For Problems 1, 2 and 3 consider the matrix A that defines the linear system $\mathbf{x}' = A\mathbf{x}$. Find the general solution and also sketch a phase portrait (with sufficiently many trajectories) Also state the name of the critical point $(0,0)$, and state whether it is unstable, stable or asymptotically stable. What is the difference between stable and asymptotically stable?

1. $A = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$

ANS. The characteristic polynomial is $(-2-r)(-2-r)-(-9) = (r+2)^2+9$. A complex eigenvalue is: $-2-3i$. Then $A-(-2-3i)I = \begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix}$ and hence the corresponding eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$. So a complex solution is

$$e^{(-2-3i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{-2t}(\cos 3t - i\sin 3t)\begin{pmatrix} i \\ 1 \end{pmatrix} =$$

We obtain real solutions by taking the real and imaginary parts of the above:

$$\mathbf{x}_1 = e^{-2t}\begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \qquad \mathbf{x}_2 = e^{-2t}\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

The origin is a spiral for this system, which is asymptotically stable. Also, since $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ the trajectory is traced out in the clockwise direction.

2. $A = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}$

ANS. The characteristic polynomial is $(2-r)(2-r)-(-2) = (r-2)^2+2$. A complex eigenvalue is: $2-\sqrt{2}i$. Then $A-(2-\sqrt{2}i)I = \begin{pmatrix} \sqrt{2}i & -2 \\ 1 & \sqrt{2}i \end{pmatrix}$ and hence the corresponding eigenvector is $\begin{pmatrix} \sqrt{2} \\ i \end{pmatrix}$. So a complex solution is

$$e^{2-\sqrt{2}it}\begin{pmatrix} 2 \\ \sqrt{2}i \end{pmatrix} = e^{2t}(\cos\sqrt{2}t - i\sin\sqrt{2}t)\begin{pmatrix} 2 \\ \sqrt{2}i \end{pmatrix} =$$

We obtain real solutions by taking the real and imaginary parts of the above:

$$\mathbf{x}_1 = e^{2t}\begin{pmatrix} 2\cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{pmatrix} \qquad \mathbf{x}_2 = e^{2t}\begin{pmatrix} -2\sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t \end{pmatrix}$$

The origin is a spiral for this system, which is unstable. Also, since $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ the trajectory is traced out in the counter-clockwise direction.

3. $A = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$

ANS. The characteristic polynomial is $(1-r)(-1-r)+2 = r^2+1$. A complex eigenvalue is: $-i$. Then $A+i)I = \begin{pmatrix} -1+i & 1 \\ -2 & 1+i \end{pmatrix}$ and hence the corresponding eigenvector is $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. So a complex solution is

$$e^{-it}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = (\cos t - i\sin t)\begin{pmatrix} 1 \\ 1-i \end{pmatrix} =$$

We obtain real solutions by taking the real and imaginary parts of the above:

$$\mathbf{x}_1 = \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} -\sin t \\ -\cos t - \sin t \end{pmatrix}$$

The origin is a center for this system, which is stable. Also, since $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ the trajectory is traced out in the clockwise direction.