We deal here with mixtures, usually salt in water, (aka brine), that have the property that when the solid is disolved in the liquid, then the volume of the latter is not changed.

Remember that concentration of the solid in the liquid defined to be the quantity (or mass) of the solid divided by the volume of the liquid.

We start with a salt water mixture in a large tank that is constantly being mixed to keep it homogeneous. A mixture with concentration of 3 g salt per liter water flows in at the rate of 4 l/min. The homogeneous mixture flows out the same rate of 4 l/min. Initially there are 400 g disolved in 100 liters of water in the tank. Find an expression for $Q(t)$, the quantity of salt in the tank at time t.

Before seeking an ODE for $Q(t)$ we arrange the data given above as follows:
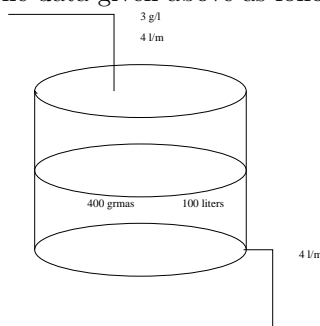
We now can write the ODE for $Q(t)$. Note that $Q' = $ rate salt goes in - rate salt goes out = flow rate in times concentration - flow rate out times concentration. So

$$Q' = \frac{3\,\text{liter}}{\text{min}}\frac{4\,\text{g}}{\text{liter}} - \frac{4\text{liter}}{\text{min}}\frac{Q\,\text{g}}{100\,\text{liter}}$$

So the ODE and initial condition are:

$$Q' = 12 - \frac{4Q}{100} \qquad Q(0) = 400$$

We can plot the direction field in the tQ-plane we start by plotting line segments with appropriate slopes along the Q-axis. Since the equation does not explicitly involve the variable t, the remainder of the direction field is obtain by translating the line segments to the right. Along the Q axis the line segments slope upward when Q is less than 300 and downward when Q is greater than 300. So we see that the quantity of salt is increasing whenever it is below 300 and decreasing whenever it is above 300. In this case the initial value is 400; so we expect the solution to always be decreasing.

Click here for the direction field. We now find an explicit formula for $Q(t)$ Since the ODE is linear we seek an integrating factor. Since $p(t) = 3/200$, the integrating factor is $exp(t/25)$. Therefore,

$$(e^{t/25}Q)' = 12e^{t/25}$$

and integrating both sides gives

$$e^{t/25}Q = 300e^{t/25} + C$$

Since $Q(0) = 400$, by setting $t = 0$ in the above, we obtain $C = 100$ and solving for Q gives

$$Q = 300 + 100e^{-t/25}$$

Indeed, after a long time the quantity of salt in the tank is approximately 300 g.