Math 251 Sections 1,2,10 FAQ 12/02/09

1. For what values of p and q does the following function solve Laplace's equation?

$$u(x,y) = \sinh(px)\cos(qy)$$

ANS. $u_{xx} = p^2 u$ and $u_{yy} = -q^2 u$. Therefore we need $p^2 = q^2$, or equivalently $p = q$ for u to be a solution to Laplace's equation.

2. Find the solution of the Laplace equation on the rectangle $\{(x,y)|0 < x < 2, \quad 0 < y < 3\}$ which has the following values on the boundary:
$u(0,y) = 0$ if $0 < y < 3$
$u(x,0) = 0$ if $0 < x < 2$
$u(x,3) = 0$ if $0 < x < 2$
$u(2,y) = \sin(\frac{\pi}{3}y)$ if $0 < y < 3$

ANS.

$$u(x,y) = \frac{1}{\sinh(\frac{\pi}{3}2)} \sinh(\frac{\pi}{3}x)\sin(\frac{\pi}{3}y)$$

3. Find the solution of the Laplace equation on the rectangle $\{(x,y)|0 < x < 2, \quad 0 < y < 3\}$ which has the following values on the boundary:
$u(0,y) = \sin(\frac{\pi}{3}y)$ if $0 < y < 3$
$u(x,0) = 0$ if $0 < x < 2$
$u(x,3) = 0$ if $0 < x < 2$
$u(2,y) = 0$ if $0 < y < 3$

ANS.

$$u_{\text{new}}(x,y) = u_{\text{above}}(2-x,y)$$

4. Find the solution of the Laplace equation on the rectangle $\{(x,y)|0 < x < 2, \quad 0 < y < \pi\}$ which has the following values on the boundary:
$u(0,y) = 0$ if $0 < y < \pi$
$u(x,0) = 0$ if $0 < x < 2$
$u(x,\pi) = 0$ if $0 < x < 2$
$u(2,y) = y$ if $0 < y < \pi$

ANS. If the boundary value on the right edge of the rectangle would have been $\sin(\frac{n\pi}{\pi}y) = \sin(ny)$, then this problem would have had the following simple solution:

$$u(x,y) = \frac{1}{\sinh(2n)} \sinh(nx)\sin(ny)$$

as we saw in Problem 2. We therefore seek a sine series for the function y on the interval $[0,\pi]$:

$$b_n \;=\; \frac{2}{\pi}\int_0^\pi y\sin(ny)\,dy = \frac{2}{\pi}\left[\frac{-1}{n}y\cos(ny)\right]_0^\pi + \frac{2}{n}\int_0^\pi \cos(ny)\,dy = \frac{-2}{n}\cos(n\pi)$$

because the last integral of the cosine function is 0. So the sine series is

$$y = \sum_{n=1}^{\infty} \frac{-2}{n}\cos(n\pi)\sin(ny)$$

Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{-2}{n} \cos(n\pi) \frac{1}{\sinh(2n)} \sinh(nx) \sin(ny)$$