1. Suppose an object of mass 2 kgrams is attached to a spring with spring constant 10 Newtons/meter. Assume that there a damping device with damping constant of 8 Newton-sec/meter is attached to the object. If the object is released one meter below its equilibrium position with no initial velocity and at time $t = 3$ a constant external force of 4 Newtons is applied to the object, then find the position of the object at time t. Also, determine the position of the object after a long time.

ANS. The IVP is as follows:

$$2y'' + 8y' + 10y = 4u(t-3) \qquad y(0) = 1, y'(0) = 0$$

or a little more neatly:

$$y'' + 4y' + 5y = 2u(t-3) \qquad y(0) = 1, y'(0) = 0$$

Let $\mathcal{L}\{\mathrm{y(t)}\} = \mathrm{Y(s)}$. We start taking Laplace transforms of both sides.

$$s(sY-1) + 4(sY-1) + 5Y = 2e^{-3s}\frac{1}{s}$$

So

$$Y = \frac{s+2}{s^2+4s+5} + \frac{2}{s^2+4s+5} + e^{-3s}\frac{2}{s(s^2+4s+5)}$$

Before continuing to find the function y whose Laplace transform is Y we need to find the partial fraction expansion of the last two term on the right:

$$\begin{aligned}\frac{2}{s(s^2+4s+5)} &= \frac{2}{5}\left(\frac{1}{s} - \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}\right)\\ &= \frac{2}{5}\mathcal{L}\{1 - \mathrm{e}^{-2\mathrm{t}}\left(\cos\mathrm{t} + 2\sin\mathrm{t}\right)\}\end{aligned}$$

We now return to Y and find its inverse Laplace transform:

$$Y = \mathcal{L}\{\mathrm{e}^{-2\mathrm{t}}(\cos\mathrm{t} + 2\sin\mathrm{t}) + \frac{2}{5}\mathrm{u(t-3)}\left(1 - \mathrm{e}^{-2(\mathrm{t}-3)}\left(\cos(\mathrm{t}-3) + 2\sin(\mathrm{t}-3)\right)\right)\}$$

2. <u>Chandelier - lion - bird problem</u>. Suppose that a hungry lion and a bird are in a room and the bird is flying around the room trying to avoid being caught by the lion. The room contains nothing but a 2 kg chandelier suspended from the ceiling of the room by means of a spring, which is stretched 2.5 meter to its equilibrium position. Also assume that at time $t = 0$ the chandelier was set in motion by releasing it at 1 meter below its equilibrium position and that the effect of air resistance on the chandelier is negligible. If the spring is stretched 1.025 meter below the equilibrium position then it breaks. The .2 kg bird is getting rather tired and would like to rest on the chandelier which oscillates well beyond the reach of the lion. Can the bird find a suitable time to gently rest on the chandelier without breaking the spring and falling into the grasp of the lion? (In this problem assume that Hooke's law is valid even when the spring is stretched to its breaking point.)

ANS. Assume that the bird lands on the chandelier at $t = c$. Here we have a spring-mass system with an external force $f(t) = .2 \cdot 10u(t-c)$ (ie, no external force when $t < c$ and force equal to the mass of the bird times $g = 10$ when $t \geq c$. Also the mass of the chandelier $m = 2$ and the spring constant $k = 8$ Thus the initial value problem we have to solve is:

$$2y'' + 8y = 2u(t-c) \qquad y(0) = 1 \qquad y'(0) = 0$$

Taking the Laplace transform of both sides gies

$$s^2Y - s + 4Y = \frac{e^{-cs}}{s}$$

Hence,

$$\begin{aligned} Y &= \frac{s}{s^2+4} + \frac{e^{-cs}}{s(s^2+4)} \\ &= \frac{s}{s^2+4} + e^{-cs}\frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \\ &= \mathcal{L}\{\cos 2t + \frac{1}{4}u(t-c)(1-\cos 2(t-c))\} \end{aligned}$$

We need to examine the local maxima of $y(t)$ and determine whether or not it is possible to choose c so that they do exceeed 1.025 meters. If $t < c$, then $y(t) = \cos 2t$ and if $t \geq c$ then $y(t) = \cos 2t - \frac{1}{4}\cos 2(t-c) + \frac{1}{4}$ To ensure that the local maxima of $y(t)$ are as small as possible the bird ought to choose c so that the local maxima of $\cos 2t$ and $\cos 2(t-c)$ occur for the same values of t. This will ensure maximum cancelation between the two terms. This happens for example when $2(t-c) = 2t - 2\pi$, ie, $c = \pi$ and in this case $y(t) = \frac{3}{4}\cos 2t + \frac{1}{4}$ for $t \geq c = \pi$. So for this value of c the breaking point of the spring is not reached!

3. An object with mass 2 kgram stretches a spring 1.25 meters to equilibrium. At time $t = 0$ it is released 1 meter below its equilibrium position. At time $t = 1$ it is struck with a hammer and as a result its momentum is increased by 4 kgrams-meter/sec. Find its displacement $y(t)$ from its equilibrium position at any time $t > 0$.

ANS. Since the momentum is increased by 4 kgrams-meter/sec at $t = 1$ the external force in this problem can be expressed as $4\delta(t-1)$. The spring constant in this problems is $20/1.25 = 16$. Thus the IVP that we have to solve is

$$2y'' + 16y = 4\delta(t-1), \qquad y(0) = 1 \qquad y'(0) = 0$$

Taking Laplace transforms of both sides and solving for the unknown functions $Y(s)$ gives

$$Y = \frac{s}{s^2+8} + e^{-s}\frac{2}{s^2+8}$$

Taking inverse Laplace transforms gives:

$$Y = \mathcal{L}\{\cos\sqrt{8}\mathrm{t} + \frac{2}{\sqrt{8}}\mathrm{u}(\mathrm{t}-1)\sin\sqrt{8}(\mathrm{t}-1)\}$$

4. An object with mass 2 kg stretches a spring 4 m to equilibrium. At time $t = 0$ it is released 2 meters below its equilibrium position with an upward velocity of 3 meters/sec . At time $t = 6$ it is struck with a hammer and as a result its momentum is **decreased** by 7 kg-m/sec at that moment in time. At time $t = 8$ a constant external force of 9 Newtons is added and at $t = 10$ it is removed. Write down a differential equation with initial conditions for the displacement $y(t)$ of the object below its equilibrium position. **DO NOT SOLVE THIS EQUATION**

ANS.

$$2y'' + 5y = -7\delta(t-6) + 9(u(t-8) - u(t-10)) \qquad y(0) = 2 \quad y'(0) = -3$$

5. Chandelier - lion - monkey problem. Suppose that a hungry lion and monkey are in a room which contains nothing but a 2 kgram chandelier suspended from the ceiling by means of a spring which is stretched 2.5 meter to its equilibrium position. The chandelier oscillates well beyond the reach of the lion and hence the monkey would like to jump onto the chandelier in order to escape from the lion. We also make the following assumptions:

1. At time $t = 0$ the chandelier was set in motion by releasing it at 1 meter below its equilibrium position
2. The effect of air resistance on the chandelier is negligible.
3. If the spring is stretched 1.025 meter below the equilibrium position then it breaks.
4. The monkey's mass is .2 kg. At the instant the monkey lands on the chandelier its weight acts as an external force on the chandelier.
5. At the instant the monkey lands on the chandelier the momentum of the chandelier is increased by 1 unit.
6. Hooke's law is valid even when the spring is stretched to its breaking point.

Can the monkey choose a time $c > 0$ to jump onto the chandelier so as not to break the suspending spring?

ANS. Here we have a spring-mass system with $m = 2$ and the spring constant $k = 8$ and an external force $f(t) = .2 \cdot 10u(t-c) + \delta(t-c)$. Thus the initial value problem we have to solve is:

$$2y'' + 8y = 2u(t-c) + \delta(t-c) \qquad y(0) = 1 \qquad y'(0) = 0$$

Taking the Laplace transform of both sides gies

$$s^2Y - s + 4Y = e^{-cs}\left(\frac{1}{s} + \frac{1}{2}\right)$$

Hence, solving for Y

$$\begin{aligned}
Y &= \frac{s}{s^2+4} + e^{-cs}\left(\frac{1}{s(s^2+4)} + \frac{1/2}{s^2+4}\right) \\
&= \frac{s}{s^2+4} + e^{-cs}\left(\frac{1/4}{s} - \frac{s/4}{s^2+4} + \frac{2/4}{s^2+4}\right) \\
&= \frac{s}{s^2+4} + \frac{1}{4}e^{-cs}\left(\frac{1}{s} - \frac{s}{s^2+4} + \frac{2}{s^2+4}\right) \\
&= \mathcal{L}\{\cos 2\mathrm{t} + \frac{1}{4}\mathrm{u}(\mathrm{t}-\mathrm{c})\,(1 - (\cos 2(\mathrm{t}-\mathrm{c}) - \sin 2(\mathrm{t}-\mathrm{c})))\} \\
&= \mathcal{L}\{\cos 2\mathrm{t} + \frac{1}{4}\mathrm{u}(\mathrm{t}-\mathrm{c})\left(1 - \sqrt{2}\cos(2(\mathrm{t}-\mathrm{c}) - \frac{7\pi}{4})\right)\}
\end{aligned}$$

We now look at the oscillations of $y(t)$ when $t \geq c$ and try to choose c to make them as small as possible. When $t \geq c$

$$y(t) = \frac{1}{4} + \cos 2t - \frac{\sqrt{2}}{4}\cos(2(t-c) - \frac{7\pi}{4})$$

The oscillations here are as small as possible when $2(t-c) - \frac{7\pi}{4} = 2t$+period of cosine
ie, $-2c - \frac{7\pi}{4} = -2\pi$ or $c = \frac{\pi}{8}$.
Now with this choice of c and $t \geq c$ we have

$$y(t) = \frac{1}{4} + \left(1 - \frac{\sqrt{2}}{4}\right)\cos 2t$$

So the maximum value of $y(t)$ for $t \geq c$ is: $\frac{1}{4} + 1 - \sqrt{2}/4 \approx .897$ and the spring does not break.