Math 251 Sections 1,2,10 FAQ 9/4/09

1. Suppose that an object of mass 1kg is thrown upward at a height of 100m above the ground with an initial velocity whose magnitude is 30m/s . Assume that air resistance exerts a force whose magnitude is twice the speed of the object. (Recall that speed is the magnitude of the velocity). Write an IVP satisfied by v, the velocity of the object. Check that it is valid when the object is moving upward and when the object is moving downward.

ANS. Let v be the velocity of the object at time t. By Newton's second law: $mv' =$ the total force on the object, which is the sum of the force due to gravity and the force exerted by air resistance. Therefore,

$$mv' = -10m - 2v$$

Plugging in $m = 1$ gives

$$v' = -10 - 2v, \qquad v(0) = 30$$

When the object is moving upward, v is positive but air resistance is working in the same direction as gravity and hence should be a negative force and indeed $-2v$ is negative.
When the object is moving downward, v is negative and air resistance is working in the direction opposite gravity and hence should be a positive force and indeed $-2v$ is posiive. Therefore the ODE is always valid.

2. Draw a direction field for this ODE. What is an equilibrium solution? What is the highest speed (equals the magnitude of the velocity) for the object on its way up and what is an upper bound for the speed of the object on its way down?

ANS. An equilibrium solution is $v = -5$. Above the equilibrium solution the arrows are always pointing downward, indicating the fact that the velocity of the object is always decreasing. On the way up the velocity decreases from 30 to 0 and on the way down it decreases from 0 but never goes below -5m/s.
Also, try to check the draw the direction field with Maxima.

3. Solve for v.

ANS. The ODE for v is linear and separable equation, so you have a choice of methods to find the general solution. Let's solve it as a linear equation:

$$1v' + 2v = -10 \qquad v(0) = 30$$

The integrating factor is $\mu = e^{t/2}$ and hence

$$ve^{2t} = -5e^{2t} + C$$

Plugging in the initial condition gives

$$C = 35 \qquad v = -5 + 35e^{-2t}$$

4. Find a formula for $y(t)$ the displacement of the object above the ground at time t. Determine the velocity of the object at the moment in time that the object hits the ground. (Either use Newton's method to solve the equation $y(t) = 0$ or graph the function y using a computer/calculator to approximate the root.)

ANS. We need to solve

$$s' = v = -5 + 35e^{-2t} \qquad s(0) = 100$$

So

$$s = -5t - \frac{35}{2}e^{-2t} + \frac{235}{2}$$

To find the the object hits gound we need to find the positive solution of the following equation

$$0 = -5t - \frac{35}{2}e^{-2t} + \frac{235}{2}$$

This equation can be solved using Newton's method, guessing that a positive solution is near $t = 23.5$. A method with limited accuracy, but adequate for this course, is graphing the function s and reading the cursor's t coordinate at the t-intercepts. Another method of approximating the solution to this equation is to observe that when t gets to about 20 seconds the exponential term becomes extremely small and can be ignored. Therefore, solving $0 = -5t + \frac{235}{2}$ does approximate the solution.

5. Suppose a small tractor has a mass of 200kg, including the operator, and can maintain a constant acceleration of 6 m/s^2. If it crosses a large field where the grass resists its forward motion with a force whose magnitude is 10 times its speed, then write a differential equation for the velocity of this tractor while it crosses the field. Draw a direction field for this differential equation. Discuss how the velocity of tractor towards the end of the field depends on its velocity at the start of the field.

ANS. The force on the tractor due to its constant acceleration is 6×200 and the force due to the grass is $-10 \times v$. Therefore the total force is:

$$200v' = 6(200) - 10v$$

Or,

$$v' = 6 - v/20$$

The terminal velocity of the tractor is 120 m/s.
No matter what velocity the tractor starts with, eventually it will approach this terminal velocity as it ends its excursion across the field.