Math 251 Sections 1,2,10 FAQ 9/21/09

1. Consider the second order differential equation

$$y'' + 4y = 0$$

a. What are the roots of the characteristic polynomial of this equation?

$$r_1, r_2 = \pm 2i$$

b. What are two complex-valued solutions to this equation?

$$e^{2it}, e^{-2it}$$

c. What are the real and imaginary parts of the first function?

$$\mathrm{Re} e^{2it} = \cos 2t \qquad \mathrm{Im} e^{2it} = \sin 2t$$

d. What is the general solution to this equation?

$$y = c_1 \cos 2t + c_2 \sin 2t$$

e. How does the general solution change if instead we work with the real and imaginary parts of the second function?

ANS. We would get

$$y = c_1 \cos(-2t) + c_2 \sin(-2t) = c_1 \cos 2t - c_2 \sin 2t = d_1 \cos 2t + d_2 \sin 2t$$

which is really the same general solution.

f. Solve the following IVP for the above ODE

$$y(0) = 1, y'(0) = -1$$

ANS. Here

$$y' = -2c_1 \sin 2t + 2c_2 \cos 2t$$

Therefore plugging in $t = 0$ and $y = 1, y' = -1$ gives $1 = c_1 + 0$ and $-1 = 0 + 2c_2$. That is,

$$y = \cos 2t - \frac{1}{2} \sin 2t$$

2. Solve the following IVP:

$$y'' + 8y' + 25y = 0, \qquad y(0) = 1, y'(0) = -1$$

Also find the solution that satisfies:

$$y(2.009) = 1, \qquad y'(2.009) = -1$$

ANS.

$$r_1, r_2 = -4 \pm 3i$$
$$y_c = e^{(-4+3i)t}$$

is a complex solution. The real and imaginary parts are real solutions to the original equation:

$$y_1 = \mathrm{Re} y_c = e^{-4t} \cos 3t \qquad y_2 = \mathrm{Im} y_c = e^{-4t} \sin 3t$$

The general solution is:

$$y = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)$$

Its derivative is:

$$y' = -4y + 3e^{-4t}(-c_1 \sin 3t + c_2 \cos 3t)$$

Plugging in the initial conditions gives

$$y(0) = c_1 = 1, \qquad y'(0) = -4 + 3c_2 = -1 \qquad 3c_2 = 3 \qquad c_2 = 1$$

Therefore,

$$y = e^{-4t}(\cos 3t + \sin 3t)$$

For the second IVP we have the following solution

$$y(t - 2.007) = e^{-4(t-2.009)}(\cos 3(t - 2.009) + \sin 3(t - 2.009))$$

3. Describe the behavior of the solutions as $t \to \infty$ by examining their characterstic polynomials (without actually writing down their solutions). In each case determine how many times can you expect the solution to cross the t-axis?

$y'' - 6y' + 5y = 0$
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$y'' + 26y' + 25y = 0$
$y'' + 25y = 0$
$y'' - 25y = 0$

ANS. $y'' + 6y' + 5y = 0$

The characteristic polynomial has two negative roots. All solutions approach 0 as $t \to \infty$ A combination of two exponentials crosses t-axis at most once.
$y'' - 6y' + 5y = 0$

The characteristic polynomial has two positive roots. All nonzero solutions approach $\pm\infty$ as $t \to \infty$ A combination of two exponentials crosses t-axis at most once.
$y'' + 6y' + 25y = 0$

The characteristic polynomial $r^2 + 6r + 25$ has roots $-3 \pm 4i$. Therefore every nonzero solution has oscillations which approach 0 as $t \to \infty$. A combination of sine and cosine crosses t-axis infinitely many times.
$y'' - 6y' + 25y = 0$

The characteristic polynomial $r^2 - 6r + 25$ has roots $3 \pm 4i$. Therefore every nonzero solution has oscillations which grow progressively larger as $t \to \infty$. A combination of sine and cosine crosses t-axis infinitely many times.

$y'' + 26y' + 25y = 0$

The characteristic polynomial $r^+26r + 25 = (r + 25)(r + 1)$. Therefore every solution approaches 0 as $t \to \infty$. A combination of two exponentials crosses t-axis at most once.

$y'' + 25y = 0$

The characteristic polynomial has roots $\pm 5i$. Therefore every nonzero solution has oscillations of constant amplitude as $t \to \infty$. A combination of sine and cosine crosses t-axis infinitely many times.

$y'' - 25y = 0$

The characteristic polynomial has roots ± 5. Therefore everything is possible. Namely, it is possible that a solution approaches ∞ as $t \to \infty$. It is possible that a solution approaches $-\infty$ as $t \to \infty$. And it is possible that a solution approaches 0 as $t \to \infty$. Exactly which one of these happens depends on the initial condition the solution satisfies. A combination of two exponentials crosses t-axis at most once.