Math 251 Sections 1,2,10 FAQ 10/2/09

1. Find a particular solution of the equation $y'' - 2y' - 3y = 2e^{3t}$

ANS. The characteristic polynomial $r^2 - 2r - 3$ has roots $r_1 = 3, r_2 = -1$.
Since the constant in the exponential is a simple root. So we need to multiply our usual guess for y_p by an extra factor of t: So we plug in $y_p = Ate^{3t}$

$$\begin{aligned} -3(y_p &= Ate^{3t}) \\ -2(y_p' &= A(1+3t)e^{3t}) \\ 1(y_p'' &= A(3+3(1+3t))e^{3t}) \end{aligned}$$

Note that as expected all terms involving t cancel: $A(-3t - 6t + 9t)e^{3t} = 0$.
Equating terms not involving t gives: $A(-2+3+3) = 2$ we see that $A = 1/2$ and consequently $y_p = \frac{1}{2}te^{3t})$

2. Find a particular solution to $y'' - y' = t + e^t$

ANS. The characteristic polynomial $r^2 - r$ has roots $r_1 = 0, r_2 = 1$.
We solve two problems and remember to add our answers: $y'' - y' = te^{0t}$ and $y'' - y' = e^t$
For the first the constant 0 in the exponential is a simple root. So we need to multiply our usual guess for y_p by an extra factor of t: So we plug in $y_p = t(At + B) = At^2 + Bt$

$$\begin{aligned} 0(y_p &= At^2 + Bt) \\ -1(y_p' &= 2At + B) \\ 1(y_p'' &= 2A) \end{aligned}$$

Note that as expected all terms involving t^2 disappear when plugging in. The t term $-2At$ must be t and the constant terms $2A - B$ must be 0. Therefore $y_p = (-\frac{1}{2}t - 1)t = -\frac{1}{2}t^2 - t$
For the second the constant 1 in the exponential is a simple root. So we need to multiply our usual guess for y_p by an extra factor of t: So we plug in $y_p = Ate^t$

$$\begin{aligned} 0(y_p &= Ate^t) \\ -1(y_p' &= A(1+t)e^t) \\ 1(y_p'' &= A(1+(1+t))e^t) \end{aligned}$$

As expected the t terms cancel and the remaining terms combined have coefficient equal to A. Therefore $y_p = te^t$.

3. Find a particular solution to $y'' + 2y' + 10y = e^{-t}\cos 3t$

ANS. The characteristic polynomial $r^2 + 2r + 10$ has roots $-1 \pm 3i$.
To find a particular solution y_d to the complexified equation $L[y] = e^{(-1+3i)t}$. Since the constant in the exponential is a root of the characteristic polynomial we multiply our usual guess for y_d by an extra factor of t: in order to get the form of a particular solution; that is,

$$y_d = Ate^{(-1+3i)t}$$

Plugging this into the equation gives:

$$\begin{aligned} 10(y_c &= Ate^{(-1+3i)t}) \\ 2(y_p' &= A(1+(-1+3i)t)e^{(-1+3i)t}) \\ 1(y_p'' &= A((-1+3i) + (-1+3i)(1+(-1+3i)t))e^{(-1+3i)t}) \end{aligned}$$

Adding up all terms involving t gives $A(10 + 2(-1+3i) + (-1+3i)^2)te^{(2+3i)t} = 0$, as expected.
Equating the remaining terms gives $A(2 + 2(-1+3i)) = 1$.
Thus $A = \frac{1}{6i} = \frac{-i}{6}$ and hence $y_d = \frac{-i}{6}te^{(-1+3i)t} = \frac{-i}{6}te^{-t}(\cos 3t + i\sin 3t)$.
Finally, $y_p = \mathrm{Re} y_d = \frac{1}{6}te^t \sin 3t$

4. Find a particular solution to $y'' - 6y' + 9y = e^{3t}$

ANS. The characteristic polynomial $r^2 - 6r + 9$ has a double root 3. So we need to multiply our usual guess for y_p by an extra factor of t^2. So we plug in $y_p = t^2 Ae^{3t}$.

$$\begin{aligned} 9(y_p &= At^2e^{3t}) \\ -6(y_p' &= A(2t + 3t^2)e^{3t}) \\ 1(y_p'' &= A(2 + 6t + 3(2t + 3t^2))e^{3t}) \end{aligned}$$

As expected all t^2 terms cancel: $9t^2 - 18t^2 + 9t^2 = 0$.
As expected all t terms cancel: $-12t + 6t + 6t = 0$.
The remaining term $2A$ must be equal to 1. Therefore $y_p = \frac{1}{2}t^2e^{3t}$.

For differential equations in Problems 5 - 7, write down the form of a particular solution y_p with as few unknown constants as possible. **Do not** determine any of the constants.

5. $y'' - 6y' + 9y = te^{3t}$

ANS. The characteristic polynomial $r^2 - 6r + 9$ has a double root 3. So we need to multiply our usual guess for y_p by an extra factor of t^2. So we plug in $y_p = t^2(At + B)e^{3t}$.

6. $y'' - 6y' + 10y = t^2e^{(3+i)t}$

ANS. The characteristic polynomial $r^2 - 6r + 10$ has roots $3 \pm i$. So we need to multiply our usual guess for y_d by an extra factor of t. So we plug in $y_d = t(At^2 + Bt + C)e^{(1+i3)t}$.

7. $y'' - y' - 6y = t^3e^{2t}$

ANS. The characteristic polynomial $r^2 - r - 6$ has roots $3, -2$. Neither one matches the constant in the exponential function. So we plug in $y_p = (At^3 + Bt^2 + Ct + D)e^{2t}$.