Math 251 Sections 1,2,10 FAQ 11/12/09

1. If we differentiate term by term the Fourier series of $f(x) = x$ on the interval $[-\pi, \pi]$ (which was on the last FAQ), do we get something that has any value?

ANS. From the previous FAQ we know that the Fourier series is:

$$\frac{2}{1}\sin(x) - \frac{2}{2}\sin(2x) + \frac{2}{3}\sin(3x) + \dots$$

If we differentiate this Fourier series term by term we get

$$1 = 2\cos(x) - 2\cos(2x) + 2\cos(3x) + \dots$$

This obviously makes no sense. Set $x = 0$, $x = \pi/6$, $x = \pi/4$, etc, if you have any doubts.

2. Using the Fourier series of $f(x) = x$ on the interval $[-\pi, \pi]$. find the Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$.

ANS. In the FAQ of 11/14/07 we found that

$$x = \sum_{n=1}^{\infty} \frac{-2}{n} \cos n\pi \sin nx$$

Therefore term by term integration gives

$$\frac{x^2}{2} + C = \sum_{n=1}^{\infty} \frac{2}{n^2} \cos n\pi \cos nx$$

From this we see that

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi \cos nx$$

and we simply need to use our formula for a_0 to find the constant term:

$$a_0 = \frac{1}{\pi}\int_{\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2$$

Finally,

$$x^2 = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi \cos nx$$

3. Consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$$

Find the Fourier series of $f(x)$ on the interval $[-2, 2]$ (Why not integrate the Fourier series of a simpler function term by term? What would the simpler function be?

$$g(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

ANS. From the notes of 11/10 we have that the Fourier series of $g(x)$ is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1 - \cos n\pi)\sin\left(\frac{n\pi}{2}x\right)$$

Term by term integration gives:

$$f(x) + \text{const} = \frac{x}{2} - \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}(1 - \cos n\pi)\cos\left(\frac{n\pi}{2}x\right)$$

The presence of $\frac{x}{2}$ prevents this from being a Fourier series. So we substitute for $\frac{x}{2}$ its Fourier series which was computed in FAQ of 11/10/08:

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{-2}{n\pi} \cos n\pi \sin\left(\frac{n\pi}{2}x\right)$$

We combine these two series and rearrange terms (assuming that it is legal to do so):

$$f(x) = \text{const} - \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}(1 - \cos n\pi)\cos\left(\frac{n\pi}{2}x\right) + \frac{2}{n\pi}\cos n\pi \sin\left(\frac{n\pi}{2}x\right)$$

Finally, we need to deal with the constant in the above expression for f. By the definition of Fourier series, it is $a_0/2 = 1/2$, since a_0 is $1/L = 1/2$ times the area of a triangle, ie, $a_0 = 1$.
In other words,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}x\right) + b_n \sin\left(\frac{n\pi}{2}x\right)$$

where

$$a_0 = \frac{1}{2} \qquad a_n = \frac{2}{n^2\pi^2}(\cos n\pi - 1) \qquad b_n = \frac{-2}{n\pi}\cos n\pi$$