduces to 0 by Equation (2.3), because the directions of $(\boldsymbol{A}\psi)$ and $(\boldsymbol{A}\phi)$ are the same. But this is wrong! We can't make this replacement. The directions of $(\boldsymbol{\nabla}\psi)$ and $(\boldsymbol{\nabla}\phi)$ are different. The lesson here is to think before mentally replacing $\boldsymbol{\nabla}$ with a vector.

The other pitfall concerns the polar coordinate system. We have a set of rules that work out nicely in a rectangular coordinate system. For example, the x component of $\nabla^2 \boldsymbol{h}$ is easy to write.

$$(\nabla^2 \boldsymbol{h})_x = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) h_x = \nabla^2 h_x \tag{2.60}$$

But in a polar coordinate system, the radial component of $\nabla^2 \boldsymbol{h}$ is not easy to write. It's not $\nabla^2 h_r$. It's something more complex. The lesson here is to stick to rectangular coordinate systems and avoid trouble.

Chapter 3

Vector Integral Calculus

3.1 Vector integrals; the line integral of $\nabla\Psi$

Everything in Chapter 2 can be summarized with one rule: the operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ are the three components of the vector operator ∇. In this chapter we explain the meanings of the divergence and curl operations, and we develop three important theorems.

We begin with the fundamental theorem of calculus for one variable. The function $f(x)$ can be imagined as a curve that hovers above and below the x-axis. The function $f'(x)$ is the slope of $f(x)$. That is, $f'(x) = df/dx$. The fundamental theorem states that, given two points a and b, the total difference between $f(a)$ and $f(b)$ can be computed by adding up all the small differences df between a and b, where each small difference df is the product of a slope $f'(x) = df/dx$ and a small distance dx.

$$f(b) - f(a) = \int_a^b f'(x)dx$$

We are given a scalar field $\psi(x, y, z)$. These are the values ψ of some scalar quantity (such as temperature) at each point (x, y, z) in some three-dimensional space (such as a block of concrete). We are also given a vector field $\nabla\psi(x, y, z)$. These are the slopes $(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z})$ of scalar field ψ at each point (x, y, z).

Consider Figure 3-1, where Γ is some curve in the scalar field $\psi(x, y, z)$ that starts at point $(1) = (x_1, y_1, z_1)$ and ends at point $(2) = (x_2, y_2, z_2)$. The vector $\boldsymbol{ds} = (\partial x, \partial y, \partial z)$ is an infinitesimal line element along the curve Γ. We may write the fundamental theorem of calculus in vector form. The total difference between $\psi(1)$ and $\psi(2)$ can be computed by adding up all the small differences $\partial\psi$ between (1) and (2), where each small difference $\partial\psi$ is the scalar product of a slope $\boldsymbol{\nabla}\psi = (\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z})$ and a small distance $\boldsymbol{ds} = (\partial x, \partial y, \partial z)$.

$$\psi(2) - \psi(1) = \int_{\substack{(1) \\ \text{along } \Gamma}}^{(2)} (\boldsymbol{\nabla}\psi) \cdot d\boldsymbol{s} \tag{3.1}$$

In Figure 3-2, $(\boldsymbol{\nabla}\psi)_t$ is the component of $\boldsymbol{\nabla}\psi$ in the direction of $\Delta\boldsymbol{s}_i$. Simple trigonometry gives us $(\boldsymbol{\nabla}\psi)_t = |\boldsymbol{\nabla}\psi|\cos\theta$, where θ is the angle between $\boldsymbol{\nabla}\psi$ and $(\boldsymbol{\nabla}\psi)_t$. Δs_i is the magnitude of the vector $\Delta\boldsymbol{s}_i$. I.e., $\Delta s_i = |\Delta\boldsymbol{s}_i|$. So we have $(\boldsymbol{\nabla}\psi)_t\Delta s = |\boldsymbol{\nabla}\psi||\Delta\boldsymbol{s}_i|\cos\theta$. Now we apply the identity $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = \boldsymbol{a}\cdot\boldsymbol{b}$ and drop the i subscripts to get

$$(\boldsymbol{\nabla}\psi)_t\,\Delta s = (\boldsymbol{\nabla}\psi)\cdot\Delta\boldsymbol{s} \tag{3.2}$$

The small interval vector $\Delta\boldsymbol{R} = (\Delta x, \Delta y, \Delta z)$ was introduced in section 2-3. We now take equation (2.15), where $\Delta T = \boldsymbol{\nabla}T\cdot\Delta\boldsymbol{R}$, and replace terms to get

$$\Delta\psi_1 = (\boldsymbol{\nabla}\psi)_1\cdot\Delta\boldsymbol{s}_1$$

In figure 3-2, point a is the first point after (1) along the curve Γ. It's obvious that $\Delta\psi_1 = \psi(a) - \psi(1)$. Combining these two equations for $\Delta\psi_1$ we have

$$\Delta\psi_1 = \psi(a) - \psi(1) = (\boldsymbol{\nabla}\psi)_1\cdot\Delta\boldsymbol{s}_1 \tag{3.3}$$

We do the same thing for $\Delta\psi_2$.

$$\Delta\psi_2 = \psi(b) - \psi(a) = (\boldsymbol{\nabla}\psi)_2\cdot\Delta\boldsymbol{s}_2 \tag{3.4}$$

We add equations (3.3) and (3.4) to get the sum $\Delta\psi_1 + \Delta\psi_2$. Notice that the $\psi(a)$ terms cancel out.

$$\psi(b) - \psi(1) = (\boldsymbol{\nabla}\psi)_1\cdot\Delta\boldsymbol{s}_1 + (\boldsymbol{\nabla}\psi)_2\cdot\Delta\boldsymbol{s}_2 \tag{3.5}$$

If we add up all of the segments along Γ we get

$$\psi(2) - \psi(1) = \sum(\boldsymbol{\nabla}\psi)_i\cdot\Delta\boldsymbol{s}_i \tag{3.6}$$

It's okay to drop the parentheses around $(\boldsymbol{\nabla}\psi)$.

$$(\boldsymbol{\nabla}\psi)\cdot d\boldsymbol{s} = \boldsymbol{\nabla}\psi\cdot d\boldsymbol{s} \tag{3.7}$$

Consider the sum in equation (3.6). If we let $\Delta \boldsymbol{s}_i$ approach 0 we can replace the summation sign $\sum$ with an integral sign $\int$. This theorem is correct for any curve Γ from (1) to (2).

FUNDAMENTAL THEOREM:

$$\psi(2) - \psi(1) = \underset{\text{any curve}}{\int_{(1)}^{(2)}} \boldsymbol{\nabla}\psi \cdot d\boldsymbol{s} \tag{3.8}$$

3.2 The flux of a vector field

Suppose we have some block of material, inside of which is some closed surface S which encloses the volume V. We wish to compute a rate called *heat flux*, the total amount of heat energy that flows through surface S per unit time. There are several ways to do this. One way is to compute the flow through an infinitesimal surface area da and then integrate over the entire surface S. Another way is to split the volume into two smaller volumes, compute the flow through each volume, and then take the sum.

We write da for the area of an infinitesimal surface element. And $\boldsymbol{n}$ is a unit vector that is orthogonal to da. We have a field of heat flux vectors $\boldsymbol{h}(x, y, z)$. One of these vectors points through da at some angle θ. Let h_n be the component of $\boldsymbol{h}$ that is parallel to $\boldsymbol{n}$.

$$h_n = \boldsymbol{h} \cdot \boldsymbol{n} \tag{3.9}$$

We express the heat flow through infinitesimal surface element da.

$$\boldsymbol{h} \cdot \boldsymbol{n}\, da \tag{3.10}$$

The total amount of heat flow through the surface S is simply the integral.

$$\text{Total heat flow outward through } S = \int_S \boldsymbol{h} \cdot \boldsymbol{n}\, da \tag{3.11}$$

We can generalize equation (3.11) for any vector field. For example, consider the electric field $\boldsymbol{E}(x, y, z)$ instead of the heat field $\boldsymbol{h}(x, y, z)$. We can replace vector $\boldsymbol{h}$ with vector $\boldsymbol{E}$.

$$\text{Flux of } \boldsymbol{E} \text{ through the surface } S = \int_S \boldsymbol{E} \cdot \boldsymbol{n}\, da \tag{3.12}$$

It is important to note that this definition holds for any surface S closed or open. We have just used it for closed surfaces; we will shortly use it for open surfaces.

Consider the situation where *heat is conserved*. Energy is neither generated nor absorbed inside the block of material, and Q is the amount of heat inside the surface S. The conservation of energy law tells us that the total heat flow through S is equal to the rate of heat loss from the inside.

$$\int_S \boldsymbol{h} \cdot \boldsymbol{n}\, da = -\frac{dQ}{dt} \tag{3.13}$$

Now we shall demonstrate an interesting fact, the *principle of superposition* for fluxes in any vector field $\boldsymbol{C}$. Consider the volume V in Figure 3-4 which is cut into two smaller volumes V_1 and V_2. The outer surface S is the sum of the two smaller outer surfaces S_a and S_b. Both smaller volumes have the inner surface S_{ab} in common.

Surface S_1 is the sum of outer surface S_a and inner surface S_{ab}. The flux through surface S_1 is the sum of the fluxes through S_a and S_{ab}. The unit vector $\boldsymbol{n}_1$ points *out* of the volume V_1.

$$\text{Flux through } S_1 = \int_{S_a} \boldsymbol{C} \cdot \boldsymbol{n}\, da + \int_{S_{ab}} \boldsymbol{C} \cdot \boldsymbol{n}_1\, da \tag{3.14}$$

Surface S_2 is the sum of outer surface S_b and inner surface S_{ab}. The flux through surface S_2 is the sum of the fluxes through S_b and S_{ab}. The unit vector $\boldsymbol{n}_2$ points *out* of the volume V_2.

$$\text{Flux through } S_2 = \int_{S_b} \boldsymbol{C} \cdot \boldsymbol{n}\, da + \int_{S_{ab}} \boldsymbol{C} \cdot \boldsymbol{n}_2\, da \tag{3.15}$$

Here's what makes the principle of superpositon hold for fluxes. The two unit vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ point in opposite directions. That is, $\boldsymbol{n}_1 = -\boldsymbol{n}_2$. This makes the sum of the two fluxes through the inner surface S_{ab} equal to zero.

$$\int_{S_{ab}} \boldsymbol{C} \cdot \boldsymbol{n}_1\, da = -\int_{S_{ab}} \boldsymbol{C} \cdot \boldsymbol{n}_2\, da \tag{3.16}$$

3.3 The flux from a cube; Gauss' theorem

We shall develop an interesting identity called Gauss' Theorem. It expresses the flux of a vector field $\boldsymbol{C}$ through a surface S. To begin, we describe the flux through each face of a cube and then sum the six fluxes.

Consider the cube in Figure 3-5. The six faces are marked 1 through 6. Face 1 has an area $\Delta y \Delta z$. Since the cube is very small, we'll assume that the x-component of $\boldsymbol{C}$ is the same through each point in face 1. We call it $-C_x(1)$. We want the *outward* value of the x-component, but since the x-component of $\boldsymbol{C}$ points *into* the cube, we need the negative sign. We follow this logic for each of the six faces.

$$\begin{aligned}
\text{Flux out of 1} &= -C_x(1)\Delta y \Delta z \\
\text{Flux out of 2} &= +C_x(2)\Delta y \Delta z \\
\text{Flux out of 3} &= -C_y(3)\Delta x \Delta z \\
\text{Flux out of 4} &= +C_y(4)\Delta x \Delta z \\
\text{Flux out of 5} &= -C_z(5)\Delta x \Delta y \\
\text{Flux out of 6} &= +C_z(6)\Delta x \Delta y
\end{aligned}$$

Now the total flux out of the cube is simply the sum of these six fluxes. We group similar terms to get a sum of three differences.

$$\begin{aligned}
\text{Total flux} = & \big[C_x(2) - C_x(1)\big]\Delta y \Delta z + \\
& \big[C_y(4) - C_y(3)\big]\Delta x \Delta z + \\
& \big[C_x(6) - C_x(5)\big]\Delta x \Delta y
\end{aligned}$$

Since the cube is very small, we can replace the differences with very small numbers. For example, the difference between the two components $C_x(1)$ and $C_x(2)$ is the rate of change of $\boldsymbol{C}$ in the x-direction, which is $\partial C_x / \partial x$, times the distance between face 1 and face 2, which is Δx. We follow this logic for all three differences.

$$\begin{aligned}
C_x(2) - C_x(1) &= \frac{\partial C_x}{\partial x}\Delta x \\
C_y(4) - C_y(3) &= \frac{\partial C_y}{\partial y}\Delta y \\
C_z(6) - C_z(5) &= \frac{\partial C_z}{\partial z}\Delta z
\end{aligned}$$

Now we take the total flux equation and replace the three differences with the three derivatives and group them together. The sum of the derivatives is the divergence ($\boldsymbol{\nabla} \cdot \boldsymbol{C}$). And the product of the deltas is ΔV.

$$\begin{aligned}
\text{Total flux} &= \left(\frac{\partial C_x}{\partial x}\Delta x\right)\Delta y \Delta z + \left(\frac{\partial C_y}{\partial y}\Delta y\right)\Delta x \Delta z + \left(\frac{\partial C_z}{\partial z}\Delta z\right)\Delta x \Delta y \\
&= \left(\frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z}\right)\Delta x \Delta y \Delta z \\
&= \Big(\boldsymbol{\nabla} \cdot \boldsymbol{C}\Big)\Delta V
\end{aligned}$$

Now we can write an identity for the flux from a cube. We can say that *for an infinitesimal cube*

$$\int_{\text{cube}} \boldsymbol{C} \cdot \boldsymbol{n} \, da = \Big(\boldsymbol{\nabla} \cdot \boldsymbol{C}\Big) \Delta V \tag{3.17}$$

Now we apply the fact that we proved in Section 3-2. We subdivide some volume V into an infinite number of infinitesimal cubes and compute the flux of each. The flux of $\boldsymbol{C}$ across the surface S is the divergence of $\boldsymbol{C}$ through the entire volume V. This identity works for *any* closed surface S and volume V.

GAUSS' THEOREM:

$$\int_S \boldsymbol{C} \cdot \boldsymbol{n} \, da = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{C} \, dV \tag{3.18}$$

3.4 Heat conduction; the diffusion equation

We develop the *heat diffusion equation*. It is a differential equation in x, y, z, and t for temperature T. We begin with an equation for the "heat out" of a cube that is cooling off. We take equation (3.17) and replace vector $\boldsymbol{C}$ with heat flux vector $\boldsymbol{h}$.

$$\text{Heat out} = \int_{\text{cube}} \boldsymbol{h} \cdot \boldsymbol{n} \, da = \Big(\boldsymbol{\nabla} \cdot \boldsymbol{h}\Big) \Delta V \tag{3.19}$$

Next, we write an equation for the "heat lost" from the inside of the cube. There are neither heat sources nor heat sinks inside the cube. If the heat per unit volume is q and the volume of the cube is ΔV, then the heat lost is simply the time derivative of the total heat $q \, \Delta V$.

$$\text{Heat lost} = -\frac{d}{dt}(q \, \Delta V) = -\frac{dq}{dt} \, \Delta V \tag{3.20}$$

The conservation of energy law tells us that the heat lost from inside the cube is exactly the same as the heat out of the cube. This gives us a *differential* form that appears often in physics.

$$\begin{aligned} \text{Heat lost} &= \text{Heat out} \\ -\frac{dq}{dt} \, \Delta V &= \boldsymbol{\nabla} \cdot \boldsymbol{h} \, \Delta V \\ -\frac{dq}{dt} &= \boldsymbol{\nabla} \cdot \boldsymbol{h} \end{aligned} \tag{3.21}$$

We can also derive the *integral* form of equation (3.13). We start with Gauss' Theorem and replace vector $\boldsymbol{C}$ with heat flux vector $\boldsymbol{h}$.

$$\int_S \boldsymbol{h} \cdot \boldsymbol{n} \, da = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{h} \, dV \tag{3.22}$$

We replace the $\boldsymbol{\nabla} \cdot \boldsymbol{h}$ in the right-hand integral with $-dq/dt$ and then simplify using $\int dV = \Delta V$ and $Q = q\,\Delta V$.

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{h}\, dV = \int_V \Big(-\frac{dq}{dt} \Big)\, dV = -\frac{dQ}{dt}$$

We now consider a different case, where heat energy is generated at some point P inside a block of material. Let W represent the energy liberated per second at that point. We wish to describe the heat vector field near P. Our "physical intuition" tells us that $\boldsymbol{h}$ is radial. That is, each vector $\boldsymbol{h}$ points away from P. We imagine a spherical surface S inside the material, centered around P and with radius R. This allows us to write a simple equation for the heat flux, where $4\pi R^2$ is the area of the sphere. Note that the dot on the left-hand side is the dot product operator whereas the dot on the right-hand side is the multiplication operator. And h is the magnitude of $\boldsymbol{h}$.

$$\int_S \boldsymbol{h} \cdot \boldsymbol{n}\, da = h \cdot 4\pi R^2 \tag{3.23}$$

The right-hand side is equal to W so we can specify magnitude h in terms of W. Since both the heat vector $\boldsymbol{h}$ and the unit vector $\boldsymbol{e}_r$ are radial, we can multiply the unit vector $\boldsymbol{e}_r$ by magnitude h to get an equation for vector $\boldsymbol{h}$.

$$\begin{aligned} h &= \frac{W}{4\pi R^2} \\ \boldsymbol{h} &= \frac{W}{4\pi R^2}\boldsymbol{e}_r \end{aligned} \tag{3.24}$$

We wish to find an equation for the most general kind of heat flow, with only the condition that heat is conserved. We start with equation (2.44), which specifies the differential equation for heat conduction. Remember that heat flows "downhill" (hence the minus sign) and κ is the thermal conductivity constant.

$$\boldsymbol{h} = -\kappa\, \boldsymbol{\nabla} T \tag{3.25}$$

Now we take equation (3.21) and multiply both sides by -1. Then we use equation (3.25) to replace $\boldsymbol{h}$ with $-\kappa\, \boldsymbol{\nabla} T$. Finally, we rearrange, cancel the two negative signs, and replace $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ with the Laplacian operator ∇^2.

$$\begin{aligned} \frac{dq}{dt} &= -\boldsymbol{\nabla} \cdot \boldsymbol{h} \\ &= -\boldsymbol{\nabla} \cdot (-\kappa\, \boldsymbol{\nabla} T) \\ &= \kappa\, \nabla^2 T \end{aligned} \tag{3.26}$$

We now make the assumption that a small change in the heat energy of the material is proportional to a small change in the temperature of the material, by some constant c_v. That is, $\Delta q = c_v\ \Delta T$. We differentiate both sides with respect to time.

$$\frac{dq}{dt} = c_v \frac{dT}{dt} \tag{3.27}$$

We replace dq/dt with $(\kappa\ \nabla^2 T)$ and rearrange. We may also replace the constant κ/c_v with the diffusion symbol D.

$$\begin{aligned} \frac{dT}{dt} &= \frac{\kappa}{c_v}\nabla^2 T & (3.28) \\ &= D\ \nabla^2 T & (3.29) \end{aligned}$$

This is the *heat diffusion equation.* It is a differential equation in x, y, z, and t for temperature T. We expand it here to show everything.

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_v}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

3.5 The circulation of a vector field

If $\boldsymbol{C}$ is any vector field, we take its component along a curved line and take the integral of this component all the way around a complete loop. The integral is called the *circulation* of the vector field around the loop.

Consider Figure 3-7 with vector field $\boldsymbol{C}$ and closed loop Γ. The tangential component of $\boldsymbol{C}$ at any point along the loop is C_t. The closed loop consists of an infinite number of infinitesimal vector segments $d\boldsymbol{s} = (ds_x, ds_y, ds_z)$. The magnitude of $d\boldsymbol{s}$ at any point along the loop is ds.

Given the two vectors $\boldsymbol{C}$ and $d\boldsymbol{s}$, their dot product is $|\boldsymbol{C}||d\boldsymbol{s}|\cos\theta$. Now the tangential component $C_t = |\boldsymbol{C}|\cos\theta$. And the magnitude $ds = |d\boldsymbol{s}|$. Therefore we have the following identity for the integral all the way around Γ. The circle on the integral sign means that the integral is taken around the entire loop.

$$\oint_\Gamma C_t\ ds = \oint_\Gamma \boldsymbol{C} \cdot d\boldsymbol{s} \tag{3.30}$$

Consider Figure 3-8. We split Γ into two smaller loops Γ_1 and Γ_2. Both of these smaller loops have the segment Γ_{ab} in common. However, the directions of $d\boldsymbol{s}_1$ and $d\boldsymbol{s}_2$ are opposite each other. When we compute the circulations around the smaller loops and then add them together, the two partial circulations for Γ_{ab} cancel each other, so that the sum is the same as the circulation around the big loop Γ.

Now consider Figure 3-9. Here we have some arbitrary surface bounded by a closed loop Γ. We subdivide the surface into an infinite number of infinitesimal squares. And since the squares are so small, we may make the assumption that each square is flat, even though the surface itself may not be flat at all. So now we can calculate the circulation around Γ. It is the sum of the circulations around each of the infinitesimal squares.

3.6 The circulation around a square; Stokes' theorem

We wish to derive an equation for the circulation around a square. We'll make some assumptions in order to simplify the derivation. The surface in Figure 3-9 may not be flat, but each square is so small that we can assume it to be flat. We'll also assume that our square lies in a plane which is orthogonal to one of the Cartesian axes, say, the xy plane. Finally, we'll assume that the sides of our square are parallel to the x and y axes.

Recall that a vector field $\boldsymbol{C}$ can be thought of as the gradient of a scalar field, e.g. temperature T. Each component of the vector specifies a rate of change in one of three directions. In the case of temperature, the units for each component might be degrees per centimeter.

$$\boldsymbol{C} = \boldsymbol{\nabla} T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

Consider the square in Figure 3-10. The circulation is expressed as the sum of four terms. Each term is the product of a tangential component of $\boldsymbol{C}$ and the length of a side.

$$\oint_\Gamma \boldsymbol{C} \cdot d\boldsymbol{s} = C_x(1)\Delta x + C_y(2)\Delta y - C_x(3)\Delta x - C_y(4)\Delta y \tag{3.31}$$

We combine the two terms that contain Δx.

$$\left[C_x(1) - C_x(3)\right] \Delta x \tag{3.32}$$

The distance between side 1 and side 3 is small, and therefore the difference between $C_x(1)$ and $C_x(3)$ is small. But it's not zero. The two sides are separated in the y direction by Δy. The rate of change between the two C_x values is $\partial C_x/\partial y$. Therefore we have

$$C_x(3) = C_x(1) + \frac{\partial C_x}{\partial y} \Delta y \tag{3.33}$$

We replace the two Δx terms in equation (3.31) with this difference.

$$[C_x(1) - C_x(3)]\Delta x = -\frac{\partial C_x}{\partial y}\ \Delta x\ \Delta y \tag{3.34}$$

We apply the same logic to replace the two Δy terms.

$$C_y(2)\Delta y - C_y(4)\Delta y = \frac{\partial C_y}{\partial x}\ \Delta x\ \Delta y \tag{3.35}$$

This gives us the product of a difference and an area.

$$\left(\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right)\Delta x\ \Delta y \tag{3.36}$$

The difference term just happens to be the z component of the cross-product $(\boldsymbol{\nabla} \times \boldsymbol{C})$. And we can let area $\Delta a = \Delta x \Delta y$. This gives us

$$(\boldsymbol{\nabla} \times \boldsymbol{C})_z \Delta a$$

Now we'll eliminate the Cartesian coordinates altogether. In Figure 3-10, the z-axis is normal to our square in the xy plane. But if our square were not in the xy plane, it would still have a normal axis n. So we can replace the z and simply use n instead. This gives us

$$(\boldsymbol{\nabla} \times \boldsymbol{C})_n \Delta a$$

We replace the four terms with the single cross-product term.

$$\oint \boldsymbol{C} \cdot d\boldsymbol{s} = (\boldsymbol{\nabla} \times \boldsymbol{C})_n\ \Delta a = (\boldsymbol{\nabla} \times \boldsymbol{C}) \cdot \boldsymbol{n}\ \Delta a \tag{3.37}$$

Finally, we take the closed loop Γ, fill it in with some surface, subdivide that surface into an infinite number of infinitesimal squares, compute the circulation around each square, and then take the sum.

Stokes' Theorem:

$$\oint_\Gamma \boldsymbol{C} \cdot d\boldsymbol{s} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{C})_n\ da \tag{3.38}$$

where S is any surface bounded by Γ.

3.7 Curl-free and divergence-free fields

Consider Figure 3-13 with small loop Γ and large surface S. We now shrink the loop to nothing and create a *closed surface* — a surface with no boundry. According to Stokes' theorem, if the circulation around a loop Γ is zero, then the surface integral of $(\boldsymbol{\nabla} \times \boldsymbol{C})_n$ must also be zero. So we have a new theorem for the special case where S is a closed surface.

$$\int_{\substack{\text{any closed}\\\text{surface}}} (\boldsymbol{\nabla} \times \boldsymbol{C})_n \, da = 0 \tag{3.39}$$

Remember that $(\boldsymbol{\nabla} \times \boldsymbol{C})_n$ is the same as $(\boldsymbol{\nabla} \times \boldsymbol{C}) \cdot \boldsymbol{n}$. Now we take Gauss' theorem and replace each $\boldsymbol{C}$ with $(\boldsymbol{\nabla} \times \boldsymbol{C})$. We also replace $(\boldsymbol{\nabla} \times \boldsymbol{C}) \cdot \boldsymbol{n}$ on the left side with $(\boldsymbol{\nabla} \times \boldsymbol{C})_n$. This gives us

$$\int_{\substack{\text{closed}\\\text{surface}}} (\boldsymbol{\nabla} \times \boldsymbol{C})_n \, da = \int_{\substack{\text{volume}\\\text{inside}}} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{C}) \, dV \tag{3.40}$$

The expression on the left side is the same as in equation (3.39), which is zero. Therefore the expression on the right side must also be zero. The integrand $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{C})$ is 0, just as equation (2.49) states.

$$\int_{\substack{\text{any}\\\text{volume}}} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{C}) \, dV = 0 \tag{3.41}$$

3.8 Summary

The $\boldsymbol{\nabla}$ Operator:
The operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ can be considered as the three components of a vector operator $\boldsymbol{\nabla}$, and the formulas which result from vector algebra by treating this operator as a vector are correct.

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

The Fundamental Theorem of Calculus:
The difference of the values of a scalar field at two points is equal to the line integral of the tangential component of the gradient of that scalar along any curve at all between the first and second points.

$$\psi(2) - \psi(1) = \int_{\substack{(1)\\\text{any curve}}}^{(2)} \boldsymbol{\nabla}\psi \cdot d\boldsymbol{s} \tag{3.42}$$

Gauss' Theorem:
The surface integral of the normal component of an arbitrary vector over a closed surface is equal to the integral of the divergence of the vector over the volume interior to the surface.

$$\int\limits_{\substack{\text{closed}\\ \text{surface}}} \boldsymbol{C}\cdot\boldsymbol{n}\, da = \int\limits_{\substack{\text{volume}\\ \text{inside}}} \boldsymbol{\nabla}\cdot\boldsymbol{C}\, dV \tag{3.43}$$

Stokes' Theorem:
The line integral of the tangential component of an arbitrary vector around a closed loop is equal to the surface integral of the normal component of the curl of that vector over any surface which is bounded by the loop.

$$\oint\limits_{\text{boundry}} \boldsymbol{C}\cdot d\boldsymbol{s} = \int\limits_{\text{surface}} (\boldsymbol{\nabla}\times\boldsymbol{C})\cdot\boldsymbol{n}\, da \tag{3.44}$$