2 QFT in zero dimensions

We'll embark on our journey from the simplest possible starting point: we'll study QFT in a space–time with zero dimensions. That's a very drastic simplification, and much of the richness of QFT will be absent here. Indeed, I expect many of the ideas in this chapter will be things you've met (long) before, although perhaps in a different context. Still, you shouldn't sneer. We'll see that even this simple case contains baby versions of ideas we'll study more generally later in the course, and it will provide us with a safe playground in which to check we understand what's going on. Furthermore, it has been seriously conjectured that full, non-perturbative string theory is itself a zero–dimensional QFT (though admittedly with infinitely many fields).

2.1 Partition functions and correlation functions in $d = 0$

If our space–time M is zero–dimensional and connected, then it must be just a single point:

$$M = \{\mathrm{pt}\}\,. \tag{2.1}$$

In zero dimensions, there are no lengths, so there is no notion of a metric. Similarly, the Lorentz group is trivial, hence all its representations are trivial. In other words, all fields must be scalars: there is no notion of the 'spin' of a field, simply because there is no notion of a Lorentz transformation.

In the simplest case, a 'field' on M is a map $\phi : \{\mathrm{pt}\} \to \mathbb{R}$, or in other words just a real variable. The space $\mathcal{C}$ of all field configurations is also easy to describe: it's again just $\mathbb{R}$, because our entire universe M is just one point, so we completely specify what the field looks like by giving its value at this one point.

Now let's choose our action. In zero dimensions, there are no space–time directions along which we could differentiate our 'field', so there can be no kinetic terms. Thus, the action is just a function $S(\phi)$ of this one real variable. All that really matters is that S is chosen so that the partition function (2.2) converges, but we'll typically take $S(\phi)$ to be a polynomial (with highest term of even degree), such as

$$S(\phi) = \frac{m^2}{2}\phi^2 \qquad \text{or perhaps} \qquad S(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\,.$$

The coupling constants are just the coefficients of the various powers of ϕ in the action. The coefficients of ϕ^p with $p = 0, 1, 2$ have a slightly special status, and are known as the **vacuum energy**, the **tadpole** and the **mass** of the field, respectively, although they are also just coupling constants from our general point of view.

Because $\mathcal{C} \cong \mathbb{R}$, the path integral measure $\mathcal{D}\phi$ becomes just the standard (Lebesgue) measure $\mathrm{d}\phi$ on $\mathbb{R}$ and the partition function

$$\mathcal{Z} = \int_{\mathbb{R}} \mathrm{d}\phi\, \mathrm{e}^{-S(\phi)/\hbar}\,, \tag{2.2}$$

is just a standard integral over the real line. Similarly, correlation functions are

$$\langle f \rangle := \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} \mathrm{d}\phi\, f(\phi)\, \mathrm{e}^{-S(\phi)/\hbar} \tag{2.3}$$

where we've inserted some other function $f(\phi)$ into the basic integral. We'll assume that f is sufficiently well-behaved that the integral (2.3) still exists. In particular, f should not grow so rapidly as $|\phi| \to \infty$ as to overcome the decay of $\mathrm{e}^{-S(\phi)/\hbar}$. In practice, we'll restrict ourselves to the case that f is just a polynomial. In this $\dim(M) = 0$ case, so long as the action $S(\phi)$ is real, $\mathrm{e}^{-S/\hbar} \geq 0$ so we really can think of $\mathrm{e}^{-S/\hbar}/\mathcal{Z}$ as a **probability density** on the space of fields, with the factor of $1/\mathcal{Z}$ ensuring that the probability measure is normalized. The correlation function (2.3) is just the **expectation value** $\langle f \rangle$ of $f(\phi)$ averaged over the space of fields with this measure.

As before, what we get for $\mathcal{Z}$ depends on which action we picked, so the partition function depends on the values of the coupling constants

$$\mathcal{Z} = \mathcal{Z}(m^2, \lambda, \cdots)\,. \tag{2.4}$$

Correlation functions depend on the coefficients of the polynomial $f(\phi)$ as well as the couplings in the action. Again, we can think of the correlator as probing the response of the partition function to an infinitesimal change in the couplings in the action. For example, in the simplest case that $f(\phi) = \phi^p$ is monomial, we have formally

$$\frac{1}{p!}\langle \phi^p \rangle = -\frac{\hbar}{\mathcal{Z}}\,\frac{\partial}{\partial \lambda_p}\mathcal{Z}(m^2, \lambda_i)\bigg|_* \tag{2.5}$$

where λ_p is the coupling to $\phi^p/p!$ in the general action, and $*$ is the point in theory space where the couplings are set to their values in the specific action that appears in (2.3). Finally, as a piece of notation, I'll often write $\mathcal{Z}_0$ for the partition function in the free theory, where the couplings of all but the term quadratic in the field(s) are set to zero.

2.2 Free field theory

The simplest QFTs are free, meaning that the action is (at most) quadratic in the fields. As an example, suppose we have n fields ϕ^a with $a = 1, \ldots, n$, thought of as a map $\phi : \{\mathrm{pt}\} \to \mathbb{R}^n$. We choose the action to be the quadratic function

$$S(\phi) = \frac{1}{2}M(\phi, \phi) = \frac{1}{2}M_{ab}\,\phi^a\phi^b\,, \tag{2.6}$$

where $M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is represented by a real, positive–definite, symmetric matrix. The partition function of this free, zero–dimensional QFT is the basic Gaussian integral

$$\mathcal{Z}_0 = \int_{\mathbb{R}^n} \mathrm{d}^n\phi\,\mathrm{e}^{-M(\phi,\phi)/2\hbar} \tag{2.7}$$

with the standard (Lebesgue) measure $\mathrm{d}^n\phi$ on the space of fields, which is now just $\mathbb{R}^n$. To evaluate this, note that since M is a real symmetric matrix, its eigenvectors are orthogonal so M can be diagonalized by some orthogonal transformation $O : \mathbb{R}^n \to \mathbb{R}^n$. The path integral measure is the standard measure $\mathrm{d}^n\phi$ on $\mathbb{R}^n$, which is invariant under such an orthogonal transformation. In the basis of eigenvectors, the integral is just a product of n independent Gaussian integrals

$$\int_{\mathbb{R}} \mathrm{d}\chi\,\mathrm{e}^{-m\chi^2/2\hbar} = \sqrt{\frac{2\pi\hbar}{m}}\,, \tag{2.8}$$

where m is the eigenvalue of M. Multiplying all the contributions, we obtain

$$\mathcal{Z}_0 = \int_{\mathbb{R}^n} \mathrm{d}^n\phi \, \mathrm{e}^{-M(\phi,\phi)/2\hbar} = \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}}\,, \tag{2.9}$$

where we have written the product of eigenvalues more invariantly as the determinant. Note that M being positive–definite ensures that $\det M > 0$ and the integral exists.

We also want to compute the partition function in the presence of a source for ϕ. Thus, we include a linear source term $J_a\phi^a$ in the action:

$$S(\phi) = \frac{1}{2}M(\phi,\phi) + J\cdot\phi\,. \tag{2.10}$$

Completing the square, we have

$$\frac{1}{2}M(\phi,\phi) + J\cdot\phi = \frac{1}{2}M(\tilde{\phi},\tilde{\phi}) - \frac{1}{2}M^{-1}(J,J) \tag{2.11}$$

where $\tilde{\phi} := \phi + M^{-1}(J,\,\cdot\,)$ are some translated coordinates on $\mathbb{R}^n$. (Our assumption that M was positive–definite also guarantees that M^{-1} exists.) Since $\tilde{\phi}$ differs from ϕ by a translation, the measure $\mathrm{d}^n\tilde{\phi} = \mathrm{d}^n\phi$. Therefore, in the presence of the source J the partition function is

$$\begin{aligned}\mathcal{Z}(J) &= \int_{\mathbb{R}^n} \mathrm{d}^n\phi \, \exp\left(-\frac{1}{\hbar}\left(\frac{1}{2}M(\phi,\phi) + J\cdot\phi\right)\right) \\ &= \exp\left(\frac{1}{2\hbar}M^{-1}(J,J)\right) \int_{\mathbb{R}^n} \mathrm{d}^n\tilde{\phi}\; \mathrm{e}^{-M(\tilde{\phi},\tilde{\phi})/2\hbar} = \exp\left(\frac{1}{2\hbar}M^{-1}(J,J)\right)\mathcal{Z}_0\end{aligned} \tag{2.12}$$

where $\mathcal{Z}_0$ is the original partition function (2.9).

To see how this generalization allows us to compute correlation functions, suppose $\langle P(\phi)\rangle$ is a polynomial $P : \mathbb{R}^n \to \mathbb{R}$. By linearity of the integral, evaluation of the correlation function $\langle P(\phi)\rangle$ reduces to the case that P is a product of linear factors $\ell(\phi) = \ell\cdot\phi$, so we just need to compute

$$\langle \ell_1(\phi)\cdots\ell_p(\phi)\rangle = \frac{1}{\mathcal{Z}_0}\int_{\mathbb{R}^n} \mathrm{d}^n\phi\; \mathrm{e}^{-M(\phi,\phi)/2\hbar} \prod_{i=1}^{p} \ell_i(\phi)\,. \tag{2.13}$$

If p is odd, then the integrand is an odd function of (at least one direction of) ϕ, so vanishes when integrated over $\mathbb{R}^n$. Let's evaluate the remaining case $p = 2k$. We have that

$$\begin{aligned}\langle \ell_1(\phi)\cdots\ell_{2k}(\phi)\rangle &= \frac{1}{\mathcal{Z}_0}\int_{\mathbb{R}^n} \mathrm{d}^n\phi\; \prod_{i=1}^{2k}\ell_i(\phi)\; \mathrm{e}^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar}\Bigg|_{J=0} \\ &= \frac{(-\hbar)^{2k}}{\mathcal{Z}_0}\int_{\mathbb{R}^n} \mathrm{d}^n\phi\; \prod_{i=1}^{2k}\ell_i\cdot\frac{\partial}{\partial J}\left[\mathrm{e}^{-M(\phi,\phi)/2 - J(\phi)/\hbar}\right]\Bigg|_{J=0} \\ &= \prod_{i=1}^{2k}\ell_i\cdot\frac{\partial}{\partial J}\left[\frac{\hbar^{2k}}{\mathcal{Z}_0}\int_{\mathbb{R}^n}\mathrm{d}^n\phi\; \mathrm{e}^{-M(\phi,\phi)/2 - J(\phi)/\hbar}\right]\Bigg|_{J=0} \\ &= \hbar^{2k}\prod_{i=1}^{2k}\ell_i\cdot\frac{\partial}{\partial J}\left[\mathrm{e}^{M^{-1}(J,J)/2\hbar}\right]\Bigg|_{J=0}\end{aligned} \tag{2.14}$$

The first line is a triviality: we want to know the correlation function in the original theory where $J = 0$. In going to the second line here we differentiated the action *wrt* J to bring down each factor of ϕ, in going to the third line we note that the integrand is absolutely convergent so the order of integration and differentiation may safely be exchanged, and the final line uses the result (2.12).

Let's first think about this formula in the important special case $k = 1$, where it reduces to

$$\langle \ell_1(\phi)\, \ell_2(\phi) \rangle = \hbar\, M^{-1}(\ell_1, \ell_2) \qquad \text{or equivalently} \qquad \langle \phi^a \phi^b \rangle = \hbar\, (M^{-1})^{ab} . \tag{2.15}$$

This says that (in this free theory) the two–point function is just the inverse of (minus) the quadratic term in the exponential. In dimensions $d > 0$, we'll consider correlation functions where the fields are inserted at different points in our space–time. In this case, the coefficient M of the term quadratic in the fields is a differential operator, whose inverse M^{-1} is the **Green's function** or **propagator** of the theory. We can think of the propagator as representing the response of one field insertion to the presence of another. It'll be useful to represent this result by the picture with the solid line keeping track of the fact that the

$$\langle \phi^a \phi^b \rangle \quad = \quad \phi^a \bullet\!\!\text{———}\!\!\bullet \phi^b \quad = \quad \hbar\, (M^{-1})^{ab}$$

field insertions are joined by a copy of M^{-1}. This picture is a (rather trivial) example of **Feynman diagram**.

Now let's return to the general case of (2.14). For every derivative $\hbar\, \ell \cdot \partial/\partial J$ that acts on the exponential we get a factor of $M^{-1}(\ell, J)$. Because we'll set $J = 0$ at the end of the calculation, we can get a non–vanishing contribution to (2.14) only when exactly half the derivatives bring down such factors, while the other half then removes the J dependence in front of the exponential. We'll then be left with a product of k factors of M^{-1}, contracted into ℓ's (or having free indices) in a way that depends on how we paired up the way the derivatives act. Let σ be a way of joining the elements of the set $\{1, 2, \ldots, 2k\}$ into pairs, and let Π_{2k} denote the set of all possible (complete) pairings. Then the correlation function is

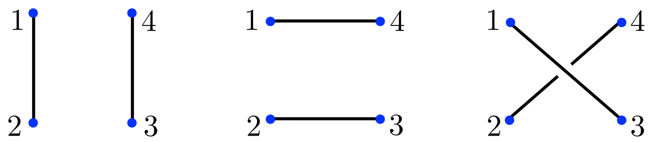
$$\langle \ell_1(\phi) \cdots \ell_{2k}(\phi) \rangle = \hbar^k \sum_{\sigma \in \Pi_{2k}} \prod_{i \in \{1, \ldots, 2k\}/\sigma} M^{-1}(\ell_i, \ell_{\sigma(i)}) , \tag{2.16}$$

in other words, a sum over products of all inequivalent ways of connecting pairs of ℓ_i using M^{-1}.

We can use our Feynman diagrams to help keep track of the possible pairings. For example, the 4-point function

$$\langle \phi^a \phi^b \phi^c \phi^d \rangle = \hbar^2 \left((M^{-1})^{ab} (M^{-1})^{cd} + (M^{-1})^{ac} (M^{-1})^{db} + (M^{-1})^{ad} (M^{-1})^{bc} \right) \tag{2.17}$$

can be represented by the Feynman diagrams



In general, there are $|\Pi_{2k}| = (2k)!/(2^k k!)$ ways of joining $2k$ elements into pairs, so the $2k$-point function receives $(2k)!/(2^k k!)$ contributions. In particular, we find

$$\langle (\ell\cdot\phi)^{2k}\rangle = \frac{(2k)!}{2^k k!}\left(\hbar M^{-1}(\ell,\ell)\right)^k \tag{2.18}$$

when all of the ℓ_i's are the same.

Our result (2.16) for the correlation function is known as **Wick's theorem** in QFT, though in the $d = 0$ context of Gaussian distributions it's called **Isserlis' theorem** by probabalists. You met Wick's theorem last term from the point of view of canonical quantization, where it arose from decomposing the field operator ϕ into creation and annihilation operators, and commuting these operators past one another. Of course, in $d = 0$, there's no sense in which the fields 'propagate' anywhere, so the Feynman diagrams are just a nifty way to keep track of the combinatorics. Also, since we're currently thinking just about free theory, our diagrams have no (internal) vertices at present.

2.3 Perturbation theory

Interesting theories involve interactions, so that the action $S(\phi)$ is not merely quadratic. In this case, integrals such as

$$\int_{\mathbb{R}^n} \mathrm{d}^n\phi\ f(\phi)\,\mathrm{e}^{-S(\phi)/\hbar} \tag{2.19}$$

become transcendental, even for simple actions $S(\phi)$ – including most of physical interest – and simple choices of $f(\phi)$. Typically, we do not know how to evaluate such integrals analytically. We may hope to approximate such integrals perturbatively by expanding around the classical limit $\hbar \to 0$. However, our integral cannot have a Taylor expansion around $\hbar = 0$, since any such Taylor expansion would have to converge for all $\hbar$ in a disc $D \subset \mathbb{C}$ centered on the origin. But if the action is chosen so that the integral converges whenever $\hbar > 0$, then (2.19) surely *diverges* if we formally attempt continue into the region $\mathrm{Re}(\hbar) < 0$. Barring numerical methods, the best we can do is to obtain an **asymptotic expansion** for such path integrals. (Recall that a series $\sum_n a_n\hbar^n$ is **asymptotic** to a function $I(\hbar)$ if, for all $N \in \mathbb{N}$,

$$\lim_{\hbar\to 0^+} \frac{1}{\hbar^N}\left| I(\hbar) - \sum_{n=0}^{N} a_n\hbar^n\right| = 0\ . \tag{2.20}$$

In other words, with fixed N, for sufficiently small $\hbar \in \mathbb{R}_{\geq 0}$ the first N terms of the series differ from the exact answer by less than $\epsilon\lambda^N$ for *any* $\epsilon > 0$. (The difference is $o(N)$). We write[12]

$$I(\hbar) \sim \sum_{n=0}^{\infty} a_n \hbar^n \qquad \text{as} \quad \hbar \to 0 \tag{2.21}$$

to mean that the series on the right is an asymptotic expansion of $I(\hbar)$ as $\hbar \to 0$. It's important to remember that the true function may differ from its asymptotic series by transcendental terms; for example, the function $\mathrm{e}^{-1/\hbar^2} \sim 0$ as $\hbar \to 0$, but clearly $\mathrm{e}^{-1/\hbar^2} \neq 0$. Thus, if we instead fix a value of $\hbar$, however small, and include more and more terms in the sum, we will eventually get worse and worse approximations to the answer. Perturbation theory thus tells us important, but not complete, information about our QFT.

Now suppose $S(\phi)$ is a smooth function that has a unique global minimum at a unique point $\phi = \phi_0 \in \mathbb{R}^n$, so that the Hessian matrix $\partial_a \partial_b S|_{\phi_0}$ is positive–definite. Then (2.19) has an asymptotic expansion

$$\int_{\mathbb{R}^n} \mathrm{d}^n\phi \; f(\phi)\, \mathrm{e}^{-S(\phi)/\hbar} \;\sim\; (2\pi\hbar)^{n/2} \frac{f(\phi_0)\, \mathrm{e}^{-S(\phi_0)/\hbar}}{\sqrt{\det(\partial_a\partial_b S|_{\phi_0})}} \left(1 + \hbar A_1 + \hbar^2 A_2 + \cdots\right) \tag{2.22}$$

as $\hbar \to 0^+$. The proof of this is known as **steepest descent** and should be familiar if you've taken a course such as Part II Asymptotic Methods[13]. The leading term in this

[12] In this course, all the asymptotic expansions we consider will be valid as $\hbar \to 0$, so we'll usually take this limit as understood.

[13] In case you didn't take such a course, here's an outline of a proof in the case of a single field: Let

$$A(\hbar) = \frac{\mathrm{e}^{+S(\phi_0)/\hbar}}{\sqrt{\hbar}} \int_a^b \mathrm{e}^{-S(\phi)/\hbar} f(\phi)\, \mathrm{d}\phi$$

and let $\epsilon \in (0, \frac{1}{2})$. Define $B(\hbar)$ in the same way as $A(\hbar)$, but where the integral is taken over the range $[\phi_0 - \hbar^{\frac{1}{2}-\epsilon}, \phi_0 + \hbar^{\frac{1}{2}-\epsilon}]$. As $\hbar \to 0$, we have that $A(\hbar) - B(\hbar)$ is smaller than $\hbar^N$ for any $N \in \mathbb{N}$. (We say the difference is **rapidly decaying** in $\hbar$.) Now let $\chi = (\phi - \phi_0)/\sqrt{\hbar}$, so

$$B(\hbar) = \int_{-\hbar^\epsilon}^{\hbar^\epsilon} \mathrm{e}^{(S(\phi_0) - S(\phi_0 + \chi\sqrt{\hbar}))/\hbar} f(\phi_0 + \chi\sqrt{\hbar})\, \mathrm{d}\chi\,.$$

Provided the action $S(\phi)$ and insertion $f(\phi)$ were smooth, the integrand of this expression is a smooth function of $\sqrt{\hbar}$ when $\hbar \geq 0$. Let $C(\hbar)$ be the same integral as for $B(\hbar)$, but with the integrand replaced by its Taylor expansion around 0 in $\sqrt{\hbar}$, modulo terms of order $\hbar^N$. Then

$$|B(\hbar) - C(\hbar)| \leq K\, \hbar^{N-\epsilon}$$

for some constant $K \geq 0$. Finally, let $D(\hbar)$ be the same as $C(\hbar)$, but where the limits of the integral are $-\infty$ and ∞. Then $D(\hbar)$ is a polynomial in $\sqrt{\hbar}$, while $C(\hbar) - D(\hbar)$ is rapidly decaying in $\hbar$. Since $D(\hbar)$ is a polynomial in $\sqrt{\hbar}$, it admits a Taylor expansion in $\sqrt{\hbar}$ modulo $\hbar^{N-\epsilon}$. Also, the coefficients of odd powers of $\sqrt{\hbar}$ in $D(\hbar)$ are given by integrals of an odd function of χ over all of $\mathbb{R}$, and hence vanish. Finally, we have

$$D(0) = \int_{\mathbb{R}} \mathrm{e}^{-\partial^2 S|_{\phi_0}\, \chi^2/2} f(\phi_0)\, \mathrm{d}\chi = \frac{\sqrt{2\pi}\, f(\phi_0)}{\sqrt{\partial^2 S|_{\phi_0}}}\,.$$

Putting all these facts together shows that

$$\int_{\mathbb{R}} \mathrm{e}^{-S(\phi)/\hbar} f(\phi)\, \mathrm{d}\phi = \mathrm{e}^{-S(\phi_0)/\hbar} \sqrt{\hbar}\, A(\hbar) \sim \sqrt{2\pi\hbar} \frac{\mathrm{e}^{-S(\phi_0)/\hbar} f(\phi_0)}{\sqrt{\partial^2 S|_{\phi_0}}} \sum_{n=0}^{\infty} A_n \hbar^n\,,$$

where $A_0 = 1$. This proves (2.22) in the case of a single field. The generalization to finitely many fields ϕ^a is straightforward. But don't worry, neither of these proofs are examinable for this course.

expansion is known as the **semi-classical** term. In particular, expanding ϕ around the classical solution ϕ_0 as $\phi^a = \phi_0^a + \delta\phi^a$, we have

$$S(\phi) = S(\phi_0) + \frac{1}{2}\partial_a\partial_b S|_{\phi_0}\delta\phi^a\,\delta\phi^b + \cdots \tag{2.23}$$

so that the leading term

$$\mathcal{Z}_0 = (2\pi\hbar)^{n/2}\frac{\mathrm{e}^{-S(\phi_0)/\hbar}}{\sqrt{\det(\partial_a\partial_b S|_{\phi_0})}} \tag{2.24}$$

in the asymptotic series of the partition function is just what we'd obtain as the partition function of a theory a purely quadratic theory. We'll see that it arises in perturbation theory from the 1-loop approximation, while terms at higher order in $\hbar$ in the series (2.22) arise from multi-loop diagrams.

Let's understand how this works in an example. Consider the $d = 0$ QFT with a single scalar field ϕ and action $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$. We need to take $\lambda > 0$ for the partition function to converge, and we'll also assume $m^2 > 0$ so that the action has a unique minimum at $\phi_0 = 0$. Then the leading term in our asymptotic expansion is

$$(2\pi\hbar)^{1/2}\,\frac{\mathrm{e}^{-S(\phi_0)/\hbar}}{\sqrt{\partial^2 S|_{\phi_0}}} = \frac{\sqrt{2\pi\hbar}}{m}\,, \tag{2.25}$$

since $S(\phi_0) = 0$ and $\partial^2 S|_{\phi_0} = m^2$. As claimed, this is just the partition function $\mathcal{Z}(m, 0)$ of the free theory. Going further, since

$$\mathcal{Z}(m^2,\lambda) = \int_{\mathbb{R}} \mathrm{d}\phi\,\mathrm{e}^{-\frac{1}{\hbar}\left(\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right)} = \int_{\mathbb{R}} \mathrm{d}\phi\,\left[\mathrm{e}^{-m^2\phi^2/2\hbar}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\lambda}{4!\hbar}\right)^n\phi^{4n}\right]\,, \tag{2.26}$$

we obtain an asymptotic series for $\mathcal{Z}(m^2,\lambda)$ by truncating to the first $N+1$ terms of this expansion, whereupon

$$\begin{aligned}
\mathcal{Z}(m^2,\lambda) &\sim \int_{\mathbb{R}} \mathrm{d}\phi\,\left[\mathrm{e}^{-m^2\phi^2/2\hbar}\sum_{n=0}^{N}\frac{1}{n!}\left(\frac{-\lambda}{4!\hbar}\right)^n\phi^{4n}\right] \\
&= \frac{\sqrt{2\hbar}}{m}\sum_{n=0}^{N}\frac{1}{n!}\left(\frac{-\hbar\lambda}{3!\,m^4}\right)^n\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-x}\,x^{2n+\frac{1}{2}-1} \\
&= \frac{\sqrt{2\hbar}}{m}\sum_{n=0}^{N}\frac{1}{n!}\left(\frac{-\hbar\lambda}{3!\,m^4}\right)^n\Gamma\!\left(2n+\frac{1}{2}\right)\,.
\end{aligned} \tag{2.27}$$

In going to the second line we substituted $x = m^2\phi^2/2\hbar$ and exchanged the order of the finite summation and integration. Note that it would not be legitimate to exchange the order of the integral and the *infinite* sum in (2.26), because the original integral does not converge if $\hbar < 0$. The final line recognizes the integral as a representation of the gamma function. (Somewhat more laboriously, this integral can be computed by repeated integration by parts.) Using the value of $\Gamma(z)$ at positive half-integers we have finally

$$\begin{aligned}\mathcal{Z}(m^2,\lambda) &\sim \frac{\sqrt{2\pi\hbar}}{m}\sum_{n=0}^{N}(-)^n\frac{\hbar^n\lambda^n}{m^{4n}}\frac{1}{(4!)^n\,n!}\frac{(4n)!}{4^n(2n)!}\\ &= \mathcal{Z}_0\left[1-\frac{\hbar\lambda}{8m^4}+\frac{35}{384}\frac{\hbar^2\lambda^2}{m^8}+\cdots\right]\end{aligned} \tag{2.28}$$

as our asymptotic series for the partition function, where $\mathcal{Z}_0 = \mathcal{Z}(m^2, 0) = \sqrt{2\pi\hbar}/m$.

Let me make a couple of remarks. Firstly, the fact that each term in the expansion of $\mathcal{Z}(m^2,\lambda)/\mathcal{Z}_0$ is proportional to $(-\hbar\lambda/m^4)^2$ is essentially fixed by dimensional analysis. The coefficient

$$\frac{1}{(4!)^n\,n!}\frac{(4n)!}{4^n(2n)!}$$

can be understood as a product of the factor $\frac{1}{(4!)^n n!}$ that comes straightforwardly from expanding the ϕ^4 term in the exponential, and the remaining factor $(4n!)/4^n(2n)!$ is the number of ways of joining $4n$ elements (the ϕ insertions) into distinct pairs; indeed, we saw in the discussion of Wick's theorem that the integral $\int \mathrm{e}^{-\phi^2/2}\,\phi^4\,\mathrm{d}\phi$ had a combinatoric interpretation in terms of pairings. Note that we can see the divergence of the perturbation series directly from these coefficients: From Stirling's approximation $n! \approx \mathrm{e}^{n\ln n}$, we see that

$$\frac{1}{(4!)^n\,n!}\frac{(4n)!}{4^n(2n)!} \approx \mathrm{e}^{n\ln n}$$

for large n. Thus these coefficients asymptotically grow faster than exponentially with n, so the series (2.28) has zero radius of convergence. It's interesting to ask whether it is possible to recover the exact value of $\mathcal{Z}(m^2,\lambda)$ from its asymptotic series. Remarkably, a technique known as **Borel resummation** allows one to achieve this, at least in certain circumstances. You're invited to explore it for this example in the problem sheets.

As a second remark, observe that $\mathcal{Z}(m^2,\lambda)$ itself should exist even if $m^2 < 0$, provided $\hbar$ and λ are strictly positive, because the exponential enhancement from the factor $\mathrm{e}^{+|m^2|\phi^2/2\hbar}$ at small ϕ is eventually suppressed by the quartic term in the action. However, the asymptotic series (2.28) is not valid in this case, as we can see from the fact that the (Gaussian) integrals in the second line of (2.27) require $m^2 > 0$ to converge. More fundamentally, the problem is that when $m^2 < 0$, the point $\phi = 0$ which we took to give the dominant contribution to the integral is now a (local) **maximum** of the action, the global minima being at $\phi_0 = \pm\sqrt{6m^2/\lambda}$. In physics terminology *we are expanding around the wrong vacuum.* Particles with $m^2 < 0$ are called **tachyons**, and they always signal an instability. Whether or not this instability is just due to a poor choice of perturbative expansion (as here), or whether the whole theory is unstable (meaning $\mathcal{Z}(m^2,\lambda)$ does not exist for $m^2 < 0$) is not always clear. The situation where the minimum of the action involves a non–zero value for some field is often associated with **spontaneous symmetry breaking**. You can learn more about this *e.g.* in the Part III course on the Standard Model.

2.3.1 Feynman diagrams

Above, we gave a combinatoric interpretation of the numerical coefficients of the asymptotic series

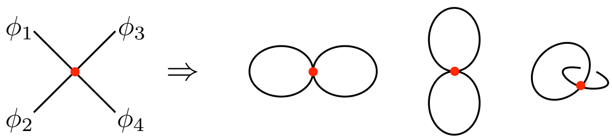
$$\mathcal{Z}(m^2,\lambda)/\mathcal{Z}_0 \sim \sum_{n=0}^{N} (-)^n \frac{\hbar^n \lambda^n}{m^{4n}} \frac{1}{(4!)^n\, n!} \frac{(4n)!}{4^n (2n)!} \tag{2.29}$$

in terms of ways of pairing up the ϕ insertions in the integral. Let's now reconsider this from the point of view of Feynman diagrams. With the action $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$ the ingredients of our Feynman diagrams are

for the propagator and vertex. Note again that the propagator is just a constant since we are in zero dimensions, while the minus sign in the vertex comes from the fact that we are expanding $e^{-S/\hbar}$.

To compute perturbation series in this theory, Feynman tells us to start by constructing all possible graphs (not necessarily connected) using this propagator and vertex. In the case of the partition function, we want vacuum graphs, *i.e.*, those with no external[14] edges. Let D_n be the set of all labelled vacuum graphs containing n vertices, and let there be $|D_n|$ elements in this set. By a *labelled* graph, I mean that individual vertices carry their own unique 'label', so that we can tell them apart. Likewise, each of the four legs emanating from a given vertex carries its own label.

Since each end of every edge in a vacuum graph is attached to a vertex, and the vertex is 4-valent in this theory, every graph in D_n must contain precisely $2n$ edges. Thus, using the propagator and vertex given above, every graph in D_n contributes a term proportional to $(-\hbar\lambda/m^4)^n$, as indeed we saw in (2.29). For example, in this theory the set D_1 consists of the three graphs



corresponding to the three possible ways to join up the four ϕ fields into pairs. Thus, the term proportional to λ receives contributions from these three individual graphs.

Joining up our labelled vertices in *every* possible way means that the set D_n may contain several elements that are identical as unlabelled topological graphs, but differ just in the labelling of their vertices or edges. For example, all three graphs displayed above are equivalent as topological graphs. Identical topological graphs correspond to identical

[14] An **internal edge** of a graph is one in which both ends of the edge are attached to vertices, which may be distinct vertices or the same. An **external edge** is an edge that is not internal, and so has at least one end not attached to a vertex.

physical processes[15], and the original integral knew nothing of our choice of labels, so in working out the perturbation series we need to remove this overcounting. To do so, observe that D_n is naturally acted on by the group $G_n = (S_4)^n \rtimes S_n$ that permutes each of the four fields present at a given vertex (n copies of the permutation group S_4 on 4 elements) and also permutes the labels of each of the n vertices (the permutation group S_n)[16]. This group has order $|G_n| = (4!)^n n!$, which is the same factor we saw before from expanding $\mathrm{e}^{-S/\hbar}$ in powers of λ. Thus the asymptotic series (2.29) may be rewritten as

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} \sim \sum_{n=0}^{N} \left(\frac{-\lambda}{m^4}\right)^n \frac{|D_n|}{|G_n|} \,. \tag{2.30}$$

In detail, the power $(-\lambda)^n$ is the contribution of the coupling constants in each graph, the power of $(1/m^2)^{2n}$ comes from the fact that any vacuum diagram with exactly n 4–valent vertices must have precisely $2n$ edges, each of which contributes a factor of $1/m^2$. The factor $|D_n|/|G_n|$ is the number of diagrams that contribute at this order, counting as equivalent those diagrams that merely permute the labels of the fields at a given vertex, or the labelling of the vertices.

There's another way to think of $|D_n|/|G_n|$ that is sometimes convenient[17]. An **orbit** Γ of G_n in D_n is a set of labeled graphs in D_n that are identical except for a relabelling of their fields and vertices, so that we can move from one labelled graph to another in the orbit using an element of G_n (*i.e.* by permuting these labels). Thus an orbit Γ is a topologically distinct graph in D_n. Let O_n be the set of such orbits Γ; that is O_n is the set of *topologically distinct* vacuum graphs on n vertices. The **orbit stabilizer theorem** says that[18]

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma \in O_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} \,, \tag{2.31}$$

where $\mathrm{Aut}\,\Gamma$ is the **stabilizer** of any element in Γ in G_n, *i.e.*, the elements of the permutation group G_n that don't alter the labelled graph. For example, if a graph in D_n contains an edge both of whose edges are attached to the same vertex, then exchanging the labelling of those fields doesn't change the labelled graph. Similarly, if a pair of vertices are connected by two (or more) propagators, then exchanging the labels of the two (or more) legs on each vertex that are joined to these propagators does not change the labelled

[15]For this statement to hold true, we need to be careful to account for all the quantum numbers, and to give precise meaning to 'topological equivalence'. For example, we often draw Feynman graphs representing matrix–valued fields as graphs whose edges are thickened into **ribbons**. Graphs that would be equivalent as line graphs but that differ by twisting of the ribbons should be counted separately (they correspond to different ways to tie up the matrix indices of the fields), and the relevant notion of topological equivalence is called **ambient isotopy**.

[16]Exercise: why is the full group the *semi-direct product* of these two subgroups?

[17]In practice, at least for the simple graphs we'll meet in this course, it's often just as quick to think through the possible ways a given topological graph Γ may be obtained by expanding out the vertices in $\mathrm{e}^{-S/\hbar}$ and joining pairs of fields by propagators, as to work out the symmetry factor $|\mathrm{Aut}\,\Gamma|$. I'll leave it to your taste.

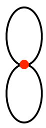
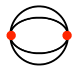
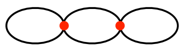
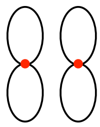
[18]If you don't know this already, you can find a nicely explained proof on Gowers's Weblog.

graph. Finally then, we can rewrite our asymptotic series (2.29) as

$$
\begin{aligned}
\frac{\mathcal{Z}}{\mathcal{Z}_0} &\sim \sum_{n=0}^{\infty}\left[\left(\frac{-\hbar\lambda}{m^4}\right)^n \sum_{\Gamma\in O_n} \frac{1}{|\mathrm{Aut}\,\Gamma|}\right] \\
&= \sum_{\Gamma} \frac{\hbar^{|e(\Gamma)|-|v(\Gamma)|}}{|\mathrm{Aut}\,\Gamma|}\,\frac{(-\lambda)^{|v(\Gamma)|}}{(m^2)^{|e(\Gamma)|}}\,,
\end{aligned}
\tag{2.32}
$$

in terms of a sum over Feynman graphs Γ, where $|v(\Gamma)|$ and $|e(\Gamma)|$ are respectively the number of vertices and edges of the graph Γ. The factor $|\mathrm{Aut}\,\Gamma|$ is often known as the **symmetry factor** of the graph.

We've rederived the Feynman rule that we should weight each topologically distinct graph by $|v(\Gamma)|$ powers of (minus) the coupling constant $-\lambda/\hbar$ and $|e(\Gamma)|$ powers of the propagator $\hbar/m^2$, then divide by the symmetry factor $|\mathrm{Aut}\,\Gamma|$ of the graph. Thus, the asymptotic expansion of the partition function is given by the Feynman diagrams

$\mathcal{Z}/\mathcal{Z}_0 = \emptyset +$  $+$  $+$  $+$  $+ \cdots$

$$
= 1 - \frac{\hbar\lambda}{8m^4} + \frac{\hbar^2\lambda^2}{48m^8} + \frac{\hbar^2\lambda^2}{16m^8} + \frac{\hbar^2\lambda^2}{128m^8} + \cdots
$$

where we include both connected and disconnected graphs, with the contribution of a disconnected graph being the product of the contributions of the two connected graphs. Notice that this requires that we assign a factor 1 to the trivial graph $\emptyset$ (no vertices or edges), which is also included as the zeroth–order term in the sum.

More generally, our theory may involve a different types of field, each associated with a propagator $1/P_a$. These fields could interact via various different vertices v_α, either joining different types of field or different powers of the same field. Let's suppose that a vertex of type α (where α labels the types and multiplicities of the fields at this vertex) has a coupling constant λ_α in the action. Then a graph Γ containing $|e_a(\Gamma)|$ edges representing propagators of the type a field and $|v_\alpha(\Gamma)|$ vertices of type α is associated with a weight factor

$$
F(\Gamma) = \prod_{a,\alpha} \frac{(-\lambda_\alpha)^{|v_\alpha(\Gamma)|}}{(P_a)^{|e_a(\Gamma)|}}
\tag{2.33}
$$

by the Feynman rules. Let $|e(\Gamma)| = \sum_a |e_a(\Gamma)|$ and $|v(\Gamma)| = \sum_\alpha |v_\alpha(\Gamma)|$ be the total number of edges and vertices in the graph, and let

$$
b(\Gamma) = |e(\Gamma)| - |v(\Gamma)|
\tag{2.34}
$$

be the difference. Since each propagator contributes a factor of $\hbar$ and each vertex a factor of $1/\hbar$, a graph with $|e(\Gamma)|$ edges and $|v(\Gamma)|$ vertices, Γ comes with a power $\hbar^{|b(\Gamma)|}$. Thus the partition function has the perturbative expansion

$$
\frac{\mathcal{Z}}{\mathcal{Z}_0} \sim \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|}\hbar^{b(\Gamma)}F(\Gamma)
\tag{2.35}
$$

as $\hbar \to 0$, where we sum this expression over both connected and disconnnected vacuum graphs, including the trivial graph with no vertices. In particular, if our action includes sources J, then the Feynman diagrams may involve a vertex which joins the fields to these external sources. If J couples to a single power of the field, then the vertex is 1-valent and is associated with a Feynman rule

 J $\qquad\qquad -J/\hbar$

whereas if J sources a composite operator involving p_a powers of the field of type a, then this vertex will be $(\sum_a p_a)$-valent, absorbing p_a factors of ϕ^a. We include such vertices in what we mean by a 'vacuum' graph, so edges that terminate on a (green) source are not considered external.

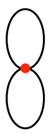
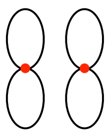
2.4 Effective actions

In this section I want to introduce the very important notion of an **effective action**, which will help us to develop a better feel for the partition function. We'll see that there are (at least) two distinct definitions that are related to eachother by a Legendre transform, very much analogous to the relation between the Helmholz (F) and Gibbs (G) free energies in statistical physics.

These effective actions will turn out to be central to our understanding of QFT in higher dimensions. The 'Helmholz' version plays a key role in Wilson's approach to renormalization, first developed in condensed matter systems, whereas the 'Gibbs' version is more closely related to the approach of Goldstone, Salam, Weinberg and Jona-Lasinio that was developed in parallel with high energy physics in mind.[19]

2.4.1 Connected graphs and a loop expansion

We start with a pragmatic observation: In computing the asymptotic expansion of $\mathcal{Z}$, we needed to take both connected and disconnected graphs into account. For example, in computing the partition function of the theory $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$ above, both

 and 

appeared (and higher powers of this and all other diagrams would occur further down the perturbative expansion). This is a duplication of effort – a disconnected graph is made up of several connected graphs, each of whose contributions we've already included. We'll now show that

$$\mathcal{W} = -\hbar \ln \mathcal{Z}\,, \tag{2.36}$$

is given asymptotically by a sum of **connected** Feynman graphs, avoiding this extra effort. In QFT, $\mathcal{W}$ is known as the **Wilsonian effective action**, and as I mentioned it's closely

[19] I'd love you to already have profound physical insight and a strong mathematical grasp of the uses and definitions of F *vs* G, but I'm a realist, so we'll try to develop these as we go along.

analogous to the Helmholz free energy in statistical physics. Knowing $\mathcal{W}$ is equivalent to knowing $\mathcal{Z}$, so of course $\mathcal{W}$ also depends on all the choices we made in setting up our QFT and in particular may depend on the sources.

To understand how $\mathcal{W}$ involves only connected graphs, suppose $\{\Gamma_j\}$ is the set of all possible connected vacuum graphs we can build using our propagators and vertices, where the label j tells us the topology of the graph. We define the product $\Gamma_1\Gamma_2$ of any two graphs Γ_1, Γ_2 to be their disjoint union, and similarly we interpret $(\Gamma_j)^n$ as the disjoint union of n copies of the same connected graph Γ_j. Any *disconnected* graph Γ is specified by a set of numbers $\{n_j\}$ (with each $n_j \in \mathbb{N}_0$) telling us how many copies of the connected graph Γ_j it contains.

Now, the symmetry factor of a disconnected graph consisting of n_1 copies of Γ_1, n_2 copies of Γ_2 *etc.* is

$$|\mathrm{Aut}(\Gamma_1^{n_1}\Gamma_2^{n_2}\cdots\Gamma_k^{n_k})| = \prod_{j=1}^{k}(n_j!)\,|\mathrm{Aut}(\Gamma_j)|^{n_j}\,, \tag{2.37}$$

because this is just a product of all the symmetry factors for the individual graph components, times a factor of $n_j!$ arising because we get an identical disconnected graph if we exchange any of the n_j copies of graph Γ_j. Also,

$$F\left(\prod_j \Gamma_j^{n_j}\right) = \prod_j F(\Gamma_j)^{n_j} \qquad \text{and} \qquad b\left(\prod_j \Gamma_j^{n_j}\right) = \sum_j n_j b(\Gamma_j)\,, \tag{2.38}$$

since the vertices and propagators contribute multiplicatively to an individual graph. Putting these facts together, we can write the partition function as

$$\begin{aligned}
\frac{\mathcal{Z}}{\mathcal{Z}_0} &\sim \sum_{\Gamma\,\in\,\mathrm{disconn}} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}\,\Gamma|}F(\Gamma) = \sum_{\{n_j\}} \frac{\hbar^{b\left(\prod_j \Gamma_j^{n_j}\right)}}{\left|\mathrm{Aut}\left(\prod_j \Gamma_j^{n_j}\right)\right|} F\left(\prod_j \Gamma_j^{n_j}\right) \\
&= \sum_{\{n_j\}}\prod_j \frac{1}{n_j!}\,\frac{\hbar^{n_j b(\Gamma_j)}}{|\mathrm{Aut}(\Gamma_j)|^{n_j}}F(\Gamma_j)^{n_j} = \prod_j\left(\sum_{n_j=0}^{\infty}\frac{1}{n_j!}\left(\frac{\hbar^{b(\Gamma_j)}}{|\mathrm{Aut}(\Gamma_j)|}F(\Gamma_j)\right)^{n_j}\right) \\
&= \prod_j\ \exp\left(\frac{\hbar^{b(\Gamma_j)}}{|\mathrm{Aut}(\Gamma_j)|}F(\Gamma_j)\right) = \exp\left(\sum_{\Gamma\,\in\,\mathrm{conn}}\frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}F(\Gamma)\right)\,.
\end{aligned} \tag{2.39}$$

Comparing with the definition (2.36) we have shown that the Wilsonian effective action is given by

$$\mathcal{W} \sim \mathcal{W}_0 - \hbar\sum_{\Gamma\,\in\,\mathrm{conn}}\frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}F(\Gamma)\,, \tag{2.40}$$

where $\mathcal{W}_0 = -\hbar\ln\mathcal{Z}_0$. As promised, the effective action has an asymptotic expansion in terms of connected graphs built from the same propagators and vertices as the partition function itself.

Euler's theorem tells us that, for a connected graph,

$$b(\Gamma) = |e(\Gamma)| - |v(\Gamma)| = \ell(\Gamma) - 1 \tag{2.41}$$

where $\ell(\Gamma)$ is the number of **loops**[20] in the graph. Comparing to (2.40) shows that an ℓ-loop connected Feynman graph contributes a term of order $\hbar^\ell$ to the expansion of the Wilsonian effective action. For this reason, the asymptotic expansion of the partition function is often known as the **loop expansion** of the QFT. We can say more by doing a little more elementary graph theory: If a vertex $v_\alpha(\Gamma)$ involves $n_{\alpha a}$ fields of type a, then for vacuum graphs

$$2|e_a(\Gamma)| = \sum_\alpha n_{\alpha a}|v_\alpha(\Gamma)|\,, \tag{2.42}$$

because each end of every edge must be attached to some vertex. Let's also suppose that these vertices all represent genuine *interactions*, so $\sum_a n_{\alpha a} > 2$ as at least three fields (possibly of different types) meet at each vertex. Then

$$\ell(\Gamma) = 1 + \sum_a |e_a(\Gamma)| - \sum_\alpha |v_\alpha(\Gamma)| = 1 + \sum_\alpha \left(-1 + \sum_a \frac{n_{\alpha a}}{2}\right)|v_\alpha(\Gamma)| > 1\,. \tag{2.43}$$

In other words, if all our vertices are at least 3-valent, then every non–trivial vacuum graph contains at least 2 loops.

Using the definition (2.24) of $\mathcal{Z}_0$, this shows that

$$\mathcal{W} \sim S(\phi_0) + \frac{\hbar}{2}\ln\det(\partial_a\partial_b S|_{\phi_0}) - \sum_{\Gamma\,\in\,\mathrm{conn}} \frac{\hbar^{\ell(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}F(\Gamma)\,, \tag{2.44}$$

where $\ell(\Gamma) \geq 2$ in each term in the final sum. We see that the leading term in $\mathcal{W}$, of order $\hbar^0$, is just the original classical action evaluated at its minimum ϕ_0. In (2.24) we saw that the term of order $\hbar$ came from expanding the action to quadratic order around the minimum, and integrating over the fluctuations. We'll see shortly that this can indeed be interpreted as a (sum of) 1-loop diagrams; as a quick plausibility check note that $\partial_a\partial_b S|_{\phi_0}\delta\phi^a\,\delta\phi^b$ can be interpreted as action consisting of purely 2-valent vertices of the form $\partial_a\partial_b S|_{\phi_0}$, and that (2.43) says that if all $n_{\alpha a} = 2$, we can only construct 1-loop graphs. As we said before, the higher order terms in the asymptotic series correspond to multi-loop diagrams[21].

I stress that the counting given above is valid for vacuum graphs in which all the vertices are at least trivalent; Feynman diagrams associated to scattering amplitudes or correlation functions, or those involving external sources corresponding to a 1-valent vertex, may come with different powers of $\hbar$ depending on the number of external states, number of field insertions in the correlator, or number of vertices involving the external source.

[20] A 'loop' is an independent 1-cycle in the sense of homology of the graph.

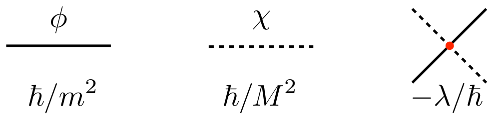
[21] I'm sorry to break the bad news, but in studying tree diagrams throughout last term's course, you weren't really doing any *quantum* field theory at all. Rather, the tree diagrams you drew were just a perturbative way to evaluate the classical action on a solution to the equations of motion. (Feynman tree diagrams are very closely related to **Picard iteration**, a standard perturbative technique to solve non-linear differential equations.) Unlike our example above, you found $S(\phi_0) \neq 0$ because you were working on a non-compact space $\mathbb{R}^{3,1}$ and demanded the fields were non–trivial in the distant past and future: *i.e.*, you computed a *scattering amplitude*. Nonetheless, the tree amplitude you obtained was purely classical — indeed, QFT should agree with classical field theory as $\hbar \to 0$.

2.4.2 Integrating out fields

Having seen that it's computed using just connected graphs, let's now try to get a feel for the physical meaning of $\mathcal{W}$. To begin, suppose we have two real–valued fields ϕ and χ, so that the space of fields is $\mathbb{R}^2$, and let the action be

$$S(\phi,\chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2 \tag{2.45}$$

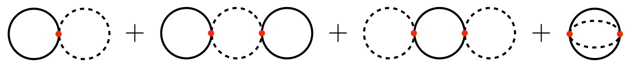
so that λ provides a coupling between the two fields. The Feynman rules are



and we may use these to compute perturbative expressions for correlation functions such as

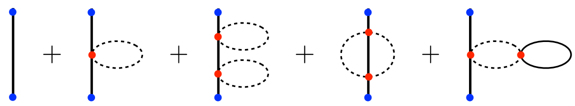
$$\langle f\rangle = \frac{1}{\mathcal{Z}}\int_{\mathbb{R}^2} \mathrm{d}\phi\,\mathrm{d}\chi\,\mathrm{e}^{-S(\phi,\chi)/\hbar}\,f(\phi,\chi)$$

in the usual way. For example, we have

$-\hbar^{-1}\mathcal{W} \sim$ 

$$= -\frac{\hbar\lambda}{4m^2M^2} + \frac{\hbar^2\lambda^2}{16m^4M^4} + \frac{\hbar^2\lambda^2}{16m^4M^4} + \frac{\hbar^2\lambda^2}{8m^4M^4}$$

as the sum of connected vacuum diagrams, and also

$\langle\phi^2\rangle =$ 

$$= \frac{\hbar}{m^2} - \frac{\lambda\hbar^2}{2m^4M^2} + \frac{\lambda^2\hbar^3}{4m^6M^4} + \frac{\lambda^2\hbar^3}{2m^6M^4} + \frac{\lambda^2\hbar^3}{4m^6M^4}$$

where the insertion of each power of ϕ is represented by a blue dot.

I want to arrive at this result in a different way. Suppose we first perform the integral over χ whilst holding ϕ fixed. In higher dimensions this step might be appropriate if, for example, $M \gg m$ so that our experiment isn't powerful enough to observe real χ production so can only measure ϕ directly. If we have no idea what χ is doing, we perform its path integral first, *i.e.*, we average over the behaviour of χ at each fixed ϕ. From this point of view, whilst performing the χ integral, the coupling $\phi^2\chi^2$ acts as a background source $J = \phi^2$ for the composite operator χ^2. The χ path integral then yields a $\mathcal{W}(\phi)$ that depends on this source:

$$\mathrm{e}^{-\mathcal{W}(\phi)/\hbar} = \int_{\mathbb{R}} \mathrm{d}\chi\,\mathrm{e}^{-S(\phi,\chi)/\hbar}\,. \tag{2.46}$$

Once we've found this $\mathcal{W}(\phi)$, we can use it in the remaining ϕ integral to compute $\langle f\rangle$ for any observable f that depends only on ϕ – *i.e.* the only quantities our low–energy experiment is able to probe. Of course there's nothing mysterious here, we're simply choosing in which order to do our integrals, writing

$$\langle f(\phi)\rangle = \frac{1}{\mathcal{Z}}\int \mathrm{d}\phi\,\mathrm{d}\chi\,\mathrm{e}^{-S(\phi,\chi)/\hbar}\,f(\phi) = \frac{1}{\mathcal{Z}}\int \mathrm{d}\phi\,\mathrm{e}^{-\mathcal{W}(\phi)/\hbar}f(\phi)\,. \tag{2.47}$$

Note that indeed $\mathcal{W}(\phi)$ plays the role of an **effective action** for the ϕ field – one in which all the quantum effects of χ are taken into account.

In general, computing $\mathcal{W}(\phi)$ has to be done perturbatively in terms of a sum of connected Feynman diagrams in the presence of the source $J = \phi^2$. However, in our toy example it's straightforward to find $\mathcal{W}(\phi)$ exactly:

$$\int_{\mathbb{R}} \mathrm{d}\chi\,\mathrm{e}^{-S(\phi,\chi)/\hbar} = \mathrm{e}^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2+\lambda\phi^2/2}} \tag{2.48}$$

and therefore

$$\mathcal{W}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\hbar}{2}\ln\left[1+\frac{\lambda}{2M^2}\phi^2\right] + \frac{\hbar}{2}\ln\frac{M^2}{2\pi\hbar}\,. \tag{2.49}$$

This is exactly what we expect from above. At constant ϕ, the original action has a unique minimum at $\chi_0 = 0$, where $S(\phi,\chi_0) = m^2\phi^2/2$, the leading term in $\mathcal{W}(\phi)$. The logarithms comes from the χ integral, which is our example is purely Gaussian. The final term in (2.49) is independent of the field ϕ; such field–independent terms are irrelevant in QFT, for example, they will cancel when we compute any correlation function normalized by the partition function of the free ($\lambda = 0$) theory. We will drop this term henceforth, but note that the fact the constant term in the action changes as we integrate out fields is actually the origin of the notorious **cosmological constant problem**.

Expanding the remaining logarithm, we write $\mathcal{W}(\phi)$ as an infinite series

$$\begin{aligned}\mathcal{W}(\phi) &= \left(\frac{m^2}{2}+\frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \cdots \\ &= \frac{m^2_{\text{eff}}}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \cdots\,.\end{aligned} \tag{2.50}$$

Thus the effect of integrating out the 'high energy' field χ is to *change* the structure of the action seen by ϕ. In particular, the mass term of the ϕ field has been shifted

$$m^2 \to m^2_{\text{eff}} = m^2 + \frac{\hbar\lambda}{2M^2}\,. \tag{2.51}$$

Even more strikingly, we've generated an infinite series of new coupling terms

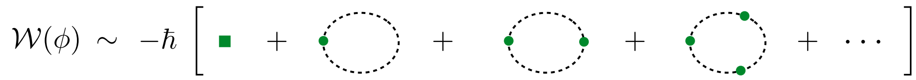
$$\lambda_4 = -\frac{3\hbar}{2}\frac{\lambda^2}{M^4}\,,\qquad \lambda_6 = 15\hbar\frac{\lambda^3}{M^6}\,,\qquad \lambda_{2k} = (-1)^{k+1}\hbar\,\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}} \tag{2.52}$$

describing self–interactions of ϕ. It's important to observe that the ϕ mass shift and new ϕ self–interactions all vanish as $\hbar \to 0$; they are *quantum* effects. Notice also that they're each suppressed by powers of the (high) mass M.

Following our general story above, it's useful to think in a little more detail about how these new couplings arise. We can perform the χ path integral using Feynman graphs, using the ingredients

$$\frac{\hbar}{M^2} \qquad\qquad -\frac{\lambda\phi^2}{2\hbar}$$

which involve the same χ propagator as before, but now account for the fact that we are treating the interaction as a source, which takes the value $-\lambda\phi^2/2$ from the point of view of the χ integral. These ingredients lead to the following perturbative construction of $\mathcal{W}(\phi)$[22]:



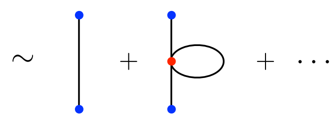
$$= \quad S(\phi) \quad + \frac{1}{2}\frac{\hbar\lambda}{2M^2}\phi^2 \;-\; \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 \;+\; \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \;\cdots$$

where in the first term $S(\phi) = S(\phi, 0)$ is the part of the original action that came straight out of the χ integral. Again since $-\hbar^{-1}\mathcal{W}(\phi)$ is the *logarithm* of the χ integral, only connected diagrams appear.

Just as we expected, the diagrammatic expansion reveals that the new interactions in $\mathcal{W}$ are generated by the χ field running around a loop, interacting with the 'source' as it goes. In our effective description that knows only about the behaviour of the ϕ field, we can no longer 'see' the χ field 'circulating' around the loop. Instead, we perceive this just as a new interaction vertex for ϕ. As promised, the fact that χ appears only quadratically in the original action (2.45) means that in this example we can only construct 1-loop diagrams from our propagator and 2-valent vertex. All these 1-loop diagrams sum up to give the logarithm we obtained by direct integration. Starting from a more generic initial action with higher valent vertices, we'd obtain contributions from higher loop graphs, each coming with a factor of $\hbar^{\ell(\Gamma)}$.

Using this effective action, we find

$$\langle\phi^2\rangle = \frac{1}{\mathcal{Z}}\int \mathrm{d}\phi\, \mathrm{e}^{-\mathcal{W}(\phi)/\hbar}\,\phi^2 \quad \sim$$



$$= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\lambda_4\hbar^2}{2m_{\text{eff}}^6} \; + \; \cdots$$

where the propagator and vertices here are the ones appropriate for the effective action $\mathcal{W}(\phi)$. Using the definition (2.51)-(2.52) of the new couplings in terms of the original λ and

[22] In evaluating these Feynman diagrams, I've kept the symmetry factors separate from the vertices and propagators – check you understand them.

M, this unsurprisingly agrees with our answer before, correct to order λ^2. However, once we had the effective action, we arrived at this answer using just two diagrams, whereas previously it required five. If we only care about a single correlation function then the work involved in first computing $\mathcal{W}(\phi)$ and then using the new set of Feynman rules to compute the low–energy correlator is roughly the same as just using the original action to compute this correlator directly. On the other hand, if we wish to compute many low–energy correlators then we're clearly better off investing a little time to work out the effective action first.

However, the real point I wish to make is this: the way we experience the world is *always* through an effective action. Naively at least, we have no idea what new physics may be lurking just out of reach of our most powerful accelerators; there may be any number of new, hitherto undiscovered species of particle, or new dimensions of space–time, or even wilder new phenomena. However, when describing low–energy physics, we should only seek to describe the behaviour of the degrees of freedom (fields) that are relevant and accessible at the energy scale at which we're conducting our experiments, *even if we happen to know what the more fundamental description is.* For example, a glass of water certainly consists of very many H_2O molecules, these molecules are bound states of atoms, each of which consist of many electrons orbiting around a central nucleus. In turn, this nucleus comprises of protons and neutrons stuck together by a strong force mediated by pions, and all these hadrons are themselves seething masses of quarks and gluons. But it would be very foolish to imagine we should describe the properties of water that are relevant in everyday life by starting from the Lagrangian for QCD.

Let me make one final comment. In the example above, we started from a very simple action in equation (2.45) and obtained a more complicated effective action (2.50) after integrating out the unobserved degree of freedom χ. A more generic case would start from a general action (invariant under $\phi \to -\phi$ and $\chi \to -\chi$ for simplicity)

$$S'(\phi, \chi) = \sum_{i,j} \frac{\lambda_{i,j}}{(2i)!\,(2j)!}\phi^{2i}\chi^{2j} \tag{2.53}$$

in which all possible even monomials in ϕ and χ are allowed. For example, we may have arrived at this action by integrating out some other field that was unknown in our above considerations. In this generic case, the effect of integrating out χ will not generate *new* interactions for ϕ — all possible even self–interactions are included anyway — but rather the values of the coupling constants $\lambda_{i,0}$ will get shifted, just as for the mass shift we saw above. In addition, because the χ path integral would now be very complicated, we can only reasonably expect to describe the shifted couplings as an asymptotic series in $\hbar$, rather than the single power of $\hbar$ we obtained above. Nonetheless, the main lesson to remember is that integrating out degrees of freedom changes the values of the coupling constants in the effective action for the remaining fields.

2.4.3 The 1PI effective action

Wilson's effective action is motivated by the idea of averaging over quantum fluctuations of high energy fields that are beyond the reach of our experimental observations, and provides us with a new action for the remaining, low energy degrees of freedom. The quantum effects of the remaining fields still need to be computed. We'd now like to construct a new type of effective action that takes account of the quantum fluctuations of the *whole* system.

You might think that this should just be $\mathcal{W}(J)$ itself: we couple our fields to sources, integrate out all the quantum fields to obtain $\mathcal{W}(J)$ and then differentiate *wrt* J to obtain correlation functions. This point of view is indeed useful if our quantum system is immersed in some background (the choice of sources) that we are able to vary. However, for an *isolated* quantum system (such as the whole Universe, or a scattering experiment performed in CERN) there is no obvious background.

We include a source term $J\phi$ in the original action and let

$$\begin{aligned} \Phi = \frac{\partial \mathcal{W}}{\partial J} &= -\frac{\hbar}{\mathcal{Z}(J)}\frac{\partial}{\partial J}\left[\int \mathrm{d}\phi\, \mathrm{e}^{-(S+J\phi)/\hbar}\right] \\ &= \frac{1}{\mathcal{Z}(J)}\int \mathrm{d}\phi\, \mathrm{e}^{-(S+J\phi)/\hbar}\,\phi = \langle\phi\rangle_J\,, \end{aligned} \tag{2.54}$$

so that Φ is the average value of the field ϕ, including all quantum effects. I emphasise that this average is computed in the presence of a source J for ϕ itself – *i.e.*, we do *not* set $J = 0$ after taking the derivative. Clearly, this means that Φ depends on what we choose for J and, conversely, if we specify a value of Φ we want to obtain then the source is fixed (at least if the relation $\Phi(J)$ is invertible).

We define the **quantum effective action** $\Gamma(\Phi)$ as the Legendre transformation

$$\Gamma(\Phi) = \mathcal{W}(J) - \Phi J \tag{2.55}$$

of the Wilsonian effective action. Note that

$$\frac{\partial\Gamma}{\partial\Phi} = \frac{\partial\mathcal{W}}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi} = \frac{\partial\mathcal{W}}{\partial J}\frac{\partial J}{\partial\Phi} - J - \Phi\,\frac{\partial J}{\partial\Phi} = -J\,. \tag{2.56}$$

The relations

$$\Phi = \frac{\partial\mathcal{W}}{\partial J} \qquad \text{and} \qquad J = -\frac{\partial\Gamma}{\partial\Phi} \tag{2.57}$$

allow us to transform between $\mathcal{W}(J)$ and $\Gamma(\Phi)$: If we are given $\mathcal{W}(J)$ as a function of J, we define Φ by $\partial\mathcal{W}/\partial J$ as above and inverting[23] this gives us $J(\Phi)$. Then $\Gamma(\Phi) = \mathcal{W}(J(\Phi)) - \Phi\, J(\Phi)$ is a function of Φ. On the other hand, if we are presented with a function $\Gamma(\Phi)$, we define J to be $-\partial\Gamma/\partial\Phi$. Inverting gives $\Phi(J)$ and hence we reconstruct $\mathcal{W}(J) = \Gamma(\Phi(J)) + \Phi(J)\,J$ as a function of J.

To understand the role of $\Gamma(\Phi)$, first note that

$$\left.\frac{\partial\Gamma}{\partial\Phi}\right|_{J=0} = 0 \tag{2.58}$$

so that the possible quantum averaged values for Φ in the *absence* of a source are just the extrema of $\Gamma(\Phi)$. This is one sense in which $\Gamma(\Phi)$ is an effective action – the extrema of

[23]The Legendre transform requires that the functions $\mathcal{W}(J)$ and $\Gamma(\Phi)$ are convex, which ensures the derivatives are monotonically non-decreasing, so that these relations are invertible. This is known to be the case in statistical mechanics, but is much less clear in the infinite dimensional context of QFT.

$\Gamma(\Phi)$ correspond to equations of motion (which, in our current $d = 0$ context will just be algebraic) with all the quantum corrections taken into account.

Let's go even further and consider a *quantum* theory defined via a (path) integral over Φ where we let $\Gamma(\Phi)$ play the role of the classical action. We define a quantity $\mathcal{W}_\Gamma(J, g)$ by

$$\mathrm{e}^{-\mathcal{W}_\Gamma(J)/g} = \int \mathrm{d}\Phi\, \mathrm{e}^{-(\Gamma(\Phi)+J\Phi)/g}\,, \tag{2.59}$$

where the parameter g plays the role of $\hbar$ – I wish to keep g separate from the original parameter $\hbar$ that is still present in the vertices of $\Gamma(\Phi)$. It follows from our previous results that $\mathcal{W}_\Gamma(J)$ can be computed in terms of a series of connected Feynman graphs, now built using the propagators and vertices that follow from $\Gamma(\Phi)$, rather than the original classical action $S(\phi)$. As before, an ℓ-loop diagram will contribute a term to $\mathcal{W}_\Gamma$ that is proportional to g^ℓ, so we can expand

$$\mathcal{W}_\Gamma(J) = \sum_{\ell=0}^{\infty} g^\ell\, \mathcal{W}_\Gamma^{(\ell)}(J) \tag{2.60}$$

where $\mathcal{W}_\Gamma^{(\ell)}(J)$ is the sum of all ℓ-loop connected Feynman graphs present in (2.59). In particular, the *tree* graphs we can construct using the propagators and vertices of the quantum effective action all appear in $\mathcal{W}_\Gamma^{(0)}(J)$. To extract these tree graphs, we take the limit $g \to 0$. In this limit, by the method of steepest descent we know that the integral (2.59) will be dominated by the minimum of the argument of the exponential, *i.e.* the value of Φ for which

$$\frac{\partial \Gamma}{\partial \Phi} = -J \tag{2.61a}$$

and that, to leading order,

$$\mathcal{W}_\Gamma(J) = \mathcal{W}_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi \tag{2.61b}$$

evaluated at this extremum. These are exactly the same equations as (2.55) & (2.57), so we see that the tree level term $\mathcal{W}_\Gamma^{(0)}(J)$ is nothing other than $\mathcal{W}(J)$. In other words, the sum of connected diagrams $\mathcal{W}(J)$ built from the classical action $S(\phi) + J\phi$ can also be obtained as a sum of *tree* diagrams using the effective action $\Gamma(\Phi) + J\Phi$.

To understand how this can be possible, note that any connected graph can be viewed as a tree whose 'vertices' are all possible **one particle irreducible** graphs. (An edge e

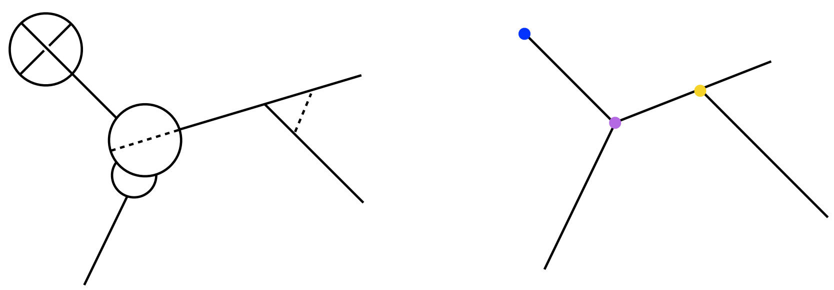


Figure 2: *Any connected graph can be viewed as a tree whose vertices are 1PI graphs.*

in a connected graph Γ is a **bridge** if $\Gamma \backslash e$ is disconnected. A connected graph is said to be one particle irreducible, or 1PI, if it does not contain any bridges.) This is simple to see: start from any connected graph and remove all bridges. The result is a product of 1PI graphs, which may be taken as vertices of a tree – see figure **??** for an example. This tells us how to compute $\Gamma(\Phi)$ perturbatively from the original action: $\Gamma(\Phi)$ consists of all possible 1PI Feynman graphs that may be constructed using the propagators and vertices in $S(\phi)$. These graphs may have arbitrarily many external lines, with each external line associated with a factor of Φ. The number of external lines in a given 1PI graph thus tells us the valency of a vertex in $\Gamma(\Phi)$.

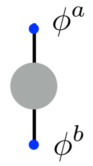
As a check on this formalism, suppose we have several fields ϕ^a, each with sources J_a and let $\mathrm{e}^{-\mathcal{W}(J_a)/\hbar} = \int \mathrm{d}^n\phi \, \mathrm{e}^{-(S(\phi^a)+J_a\phi^a)/\hbar}$. Then

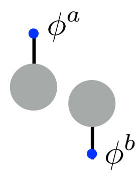
$$\begin{aligned}
-\hbar\frac{\partial^2\mathcal{W}}{\partial J_a\,\partial J_b} &= -\hbar\frac{\partial}{\partial J_a}\left[\frac{1}{\mathcal{Z}(J)}\int \mathrm{d}^n\phi\ \mathrm{e}^{-(S(\phi^c)+J_c\phi^c)/\hbar}\,\phi^b\right] \\
&= \frac{1}{\mathcal{Z}(J)}\int \mathrm{d}^n\phi\ \mathrm{e}^{-(S(\phi)+J\phi)/\hbar}\,\phi^a\phi^b \\
&\qquad - \frac{1}{\mathcal{Z}(J)^2}\left[\int \mathrm{d}^n\phi\,\mathrm{e}^{-(S(\phi^c)+J_c\phi^c)/\hbar}\,\phi^a\right]\left[\int \mathrm{d}^n\phi\,\mathrm{e}^{-(S(\phi^c)+J_c\phi^c)/\hbar}\,\phi^b\right],
\end{aligned} \tag{2.62}$$

where the terms in the final line come from letting the second derivative operator act on $1/\mathcal{Z}(J)$. In keeping with the fact that $\mathcal{W}(J)$ involves only connected graphs, we see that expression is the *connected* two–point function of the fields

$$\langle\phi^a\phi^b\rangle_J^{\text{conn}} = \langle\phi^a\phi^b\rangle_J - \langle\phi^a\rangle_J\,\langle\phi^b\rangle_J\,, \tag{2.63}$$

since any contribution to $\langle\phi^a\phi^b\rangle_J$ coming from a Feynman graph that does not somehow join together the two ϕ insertions will cancel against identical Feynman graphs in $\langle\phi^a\rangle_J\langle\phi^b\rangle_J$:

$\langle\phi^a\phi^b\rangle \quad =$ 

$+$ 

$= \ \langle\phi^a\phi^b\rangle^{\text{conn}} + \langle\phi^a\rangle\,\langle\phi^b\rangle$

We can thus view $\langle\phi^a\phi^b\rangle_{J=0}^{\text{conn}}$ as an expression for the exact propagator in the interacting theory, including not just the inverse of the kinetic term in $S(\phi)$, but also corrections due to interactions.

Using the connecting relations (2.57) we can also express this as

$$\langle\phi^a\phi^b\rangle_J^{\text{conn}} = -\hbar\frac{\partial^2\mathcal{W}}{\partial J_a\,\partial J_b} = -\hbar\frac{\partial\Phi^b}{\partial J_a} = -\hbar\left(\frac{\partial J_a}{\partial\Phi^b}\right)^{-1} = \hbar\left(\frac{\partial^2\Gamma}{\partial\Phi^b\,\partial\Phi^a}\right)^{-1}. \tag{2.64}$$

This shows that the exact propagator in our interacting theory is indeed given by $\hbar$ times the inverse of the quadratic term in the quantum effective action. Differentiating further allows us to see that the connected n–point functions $\langle\phi^a\phi^b\cdots\phi^d\rangle_J^{\text{conn}}$ of the fields are exactly the tree graphs we obtain by connecting the 1PI vertices in $\Gamma(\Phi)$ with these exact propagators. (We're usually interested in the case that there is no background source, so we set $J_a = 0$ at the end of the day.)

2.5 Fermions and Grassmann variables

Realistic theories contain **fermions**. In higher dimensions, the spin–statistics theorem says that for a unitary theory, fermions must have half–integral spin. However, in $d = 0$ there is no notion of spin, much less a spin–statistics theorem, and fermionic 'fields' are simply **Grassmann numbers**. These are a set of n elements $\{\theta^a\}$ obeying the algebra

$$\theta^a\theta^b = -\theta^b\theta^a \qquad \text{and} \qquad \theta^a\phi^b = \phi^b\theta^a \quad \text{for all } \phi^b \in \mathbb{C}\,. \tag{2.65}$$

Thus, Grassmann variables *anticommute* with eachother and commute with any bosonic variable. In particular, this implies $\theta^a\theta^a = -\theta^a\theta^a = 0$ for each a (no sum). This property means that any function of a finite number of Grassmann variables has a finite expansion

$$F(\theta) = f + \rho_a\,\theta^a + \frac{1}{2!}g_{a_1a_2}\,\theta^{a_1}\theta^{a_2} + \cdots + \frac{1}{n!}h_{a_1a_2\cdots a_n}\,\theta^{a_1}\theta^{a_2}\cdots\theta^{a_n}\,, \tag{2.66}$$

where we can take the coefficients to be totally antisymmetric, *e.g.* $g_{a_1a_2} = -g_{a_2a_1}$.

We can also define differentiation and integration for Grassmann variables. For differentiation we have

$$\frac{\partial}{\partial\theta^a}\theta^b + \theta^b\frac{\partial}{\partial\theta^a} = \delta^b{}_a \tag{2.67}$$

so that the derivative operator itself anticommutes with the variables. Since any function of a single Grassmann variable θ is of the form $f + \rho\,\theta$, we only have to define $\int \mathrm{d}\theta$ and $\int \mathrm{d}\theta\,\theta$. We ask that our definition be translationally invariant, so that

$$\int \mathrm{d}\theta\,(\theta + \eta) = \int \mathrm{d}\theta\,\theta \tag{2.68}$$

and this implies

$$\int \mathrm{d}\theta\,1 = 0\,. \tag{2.69a}$$

We then choose to normalise our integration measure such that

$$\int \mathrm{d}\theta\,\theta = 1\,. \tag{2.69b}$$

These rules are often known as **Berezin integration**. Note that these definitions imply

$$\int \mathrm{d}\theta\,\frac{\partial}{\partial\theta}F(\theta) = 0 \tag{2.70}$$

since the derivative removes the single power of θ that can appear in $F(\theta)$. This allows us to integrate by parts, provided due care is taken of signs.

If we have n Grassmann variables θ^a, repeated application of the above rules shows that the only non–vanishing integral is one whose integrand involves exactly one power of every θ^a. Specifically, we have

$$\int \mathrm{d}^n\theta\;\theta^1\theta^2\cdots\theta^{n-1}\theta^n = \int \mathrm{d}\theta^n\,\mathrm{d}\theta^{n-1}\,\cdots\,\mathrm{d}\theta^1\;\theta^1\theta^2\cdots\theta^n = 1 \tag{2.71}$$

and, in general

$$\int \mathrm{d}^n\theta \ \theta^{a_1}\theta^{a_2}\cdots\theta^{a_n} = \epsilon^{a_1a_2\cdots a_n} \tag{2.72}$$

with the sign coming from ordering the θs. Suppose we write $\theta'^a = N^a_{\ b}\theta^b$ for some $N \in GL(n;\mathbb{C})$. Then, by linearity

$$\begin{aligned}\int \mathrm{d}^n\theta \ \theta'^{a_1}\theta'^{a_2}\cdots\theta'^{a_n} &= N^{a_1}_{\ b_1}N^{a_2}_{\ b_2}\cdots N^{a_n}_{\ b_n}\int \mathrm{d}^n\theta \ \theta^{b_1}\theta^{b_2}\cdots\theta^{b_n} \\ &= N^{a_1}_{\ b_1}N^{a_2}_{\ b_2}\cdots N^{a_n}_{\ b_n}\epsilon^{b_1b_2\cdots b_n} \\ &= \det(N)\,\epsilon^{a_1a_2\cdots a_n} = \det N\int \mathrm{d}^n\theta' \ \theta'^{a_1}\theta'^{a_2}\cdots\theta'^{a_n}\,.\end{aligned} \tag{2.73}$$

Thus we see that for Berezin integration

$$\theta'^a = N^a_{\ b}\theta^b \qquad \Rightarrow \qquad \mathrm{d}^n\theta = \det(N)\,\mathrm{d}^n\theta' \tag{2.74}$$

where the Jacobian of the change of variables appears upside down (and without a modulus sign) compared to the standard, bosonic rule $\mathrm{d}^n\phi = \mathrm{d}^n\phi'/|\det N|$ if $\phi'^a = N^a_{\ b}\phi^b$.

2.5.1 Fermionic free field theory

Let's suppose our $d = 0$ QFT involves two fermionic fields, $\{\theta^1, \theta^2\}$. The action is a bosonic quantity, so each term has to involve and even number of fermions. Consequently, the only non–constant action we can write down is

$$S(\theta) = \frac{1}{2}A\theta^1\theta^2\,, \tag{2.75}$$

because since $(\theta^1)^2 = 0 = (\theta^2)^2$ there is no way to introduce a non–trivial interaction. The partition function is then

$$\mathcal{Z}_0 = \int \mathrm{d}^2\theta \ \mathrm{e}^{-S(\theta)/\hbar} = \int \mathrm{d}^2\theta\left(1 - \frac{A}{2\hbar}\theta^1\theta^2\right) = -\frac{A}{2\hbar} \tag{2.76}$$

using the fact that the expansion of $\mathrm{e}^{-S(\theta)/\hbar}$ truncates at the first non–trivial term, and the rule (2.71) of Berezin integration. More generally, if we have $2m$ fermionic fields θ^a described by the quadratic action

$$S(\theta) = \frac{1}{2}A_{ab}\theta^a\theta^b \tag{2.77}$$

where A is an antisymmetric matrix, the partition function is given by the Berezin integral

$$\begin{aligned}\mathcal{Z}_0 &= \int \mathrm{d}^{2m}\theta \ \mathrm{e}^{-A(\theta,\theta)/2\hbar} = \int \mathrm{d}^{2m}\theta \ \sum_{n=0}^{m}\frac{(-)^n}{(2\hbar)^n n!}(A_{ab}\theta^a\theta^b)^n \\ &= \frac{(-)^m}{(2\hbar)^m m!}\int \mathrm{d}^{2m}\theta \ A_{a_1a_2}A_{a_3b_4}\cdots A_{a_{2m-1}a_{2m}}\,\theta^{a_1}\theta^{a_2}\cdots\theta^{a_{2m-1}}\theta^{a_{2m}} \\ &= \frac{(-)^m}{(2\hbar)^m m!}\epsilon^{a_1a_2\cdots a_{2m-1}a_{2m}}A_{a_1a_2}\cdots A_{a_{2m-1}a_{2m}}\,,\end{aligned} \tag{2.78}$$

where we note that only the m^{th} term of the expansion can contribute. The **Pfaffian** of an antisymmetric matrix A is given by[24]

$$\operatorname{Pfaff}(A) = \epsilon^{a_1 a_2 \cdots a_{2m-1} a_{2m}} A_{a_1 a_2} \cdots A_{a_{2m-1} a_{2m}} \tag{2.79}$$

and in the first problem set, I ask you to use Grassmann variables to show that $(\operatorname{Pfaff} A)^2 = \det A$.

In summary, the partition function of $n = 2m$ free fermions can be written as

$$\mathcal{Z}_0 = \pm\sqrt{\frac{\det(A)}{\hbar^n}}\,, \tag{2.80}$$

whereas for n free bosons we had $\mathcal{Z}_0 = \sqrt{(2\pi\hbar)^n/\det(M)}$ with M a symmetric matrix. Except for a numerical factor (which we could in any case include in the normalization of the measure), the fermionic result is just the inverse of the bosonic one. We'll see various important consequences of this fact later.

We may also consider the partition function in the presence of sources. Since we want the action to be bosonic, the source itself must now be fermionic and we denote it by η. Let

$$S(\theta,\eta) = \frac{1}{2}A_{ab}\theta^a\theta^b + \eta_a\theta^b\,. \tag{2.81}$$

Completing the square as before gives[25]

$$S(\theta,\eta) = \frac{1}{2}\left(\theta^a + \eta_c(A^{-1})^{ca}\right)A_{ab}\left(\theta^b + \eta_d(A^{-1})^{db}\right) + \frac{1}{2}\eta_a(A^{-1})^{ab}\eta_b\,, \tag{2.82}$$

so using the translational invariance of the measure $\mathrm{d}^n\theta$, the partition function in the presence of sources is

$$\mathcal{Z}_0(\eta) = \exp\left(-\frac{1}{2\hbar}A^{-1}(\eta,\eta)\right)\mathcal{Z}_0(0)\,. \tag{2.83}$$

As before this allows us to compute correlation functions of the fermion fields. As an example, the two–point function

$$\langle\theta^a\theta^b\rangle = \frac{\hbar^2}{\mathcal{Z}_0(0)}\frac{\partial^2\mathcal{Z}_0(\eta)}{\partial\eta_a\partial\eta_b}\bigg|_{\eta=0} = \hbar\,(A^{-1})^{ab}\,. \tag{2.84}$$

which is just the inverse of the kinetic term for the θs and plays the role of the 'propagator' in this $d = 0$ theory. Notice that this propagator is the same (not the inverse) of the propagator we'd obtain in the bosonic theory, except that for fermions the matrix A (and hence A^{-1}) must be antisymmetric, whereas for bosons M^{-1} was symmetric.

The fact that functions of a finite number of Grassmann variables can always be represented as polynomials means that in $d = 0$, we never need use perturbation theory

[24] For example, $\operatorname{Pfaff}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a\,.$

[25] It's a good exercise to go through this and check you're comformtable with all the signs, both here and in the calculation of the two–point function below.

to evaluate fermionic path integrals: It's always possible to perform finitely many Berezin integrations exactly. Nonetheless, for a nonlinear theory such as

$$S(\theta) = \frac{1}{2}A_{ab}\theta^a\theta^b + \frac{1}{4!}\lambda_{abcd}\,\theta^a\theta^b\theta^c\theta^d \tag{2.85}$$

we can, if we choose, construct Feynman diagrams with propagator $\hbar A^{-1}$ and vertex $-\lambda_{abcd}/\hbar$. We then construct Feynman diagrams with these ingredients in just the same way as for the bosonic theory. In higher dimensional QFT, fermions will be described by Grassmann valued fields, so we'll have infinitely many Grassmann variables over which to integrate (we'll understand this better later). With infinitely many Grassmann variables, the situation for fermions is really no different from bosons, in the sense that in both cases it is usually necessary to work perturbatively and compute an asymptotic series approximation to the full path integral.

2.5.2 Supersymmetry and localization

For a generic QFT, the asymptotic series is as good a representation of the partition function (or correlation functions) as we can hope for, barring numerics. However, if the action is of a very special type, it may sometimes possible to evaluate the partition function and even certain correlation functions *exactly*. There are many mechanisms by which this might happen; this section gives a toy model of one of them, known as **localization** in supersymmetric theories.

Let's take a theory where that in addition to our bosonic field ϕ, we have two fermionic fields ψ_1 and ψ_2. With a zero–dimensional space–time, the space of fields is just $\mathbb{R}^{1|2}$. Given an action $S(\phi,\psi_i)$ the partition function is, as usual,

$$\mathcal{Z} = \int \frac{\mathrm{d}\phi\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2}{\sqrt{2\pi}}\,\mathrm{e}^{-S(\phi,\psi_i)} \tag{2.86}$$

where I've thrown a factor of $1/\sqrt{2\pi}$ into the measure for later convenience. Generically, we'd have to be content with a perturbative evaluation of $\mathcal{Z}$, using Feynman diagrams formed from edges for the ϕ and ψ_i fields, together with vertices from all the different vertices that appear in our action. For a complicated action, even low orders of the perturbative expansion might be difficult to compute in general.

However, let's suppose the action takes the special form

$$S(\phi,\psi_1,\psi_2) = \frac{1}{2}(\partial h)^2 - \psi_1\psi_2\,\partial^2 h \tag{2.87}$$

where $h(\phi)$ is some ($\mathbb{R}$-valued) polynomial in ϕ and ∂h is its derivative *wrt* ϕ. Note that there can't be any terms in S involving only one of the fermion fields since this term would itself be fermionic. There also can't be higher order terms in the fermion fields since $\psi_i^2 = 0$ for a Grassmann variable, so the only thing special about this action is the relation between the purely bosonic piece and the second term involving $\psi_1\psi_2$.

Now consider the transformations

$$\delta\phi = \epsilon_1\psi_1 + \epsilon_2\psi_2\,, \qquad \delta\psi_1 = \epsilon_2\partial h\,, \qquad \delta\psi_2 = -\epsilon_1\partial h \tag{2.88}$$

where ϵ_i are fermionic parameters. These are supersymmetry transformations in this zero–dimensional context; take the Part III Supersymmetry course to meet supersymmetry in higher dimensions. The most important property of these transformations is that they are *nilpotent*[26]. Under (2.88) the action (2.87) transforms as

$$\delta S = \partial h \, \partial^2 h(\epsilon_1 \psi_1 + \epsilon_2 \psi_2) - (\epsilon_2 \partial h)\psi_2 \, \partial^2 h - \psi_1(-\epsilon_1 \partial h)\partial^2 h = 0 \tag{2.89}$$

and is thus invariant — this is what the special relation between the bosonic and fermionic terms in S buys us. (To obtain this result we used the fact that Grassmann variables anticommute.) It's also true that the integral measure $\mathrm{d}\phi \, \mathrm{d}^2\psi$ is likewise invariant; I'll leave this too as an exercise.

Supersymmetric QFTs are drastically simpler than generic ones, especially in zero dimensions. Let $\delta\mathcal{O}$ be the supersymmetry variation of some operator $\mathcal{O}(\phi, \psi_i)$ and consider the correlation function $\langle \delta\mathcal{O} \rangle$. Since $\delta S = 0$ we have

$$\langle \delta\mathcal{O} \rangle = \frac{1}{\mathcal{Z}_0} \int \mathrm{d}\phi \, \mathrm{d}^2\psi \, \mathrm{e}^{-S} \, \delta\mathcal{O} = \frac{1}{\mathcal{Z}_0} \int \mathrm{d}\phi \, \mathrm{d}^2\psi \, \delta \left(\mathrm{e}^{-S}\mathcal{O}\right) . \tag{2.90}$$

The supersymmetry variation here acts on both ϕ and the fermions ψ_i in $\mathrm{e}^{-S}\mathcal{O}$. But if it acts on a fermion ψ_i then the resulting term does not contain that ψ_i and hence cannot contribute to the integral because $\int \mathrm{d}\psi \, 1 = 0$ for Grassmann variables. On the other hand, if it acts on ϕ then while the resulting term may survive the Grassmann integral, it is a total derivative in the ϕ field space. Thus, provided $\mathcal{O}$ does not disturb the decay of e^{-S} as $|\phi| \to \infty$, any such correlation function must vanish, $\langle \delta\mathcal{O} \rangle = 0$.

In particular, if we choose $\mathcal{O}_g = \partial g \, \psi_1$ for some $g(\phi)$, then setting the parameters $\epsilon_1 = -\epsilon_2 = \epsilon$ we have

$$0 = \langle \delta\mathcal{O}_g \rangle = \epsilon \langle \partial g \, \partial h - \partial^2 g \, \psi_1 \psi_2 \rangle \, . \tag{2.91}$$

The significance of this is that the quantity $\partial g \, \partial h - \partial^2 g \, \psi_1 \psi_2$ is the first–order change in the action under the *deformation* $h \to h+g$, again so long as g does not alter the behaviour of h as $|\phi| \to \infty$. The fact that $\langle \delta\mathcal{O}_g \rangle = 0$ tells that the partition function $\mathcal{Z}[h]$, which we might think depends on all the couplings in the vertices in the polynomial h, is in fact largely insensitive to the detailed form of h because we can deform it by any other polynomial of the same degree or lower. The most important case is if we choose g to be proportional to h, then our deformation just rescales $h \to (1 + \lambda)h$ and so we see that $\mathcal{Z}[h]$ is independent of the overall scale of h. By iterating this procedure, we can imagine rescaling h by a large factor so that the bosonic part of the action $(\partial h)^2/2 \to \Lambda^2(\partial h)^2/2$. As $\Lambda \to \infty$, the factor e^{-S} exponentially suppresses any contribution to $\mathcal{Z}$ except from an infinitesimal neighbourhood of the critical points of h where $\partial h = 0$. This phenomenon is known as **localization** of the path integral.

It's now straightforward to work out the partition function. Near any such critical point ϕ_* we have

$$h(\phi) = h(\phi_*) + \frac{c_*}{2}(\phi - \phi_*)^2 + \cdots \tag{2.92}$$

[26] That is, $\delta_1^2 = 0$, $\delta_2^2 = 0$ and $[\delta_1, \delta_2] = 0$, where δ_1 is the transformation with parameter $\epsilon_2 = 0$, *etc.*. You should check this from (2.88) as an exercise!

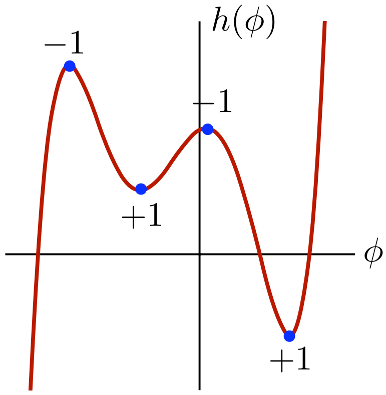


Figure 3: *The supersymmetric path integral receives contributions just from infinitesimal neighbourhoods of the critical points of $h(\phi)$. These alternately contribute ± 1 according to whether they are minima or maxima.*

where $c_* = \partial^2 h(\phi_*)$, so the action (2.87) becomes

$$S(\phi, \psi_i) = \frac{c_*^2}{2}(\phi - \phi_*)^2 + c_* \psi_1 \psi_2 + \cdots . \tag{2.93}$$

The higher order terms will be negligible as we focus on an infinitesimal neighbourhood of ϕ_*. Expanding the exponential in Grassmann variables the contribution of this critical point to the partition function is

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\phi \, \mathrm{d}^2\psi \, \mathrm{e}^{-c_*(\phi-\phi_*)^2/2} \left[1 - c_* \psi_1 \psi_2\right] &= \frac{c_*}{\sqrt{2\pi}} \int \mathrm{d}\phi \, \mathrm{e}^{-c_*(\phi-\phi_*)^2} \\ &= \frac{c_*}{\sqrt{c_*^2}} = \mathrm{sgn}\left(\partial^2 h|_*\right) . \end{aligned} \tag{2.94}$$

Summing over all the critical points, the full partition function thus becomes

$$\mathcal{Z}[h] \;=\; \sum_{\phi_* : \partial h|_{\phi_*} = 0} \mathrm{sgn}\left(\partial^2 h|_{\phi_*}\right) \tag{2.95}$$

and, as expected, is largely independent of the detailed form of h. In fact, if h is a polynomial of odd degree, then $\partial h = 0$ must have an even number of roots with $\partial^2 h$ being alternately > 0 and < 0 at each. Thus their contributions to (2.95) cancel pairwise and $\mathcal{Z}[h_{\mathrm{odd}}] = 0$ identically. On the other hand, if h has even degree then it has an odd number of critical points and we obtain $\mathcal{Z}[h_{\mathrm{ev}}] = \pm 1$, with the sign depending on whether $h \to \pm\infty$ as $|\phi| \to \infty$. (See figure 3.)

The fact that the partition function is so simple in this class of theories is a really remarkable result! To reiterate, we've found that for any form of polynomial $h(\phi)$, the partition function $\mathcal{Z}[h]$ is always either 0 or ± 1. If we imagined trying to compute $\mathcal{Z}[h]$ perturbatively, then for a non–quadratic h we'd still have to sum infinitely diagrams using the vertices in the action. In particular, we could certainly draw Feynman graphs Γ with arbitrarily high numbers of loops involving both ϕ and ψ_i fields, and these graphs would each contribute to the coefficient of some power of the coupling constants in the perturbative expansion. However, by an apparent miracle, we'd find that these graphs always cancel themselves out; the net coefficient of each such loop graph would be zero with the contributions from graphs where either ϕ or $\psi_1\psi_2$ run around the loop contributing with opposite sign. The reason for this apparent perturbative miracle is the localization property of the supersymmetric integral.

In supersymmetric theories in higher dimensions, complications such as spin mean the cancellation can be less powerful, but it is nonetheless still present and is responsible for making supersymmetric quantum theories 'tamer' than non–supersymmetric ones. As an important example, diagrams where the Higgs particle of the Standard Model runs around a loop can have the effect of destabilizing the mass of the Higgs, sending it up to a very high scale. (We'll understand this later on.) Until very recently, many physicists believed in the existence of a hypothesized supersymmetric partner to the Higgs that would cancel these dangerous loop diagrams, protecting the mass of the Higgs and thereby providing a rationale why the natural energy scale of the weak interactions is so much lower than the Planck scale. The ultimate mechanism for this cancellation would be just what we've seen above, though it's power is filtered through the layers of a much more complicated theory. Experiment has now shown that supersymmetry – if it is relevant to Nature at all – is not responsible for looking after the Higgs mass in this way[27].

I also want to point out that localization is useful for calculating much more than just the partition function. For $i \in \{1, 2, 3, \ldots\}$ suppose that $\mathcal{O}_i(\phi, \psi_i)$ is an operator that obeys $\delta\mathcal{O}_i = 0$, *i.e.* each operator is invariant under supersymmetry transformations (2.88). Then the (unnormalized) correlation function

$$\left\langle \prod_i \mathcal{O}_i \right\rangle = \int \frac{\mathrm{d}\phi\, \mathrm{d}^2\psi}{\sqrt{2\pi}}\, \mathrm{e}^{-S} \prod_i \mathcal{O}_i \tag{2.96}$$

again localizes to the critical points of h. Once again, this is because deforming $h \to h+g$ leaves the correlator invariant since the deformation affects the correlation function as

$$\left\langle \prod_i \mathcal{O}_i \right\rangle \overset{h\to h+g}{\longrightarrow} \left\langle \delta\mathcal{O}_g \prod_i \mathcal{O}_i \right\rangle = \left\langle \delta \left(\mathcal{O}_g \prod_i \mathcal{O}_i \right) \right\rangle = 0 \tag{2.97}$$

which vanishes by the same arguments as before. Here, we used the fact that $\delta\mathcal{O}_i = 0$ to write the operator on the *rhs* as a total derivative.

Of course, if any of the $\mathcal{O}_i$ are already of the form $\delta\mathcal{O}'$, so that this $\mathcal{O}_i$ is itself the supersymmetry transformation of some $\mathcal{O}'$, then $\langle \prod \mathcal{O}_i \rangle = 0$ which is not very interesting. The interesting operators are those which are δ-closed ($\delta\mathcal{O} = 0$) but not δ-exact ($\mathcal{O} \neq \delta\mathcal{O}'$). These operators describe the **cohomology** of the nilpotent operator δ. This is the starting–point for much of the mathematical interest in QFT: we can build supersymmetric QFTs that compute the cohomology of interesting spaces. For example, Donaldson's theory of

[27] Whether *anything* protects the Higgs mass, or whether it is just fine-tuned, is currently one of the outstanding mysteries of Beyond the Standard Model physics.

invariants of 4–manifolds that are homeomorphic but not diffeomorphic, and the Gromov–Witten generalization of intersection theory can both be understood as examples of (higher–dimensional) supersymmetric QFTs where the localization / cancellation is precise. In the absence of experimental evidence for a supersymmetric extension of the Standard Model, the close connections between supersymmetric QFTs and deep mathematics and the fact that supersymmetry helps tame otherwise intractable path integrals now provide the main reasons for studying supersymmetry.

Finally, let me remark that we'll also meet essentially the same localization idea again in a slightly different context later in this course when we study BRST quantization of gauge theories.