Math 251 Sections 1,2,10 FAQ 12/03/09

1. The functions $r^n \cos(n\theta)$, $r^n \sin(n\theta)$ for any nonnegative integer n are solutions of the Laplace equation in polar coordinates. Find the solution to the Dirichlet problem with boundary values

$$F(\theta) = 19.1 - 3\sin(4\theta) + 5\cos(11\theta) + 6\sin(19\theta)$$

on the boundary of the unit disk. That is, find the solution the of the Laplace equation on the unit disk $\{(r,\theta) \mid r < 1\}$ which at the points $\{(1,\theta) \mid -\pi < \theta < \pi\}$ on the boundary of the unit disk has the property $u(1,\theta) = F(\theta)$.

ANS. The solution is:

$$u(r,\theta) = 19.1 - 3r^4\sin(4\theta) + 5r^{11}\cos(11\theta) + 6r^{19}\sin(19\theta)$$

2. Do Problem 1 with $F(\theta)$ given the following:

$$F(\theta) = \begin{cases} 0 & \text{if} \quad -\pi < \theta < 0 \\ 40 & \text{if} \quad \theta \geq 0 \end{cases}$$

ANS. The solution is:

$$u(r,\theta) = \frac{a_o}{2} + \sum_1^\infty r^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$

where a_0, a_n, b_n are the Fourier coefficients of $F(\theta)$.

We easily find that $a_0 = \frac{1}{\pi}$area below graph $= \frac{1}{\pi}40\pi = 40$. Since $F(\theta) - 20$ is an odd function, we see that all other $a_n = 0$. Finally,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_\pi^\pi F(\theta)\sin n\theta\, d\theta \\ &= \frac{40}{\pi}\int_0^\pi \sin n\theta\, d\theta \\ &= \frac{-40}{n\pi}[\cos n\theta]_0^\pi \\ &= \frac{40}{n\pi}(1 - \cos n\pi) \end{aligned}$$

3. If in Problem 1 $F(\theta)$ is equal to e^θ then find the value of $u(0,0)$.

ANS.

$$u(0,0) = \frac{a_0}{2} = \frac{1}{2}\frac{1}{\pi}\int_{-\pi}^{\pi} e^\theta\, d\theta = \frac{1}{\pi}\sinh\pi$$

Or, if one prefers not to invoke sinh, then $u(0,0) = \frac{1}{2\pi}\left(e^\pi - e^{-\pi}\right)$

4. If all we know about $F(\theta)$ in Problem 1 is that $F(\theta) \geq 6$ and that $u(0,0) = 6$, then determine $u(1/2, \pi/4)$.

ANS. We saw in the Problem 3 that $u(0,0)$ is the average value of $F(\theta)$. If a function has an average value of 6 and it is always ≥ 6, then it must always be exactly 6. So $F(\theta) = 6$ for all θ. But we know the Fourier series of 6 is 6. Therefore $u(r,\theta) = 6$ for for $0 < r < 1$ and $-\pi < \theta \leq \pi$. So $u(1/2, \pi/4) = 6$.