Math 251 Sections 1,2,10 FAQ 12/07/09

1. For each of the following partial differential equations determine whether or not it is possible to find a solution by assuming that a solution is of the form $u(x,t) = X(x)T(t)$. If it is, then find a solution.

a. $u_t + u_x = u$

ANS. Assume that $u(x,t) = X(x)T(t)$. Then plugging into the equation we obtain

$$X(x)T'(t) + X'(x)T(t) = X(x)T(t)$$

We now try to separate the functions involving the variables:

$$X(x)T'(t) = X(x)T(t) - X'(x)T(t)$$

$$\frac{T'(t)}{T(t)} = 1 - \frac{X'(x)}{X(x)}$$

Since a function of t is equal to a function of x in the above equation, both must be a constant; call it λ. Thus we obtain two equations, each one very easy to solve:

$$\frac{T'(t)}{T(t)} = \lambda, \qquad \lambda = 1 - \frac{X'(x)}{X(x)}$$

Thus

$$T(t) = e^{\lambda t}, \qquad X(x) = e^{(1-\lambda)x}$$

and finally

$$u(x,t) = e^{\lambda t}e^{(1-\lambda)x}$$

Also any constant multiple of $u(x,t)$ is a solution.

b. $u_t + u_x = t$

ANS. Assume that $u(x,t) = X(x)T(t)$. Then plugging into the equation we obtain

$$X(x)T'(t) + X'(x)T(t) = t$$

We now try to separate the functions involving the variables:

$$X(x)T'(t) = t - X'(x)T(t)$$

And at this point we are stuck! No algebraic manipulation will put an equal sign between all the stuff involving t's and all the stuff involving x's.

2. Consider the PDE

$$u_{xx} + u_t = 0$$

on the infinite rectangle $0 < x < \pi$, $0 < t$

a. Use the technique of separation of variables to discover ALL nonzero solutions of the form $u(x,t) = X(x)T(t)$ which satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $0 < t$.

ANS. We first try to separate variables:

$$X''(x)T(t) + X(x)T'(t) = 0$$

or written in a more abbreviated form:

$$X''T + XT' = 0$$

This is pretty easy:

$$\frac{X''}{X} = -\frac{T'}{T}$$

We conclude that each side of the above formula is a constant $-\lambda$ (It is good to use $-\lambda$ to denote the constant so that the ODE for which we have a two point boundary value problem looks like the one we analyzed last time.)

$$\frac{X''}{X} = -\lambda \qquad \text{and} \qquad -\frac{T'}{T} = -\lambda$$

We look at the second ODE and see that any nonzero solution must have the form $T = C_1 e^{\lambda t}$. This means $T(t)$ is not zero for every value of t. Therefore the boundary conditions $u(0,t) = u(\pi,t) = 0$ for any $t > 0$ translate into the following boundary value problem for X:

$$X'' + \lambda X = 0 \qquad X(0) = 0, \quad X(\pi) = 0$$

From our previous work we know that the eigenvalue eigenfunction pairs for this problem are

$$\lambda = n^2 \qquad X(x) = \sin nx$$

where n is a positive integer. Therefore solutions to the given boundary value problem are:

$$u(x,t) = C_1 \sin nx e^{n^2 t}$$

This means that a building block for this PDE with boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$ for all $t > 0$ is $\sin nx e^{n^2 t}$

b. Find the solution that satisfies the additional condition $u(x,0) = x$ for $0 < x < \pi$. (Hint: You may need to use the answers to some of the Fourier series problems on previous FAQ's).

ANS. Setting $t = 0$ in the expression derived in part (a) gives

$$u(x,0) = C_1 \sin nx$$

Believing that a series of solutions of this PDE is also a solution, we seek a sine series for x on $0 < x < \pi$. This can be found in a previous FAQ:

$$\sum_{n=1}^{\infty} \frac{-2}{n} \cos n\pi \sin nx$$

Therefore, it is easy to check that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-2}{n} \cos n\pi \sin nx e^{n^2 t}$$

satisfies all the additional boundary conditions, including $u(x,0) = x$ for $0 < x < \pi$.