1. Use Laplace transforms to solve the following IVP: $y' + 4y = u(t-1) \quad y(0) = 2$

ANS. Let $\mathcal{L}\{\text{y(t)}\} = \text{Y(s)}$. We start taking Laplace transforms of both sides:

$$sY - 2 + 4Y = e^{-s}\frac{1}{s}$$

So

$$\begin{aligned}
Y &= \frac{2}{s+4} + e^{-s}\frac{1}{s(s+4)} \\
&= \frac{2}{s+4} + e^{-s}\left(\frac{1/4}{s} - \frac{1/4}{s+4}\right) \\
&= \mathcal{L}\{2e^{-4t} + u(t-1)\left(\frac{1}{4} - \frac{1}{4}e^{-4(t-1)}\right)\}
\end{aligned}$$

2. Use Laplace transforms to solve the following IVP: $y'' + 4y' + 3y = u(t-2) \quad y(0) = 1, \quad y'(0) = 2$

ANS. Let $\mathcal{L}\{\text{y(t)}\} = \text{Y(s)}$. We start taking Laplace transforms of both sides:

$$s(sY - 1) - 2 + 4(sY - 1) + 3Y == e^{-2s}\frac{1}{s}$$

So

$$\begin{aligned}
Y &= \frac{s+6}{s^2+4s+3} + e^{-2s}\frac{1}{s(s^2+4s+3)} \\
&= \frac{5/2}{s+1} - \frac{3/2}{s+3} + e^{-2s}\left(\frac{1/3}{s} - \frac{1/2}{s+1} + \frac{1/6}{s+3}\right) \\
&= \mathcal{L}\{\frac{5}{2}e^{-t} - \frac{3}{2}e^{-3t} + u(t-2)\left(1/3 - \frac{1}{2}e^{-(t-2)} + \frac{1}{6}e^{-3(t-2)}\right)\}
\end{aligned}$$

3. Use Laplace transforms to solve the following IVP: $y'' + 4y' + 5y = tu(t-3) \quad y(0) = 1, \quad y'(0) = 2$

ANS. Let $\mathcal{L}\{\text{y(t)}\} = \text{Y(s)}$. We start taking Laplace transforms of both sides. In order to do that we need to replace $tu(t-3)$ on the right hand side by $(t-3)u(t-3) + 3u(t-3)$.

$$s(sY - 1) - 2 + 4(sY - 1) + 5Y = e^{-3s}\frac{1}{s^2} + 3e^{-3s}\frac{1}{s}$$

So

$$Y = \frac{s+6}{s^2+4s+5} + e^{-3s}\frac{1}{s^2(s^2+4s+5)} + e^{-3s}\frac{3}{s(s^2+4s+5)}$$

Before continuing to find the function y whose Laplace transform is Y we need to find the partial fraction expansion of the last two terms on the right. And, we come up with

$$\begin{aligned}
\frac{1}{s^2(s^2+4s+5)} + \frac{3}{s(s^2+4s+5)} &= -\frac{11\,s+49}{25\,(s^2+4\,s+5)} + \frac{11}{25\,s} + \frac{1}{5\,s^2} \\
&= -\frac{11\,(s+2)+27}{25\,(s^2+4\,s+5)} + \frac{11}{25\,s} + \frac{1}{5\,s^2} \\
&= -\frac{11\,(s+2)}{25\,(s^2+4\,s+5)} - \frac{27}{25\,(s^2+4\,s+5)} + \frac{11}{25\,s} + \frac{1}{5\,s^2}
\end{aligned}$$

We recognize that the right hand side of this formula is the following Laplace transform:

$$\mathcal{L}\{\frac{1}{25}\mathrm{e}^{-2t}\left(-11\cos\mathrm{t}-27\sin\mathrm{t}\right)+\frac{11}{25}+\frac{1}{5}\mathrm{t}\}$$

We now return to Y and find its inverse Laplace transform:

$$\begin{aligned}
Y &= \frac{s+2}{s^2+4s+5}-\frac{4}{s^2+4s+5}+e^{-3s}\frac{1}{5}\left(-\frac{11\,(s+2)}{25\,(s^2+4\,s+5)}-\frac{27}{25\,(s^2+4\,s+5)}+\frac{11}{25\,s}+\frac{1}{5\,s^2}\right)\\
&= \mathcal{L}\{e^{-2t}\cos t-4e^{-2t}\sin t+\frac{1}{5}u(t-3)\left(\frac{1}{25}e^{-2(t-3)}\left(-11\cos(t-3)-27\sin(t-3)\right)+\frac{11}{25}+\frac{1}{5}(t-3)\right)\}
\end{aligned}$$

4. Use Laplace transforms to solve the following IVP: $y''+4y'+4y=u(t-5)e^{-2t+10}$ $y(0)=1,\quad y'(0)=2$

ANS. Let $\mathcal{L}\{\mathrm{y(t)}\}=\mathrm{Y(s)}$. We start taking Laplace transforms of both sides:

$$s(sY-1)-2+4(sY-1)+4Y=e^{-5s}\frac{1}{s+2}$$

So

$$\begin{aligned}
Y &= \frac{s+6}{(s+2)^2}+e^{-5s}\frac{1}{(s+2)^3}\\
&= \frac{1}{s+2}+\frac{4}{(s+2)^2}+e^{-5s}\frac{1}{(s+2)^3}\\
&= \mathcal{L}\{e^{-2t}+4te^{-2t}+u(t-5)\left(\frac{1}{2}(t-5)^2e^{-2(t-5)}\right)\}
\end{aligned}$$