1. Assume that the thermal diffusivity of a thin metal rod $\alpha^2 = 0.9$ and the length of the rod is 3 cm. Assume that now both ends of the rod are insulated and that the initial temperature distribution $f(x) = 5\cos(\frac{\pi}{3}x)$. Find the temperature $u(x,t)$ of the rod at any time $t > 0$.

ANS.

$$u(x,t) = 5\cos(\frac{\pi}{3}x)e^{-0.9(\pi/3)^2 t}$$

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2. Now assume that the rod in Problem 1 has initial temperature distribution $f(x) = 20$ if $0 < x < 1$ and $f(x) = 0$ if $1 < x < 3$ and Determine the temperature $u(x,t)$ of the rod at any time $t > 0$.

ANS. We first need to find a cosine series for $f(x)$:

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{40}{3}\int_0^1 dx = \frac{40}{3}$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos(\frac{n\pi}{3}x)\,dx = \frac{40}{3}\int_0^1 \cos(\frac{n\pi}{3}x)\,dx = \frac{40}{n\pi}\left[\sin(\frac{n\pi}{3}x)\right]_0^1 = \frac{40}{n\pi}\sin(\frac{n\pi}{3})$$

So

$$a_1 = \frac{20}{1\pi}\sqrt{3} \qquad a_2 = \frac{20}{2\pi}\sqrt{3} \qquad a_3 = 0 \qquad q_4 = \frac{20}{4\pi}\sqrt{3} \qquad \ldots$$

And we can now write the solution

$$u(x,t) = \frac{20}{3} + \frac{20}{1\pi}\sqrt{3}\cos(\frac{1\pi}{3}x)e^{-0.9(\pi/3)^2 t} + \frac{20}{2\pi}\sqrt{3}\cos(\frac{2\pi}{3}x)e^{-0.9(2\pi/3)^2 t} + 0 - \frac{20}{4\pi}\sqrt{3}\cos(\frac{4\pi}{3}x)e^{-0.9(4\pi/3)^2 t} + \ldots$$

3. Approximately what is the temperature $u(x,t)$ at any point x in the rod in Problem 2 after a long time?

ANS. After a long time the temperature at any point of an insulated rod approaches the initial average temperature which does not change.

$$\frac{a_0}{2} = \frac{40}{3}\frac{1}{2} = \frac{20}{3}$$