Math 251 Sections 1,2,10 FAQ 9/23/09

1. Solve the initial value problem $y'' - 6y' + 9y = 0 \qquad y(0) = 1, y'(0) = 1$.

ANS. The characteristic polynomial is $r^2 - 6r + 9 = (r-3)^2$. Therefore, a fundamental set is $y_1 = e^{3t} and y_2 = ty_1$. The general solution is

$$y = e^{3t}(c_1 + c_2 t)$$

$$y' = 3y + e^{3t}c_2$$

Setting $t = 0$ gives $c_1 = 1$ and $1 = 3 + c_2$ That is, $c_2 = -2$. So

$$ye^{3t}(1 - 2t)$$

2. How many times can a solution of $y'' - 6y' + 9y = 0$ cross the t-axis?

ANS. The general solution is

$$y = e^{3t}(c_1 + c_2 t)$$

This can is equal to zero for one value of t if c_2 happens to be nonzero. Otherwise it is never zero.

3. What does a solution of of $y'' + 6y' + 9y = 0$ approach as $t \to \infty$?

ANS. The general solution is

$$y = e^{-3t}(c_1 + c_2 t)$$

By L'Hospital's rule every solution appoaches zero as approach as $t \to \infty$.

4. Describe the behavior of the solutions as $t \to \infty$ by examining their characterstic polynomials (without actually writing down their solutions). In each case state determine how many times can you expect the solution to cross the t-axis?

$y'' - 36y = 0$
$y'' + 36y = 0$
$y'' - 12y' + 36y = 0$
$y'' + 12y' + 36y = 0$
$y'' - 12y' + 100y = 0$
$y'' + 12y' + 100y = 0$
$y'' - 15y' + 36y = 0$
$y'' + 15y' + 36y = 0$

ANS. $y'' - 36y = 0$
The characteristic polynomial has roots ± 6. Therefore everything is possible. Namely, it is possible that a solution approaches ∞ as $t \to \infty$. It is possible that a solution approaches $-\infty$ as $t \to \infty$. And it is possible that a solution approaches 0 as $t \to \infty$. Exactly which one of these happens depends on the initial condition the solution satisfies. A combination of two exponentials crosses t-axis at most once.

$y'' + 36y = 0$
The characteristic polynomial has roots $\pm 5i$. Therefore every nonzero solution has oscillations of constant amplitude as $t \to \infty$. Combination of sine and cosine crosses t-axis infinitely many times.

$y'' + 12y' + 36y = 0$
The characteristic polynomial has a double negative root -6. The general solution is

$$y = e^{-6t}(c_1 + c_2 t)$$

which by l'Hospital's rule approaches zero at $t \to \infty$.

$y'' - 12y' + 36y = 0$
The characteristic polynomial has a double root 6. The general solution is

$$y = e^{6t}(c_1 + c_2 t)$$

which approaches $\pm\infty$ except when both c_1 and c_2 are zero. Since a linear function crosses the t-axis at most once, all solutions cross the t-axis at most once.

$y'' + 12y' + 100y = 0$
The characteristic polynomial $r^2 + 12r + 100$ has roots $-6 \pm 8i$. Therefore every nonzero solution has oscillations which approach 0 as $t \to \infty$. A combination of sine and cosine crosses t-axis infinitely many times and the exponential does not affect these

$y'' - 12y' + 100y = 0$
The characteristic polynomial $r^2 - 12r + 100$ has roots $6 \pm 8i$. Therefore every nonzero solution has oscillations which grow progressively larger as $t \to \infty$. A combination of sine and cosine crosses t-axis infinitely many times and the exponential does not affect these.

$y'' + 15y' + 36y = 0$
The characteristic polynomial has two negative roots. All solutions approach 0 as $t \to \infty$ A combination of two exponentials crosses t-axis at most once.

$y'' - 15y' + 36y = 0$
The characteristic polynomial has two positive roots. All nonzero solutions approach $\pm\infty$ as $t \to \infty$ A combination of two exponentials crosses t-axis at most once.