$$4a + 3 = 3a + 4$$

The above is an algebraic equation. The symbol a represents an unknown real number. We all know how to find that number: $a = 1$

$$y' + y = 2e^t$$

The above is a differential equation, labled DE for short. The symbol y represents an unknown function of the independent variable t, y' its derivative with respect to t. (In this course we frequently abbreviate the familiar notation for a function: $y = y(t)$ and simply denote by it by y and denote its derivative with respect to t by y').

The solution is

$$y = e^t + Ce^{-t}$$

for an arbitrary constant C. We check that it actually is a solution:

$$
\begin{aligned}
y &= e^t + Ce^{-t} \\
y' &= e^t - Ce^{-t} \\
y + y' &= 2e^t
\end{aligned}
$$

Some other examples of DE's

$$y' = 0, \qquad y' = \sec(t), \qquad y' = y$$

The first one was solved in Math 140, last two solved in Math 141.

It is worthwhile to remind you here that a solution to the first DE is $y = 1$, or, in general, $y = C$, where C is any constant. Also, observe that the solution $y = 1$ to the DE is conceptually a different object than the solution $a = 1$ to the algebraic equation. Perhaps, the need for this warning is necessitated by the shorthand of suppressing the symbol t in the notation $y(t) = 1$ for the constant function 1; that is, the function y which assigns 1 to each real number t.

Defintion: A DE is an equation involving an unknown function of one or more variables and at least one of its derivatives.

Comments: 1. The highest order derivative appearing is called the **order** of the eqn.

2. The independent variable(s) of the unknown function may or may not appear in the the DE. But we are always aware of them.

3. We will consider 1st and 2nd order DE's in this course.

4. For 1st order, we will always assume that the equation can be rewritten $y' = f(t, y)$, although that may not always be the case outside this course. Later we will make analogous assumptions about the higher order ODE's we consider.

Let us now look at the DE $y' = t - y$ and determine the slope of the line tangent to the graph of the solution at the point $(3, 2)$! One does not need to solve the DE in order to do this. It is VERY EASY! Just plug the values $t = 3$ and $y = 2$ into the DE and get $3 - 2 = 1$.

This leads to the idea of Direction Field (aka Slope Field): At each point among a bunch of points, plot the slope to the graph of the soln at the point. This gives the following:
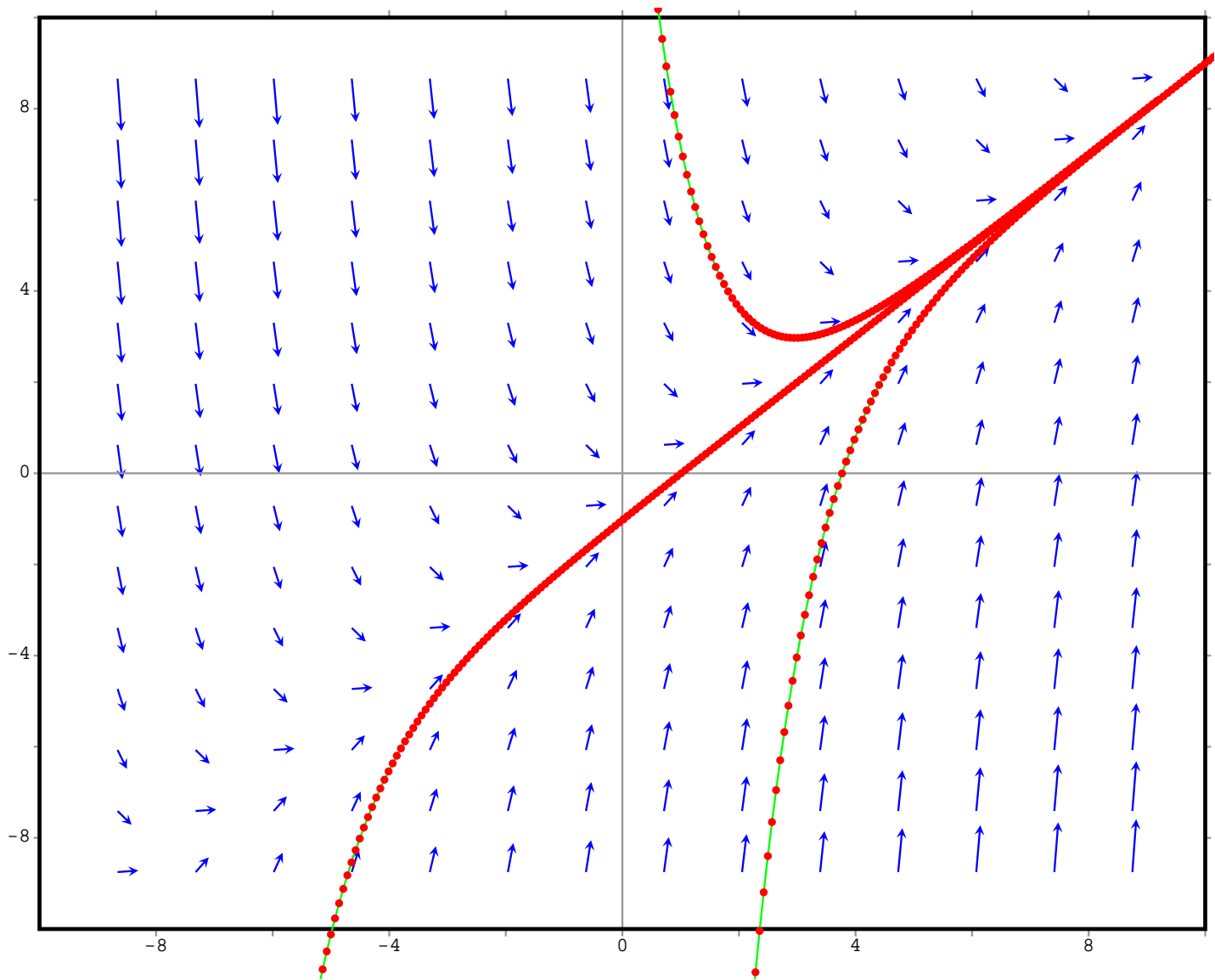
Looking at the direction field, we can guess the rough graph of any soln of this DE. The graph of a solution of an DE is called an **integral curve**, or **trajectory.**

The process of plotting a direction field is rather tedious. Fortunately there is ample computer software to help out. One such piece of software is the Computer Algebra System (CAS) called Maxima. This is open source software and is available without charge on the internet. A few words about getting it and running it are appended at the end of this lesson.

Although in general the plotting of a direction field is tedious, in one noteworthy case it is rather easy. This is the case of an **autnomous DE** direction field. The definition of autonomous DE is one with no no explicit mention of t, the independent variable in the formula for the DE. For example, in the DE $y' = 3 - y$, y and y' are both understood to be functions of t but t does not appear in the formula $3 - y$ for y'. Here we simply plot some arrows with the correct slopes along the y-axis and then repeatedly shift them horizontally. Of course, one must take care to shift only horizontally and not to disturb the slope of the arrow in the process.
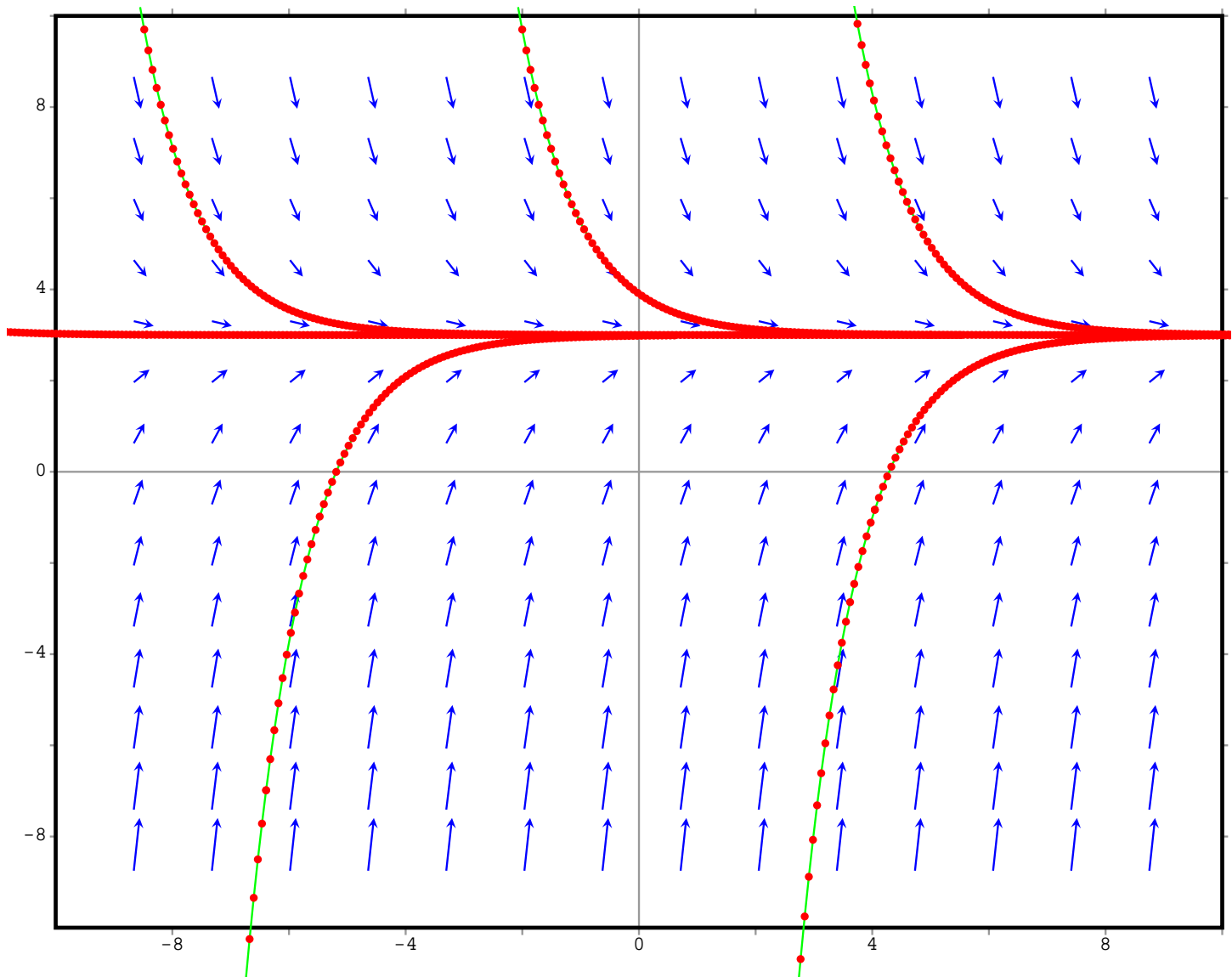
Note there is a solution whose graph is a horizonatal line: $y = 3$ Such solutions are called **equilibrium solutions**. An **equilibrium solution** is a constant solution and they can be found by finding value(s) of y that make the right hand side of the DE equal 0. Such values are called **critical values (or pts)**. There is a slight distinction between equilibrium solution and critical values. An equilibrium solution is a function t, specifically the **constant function** whose value is a critical value. Also note that eventually every soln of this DE approaches 3. This is the long time behavior of the DE. Other DE's may exhibit different long time behaviors.

Let's solve a problem:

Without intervention, the population of mice in a certain neighborhood increases at a rate equal to 2 times the population. Now assume that owl's intervene to reduce the population by 100 in a given unit of time, which we assume to a month in this problem. The question we ask is: Can we determine whether or not the population will be zero after a period of time.

Let's use direction fields to analyze this problem.. Let P denote the population at time t months. Then without intervention rate of change of population is: $P' = 2P$ and with intervention it is: $P' = 2P - 100$. The Direction Field for this DE is
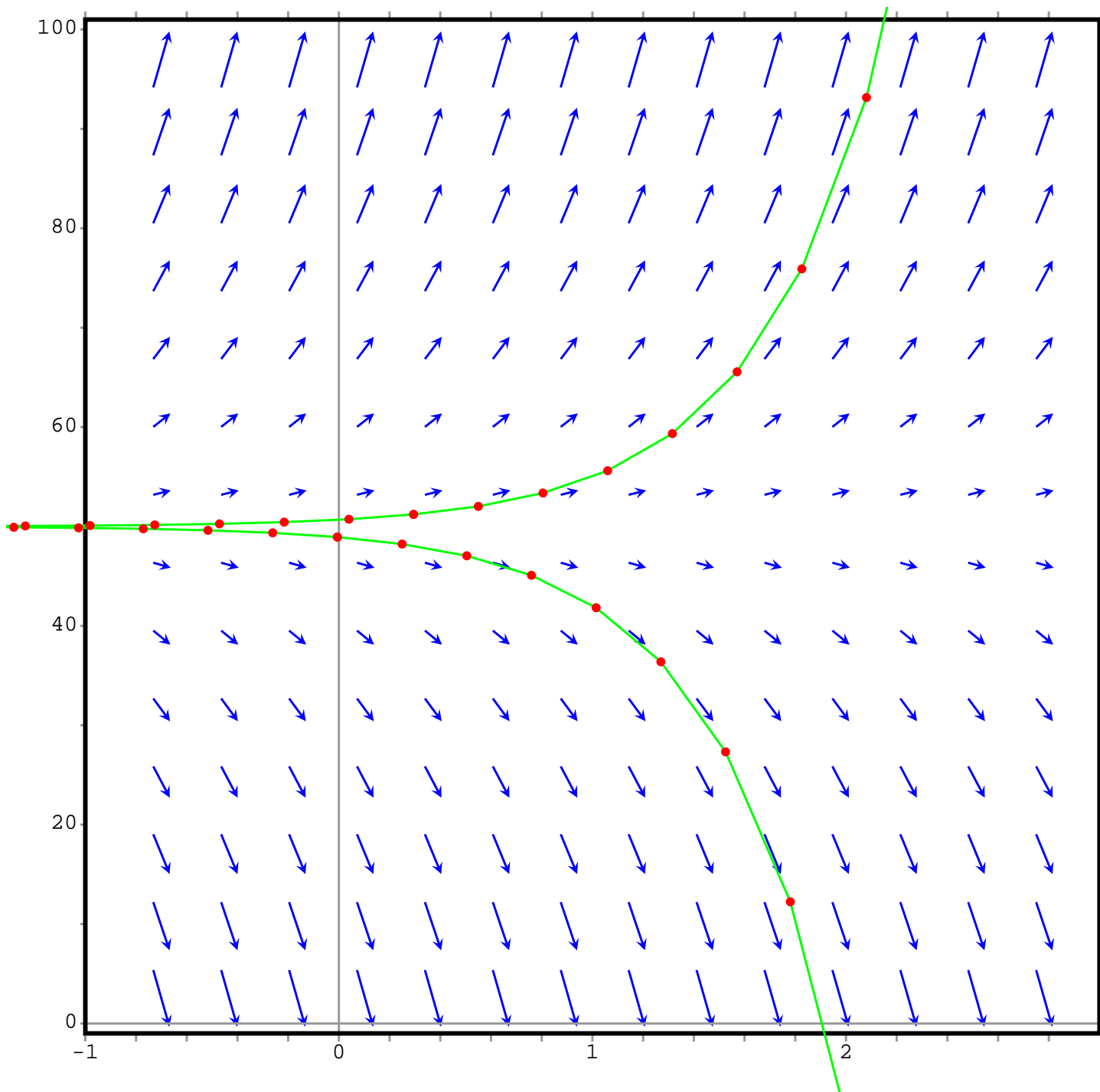
Note that if at time $t = 0$ $P(0) = 50$ then $P(t) = 50$ However, if $P(0) = 49$, or less, then eventually there will be no mice in the neighborhood. In fact, it looks like it will take somewhat less than 2 months for that to happen. If one needs an exact time when that will happen, then we need to come up with an explicit formula for $P(t)$ and this we will do next time.

Maxima CAS

At various points it this course your ability to visualize concepts will be enhanced by CAS. On the the Math 251 website there is a link to downloading Maxima for the Windows operating system. Maxima is also available for Linux and OSX; ask me for some direction on how to get Maxima if you use one of these.

The installation of Maxima under Windows is completely self explanatory. Two different graphical users interfaces are

installed. The default installation choices sets an icon on the desktop only for one of them wxMaxima. For beginners this interface has the advantage of providing many of Maxima's functions via a menu system. However, full use of Maxima is easier with the user interface called xMaxima.

To help with this lesson you will want to work with Maxima's direction field plotting feature, plotdf. Start **wxMaxima** by clicking the Desktop Icon. Since `plotdf` is not yet available as a menu item, you will have to start it by hand. As soon as wxMaxima starts, click on its window and press enter. This will open a spot in the window for text-entry. Assuming that you wish to see the direction field of $y' = t - y$ enter the right-hand-side of this DE by typing the following text into the text-entry area:

`plotdf(t-y,[t,y])`

and press SHIFT ENTER when you are done typing. (Notice the semi-colon that is automatically added when you do this. Maxima requires that every statement to it be ended either with a semi-colon or dollar sign. Maxima takes a second or to execute the command. Then wxMaxima places the graphical output in a new window with its own menu system. In the configure menu the new window you will find options for modifying the graphical output. Among the options you will find parameters the parameters, xcenter, yxcenter, xradius, yradius, which control the position and size of the graphics window and which may have to be altered if, as is frequently the case, the default choice does not give a reasonable facsimile of the direction field.

You need to be aware that Maxima does not understand math symbols the way human beings normally write them. It understands "computer" math. Therefore, when asking Maxima to plot a direction field for $y' = 2y - 50$ one must enter:`plotdf(2*y - 50,[t,y]);` Entering `plotdf(2y - 50,[t,y])` will evoke an error message from Maxima because Maxima simply does not understand juxtaposition of symbols.