For each of the following nonlinear system of autonomous 1st order ODE's do the following:
a. Find all the critical points of the nonlinear system.
b. In a neighborhood of each critical point approximate the nonlinear system by a linear system.
c. Determine the name and the stability of the the critical points of each of the linear approximations.
d. Sketch a phase portrait for the original nonlinear system.
e. Which critical points of the original systems cannot have the type of their stability determined with certainty by these approximations?

1. The nonlinear system:

$$\begin{aligned} x' &= x + x^2 + y^2 \\ y' &= y - xy \end{aligned}$$

ANS. From the second equation we see that we have two sorts of critical points $\begin{pmatrix} 1 \\ ? \end{pmatrix}$ or $\begin{pmatrix} ! \\ 0 \end{pmatrix}$

From the first equation we see that $x = 1$ is not possible and if $y = 0$ we are left with $x + x^2 = 0$ which leaves two possibilities: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the linearization is

$$\begin{aligned} x' &= x \\ y' &= y \end{aligned}$$

which is an unstable proper node.
At $\mathbf{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ we set $\begin{matrix} u = x + 1 \\ v = y \end{matrix}$. Substituting this into the original equation gives

$$\begin{aligned} u' &= (u-1) + (u-1)^2 + v^2 \approx -u \\ v' &= v - (u-1)v \approx 2v \end{aligned}$$

The eigenvalues of this system are -1 and 2. This critical point is a saddle for the linearized system.
To see the phase portrait enter the system into the phase portrait applet.

2. The nonlinear system:

$$\begin{aligned} x' &= x - \frac{1}{2}xy \\ y' &= -\frac{3}{4}y + \frac{1}{4}xy \end{aligned}$$

ANS. Setting

$$\begin{aligned} 0 &= x - \frac{1}{2}xy = \frac{1}{2}x(2-y) \\ 0 &= -\frac{3}{4}y + \frac{1}{4}xy = \frac{1}{4}y(-3+x) \end{aligned}$$

gives two critical points $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the linearization is $\begin{matrix} x' = x \\ y = -\frac{3}{4}y \end{matrix}$ which has eigenvalues $r_1 = 1$ and $r_2 = -3/4$ which have opposite sign.
Hence the origin is a saddle for for the linearized system. At $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ we set $\begin{matrix} u = x - 3 \\ v = y - 2 \end{matrix}$. Substituting this

into the original equation gives

$$\begin{aligned} u' &= \frac{1}{2}(u+3)(-v) \approx -\frac{3}{2}v \\ v' &= \frac{1}{4}(v+2)u \approx \frac{1}{2}u \end{aligned}$$

The eigenvalues of this system are purely imaginary. Thus this critical point is a center for the linearized system. To see the phase portrait enter the system into the phase portrait applet.