3 QFT in one dimension (= QM)

In one dimension there are two possible compact (connected) manifolds M: the circle S^1 and the interval I. We will parametrize the interval by $t \in [0, T]$ so that $t = 0$ and $t = T$ are the two point–like boundaries, while we will parametrize the circle by $t \in [0, T)$ with the identification $t \cong t + T$.

The most important example of a field on M is a map $x : M \to N$ to a Riemannian manifold (N, g) which we will take to have dimension n. That is, for each point t on our 'space–time' M, $x(t)$ is a point in N. It's often convenient to describe N using coordinates. If an open patch $U \subset N$ has local co-ordinates x^a for $a = 1, \ldots, n$, then we let $x^a(t)$ denote the coordinates of the image point $x(t)$. More precisely, $x^a(t)$ are the pullbacks to M of coordinates on U by the map x.

With these fields, the standard choice of action is

$$S[\phi] = \int_M \left[\frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b + V(x) \right] \mathrm{d}t \,, \tag{3.1}$$

where $g_{ab}(x)$ is the pullback to M of the Riemannian metric on N and $\dot{x}^a = dx^a/dt$. We have also included in the action a choice of function $V : N \to \mathbb{R}$, or more precisely the pullback of this function to M, which is independent of worldline derivatives of x. In writing this action we have chosen one–dimensional metric on M to be just the flat Euclidean metric $\delta_{tt} = 1$. Under a small variation δx of x we have

$$\begin{aligned} \delta S &= \int_M \left[g_{ab}(x) \dot{x}^a \dot{\delta x}^b + \frac{1}{2} \frac{\partial g_{ab}(x)}{\partial x^c} \delta x^c \, \dot{x}^a \dot{x}^b + \frac{\partial V(x)}{\partial x^c} \, \delta x^c \right] \mathrm{d}t \\ &= \int_M \left[-\frac{d}{dt} \left(g_{ac}(x) \dot{x}^a \right) + \frac{1}{2} \frac{\partial g_{ab}(x)}{\partial x^c} \dot{x}^a \dot{x}^b + \frac{\partial V(x)}{\partial x^c} \right] \delta x^c \, \mathrm{d}t \end{aligned} \tag{3.2}$$

and requiring that this vanishes for arbitrary $\delta\phi(t)$ gives the Euler–Lagrange equations

$$\frac{d^2 x^a}{dt^2} + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = g^{ab}(x) \frac{dV}{dx^b} \tag{3.3}$$

where $\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$ is the Levi–Civita connection on N, again pulled back to the worldline.

The standard interpretation of all this is to image an arbitrary map $x(t)$ describes a possible *trajectory* a particle might in principle take as it travels through the space N. (See figure 2.) In this context, N is called the **target space** of the theory, while M (or its image $x(M) \subset N$) is known as the **worldline** of the particle. The field equation (3.3) says that when $V = 0$, classically the particle travels along a geodesic in (N, g). V itself is then interpreted as a (non–gravitational) potential[4] through which this particle moves.

[4]The absence of a minus sign on the *rhs* of (3.3) is probably surprising, but follows from the action (3.1). This is actually the *correct* sign with a Euclidean worldsheet, because under the Wick rotation $t \to it$ back to a Minkowski signature worldline, the *lhs* of (3.3) acquires a minus sign. In other words, in Euclidean time $\mathbf{F} = -m\mathbf{a}$!

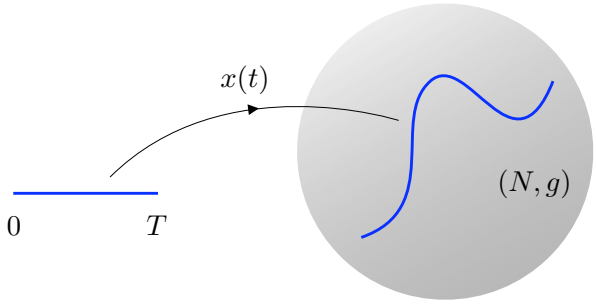


Figure 2: *The theory* (3.1) *describes a map from an abstract worldline into the Riemannian target space* (N, g)*. The corresponding one–dimensional QFT can be interpreted as single particle Quantum Mechanics on* N*.*

From this perspective, it's natural to think of the *target* space N as being the world in which we live, and computing the path integral for this action will lead us to single particle Quantum Mechanics, as we'll see below. However, we're really using this theory as a further warm–up towards QFT in higher dimensions, so I want you to also keep in mind the idea that the *worldline* M is actually 'our space–time' in a one–dimensional context, and the target space N can be some abstract Riemannian manifold unrelated to the space we see around us. For example, at physics of low–energy pions is described by a theory of this general kind, where M is our Universe and N is the group manifold SU(3).

3.1 Quantum Mechanics

The usual way to do Quantum Mechanics is to pick a Hilbert space $\mathcal{H}$ and a Hamiltonian H, which is a Hermitian operator $H : \mathcal{H} \to \mathcal{H}$. In the case relevant above, the Hilbert space would be $L^2(N)$, the space of square–integrable functions on N, and the Hamiltonian would usually be

$$H = \frac{1}{2}\Delta + V\,, \qquad \text{where} \qquad \Delta := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\left(\sqrt{g}g^{ab}\frac{\partial}{\partial x^b}\right) \tag{3.4}$$

is the Laplacian acting on functions in $L^2(N)$. The amplitude for the particle to travel from an initial point $y_0 \in N$ to a final point $y_1 \in N$ in Euclidean time T is given by

$$K_T(y_0, y_1) = \langle y_1 | \mathrm{e}^{-HT} | y_0 \rangle\,, \tag{3.5}$$

which is known as the **heat kernel**. (Here I've written the *rhs* in the Heisenberg picture, which I'll use below. In the Schrödinger picture where states depend on time we would instead write $K_T(y_0, y_1) = \langle y_1, T | y_0, 0 \rangle$.)

The heat kernel is a function on $I \times N$ which may be defined to be the solution of the pde

$$\frac{\partial}{\partial t}K_t(x, y) + HK_t(x, y) = 0 \tag{3.6}$$

subject to the initial condition that $K_0(x, y) = \delta(x - y)$. I remind you that we're in Euclidean worldline time here, and in units where $\hbar = 1$ here. Rotating to Minkowski

signature by sending $t \to \mathrm{i}t$ and restoring the $\hbar$ gives instead

$$\mathrm{i}\hbar \frac{\partial}{\partial t} K_{\mathrm{i}t}(x,y) = H K_{\mathrm{i}t}(x,y) \tag{3.7}$$

that we recognize as Schrödinger's equation. In the simplest example where $N \cong \mathbb{R}^n$ with flat metric $g_{ab} = \delta_{ab}$ and vanishing potential $V = 0$, the heat kernel

$$K_T(x,y) = \frac{1}{(2\pi T)^{n/2}} \exp\left(-\frac{|x-y|^2}{2T}\right) \tag{3.8}$$

where $|x-y|$ is the Euclidean distance between x and y.

As you learned last term, Feynman showed that this heat kernel could also be represented as a path integral. The usual idea is to break the time interval T into N chunks, each of duration $\Delta t = T/N$. We can then write

$$\begin{aligned}
\langle y_1 | \mathrm{e}^{-HT} | y_0 \rangle &= \langle y_1 | \mathrm{e}^{-H\Delta t}\, \mathrm{e}^{-H\Delta t} \cdots \mathrm{e}^{-H\Delta t} | y_0 \rangle \\
&= \int \mathrm{d}^n x_1 \cdots \mathrm{d}^n x_{N-1}\, \langle y_1 | \mathrm{e}^{-H\Delta t} | x_{N-1} \rangle \cdots \langle x_2 | \mathrm{e}^{-HT} | x_1 \rangle \langle x_1 | \mathrm{e}^{-H\Delta t} | y_0 \rangle \\
&= \int \prod_{i=1}^{N-1} \mathrm{d}^n x_i\, K_{\Delta t}(y_1, x_{N-1}) \cdots K_{\Delta t}(x_2, x_1)\, K_{\Delta t}(x_1, y_0)\,.
\end{aligned} \tag{3.9}$$

In the second line here we have inserted the identity operator $\int \mathrm{d}^n x_i\, |x_i\rangle\langle x_i|$ on $\mathcal{H}$ in between each evolution operator; in the present context this can be understood as the concatentation identity

$$K_{t_1+t_2}(x_3, x_1) = \int \mathrm{d}^n x_2\; K_{t_2}(x_3, x_2)\, K_{t_1}(x_2, x_1) \tag{3.10}$$

obeyed by convolutions of the heat kernel.

Now, while the flat space expression (3.8) for the heat kernel does not hold when g_{ab} is a more general Riemannian metric on N, in fact it is (almost) correct in the limit of small times. More precisely, it can be shown that the heat kernel always has the asymptotic form

$$\lim_{\Delta t \to 0} K_{\Delta t}(x,y) \sim \frac{1}{(2\pi\Delta t)^{n/2}} a(x) \exp\left(-\frac{d(x,y)^2}{2\Delta t}\right) \tag{3.11}$$

for small t, where $d(x,y)$ is the geodesic distance between x and y measured using the metric g, and where $a(x)$ is some polynomial in the Riemann curvature tensor that we won't need to be specific about. Therefore, splitting our original time interval $[0,T]$ into very many pieces of very short duration $\Delta t = T/N$ gives

$$\langle y_1 | \mathrm{e}^{-HT} | y_0 \rangle = \lim_{N\to\infty} \left(\frac{1}{2\pi\Delta t}\right)^{\frac{nN}{2}} \int \prod_{i=1}^{N-1} \mathrm{d}^n x_i\, a(x_i)\, \exp\left[-\frac{\Delta t}{2}\left(\frac{d(x_{i+1}, x_i)}{\Delta t}\right)^2\right] \tag{3.12}$$

as an expression for the heat kernel.

This more or less takes us to the path integral. *If* it is sensible to take the limits, then we can take

$$\mathcal{D}x \stackrel{?}{:=} \lim_{N\to\infty} \left(\frac{1}{2\pi\Delta t}\right)^{\frac{nN}{2}} \prod_{i=1}^{N-1} \mathrm{d}^n x_i \, a(x_i) \tag{3.13}$$

to be the path integral measure. Similarly, *if* the trajectory is at least once differentiable then $(d(x_{i+1}, x_i)/\Delta t)^2$ converges to $g_{ab}\dot{x}^a\dot{x}^b$ and we can write

$$\lim_{N\to\infty} \prod_{i=1}^{N-1} \exp\left[-\frac{\Delta t}{2}\left(\frac{d(x_{i+1}, x_i)}{\delta t}\right)^2\right] = \exp\left[-\frac{1}{2}\int_0^T g_{ab}\,\dot{x}^a\dot{x}^b\,\mathrm{d}t\right] \tag{3.14}$$

which recovers the action (3.1), with $V = 0$. (A more general heat kernel can be used to incorporate a non–zero potential.)

We'll investigate these limits further below. Accepting them for now, combining (3.13) & (3.14) we obtain the path integral expression

$$\langle y_1|\mathrm{e}^{-HT}|y_0\rangle = \int_{C_T[y_0,y_1]} \mathcal{D}x \, \exp\left[-\frac{1}{2}\int_0^T g_{ab}\,\dot{x}^a\dot{x}^b\,\mathrm{d}t\right], \tag{3.15a}$$

or in other words, the heat kernel can formally be written as

$$K_T(y_0, y_1) = \int_{C_T[y_0,y_1]} \mathcal{D}x \; \mathrm{e}^{-S}. \tag{3.15b}$$

The integrals in these expressions are to be taken over the space $C_T[y_0, y_1]$ of all continuous maps $x : I \to N$ that are constrained to obey the boundary conditions $x(0) = y_0$ and $x(T) = y_1$.

3.1.1 The partition function

The partition function on the circle can likewise be given and interpretation in the operator approach to Quantum Mechanics. Tracing over the Hilbert space gives

$$\mathrm{Tr}_{\mathcal{H}}(\mathrm{e}^{-TH}) = \int \mathrm{d}^n y \, \langle y|\mathrm{e}^{-HT}|y\rangle = \int_N \mathrm{d}^n y \int_{C_T[y,y]} \mathcal{D}x \; \mathrm{e}^{-S} \tag{3.16}$$

using the path integral expression (3.15b) for the heat kernel. The path integral here is (formally) taken over all continuous maps $x : [0, T] \to N$ such that the endpoints are both mapped to the same point $y \in N$. We then integrate y everywhere over N[5], erasing the memory of the particular point y. This is just the same thing as considering all continuous maps $x : S^1 \to N$ where the worldline has become a circle of circumference T. This shows that

$$\mathrm{Tr}_{\mathcal{H}}(\mathrm{e}^{-TH}) = \int_{C_{S^1}} \mathcal{D}x \, \mathrm{e}^{-S} = \mathcal{Z}_{S^1}[N, g, V], \tag{3.17}$$

which is nothing but the partition function on S^1. In higher dimensions this formula will be the basis of the relation between QFT and Statisical Field Theory, and is really the origin of the name 'partition function' for $\mathcal{Z}$ in physics.

[5]In flat space, the heat kernel (3.8) obeys $K_T(y, y) = K_T(0, 0)$ so is independent of y. Thus if $N \cong \mathbb{R}^n$ with a flat metric, this final y integral does not converge. It will converge if N is compact, say by imposing that we live in a large box, or on a torus *etc.*.

3.1.2 Operators and correlation functions

As in zero dimensions, we can also use the path integral to compute correlation functions of operators.

A **local operator** is one which depends on the field only at one point of the worldline. The simplest types of local operators come from functions on the target space. If $\mathcal{O} : N \to \mathbb{R}$ is a real–valued function on N, let $\hat{\mathcal{O}}$ denote the corresponding operator on $\mathcal{H}$. Then for any fixed time $t \in (0, T)$ we have

$$\langle y_1|\hat{\mathcal{O}}(t)|y_0\rangle = \langle y_1|\mathrm{e}^{-H(T-t)}\,\hat{\mathcal{O}}\,\mathrm{e}^{-Ht}|y_0\rangle \tag{3.18}$$

in the Heisenberg picture. Inserting a complete set of $\hat{\mathcal{O}}(x)$ eigenstates $\{|x\rangle\}$, this is

$$\begin{aligned}\int \mathrm{d}^n x\, \langle y_1|\mathrm{e}^{-H(T-t)}\,\hat{\mathcal{O}}(x)|x\rangle\,\langle x|\mathrm{e}^{-Ht}|y_0\rangle &= \int \mathrm{d}^n x\, \mathcal{O}(x)\,\langle y_1|\mathrm{e}^{-H(T-t)}|x\rangle\,\langle x|\mathrm{e}^{-Ht}|y_0\rangle \\ &= \int \mathrm{d}^n x\, \mathcal{O}(x)\, K_{T-t}(y_1, x)\, K_t(x, y_0)\,,\end{aligned} \tag{3.19}$$

where we note that in the final two expressions $\mathcal{O}(x)$ is just a number; the eigenvalue of $\hat{\mathcal{O}}$ in the state $|x\rangle$.

Using (3.15b), everything on the *rhs* of this equation can now be written in terms of path integrals. We have

$$\begin{aligned}\langle y_1|\mathrm{e}^{-H(T-t)}\,\hat{\mathcal{O}}\,\mathrm{e}^{-Ht}|y_0\rangle &= \int \mathrm{d}^n x_t \left[\int_{C_{T-t}[y_1,x_t]} \mathrm{e}^{-S} \;\times\; \mathcal{O}(x_t) \;\times\; \int_{C_t[x,y_0]} \mathrm{e}^{-S}\right] \\ &= \int_{C_T[y_1,y_0]} \mathcal{D}x\, \mathrm{e}^{-S}\, \mathcal{O}(x(t))\,,\end{aligned} \tag{3.20}$$

where to we again note that integrating over all maps $x : [0, t] \to N$ with endpoint $x(t) = x_t$, then over all maps $x : [t, T] \to N$ with initial point $x(t)$ again fixed to x_t and finally integrating over all points $x_t \in N$ is the same thing as integrating over all maps $x : [0, T] \to N$ with endpoints y_0 and y_1.

More generally, we can insert several such operators. If $0 < t_1 < t_2 < \ldots < t_n < T$ then exactly the same arguments give

$$\begin{aligned}\langle y_1|\hat{\mathcal{O}}_n(t_n)\cdots\hat{\mathcal{O}}_1(t_2)\,\hat{\mathcal{O}}_1(t_1)|y_0\rangle &= \langle y_1|\mathrm{e}^{-H(T-t_n)}\hat{\mathcal{O}}_n(x)\cdots\hat{\mathcal{O}}_2(x)\,\mathrm{e}^{-H(t_2-t_1)}\,\hat{\mathcal{O}}_1(x)\,\mathrm{e}^{-Ht_1}|y_0\rangle \\ &= \int_{C_T[y_0,y_1]} \mathcal{D}x\, \mathrm{e}^{-S} \prod_{i=1}^{n} \mathcal{O}_i(x(t_i))\end{aligned} \tag{3.21}$$

for the n–point correlation function. The hats on the $\hat{\mathcal{O}}_i$ remind us that the *lhs* involves *operators* acting on the Hilbert space $\mathcal{H}$. The objects $\mathcal{O}_i$ inside the path integral are just ordinary functions, evaluated at the point $x(t_i) \in N$ [6].

Notice that in order to run our argument, it was very important that the insertion times t_i obeyed $t_i < t_{i+1}$: we would not have been able to interpret the *lhs* in the Heisenberg

[6] A more precise statement would be that they are functions on the space of fields $C_T[y_0, y_1]$ obtained by pullback from a function on N by the evaluation map at time t_i.

picture had this not been the case[7]. On the other hand, the insertions $\mathcal{O}_i(x(t_i))$ in the path integral are just functions and have no notion of ordering. Thus the expression on the right doesn't have any way to know which insertion times was earliest. For this to be consistent, for a general set of times $\{t_i\} \in (0,T)$ we must actually have

$$\int_{C_T[y_0,y_1]} \mathcal{D}x \left(\mathrm{e}^{-S} \prod_{i=1}^{n} \mathcal{O}_i(x(t_i)) \right) = \langle y_1 | \mathcal{T}\{ \prod_i \hat{\mathcal{O}}_i \} | y_0 \rangle \tag{3.22}$$

where the symbol $\mathcal{T}$ on the *rhs* is defined by

$$\begin{aligned} \mathcal{T}\hat{\mathcal{O}}_1(t_1) &:= \mathcal{O}_1(t_1)\,, \\ \mathcal{T}\{\hat{\mathcal{O}}_1(t_1)\,\hat{\mathcal{O}}_2(t_2)\} &:= \Theta(t_2-t_1)\,\hat{\mathcal{O}}_2(t_2)\,\hat{\mathcal{O}}_1(t_1) + \Theta(t_1-t_2)\,\hat{\mathcal{O}}_1(t_1)\,\hat{\mathcal{O}}_2(t_2)\,, \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \end{aligned} \tag{3.23}$$

and so on, where $\Theta(t)$ is the Heaviside step function. By construction, these step functions mean that the *rhs* is now completely symmetric with respect to a permutation of the orderings. However, for any given choice of times t_i, only one term on the *rhs* can be non–zero. In other words, insertions in the path integral correspond to the **time–ordered product** of the corresponding operators in the Heisenberg picture.

The derivative terms in the action play an important role in evaluating these correlation functions. For suppose we had chosen our action to be just a potential term $\int V(x(t))\,\mathrm{d}t$, independent of derivatives $\dot{x}(t)$. Then, regularizing the path integral by dividing M into many small intervals as before, we would find that neighbouring points on the worldline completely decouple: unlike in (3.12) where the geodesic distance $d(x_{i+1},x_i)^2$ in the heat kernel provides cross–terms linking neighbouring points together, we would obtain simply a product of independent integrals at each time step. Inserting functions $\mathcal{O}_i(x(t_i))$ that are likewise independent of derivatives of x into such a path integral would not change this conclusion. Thus, without the derivative terms in the action, we would have

$$\langle \mathcal{O}_1(t_1)\,\mathcal{O}_2(t_2) \rangle = \langle \mathcal{O}_1(t_1) \rangle \, \langle \mathcal{O}_2(t_2) \rangle \tag{3.24}$$

for all such insertions. In other words, there would be no possible non–trivial correlations between objects at different points of our (one–dimensional) Universe. This would be a very boring world: without derivatives, the number of people sitting in the lecture theatre would have nothing at all to do with whether or not a lecture was actually going on, and what you're thinking about right now would have nothing to do with what's written on this page.

This conclusion is a familiar result in perturbation theory. The kinetic terms in the action allow us to construct a **propagator**, and using this in Feynman diagrams enables us to join together interaction vertices at different points in space–time. As the name suggests, we interpret this propagator as a particle *traveling* between these two space–time interactions and the ability for particles to move is what allows for non–trivial correlation functions. Here we've obtained the same result directly from the path integral.

[7]Exercise: explain what goes wrong if we try to compute $\langle y_1 | \mathrm{e}^{+TH} | y_1 \rangle$ with $T > 0$.

A wider class of local path integral insertions depend not just on x but also on its worldline derivatives $\dot{x}, \ddot{x}$ *etc.*. In the canonical framework, with Lagrangian L we have

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab}\dot{x}^b \tag{3.25}$$

where the last equality is for our action (3.1). Thus we might imagine replacing the *function* $\mathcal{O}(x^a, \dot{x}^a)$ of x and its derivative in the path integral by the *operator* $\mathcal{O}(\hat{x}^a, g^{ab}(\hat{x})\hat{p}_b)$ in the canonical framework.

Now, probably the first thing you learned in Quantum Mechanics was that $[\hat{x}^a, \hat{p}_b] \neq 0$, so at least for generic functions the replacement

$$\mathcal{O}(x^a, \dot{x}^a) \to \mathcal{O}(\hat{x}^a, \hat{g}^{ab}\hat{p}_b)$$

is plagued by ordering ambiguities. For example, if we represent p_a by[8] $-\partial/\partial x^a$, then should we replace

$$g_{ab}\, x^a \dot{x}^b \to -x^a \frac{\partial}{\partial x^a}$$

or should we take

$$g_{ab}\, x^a \dot{x}^b \to -\frac{\partial}{\partial x^a} x^a = -n - x^a \frac{\partial}{\partial x^a}$$

or perhaps something else? Even in free theory, we need to make a **normal ordering prescription** among the x's and p's to define what a composite operator means[9].

From the path integral perspective, however, something smells fishy here. I've been emphasizing that path integral insertions $\mathcal{O}(x, \dot{x})$ are just ordinary *functions*, not operators. How can two ordinary functions fail to commute? To understand what's going on, we'll need to look into the definition of our path integral in more detail.

3.2 The continuum limit

In writing down the basic path integral (3.15b), we assumed it made sense to take the limit

$$\mathcal{D}x \stackrel{?}{=} \lim_{N\to\infty} \prod_{i=1}^{N} \left[\frac{1}{(2\pi\Delta t)^{n/2}}\, \mathrm{d}^n x_i\, a(x_i) \right] \tag{3.26a}$$

to construct a measure on the space of fields. We also assumed it made sense to write

$$S[x] \stackrel{?}{=} \lim_{N\to\infty} \sum_{n=1}^{N-1} \Delta t\, \frac{1}{2} \left(\frac{x_{n+1} - x_n}{\Delta t} \right)^2 \tag{3.26b}$$

as the continuum action (here for a free particle).

Alternatively, instead of splitting the interval $[0, T]$ into increasingly many pieces, another possible way to define a regularized path integral starts by expanding each component of the field $x(t)$ as a Fourier series

$$x^a(t) = \sum_{k\in\mathbb{Z}} \tilde{x}^a_k\, \mathrm{e}^{2\pi \mathrm{i} t/T}\,.$$

[8] The absence of a factor of i on the *rhs* here is again a consequence of having a Euclidean worldline.

[9] And even there we may not be able to make a consistent choice. Read about the Groenewald–Van Hove theorem if you want sleepless nights.

We now regularize by truncating this to a finite sum with $|k| \leq N$. The (free) action for the truncated field is

$$S = \frac{2\pi}{T} \sum_{|k|\leq N} k^2 \, \delta_{ab} \, \tilde{x}^a_k \, \tilde{x}^b_k \tag{3.27}$$

and depends only on the Fourier coefficients. We might now try to define the path integral measure as the limit

$$\mathcal{D}x \stackrel{?}{=} \lim_{N\to\infty} \prod_{k=-N}^{N} \frac{\mathrm{d}^n \tilde{x}_k}{(2\pi)^{n/2}} \tag{3.28a}$$

as an integral over more and more of these Fourier modes with higher and higher frequencies. The continuum action would then be taken to be the infinite series

$$S[x] \stackrel{?}{=} \lim_{N\to\infty} \frac{2\pi}{T} \sum_{|k|\leq N} k^2 \, \delta_{ab} \, \tilde{x}^a_k \, \tilde{x}^b_k \tag{3.28b}$$

which we hope converges.

The obvious question to ask is whether the limits in (3.26a) & (3.26b) or in (3.28a) & (3.28b) actually exist. Perhaps the single most important fact in QFT is that the answer to this question is "No!".

3.2.1 The path integral measure

To prove this, let's keep things simple and work just with the case that $N \cong \mathbb{R}^n$ with a flat metric, so that the space of fields is naturally an infinite dimensional *vector* space, where addition is given by pointwise addition of the fields at each t on the worldline.

We'll start with the measure. In fact, it's easy to prove that *there is no non–trivial Lebesgue measure on an infinite dimensional vector space*. To see this, first recall that for finite dimension D, $\mathrm{d}\mu$ is a **Lebesgue measure** on $\mathbb{R}^D$ if it assigns a strictly positive volume $\mathrm{vol}(U) = \int_U \mathrm{d}\mu > 0$ to every non–empty open set $U \subset \mathbb{R}^D$, if $\mathrm{vol}(U') = \mathrm{vol}(U)$ whenever U' may be obtained from U by translation, and finally if for every $x \in \mathbb{R}^D$ there exists at least one open neighbourhood U_x containing x for which $\mathrm{vol}(U_x) < \infty$. The standard example is of course $\mathrm{d}\mu = \mathrm{d}^D x$. Now let $C_x(L)$ denote the open (hyper)cube centered on x and of side length L. This cube contains 2^D smaller cubes $C_{x_n}(L/2)$ of side length $L/2$, all of which are disjoint. Then

$$\mathrm{vol}(C_x(L)) \geq \sum_{n=1}^{2^D} \mathrm{vol}(C_{x_n}(L/2)) = 2^D \, \mathrm{vol}(C_x(L/2)) \tag{3.29}$$

where the first inequality uses the fact that the measure is positive–definite, and the final equality uses translational invariance. We see that as $D \to \infty$, the only way the *rhs* can remain finite is if $\mathrm{vol}(C_x(L/2)) \to 0$ for any finite L. So the measure must assign zero volume to any infinite dimensional hypercube. Finally, provided our vector space V is *countably* infinite (which both the Fourier series and discretized path integral make plain), we can cover *any* open $U \subset V$ using at most countably many such cubes $C(L/2)$, so $\mathrm{vol}(U) = 0$ for any U and the measure must be identically zero.

3.2.2 Discretization and non–commutativity

The question of whether the discretized action itself converges will shed light on the puzzle of how x and p might not commute in the path integral. It suffices to consider the simplest case of a free particle in one dimension, so choose $N = \mathbb{R}$ and $V = 0$. Then if $0 < t_- < t < t_+ < T$ we have

$$\int_{C_T[y_0,y_1]} \mathcal{D}x \; \mathrm{e}^{-S}\, x(t)\, \dot{x}(t_-) = \langle y_1 | \mathrm{e}^{-H(T-t_+)}\, \hat{x}\, \mathrm{e}^{-H(t_+-t)}\, \hat{p}\, \mathrm{e}^{-Ht} | y_0 \rangle \,, \tag{3.30a}$$

when the insertion of x is later than that of $\dot{x}$, and

$$\int_{C_T[y_0,y_1]} \mathcal{D}x \; \mathrm{e}^{-S}\, x(t)\, \dot{x}(t_+) = \langle y_1 | \mathrm{e}^{-H(T-t)}\, \hat{p}\, \mathrm{e}^{-H(t-t_-)}\, \hat{x}\, \mathrm{e}^{-Ht_-} | y_0 \rangle \tag{3.30b}$$

when x is inserted at an earlier time than $\dot{x}$. Taking the limits $t_+ \to t$ from above and $t_- \to t$ from below, the difference between the *rhs* of (3.30a) & (3.30b) is

$$\langle y_1 | \mathrm{e}^{-H(T-t)}\, [\, \hat{x}, \hat{p}\,]\, \mathrm{e}^{-Ht} | y_0 \rangle \;=\; \langle y_1 | \mathrm{e}^{-HT} | y_0 \rangle \tag{3.31}$$

which does not vanish. By contrast, the difference of the *lhs* seems to be automatically zero. What have we missed?

In handling the *lhs* of (3.30a)-(3.30b) we need to be careful. If we regularize the path integral by discretizing $[0, T]$, chopping it into chunks of width Δt, then we cannot pretend we are bringing the x and $\dot{x}$ insertions any closer to each other than Δt without also taking account of the discretization of the whole path integral. Thus we replace

$$\lim_{t_- \uparrow t} [x(t)\, \dot{x}(t_-)] - \lim_{t_+ \downarrow t} [x(t)\, \dot{x}(t_+)]$$

by the discretized version

$$x_t\; \frac{x_t - x_{t-\Delta t}}{\Delta t} \;-\; x_t\; \frac{x_{t+\Delta t} - x_t}{\Delta t} \tag{3.32}$$

where we stop the limiting procedure as soon as x coincides with any part of the discretized derivative. The order of the factors of x_t and $x_{t\pm\Delta t}$ here doesn't matter; they're just ordinary integration variables.

Now consider the integral over x_t. Apart from the insertion of (3.32), the only dependence of the discretized path integral on this variable is in the heat kernels $K_{\Delta t}(x_{t+\Delta t}, x_t)$ and $K_{\Delta t}(x_t, x_{t-\Delta t})$. Using the explicit form of these kernels in flat space we have

$$\begin{aligned} &\int \mathrm{d}x_t\, K_{\Delta t}(x_{t+\Delta t}, x_t) \left(x_t\; \frac{x_t - x_{t-\Delta t}}{\Delta t} \;-\; x_t\; \frac{x_{t+\Delta t} - x_t}{\Delta t} \right) K_{\Delta t}(x_t, x_{t-\Delta t}) \\ &\qquad = \; -\int \mathrm{d}x_t\, x_t\, \frac{\partial}{\partial x_t} \Big(K_{\Delta t}(x_{t+\Delta t}, x_t)\, K_{\Delta t}(x_t, x_{t-\Delta t}) \Big) \\ &\qquad = \; \int \mathrm{d}x_t\, K_{\Delta t}(x_{t+\Delta t}, x_t)\, K_{\Delta t}(x_t, x_{t-\Delta t}) \;=\; K_{2\Delta t}(x_{t+\Delta t}, x_{t-\Delta t}) \end{aligned} \tag{3.33}$$

where the second step is a simple integration by parts and the final step uses the concatenation property (3.10). The integration over x_t thus removes all the insertions from the

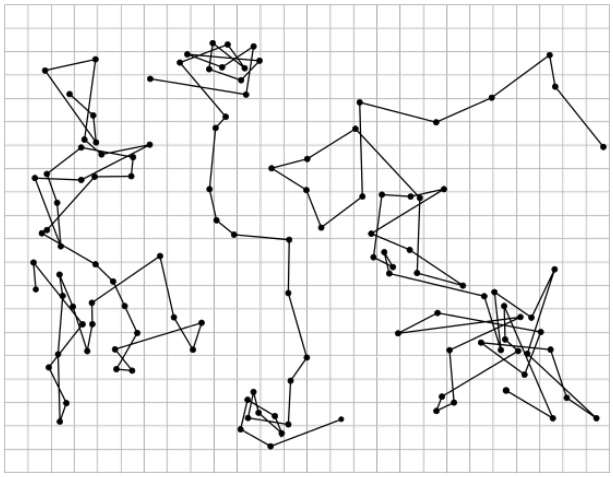


Figure 3: *Stimulated by work of Einstein and Smoluchowski, Jean–Baptiste Perron made many careful plots of the locations of hundreds of tiny particles as they underwent Brownian motion. Understanding their behaviour played a key role in confirming the existence of atoms. A particle undergoing Brownian motion moves an average (rms) distance of* $\sqrt{t}$ *in time* t*, a fact that is responsible for non–trivial commutation relations in the (Euclidean) path integral approach to Quantum Mechanics.*

path integral, and the remaining integrals can be done using concatenation as before. We are thus left with $K_T(y_1, y_0) = \langle y_1 | \mathrm{e}^{-HT} | y_0 \rangle$ in agreement with the operator approach.

There's an important point to notice about this calculation. Had we assumed the path integral included only maps $x : [0, T] \to N$ that are everywhere *differentiable*, rather than merely continuous, then the limiting value of (3.32) would necessarily vanish when $\Delta t \to 0$, contradicting the operator calculation. Non–commutativity arises in the path integral approach to Quantum Mechanics precisely because we're forced to include *non–differentiable* paths, *i.e.* our map $x \in C^0(M, N)$ but $x \notin C^1(M, N)$. But because our path integral includes non–differentiable maps we cannot assign any sensible meaning to $\lim_{\Delta t \to 0} (x_{t+\Delta t} - x_t)/\Delta t$ and the continuum action also fails to exist.

This non–differentiability is the familiar stochastic ('jittering') behaviour of a particle undergoing Brownian motion. It's closely related to a very famous property of random walks: that after a times interval t, one has moved through a net distance proportional to $\sqrt{t}$ rather than $\propto t$ itself. More specifically, averaging with respect to the one–dimensional heat kernel

$$K_t(x, y) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-(x-y)^2/2t} \,,$$

in time t, the mean squared displacement is

$$\langle (x-y)^2 \rangle = \int_{-\infty}^{\infty} K_t(x, y)\, (x-y)^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} K_t(u, 0)\, u^2 \, \mathrm{d}u = t \tag{3.34}$$

so that the rms average displacement from the starting point after time t is $\sqrt{t}$. Similarly,

our regularized path integrals yield a finite result because the average value of

$$x_{t+\Delta t}\,\frac{x_{t+\Delta t}-x_t}{\Delta t} - x_t\,\frac{x_{t+\Delta t}-x_t}{\Delta t} \;=\; \Delta t\left(\frac{x_{t+\Delta t}-x_t}{\Delta t}\right)^2,$$

which for a differentiable path would vanish as $\Delta t \to 0$, here remains finite.

3.2.3 Non–trivial measures?

The requirement that the measure be translationally invariant played an important role in the proof that the naive path integral measure $\mathcal{D}x$ doesn't exist. Do we really need this requirement? In fact, in one dimension, while neither $\mathcal{D}x$ nor $S[x]$ themselves have any continuum meaning, the limit

$$\mathrm{d}\mu_{\mathrm{W}} := \lim_{N\to\infty}\left[\prod_{i=1}^{N}\frac{\mathrm{d}^n x_{t_i}}{(2\pi\Delta t)^{n/2}}\;\exp\left[-\frac{\Delta t}{2}\left(\frac{x_{t_{i+1}}-x_{t_i}}{\Delta t}\right)^2\right]\right] \tag{3.35}$$

of the standard measures $\mathrm{d}^n x_{t_i}$ on $\mathbb{R}^n$ at each time–step *together with* the factor e^{-S_i} does exist. The limit $\mathrm{d}\mu_{|rmW}$ is known as the **Wiener measure** and, as you might imagine from our discussion above, it plays a central role in the mathematical theory of Brownian motion. Notice that the presence of the factor e^{-S_i} means that this measure is certainly not translationally invariant in the fields, avoiding the no–go theorem. For Bryce de Wit, the competition between the efforts of e^{-S} to damp out the contribution of wild field configurations and $\mathcal{D}x$ to concentrate on such fields was poetically "The eternal struggle between energy and entropy.". Wiener's result means that in one dimension the contest is beautifully balanced.

In higher dimensions the situation is less clear. Certainly, the naive path integral measure does *not* exist. It is *believed* that Quantum Field Theories that are **asymptotically free** do have a sensible continuum limit, for reasons we'll see later in the course. The most important example of such a QFT is Yang–Mills theory in four dimensions: every physicist believes this exists, but you can still pick up $1,000,000 from the Clay Institute for actually proving[10] it.

Perhaps more surprisingly, there are plenty of very important field theories for which a continuum path integral measure, of any sort, almost certainly does *not* exist. The most famous example is General Relativity, but it is also true of both Quantum Electrodynamics (QED) and very likely even the Standard Model. Yet planets orbit around the Sun and satellites orbit around the Earth in exquisite agreement with the predictions of General Relativity, QED is the arena for the most accurate scientific measurements ever carried out, and the Standard Model is the Crown Jewel in our understanding of Nature at the subatomic level. Clearly, not having a well–defined continuum limit does *not* mean these theories are so hopelessly ill–defined as to be useless. On the contrary, we can define *effective* quantum theories in all these cases that make perfect sense: we just restrict ourselves to taking the path integral only over low–energy modes, or over some discretized

[10] Terms and conditions apply; see here for details.

version (such as putting the theory on a lattice). So long as we probe these theories within their domain of validity, they make powerful, accurate predictions. What lies beyond may not even be a QFT at all, but something else, perhaps String Theory.

3.3 Locality and Effective Quantum Mechanics

To appreciate the notion of an effective field theory in a simple setting, let's consider what happens in one dimension.

We imagine we have two different fields x and y on the same worldline, which we'll take to be a circle to avoid complications with end–points. I'll choose the action to be

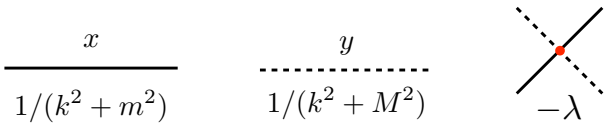
$$S[x,y] = \int_{S^1} \left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + V(x,y)\right] \mathrm{d}t \tag{3.36}$$

where the potential

$$V(x,y) = \frac{1}{2}(m^2x^2 + M^2y^2) + \frac{\lambda}{4}x^2y^2 \tag{3.37}$$

allows the two fields to interact. In terms of the one–dimensional QFT, x and y look like interacting fields with masses m and M, while from the point of view of the target space $\mathbb{R}^2$ you should think of them as two harmonic oscillators with frequencies m and M, coupled together in a particular way. Of course, this coupling has been chosen to mimic what we did in section 2.6 in zero dimensions.

If we are interested in perturbatively computing correlation functions of (local) operators that are independent of $y(t)$, for example $\langle x(t_2)x(t_1)\rangle$, then we could proceed by directly using (3.36) to construct Feynman diagrams. We'd find ingredients



where k is the one–dimensional worldline momentum (which would be quantized in units of the inverse circumference of the circle). On the other hand, we learned in section 2.6 that for such a class of observables, it is expendient to first construct an effective action by integrating out the y field directly. We expect this effective action to contain infinitely many new self–interactions of x which together take into account the effect of the unobserved y field.

Let's repeat that calculation here. As far as the path integral over $y(t)$ is concerned, x is just a fixed background field so we have formally

$$\int \mathcal{D}y \ \exp\left[-\frac{1}{2}\int_0^T y\left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2}x^2\right)y^2\right] = \det\left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2}x^2\right) \tag{3.38}$$

where I've imposed the boundary conditions $y\dot{y}|_{t=0,T} = 0$ on $y(t)$. Accordingly, the effective action for x is

$$S_{\text{eff}}[x] = \int_0^T \left[\frac{1}{2}\dot{x}^2 + \frac{m^2}{2}x^2\right] \mathrm{d}t - \operatorname{tr}\ln\left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2}x^2\right) \tag{3.39}$$

where we've used the identity $\ln\det \mathrm{A} = \mathrm{tr}\ln \mathrm{A}$ (which holds provided A is a trace–class operator; don't worry if you don't know what this means).

Remarkably, *the effective action for x is non–local!* To see this, suppose $G(t,t')$ is the worldline Green's function that obeys

$$\left(\frac{d^2}{dt^2} - M^2\right) G(t,t') = \delta(t-t') \tag{3.40}$$

and so is the inverse of the operator $d^2/dt^2 - M^2$ on the circle. Explicitly one has

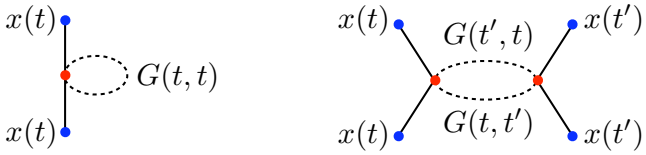
$$G(t,t') = \frac{1}{2M}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-M|t-t'+\beta k|} \tag{3.41}$$

where $\beta = 1/T$ is the inverse circumference and k represents the momentum modes. Now, using $\mathrm{tr}\ln(\mathrm{AB}) = \mathrm{tr}\,(\ln \mathrm{A} + \ln \mathrm{B}) = \mathrm{tr}\ln \mathrm{A} + \mathrm{tr}\ln \mathrm{B}$ we have

$$\begin{aligned}
&\mathrm{tr}\ln\left(-\frac{d^2}{dt^2} + M^2 + \frac{x^2}{2}\right) - \mathrm{tr}\ln\left(-\frac{d^2}{dt^2} + M^2\right) = \mathrm{tr}\ln\left(1 - \lambda\left(\frac{d^2}{dt^2} - M^2\right)^{-1}\frac{x^2}{2}\right) \\
&\quad = -\frac{\lambda}{2}\int_{S^1} \mathrm{d}t\, G(t,t)\, x^2(t) - \frac{\lambda^2}{8}\int_{S^1\times S^1} \mathrm{d}t\,\mathrm{d}t'\, G(t',t)\, x^2(t)\, G(t,t')\, x^2(t') + \cdots \\
&\quad = -\sum_{n=1}^{\infty}\frac{\lambda^n}{2^n n}\int_{(S^1)^n} \mathrm{d}t_1\cdots\mathrm{d}t_n\; G(t_n,t_1)\, x^2(t_1)\, G(t_1,t_2)\, x^2(t_2)\cdots G(t_{n-1},t_n)\, x^2(t_n)
\end{aligned} \tag{3.42}$$

where the second term on the *lhs* is a divergent, but $x(t)$ independent constant. Integrating out y has indeed generated an infinite series of new interactions for $x(t)$, but except for the $\mathcal{O}(\lambda)$ term, these interactions are now *non–local*!

Again, it's instructive to see why this non–locality has arisen. The first two terms in the series (3.42) represent the Feynman diagrams



that arise in the perturbative evaluation of the y path integral. Unlike the trivial case of zero dimensions, here the y field is dynamical; in particular it has its own worldline propagator $G(t,t')$ that allows it to move around on the worldline. (Note that in these diagrams, I've drawn the external blue vertices at different places on the page just for clarity. Each external x field in the diagram on the left resides at the point $t \in S^1$, while the four xs on the right live pairwise at points t and t'.)

Non–locality is generally bad news in physics: the equations of motion we'd obtain from $S_{\mathrm{eff}}[x]$ would be integro–differential equations stating that in order to work out the behaviour of the field x here, we first have to add up what it's doing everywhere else in the (one–dimensional) Universe. But we don't want the results of our experiment in CERN to

depend on what Ming the Merciless may or may not be having for breakfast over on the far side of the Galaxy. So how bad is it here?

From the explicit form (3.41) of the Green's function we see that $G(t,t')$ decays exponentially quickly when $t \neq t'$, with a scale set by the inverse mass M^{-1} of y. This suggests that the effects of non–locality will be small *provided we restrict attention to fields whose derivatives vary slowly on timescales* $\sim M^{-1}$. More specifically, expanding $x(t)$ we have

$$\begin{aligned}
&\int \mathrm{d}t\,\mathrm{d}t'\, G(t,t')^2\, x^2(t)\, x^2(t') \\
&= \int \mathrm{d}t\,\mathrm{d}t'\, G(t,t')^2\, x^2(t) \left[x^2(t) + 2x(t)\dot{x}(t)(t-t') + \left(\dot{x}^2(t) + \frac{1}{2}x(t)\ddot{x}(t)\right)(t-t')^2 + \cdots \right] \\
&= \int \mathrm{d}t\ \left[\frac{\alpha}{M} x^4(t) + \frac{\beta}{M^3}\left(x^2\dot{x}^2 + \frac{1}{2}x^2\ddot{x}\right) + \frac{\gamma}{M^5}\,(\text{four derivative terms}) + \cdots \right] .
\end{aligned} \tag{3.43}$$

In going to the last line we have performed the t' integral. To do so, note that the Green's function $G(t,t')$ depends on t' only through the dimensionless combination $u = M(t-t')$. Thus we replace the factor $(t-t')^p$ in the p^{th} order term in the Taylor expansion by $(u/M)^p$ and change variables $\mathrm{d}t' = \mathrm{d}u/M$ to integrate over the dimensionless quantity u. In particular, the infinite series of dimensionless constants $\alpha, \beta, \gamma, \cdots$ are just some dimensionless numbers – their precise values don't matter for the present discussion.

The important point is that every new derivative of x in these vertices is *suppressed* by a further power of the mass M of the y field. Thus, so long as $\dot{x}, \ddot{x}, \dddot{x}, \ldots$ are all small in units of M^{-1}, we should have a controllable expansion. In particular, if we truncate the infinite Taylor expansion and the infinite expansion (3.42) at any finite order, we will regain an apparently local effective action. This truncation is justified provided we restrict to processes where the momentum of the x field is $\ll M$.

However, once we start to probe energies $\sim M$ something will go badly wrong with our truncated theory. Assuming the original action (3.36) defined a unitary theory (at least in Minkowski signature), simply performing the exact path integral over y must preserve unitary. This is because we haven't yet made any approximations, just taken the first step to performing the full $\mathcal{D}x\,\mathcal{D}y$ path integral. All the possible states of the y field are still secretly there, encoded in the infinite series of non–local interactions for x. However, the approximation to keep just the first few terms in S_{eff} *can't* be unitary, because we're rejecting by hand various pieces of Feynman diagrams: we're throwing away some of the things y might have been doing.

The weak interactions are responsible for many important things, from the formation of light elements such as deuterium in the early Universe, to powering stars such as our Sun, to the radioactive β-decay of ^{14}C used in radiocarbon dating. Since the 1960s physicists have known that these weak interactions are mediated by a field called the W–boson and in 1983, the UA1 experiment at CERN discovered this field and measured its mass to be $M_{\text{W}} \simeq 80\,\text{GeV}$. Typically, β-decay takes place at much lower energies, so to describe them it makes sense to integrate out the dynamics of the W boson leaving us with an effective action for the proton, neutron, electron and neutrino that participate in the interaction.

This effective action contains an infinite series of terms, suppressed by higher and higher powers of the large mass $M_{\rm W}$. Truncating this infinite effective action to its first few terms leads to Fermi's theory of β-decay which gives excellent results at low energies. However, if ones extrapolates the results obtained using this truncated action to high energies, one finds a violation of unitarity. The non–unitarity in Fermi's theory is what lead physicists to suspect the existence of the W–boson in the first place.

3.4 The worldline approach to perturbative QFT

In this chapter, we've been studying the case of QFT in 1 space–time dimension as a warm–up for the higher–dimensional QFTs we'll meet later on. Before proceeding, I'd like to point out an alternative approach to perturbative QFT that was invented by Feynman.

Let's start by considering maps $x : [0, T] \to \mathbb{R}^n$ with the free action

$$S[x] = \int_0^T \mathrm{d}t \, \left[\frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b + \frac{m^2}{2}\right] = \frac{m^2}{2}T + \int_0^T \mathrm{d}t \, \frac{1}{2}\dot{x}^2 \,, \tag{3.44}$$

where I've included a constant term $m^2/2$ in the Lagrangian. This may seem like a strange step; the constant term does not affect the dynamics of the field x in any way. Indeed, the path integral over x becomes

$$\int_{C_I[x,y]} \mathcal{D}x \, \mathrm{e}^{-S} = \mathrm{e}^{-Tm^2/2} \, \langle y|\mathrm{e}^{-HT}|x\rangle \tag{3.45}$$

with the constant term in the action providing just an overall factor. Its true purpose will be revealed below.

With this action, the momentum conjugate to the field x^a is $p_a = \partial L/\partial\dot{x}^a = \dot{x}_a$, so the Hamiltonian is $H = p_a\dot{x}^a - L = p_ap^a/2$, as expected for a free particle on $\mathbb{R}^n$. Therefore, by inserting complete sets of momentum eigenstates, we have

$$\begin{aligned} \langle y|\mathrm{e}^{-HT}|x\rangle &= \int \mathrm{d}^n p \, \mathrm{d}^n q \, \langle y|p\rangle \, \langle p|\mathrm{e}^{-HT}|q\rangle \, \langle q|x\rangle \\ &= \int \frac{\mathrm{d}^n p}{(2\pi)^n} \, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} \, \mathrm{e}^{-Tp^2/2} \end{aligned} \tag{3.46}$$

and so the path integral becomes

$$\int_{C_I[x,y]} \mathcal{D}x \, \mathrm{e}^{-S} = \int \frac{\mathrm{d}^n p}{(2\pi)^n} \, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} \, \mathrm{e}^{-T(p^2+m^2)/2} \ . \tag{3.47}$$

(An alternative way to obtain the same result is to write the flat space heat kernel (3.8) as its inverse Fourier transform.) Feynman noticed that if we integrate this expression over all possible lengths $T \in (0, \infty)$ of our worldline, then we obtain

$$\int_0^\infty \mathrm{d}T \int_{C_I[x,y]} \mathcal{D}x \, \mathrm{e}^{-S} = 2 \int \frac{\mathrm{d}^n p}{(2\pi)^n} \, \frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{p^2 + m^2} \tag{3.48}$$

which is the Fourier transform of the momentum space propagator $1/(p^2 + m^2)$ for a scalar field $\Phi(x)$ of mass m on the *target* space $\mathbb{R}^n$.

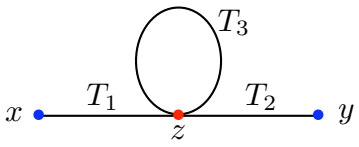
Feynman now realized that one could describe several such particles interacting with one another if one replaced the world*line* I by a world*graph* Γ. For example, to obtain a perturbative evaluation of the r–point correlation function

$$\langle \Phi(x_1)\Phi(x_2)\dots\Phi(x_r)\rangle$$

of a massive scalar field $\Phi(x)$ in $\lambda\Phi^4$ theory on $\mathbb{R}^n$, one could start by considering a 1–dimensional QFT living on a 4–valent graph Γ with r end–points. This QFT is described by the action (3.44), where x is constrained to map each end–point of the graph to a different one of the Φ insertion points $x_i \in \mathbb{R}^n$. We assign a length T_e to each edge e of the graph, which in this context is often known as a **Schwinger parameter** of the graph. We now take the path integral over all maps $x : \Gamma \to \mathbb{R}^n$ and integrate over the Schwinger parameters of each edge.

Part of what is meant by an 'integral over all maps $x : \Gamma \to \mathbb{R}^n$' includes an integral over the location in $\mathbb{R}^n$ to which each vertex of Γ is mapped. When we perform this integral, the factors of $\mathrm{e}^{\mathrm{i}p\cdot(x-y)}$ in the path integral (3.47) for each edge lead a to target space momentum conserving δ–function at each vertex. As in (3.48), integrating over the Schwinger parameters generates a propagator $1/(p^2+m^2)$ for each edge of the graph. Thus, after including a factor of $(-\lambda)^{|v(\Gamma)|}$ and dividing by the symmetry factor of the graph, our 1–dimensional QFT has generated the same expression as we would have obtained from Feynman rules for $\lambda\Phi^4$ on $\mathbb{R}^n$.

For example, the 4–valent graph with two end–points shown here:



corresponds to the path integral expression

$$\begin{aligned}
&\frac{-\lambda}{4}\int_0^\infty \mathrm{d}T_1 \int_{C_{T_1}[x,z]} \mathcal{D}x\, \mathrm{e}^{-S} \times \int_0^\infty \mathrm{d}T_2 \int_{C_{T_2}[y,z]} \mathcal{D}x\, \mathrm{e}^{-S} \times \int_0^\infty \mathrm{d}T_3 \int_{C_{T_3}[z,z]} \mathcal{D}x\, \mathrm{e}^{-S} \\
&\qquad = \frac{-\lambda}{4}\int \mathrm{d}^n z\, \frac{\mathrm{d}^n p}{(2\pi)^n}\frac{\mathrm{d}^n q}{(2\pi)^n}\frac{\mathrm{d}^n \ell}{(2\pi)^n}\,\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-z)}}{p^2+m^2}\,\frac{\mathrm{e}^{\mathrm{i}q\cdot(y-z)}}{q^2+m^2}\,\frac{\mathrm{e}^{\mathrm{i}\ell\cdot(z-z)}}{\ell^2+m^2} \\
&\qquad = \frac{-\lambda}{4}\int \frac{\mathrm{d}^n p}{(2\pi)^n}\frac{\mathrm{d}^n \ell}{(2\pi)^n}\,\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{(p^2+m^2)^2\,(\ell^2+m^2)}\,.
\end{aligned} \tag{3.49}$$

This is the same order λ contribution to the 2–point function $\langle\Phi(x)\Phi(y)\rangle$ that we'd obtain from (Fourier transforming) the momentum space Feynman rules for $\lambda\Phi^4$ theory, with the graph treated as a Feynman graph in $\mathbb{R}^n$ rather than a one–dimensional Universe.

To obtain the full perturbative expansion of $\langle\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle$ we now sum over all graph topologies appropriate to our 4–valent interaction. Thus

$$\langle\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle = \sum_\Gamma \frac{(-\lambda)^{|v(\Gamma)|}}{|\mathrm{Aut}\,\Gamma|}\int_0^\infty \mathrm{d}^{|e(\Gamma)|}T \int_{C_\Gamma[x_1,x_2,\dots,x_n]} \mathcal{D}\phi\; \mathrm{e}^{-S_\Gamma[\phi]}\,, \tag{3.50}$$

where $|e(\Gamma)|$ and $|v(\Gamma)|$ are respectively the number of edges and vertices of Γ.

This worldline approach to perturbative QFT is close to the way Feynman originally thought about the subject, presenting his diagrams at the Pocono Conference of 1948. The relation of this approach to higher (four) dimensional QFT as we usually think about it was worked out by Dyson a year later, long before people used path integrals to compute anything in higher dimensions. Above, we've described just the simplest version of this picture, relevant for a scalar theory on the target space. There are more elaborate $D = 1$ QFTs that would allow us to obtain target space Quantum Mechanics for particles with spin, and we could also allow for more interesting things to happen at the interaction vertices of our worldgraphs. In this way, one can build up worldline approaches to many perturbative QFTs. This way of thinking can still be useful in practical calculations today, and still occasionally throws up conceptual surprises, but we won't pursue it further in this course.

Finally, I can't resist mentioning that what we've really been studying in this section is not merely one–dimensional QFT, but one–dimensional Quantum Gravity. In one dimension a Riemannian metric is just a 1×1 matrix $g_{tt}(t)$ with positive eigenvalues; in other words, a positive number at each point t of the worldline. General coordinate transformations act on this 1×1 matrix as

$$g_{tt} \to g_{t't'} = \frac{dt}{dt'}\frac{dt}{dt'}\, g_{tt} = \left(\frac{dt}{dt'}\right)^2 g_{tt}\,, \tag{3.51}$$

and so can be used to rescale the value of the metric to anything we like. Throughout this chapter, we've implicitly been using a coordinate system t on the worldline in which the worldline metric had been fixed to 1, which we're always free to do. The proper length T of the worldline interval I can be written

$$T = \int_I \mathrm{d}t\, \sqrt{g_{tt}} = \int_I \mathrm{d}t'\, \sqrt{g_{t't'}} \tag{3.52}$$

and is invariant under the general coordinate transform (3.51). In fact, in one dimension the length T is the *only* diffeomorphism invariant of a Riemannian metric, essentially because there is no 'room' for any sort of curvature. Thus, the integral over the lengths of all the edges of our graph in (3.50) is best thought of as an integral over the space of all possible Riemannian metrics on Γ, up to diffeomorphism invariance. Rather grandly, this is known as the **moduli space** of Riemannian metrics on Γ. Furthermore, in summing over graphs Γ we were really summing over the topological type of our one dimensional Universe. Notice that the vertices of our graphs are singularities of the one–dimensional Riemannian manifold, so we're allowing our Universe to have such wild (even non–Hausdorff) behaviour. So for fixed lengths T_e the path integral over $x(t)$ is the 'matter' QFT on a fixed background space Γ, while the integral over the lengths of edges in Γ together with the sum over graph topologies is Quantum Gravity.

This picture is also very close to perturbative String Theory. There, as you'll learn if you're taking the Part III String Theory course, the worldgraph Γ is replaced by a two dimensional worldsheet (Riemann surface) Σ, the $D = 1$ worldline QFT replaced by a

$D = 2$ worldsheet CFT[11]. Likewise, the integral over the moduli space of Riemannian metrics on Γ becomes an integral over the moduli space of Riemann surfaces, and finally the sum over graphs is replaced by a sum over the topology of the Riemann surface. We know that the worldgraph approach to QFT only captures some aspects of perturbation theory, and in the following chapters we'll see that deeper insight is provided by QFT proper. Asking whether there's a similarly deeper approach to String Theory will take you to the mystic shores of String Field Theory, about which very little is known.

[11] CFT = Conformal Field Theory.