Consider the linear system $\mathbf{x}' = A\mathbf{x}$. In Problems 1 -4 suppose that we general solution of the equation has already been found. Then state the name that is associated to the critical point $(0, 0)$, sketch the phase portrait, and state whether the critical point is unstable, asymptotically stable. Also, sketch a rough graph of $x(t)$, $y(t)$, for $\mathbf{x}$ satifying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Finally, consider the IVP $\mathbf{x}(0) = \begin{pmatrix} \alpha \\ 9 \end{pmatrix}$. Find α so that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. (Note it may be the case that there is no α, there is one α, or there are infinitely many α for which this is true.)

1. $\mathbf{x} = c_1 e^{3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ANS. Since the eigenvalues are distinct and positive, the origin is an unstable node for this system. To see the phase portrait use the phase portrait applet to produce the phase portrait for the system with $A = \begin{pmatrix} 5/2 & -1/4 \\ -1 & 5/2 \end{pmatrix}$
Clearly, no α gives $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

2. $\mathbf{x} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ANS. Since the eigenvalues have opposite sign, the origin is a saddle point which is always unstable. To see the phase portrait use the phase portrait applet to produce the phase portrait for the system with $A = \begin{pmatrix} -1/2 & 5/4 \\ 5 & -1/2 \end{pmatrix}$
The only $\mathbf{x}(0)$ for which $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ are the ones which are on the line in the direction of the eigenvector corresponding to the negative eigenvalue. Therefore we need to choose α so that $\frac{\alpha}{9} = \frac{1}{-2}$ That is, $\alpha = -\frac{9}{2}$.

3. $\mathbf{x} = c_1 e^{3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ANS. Since the eigenvalues have opposite sign, the origin is a saddle point which is always unstable. To see the phase portrait use the phase portrait applet to produce the phase portrait for the system with $A = \begin{pmatrix} 1/2 & -5/2 \\ -5/2 & 1/2 \end{pmatrix}$
The only $\mathbf{x}(0)$ for which $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ are the ones which are on the line in the direction of the eigenvector corresponding to the negative eigenvalue. Therefore we need to choose α so that $\frac{\alpha}{9} = \frac{1}{2}$ That is, $\alpha = \frac{9}{2}$.

4. $\mathbf{x} = c_1 e^{-3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

ANS. Since the eigenvalues are both negative, the origin is a node and it is asymptotically stable. To see the phase portrait use the phase portrait applet to produce the phase portrait for the system with $A = \begin{pmatrix} -5/2 & 1/2 \\ 1/2 & -5/2 \end{pmatrix}$
All solutions to this system approach $\mathbf{0}$ as $t \to \infty$. Therefore α can be chosen to be any number.

Addendum on how to find the matrix A given its eigenvalues and eigenvectors for students who have already taken Math 220:

```
We view the collection of all two dimensional vectors as an abstract vector space V and recall
that a 2x2 matrix A acting on vectors can be viewed as a linear transformation T:V --> V.

A linear transformation has a matrix representation in terms of ANY choice of basis for the
vector space V.

What is the matrix representation of T with respect to the standard basis?  That is easily
seen to be just the original matrix A.
```

The two eigenvectors of A are not multiples of each other since their eigenvalues are not equal (by applying a Math 220 Theorem). Hence the two eigenvectors

```
             xi_1    and     xi_2
```

also form a basis for our vector space V.

What is the matrix representation of T with respect to this basis? Well, since

```
    (A - r_i I)xi_i = zerovect.  (for i = 1,2)
```

by multiplying and transposing we see that

```
  A xi_i = r_i xi_i   (for i = 1,2)
```

ie, A applied to either one of the eigenvectors just multiplies that eigenvector by the corresponding eigenvalue.

So let B be the matrix representation of T with respect to the basis consisting of the two eigenvectors then we see that B is just the diagonal matrix

```
           [ xi_1  0 ]
   B =     [         ]
           [  0  xi_2]
```

Now let P be the matrix whose columns consist of the (column) eigenvectors then the Change of Basis Formula in Math 220 (whose proof is very simple) says:

```
          A = PBP^{-1}
```

where P^{-1} denotes the inverse of the matrix P.

Now let's apply this to Problem #1 on FAQ from 10/26/09.
There the eigenvalues are 3 and 2 and the corresponding eigenvectors are:

```
             [-1 ]    [ 1 ]
     [   ]    [   ]
     [ 2 ]    [ 2 ]
```

So the matrix P is:

```
              [-1  1 ]
   P =        [      ]
      [ 2  2 ]
```

Now let's use Maxima to check out this whole idea.

```
Maxima 5.18.0 http://maxima.sourceforge.net
Using Lisp SBCL 1.0.25
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
```

```
                                    done
(%i1) load(eigen);
(%o1)        /usr/local/share/maxima/5.18.0/share/matrix/eigen.mac
(%i2)  B:matrix([3, 0],[0,2]);
                                       [ 3  0 ]
(%o2)                                  [      ]
                                       [ 0  2 ]
(%i3)  P:matrix([-1,1],[2,2]);
                                      [ - 1  1 ]
(%o3)                                 [        ]
                                      [  2   2 ]

(%i4) A:P.B.invert(P);
                                     [  5     1 ]
                                     [  -   - - ]
                                     [  2     4 ]
(%o4)                                [          ]
                                     [       5  ]
                                     [ - 1   -  ]
                                     [       2  ]
(%i5) eigenvectors(A);
(%o5)                [[[3, 2], [1, 1]], [1, - 2], [1, 2]]
------------------------------------------------------------------
```

So the mathematical theory works!!

NOTE the [3,2] in the above output by Maxima consists of the eigenvalues of A; the [1, - 2], [1, 2] are the eigenvectors of A; the [1, 1] are the multiplicities of the eigenvalues [3,2] (obviously [1,1] because there are no repeated eigenvalues in this problem).

PS: If you would like to get computer help with doing linear algebra then DON'T use Maxima. Use Octave.