Math 251 Sections 1,2,10 FAQ 8/26/09

1. Suppose that y is an UNKNOWN function of t. Some of the following indefinite integrals can be expressed by a formula involving y and possibly the variable t and some cannot. Find this formula in the cases where it is possible.

$$\int y\,dt \qquad \int y^{-2}y'\,dt \qquad \int e^y\,dt \qquad \int \frac{y'}{y}\,dt$$

ANS. Using simple substitution :

$$\int y^{-2}y'\,dt = \frac{-1}{y} + C \qquad \int \frac{y'}{y}\,dt = \ln y + C$$

Since a factor of y' in the other two integrands, this substitution cannot be performed and hence there is no way to perform the integration without additional knowledge concerning the function y.

2. For each following DE's determine its order and determine if it is a PDE or an ODE.

$$\sin t = \left(\frac{dy}{dt}\right)^2 \qquad y' = 2y'' \qquad \frac{d(y')}{dt} + y^3 = y \qquad \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial t \partial y} \qquad y + \frac{1}{y} = y^2 \qquad f_t f_y = (f_t)^2 + (f_y)^2$$

ANS.
$\sin t = \left(\frac{dy}{dt}\right)^2$ is a 1st order ODE
$y' = 2y''$ is a 2nd order ODE
$\frac{d(y')}{dt} + y^3 = y$ is a 2nd order ODE
$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial t \partial y}$ is a 2nd order PDE
$y + \frac{1}{y} = y^2$ is not a DE at all
$f_t f_y = (f_t)^2 + (f_y)^2$ is a first order PDE

3. Find a formula for the general solution of the differential equation:

$$y' + 2 = 3y$$

ANS. This is a separable ODE: $y' = 3y - 2$. So we "Divide and Integrate":

$$\int \frac{y'}{3y-2}\,dt = \int 1\,dt$$

$$\frac{1}{3}\ln|3y-2| = t + C$$

$$|3y-2|^{1/3} = e^{t+C} = e^C e^t$$

$$|3y-2| = e^{3C}e^{3t}$$

$$3y = 2 + \pm e^{3C}e^{3t}$$

$$y = \frac{2}{3} + \pm e^{3C}e^{3t} = \frac{2}{3} + C_1 e^{3t}$$

Here C_1 cannot be zero! Since $y = 2/3$ is a solution to this ODE, which is lost as a result of the division in the first step, we allow C_1 to be zero after all in order to incorporate the "lost solution" in to the final formula for the solutions to this ODE:

$$y = \frac{2}{3} + \pm C_1 e^{3t}$$

Now let's try this procedure the book's way, i.e., divide the separable ODE by $y - \frac{2}{3}$:

$$\int \frac{y'}{y - \frac{2}{3}}\,dt = \int 3\,dt$$

$$|y-\frac{2}{3}|=e^{3t+C}$$
$$y-\frac{2}{3}=\pm e^{3C}e^{3t}$$
$$y=\frac{2}{3}+C_1e^{3t}$$

Which agrees with the first calculation of the solution.

4. Find the solution of the above equation which satisfies the initial condition $y(0)=4$ and determine $\lim_{t\to\infty}y(t)$.

ANS. We plug in $t=0$ and $y=4$ and see that

$$y=\frac{2}{3}+\frac{10}{3}e^{3t}$$

Oviously the $\lim_{t\to\infty}y(t)=\infty$, which means that the limit does not exist because the function $y(t)$ becomes progressively larger without bound at the variable t becomes progressively larger.

5. Solve the initial value problem: $y'=\frac{9t^2}{2y}$, $y(0)=-9$. What is the domain of the solution?

ANS. Multiply both sides by the function of $2y$:

$$2yy'=9t^2$$

Integrate both sides with respect to t:

$$\int 2yy'\,dt=\int 9t^2dt$$

The general solution is

$$y^2=3t^3+C$$

Plugging in the initial condition gives: $81=C$. This solution is simple enough to solve explicitly for y:

$$y=-\sqrt{3t^3+81}$$

The domain of this solution is the set of real numbers t for which $3t^3+81\geq 0$, i.e., all real numbers.

6. Find the general solution of the differential equation

$$y'=\frac{ty}{y^2+1}$$

ANS. Express right hand side as function of t multiplied by a function of y.

$$y'=t\frac{y}{y^2+1}$$

Divide both sides by the function of y:

$$\frac{y^2+1}{y}y'=t$$

Integrate both sides with respect to t:

$$\int\frac{y^2+1}{y}y'\,dt=\int t\,dt$$

Algebraically simplify the integrand involving y:

$$\int(y+y^{-1})y'\,dt=\int t\,dt$$

$$\frac{1}{2}y^2+\ln|y|=\frac{1}{2}t^2+C$$

It appears that this equation involving y and t cannot be solved for y in terms of t. The equation is an implicit representation of y as a function of t. To write the entire general solution to the differential equation you must go back and check what assumption was made in the step involving division. The assumption was $y\neq 0$. But clearly $y=0$ solves the equation. So the general solution consists of two formulas:

$$\frac{1}{2}y^2+\ln|y|=\frac{1}{2}t^2+C \qquad \text{and} \qquad y=0$$