Math 251 Sections 1,2,10 FAQ 11/04/09

1. For the following nonhomogeneous linear system find the critical point $\mathbf{x}_0$, introduce the new variable $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$, solve the homogeneous linear system $\mathbf{u}' = A\mathbf{u}$ and draw a phase portrait in the xy-plane.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

ANS. We easily see that a critical point x_0 of this system is $\mathbf{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. We introduce the new variable $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{x} - \mathbf{x}_0 = \begin{pmatrix} x+1 \\ y \end{pmatrix}$. In terms of this variable the original system is

$$\mathbf{u}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{u} = A\mathbf{u}$$

This matrix has characteristic polynomial is $(-2-r)(-2-r) - 1 = r^2 + 4r + 3 = (r+1)(r+3)$. So it has distinct eigenvalues -1 and -3 and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ Hence the critical point x_0 of the original system is an asymptotically stable node. Click here for the phase portrait plotter.

2. For the following nonhomogeneous linear system find the critical point $\mathbf{x}_0$, introduce the new variable $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$, solve the homogeneous linear system $\mathbf{u}' = A\mathbf{u}$ and draw a phase portrait in the xy-plane.

$$\mathbf{x}' = \begin{pmatrix} -2 & 3 \\ 4 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 \\ -8 \end{pmatrix}$$

ANS. We easily see that a critical point x_0 of this system is $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We introduce the new variable $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{x} - \mathbf{x}_0 = \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$. In terms of this variable the original system is

$$\mathbf{u}' = \begin{pmatrix} -2 & 3 \\ 4 & 2 \end{pmatrix} \mathbf{u} = A\mathbf{u}$$

This matrix has characteristic polynomial is $(-2-r)(2-r) - 12 = r^2 - 16$. So it has distinct eigenvalues 4 and -4 and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ Hence the critical point x_0 of the original system is an unstable saddle. Click here for the phase portrait plotter.

3. For the following nonhomogeneous linear system find the critical point $\mathbf{x}_0$, introduce the new variable $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$, solve the homogeneous linear system $\mathbf{u}' = A\mathbf{u}$ and draw a phase portrait in the xy-plane.

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

ANS. We easily see that a critical point x_0 of this system is $\mathbf{x}_0 = \begin{pmatrix} 7/5 \\ 1/5 \end{pmatrix}$. We introduce the new variable $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{x} - \mathbf{x}_0 = \begin{pmatrix} x-7/5 \\ y-1/5 \end{pmatrix}$. In terms of this variable the original system is

$$\mathbf{u}' = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} \mathbf{u} = A\mathbf{u}$$

This matrix has characteristic polynomial is $(-2-r)(-2-r) + 1 = (-r-2)^2 + 1$. So it has complex eigenvalues $-2 \pm i$. and the corresponding eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$. Hence the critical point x_0 of the original system is an asymptotically spiral. Click here for the phase portrait plotter.

The solution to this system is

$$\mathbf{x} = \mathbf{u} + \begin{pmatrix} 7/5 \\ 1/5 \end{pmatrix}$$

where

$$\mathbf{u} = c_1 \mathrm{Re}\mathbf{u}_d + c_1 \mathrm{Im}\mathbf{u}_d$$

and

$$\mathbf{u}_d = e^{(-2+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and

$$\mathbf{u}_1 = e^{-2t} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \qquad \mathbf{u}_2 = e^{-2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$