1. What is the Laplace transform of t? What is the Laplace transform of e^{-2t}? Find the Laplace transform of te^{-2t}. (Can you do it in two ways?)

ANS. The Laplace transform of t is $\frac{1}{s^2}$.
The Laplace transform of e^{-2t} is $\frac{1}{s+2}$.

To find the Laplace transform of te^{-2t} do not multiply the above two Laplace transforms.

Use the formula for Laplace transform of t times a function:

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2} \qquad \mathcal{L}\{te^{-2t}\} = -\frac{d}{ds}\frac{1}{(s+2)^2} = \frac{1}{(s+2)^2}$$

Or, you can use the shift formula

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$$

2. Find the inverse Laplace transform of $\frac{1}{s^3}$, $\frac{1}{s^4}$ and of $\frac{1}{(s-2)^3}$.

ANS. We know that

$$\mathcal{L}\{1\} = \frac{1}{s}$$

Therefore,

$$\mathcal{L}\{t\} = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$$

and therefore,

$$\mathcal{L}\{t^2\} = -\frac{d}{ds}\frac{1}{s^2} = \frac{2}{s^3}$$

So we see that

$$\mathcal{L}\{\frac{t^2}{2}\} = \frac{1}{s^3}$$

From the above we also see that

$$\mathcal{L}\{e^{2t}\frac{t^2}{2}\} = \frac{1}{(s-2)^3}$$

and

$$\mathcal{L}\{e^{2t}\frac{t^3}{6}\} = \frac{1}{(s-2)^4}$$

3. Solve the following IVP using Laplace transforms:

$$y'' - 3y' + 2y = e^t \qquad y(0) = 0 \qquad y'(0) = 1$$

(Can you solve this problem using undetermined coefficients?)

ANS. Let $Y = \mathcal{L}\{y\}$. Then taking Laplace transforms of both sides gives:

$$s(sY - 0) - 1 - 3(sY - 0) + 2Y = \frac{1}{s-1}$$

$$s^2Y - 1 - 3sY + 2Y = \frac{1}{s-1}$$

Solving for Y gives

$$Y = \frac{1}{(s-1)(s-2)} + \frac{1}{(s-1)^2(s-2)}$$

We use partial fraction to rewrite the right hand side: 1

$$\frac{1}{(s-1)(s-2)} + \frac{1}{(s-1)^2(s-2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}$$

We find that $A = -2$, $B = -1$ and $C = 2$ the right hand side is

$$\frac{2}{s-2} - \frac{2}{s-1} - \frac{1}{(s-1)^2}$$

Therefore the solution is

$$y = 2e^{2t} - 2e^t - te^t$$

4. Find the inverse Laplace transform of $\frac{s}{(s^2+4)^2}$.

ANS. This function appears to be related to the derivative of

$$\frac{2}{s^2+4} = \frac{2}{s^2+2^2}$$

Indeed,

$$\frac{d}{ds}\frac{2}{s^2+4} = \frac{-4s}{(s^2+4)^2}$$

So,

$$\mathcal{L}\{\frac{1}{4}t\sin 2t\} = -\frac{-s}{(s^2+2^2)^2}$$

5. Use Laplace transforms to solve the following IVP: $y'' + 4y = \cos 2t \quad y(0) = 0, \quad y'(0) = 1$ (Is there any resonance here?)

ANS. Let $\mathcal{L}\{\mathrm{y(t)}\} = \mathrm{Y(s)}$. We start taking Laplace transforms of both sides by observing that $\mathcal{L}\{\mathrm{y'(t)}\} = \mathrm{sY}$. We obtain

$$s(sY) - 1 + 4Y = \frac{s}{s^2+4}$$

Solving for Y gives

$$Y = \frac{1}{s^2+2^2} + \frac{s}{(s^2+2^2)^2}$$

It is easy to recognize the function which Laplace transforms into the first term on the right side of the above equation and from Problem 4 we can recognize the inverse Laplace transform of Therefore, the solution is

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t = (\frac{1}{2} + \frac{1}{4}t)\sin 2t$$