Math 251 Sections 1,2,10 FAQ 12/04/09

1. Find all eigenvalues and eigenfunctions of the following boundary value problem:

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(2) = 0$$

What if 2 is replace by $L > 0$?

ANS. If $\lambda < 0$ then setting $\omega = \sqrt{-\lambda}$ gives the general solution is

$$X = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$$

Then

$$X' = \omega(c_1 \sinh(\omega x) + c_2 \cosh(\omega x))$$

Setting $x = 0$ in X' gives $c_2 = 0$ and finally setting $x = 2$ in X' gives $\omega c_1 \sinh(\omega 2) = 0$ Since sinh is zero only at $t = 0$, we conclude that c_1 must be zero. Thus there are no eigenvalues when $\lambda < 0$.

If $\lambda = 0$ then the general solution is $X = mt + b$ and then plugging in the boundary condition in to X' shows that $m = 0$. Therefore, $\lambda = 0$ is an eigenvalue and any nonzero constant is the corresponding eigenfunction.

If $\lambda > 0$ then setting $\omega = \sqrt{\lambda}$ the general solution is:

$$X = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

and

$$X' = \omega(-c_1 \sin(\omega x) + c_2 \cos(\omega x))$$

Setting $t = 0$ in X' gives c_2. We now set $t = 2$ in X' and we see that $-\omega c_1 \sin(2\omega) = 0$. In order to avoid having also $c_1 = 0$ we need to have $2\omega = n\pi$, n any nonnegative integer. Thus we obtain the following eigenvalues and eigenfunctions:

$$\lambda = \left(\frac{n\pi}{2}\right)^2 \qquad \cos\left(\frac{n\pi}{2}x\right)$$

If 2 is replace by L then we obtain the following eigenvalues and eigenfunctions:

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad \cos\left(\frac{n\pi}{L}x\right)$$

The eigenfunctions in this problem appeared in when we were finding the temperature of a rod with insulated ends.

2. Find all eigenvalues and eigenfunctions of the following boundary value problem:

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(2) = 0$$

ANS. If $\lambda < 0$ then setting $\omega = \sqrt{-\lambda}$ gives the general solution is

$$X = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$$

Then

$$X' = \omega(c_1 \sinh(\omega x) + c_2 \cosh(\omega x))$$

Setting $x = 0$ in X' gives $c_2 = 0$ and finally setting $x = 2$ in X gives $c_1 \cosh(\omega 2) = 0$ Since cosh is never zero, we conclude that c_1 must be zero. Thus there are no eigenvalues when $\lambda < 0$.

If $\lambda = 0$ then the general solution is $X = mx + b$ and then setting $x = 0$ in X' gives $m = 0$ and setting $x = 2$ in X gives $b = 0$. Therefore, $\lambda = 0$ is not an eigenvalue.

If $\lambda > 0$ then setting $\omega = \sqrt{\lambda}$ the general solution is:

$$X = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

and

$$X' = \omega(-c_1 \sin(\omega x) + c_2 \cos(\omega x))$$

Setting $t = 0$ in X' gives $c_2 = 0$. We now set $t = 2$ in X and obtain that $c_1 \cos(2\omega) = 0$. In order to avoid having $c_1 = 0$ we need to have $2\omega = \pi(n + 1/2)$, n any nonnegative integer. Therefore we obtain the following eigenvalues and eigenfunctions:

$$\lambda = \left(\frac{\pi(2n+1)}{4}\right)^2 \qquad \cos\left(\frac{\pi(2n+1)}{4}x\right)$$

3. Consider the following initial value and boundary value problems for certain second order linear constant coefficient ODE's. Determine which have a unique solution and which do not.

$$\begin{array}{rrcll}
\text{i.} & X'' + 11X & = & 0, & X(0) = 1, \quad X'(0) = 2 \\
\text{ii.} & X'' - 12X & = & 0, & X(0) = -1, \quad X'(0) = 2 \\
\text{iii.} & X'' + 13X & = & 0, & X(0) = -3, \quad X(21) = 4 \\
\text{iv.} & X'' - 14X & = & 0, & X(0) = 3, \quad X'(21) = -4 \\
\text{v.} & X'' + 9X & = & 0, & X'(0) = -5, \quad X'(\pi) = 6 \\
\text{vi.} & X'' - 16X & = & 0, & X'(0) = 5, \quad X'(\pi) = -6
\end{array}$$

ANS. Problems i. and ii. are initial value problems. They always have unique solutions for 2nd order constant coefficient homogeneous linear ODE's.

For iii., the eigenvalues corresponding to the $L = 21$ are $\left(\frac{n\pi}{21}\right)^2$. Since $\lambda = 13$ is not one of them, no solution other than 0 satisfies zero boundary conditions. Therefore by the superposition principle a solution satisfying these boundary conditions is unique. It is easy to write the general solution and solve for c_1 and c_2 to find the solution that satisfies the given boundary conditions.

For iv., the eigenvalues corresponding to the $L = 21$ are positive. So $\lambda = -14$ is not one of them. Therefore no solution of this ODE other than 0 satisfies zero boundary conditions. Therefore by superposition principle a solution satisfying these boundary conditions is unique. It is easy to write the general solution and solve for c_1 and c_2 to find the solution that satisfies the given boundary conditions.

For v., the eigenvalues corresponding to the $L = \pi$ are $\left(\frac{n\pi}{\pi}\right)^2 = n^2$. The $\lambda = 9$ is one of them. So the solution with zero boundary conditions is not unique. However, the general solution is $X = c_1 \cos 3x + c_2 \sin 3x$ and plugging the boundary conditions into $X- = -3c_1 \sin 3x + 3c_2 \cos 3x$ shows that there is no solution satisfying the given boundary conditions.

The λ in vi. is negative but the eigenvalues are positive. Therefore no solution other than 0 satisfies zero boundary conditions. Therefore the solution satisfying these boundary conditions is unique. It is easy to write the general solution and solve for c_1 and c_2 to find the solution that satisfies the given boundary conditions.

The point of this problem is to demonstrate that the uniqueness of the solution with zero boundary conditions is connected with the existence of solutions to the 2-point boundary value problem with nonzero solutions.