Math 251 Sections 1,2,10 FAQ 11/30/09

1. Suppose that a string of length 2 has its ends clamped so that $a^2 = 3$. If its initial displacement is $5\sin(4\pi x)$ and its initial velocity is zero, then what is its displacement at any $t > 0$. What is the first value of t_0 so that $u(x, t_0) = 0$? What is the displacement of the string at time $t = \dfrac{1}{12\sqrt{3}}$. Is it just a constant times the initial displacment?

ANS. The solution is $5\sin(4\pi x)\cos(\sqrt{3}4\pi t)$. The first value of t_0 so that $u(x, t_0) = 0$ is determined by setting $\sqrt{3}4\pi t_0 = \pi/2$, ie, $t_0 = 1/(8\sqrt{3})$. The displacement of the string at time $t = \dfrac{1}{12\sqrt{3}}$ is

$$5\sin(4\pi x)\cos(\frac{\pi}{3}) = \frac{1}{2}5\sin(4\pi x)$$

which is just 1/2 of the original displacement.

2. Assume the string in Problem 1 has initial displacement is $f(x) = 1 - |1 - x|$ and initial velocity equal to zero. Draw a sketch of the displacement of the string at time $t = \dfrac{1}{12\sqrt{3}}$. You need to use the Java applet for plotting solutions of the wave equation to answer this part of this problem. Is it just a constant times the initial displacment?

ANS. We need to find a sine series for the function

$$f(x) = 1 - |1 - x| = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ 2 - x & \text{if} \quad 1 < x < 2 \end{cases}$$

And we have the following formula for b_n:

$$\begin{aligned} b_n &= \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)\,dx \\ &= \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)\,dx + \int_1^2 (2-x)\sin\left(\frac{n\pi}{2}x\right)\,dx \end{aligned}$$

The above calculations involve two integration by parts For the first one we obtain:

$$\begin{aligned} \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)\,dx &= -\left(\frac{2}{n\pi}\right)\left[x\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \left(\frac{2}{n\pi}\right)\int_0^1 \cos\left(\frac{n\pi}{2}x\right)\,dx \\ &= -\left(\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

For the second one we obtain:

$$\begin{aligned} \int_1^2 (2-x)\sin\left(\frac{n\pi}{2}x\right)\,dx &= -\left(\frac{2}{n\pi}\right)\left[(2-x)\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 - \left(\frac{2}{n\pi}\right)\int_1^2 \cos\left(\frac{n\pi}{2}x\right)\,dx \\ &= \left(\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Adding these two finally gives us:

$$b_n = 2\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

3. Assume the elastic string in Problem 1 has initial displacement zero and initial velocity given by $g(x) = 3\sin(4\pi x)$, then what is its displacement at any $t > 0$? What is the first value of t_0 so that $u(x, t_0) = 0$?

ANS. We would like to try $3\sin(4\pi x)\sin(\sqrt{3}4\pi t)$ as the solution. However, when we differentiate with respect to t and set $t = 0$ we are left with an extra factor of $\sqrt{3}4\pi$. So the correct formula for the displacement is

$$u(x,t) = \frac{3}{\sqrt{3}4\pi}\sin(4\pi x)\sin(\sqrt{3}4\pi t)$$

To find the first value of t_0 so that $u(x, t_0) = 0$ we solve $\sqrt{3}4\pi t_0 = \pi$. That is, $t_0 = (\sqrt{3}4)^{-1}$

4. For the string in Problem 1 verify that the solution can be rewritten as:

$$u(x,t) = \frac{1}{2}(f(x+at) + f(x-at))$$

ANS. The solution to Problem 1 is: $u(x,t) = 5\sin(4\pi x)\cos(\sqrt{3}4\pi t)$. We use the identity:

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

to write $u(x,t)$

$$u(x,t) = \frac{5}{2}\left(\sin\left(4\pi(x+\sqrt{3}t)\right) + \sin\left(4\pi(x-\sqrt{3}t)\right)\right) = \frac{5}{2}(f(x+\sqrt{3}t) + f(x-\sqrt{3}t))$$

where $f(x) = \sin(4\pi x)$.