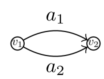
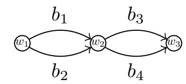
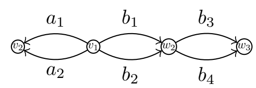
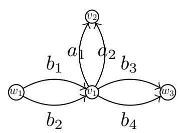
1 Gluing vertices

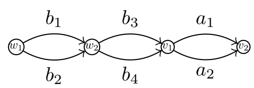
Example: glue the 1-dimensional quiver 

to the 2-dimensional quiver 

. There are 6 possible ways to do this.

- $v_1 = w_1$. In this case, we get the quiver: 

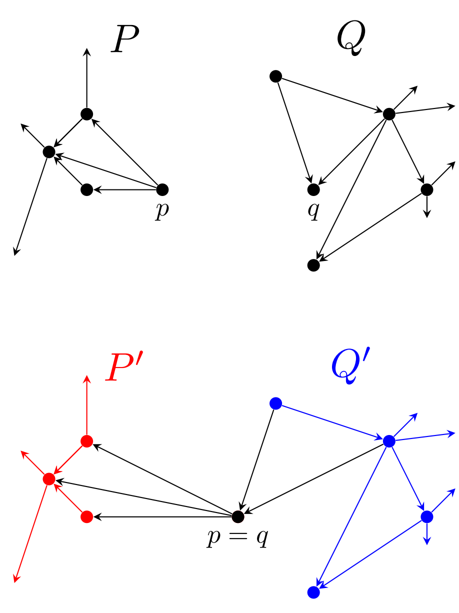
- $v_1 = w_2$ 

- $v_1 = w_3$ 

- $v_2 = w_1$ Similar to $v_1 = w_3$
- $v_2 = w_2$ Similar to $v_1 = w_2$
- $v_2 = w_3$ Similar to $v_1 = w_1$

Claim: joining 2 acyclic quivers by identifying a vertex yields an acyclic quiver

Let P, Q be two quivers, and S be the directed graph obtained by identifying vertex $p \in P$ with vertex $q \in Q$. Note that S is connected, and $S \setminus \{p = q\}$ is the disjoint union of sets $P' \subset P$ and $Q' \subset Q$.



Now, assume that there is indeed a cycle in S. Note that this cycle must contain the vertex $p = q$, (otherwise, we could isolate this cycle in either $P' \subset P$ or $Q' \subset Q$ and both of these quivers are, by assumption, acyclic)

Since, however, $p = q$ is the only vertex connecting the two "lobes" of S, this means that the cycle containing $p = q$ can be simplified to a cycle contained entirely in P or in Q. (BWOC, sps that the cycle contains loop components in both P and Q. Then since these two quivers are disjoint except for $p = q$, then they both must contain $p = q$, and the arrows and vertices for each loop component are contained entirely in the respective quivers P, Q. Thus P contains a cycle, as does Q. A contradiction)

Claim: $dim(S) = dim(P) + dim(Q)$.

Recall: $dim(S) = |S_1| - |S_0| + 1$. Now, since this method of gluing results in a quiver with all of the arrows from both, and all vertices from both with two identified together, we have:

$$|S_1| = |P_1| + |Q_1|, \quad |S_0| = |P_0| + |Q_0| - 1$$

Then

$$|S_1| - |S_0| + 1 = \big(|P_1| + |Q_1|\big) - \big(|P_0| + |Q_0| - 1\big) + 1$$

Rearranging, we get:

$$dim(S) = |S_1| - |S_0| + 1 = \big(|P_1| - |P_0| + 1\big) + \big(|Q_1| - |Q_0| + 1\big) = dim(P) + dim(Q)$$

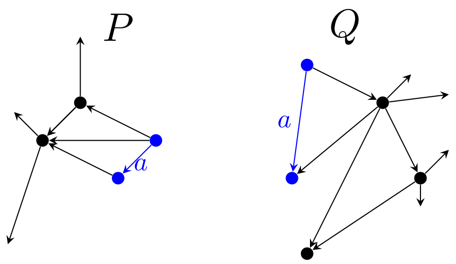
2 Gluing arrows

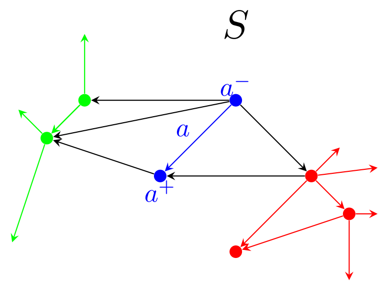
Example Glue the 1-dimesional quiver $\begin{tikzcd} v_1 \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] & v_2 \end{tikzcd}$ to the 2-dimensional one by identifying an arrow. There are 8 possible gluings: $a_1 \in \{b_1, ..., b_4\}$ and $a_2 \in \{b_1, ...b_4\}$

For the case $a_1 = b_1$, we have then: $\begin{tikzcd} v_1 \arrow[r, bend left=60, "a_2"] \arrow[r, "a_1/b_1"] \arrow[r, bend right, "b_2"'] & v_2 \arrow[r, bend left, "b_3"] \arrow[r, bend right, "b_4"'] & w_3 \end{tikzcd}$

Claim: joining 2 acyclic quivers by identifying an arrow from each (following orientation) yields an acyclic quiver.

Let P, Q be two acyclic quivers, and S be the directed graph obtained by identifying arrow $a_1 \in P$ with arrow $a_2 \in Q$. For simplicity, we will now call this arrow as simply $a \in S_1 \cap P_1 \cap Q_1$.





Denote by a^+ and a^- the head and tail resp of arrow a.

Then if the ordered colletion of arrows $C = \{\alpha_1, \alpha_2, ...\}$ is a cycle in S with vertices $V = \{v_1, ..., v_{n-1}, v_n = v_1\}$, we have the following observation:

$$a^{\pm} \in V.$$

Note: if only one of the $a^{\pm}$ is contained in the cycle, then we are in a precisely similar situation as that with identifying a single vertex to glue.

Alternatively, if neither $a^{\pm}$ is contained in the cycle, then we could write $C \subset S \setminus N_a$ where N_a is the set of all edges with endpoints one of $a^{\pm}$. Again, the acyclic nature of P and Q make this impossible ($S \setminus N_a$ is composed of two disjoint quivers, one $\subset P$ and one $\subset Q$).

Therefore V must contain both the head and the tail of the arrow a.

WLOG assume that C contains no loops. Now, we can write the ordered vertices in C as:

$$a^-, p_1, ..., p_i, a^+, q_1, ..., q_j, a^-$$

where the $p_k \in P$ and $q_k \in Q$. (otherwise C would contain a loop). Then note that S also contains the cycle

$$a^-, a^+, q_1, ..., q_j, a^-,$$

since $a \in S_1$. But note that $a^{\pm} \subset Q_0$, since $a \in Q_1$. Therefore Q contains the cycle given above, a contradiction.

Therefore S does not contain any cycle.

Claim: $dim(S) = dim(P) + dim(Q)$.

By simple observation, we see that

$$|S_1| = |P_1| + |Q_1| - 1$$

and similarly, since the head of each arrow is identified together and the tail of each is identified together,

$$|S_0| = |P_0| + |Q_0| - 2$$

Therefore:

$$dim(S) = |S_1| - |S_0| + 1 = |P_1| + |Q_1| - 1 - |P_0| - |Q_0| + 2 + 1 = \big(|P_1| - |P_0| + 1\big) + \big(|Q_1| - |Q_0| + 1\big)$$