1. A salt water mixture in a large tank is constantly being mixed to keep it homogeneous. A mixture with concentration of 1 kg salt per liter water flows in at the rate of 3 l/min. The homogeneous mixture flows out the same rate of 3 l/min. Initially there are 300 kg disolved in 200 liters of water in the tank. Let $Q(t)$ be the quantity of salt in the tank at time t. Write down a differential equation for $Q(t)$ and draw a direction field for it.

ANS. Let $Q = Q(t)$ be the quantity of salt in the tank at time t.

Then $Q' =$ rate salt goes in - rate salt goes out
= flow rate in times concentration - flow rate out times concentration.
So

$$Q' = \frac{3\,\text{liter}}{\text{min}}\frac{1\,\text{kg}}{\text{liter}} - \frac{3\text{liter}}{\text{min}}\frac{Q\,\text{kg}}{200\,\text{liter}}$$

So the ODE and initial condition are:

$$Q' = 3 - \frac{3Q}{200} \qquad Q(0) = 300$$

To plot the direction field in the tQ-plane we start by plotting line segments with appropriate slopes along the Q-axis. Since the equation does not explicitly involve the variable t, the remainder of the direction field is obtain by translating the line segments to the right. Along the Q axis the line segments slope upward when Q is less than 200 and downward when Q is greater than 200. So we see that the quantity of salt is increasing whenever it is below 200 and decreasing whenever it is above 200. In this case the initial value is 300; so we expect the solution to always be decreasing.
Click here to download Maxima to your computer and plot the direction field with the instructions load(plotdf); plotdf $3 * (1 - y/200)$;

2. Is the concentration of the salt in the mixture going up or down? After a very long time what will the concentration of salt in the tank be? (The above are more difficult when the volume of mixture is not constant.)

ANS. From the direction field above it is obvious that every solution tends to the equilibrium solution $Q = 200$. Here the volume of mixture in the tank is constant, so we can easily talk about the concentration of the salt and see that it tends to 1 regardless what the initial concentration of the mixture is. Note that this is the concentration of the mixture flowing into the tank.

3. Find an explicit formula for $Q(t)$

ANS. Since the ODE is linear we seek an integrating factor. Since $p(t) = 3/200$, the integrating factor is $exp(3t/200)$. Therefore,

$$(e^{3t/200}Q)' = 3e^{3t/200}$$

and integrating both sides gives

$$e^{3t/200}Q = 200e^{3t/200} + C$$

Since $Q(0) = 300$, by setting $t = 0$ in the above, we obtain $C = 100$ and solving for Q gives

$$Q = 200 + 100e^{-3t/200}$$

4. How long will it take for the concentration to reach 1.25 kg/l? What is the quantity of salt at this time?

ANS. Since the volume is always 200l, to find the quantity of salt when the concentration is 1.25 kg/l we simply multiply:

$$200\text{l} \times 1.25\text{kg/l} = 250 \text{ kg}$$

So we solve

$$250 = 200 + 100e^{-3t/200}$$

for t. We get $1/2 = exp(-3t/200)$ and hence $t = (200/3)\ln(2)$.

5. Suppose that homogeneous mixture flows out of the above tank at the rate of 1 l/min and that the total capacity of the tank is 500 l. How much salt will there be in the tank at the moment it is completely filled? (Find a formula for the volume of mixture in the tank at time t and rewrite the ODE for Q.)

ANS. In this case the volume of mixture in the tank at any time t varies according to the following formula:

$$V(t) = 200 + 2t$$

and therefore the differential equation satisfied by $Q(t)$ is altered as follows:

$$Q' = 3 - \frac{Q}{200+2t} \qquad Q(0) = 300$$

$$Q' + \frac{1}{2}\frac{Q}{100+t} = 3 \qquad Q(0) = 300$$

Now the integrating factor becomes $\exp(\frac{1}{2}\ln(100+t)) = (100+t)^{1/2}$ and we obtain:

$$((100+t)^{1/2}Q)' = 3(100+t)^{1/2}$$

By integration we obtain

$$(100+t)^{1/2}Q = 2(100+t)^{3/2} + C$$

and setting $t = 0$ gives

$$C = 100^{1/2}300 - 2(100)^{3/2} = 100^{3/2}$$

Therefore,

$$Q = 2(100+t) + 100^{3/2}(100+t)^{-1/2}$$

The tank fills when $200 + 2t = 500$, i.e., $t = 150$, and we find that

$$Q(150) = 500 + 100(\frac{2}{5})^{1/2}\text{kgrams}$$