1. Consider the following systems of ODE's:
i.
i. $x' = y, \quad y' = -5\sin x - 2y$
and
ii. $x' = y, \quad y' = -\sin x$.
Describe the critical point each system has at $\mathbf{x} = \mathbf{0}$. Which one represents a pendulum and which one represents a damped pendulum with angular displacement x? (HINT:$\sin x \approx x$ for small x.)

ANS. For the first system

$$A = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$$

The characteristic polynomial is $r^2 + 2r + 5$ which has complex roots with negative real part. The origin is an asymtotically stable spiral.
For the second one

$$A = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

The characteristic polynomial is $r^2 + 1$ which has purely imaginary roots. The origin is a stable center.
Therefore, the first system represents a damped pendulum and the second an undamped one.

2. Suppose that $x = x(t)$ is the angular displacement of pendulum which is damped by air resistance and that y is the derivative. Also assume that x and y satisfy the following system of ODE's:

$$x' = 5y, \qquad y' = -9\sin x - \frac{1}{5}y$$

The 2×2 nonlinear first order system of ODE's satisfied by x and y has critical points at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$.

a. What is the type of each critical point.

ANS. At $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$A = \begin{pmatrix} 0 & 5 \\ -9 & -1/5 \end{pmatrix}$$

The characteristic polynomial is $r^2 + r/5 + 45$ which has complex roots with negative real part. The origin is an asymtotically stable spiral.

At $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$:

$$A = \begin{pmatrix} 0 & 5 \\ 9 & -1/5 \end{pmatrix}$$

The characteristic polynomial is $r^2 + r/5 - 45$ which has two real roots with opposite signs (approximately 20.0 and -20.2). This critical point is a saddle.

b. What term(s) in the above system represents the damping force?

ANS. $-\frac{1}{5}y$

c. If damping is removed, then what are the critical points for x between 0 and $3\pi/2$? And, what are their types?

ANS. The critical points remain at the same locations. The type of the one at origin changes from asymptotically stable spiral to a stable center.
3. Again consider the damped pendulum in Problem 2. Use Professor Mansfield's phase portrait applet to determine the smallest initial angular velocity applied to the pendulum at zero initial angular displacement that will send the pendulum over the top (ie, that will allow the angular displacement to exceed π).

ANS. Approximately 2.5 m/s.