If $a = 2 + 3i$, $b = 4 + 5i$, then find Re a, Im a, $a + b$, $5a$, ab, $\bar{a}$, $a\bar{a}$, $|a|$, $1/a$, b/a.

ANS. Re $a = 2$ Im $a = 3$ $a + b = 6 + 8i$, $5a = 10 + i15$ $ab = -7 + i22$ $\bar{a} = 2 - 3i$, $a\bar{a} = 13$, $|a| = \sqrt{13}$, $1/a = \frac{1}{13}(2 - 3i)$ $b/a = \frac{1}{13}(2 - 3i)(4 + 5i) = \frac{1}{13}(23 - 2i)$

2. Check the identity $\arg(ab) = \arg a + \arg b$ for a and b in Problem 1. What does this identity say when $a = b = -i$? Perhaps it would be more correct to write $\arg(ab) \mod (2\pi) = \arg a \mod (2\pi) + \arg b \; mod(2\pi)$.

ANS. $\arg a = arctan(1.5) \approx .98279$
$\arg b = arctan(1.25) \approx .89606$
$\arg ab =$ 2nd quadrant angle whose tangent is $-22/7$
$\arg ab = \pi - arctan(22/7) \approx 1.8788$

If $a = b = -i$, then $\arg a = \arg b = 3\pi/2$ and $\arg(ab) = \arg(-1) = \pi$. It is hard to see how 3π can be equal to π unless one is allowed to add a multiple of 2π whenever it is convenient to do so.

3. Express the following complex numbers in the form $\alpha + i\beta$: $b = e^{\ln 2 + i(2\pi/3)}$.

ANS. According to the defintion of complex exponential

$$e^{\ln 2 + i(2\pi/3)} = e^{\ln 2} e^{i}(2\pi/3) = 2(\cos(2\pi/3) + i \sin(2\pi/3)) = 2(\frac{-1}{2} + i\frac{\sqrt{3}}{2})$$

NOTE: The number $\frac{a}{|a|}$ must have the same argument as a and that the argument of the number $\frac{a}{|a|}$ is $\frac{2\pi}{3}$. Looking back at b we now see that a and b and that the argument of $e^{i}(2\pi/3)$ is the same as that of b and hence it is also $\frac{2\pi}{3}$.

4. What is the derivative of the complex valued function $h(t) = e^{(1+2i)t}$? What are the real and imaginary parts of $h(t)$? Find the derivatives of the real and imaginary parts of $h(t)$. How are they related to the derivative of $h(t)$?

ANS. If we use the standard formula for differentiating an exponential to find the derivative of $h(t) = e^{(1+2i)t}$ then we get $h'(t) = (1 + 2i)e^{(1+2i)t}$.
To find the real and imaginary parts, we write
$h(t) = e^{(1+2i)t} = e^t e^{2it} = e^t(\cos 2t + i \sin 2t) = e^t \cos 2t + ie^t \sin 2t$. The derivative of the real part of $h(t)$ is $e^t \cos 2t - 2e^t \sin 2t$ and the derivative of the imaginary part of $h(t)$ is $e^t \sin 2t + 2e^t \cos 2t$ These two derivatives agree with the real and imaginary parts of the expression for $h'(t)$ given above.

5. a. Find a complex number $\gamma = \alpha + i\beta$ such that $e^\gamma = 17(\frac{1}{2} - i\frac{\sqrt{3}}{2})$

ANS. Recall the definition $e^\gamma = e^\alpha(\cos \beta + i \sin \beta)$. So we need to choose α so that $e^\alpha = 17$. ie, $\alpha = \ln 17$. We also need to choose β subject to two conditions; $\cos \beta = 1/2$ and $\sin \beta = -\sqrt{3}/2$. Thus β needs to be a 4th quadrant angle with reference angle equal to $\pi/3$. ie, $\beta = 5\pi/3$.

b. Find a complex-valued function $g(t)$ whose real part is $e^{2t} \cos(3t)$.

ANS. $g(t) = \text{Re} e^{(2+3i)t}$

c. Find a complex-valued function $h(t)$ whose imaginary part is $e^{4t} \sin(5t)$.

ANS. $h(t) = \text{Im} e^{(4+5i)t}$

d. Find a complex-valued function $f(t)$ whose imaginary part is $e^{6t} \sin(7t)$.

ANS. Since $e^{(6+7i)t} = e^{6t}(\cos(7t) + i \sin(7t))$, we see that
$e^{6t} \sin(7t) = \text{Im}(e^{(6+7i)t})$.