1. Consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 & \text{if} \quad 1 \le x < 2 \\ 0 & \text{if} \quad 2 \le x \end{cases}$$

Find the sine series of $f(x)$ and the cosine series for $f(x)$ on the interval $[0, 3]$
Use summation notation to express your answer.

ANS.

$$a_0 = \frac{1}{3} \quad \text{area} \quad \text{below} \quad f_e(x) = \frac{2}{3}$$

$$\begin{aligned}
a_n &= \frac{1}{3}\int_{-3}^{3} f_e(x)\cos\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{2}{3}\int_{0}^{3} f(x)\cos\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{2}{3}\int_{1}^{2} f(x)\cos\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{2}{3}\int_{1}^{2} 1\cos\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{2}{3}\frac{3}{n\pi}\left[\sin\left(\frac{n\pi}{3}x\right)\right]_1^2 \\
&= \frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{3}2\right) - \sin\left(\frac{n\pi}{3}\right)\right)
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{3}\int_{-3}^{3} f_o(x)\sin\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{2}{3}\int_{0}^{3} f(x)\sin\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{2}{3}\int_{1}^{2} \sin\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{-2}{3}\frac{3}{n\pi}\left[\cos\left(\frac{n\pi}{3}x\right)\right]_1^2 \\
&= \frac{-2}{n\pi}\left(\cos\left(\frac{n\pi}{3}2\right) - \cos\left(\frac{n\pi}{3}\right)\right)
\end{aligned}$$

Using summation notation the cosine series is:

$$\frac{1}{3} \quad + \quad \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{3}2\right) - \sin\left(\frac{n\pi}{3}\right)\right)\cos\left(\frac{n\pi}{3}x\right)$$

Using summation notation the sine series is:

$$\sum_{n=1}^{\infty} \frac{-2}{n\pi}\left(\cos\left(\frac{n\pi}{3}2\right) - \cos\left(\frac{n\pi}{3}\right)\right)\sin\left(\frac{n\pi}{3}x\right)$$

2. Assume that the thermal diffusivity of a thin metal rod $\alpha^2 = 1.9$ and the length of the rod is 3 cm. Assume that the initial temperature of the rod is given by 50 times the function $f(x)$ given in Problem 1 and that the left and right ends of the rod are placed in ice water. Find the temperature $u(x,t)$ of the rod at any time $t > 0$. Approximately what will the temperature of the rod be at $x = 1$ after a long time?

ANS.

$$u(x,t) = 50\sum_{n=1}^{\infty} \frac{-2}{n\pi}\left(\cos\left(\frac{n\pi}{3}2\right) - \cos\left(\frac{n\pi}{3}\right)\right)\sin\left(\frac{n\pi}{3}x\right)e^{-1.9(n\pi/3)^2 t}$$

After a long time the temperature of the entire rod will be approximately that of the ice water.

3. Again assume that the thermal diffusivity of a thin metal rod $\alpha^2 = 1.9$ and the length of the rod is 3 cm. Also assume that the initial temperature of the rod is given by the function

$$g(x) = \begin{cases} 5 + 10x & \text{if} \quad x < 1 \\ 15 + 10x & \text{if} \quad 1 \leq x < 2 \\ 5 + 10x & \text{if} \quad 2 \leq x \end{cases}$$

Finally assume that the left end of the rod is held at 5^o and the right is held at 35^o. Find the temperature $u(x,t)$ of the rod at any time $t > 0$. What will the temperature of the rod be at $x = 2$ after a long time?

ANS. Since the ends of the rod are held at 5^o and at 35^o we seek the steady state solution $u_{\text{steady-state}}(x,t) = 5 + 10x$ for these temperatures at the ends and write our solution as $u(x,t) = u_{\text{steady-state}}(x,t) + u_{\text{transient}}(x,t)$ The transient solution will have 0^o at the ends and initial temperature equal to the given temperature $g(x)$ minus the steady state (initial) temperature:

$$g(x) - (5 + 10x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 10 & \text{if} \quad 1 \leq x < 2 \\ 0 & \text{if} \quad 2 \leq x \end{cases}$$

We see that this is exactly 10 times the $f(x)$ given in Problem 1. Therefore,

$$u(x,t) = u_{\text{steady-state}}(x,t) + u_{\text{transient}}(x,t) = 5 + 10x + 10 \sum_{n=1}^{\infty} \frac{-2}{n\pi} \left(\cos\left(\frac{n\pi}{3} 2\right) - \cos\left(\frac{n\pi}{3}\right) \right) \sin\left(\frac{n\pi}{3} x\right) e^{-1.9(n\pi/3)^2 t}$$

4. Assume that the thermal diffusivity of a thin metal rod $\alpha^2 = 1.9$ and the length of the rod is 3 cm. Assume that now both ends of the rod are insulated and that the initial temperature distribution by 36 times $f(x)$ given in Problem 1. Find the temperature $u(x,t)$ of the rod at any time $t > 0$. What will the temperature be after a long time at $x = 2$?

ANS. When the ends are insulated we need to use the cosine series for $f(x)$, which we found in Problem 1:

$$u(x,t) = 12 \quad + \quad 36 \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\sin\left(\frac{n\pi}{3} 2\right) - \sin\left(\frac{n\pi}{3}\right) \right) \cos\left(\frac{n\pi}{3} x\right) e^{-1.9(n\pi/3)^2 t}$$

The steady state solution to this insulated ends problem is 12. That is the temperature the entire rod approaches after a long time. It is also the average temperature of the rod at any time t .