1. Consider the linear first order ODE: $y' + py = g$, where p and g are functions of t. It is possible to multiply this equation by a function μ to make the left hand side into the derivative of a product. μ called an **integrating** factor. What is the procedure for discovering μ? What is the final formula for μ which should be memorized?

ANS. We want to have $(y' + py)\mu = (y\mu)'$. Multiplying through and applying the product rule gives:

$$\mu y' + \mu p y = \mu' y + \mu y'$$

Since we may assume that $y \neq 0$ we may cancel the y from both sides to obtain

$$\mu p = \mu'$$

We recognize this as being a separable ODE for the unknown function μ. We can easily solve this using our knowledge of separable ODE's.

$$\frac{\mu'}{\mu} = p$$

We integrate both sides with respect to t:

$$\ln \mu = \int p\, dt$$

and we then see that an integrating factor μ is:

$$\mu = e^{\int p\, dt}$$

(When we perform the integration we can ignore the constant of integration since we are looking for one integrating factor and have no need for the most general one.) Remember this and the fact that the product of μ with the left hand side is:

$$\mu y' + \mu p y = (\mu y)'$$

Each of Problems 2-5 involves a linear first order ODE. Solve it by finding an integrating factor and then multiplying the ODE through by it. **2.** Solve the following ODE: $(t-5)y' - 2y = (t-5) \qquad (t < 5)$

ANS. Before we try to solve this differential equation, we make the coefficient of y' equal 1 by dividing through by $(t-5)$: $y' - \frac{2}{(t-5)}y = 1$. The integrating factor is

$$\mu = e^{-\int \dfrac{2}{t-5}dt} = e^{-2\ln|t-5|)} = e^{\ln(t-5)^{-2}} = \frac{1}{(t-5)^2}$$

Multiplying through by μ gives:

$$\left(\frac{1}{(t-5)^2}y\right)' = \frac{1}{(t-5)^2}$$

Integrating both sides gives

$$\frac{1}{(t-5)^2}y = \frac{-1}{(t-5)} + C$$

Since it is so easy to solve for y, let's do it:

$$y = -(t-5) + C(t-5)^2$$

3. Solve the following IVP: $ty' - y = t^2 \quad y(1) = 2$.

ANS. Before we try to solve this differential equation, we make the coefficient of y' equal 1 by dividing through by t: $y' - \frac{1}{t}y = t$. The integrating factor is

$$\mu = e^{-\int \dfrac{1}{t}dt} = e^{-\ln t} = e^{\ln t^{-1}} = \frac{1}{t}$$

Multiplying through by μ gives:

$$\left(y\frac{1}{t}\right)' = 1$$

Integrating both sides gives

$$y\frac{1}{t} = t + C \qquad\qquad y = t^2 + Ct$$

Finally, plugging in the initial conditions gives:

$$2 = 1 + C \qquad\qquad C = 1 \qquad\qquad y = t^2 + t$$

4. Find the general solution of the differential equation: $(t+1)y' + 2y = t$.

ANS. Before we try to solve this differential equation, we make the coefficient of y' equal 1 by dividing through by $t+1$: $y' + \frac{2}{t+1}y = \frac{t}{t+1}$. The integrating factor is

$$\mu = e^{\int \frac{2}{t+1}dt} = e^{2\ln(t+1)} = e^{\ln(t+1)^2} = (t+1)^2$$

Multiplying through by μ gives:

$$\left(y(t+1)^2\right)' = t^2 + t$$

And integrating both sides gives:

$$y(t+1)^2 = \frac{2t^3 + 3t^2 + C}{6}$$

Therefore

$$y = \frac{2t^3 + 3t^2 + C}{6(t+1)^2}$$

5. Find the general solution of the differential equation: $(\sin t)y' + (\cos t)y = \csc^2 t$.

ANS. Before we try to solve this differential equation, we make the coefficient of y' equal 1 by dividing through by $\sin t$:

$$y' + \frac{\cos t}{\sin t}y = \csc^3 t$$

The integrating factor is

$$\mu = \exp\left(\int \frac{\cos t}{\sin t}dt\right) = e^{\ln \sin t} = \sin t$$

This tells us that the operation of dividing through by $\sin t$ was unnecessary and that the left hand side of the ODE, as it appears in the statement of the problem, is the derivative of the $(y \sin t)'$: That is

$$(y\sin t)' = \csc^2 t$$

Integrating both sides gives

$$y\sin t = -\cot t + C \qquad\qquad y = -\csc t\cot t + C\csc t$$