1. Find a particular solution to $y'' - 2y' - 3y = t^2$

ANS. We guess that $y_p = At^2 + Bt + C$ will solve the equation. Plugging in:

$$\begin{aligned} -3(y_p &= At^2 + Bt + C) \\ -2(y_p' &= 2At + B) \\ 1(y_p'' &= 2A) \end{aligned}$$

Therefore,

$$\begin{aligned} -3A &= 1 \\ -3B - 4A &= 0 \\ 2A - 2B - 3C &= 0 \end{aligned}$$

We see that $y_p = \frac{-1}{3}t^2 + \frac{4}{9}t + \frac{1}{3}(\frac{-2}{3} + \frac{-8}{9}) = \frac{-1}{3}t^2 + \frac{4}{9}t - \frac{14}{27}$.

2. Find the solution to $y'' - 2y' - 3y = 2e^t$ which solves the IVP: $y(0) = 1,\ y'(0) = 0$

ANS. We first find a particular solution $y_p = Ae^t$ that solve this nonhomogeneous ODE. Then we find the complementary solution y_c which is the general solution of the associated homogeneous ODE. Adding these two together gives the general solution to the nonhomogeneous ODE: $y = y_c + y_p = c_1e^{3t} + c_2e^{-t} + y_p$. Plugging in:

$$\begin{aligned} -3(y_p &= Ae^t) \\ -2(y_p' &= Ae^t) \\ 1(y_p'' &= Ae^t) \end{aligned}$$

Therefore, $(-3A - 2A + A)e^t = 2e^t$ and hence $A = \frac{-1}{2}$ and $y_p = \frac{-1}{2}e^t$
Therefore $y = c_1e^{3t} + c_2e^{-t} - \frac{1}{2}e^t$ and $y' = 3c_1e^{3t} - c_2e^{-t} - \frac{1}{2}e^t$ Plugging in $t = 0,\ y = 1,\ y' = 0$ gives
$1 = c_1 + c_2 - 1/2$ and
$0 = 3c_1 - c_2 - 1/2$
Adding these gives $1 = 4c_1 - 1$, ie, $c_1 = 1/2$ and then $c_2 = 1$.
The solutions is $y = \frac{1}{2}e^{3t} + e^{-t} - \frac{1}{2}e^t$

3. Find a particular solution to $y'' - 2y' - 3y = (t + 3)e^{2t}$

ANS. We expect a solution of the form $y_p = (At + B)e^{2t}$ Plugging in:

$$\begin{aligned} -3(y_p &= (At + B)e^{2t}) \\ -2(y_p' &= (A + 2(B + At))e^{2t})) \\ 1(y_p'' &= 2A + 2(A + 2(B + At))e^{2t}) \end{aligned}$$

Setting all terms involving the factor t equal gives $-3A - 4A + 4A = 1$ and hence $A = -1/3$. Now all terms not involving the factor t must add up to two. Hence $-3B - 2A - 4B + 2A + 2A + 4B = 3$. This means that $3 = -3B + 2A$, ie, $3B = 2A - 3 = -2/3 - 3$. So $B = -11/9$. Therefore, $y_p = \frac{1}{9}(-3t + 11)e^{2t}$