1. Consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 \le x < -1 \\ -1 & \text{if} \quad -1 \le x \le 0 \\ 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{if} \quad 1 \le x \le 2 \end{cases}$$

What is the smallest positive period of the Fourier series of $f(x)$ on $[-2,2]$? To what values does the Fourier series of $f(x)$ on $[-2,2]$ converge at $x = 14, 15, 16, 17, 18$

ANS. Before answering the question the three assumptions in the Fourier Convegence Theorem have to be verified.
I. Obviously the function $f(x)$ and its derivative are p.w. continuous. $f(x)$.
II. $f(x)$ must be defined for all x and be periodic with period $2L$; here, that is 4 . **III.** At all points x the function $f(x)$ must be equal to the average of its 1-sided limits. So we may need to redefine $f(x)$ at $x = 0, \pm 1, \pm 2, \ldots$.
At $x = \pm 2$ the function is continuous.
At $x = -1$ the function should be redefined to be $-1/2$.
At $x = 0$ the function should be redefined to be 0.
At $x = 1$ the function should be redefined to be $1/2$.

Now the $s_n(x)$ converge to $f(x)$ for every x. Since $x = 14$ is a bit out of sight, we observe that the partial sums at $x = 14$ to the same thing as at $x = -2$; there they converge to 0.
The partial sums at $x = 15$ to the same thing as at $x = -1$; there they converge to -1/2.
The partial sums at $x = 16$ to the same thing as at $x = 0$; there they converge to 0.
The partial sums at $x = 17$ to the same thing as at $x = -1$; there they converge to $1/2$.
The partial sums at $x = 18$ to the same thing as at $x = -2$; there they converge to 0.

2. If $s_n(x)$ denotes the n-th partial sum of the Fourier series in the above problem, then Gibbs' phenomenon states that in any open interval containing a jump discontinuity of $f(x)$ eventually all the $s_n(x)$ overshoot and undershoot $f(x)$ by a fixed percentage of the jump of $f(x)$. Use the Fourier series plotter on the Math 251 webpage to discover approximately the percentage discovered by Professor Gibbs.

ANS. Go to the Partial Sum Plotter applet. Remember that this applet requires that you use t as the independent variable instead of x. The step function $u(t)$ is recognized by the computer if you type in the symbols $(0 < t)$. So if you need $u(-t)$ you could type in $(0 < (-t))$ or $(t < 0)$. If you need $u(t-c)$, then you would type in $(c < t)$. By plotting the partial sums for $n = 2, 4, 8, 16, 32$ you should be able to see Gibbs' phenomenon and estimate the fixed percentage which Gibbs' discovered by moving the mouse to various points on the graph on the screen.

3. For the function in Problem 1, circle the statements among the following that are false about the partial sums of its Fourier series:

i. For all sufficiently large n: $s_n(x) \le 1.001$ for every x in the interval $[.25, .75]$

ANS. This interval contains no discontinuities of $f(x)$. The error (that is the difference between $f(x)$ and $s_n(x)$ can be made uniformly small on this interval by choosing n sufficiently large.

ii. For all sufficiently large n: $s_n(.001) \le 1.001$

ANS. At any fixed point in the interval the error can be made arbitarily small by choosing n sufficiently large.

iii. For all sufficiently large n: $s_n(x) \le 1.12$ for every x in the interval $[-.001, .001]$

ANS. The total jump at $x = 0$ is 2. This interval contains the discontinuity. Gibb's phenomenon says that the overshoot greater than approximately .16 in this interval for all sufficiently large n.

iv. For all sufficiently large n: $s_n(x) \le 1.18$ for every x in the interval $[-.001, .001]$

ANS. However, the overshoot can be made less than .18 by choosing n sufficiently large.

v. For all sufficiently large n: $s_n(x) \ge -1.12$ for every x in the interval $[-.001, .001]$

ANS. The answer is the same as to part **iii.** as Gibb's phenomenon does not distinguis between overshoot and undershoot.

4. Find the Fourier series of the function f(x) on $[-2,2]$ defined in Problem 1.

ANS. Since the function $f(x)$ is odd, $a_0 = 0$ and all $a_n = 0$ with $n > 0$

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)\,dx \\
&= \frac{2}{2}\int_{0}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)\,dx = \int_0^1 1\sin\left(\frac{n\pi}{2}x\right)\,dx \\
&= \frac{-2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right)-1\right) \\
b_1 &= \frac{2}{1\pi}, \quad b_2 = \frac{4}{2\pi}, \quad b_3 = \frac{2}{3\pi}, \quad b_4 = 0
\end{aligned}
$$

$$
\begin{aligned}
x = \tfrac{0}{2} \quad & + 0\cos\left(\frac{1\pi}{2}x\right) + \frac{2}{1\pi}\sin\left(\frac{1\pi}{2}x\right) \\
& + 0\cos\left(\frac{2\pi}{2}x\right) - \frac{4}{2\pi}\sin\left(\frac{2\pi}{2}x\right) \\
& + 0\cos\left(\frac{3\pi}{2}x\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi}{2}x\right) \\
& + 0\cos\left(\frac{4\pi}{2}x\right) + 0\sin\left(\frac{4\pi}{2}x\right) + \dots
\end{aligned}
$$