Math 251 Sections 1,2,10 FAQ 11/09/09

1. Determine smallest positive period for the following functions:

$$\sin\left(\frac{2\pi}{3}x\right) \qquad \cos\left(\frac{7}{11}x\right)$$

ANS. We solve $\frac{2\pi}{3}P = 2\pi$ to obtain period $P = 3$
We solve $\frac{7}{11}P = 2\pi$ to obtain period $P = \frac{22\pi}{7}$

2.
Consider the functions $f(x)$ and $g(x)$ given below. It is easy to find the Fourier series on the interval $[-3, 3]$ for one of them and not so easy for the other one. Find the easy one.

$$f(x) = 4\sin(\frac{3\pi}{2}x) \qquad g(x) = \frac{1}{4} + 2\cos(\pi x)$$

ANS. We try to see whether the sine in $f(x)$ or the cosine in $g(x)$ fits the pattern of a Fourier series on the interval $[-3, 3]$ That is, whether

$$\frac{3\pi}{2} = \frac{n\pi}{3}$$

for a positive integer n or whether

$$\pi = \frac{n\pi}{3}$$

for a positive integer n. The answer to the former is no and the answer to the latter is yes if $n = 3$. Therefore $g(x)$ is already in the form of a Fourier series and nothing needs to be done. What would need to be done to find the Fourier for $f(x)$ on $[3, -3]$? What if the interval would be changed to $[-4, 4]$?

3. Consider the function

$$f(x) = u(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

Find the Fourier series of $f(x)$ on the interval $[-3, 3]$
Use summation notation to express your answer.

ANS.

$$\begin{aligned}
a_0 &= \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_{0}^{3} 1\,dx = 1 \\
a_n &= \frac{1}{3}\int_{0}^{3} 1\cos\left(\frac{n\pi}{3}x\right)\,dx = \frac{1}{3}\frac{3}{n\pi}\left[\sin\left(\frac{n\pi}{3}x\right)\right]_0^3 = 0 \\
b_n &= \frac{1}{3}\int_{0}^{3} 1\sin\left(\frac{n\pi}{3}x\right)\,dx = \frac{-1}{3}\frac{3}{n\pi}\left[\cos\left(\frac{n\pi}{3}x\right)\right]_0^3 = \frac{-1}{n\pi}(\cos n\pi - \cos 0) \\
b_1 &= \frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{2}{3\pi}
\end{aligned}$$

Using summation notation

$$\frac{1}{2} \quad + \quad \sum_{n=1}^{\infty} \frac{1}{n\pi}(1 - \cos n\pi)\sin\left(\frac{n\pi}{3}x\right)$$

4. Find the Fourier series of the function $f(x) = \sin(4\pi x)$ on the interval $[-2, 2]$.

ANS. We plug in $f(x)$ into the formulas for the Fourier coefficients and then use the formulas from the hand out on trig integrals:

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} \sin(4\pi x)\ dx = 0 \\
a_n &= \frac{1}{2}\int_{-2}^{2} \sin(4\pi x)\cos\left(\frac{n\pi}{2}x\right)\ dx = 0 \\
b_n &= \frac{1}{2}\int_{-2}^{2} \sin(4\pi x)\sin\left(\frac{n\pi}{2}x\right)\ dx = \begin{cases} 0 & \text{if} \quad n \neq 8 \\ 1 & \text{if} \quad n = 8 \end{cases}
\end{aligned}$$

5. Find the Fourier series of $f(x) = x$ on the interval $[-2, 2]$
Use summation notation to express your answer.

ANS. Here it is best to start by writing down some formulas that are easily found using integration by parts:

$$\begin{aligned}
\int x\cos rx\,dx &= \frac{1}{r}x\sin rx + \frac{1}{r^2}\cos rx \\
\int x\sin rx\,dx &= \frac{-1}{r}x\cos rx + \frac{1}{r^2}\sin rx
\end{aligned}$$

Now onto computing Fourier coefficients. Remember that the definite integral of an odd function over a symmetric interval is 0.

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} x\,dx = 0 \\
a_n &= \frac{1}{2}\int_{-2}^{2} x\cos\left(\frac{n\pi}{2}x\right)\ dx = 0 \\
b_n &= \frac{1}{2}\int_{-2}^{2} x\sin\left(\frac{n\pi}{2}x\right)\ dx \\
&= \int_{0}^{2} x\sin\left(\frac{n\pi}{2}x\right)\ dx \\
&= \left[\frac{-2}{n\pi}x\cos\left(\frac{n\pi}{2}x\right) + \frac{2^2}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}x\right)\right]_0^2 \\
&= \frac{-4}{n\pi}\cos(n\pi) \\
b_1 &= \frac{4}{\pi}, \quad b_2 = \frac{-4}{2\pi}, \quad b_3 = \frac{4}{3\pi}
\end{aligned}$$

Thus the Fourier series is

$$\frac{4}{\pi}\sin\left(\frac{n\pi}{2}x\right) - \frac{4}{2\pi}\sin\left(\frac{n\pi}{2}x\right) + \frac{4}{3\pi}\sin\left(\frac{n\pi}{2}x\right) + \ldots$$

Using summation notation it is:

$$\sum_{n=1}^{\infty} \frac{-4}{n\pi}\cos n\pi \sin\left(\frac{n\pi}{2}x\right)$$

6. Find the Fourier series of $f(x) = x$ on the interval $[-\pi, \pi]$
Use summation notation to express your answer.

ANS.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx \\
&= \frac{2}{\pi}\left[\frac{-1}{n}x\cos nx + \frac{2}{n^2}\sin nx\right]_0^{\pi} = \frac{-2}{n}\cos n\pi \\
b_1 &= \frac{2}{1}, \quad b_2 = \frac{-2}{2}, \quad b_3 = \frac{2}{3}
\end{aligned}
$$

Thus the Fourier series is

$$\frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x + \ldots$$

Using summation notation it is:

$$\sum_{n=1}^{\infty} \frac{-2}{n}\cos n\pi \sin nx$$