Perspectives on Functions

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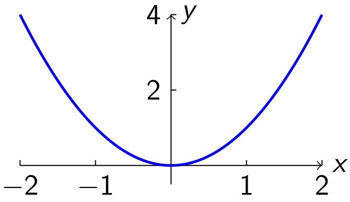
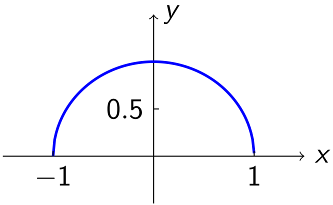
Definition

A function $f(x)$ is an assignment of a real number x to some other unique real number.

- $f(x) = x^2$



- $f(x) = \sqrt{1 - x^2}$



Definition: The *domain* of a function is the largest set on which the function is defined. The *range* of a function is the set of all points which are witnessed by the function.

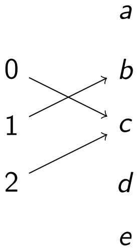
Example Question: Find the domain and range of the function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Solution: In the denominator we have a square root whose argument cannot be negative, so $1 - x^2 \geq 0$. Solving this gives $-1 \leq x \leq 1$. This is the only restriction, so the domain is $[-1, 1]$. For the range, note that every term of the function is non-negative, and it can never be zero, so $f(x) > 0$ for any value of x. The function is smallest when the denominator is largest, which occurs when $x = 0$, so the minimum of f is $f(0) = 1$. As x gets closer to 0, the denominator becomes arbitrarily small, meaning that f gets arbitrarily big, thus the range is $[1, \infty)$.

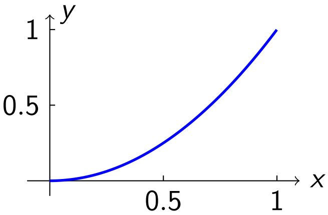
Definition

Given two sets A and B, a function is a map $f : A \to B$ such that $f(a) \in B$ for all $a \in A$. In this case, A is said to be the *domain*, while B is said to be the *codomain*.

- Let $A = \{0, 1, 2\}$ and $B = \{a, b, c, d\}$, and define the function by the following map:



- Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$



Definition: A function $f : A \to B$ is said to be *injective* if whenever $f(x) = f(y)$ then $x = y$.

Example Question: Suppose that $f : B \to C$ and $g : A \to B$ are functions such that $f \circ g : A \to C$ is injective. Show that g is injective.

Solution: We are told that $f \circ g$ is injective, which means that if $f(g(x)) = f(g(y))$ then $x = y$. We want to show that if $g(x) = g(y)$ then $x = y$. So suppose $g(x) = g(y)$, and apply f to both sides, so that $f(g(x)) = f(g(y))$. But since $f \circ g$ is injective, it must be the case that $x = y$.

Definition(s)

Definition: The *cartesian product* of two sets A and B is the set $A \times B = \{(a, b) : a \in A, b \in B\}$.

Definition: A *binary relation* on the sets A and B is a subset $R \subseteq A \times B$.

Definition: A function $f : A \to B$ is a binary relation R on the sets A and B such that

1. if $(x, y), (x, z) \in R$ then $y = z$. Here A is the *domain* and B is the *codomain*;
2. for every $x \in A$ there exists a $y \in B$ such that $(x, y) \in R$.

- For example, on the set $\{0, 1, 2\} \times \{a, b, c, d, e\}$ we can define a function via the relation $R = \{(0, c), (1, b), (2, c)\}$. We often write this as $f(0) = c, f(1) = b, f(2) = c$.
- Define a function on $[0, 1] \times \mathbb{R}$ by $R = \{(x, x^2) : x \in [0, 1]\}$, often written $f(x) = x^2$.

Definition: A function $f : B \to C$ is said to be *monic* if whenever $g_1, g_2 : A \to B$ satisfy $f \circ g_1 = f \circ g_2$ then $g_1 = g_2$.

Example Question: Show that a function is injective if and only if it is monic.

Solution: Suppose that f is a monomorphism, and that $b_1, b_2 \in B$ satisfy $f(b_1) = f(b_2)$. Define the constant functions $g_i : B \to B$ be the constant function $g_i(x) = b_i$. Now $f(g_1(x)) = f(b_1) = f(b_2) = f(g_2(x))$ for all $x \in B$, so by assumption $g_1(x) = g_2(x)$ for all $x \in B$, which in turn shows that $b_1 = b_2$.

Conversely, suppose that f is injective. Injective functions are left-invertible, so choose one such inverse $h : C \to B$ such that $h \circ f = \mathrm{id}_B$. Let $g_1, g_2 : A \to B$ be any to functions such that $f \circ g_1 = f \circ g_2$, and post-compose by h to get

$$\begin{aligned} h \circ (f \circ g_1) &= (h \circ f) \circ g_1 = \mathrm{id}_B \circ g_1 = g_1 \\ &= h \circ (f \circ g_2) && \text{by assumption} \\ &= (h \circ f) \circ g_2 = \mathrm{id}_B \circ g_2 = g_2. \end{aligned}$$

This shows that $g_1 = g_2$, allowing us to conclude that f is a monomorphism as required.