Math 251 Sections 1,2,10 FAQ 9/09/09

DO NOT TRY TO SOLVE the ODEs appearing in problems 1-6. For each of the initial value problems state whether or not there is a guarantee that a unique solution exists. Also state whether or not there is a guarantee about a specific interval of existence for the solution. If there is such a guarantee, then what is that interval? Make sure you indicate how the hypotheses of the guarantee is verified.

1. $(1-t^2)y' = 3t^2 - 2ty, \quad y(0) = 1$

ANS.
This is a linear equation with $p(t) = 2t/(1-t^2)$ and $g(t) = 3t^2/(1-t^2)$. The functions $p(t)$ and $q(t)$ are continuous except at $t = \pm 1$. The point t_0 is in the interval $(-1, 1)$. Therefore the solution is guaranteed to exist and to be unique in this interval.

2. $(1-t^2)y' = 3t^2 - 2ty, \quad y(-16) = 0$

ANS. This is a linear equation with $p(t) = 2t/(1-t^2)$ and $g(t) = 3t^2/(1-t^2)$. The functions $p(t)$ and $q(t)$ are continuous except at $t = \pm 1$. The point t_0 is in the interval $(-\infty, -1)$. Therefore the solution is guaranteed to exist and to be unique in this interval.

3. $(1-t^2)y' = 3t^2 - 2ty, \quad y(2) = 3$

ANS. This is a linear equation with $p(t) = 2t/(1-t^2)$ and $g(t) = 3t^2/(1-t^2)$. The functions $p(t)$ and $q(t)$ are continuous except at $t = \pm 1$. The point t_0 is in the interval $(1, \infty)$. Therefore the solution is guaranteed to exist and to be unique in this interval.

4. $(1-t^2)y' = 3t^2 - 2ty^{1/5}, \quad y(-2) = 1$

ANS. This is a nonlinear equation $y' = f(t, y)$ with

$$f(t, y) = \frac{3t^2 - 2ty^{1/5}}{1-t^2}$$

. Also

$$\frac{\partial f}{\partial y} = \frac{-2t/5}{y^{4/5}(1-t^2)}$$

. Both of these are continuous in an open rectangle containing the point $(-2, 1)$ in the ty-plane. Therefore the IVP has a unique solution in some interval I containing $t_0 = -2$. We cannot determine the size of this interval without actually solving the IVP.

5. $(1-t^2)y' = 3t^2 - 2ty^{1/5}, \quad y(2) = 0$

ANS. This is a nonlinear equation $y' = f(t, y)$ with both $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ as in the previous problem. However, $\frac{\partial f}{\partial y}(t, y)$ in not continuous in a neighborhood of $(2, 0)$. Therefore none of the guarantees we have had in this course apply.

6. $(1-t^2)y' = 3t^2 - 2ty^2, \quad y(-2) = 0$

ANS. This is a nonlinear equation $y' = f(t, y)$ with

$$f(t, y) = \frac{3t^2 - 2ty^2}{1-t^2}$$

. Also

$$\frac{\partial f}{\partial y} = \frac{-4ty}{1-t^2}$$

. Both of these are continuous in an open rectangle containing the point $(-2, 0)$ in the ty-plane Therefore the IVP has a unique solution in some interval I containing $t_0 = -2$. We cannot determine the size of this interval without actually solving the IVP.

7. If a first order linear ODE is also autonomous, then explain why the domain of the solution is $(-\infty, \infty)$. Also, explain why the graphs of two different solutions y_1 and y_2 do not intersect.

ANS. In general a first order linear ODE has the form $y' = -py + g$ where p and g are constants. If the ODE is also autonomous, then t does not appear in the formulas for p and g and hence p and g are constant functions of t. Constant functions are everywhere continuous. In the absence of points of discontinuity for the r.h.s of the ODE, the existence guarantee for first order linear ODE's says that the domain of a solution that satisfies any initial condition is $(-\infty, \infty)$.
Now assume that the solutions y_1 and y_2 did intersect at the point (t_0, y_0). We apply the uniqueness with that point as the initial condition to the ODE, and conclude that y_1 and y_2 are the same solution, which contradicts our assumtion that they are "two different solutions".

8. More generally if for a first order autononmous ODE $y' = f(y)$ we know that both $f(y)$ and $\dfrac{df}{dy}$ are everywhere continuous then what can you say about the domains of solutions and the possibility of two different solutions having graphs that intersect?

ANS. Everything we said in the solution to the previous problem after we observed that there are no discontinuities of p and g applies here because even though the ODE is nonlinear we are assuming that "both $f(y)$ and $\dfrac{df}{dy}$ are everywhere continuous".