Math 251 Sections 1,2,10 FAQ 11/16/09

1. Assume that the thermal diffusivity of a thin metal rod $\alpha^2 = 2$ and the length of the rod is 3 cm. Assume that the initial temperature of the rod is given by

$$f(x) = 20 \sin \pi x - 30 \sin 4\pi x$$

and that the left and right ends of the rod are placed in ice water. Find the temperture $u(x, t)$ of the rod at any time $t > 0$.

ANS.

$$u(x, t) = 20 \sin(\pi x)e^{-2(\pi)^2 t} - 30 \sin(4\pi x)e^{-2(4\pi)^2 t}$$

2. Approximately what is the temperature at the midpoint of the rod in Problem 1 after a long time? Approximately how long does it take for the temperature at the midpoint of the rod to reach -10^{o}C?

ANS. After a long time the temperature of the entire rod is nearly zero. At the midpoint of the rod $x = 3/2$ for every $t \geq 0$, $u(3/2, t) = -20e^{-2(\pi)^2 t}$. We need to find when this is equal to -10. We find $-2(\pi)^2 t = -\ln(2)$ So

$$t = \frac{\ln(2)}{2\pi^2}$$

3. Assume that the rod in Problem 1 is initially heated uniformly to 50^{o}C and that its ends are then put in ice water. Find the temperature $u(x, t)$ of the rod at any time $t > 0$.

ANS. We first need to find the sine series of $f(x) = 50$ on the interval $[0, 3]$ We did nearly this in class on the first day of Fourier series:

$$\begin{aligned}
b_n &= \frac{2}{3}\int_0^3 f(x) \sin\left(\frac{n\pi}{3}x\right) dx \\
&= \frac{100}{3}\int_0^3 \sin\left(\frac{n\pi}{3}x\right) dx \\
&= -\frac{100}{n\pi}\left[\cos\left(\frac{n\pi}{3}x\right)\right]_0^3 = -\frac{100}{n\pi}(\cos(n\pi) - 1) \\
b_1 &= \frac{200}{1\pi}, \quad b_2 = 0, \quad b_3 = \frac{200}{3\pi},
\end{aligned}$$

The sine series of $f(x)$ on $[0, 3]$ is

$$\sum_{n=1}^{\infty} \frac{100}{n\pi}(1 - \cos(n\pi)) \sin\left(\frac{n\pi}{3}x\right)$$

The first three terms of the sine series is:

$$f(x) = \frac{200}{1\pi} \sin\left(\frac{1\pi}{3}x\right) + 0 \sin\left(\frac{2\pi}{3}x\right) + \frac{200}{3\pi} \sin\left(\frac{3\pi}{3}x\right) + \ldots$$

Therefore the solution to the boundary value problem is:

$$\sum_{n=1}^{\infty} \frac{100}{n\pi}(1 - \cos(n\pi)) \sin\left(\frac{n\pi}{3}x\right) e^{-2(n\pi/3)^2 t}$$

The first three terms of the solution are:

$$u(x, t) = \frac{200}{1\pi} \sin\left(\frac{1\pi}{3}x\right) e^{-2(1\pi/3)^2 t} + 0 \sin\left(\frac{2\pi}{3}x\right) e^{-2(2\pi/3)^2 t} + \frac{200}{3\pi} \sin\left(\frac{3\pi}{3}x\right) e^{-2(3\pi/3)^2 t} + \ldots$$