Math 251 Sections 1,2,10 FAQ 11/18/09

1. Assume that the thermal diffusivity of a thin metal rod $\alpha^2 = 2$ and the length of the rod is 3 cm. Assume that the left and right ends of the rod are held at 10^oC and 100^oC, respectively. Also assume that the initial temperature distribution $f(x)$ is a linear function. Find the temperture $u(x,t)$ of the rod at any time $t > 0$.

ANS. The $f(x) = 10 + 30x$ and $u(x,t) = f(x)$

2. Now assume that the rod in Problem 1 has initial temperature distribution $f(x) = 30$ if $0 < x < 3$. Also suppose that the left end of the rod is kept at 45^0C and the right end of the rod is kept at 15^oC

This problem can be broken into two problems. Indicate how this is done by completing the following table:

	Initial Temp	Temp at $x = 0$	Temp at $x = L$
This Problem	30	45	15
=			
Steady State	45 -10x	45	15
+			
Transient	10x - 15	0	0

3. Find the temperature $u(x,t)$ of the rod in Problem 2 for $t > 0$

ANS. The steady state solution is $u_{\text{steady}}(x) = -10x + 45$. The transient solution is The solution is $u(x,t) = u_{\text{steady}}(x) + u_{\text{transient}}(x,t)$ where $u_{\text{transient}}(x,t)$ is the solution of the ends in ice water problem with initial temperature $10x - 15$. We seek a sine series $\sum_1^\infty b_n \sin(\frac{n\pi}{3}x)$ for $10x - 15$ on $[0,3]$ and then $u_{\text{transient}}(x,t) = \sum_1^\infty b_n \sin(\frac{n\pi}{3}x)e^{-2(n\pi/3)^2 t}$

The only thing that remains to be done is to find the b_n which is a straightforward calculation:

$$\begin{aligned}
b_n &= \frac{1}{3}\int_{-3}^{3} f_o(x)\sin\left(\frac{n\pi}{3}x\right)\,dx \\
&= \frac{2}{3}\int_{0}^{3} (10x-15)\sin\left(\frac{n\pi}{3}x\right)\,dx \\
&= -\frac{2}{3}\frac{3}{n\pi}\left(\left[(10x-15)\cos\left(\frac{n\pi}{3}x\right)\right]_0^3 - \int_0^3 10\cos\left(\frac{n\pi}{3}x\right)\,dx\right) \\
&= -\frac{2}{n\pi}\left(\left[(10x-15)\cos\left(\frac{n\pi}{3}x\right)\right]_0^3\right) \\
&= -\frac{2}{n\pi}\left(15\cos(n\pi) + 15\right)
\end{aligned}$$

4. For the rod in Problem 2, the approximate temperature at $x = \sqrt{2}$ for large t.

ANS. Approximately $-10\sqrt{2} + 45$