Variations of geometric invariant quotients for pairs, a computational approach

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Abstract

We study, from a computational viewpoint, the GIT compactifications of pairs formed by a hypersurface and a hyperplane. We provide a general setting and algorithms to calculate all polarizations which give different GIT quotients, the finite number of one-parameter subgroups required to detect the lack of stability, and all maximal orbits of non stable pairs. Our algorithms have been fully implemented in Python for all dimensions and degrees. We applied our work with the case of cubic surfaces and their anti-canonical divisors in a sequel article.

1 Introduction

The construction and study of moduli spaces is a central subject in algebraic geometry. Geometric Invariant Theory (GIT) is a foundational tool for the study of particular cases. For example, it has been applied to study hypersurfaces [Sha81, Gal13, All03, Laz09a]; and it is a first step towards constructing the moduli space of del Pezzo surfaces admitting a Kähler–Einstein metric [OSS12]. In general, a GIT quotient depends of a choice of a line bundle in a parameter space; and any two compactifications, with the exceptions of some limit cases, are related by birational transformations (see [Tha96], [DH98]).

In this article, we consider the GIT quotients parametrizing pairs (X, H) where $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree d and $H \subset \mathbb{P}^{n+1}$ is a hyperplane. This is a natural setting to consider pairs (X, D) where $D = X \cap H$ is a hyperplane section. Our work generalizes R. Laza's description of the GIT quotients parametrizing pairs (C, L) where $C \subset \mathbb{P}^2$ is a plane curve and $L \subset \mathbb{P}^2$ is a line [Laz09b]. This construction is critical to understand the deformation theory of the N_{16} singularity and the KSBA compact moduli space of K3 surfaces of degree two [Laz12]. In [GMG15a], we apply the

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present work to describe all GIT compactifications of pairs (S, C) where S is a cubic surface and $C \in |-K_S|$ is an anticanonical divisor.

Let $\mathcal{R}_{n,d}$ be the parameter space of pairs (X, H). It can be shown that there is a one-dimensional space of stability conditions $t \in [0, t_{n,d}]$ corresponding to polarizations of $\mathcal{R}_{n,d}$ (see Section 2). Then, by general VGIT theory, there is a finite number of values $t_i \in \mathbb{Q}_{\geqslant 0}$ known as *GIT walls* where $0 = t_0 < t_1 < \cdots < t_{n,d}$ and segments (t_i, t_{i+1}) known as *GIT chambers*. Two GIT quotients are isomorphic if and only if their linearizations belong to either the same GIT chamber or wall. In particular, there is a finite number of non-isomorphic GIT quotients $\overline{M}^{GIT}_{n,d,t}$ corresponding to values $t = t_i$ and to any $t \in (t_i, t_{i+1})$.

The stability of any pair (X, H) depends of t. We will show that (X, H) is t-semistable if and only if $t \in [t_i, t_j]$ for some walls t_i, t_j. We will further show that if (X, H) is t-stable for some t then (X, H) is t-stable if and only if $t \in (t_i, t_j)$. The interval $[t_i, t_j]$ is the *interval of stability* of (X, H).

The characterization of these pairs and their GIT quotients is accomplished in four steps: First, we find all GIT walls, which allows us to identify the linearizations associated to non-isomorphic GIT quotients.

Theorem 1.1. *All GIT walls $\{t_0, \ldots, t_{n,d}\}$ are obtained as a subset of the output given by Algorithm 3; and they are contained in the interval $[0, t_{n,d}]$ where $t_{n,d} = \frac{d}{n+1}$. Every pair (X, H) has an interval of stability $[a, b]$ with $a, b \in \{t_0, \ldots, t_{n,d}\}$.*

In particular, the above theorem allows us to obtain several compactifications of families of pairs for examples which are geometrically important such as quintic surfaces and their canonical divisors, or cubic surfaces and their anticanonical divisors.

Corollary 1.2. *Assume that $\mathbb{K}$ is algebraically closed,* $\mathrm{char}(\mathbb{K}) = 0$*, the locus of stable points is not empty and $d \geqslant 3$. Then*

$$\dim \overline{M}^{GIT}_{n,d,t} = \binom{n+d+1}{d} - n^2 - 3n - 3.$$

Each $\overline{M}^{GIT}_{n,d,t}$ is a compactification of the space of log smooth pairs (X, D) described above.

The second step is to obtain a finite list of one-parameter subgroups $S_{n,d}$ which determine the stability of *any* pair (X, H) for *every* value of t.

Theorem 1.3. *There is a finite set $S_{n,d}$ of one-parameter subgroups, independent of t (see Definition 3.1 for $S_{n,d}$), such that the pair (X, H) is not t-stable (respectively not t-semistable) if and only if there exists $g \in SL(n, \mathbb{K})$ satisfying*

$$\mu_t(X, H) = \max_{\lambda \in S_{n,d}} \{\mu_t(g \cdot X, g \cdot H, \lambda)\} \geqslant 0 \qquad (\textit{respectively } > 0)$$

where μ_t is the Hilbert-Mumford function defined in Lemma 2.2.

Once a set of coordinates is fixed, any pair (X, H) can be determined by homogeneous polynomials F and F' of degrees d and 1, respectively. The pair (X, H) has an associated pair of sets of monomials, namely those which appear with non-zero coefficents in F and F'. The third step is to find sets of monomials $N_t^{\oplus}(\lambda_k, x_i)$ such that the equations of any non-t-stable pair (X, H), in some coordinate system, is a linear combination of the monomials in $N_t^{\oplus}(\lambda_k, x_i)$. A similar procedure follows for t-unstable pairs, where the relevant sets of monomials are $N_t^{+}(\lambda_k, x_i)$. Each of these sets is completely characterized by $\lambda \in S_{n,d}$, a monomial x_i and the parameter t.

Theorem 1.4. *Let $t \in [0, t_{n,d}]$. A pair (X, H) is not t-stable (t-unstable, respectively), if and only there exists $g \in SL(n, \mathbb{K})$ such that the set of monomials associated to $(g \cdot X, g \cdot H)$ is contained in a pair of sets $N_t^{\oplus}(\lambda, x_i)$ ($N_t^{+}(\lambda, x_i)$ respectively).*

Furthermore, the sets $N_t^{\oplus}(\lambda, x_i)$ and $N_t^{+}(\lambda, x_i)$ which are maximal *with respect to the containment order of sets satisfy the following:*

(i) They are computed by Algorithm 4.

(ii) They define families of non-t-stable pairs (t-unstable pairs, respectively) in $\mathcal{R}_{n,d}$.

Any pair $(g \cdot X, g \cdot H)$ as above belongs to one of these families.

These results allow us to identify non-t-stable pairs and these are either strictly t-semistable or t-unstable.

The fourth step is to distinguish these two from one another. For that purpose, we introduce the Centroid Criterion which gives a polyhedral interpretation of stability. Indeed, a pair (X, H) determines a convex polytope $\overline{\mathrm{Conv}_t(X, H)}$ and the parameter t determines a point $\mathcal{O}_t$ in affine space (for details see Section 4).

Lemma 1.5 (Centroid Criterion)**.** *Let $t \in \mathbb{Q}_{\geqslant 0}$. A pair (X, H) is t-semistable (respectively t-stable) if and only if $\mathcal{O}_t \in \overline{\mathrm{Conv}_t(X, H)}$ (respectively $\mathcal{O}_t \in$* $\mathrm{Int}\left(\mathrm{Conv}_t(X, H)\right)$*).*

1.1 Organization of the article

In Section 1.2 we introduce our notation. In Section 2 we describe in detail the variational Geometric Invariant Theory problem that we consider. In Section 3 we prove Theorem 1.3. The goal of Section 4 is to give the precise definitions used in Lemma 1.5 and prove it, together with Theorem 1.1. In Section 5 we prove Theorem 1.4. The algorithms are described in Section 6.

1.2 Conventions and notation

We work over an algebraically closed field $\mathbb{K}$. Let $G = \mathrm{SL}(n+2, \mathbb{K})$. Let $T \subset G$ be a fixed maximal torus. The choice of $T \cong (\mathbb{K}^*)^{n+2}$ induces a lattice of characters $M = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{G}_m) \cong \mathbb{Z}^{n+2}$ and a lattice of one-parameter subgroups $N = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{G}_m, T) \cong \mathbb{Z}^{n+2}$, with a natural pairing:

$$\langle -, - \rangle \colon M \times N \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}.$$

We choose projective coordinates $(x_0 : \cdots : x_{n+1})$ in $\mathbb{P}^{n+1}$ such that T is diagonal. Given a one-parameter subgroup $\lambda \colon \mathbb{G}_m \cong \mathbb{K}^* \to T \subset G$ in M, we say it is *normalized* ([Muk03, §7.2(b)]) if

$$\lambda(s) = \mathrm{Diag}(s^{r_0}, \dots, s^{r_{n+1}}) := \begin{pmatrix} s^{r_0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s^{r_{n+1}} \end{pmatrix},$$

such that $r_0 \geqslant \cdots \geqslant r_{n+1}$ and $\sum r_i = 0$. Denote by Ξ_k the set of all monomials of degree k in variables $x_0, \dots, x_{n+1}$. Since each monomial in Ξ_k can be identified with a character $\mathcal{X}^a \in M$ of weight k, we can see the pairing $\langle -, - \rangle$ of one-parameter subgroups with monomials as:

$$\langle x_0^{d_0} \cdots x_{n+1}^{d_{n+1}}, \mathrm{Diag}(s^{r_0}, \dots, s^{r_{n+1}}) \rangle = \langle \mathcal{X}^a, \lambda \rangle = \sum d_i \cdot r_i \in \mathbb{Z},$$

where $a = (d_0, \dots, d_{n+1}) \in (\mathbb{Z}_{\geqslant 0})^{n+2}$, $\sum d_i = k$.

Let X be a hypersurface of degree d defined by polynomials $F = \sum c_I x^I$ with $I = (d_0, \dots, d_{n+1})$ and let H be a hyperplane defined by $\sum h_i x_i$ where $c_I, h_i \in \mathbb{K}$. We define their *associated sets of monomials* $(\mathcal{X}, \mathcal{H})$ as the pair of sets:

$$\mathcal{X} = \{x^I \in \Xi_d \mid c_I \neq 0\}, \qquad \mathcal{H} = \{x_i \in \Xi_1 \mid h_i \neq 0\}.$$

Let λ be a normalized one-parameter subgroup of G. By definition [MFK94, 21, pg. 81], the *Hilbert-Mumford function* is

$$\mu(X, \lambda) := \min\{\langle I, \lambda \rangle \mid c_I \neq 0\}.$$

Note that for fixed X, the function $\mu(X, -)$ is piecewise linear. Finally, there is a natural partial order on Ξ_k which we call *Mukai order* [Muk03, Lemma 7.18]: given $v, m \in \Xi_k$,

$$v \leqslant m \iff \langle v, \lambda \rangle \leqslant \langle m, \lambda \rangle,$$

for all normalized one-parameter subgroups λ. Under this order there is a unique *maximal element* x_0^k and unique *minimal element* x_{n+1}^k in Ξ_k. In the special case when $k = 1$, the Mukai order is a total one.

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2 VGIT Setting

Let $\mathcal{R} = \mathcal{R}_{n,d}$ be the parameter scheme of pairs (X, H) given by

$$\mathcal{R}_{n,d} = \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))) \times \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(1)))) \cong \mathbb{P}^N \times \mathbb{P}^{n+1},$$

where $N = \binom{n+1+d}{d} - 1$.

Lemma 2.1. *The set of G-linearizable line bundles $\mathrm{Pic}^G(\mathcal{R})$ is isomorphic to $\mathbb{Z}^2$. Then a line bundle $\mathcal{L} \in \mathrm{Pic}^G(\mathcal{R})$, is ample if and only if*

$$\mathcal{L} = \mathcal{O}(a, b) := \pi_1^*(\mathcal{O}_{\mathbb{P}^N}(a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^{n+1}}(b)) \in \mathrm{Pic}^G(\mathcal{R}),$$

where π_1 and π_2 are the natural projections on $\mathbb{P}^N$ and $\mathbb{P}^{n+1}$, respectively and $a, b > 0$.

Proof. Let $\pi_1 : \mathcal{R} \to \mathbb{P}^N$, $\pi_2 : \mathcal{R} \to \mathbb{P}^{n+1}$ be the natural projections. The action of G on Ξ_d and Ξ_1 induces a natural action on $\mathcal{R} \cong \mathbb{P}^N \times \mathbb{P}^{n+1}$, which preserves the fibers. Hence we have an action of G on both $\mathbb{P}^N$ and $\mathbb{P}^{n+1}$ and π_1, π_2 are morphisms of G-varieties. Recall there is an exact sequence (see [Dol03, Theorem 7.2]):

$$0 \longrightarrow \mathcal{X}(G) \longrightarrow \mathrm{Pic}^G(\mathcal{R}) \longrightarrow \mathrm{Pic}(\mathcal{R}) \longrightarrow \mathrm{Pic}(G),$$

where $\mathcal{X}(G)$ is the kernel of the forgetful morphism $\mathrm{Pic}^G(\mathcal{R}) \to \mathrm{Pic}(\mathcal{R})$. Since $\mathcal{X}(G) = \{1\}$ and $\mathrm{Pic}(G) = \{1\}$ by [Dol03, Chapter 7.2] then $\mathrm{Pic}^G(\mathcal{R}) \cong \mathrm{Pic}(\mathcal{R})$. Moreover, given that $\mathrm{Pic}^G(\mathcal{R}) \subseteq \mathrm{Pic}(\mathcal{R})^G \subset \mathrm{Pic}(\mathcal{R})$, were $\mathrm{Pic}(\mathcal{R})^G$ is the group of G-invariant line bundles, there result follows from

$$\mathrm{Pic}^G(\mathcal{R}) \cong \mathrm{Pic}(\mathcal{R})^G \cong \mathrm{Pic}(\mathcal{R}) \cong \pi_1^*(\mathrm{Pic}(\mathbb{P}^N)) \times \pi_2^*(\mathrm{Pic}(\mathbb{P}^{n+1})) \cong \mathbb{Z} \times \mathbb{Z}.$$

□

For $\mathcal{L} \cong \mathcal{O}(a, b)$, the GIT quotient is defined as:

$$\overline{M}^{GIT}_{n,d,t} = \mathrm{Proj} \bigoplus_{m \geqslant 0} H^0(\mathcal{R}, \mathcal{L}^{\otimes m})^G,$$

where $t = \frac{b}{a}$. Next, we explain why it is enough to consider t instead of a and b. The main tool to understand variations of GIT from a computational viewpoint is the Hilbert-Mumford numerical criterion which in our particular case has the following form.

Lemma 2.2. *Given an ample $\mathcal{L} \cong \mathcal{O}(a,b) \in \mathrm{Pic}^G(\mathcal{R})$, let (X,H) be a pair parametrized by $\mathcal{R}$, and let λ be a normalized one-parameter subgroup. The* Hilbert-Mumford function *(see [MFK94, Def 2.2]), is $\mu^{\mathcal{L}}((X,H),\lambda) = a\mu_t(X,H,\lambda)$ where $t = \frac{b}{a} \in \mathbb{Q}_{\geqslant 0}$ and*

$$\begin{aligned}\mu_t(X,H,\lambda) &\coloneqq \mu(X,\lambda) + t\mu(H,\lambda)\\ &= \min\{\langle I,\lambda\rangle \mid x^I \in \mathcal{X}\} + t\min\{r_i \mid x_i \in \mathcal{H}\}.\end{aligned}$$

Proof. By [MFK94, pg 49.], for fixed (X,H) and λ, $\mu^{\mathcal{L}}\colon \mathrm{Pic}^G(\mathcal{R}) \to \mathbb{Z}$ is a group homomorphism. Moreover, given any G-equivariant morphism of G-varieties $\pi\colon \mathcal{R} \to Y$, we have that $\mu^{\pi^*\mathcal{L}}((X,H),\lambda) = \mu^{\mathcal{L}}(\pi(X,H),\lambda)$. Applying these two properties, the result follows from:

$$\begin{aligned}\mu^{\mathcal{O}(a,b)}((X,H),\lambda) &= \mu^{\pi_1^*\mathcal{O}_{\mathbb{P}^N}(a)\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^{n+1}}(b)}((X,H),\lambda)\\ &= \mu^{\pi_1^*\mathcal{O}_{\mathbb{P}^N}(a)}((X,H),\lambda) + \mu^{\pi_2^*\mathcal{O}_{\mathbb{P}^{n+1}}(b)}((X,H),\lambda)\\ &= a\mu^{\mathcal{O}_{\mathbb{P}^N}(1)}(X,\lambda) + b\mu^{\mathcal{O}_{\mathbb{P}^{n+1}}(1)}(H,\lambda) = a\mu_t(X,H,\lambda).\ \square\end{aligned}$$

Remark 2.3. Let (X,H) and (X',H') be such that $(\mathcal{X},\mathcal{H}') = (\mathcal{X},\mathcal{H})$. Then, $\mu_t(X,H,\lambda) = \mu_t(X',H',\lambda)$.

Definition 2.4. Let $t \in \mathbb{Q}_{\geqslant 0}$. The pair (X,H) is *t-stable* (resp. *t-semistable*) if $\mu_t(X,H,\lambda) < 0$ (resp. $\mu_t(X,H,\lambda) \leqslant 0$) for all non-trivial one-parameter subgroups λ of G. A pair (X,H) is *t-unstable* if it is not t-semistable. A pair (X,H) is *strictly t-semistable* if it is t-semistable but not t-stable.

3 Stratification of the space of stability conditions

In this section, we fix a maximal torus T of one-parameter subrgroups of G and a coordinate system of $\mathbb{P}^n$ such that T is diagonal in G.

Definition 3.1. The *fundamental set $S_{n,d}$ of one-parameter subgroups $\lambda \in T$* consists of all elements $\lambda = \mathrm{Diag}(s^{r_0},\dots,s^{r_{n+1}})$ where $(r_0,\dots,r_{n+1}) = c(\gamma_0,\dots,\gamma_{n+1}) \in \mathbb{Z}^{n+1}$ satisfying the following:

(1) $\gamma_i = \frac{a_i}{b_i} \in \mathbb{Q}$ such that $\gcd(a_i,b_i) = 1$ for all $i = 0,\dots,n+1$ and $c = \mathrm{lcm}(b_0,\dots,b_{n+1})$.

(2) $1 = \gamma_0 \geqslant \gamma_1 \geqslant \gamma_{n+1} = -1 - \sum_{i=1}^{n}\gamma_i$.

(3) $(\gamma_0,\dots,\gamma_{n+1})$ is the unique solution of a consistent linear system given by n equations chosen from the union of the following sets:

$$\begin{gathered}\mathrm{Eq}(n,d) \coloneqq \{\gamma_i - \gamma_{i+1} = 0 \mid i = 0,\dots,n\} \cup \\ \left\{\sum_{i=0}^{n+1}(d_i - \bar{d}_i)\gamma_i = 0 \mid d_i,\bar{d}_i \in \mathbb{Z}_{\geqslant 0} \text{ for all } i \text{ and } \sum_{i=0}^{n+1} d_i = \sum_{i=0}^{n+1}\bar{d}_i = d\right\}.\end{gathered} \tag{1}$$

The set $S_{n,d}$ is finite since there are a finite number of monomials in Ξ_d.

Proof of Theorem 1.3. Let $R^{ns}_{T_t}$ be the non-t-stable loci of $\mathcal{R}$ with respect to a maximal torus T; and let $\mathcal{R}^{ns}$ be the non t-stable loci of $\mathcal{R}$. By [Dol03, p 137.]), the following holds

$$\mathcal{R}^{ns} = \bigcup_{T_i \subset G} R^{ns}_{T_i}.$$

Let (X', H') be a pair which is not t-stable. Then, $\mu_t(X', H', \rho) \geqslant 0$ for some $\rho \in T'$ in a maximal torus T' which may be *different* from T. All the maximal tori are conjugate to each other in G, and by [Dol03, Exercise 9.2.(i)] the following holds:

$$\mu_t((X', H'), \rho) = \mu_t(g \cdot (X', H'), g\rho g^{-1}).$$

Then, there is $g_0 \in G$ such that $\lambda := g_0 \rho g_0^{-1}$ is normalized and $(X, H) := g_0 \cdot (X', H')$ has coordinates in our coordinate system such that $\mu_t(X, H, \lambda) \geqslant 0$. In these coordinate system one-parameter subgroups form a closed convex polyhedral subset Δ of dimension $n+1$ in $M \otimes \mathbb{Q} \cong \mathbb{Q}^{n+2}$ (in fact Δ is a standard simplex). Indeed, this is the case since for any normalized one-parameter subgroup, $\lambda = \mathrm{Diag}(s^{r_0}, \ldots, s^{r_{n+1}})$, $\sum r_i = 0$ and $r_i - r_{i+1} \geqslant 0$.

By Lemma 2.2, for any fixed t, X and H, the function $\mu_t(X, H, -) : M \otimes \mathbb{Q} \to \mathbb{Q}$ is piecewise linear. The critical points of μ_t (i.e. the points where μ_t fails to be linear) correspond to those points in $M \otimes \mathbb{Q}$ where $\langle I, \lambda \rangle = \langle \overline{I}, \lambda \rangle$ for $I = (d_0, \ldots, d_{n+1})$, $\overline{I} = (\overline{d_0}, \ldots, \overline{d_{n+1}})$ representing monomials of degree d of the form $f = \sum f_I x^I$ defining f. Since $\langle -, - \rangle$ is bilinear that is equivalent to say that $\langle I - \overline{I}, \lambda \rangle = 0$. These points define a hyperplane in $M \otimes \mathbb{Q}$ and the intersection of this hyperplane with Δ is a simplex $\Delta_{I,\overline{I}}$ of dimension n.

The function $\mu_t(X, H, -)$ is linear on the complement of the hyperplanes defined by $\langle I - \overline{I}, \lambda \rangle = 0$. Hence its minimum is achieved on the boundary, i.e. either on $\partial\Delta$ or on $\Delta_{I,\overline{I}}$ which are all convex polytopes of dimension n. We can repeat this reasoning by finite inverse induction on the dimension until we conclude that the minimum of $\mu_t(X, H, -)$ is achieved at one of the vertices of Δ or $\Delta_{I,\overline{I}}$. But these correspond precisely, up to multiplication by a constant, to the finite set of one-parameter subgroups in $S_{n,d}$. ☐

Corollary 3.2. *Let $(X, H) \in \mathcal{R}$ and*

$$a = \min\{t \in \mathbb{Q}_{\geqslant 0} \mid \mu_t(g \cdot (X, H), \lambda_i) \leqslant 0 \textit{ for all } \lambda_i \in S_{n,d},\ g \in G\},$$
$$b = \max\{t \in \mathbb{Q}_{\geqslant 0} \mid \mu_t(g \cdot (X, H), \lambda_i) \leqslant 0 \textit{ for all } \lambda_i \in S_{n,d},\ g \in G\}.$$

If (X, H) is t-semistable for some $t \in \mathbb{Q}_{\geqslant 0}$, then

(i) (X, H) is t-semistable if and only if $t \in [a, b] \cap \mathbb{Q}_{\geqslant 0}$,

(ii) if (X, H) is t-stable for some t, then (X, H) is t-stable for all $t \in (a, b) \cap \mathbb{Q}_{>0}$.

We will call $[a, b]$ *the* interval of stability *of the pair* (X, H)*. We say* $[a, b] = \emptyset$ *if* (X, H) *is* t*-unstable for all* $t \in \mathbb{Q}_{\geqslant 0}$.

Proof. Recall that $S_{n,d}$ is a finite set, by Theorem 1.3. Moreover, the pair (X, H) is t-(semi)stable if and only if

$$\mu_t(X, H) = \max_{\substack{\lambda_i \in S_{n,d} \\ g \in G}} \{\mu_t(g \cdot (X, H), \lambda_i)\} < 0 \quad (\leqslant 0)$$

Notice that each of the functions $\mu_t(g \cdot (X, H), \lambda_i)$ is affine on t and that there are only a finite number of such functions to consider in the definition of $\mu_t(X, H)$. Indeed, the last statement follows from observing that μ_t depends only of λ_i (finite number of choices in $S_{n,d}$ to consider) and the monomials with non-zero coefficients in the polynomials defining $g \cdot (X, H)$. But there are only a finite number of such subsets of those monomials, since $\mathcal{P}(\Xi_d) \times \mathcal{P}(\Xi_1)$ is finite.

To see that $b < \infty$, observe that any hyperplane in $\mathbb{P}^{n+1}$ is conjugate by an element of G to the hyperplane given by $\{x_0 = 0\}$. Let $r = (1, 0, \ldots, 0, -1) \in \mathbb{Z}^{n+2}$ and $\lambda = \mathrm{Diag}(s^r) \in S_{n,d}$. Then $\mu(\{x_0 = 0\}, \lambda) = 1 > 0$. Hence, for $t \gg 0$, we have that $\mu_t(X, D) > 0$ as each $\mu_t(g \cdot (X, D), \lambda)$ is piecewise affine. We conclude that if (X, D) is not t-semistable for some $t \in \mathbb{Q}_{\geqslant 0}$, then

$$[a, b] = \bigcap_{\substack{\lambda_i \in S_{n,d} \\ g \in G}} \{t \mid \mu_t(g \cdot (X, H), \lambda_i) \leqslant 0\}$$

is a bounded interval, as it is an intersection of a finite number of intervals. This proves (i).

For (ii), notice that (X, H) being t-stable for some t_0 is equivalent to the functions $\mu_{t_0}(g \cdot (X, H), \lambda_i)$ being always strictly negative. Then, the statement follows because $\mu_t(g \cdot (X, H), \lambda_i)$ are affine functions, and $[a, b]$ is a compact interval. □

4 Centroid Criterion

Theorem 1.3 allows us to detect the lack of t-stability of a G-orbit by having to consider only a finite number of one-parameter subgroups, precisely those in $S_{n,d}$. This is a necessary step to describe all t-unstable pairs in $\mathcal{R}$ algorithmically. However, sometimes it is convenient to decide on the t-stability of a given pair (X, H) without comparing to all the elements in

$S_{n,d}$. For this purpose and to shorten the proof of Theorem 1.1, we developed the Centroid Criterion, for which we need to introduce extra notation. Fix $t \in \mathbb{Q}_{\geqslant 0}$. We have a map $\mathrm{disc}_t \colon \Xi_d \times \Xi_1 \to M \otimes \mathbb{Q} \cong \mathbb{Q}^{n+2}$, defined as

$$\mathrm{disc}_t(x_0^{d_0} \cdots x_{n+1}^{d_{n+1}}, x_j) = (d_0, \dots, d_{j-1}, d_j + t, d_{j+1} \dots d_{n+1}).$$

The image of disc_t is supported on the first quadrant of the hyperplane

$$H_{n,d,t} = \left\{ (y_0, \dots, y_{n+1}) \in \mathbb{Q}^{n+2} \,\middle|\, \sum_{i=0}^{n+1} y_i = d + t \right\}.$$

We define the set $\mathrm{Conv}_t(X, H)$ as the convex hull of

$$\{\mathrm{disc}_t(v, b) \mid v \in \mathcal{X},\ b = \min(\mathcal{H})\} \subset H_{n,d,t},$$

where the minimum is for the Mukai order in Ξ_1, which is a total order (see Section 1.2). Observe that $\overline{\mathrm{Conv}_t(X, H)}$ is a convex polytope.

Given $t \in \mathbb{Q}_{\geqslant 0}$, we define the *$t$-centroid* as

$$\mathcal{O}_t = \mathcal{O}_{n,d,t} = \left(\frac{d+t}{n+2}, \dots, \frac{d+t}{n+2} \right) \in H_{n,d,t} \subset \mathbb{Q}^{n+2}.$$

Proof of Lemma 1.5. First we note that (X, H) is t-semistable (t-stable, respectively) if and only if $(X, X \cap \{\min(\mathcal{H}) = 0\})$ is t-semistable (t-stable, respectively). Indeed, let $x_k = \min(\mathcal{H})$. Given any one-parameter subgroup $\lambda \in N$ we have

$$\begin{aligned} \mu_t((X, H), \lambda) &= \mu(\mathcal{X}, \lambda) + t \min_{b \in \mathcal{H}} \{\langle b, \lambda \rangle\} \\ &= \mu(\mathcal{X}, \lambda) + t \langle x_k, \lambda \rangle = \mu_t((X, X \cap \{x_k = 0\}), \lambda). \end{aligned}$$

Hence, we may assume $D = X \cap \{x_k = 0\}$. Suppose $\mathcal{O}_t \notin \overline{\mathrm{Conv}_t(X, H)}$, then there is an affine function $\phi \colon \mathbb{R}^{n+2} \to \mathbb{R}$ such that $\phi|_{\overline{\mathrm{Conv}_t(X,H)}}$ is positive and $\phi(\mathcal{O}_t) = 0$. In fact, since the vertices of $\mathrm{Conv}_t(X, H)$ have rational coefficients, we can choose ϕ to have integral coefficients. Write

$$\phi(y_0, \dots, y_{n+1}) = \sum_{i=0}^{n+1} a_i y_i + l$$

For $\mathrm{disc}_t(x^{d_0} \cdots x^{d_{n+1}}, x_k) = (d_0, \dots, d_k + t, \dots, d_{n+1}) \in \overline{\mathrm{Conv}_t(X, H)}$ we have

$$\sum_{i=0}^{n+1} a_i d_i + t a_k + l > 0,$$

and since $\phi(\mathcal{O}_t) = 0$, we obtain $\frac{d+t}{n+2}\sum_{i=0}^{n+1} a_i + l = 0$. Let $p = -\frac{l}{d+t} \in \mathbb{Q}$ and choose $m \in \mathbb{Z}_{\geqslant 0}$ such that $mp \in \mathbb{Z}$. Let

$$\lambda(s) = \begin{pmatrix} s^{m(a_0-p)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s^{m(a_{n+1}-p)} \end{pmatrix} \in N.$$

Hence

$$\begin{aligned} \mu_t((X,H),\lambda) &= \min_{\prod_i x_i^{d_i} \in \mathcal{X}} \left\{ \sum_{i=0}^{n+1} m(a_i - p)d_i \right\} + tm(a_k - p) \\ &= m \left(\min_{\prod_i x_i^{d_i} \in \mathcal{X}} \left\{ \left(\sum_{i=0}^{n+1} a_i d_i \right) + ta_k \right\} - p(d+t) \right) \\ &= \min_{v \in \overline{\mathrm{Conv}_t(X,H)}} \phi(v) > 0. \end{aligned}$$

Hence (X,H) is not t-semistable. We have shown that if (X,H) is t-semistable, then $\mathcal{O}_t \in \overline{\mathrm{Conv}_t(X,H)}$. The proof of the statement when (X,H) is t-stable is similar; in the above reasoning we only need to swap $\overline{\mathrm{Conv}_t(X,H)}$ by $\mathrm{Int}(\mathrm{Conv}_t(X,H))$ and the strict inequalities by $\geqslant 0$.

Conversely, suppose that (X,H) is not t-semistable. Then there is a normalized one-parameter subgroup $\lambda = \mathrm{Diag}(s^{a_0}, \dots, s^{a_{n+1}}) \in N$ with $\sum a_i = 0$ and such that

$$\begin{aligned} 0 < \mu_t(X,H,\lambda) &= \min_{\prod_i x_i^{d_i} \in \mathcal{X}} \left\{ \sum_{i=0}^{n+1} d_i a_i \right\} + ta_k \\ &= \min_{\prod_i x_i^{d_i} \in \mathcal{X}} \left\{ a_0 d_0 + \cdots + a_k(d_k + t) + \cdots a_{n+1} d_{n+1} \right\}. \end{aligned}$$

Let $\phi(y_0, \dots, y_{n+1}) = \sum a_i y_i$. By convexity, we have that $\phi|_{\overline{\mathrm{Conv}_t(X,H)}} > 0$. On the other hand $\phi(\mathcal{O}_t) = \sum a_i \frac{d+t}{n+2} = \frac{d+t}{n+2} \sum a_i = 0$. Hence $\mathcal{O}_t \notin \overline{\mathrm{Conv}_t(X,H)}$. The proof for t-stability is similar. $\square$

Lemma 4.1. *Let $(X,H) \in \mathcal{R}$. Suppose that its interval of semi-stability $[a,b]$ is not empty. Then*

(i) $a = 0$ if and only if X is a GIT-semistable hypersurface of degree d.

(ii) $b \leqslant t_{n,d} = \frac{d}{n+1}$

(iii) The pair (X,H) is $t_{n,d}$-semistable if and only if $X \cap H$ is a semistable hypersurface of degree d in $H \cong \mathbb{P}^n$.

Proof. The first statement holds because the Hilbert-Mumford function at $t = 0$ coincides with the Hilbert-Mumford function for hypersurfaces, and the natural projection $\mathcal{R} \to \mathbb{P}^{n+1}$ is G-invariant.

For part (ii), suppose that $t > t_{n,d}$. Without loss of generality, we can suppose that the equations of any pair (X, H) are given by

$$X = (F(x_0, \dots, x_{n+1}) = 0), \qquad H = (x_0 = 0).$$

Let $\lambda = (n+1, -1, \dots, -1)$, then

$$\mu_{t_{n,d}}((X, H), \lambda) > -d + \frac{d}{n+1}(n+1) = 0$$

holds and (X, H) is t-unstable. Therefore $b \leqslant t_{n,d}$.

Next, we discuss (iii). Suppose that $Y_0 := X \cap H$ is unstable. We can select a coordinate system such that

$$Y_0 := (p_d(x_1, \dots, x_{n+1}) = 0), \qquad H := \{x_0 = 0\},$$

and Y_0 is unstable in the coordinate system $\{x_1, \dots, x_{n+1}\}$. By using Mukai's order of monomials, we claim that among all possible pairs (X, H) such that $Y_0 = X \cap H$, the pair $(\tilde{X}, H)$ given by

$$\tilde{X} := p_d(x_1, \dots, x_{n+1}) + x_{n+1}^{d-1} x_0, \qquad H := (x_0 = 0), \tag{2}$$

will minimize the Hilbert-Mumford function for any normalized one-parameter subgroup, because

$$\mu(\tilde{X}, \lambda) = \min\{\mu(X, \lambda) \mid X \cap H = Y_0\}.$$

Indeed, any other X with $X \cap H = Y_0$ differs from $\tilde{X}$ by a monomial involving the variable x_0. Then, we observe that any other monomial divided by x_0 is greater than $x_0 x_{n+1}^{d-1}$ in Mukai's order. As a consequence if $\mu_{t_{n,d}}(\tilde{X}, H, \lambda) > 0$, then any pair (X, H) with $X \cap H = Y_0$ is $t_{n,d}$-unstable. Next, we use the Centroid Criterion (Lemma 1.5). By hypothesis, Y_0 is not a semistable hypersurface in $\mathbb{P}^n$. Then the convex hull of its monomials does not contain the point $\left(\frac{d}{n+1}, \dots, \frac{d}{n+1}\right) \in \mathbb{R}^{n+1}$. If we consider the monomials of Y_0 as monomials in $\mathbb{K}[x_0, \dots, x_{n+1}]$, then the convex hull of the monomials of $\tilde{X}$ does not contain the point $\left(0, \frac{d}{n+1}, \dots, \frac{d}{n+1}\right) \in \mathbb{R}^{n+2}$. Notice that this implies that $\overline{\mathrm{Conv}_{t_{n,d}}(\tilde{X}, (x_0 = 0))}$ does not contain the point

$$\mathcal{O}_{n,d,t_{n,d}} = \left(0, \frac{d}{n+1}, \dots, \frac{d}{n+1}\right) + \frac{d}{n+1}(1, 0, \dots, 0),$$

and by the Centroid Criterion (X, H) is $t_{n,d}$-unstable. To see the last assertion, notice that $\overline{\mathrm{Conv}_{t_{n,d}}(\tilde{X}, (x_0 = 0))}$ is the convex hull of

$$V = \{\mathrm{disc}_{t_{n,d}}(m, x_0) \mid m \text{ is a monomial in } p_d\} \subset P := \{y_0 = t_{n,d}\} \subset \mathbb{R}^{n+2},$$

and the point $q := (1+t_{n,d}, \dots, 0, d-1) \notin P$. Therefore $\overline{\mathrm{Conv}_{t_{n,d}}(\tilde{X}, (x_0 = 0))}$ is a pyramid with base V and vertex q. Since $\mathcal{O}_{t_{n,d}} \in P \backslash V$, the claim follows.

Next suppose that (X, H) is unstable. Then, by the Centroid Criterion there is a coordinate system such that $\overline{\mathrm{Conv}_t(X, H)}$ does not contain the centroid $\mathcal{O}_{n,d,\frac{d}{n+1}}$. By using the Mukai order as in the previous case, we may assume $H = \{x_i = 0\}$. Let

$$v := \left(\frac{d}{n+1}, \dots, \frac{d}{n+1}, 0, \frac{d}{n+1}, \dots, \frac{d}{n+1}\right),$$

where the value 0 corresponds to the i-th entry. Observe $v \notin \overline{\mathrm{Conv}_0(X, H)}$, since otherwise

$$\mathcal{O}_{n,d,t_{n,d}} = \mathrm{disc}_{\frac{d}{n+1}}(v, x_i) \in \overline{\mathrm{Conv}_{t_{n,d}}(X, H)}.$$

The monomials in the polynomial defining $X \cap (x_i = 0)$ are precisely those monomials m_j in the polynomial defining X with exponents of the form $a_j = (d_0^j, \dots, d_{i-1}^j, 0, d_{i+1}^j, \dots d_{n+1}^j)$. Those monomials correspond to the points generating a face F of $\overline{\mathrm{Conv}_{t_{n,d}}(X, H)}$, namely the convex hull of points $(d_0^j, \dots, d_{i-1}^j, t_{d,n}, d_{i+1}^j, \dots d_{n+1}^j)$. The projection F_P of F onto the hyperplane $P = \{y_i = 0\} \subset \mathbb{R}^{n+2}$ gives us that $v \notin F_P$ since $F_P \subseteq \overline{\mathrm{Conv}_0(X, H)}$. But F_P corresponds to $\overline{\mathrm{Conv}_0(X \cap H)}$ and $v = \mathcal{O}_{n-1,d,0}$, so by the Centroid Criterion $X \cap H$ is unstable. $\square$

Lemma 4.2. *The set of walls $t_i \in \mathbb{Q}_{\geqslant 0}$ corresponds to a subset of the finite set*

$$\left\{ t = -\frac{\langle m, \lambda \rangle}{\langle \overline{m}, \lambda \rangle} \;\middle|\; m \in \Xi_d,\ \overline{m} \in \Xi_1, \lambda \in S_{n,d} \right\}.$$

Proof. Notice that given any wall t_i there is at least a pair (X, H) such that

$$\mu_t(X, H) = \max_{\lambda \in S_{n,d}} \{\mu_t((X, H), \lambda)\}$$

satisfies $\mu_t(X, H) \leqslant 0$ for $t \leqslant t_i$ and $\mu_t(X, H) > 0$ for $t > t_i$. Hence $\mu_t(X, H) = 0$ for $t = t_i$ since μ_t is continuous. The result follows from Theorem 1.3 and Remark 2.3. $\square$

Proof of Theorem 1.1. By Remark 2.3 and the fact $\mathcal{P}(\Xi_k) \times \mathcal{P}(\Xi_1)$ is a finite set, there is a finite number of possible intervals of stability, say $[a_j, b_j]$. Hence $t_i \in \bigcup_j \{a_j, b_j\}$ and Lemma 4.1 implies that all $b_i \leqslant t_{n,d}$. The Theorem follows from Lemma 4.2. $\square$

Proof of Corollary 1.2. From [OS78, Theorem 2.1], any hypersurface X of degree $d \geqslant 3$ has $\dim(\mathrm{Aut}(X)) = 0$. Hence, For any log smooth pair $p = (X, D) \in \mathcal{R}$, its stabilizer G_p satisfies

$$0 \leqslant \dim(G_p) = \dim(G_X \cap G_D) \leqslant \dim(G_X) \leqslant \dim(\mathrm{Aut}(X)) = 0,$$

where the last equality follows from [OS78, Theorem 2.1]. The result follows from the following identity (see [Dol03, Corollary 6.2]):

$$\begin{aligned}\dim\left(\overline{M}^{GIT}_{n,d,t}\right) &= \dim(\mathcal{H}) - \dim(G) + \min_{p\in\mathcal{H}} \dim G_p = \\ &= \left(\binom{n+1+d}{d} - 1 + (n+1)\right) - \left((n+2)^2 - 1\right).\end{aligned}$$

We are left to proof the following claim: any pair (X, D) such that $D \not\subset X$ and $X \cap D$ is smooth is t-semistable for all $t \in [0, t_{n,d}]$. Since smooth hypersurfaces are GIT semistable, the claim follows from Lemma 4.1. ☐

5 Families of t-unstable pairs

In this section we determine, for a given t, the set of monomials that characterize non-t-stable and t-unstable pairs.

Definition 5.1. Fix $t \in [0, t_{n,d}]$, and let λ be a normalized one-parameter subgroup. A non empty pair of sets $A \subset \Xi_d$ and $B \subset \Xi_1$ is a *maximal t-(semi)destabilized pair* (A, B) with respect to λ if the following conditions hold:

(i) Each pair $(v, m) \in A \times B$ satisfies $\langle v, \lambda\rangle + t\langle m, \lambda\rangle > 0$ ($\geqslant 0$, respectively).

(ii) If there is another pair of sets $\tilde{A} \subset \Xi_d$, $B \subset \Xi_1$ such that $A \subseteq \tilde{A}$, $B \subseteq \tilde{B}$ and for all $(v, m) \in \tilde{A} \times \tilde{B}$ the inequality $\langle v, \lambda\rangle + t\langle m, \lambda\rangle > 0$ ($\geqslant 0$, respectively) holds, then $\tilde{A} = A$ and $\tilde{B} = B$.

Lemma 5.2. *Given a one-parameter subgroup λ any maximal t-(semi)-destabilized pair with respect to λ can be written as*

$$N^+_t(\lambda, x_i) := (V^+_t(\lambda, x_i), B^+(x_i))$$
$$\textit{(respectively } N^\oplus_t(\lambda, x_i) := (V^\oplus_t(\lambda, x_i), B^\oplus(x_i))\textit{)}$$

where $x_i \in \Xi_1$ and

$$V^+_t(\lambda, x_i) := \{v \in \Xi_d \mid \langle v, \lambda\rangle + t\langle x_i, \lambda\rangle > 0\},\ B^+(x_i) := \{m \in \Xi_1 \mid m \geqslant x_i\},$$
$$V^\oplus_t(\lambda, x_i) := \{v \in \Xi_d \mid \langle v, \lambda\rangle + t\langle x_i, \lambda\rangle \geqslant 0\},\ B^\oplus(x_i) := \{m \in \Xi_1 \mid m \geqslant x_i\}.$$

Proof. Let (A, B) be a maximal t-semidestabilized pair with respect to λ. Let $x_i := \min(B)$. By Mukai's order,

$$\langle v, \lambda\rangle + t\langle m, \lambda\rangle \geqslant \langle v, \lambda\rangle + t\langle x_i, \lambda\rangle \geqslant 0, \qquad \text{for all } (v, m) \in (A, B).$$

Then $(A, B) \subseteq N^\oplus_t(\lambda, x_i)$ and the maximality condition implies $(A, B) = N^\oplus_t(\lambda, x_i)$. In particular, this proves that $N^\oplus_t(\lambda, x_i)$ is a maximal t-semidestabilized pair with respect to λ. The proof for maximal t-destabilized pairs is similar, exchanging the inequalities for strict inequalities. ☐

Proof of Theorem 1.4. Suppose (X, H) is a t-unstable pair (a t-semistable pair, respectively). By Theorem 1.3 there is $g \in G$ and $\lambda \in S_{n,d}$ such that:

$$\mu_t(g \cdot X, g \cdot H), \lambda) > 0 \ (\geqslant 0, \text{respectively}).$$

Then, every $(v, m) \in (g \cdot \mathcal{X}, g \cdot \mathcal{H})$ satisfies $\langle v, \lambda \rangle + t\langle m, \lambda \rangle > 0$ ($\geqslant 0$, respectively). By the definition of maximal t-(semi)stable pairs and Lemma 5.2, $g \cdot \mathcal{X} \subseteq V_t^+(\lambda, x_i)$ and $g \cdot \mathcal{H} \subseteq B^+(x_i)$ ($g \cdot \mathcal{X} \subseteq V_t^\oplus(\lambda, x_i)$ and $g \cdot \mathcal{H} \subseteq B^\oplus(x_i)$, respectively) hold for some $\lambda \in S_{n,d}$ and some $x_i \in \Xi_1$. Choosing the maximal pairs of sets $N_t^\oplus(\lambda, x_i)$ under the containment order where $\lambda \in S_{n,d}$ and $x_i \in \Xi_1$, we obtain families of pairs whose coefficients belong to maximal t-(semi)destabilized sets. This computation is performed by Algorithm 4. □

6 Algorithms

In this section we describe an algorithm to find all walls t_i (and possibly a few false walls, see Remark 6.1) as well as the maximal t-(semi)destabilized pairs which characterize the families of pairs in Theorem 1.4 for any wall $t = t_i$ and any chamber $t \in (t_i, t_{i+1})$. Our algorithm has a modular structure, reflecting its full implementation in Python (see [GMG15b]). In all the applications of geometric interest considered so far (essentially surfaces in $\mathbb{P}^3$) the whole program took seconds to run on an average desktop computer.

6.1 Top level algorithm

Algorithm 1 describes the steps to compute all maximal t-(semi)-destabilized pairs for each relevant t, referring to other algorithms in each case.

Algorithm 1 Top level algorithm.

Require: $n \geqslant 2 \vee d \geqslant 1$
  Compute $S_{n,d}$ (Algorithm 2).
  Compute $Walls = Walls(S_{n,d}) \leftarrow \{t_0 = 0, \dots, t_f = \frac{d}{n+1}\}$ (Algorithm 3).

  for all $t_i \in Walls$ or $t = \frac{t_i + t_{i+1}}{2}$ where $t_i, t_{i+1} \in Walls$ **do**
    Compute Maximal t-(semi)destabilized pairs. (Algorithm 4).
  end for
  Partially remove redundant $t \in Walls$ (Algorithm 7).

6.2 Computation of $S_{n,d}$ and walls

Recall the set of affine linear equations $\mathrm{Eq}(n, d)$ in (1) used to define $S_{n,d}$. The computation of $S_{n,d}$ follows at once from its definition: solving a finite number of linear systems of $n+2$ variables obtained as subsets of $\mathrm{Eq}(n, d)$. Hence Algorithm 2 is straight forward.

Algorithm 2 Computation of $S_{n,d}$.

$S_{n,d} = \emptyset$.
for all $K \in \mathcal{P}(Eq(n,d))$ such that $|K| = n+2$ and $\{\sum \gamma_i = 0\} \in K$ **do**
  if K has a unique solution **then**
    $S_{n,d} \leftarrow \text{solution}(K)$.
  end if
end for

The computation of the set of candidate walls (see Remark 6.1), Lemma 4.2 makes Algorithm 3 a straight forward computation considering all possible elements in $S_{n,d} \times \Xi_d \times \Xi_1$.

Algorithm 3 Computation of walls.

$\text{Walls} = \emptyset$.
for all $\lambda \in S_{n,d} \wedge v \in \Xi_d \wedge b \in \Xi_1$ **do**
  $\text{Walls} \leftarrow \frac{-\langle v, \lambda \rangle}{\langle b, \lambda \rangle}$.
end for

6.3 Computation of maximal t-(semi)destabilized pairs

Algorithm 4 computes the pairs of sets $N_t^{\oplus}(\lambda, x_i)$. According to Theorem 1.4 and Lemma 5.2, given $t \in \mathbb{Q}_{\geqslant 0}$, we first need to compute all the pairs of sets $N_t^{\oplus}(\lambda, x_i)$ as defined in Lemma 5.2 for all $\lambda \in S_{n,d}$ and $x_i \in \Xi_1$ whether they are maximal or not (these sets are stored in Pairs_t). Then, we need to find which of them are maximal in the containment order (these are stored in $\mathrm{MaximalPairs}_t$).

Notice that any $N_t^{+}(\lambda, x_i)$ which is maximal satisfies that $N_t^{+}(\lambda, x_i) = N_t^{\oplus}(\lambda, x_i)$. Hence, we need to decide which $N_t^{\oplus}(\lambda, x_i) \in \mathrm{MaximalPairs}_t$ are strictly t-semistable or t-unstable. This is easily done using the Centroid Criterion (Lemma 1.5), implemented in $\mathrm{CentroidInBoundary}((V, B))$. The latter is described in Algorithm 6.

An important part of Algorithm 4 consists on finding $N_t^{\oplus}(\lambda, x_i)$ for given $\lambda \in S_{n,d}$, $t \in \mathbb{Q}_{\geqslant 0}$ and $x_i \in \Xi_1$. This is carried out by Algorithm 5. Since the Mukai order for Ξ_1 is total, in $N_t^{\oplus}(\lambda, x_i) = (V, B)$, the set B consists of all $x_j \in \Xi_1$ such that $x_j \geqslant i$, i.e such that $j \leqslant i$.

On the other hand V consists of all $v \in \Xi_d$ such that $\langle \lambda, v\rangle + t\langle \lambda, x_j\rangle \geqslant 0$ for all $x_j \in \Xi_1$. But since for $x_j \in \Xi_1$, $x_j \geqslant x_i$, and $x_i \in B$, we have that $v \in V$ if and only if $\langle \lambda, v\rangle + t\langle \lambda, x_j\rangle \geqslant \langle \lambda, v\rangle + t\langle \lambda, x_i\rangle \geqslant 0$.

Algorithm 4 Computation of maximal t-(semi)destabilized pairs.

```
Pairs_t = ∅.
for λ ∈ S_{n,d} do
  for x_i ∈ Ξ_1 do
    Compute N_t^⊕(λ, x_i) = (V^⊕(λ, x_i), B^⊕(x_i)) (Algorithm 5).
    Pairs_t ← N_t^⊕(λ, x_i) = (V^⊕(λ, x_i), B^⊕(x_i)).
  end for
end for
Pairs_t = ∅.
for (V, B) ∈ Pairs_t do
  flag = 1.
  for (V', B') ∈ Pairs_t do
    if V ⊊ V' ∨ B ⊊ B' then
      flag = 0.
    end if
  end for
  if flag = 1 then
    if CentroidInBoundary((V, B)) then
      (V, B) is strictly t-semistable.
    else
      (V, B) is t-unstable.
    end if
    Pairs_t ← (V, B).
  end if
end for
```

Algorithm 5 Computation of $N_t^{\oplus}(\lambda, x_i)$.

```
N_t^⊕(λ, x_i) = (V, B) = (∅, ∅).
for x_j ∈ Ξ_1 do
  if j ⩽ i then
    B ← x_j.
  end if
end for
for v ∈ Ξ_d do
  if ⟨λ, v⟩ + t⟨λ, x_i⟩ ⩾ 0 then
    V ← v.
  end if
end for
```

6.3.1 Centroid Criterion

The implementation of the Centroid Criterion (Lemma 1.5) is equivalent to decide if the t-centroid belongs to the interior of a closed convex polyhedral set, its boundary or its complement. Indeed, $\mathrm{Conv}_t(X, H)$ is a simplex of dimension $n+1$ in $M \otimes \mathbb{Q} \cong \mathbb{Q}^{n+2}$. This is a classic problem which can be solved using the Simplex Algorithm.

Algorithm 6 CentroidInBoundary((V, B)) decides if (V, B) is t-semistable or t-unstable.

if $\mathcal{O}_t \in \partial\mathrm{Conv}_t(V, B)$ **then**
  return true
else
  return false
end if

6.4 Detection of redundant walls

Remark 6.1 (On false walls)**.** The values given by Lemma 4.2 to estimate the walls may include values of t which are not real walls but false walls. A value t_i is a *false wall* if and only if $\overline{M}^{GIT}_{t_i-\epsilon} \cong \overline{M}^{GIT}_{t_i+\epsilon}$. We detect false walls by comparing maximal t-(semi)destabilized pairs for $t < t_i$ and $t > t_i$. This detection is done in two steps. In the first one, we compare if the monomials describing maximal t-(semi)destabilized pairs obtained in Algorithm 4 for $t < t_i$ and $t > t_i$ are the same. If that is the case, then t_i is a false wall. This method is carried out by Algorithm 7. In the case studied in [GMG15a] $(n = 2, d = 3)$, this step detected all false walls.

However, it may be the case that there are maximal t-(semi)destabilized pairs $N^{\oplus}_{t-\epsilon}(\lambda_k, x_i) \neq N^{\oplus}_{t+\epsilon}(\lambda_l, x_j)$. In that case t_i may still be a false wall if there is an element $g \in G$ that identifies both families. The only way to detect this phenomenon is to classify the pairs in each t-(semi)destabilized pair S and compare if these change when crossing the (possibly false) wall t_i. This can only be done doing an analysis of the geometric properties of the pair, which is dependent of the values of n and d.

Algorithm 7 Partial detection of redundant walls.

Walls $= \{t_0 = 0, \dots, t_f\}$.
for all $t_i, t_{i+1} \in$ Walls **do**
  $t = \frac{t_i + t_{i+1}}{2}$.
  if $\mathrm{Pairs}_{t_i} = \mathrm{Pairs}_t$ **then**
    t is redundant.
  end if
end for

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