1. Consider the following nonhomogeneous differential equation with initial conditions:

$$L[y] = y'' - y = 1 \quad y(0) = 1, y'(0) = 0$$

Solve it using the following steps:

a. Verify that the function $y_p = -1$ is a solution to the equation

$$L[y] = 1$$

ANS. Since $y_p'' = 0$, $L[y_p] = y_p'' - y_p = 0 - (-1) = 1$.

b. Find the general solution $y_c = c_1 y_1 + c_2 y_2$ of the associated homogeneous equation

$$L[y] = 0$$

ANS. The characteristic polynomial $r^2 - 1 = 0$. Therefore, $y_c = c_1 e^t + c_2 e^{-t}$ is the general solution of the associated homogeneous equation.

c. Show that $y_c + y_p$ is a solution to $L[y] = 1$.

ANS.

$$L[y_c] + L[y_p] = 0 + 1 = 1$$

d. Find the constants c_1, c_2 so that $y_c + y_p$ matches the given initial conditions.

ANS.

$$(y_c + y_p)(0) = c_1 + c_2 - 1 = 1$$
$$(y_c' + y_p')(0) = c_1 - c_2 - 0 = 0$$

Adding these two equation gives $2c_1 = 2$, $c_1 = 1$ and hence from the second equation $c_2 = c_1 = 1$.

2. Consider the following linear ODE with nonconstant coefficients:

$$t^2 y'' - 2ty' + 2y = 0, \ t > 0$$

It is fairly obvious that $y_1 = t$ is a solution of this ODE. Let y_2 another solution which is not a constant multiple of y_1 such that $W(y_1, y_2)(1) = 2$.

a. Use Abel's formula to find $W(y_1, y_2)(t)$.

ANS. Note that we must divide through by t^2 Since $p(t) = -2/t$, we have $W(y_1, y_2)(t) = Ce^{2\ln t} = Ct^2$. From the value of the Wronskian when $t = 1$, we see that $C = 2$. Hence $W(y_1, y_2)(t) = 2t^2$.

b. Use this information to find a first order ODE to which y_2 solves.

ANS. We have $W(t, y_2) = det \begin{pmatrix} t & y_2 \\ 1 & y_2' \end{pmatrix} = ty_2' - y_2 = 2t^2$

c. Find y_2.

ANS. The ODE $ty_2' - y_2 = 2t^2$ is a linear ODE for the unknown function y_2. First divide by t to get $y_2' - t^{-1} y_2 = 2t$. Since $p = -t^{-1}$ the integrating factor is $\mu = t^{-1}$. Therefore, $(y_2/t)' = 2$ and $(y_2/t) = 2t + D$ and hence $y_2 = 2t^2 + Dt$, or $y_2 = 2t^2 + Dt$. Any constant D works here. So why not take $D = 0$ and multiply y_2 by $1/2$ to get $y_2 = t^2$.