

This is Problem 4 from yesterday's FAQ.

4. Find the general solution to the ODE: $t^2 y'' + 3ty' + y = 0$.

ANS. The indicial polynomial is $r(r-1) + 3r + 1$ which has a double root $r_1 = -1$. Thus we see that $y_1 = t^{-1}$ solves this ODE. However, at this point we either remember that multiplying the solution y_1 by $\ln t$ in the double roots case gives a second solution y_2 or we can go through the procedure using Abel's formula and the definition of Wronskian to get a simple first order linear ODE for the second solution.

Let's repeat the latter procedure instead of relying on our memories. For the purpose of using Abel's formula we need to rewrite the ODE so that the coefficient of y'' is 1:

$$y'' + 3t^{-1}y' + t^{-2}y = 0$$

Then $p = 3t^{-1}$ and by Abel's formula:

$$W = \exp\left(-\int 3t^{-1} dt\right) = C_1 t^{-3}$$

We choose $C_1 = 1$ because any single second solution not a multiple of the first suffices.

Now according to the definition of the Wronskian of $y_1 = t^{-1}$ and y_2 is

$$W(t, y_2) = \det \begin{pmatrix} t^{-1} & y_2 \\ -t^{-2} & y_2' \end{pmatrix} = t^{-1}y_2' + t^{-2}y_2 = t^{-3}$$

This gives an easily solved first order linear ODE for y_2 :

$$y_2' + \frac{1}{t}y_2 = t^{-2}$$

The integrating factor for this linear ODE is $\mu = t$ Therefore

$$(y_2 t)' = t^{-1}$$

We have

$$y_2 = \ln t \, t^{-1} + C t^{-1}$$

or simply $y_2 = \ln t \, t^{-1}$ and the general solution of the original ODE is

$$y = c_1 \ln t \, t^{-1} + c_2 t^{-1}$$