1. Find a particular solution to $y'' - y' + 4y = \sin 2t$

ANS. Find a particular solution y_d to the complexified equation $L[y] = e^{2it}$ and take its imaginary part to be y_p. The method of undetermined coefficients tells us to try:

$$y_d = Ae^{2it}$$

Plugging this into the equation gives:

$$\begin{aligned} 4(y_d &= Ae^{2it}) \\ -1(y_d' &= A2ie^{2it}) \\ 1(y_d'' &= A(-4)e^{2it}) \end{aligned}$$

Adding up all terms involving t gives $A(4 - 2i - 4) = 1$ Thus $A = \frac{1}{-2i} = \frac{i}{2}$ and hence $y_d = \frac{i}{2}(\cos 2t + i \sin 2t)$.
Finally, $y_p = \text{Im} y_d = \frac{1}{2}\cos(2t)$

2. Find a particular solution to $y'' - 2y' - y = 2\cos t$

ANS. Find a particular solution y_d to the complexified equation $L[y] = 2e^{it}$ and take its real part to be y_p. The method of undetermined coefficients tells us to try:

$$y_d = Ae^{2it}$$

Plugging this into the equation gives:

$$\begin{aligned} -1(y_d &= Ae^{it}) \\ -2(y_d' &= Aie^{it}) \\ 1(y_d'' &= A(-1)e^{it}) \end{aligned}$$

Adding up all terms involving t gives $A(-1-2i-1) = 2$ Thus $A = \frac{2}{-2-2i} = \frac{-1+i}{2}$ and hence $y_d = \frac{-1+i}{2}(\cos t + i \sin t)$.
Finally, $y_p = \text{Re} y_d = \frac{1}{2}(-\cos t - \sin t)$

3. Find a particular solution to $y'' - 2y' + y = e^t \sin t$

ANS. Find a particular solution y_d to the complexified equation $L[y] = e^{(1+i)t}$ and take its imaginary part to be y_p. The method of undetermined coefficients tells us to try:

$$y_d = Ae^{(1+i)t}$$

Plugging this into the equation gives:

$$\begin{aligned} 1(y_d &= Ae^{(1+i)t}) \\ -2(y_d' &= A(1+i)e^{(1+i)t}) \\ 1(y_d'' &= A(2i)e^{(1+i)t}) \end{aligned}$$

Adding up all the above and equating them with the expected right hands side gives: $A(1 - 2(1+i) + 2i) = 1$
Thus $A = -1$ and hence $y_d = -e^t(\cos t + i \sin t)$.
Finally, $y_p = \text{Im} y_c = -e^t \sin t$

4. Find a particular solution to $y'' + 2y' + y = te^{-t}\cos t$

ANS. Find a particular solution y_d to the complexified equation $L[y] = te^{(-1+i)t}$ and take its real part to be y_p.

$$y_d = (At + B)e^{(-1+i)t}$$

Plugging this into the equation gives:

$$\begin{aligned} 1(y_d &= (At + B)e^{(-1+i)t}) \\ 2(y_d' &= (A + (At + B)(-1+i))e^{(-1+i)t}) \\ 1(y_d'' &= (A(-1+i) + (-1+i)(A + (At + B)(-1+i)))e^{(-1+i)t}) \end{aligned}$$

The above expression is complicated. So we approach it in two parts. We first look at all the terms involving t and set them equal to $te^{(-1+i)t}$ and the sum of all remaining terms we set equal to zero. Adding up all terms involving t gives the equation $A + 2A(-1+i) + A(-1+i)^2 = 1$.
Simplifying this gives $A(1 - 2 + 2i - 2i) = 1$, ie, $A = -1$. Now, equating the remaining terms to zero gives $B + 2A + 2B(-1+i) + A(-1+i) + (-1+i)A + B(-1+i)^2 = 0$.
Fortunately, we already know $A = -1$, and that simplifies the equation to: $B - 2 + 2B(-1+i) - (-1+i) - (-1+i) + B(-2i) = 0$.
$B - 2 - 2B + 2iB + 2 - 2i - 2iB = 0$.
$B = -2i$
Therefore

$$y_d = (-t - 2i)e^{(-1+i)t} = (-t - 2i)e^{-t}e^{it} = (-t - 2i)e^{-t}(\cos t + i \sin t)$$

And finally,

$$y_p = \mathrm{Re} y_d = e^{-t}(-t \cos t + 2 \sin t)$$

5. Find the form of a particular solution to the nonhomogeneous ODE (but do not solve for the constants). $y'' - 6y' + 9y = t \cos 3t + t^2 e^{-t} \cos 3t + e^{2t} + t^3$

ANS. We split this into four problems:

$$y'' - 6y' + 9y = t \cos 3t \quad y'' - 6y' + 9y = t^2 e^{-t} \cos 3t \quad y'' - 6y' + 9y = e^{2t} \quad y'' - 6y' + 9y = t^3$$

The complexified ODE's are

$$y'' - 6y' + 9y = te^{3it} \quad y'' - 6y' + 9y = t^2 e^{-(1+3i)t} \quad y'' - 6y' + 9y = e^{2t} \quad y'' - 6y' + 9y = t^3$$

The solutions are

$$y_d = (At + B)e^{3it} \quad y_d = (Ct^2 + Dt + E)e^{-(1+3i)t} \quad y_p = Fe^{2t} \quad y_p = (Ht^3 + It^2 + Jt + K)$$

Finally taking real and imaginary parts as appropriate gives

$$y_p = \mathrm{Re}(At + B)e^{3it} \quad y_p = \mathrm{Re}(Ct^2 + Dt + E)e^{-(1+3i)t} \quad y_p = Fe^{2t} \quad y_p = (Ht^3 + It^2 + Jt + K)t^3$$

The general form of the solution to our problem is the sum of the above four.