Math 251 Sections 1,2,10 FAQ 10/23/09

1. Rewrite the following linear system using matrix notation. Find the general solution. Also find the solution satisfying the given initial conditions

$$\begin{aligned} x' &= 5x - y, \qquad x(0) = 2 \\ y' &= 3x + y, \qquad y(0) = -1 \end{aligned}$$

ANS. The given equation is equivalent to

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

The characteristic equation of A is

$$0 = \det(A - rI) = \begin{vmatrix} 5-r & -1 \\ 3 & 1-r \end{vmatrix} = r^2 - 6r + 5 - (-3) = (r-2)(r-4)$$

We have two real eigenvalues $r_1 = 2$, $r_2 = 4$. Now

$$\mathbf{0} = (A - 2I)\xi = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\xi \quad \text{gives} \quad \xi_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

as the first eigenvector. Also,

$$\mathbf{0} = (A - 4I)\xi = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\xi \quad \text{gives} \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as the second eigenvector. Therefore the general solution is: $c_1 e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ To find the solution pair $(x(t), y(t))$ satisfying $x(0) = 2$, $y(0) = -1$ we need to find c_1 and c_2 such that $\begin{pmatrix} c_1 + c_2 = 2 \\ 3c_1 + c_2 = -1 \end{pmatrix}$ ie, $c_1 = -3/2, c_2 = 7/2$. Therefore

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{3}{2}e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{7}{2}e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}e^{2t} + \frac{7}{2}e^{4t} \\ -\frac{9}{2}e^{2t} + \frac{7}{2}e^{4t} \end{pmatrix}$$

2. Find the general solution of the following autonomous, linear. homogeneous system.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}$$

ANS.

The characteristic equation of A is

$$0 = \det(A - rI) = \begin{vmatrix} -2-r & 1 \\ -5 & 4-r \end{vmatrix} = r^2 - 2r - 8 - (-5) = (r-3)(r+1)$$

We have two real eigenvalues $r_1 = 3$, $r_2 = -1$. Now

$$\mathbf{0} = (A - 3I)\xi = \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix}\xi \quad \text{gives} \quad \xi_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

as the first eigenvector. Also,

$$\mathbf{0} = (A + I)\xi = \begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix}\xi \quad \text{gives} \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as the second eigenvector. Therefore the general solution is:

$$c_1 e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3. Consider the following second order linear constant coefficient homogeneous ODE with initial conditions:

$$y'' - 3y' + 2y = 0 \qquad y(0) = 3, \quad y'(0) = 5$$

Convert it to an IVP for a system of two first order ODE's and solve that problem. Compare it to the answer you get by solving the original problem. Do you see why the term "characteristic polynomial" can be used when working with systems without ambiguity?

ANS. Set $x = y'$, then $y' = x$ and $x' = 3y' - 2y = 3x - 2y$. Using matrix notation the system can be rewritten as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \mathbf{x} \qquad x(0) = 5, \quad y(0) = 3$$

The characteristic equation of A is

$$0 = \det(A - rI) = \begin{vmatrix} 3-r & -2 \\ 1 & -r \end{vmatrix} = r^2 - 3r + 2 = (r-2)(r-1)$$

This of course coincides with characteristic polynomial of the second order ODE. We have two real eigenvalues $r_1 = 2$, $r_2 = 1$. Now

$$\mathbf{0} = (A - 2I)\xi = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \xi \quad \text{gives} \quad \xi_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

as the first eigenvector. Also,

$$\mathbf{0} = (A - I)\xi = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \xi \quad \text{gives} \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as the second eigenvector. Therefore the general solution is:

$$c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Finally solving for the IVP:2

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

and hence $c_1 = 2$ $c_2 = 1$ and

$$\mathbf{x} = 2e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We conclude that

$$y = 2e^{2t} + e^{t}$$

which agrees with the solution we obtain solving the second order ODE without converting it into a system.

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