Math 251 Sections 1,2,10 FAQ 11/13/09

1. Consider the function

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad 1 \le x < 3 \end{cases}$$

Find a sine series for this function on interval $[0,3]$. What is $\lim_{n\to\infty} s_n(-1)$? What is $\lim_{n\to\infty} s_n(-1/2)$?

ANS. The sine series is the Fourier series of the odd extension of $f(x)$ to the full interval $[-3,3]$.

$$f_o(x) = \begin{cases} 0 & \text{if} \quad -3 \le x < -1 \\ -1 & \text{if} \quad -1 \le x < 0 \\ 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad 1 \le x < 3 \end{cases}$$

The Fourier series of $f_o(x)$ on $[-3,3]$ contains only sine terms?

$$\begin{aligned} b_n &= \frac{1}{3}\int_{-3}^{3} f_o(x)\sin\left(\frac{n\pi}{3}x\right)\,dx = \frac{2}{3}\int_{0}^{3} f(x)\sin\left(\frac{n\pi}{3}x\right)\,dx \\ &= \frac{2}{3}\int_{0}^{1} 1\sin\left(\frac{n\pi}{3}x\right)\,dx = -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{3}x\right)\right]_0^1 = -\frac{2}{n\pi}(\cos\left(\frac{n\pi}{3}\right) - 1) \\ b_1 &= \frac{2}{1\pi}\frac{1}{2}, \quad b_2 = \frac{2}{2\pi}\frac{3}{2}, \quad b_3 = \frac{2}{3\pi}2, \end{aligned}$$

The first three terms of the sine series is:

$$\frac{1}{\pi}\sin\left(\frac{1\pi}{3}x\right) + \frac{3}{2\pi}\sin\left(\frac{2\pi}{3}x\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi}{3}x\right) + \dots$$

If f_o is defined as required by the Fourier Convergence Theorem then we obtain $\lim_{n\to\infty} s_n(-1) = -1/2$ and $\lim_{n\to\infty} s_n(-1/2) = -1$.

2. Find a cosine series for the function in Problem 1 on interval $[0,3]$. What is $\lim_{n\to\infty} s_n(-1)$? What is $\lim_{n\to\infty} s_n(-1/2)$?

ANS. The cosine series is the Fourier series of the even extension of $f(x)$ to the full interval $[-3,3]$. It is the Fourier series of

$$f_e(x) = \begin{cases} 0 & \text{if} \quad -3 \le x < -1 \\ 1 & \text{if} \quad -1 \le x < 1 \\ 0 & \text{if} \quad 1 \le x < 3 \end{cases}$$

on $[-3,3]$?

$$a_0 = \frac{1}{3}\int_{-3}^{3} f_e(x)\,dx = \frac{2}{3}\int_{0}^{3} f(x)\,dx = \frac{2}{3}$$

$$\begin{aligned} a_n &= \frac{1}{3}\int_{-3}^{3} f_e(x)\cos\left(\frac{n\pi}{3}x\right)\,dx = \frac{2}{3}\int_{0}^{3} f(x)\cos\left(\frac{n\pi}{3}x\right)\,dx \\ &= \frac{2}{3}\int_{0}^{1} 1\cos\left(\frac{n\pi}{3}x\right)\,dx = \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{3}x\right)\right]_0^1 \\ &= \frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right) \\ a_1 &= \frac{1}{1\pi}\sqrt{3}, \qquad a_2 = \frac{1}{2\pi}\sqrt{3}, \qquad a_3 = 0 \end{aligned}$$

The first four terms of the cosine series is:

$$\frac{1}{3} + \frac{1}{\pi}\sqrt{3}\cos\left(\frac{1\pi}{3}x\right) + \frac{1}{2\pi}\sqrt{3}\cos\left(\frac{1\pi}{3}x\right) + 0\cos\left(\frac{3\pi}{3}x\right) + \dots$$

If we define f_e as required by the Fourier Convergence Theorem we obtain $\lim_{n\to\infty} s_n(-1) = 1/2$ and $\lim_{n\to\infty} s_n(-1/2) = 1$.

3. Find a sine series for $f(x) = x$ on $[0, 2]$. What is $\lim_{n\to\infty} s_n(2)$?

ANS. Since x is odd on $[-2, 2]$, the Fourier series we found for x on $[-2, 2]$ is already a sine series. Nothing needs to be done. From the periodicity of f_o we see that $\lim_{n\to\infty} s_n(2) = 0$?

4. Find a cosine series for $f(x) = x$ on $[0, 2]$ What is $\lim_{n\to\infty} s_n(-2)$?

ANS.

$$a_0 = \frac{1}{2}\int_{-2}^{2} f_e(x)\,dx = \int_0^2 x\,dx = 2$$

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f_e(x)\cos\left(\frac{n\pi}{2}x\right)\,dx \\
&= \int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right)\,dx \\
&= \int_0^2 x\cos\left(\frac{n\pi}{2}x\right)\,dx \\
&= \frac{2}{n\pi}\left[x\sin\left(\frac{n\pi}{2}x\right)\right]_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi}{2}x\right)\,dx \\
&= 0 - \left(\frac{2}{n\pi}\right)^2\left[\cos\left(\frac{n\pi}{2}x\right)\right]_0^2 \\
&= \frac{2^2}{n^2\pi^2}\left(\cos(n\pi) - 1\right) \\
a_1 &= 2\frac{2^2}{1^2\pi^2} \\
a_2 &= 0 \\
a_3 &= 2\frac{2^2}{3^2\pi^2}
\end{aligned}$$

From the periodicity of f_e we see that $\lim_{n\to\infty} s_n(-2) = 2$?

5. In the problems above we have seen that a p.w. continuous function defined on $[0, L]$ has a sine series and a cosine series. This seems to contradict the fact that the Fourier series of a function is unique. Explain why it does not.

ANS. A function on $[-L, L]$ has a unique Fourier series on $[-L, L]$.
The sine series is the Fourier series of the odd extension f_o to $[-L, L]$.
The cosine series is the Fourier series of the even extension to $[-L, L]$. The even and odd extensions are not the same function on $[-L, L]$.
So there is no contradiction!