1 Toric Geometric Invariant Theory(GIT)

Main idea: given an exact sequence of abelian groups:

$$0 \to L \to \mathbb{Z}^n \to A \to 0$$

we can obtain an algebraic variety through a long procedure that elucidates the connection between the subgroup L and the variety as a zero set corresponding to a set of polynomials.
n.b. As the sequence is exact, we see that L(which is called a *lattice*) is infinite. Else, we would have an element that had finite order, and so the map into $\mathbb{Z}^n$ would not be injective.

Now, given te short exact sequence $0 \to L \to \mathbb{Z}^n \to A \to 0$, we can act upon this with $Hom(\cdot, \mathbb{C}^*)$ to obtain:

$$0 \to Hom(A, \mathbb{C}^*) \to Hom(\mathbb{Z}^n, \mathbb{C}^*) \to Hom(L, \mathbb{C}^*) \to 0$$

which is again exact, as $\mathbb{C}^*$ is a divisible group.

Note that $\forall \varphi \in Hom(\mathbb{Z}^n, \mathbb{C}^*)$, we can obtain $\tilde{\varphi} \in Hom(L, \mathbb{C}^*)$ by pulling back using the map $L \to \mathbb{Z}^n$:

$$\begin{tikzcd}
L \arrow[r] \arrow[dr, "\tilde{\varphi}"'] & \mathbb{Z}^n \arrow[d, "\varphi"] \\
 & \mathbb{C}^*
\end{tikzcd}$$

Recall that $Hom(\mathbb{Z}^n, \mathbb{C}^*) \simeq (\mathbb{C}^*)^n$, and so we have

$$0 \to Hom(A, \mathbb{C}^*) \to (\mathbb{C}^*)^n \to Hom(L, \mathbb{C}^*) \to 0$$

Example: Let $L \subseteq \mathbb{Z}^3$ be given by $(1,1,1)\mathbb{Z} + (1,3,5)\mathbb{Z}$. i.e. the integer scaled "lines" from the two "lines" (which in $\mathbb{Z}^3$ consist only of the lattice points that coincide with the line in continuous space) in $\mathbb{Z}^3$, one that is the diagonal through the point (1,1,1), the other of which goes through (1,3,5) with a slope of $1/\sqrt{35}$, $3/\sqrt{35}$, and $5/\sqrt{35}$ in the x, y, and z-directions respectively.
Then $A = cok(L \hookrightarrow \mathbb{Z}^3) = \mathbb{Z}^3/L$. Now, to get a better understanding of A, we can look at this as an iterated quotient: first, we see that the line $(1,1,1)\mathbb{Z}$ induces an equivalence of values along that line in 3-space.

and so A is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Now, since $(\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ and $Hom(A, \mathbb{C}^*)$ is a subgroup of $(\mathbb{C}^*)^n$, then

$$Hom(A, \mathbb{C}^*) \circlearrowright \mathbb{C}^n.$$

Example: Recall: $Hom(\mathbb{Z}^3, \mathbb{C}^*) \simeq (\mathbb{C}^*)^3$, and we have

$$(\mathbb{C}^*)^3 \to Hom(L, \mathbb{C}^*) \to 0.$$

So:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[dr, "\tilde{\varphi}"'] & \mathbb{Z}^3 \arrow[d, "\varphi"] \\
 & & \mathbb{C}^*
\end{tikzcd}$$

Given $\sigma \in \mathbb{C}^*$, want to look at its preimage in L. This gives us the pullback map $\tilde{\phi}$ for any $\phi \in Hom(\mathbb{Z}^3, \mathbb{C}^*)$.
So $(1,1,1)\mathbb{Z} + (1,3,5)\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \xrightarrow{\varphi} \mathbb{C}^*$. And, since any function can be defined in terms of its values on the three bases, this can be written:
Given $\big(\varphi(1,0,0), \varphi(0,1,0), \varphi(0,0,1)\big) \in (\mathbb{C}^*)^3$, want its image in $(\mathbb{C}^*)^3 \to Hom(A, \mathbb{Z})$.

Then $Hom(A, \mathbb{C}^*)$ is the kernel of the map $(\mathbb{C}^*)^3 \to Hom(L, \mathbb{C}^*)$, which we can now compute, since we know how the map itself looks, as in the example above.

Then write the action of $Hom(A, \mathbb{C}^*)$ on $\mathbb{C}^3$ as an action on $\mathbb{C}[x_1, ..., x_n]$, and take the subring

$$S^0 \subseteq \mathbb{C}[x_1, ..., x_n]$$

of invariants w.r.t. the action of $Hom(A, \mathbb{C}^*)$.

Thus we have a map:

$$0 \to I \to \mathbb{C}[x_1, ..., x_n] \to S^0 \to 0$$

where this I is the ideal that corresponds to the kernel of the map.

Then I induces a variety in $\mathbb{C}^3$ given by

$$Spec\,(\mathbb{C}^n//_0 Hom(A, \mathbb{C}^*))$$

So the question now is: what does this variety do?

1.1 Review

The procedure we have now can be summarized as follows: Given a lattice L, we can form a short exact sequence:

$$0 \to L \to \mathbb{Z}^n \to A \to 0$$

and follow the steps below to obtain a variety $X(L) \subset \mathbb{C}^m$, usually denoted $Spec(\mathbb{C}[x_1, ..., x_n]^G)$ where $G = Hom(A, \mathbb{C}^*)$.

1. $0 \to L \to \mathbb{Z}^n \to A \to 0$

2. Apply Hom functor:

$$0 \to Hom(A, \mathbb{C}^*) \to \mathbb{Z}^n\mathbb{C}^*) \simeq (\mathbb{C}^*)^n$$

3. Recall $(\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$, so $G = Hom(A, \mathbb{C}^*)^n$ acts on $\mathbb{C}^n$, as it is a subgroup of $(\mathbb{C}^*)^n$ (as shown by the short exact sequence from prev step)

4. Find the invariant ring $\mathbb{C}[x_1, ...x_{,n}]^G = \{f(x_1, ...x_n) : g \circ f = f \ \forall g \in G\}$ and can see that this is a finitely generated ring, so have a finite set of monomials $\overline{x}^{I_1}, ..., \overline{x}^{I_m}$ s.t.

$$\mathbb{C}[x_1, ..., x_n]^G = \mathbb{C}[\vec{x}^{I_1}, ..., \vec{x}^{I_m}]$$

(Here there are relations between the $\vec{x}^I$ terms)

5. Now can take the kernel I of the map $\mathbb{C}[x_1, ...x_n] \to \mathbb{C}[\vec{x}^{I_1}, ...\vec{x}^{I_m}]$ to form a short exact sequence:

$$0 \to I \to \mathbb{C}[z_1, ...z_n] \to \mathbb{C}[\vec{x}^{I_1}, ...\vec{x}^{I_m}]$$

and this ideal I cna be generated by a finite number of polynomials

$$I = \langle g_1(z_1, ..., z_m), ..., g_k(z_1, ..., z_m)\rangle$$

6. Take the zero set associated to I:

$$V(I) = \{(z_1, ...z_m) \in \mathbb{C}^m : g_i(\vec{z}) = 0 \ \forall i\}$$

2 Another approach

Let $(S, +)$ be an abelian semigroup. Then one can associate a ring to it as follows:

$$\mathbb{C}[S] = \mathbb{C}[x^s : s \in S]$$

where we define $x^s \cdot x^{\tilde{s}} = x^{s+\tilde{s}}$ for all $s, \tilde{s} \in S$. Suppose (for well-definedness) that $(S, +) \subset (\ N, +)$, and further suppose that S is finitely generated. i.e. $S = \langle s_1, ..., s_m\rangle$. Then

$$\mathbb{C}[S] = \mathbb{C}[s_1, ...s_m]$$

where the s_i might not be independent of each other. Then, taking the map from $\mathbb{C}[w_1, ..., w_m]$, the polynomial ring on m *independent* variables to $\mathbb{C}[s_1, ..., s_m]$ by $w_i \to s_i$, note that, as there are possible relations between the s_i, this induces a kernel, and so we have a short exact sequence:

$$0 \to I_S \to \mathbb{C}[w_1, ..., w_m] \to \mathbb{C}[s_1, ..., s_m]$$

and the zero set of this ideal forms the variety:

$$Spec(\mathbb{C}[S]) = V(I_S) \subset \mathbb{C}^m$$

Claim: Given $0 \to L \to \mathbb{Z}^n \to A \to 0$, then

$$Spec\left(\mathbb{C}[x_1, ...x_m]^{Hom(A,\mathbb{C}^*)}\right) \simeq Spec(\mathbb{C}[L \cap \mathbb{N}^n])$$

Now, given the lattice and associated short exact sequence, can define a **multigrading** in $\mathbb{C}[x_1, ...x_n]$ e.g. for $L = \mathbb{Z}^2 \subset \mathbb{Z}^3$,

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z}^2 & \to & \mathbb{Z}^3 & \to & \mathbb{Z} & \to 0 \\ & (1,0,-1) & & (1,0,0) & \to & 1 & \\ & (1,-1,0) & & (0,1,0) & \to & 1 & \\ & (0,1,-1) & & (0,0,1) & \to & 1 & \end{array}$$

Then $deg(\vec{x}^{(a_1,a_2,a_3)} = \varphi(a_1,a_2,a_3) = a_1\varphi(1,0,0) + a_2\varphi(0,1,0) + a_3\varphi(0,0,1)$. This gives the traditional grading, but we could define the map differently. e.g.

$$\begin{array}{cccc} \mathbb{Z}^2 & \to & \mathbb{Z} & \to 0 \\ (1,0) & \to & 1 & \\ (0,1) & \to & -1 & \end{array}$$

Note that a homogeneous polynomial with this grading would look something like $x^3 + y^2x^5$. Note, we could even have had the map $\mathbb{Z}^2 \to \mathbb{Z}_3 \to 0$, where $x^2 = x^{2+3n}$ for all $n \in \mathbb{Z}$

Claim: $f \in \mathbb{C}[x_1, ..., x_n]^G$ iff $deg_A(f) = 0$.

So L are the elements "of degree 0" with respect to the multigrading.

Example:
$L = \{(n_1, n_2) : n_1 + n_2 = 3k, k \in \mathbb{Z}\}$, then we have the ses:

$$0 \to L \to \mathbb{Z}^2 \to A \to 0$$

where $A \simeq \mathbb{Z}/3\mathbb{Z}$, and the multigrading on the ring:

$$\mathbb{C}[x_1, x_2] \to \mathbb{Z}_3 \to 0$$

s.t. $deg_A(x_1) = w$, $deg_A(x_2) = w$, and $w^3 = 1$. So the homogeneous monomials of degree 0 are: $x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$.

3 Generalizing for the cases with singularities

We now have a procedure (2 really) whereby, given a lattice, one can obtain a toric variety associated to that lattice. It turns out that very often this variety is trivial(a point), and so we need to introduce some new machinery to identify in which cases this occurs, and also to introduce a modification which will produce varieties that are not trivial.

Theorem: Let $S = \mathbb{C}[x_1, ..., x_n]$. TFAE:

1. $\exists a \in A$ such that

   $$S_a := \{x^{I_1} \in S \mid: deg(x^{I_1}) = a\}$$

   is a finite dimensional vector space

2. $S_0 \simeq \mathbb{C}$

3. $L \cap \mathbb{N}^n = \{0\}$.

Definition: A is a **positive grading** if $S_0 \simeq \mathbb{C}$.
Example: $\mathbb{Z}^2 \to \mathbb{Z} \to 0$ by $x \mapsto 1$, $y \mapsto -1$, then $L = \{(n,m) : n - m = 0\}$
Theorem: If A is a positive grading, then $L \cap \mathbb{N}^n = \{0\}$ so $\mathbb{C}[0] = \mathbb{C}$ and $Spec(\mathbb{C}) =$ point.

Hence the need for a generalization. It turns out that the idea of multigrading admits precisely the tool needed to produce nontrivial varieties.

Note: $S_{na} \times S_{ma} = S_{(m+n)a}$ gives the infinite direct sum

$$\bigoplus_{n \geq 0} S_{na}$$

a ring structure, and so can set

$$S_{(a)} = \bigoplus_{n \geq 0} S_{na}$$

a finitely generated ring, and so we have a map

$$\mathbb{C}[z_1, ... z_m] \to S_{(a)}, \quad \text{by} \quad z_i \mapsto x^{I_i}$$

and so there is an ideal associated to this map:

$$0 \to I \to \mathbb{C}[z_1, ..., z_m] \to S_{(a)}$$

and the zero set of I is a variety!(?)

Another way of seeing this is to see $S_{(a)} = \mathbb{C}[x^{I_1}, ... x^{I_m}]$ where the x^{I_i} are the generators, and so we have a map from an open set $x_1, ..., x_n \in (\mathbb{C}^*)^n \mapsto (\vec{x}^{I_1}, ... x^{I_m})$ which is contained in some open $U \subset \mathbb{CP}^m$ and can take the closure of the image.
Theorem: There is a map $Proj(S_{(a)}) \to Spec(S_{(0)})$.

3.1 When do we have a compact variety

Remark: $\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim$ the standard equivalence class $\vec{x} \sim \vec{y}$ iff $\exists s \in \mathbb{C}^*$ s.t. $\vec{x} = s\vec{y}$. And $\mathbb{CP}^n$ is a smooth variety.

One consequence of this is as follows: let $f(x_1, ..., x_{n+1})$ be a polynomial s.t. $\forall s \in \mathbb{C}^*$, $f(sx_1, ... sx_{n+1}) = s^m f(\vec{x})$. Then if $\vec{p} \sim \vec{q}$ and $f(\vec{p}) = 0$, then $f(\vec{q}) = 0$ also, so it is well-defined to say

$$f([p]) = 0 \quad \text{for } [p] \in \mathbb{CP}^n$$

and say that f is **homogeneous**. Further, let $I \subset \mathbb{C}[x_1, ..., x_{n+1}]$. Recall that every ideal has a finite number of generators

$$I = \langle f_1, ..., f_m \rangle$$

Then if f_i is homogeneous for all i, then I defines a variety in $\mathbb{CP}^n$ and denote it

$$Proj\left(\frac{\mathbb{C}[x_1, ... x_n]}{I}\right)$$

The difference between this and the other variety construction is (intuitively): $Proj(S/I)$ is compact, whereas $Spec(S/I)$ is not always.

$Proj$ is a section, and so only well-defined if I is homogeneous.
Remark: $f(x_1, ..., x_{n+1})$ is homogeneous iff $f(x_1, ... x_{n+1}) = \sum_{I \in \mathbb{N}_{n+1}} C_I x^I$ where $deg(x^I)$ is constant over I.
However, we have multigrading machinery, so can define a more general $\mathbb{CP}^n$ for such gradings.

In conclusion, given a homogeneous ideal I, we have a compact variety in some $\mathbb{CP}^n$ which is denoted $Proj(S/I)$.

3.2 Review

We have the ses $0 \to L \to \mathbb{Z}^{n+1} \to A \to 0$, then if $L \cap \mathbb{N}^{n+1} = \{0\}$ or equivalently if $S_0 = \mathbb{C}$, then

$$Spec\left(\frac{S}{I}\right) = pt$$

However, if we fix $a \in A$, can construct a ring

$$S_{(a)} = \bigoplus_n S_{(na)}$$

and an exact sequence

$$0 \to I_{(a)} \to \mathbb{C}[z_1, ..., z_m] \to S_{(a)} \to 0$$

and the variety

$$Proj(S_{(a)}) = Proj\left(\frac{\mathbb{C}[z_1, ..., z_m]}{I_{(a)}}\right)$$

Question: How do we know if $I_{(a)}$ is homogeneous? Answer: look at the I_i for $S_{(a)} \simeq \mathbb{C}[\vec{x}^{I_1}, ..., \vec{x}^{I_m}]$.
Example: Take the quiver below, and let $A = \left\{(\theta_1, \theta_2) \in \mathbb{Z}^{Q_0} \mid \theta_1 + \theta_2 = 0\right\}$. Consider $\mathbb{Z}^{Q_1} \to A$ by inc, the inclusion map. Then $inc(1,1) = (-2,2)$. Let $Cir(Q) := ker(inc)$. Then in our case, this will be

$$\begin{tikzcd} v_1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & v_2 \end{tikzcd}$$

$$(1,0) \mapsto (-1,1), \quad (0,1) \mapsto (-1,1)$$

and so $Cir(Q) = \{(n,-n) : n \in \mathbb{Z}\} = (1,-1)\mathbb{Z}$.

Recall: $0 \to Hom(A, \mathbb{C}^*) \to Hom(\mathbb{Z}^2, \mathbb{C}^*) \to Hom(L, \mathbb{C}^*) \to 0$ and $G = Hom(A, \mathbb{C}^*) \subset (C^*)^2$. Want to know how this containment works. Let $t_1 = Im(1,0)$, $t_2 = Im(0,1)$. Then

$$Hom(\mathbb{Z}^2, \mathbb{C}^*) \to Hom(L, \mathbb{C}^*) \text{ by } (t_1, t_2) \mapsto t_1^2 t_2^{-2}$$

and so $G = \{(t_1, t_2) : t_1^2 t_2^{-2} = 1\} = \{(t_1, t_2) \in (\mathbb{C}^*)^2 : t_1^2 = t_2^2\} = \{(t_1, t_2) \in (\mathbb{C}^*)^2 : t_1 = \pm t_2\}$

4 From a polytope to Variety

Let $\Lambda \subset \mathbb{R}^n$ be a lattice, i.e. $\Lambda \simeq \mathbb{Z}^n$. We say P is a **lattice polytope** if every vertex of P belongs to Λ.

Let P be a lattice polytope of dimension $n-d$. Will next describe the mechanism for producing a variety X_P.

1. $\forall$ lattice polytope P can be written

$$P = \{\vec{x} \in \mathbb{R}^{n-d} : \ \vec{v}_i \cdot \vec{x} \geq -w_i\}$$

   e.g.

2. Given $\vec{v}_i \in \mathbb{R}^{n-d}$, $\vec{w} \in \mathbb{R}^n$, define a map

$$\varphi : \mathbb{R}^{n-d} \to \mathbb{R}^n \text{ by } x \mapsto (\vec{v}_1 \cdot \vec{x}, ..., \vec{v}_n \cdot \vec{x})$$

   Note that $V := \varphi(\mathbb{R}^{n-d})$ is an $(n-d)$-dimensional vector subspace and $\varphi(P) \simeq P$ is given by

$$V \cap \mathbb{R}^n_{\geq -w_i}$$

   Let $L := V \cap \mathbb{Z}^n \subset \mathbb{R}^n$. and so have

$$0 \to L \to \mathbb{Z}^n \to A \to 0$$

   and an $\vec{a} \in A$ s.t. $\vec{a} = im(\vec{w})$. e.g.

3. Recall previous discussion with $\mathbb{C}^n //_a Hom(A, \mathbb{C}^*)$.
4. Explicitly find the ideal associated to $\mathbb{C}^n //_a Hom(A, \mathbb{C}^*)$. This needs the same generators of the ring $S_{(a)} = \oplus S_{n\vec{a}}$ where $S_{d\vec{a}} = \{\vec{m} \in \mathbb{C}[z_1, ..., z_m] : \ deg_A(\vec{m}) = \vec{a}d\}$.

So given a polytope P, can find a complex variety X_p, and given a projective toric variety, the polytope is unique up to scaling and $GL(n, \mathbb{Z})$.