Math 251 Sections 1,2,10 FAQ 9/3/09

1. A salt water mixture in a large tank is constantly being mixed to keep it homogeneous. A mixture with concentration of 1 kg salt per liter water flows in at the rate of 3 l/min. The homogeneous mixture flows out the same rate of 1 l/min. Initially there are 300 kg disolved in 200 liters of water in the tank. If the capacity of the tank 500 l, then find the amount of salt in the tank when the tank becomes completely full using the following steps:

a. Find a formula for the volume of mixture in the tank at any time t. When will the tank be completely full?

ANS. The volume of mixture in the tank at any time t varies according to the following formula:

$$V(t) = 200 + 2t$$

The tank fills when $V(t) = 500$, which means $t = 150$

b. Write an ODE for the volume $V(t)$ of mixture in the tank at any time t.

ANS.

$$Q' = (3)(1) - \frac{Q}{200 + 2t} \qquad Q(0) = 300$$

c. If the tank were filled from the inlet pipe without allowing any mixture to flow out, then what would be the quantity of salt in the tank.

ANS. 300 kg + 300kg = 600kg

d. If the tank were filled from the inlet pipe starting from completely empty, then what would be the quantity of salt in the tank.

ANS. 500 kg

e. Give a ball park figure for the answer to this problem, without solving the ODE found in **b.**

ANS. Between 500 and 300 kg.

f. Solve the ODE found in **b.** and evaluate the solution at the time when the tank fills completely.

ANS. To find the integrating factor for this ODE we rewrite it as

$$Q' + \frac{1}{2}\frac{Q}{100 + t} = 3 \qquad Q(0) = 300$$

We see that $p = \frac{1}{2}\frac{1}{100+t}$ and hence the integrating factor for this ODE is $\exp(\frac{1}{2}\ln(100 + t)) = (100 + t)^{1/2}$ and we obtain:

$$(Q(100 + t)^{1/2})' = 3(100 + t)^{1/2}$$

By integrating both sides with respect to t we obtain

$$Q(100 + t)^{1/2} = 2(100 + t)^{3/2} + C$$

and setting $t = 0$, $Q = 300$ gives

$$C = 100^{1/2}300 - 2(100)^{3/2} = 100^{3/2}$$

Therefore,

$$Q = 2(100 + t) + 100^{3/2}(100 + t)^{-1/2}$$

The tank fills when $200 + 2t = 500$, i.e., $t = 150$, and we find that

$$Q(150) = 500 + 100(\frac{2}{5})^{1/2}\text{kgrams} \approx 563.25\text{kgrams}$$

2. A tank with a capacity of 400 liters contains 200 liters of water and 200 grams of salt. Mixture with a concentration of 100 grams of salt per liter flows in at the rate of 3 liters/min and well mixed mixture flows out at the rate of 1 liter/men. Find the amount of salt in the tank at the point in time when the tank is completely full.

ANS.

$$V(t) = 200 + 2t$$

$$Q' = (3)(100) - \frac{Q}{200+2t}(1) \qquad Q(0) = 200$$

$$Q' + \frac{Q}{200+2t} = 300$$

The integrating factor is $(t+100)^{1/2}$.

$$(Q(t+100)^{1/2})' = 300(t+100)^{1/2}$$

$$Q(t+100)^{1/2} = 200(t+100)^{3/2} + C$$

Set $t = 0$ and $Q = 200$ to obtain

$$200(100)^{1/2} = 200(100)^{3/2} + C$$

That is,

$$2 \times 10^3 = 2 \times 10^5 + C \qquad C = 2(10^3 - 10^5)$$

$$Q = 200(t+100) + \frac{2(10^3 - 10^5)}{(t+100)^{1/2}}$$

The tank fills compteletly when $t = 100$ Therefore $Q(100) = 4 \times 10^4 + \sqrt{2} \times (10^2 - 10^4) = (4 - \sqrt{2}) \times 10^4 + \sqrt{2} \times 10^2$.