

ISyE 6739 Homework 2

Patrick Gardocki

2023-05-23

3-109

X = number of passengers that do not show up

X follows a binomial distribution

a)

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^4 \binom{125}{x} (0.1)^x (0.9)^{125-x} = 0.9961$$

```
1 - pbinom(4,125,0.1)
```

```
## [1] 0.9961414
```

b)

$$P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{x=0}^5 \binom{125}{x} (0.1)^x (0.9)^{125-x} = 0.9885$$

```
1 - pbinom(5,125,0.1)
```

```
## [1] 0.9885678
```

3-173

a)

ED = number of emergency department visits per day $\lambda T = 1.8$

$$P(ED > 5) = 1 - \sum_{x=0}^5 \frac{e^{-1.8} (1.8)^x}{x!} = 0.0103$$

```
1 - ppois(5,1.8)
```

```
## [1] 0.01037804
```

b)

EW = number of emergency department visits per week $\lambda T = 1.8 \times 7 = 12.6$

$$P(ED < 5) = \sum_{x=0}^4 \frac{e^{-12.6}(12.6)^x}{x!} = 0.00497$$

```
ppois(4,12.6)
```

```
## [1] 0.004979028
```

c)

ET = number of visits in T days $\lambda T = 1.8T$

$$P(ET \geq 1) = 1 - P(ET = 0) = 1 - \frac{e^{-1.8T}(1.8T)^0}{0!}$$
$$e^{-1.8T} = 0.01 = \ln(e^{-1.8T}) = \ln(0.01) \rightarrow T = 2.558 \text{ days}$$

```
log(0.01)/-1.8
```

```
## [1] 2.558428
```

d)

$$\text{Solve for } \lambda. P(ED > 5) = 0.1 = 1 - \sum_{x=0}^5 \frac{e^{-\lambda}(\lambda)^x}{x!}$$

4-4

$$f(x) = \frac{2}{x^3} \text{ for } x > 1$$

a)

$$P(X < 2) = \int_1^2 \frac{2}{x^3} dx = \left(\frac{-1}{x^2} \right) \Big|_1^2 = 0.75$$

```
integrand <- function(x) {2/x^3}
integrate(integrand,1,2)
```

```
## 0.75 with absolute error < 8.3e-15
```

b)

$$P(X > 5) = \int_5^\infty \frac{2}{x^3} dx = \left(\frac{-1}{x^2} \right) \Big|_5^\infty = 0.04$$

```
integrand <- function(x) {2/x^3}
integrate(integrand,5,Inf)
```

```
## 0.04 with absolute error < 5.8e-07
```

c)

$$P(4 < X < 8) = \int_4^8 \frac{2}{x^3} dx = \left(\frac{-1}{x^2} \right) \Big|_4^8 = 0.0468$$

```
integrand <- function(x) {2/x^3}  
integrate(integrand,4,8)
```

```
## 0.046875 with absolute error < 5.2e-16
```

d)

$$P(X < 4 \text{ or } X > 8) = \int_1^4 \frac{2}{x^3} dx + \int_8^\infty \frac{2}{x^3} dx = 0.9531$$

```
integrand <- function(x) {2/x^3}  
a <- integrate(integrand,1,4)  
b <- integrate(integrand,8,Inf)  
a[[1]] + b[[1]]
```

```
## [1] 0.953125
```

e)

Find x , where $P(X < x) = 0.95$.

$$P(X < x) = \int_1^x \frac{2}{x^3} dx = \left(\frac{-1}{x^2} \right) \Big|_1^x = \left(\frac{-1}{x^2} \right) - (-1) = 0.95$$
$$\therefore x = 4.4721$$

4-82

Let W be required water in million gallons. Let S be storage capacity of 350 million gallons.

$\mu = 310$ million; $\sigma = 45$ million

$X \sim N(310, 45^2)$

a)

$$P(W > S) = 1 - P(X \leq S) = 1 - \Phi\left(\frac{350-310}{45}\right) = 0.1870$$

```
pnorm(310,350,45)
```

```
## [1] 0.1870314
```

b)

$P(W > w) = P(Z > \frac{w-310}{45}) = 1 - \Phi(\frac{w}{45}) = 0.01$
 $\Phi(\frac{w-310}{45}) = 0.99 \rightarrow \frac{w-310}{45} = 2.326$ (From Cumulative Standard Normal Table)
 $x = 414.685$ million gallons

```
qnorm(0.99)*45+310
```

```
## [1] 414.6857
```

c)

$P(W < w) = P(Z < \frac{w-310}{45}) = \Phi(\frac{w-310}{45}) = 0.05$
 $\Phi(\frac{w-310}{45}) = 0.05 \rightarrow \frac{w-310}{45} = -1.644$ (From Cumulative Standard Normal Table)
 $x = 235.981$ million gallons

```
qnorm(0.05)*45+310
```

```
## [1] 235.9816
```

d)

Find μ , $X \sim N(\mu, 45^2)$
 $P(W > S) = 1 - P(X \leq S) = 1 - \Phi(\frac{350-\mu}{45}) = 0.01$
 $\Phi(\frac{350-\mu}{45}) = 0.99 \rightarrow \mu = 350 - 2.33 \times 45 = 245.15$
Given 1.4 million people, mean daily demand per person = 175.107 gallons

4-117

$\lambda = \frac{1}{E(X)} = \frac{1}{15}$ X = time until first call

a)

$$P(X > 30) = \int_{30}^{\infty} \frac{1}{15} e^{-\frac{x}{15}} dx = e^{-2} = 0.1353$$

```
integrand <- function(x) {(1/15)*exp(-x/15)}  
integrate(integrand,30,Inf)
```

```
## 0.1353353 with absolute error < 3.6e-05
```

b)

$$P(X \leq 10) = P(X > 10) \text{ Since expect 0 calls in first 10 minutes } P(X > 10) = \int_{10}^{\infty} \frac{1}{15} e^{-\frac{x}{15}} dx = e^{-2/3} = 0.5134$$

```
integrand <- function(x) {(1/15)*exp(-x/15)}
integrate(integrand,10,Inf)
```

```
## 0.5134171 with absolute error < 8e-07
```

c)

$$P(5 < X < 10) = \int_5^{10} \frac{1}{15} e^{-\frac{x}{15}} dx = e^{-1/3} - e^{-2/3} = 0.2031$$

```
integrand <- function(x) {(1/15)*exp(-x/15)}
integrate(integrand,5,10)
```

```
## 0.2031142 with absolute error < 2.3e-15
```

d)

$$P(X < x) = 0.90 \rightarrow P(X < x) = \int_0^x \frac{1}{15} e^{-\frac{x}{15}} dx = 1 - e^{-x/15} = 0.90 \rightarrow \ln(0.10) \times -15$$

$\therefore x = 34.5387 \text{ minutes}$

```
log(0.1)*-15
```

```
## [1] 34.53878
```

5-6

a)

$$P(X = x, Y = y) = \frac{\binom{60}{4-x-y} \binom{30}{y} \binom{10}{x}}{\binom{100}{4}} \text{ for } X \geq 0, Y \geq 0, X + Y \leq 4$$

The table below shows the PMF table for $x, y \in \{0, 1, 2, 3, 4\}$

```
df = data.frame()
library(kableExtra)
library(knitr)
for (i in 0:4){
  c0 = choose(60,4-i)*choose(30,0)*choose(10,i)/choose(100,4)
  c1 = choose(60,4-1-i)*choose(30,1)*choose(10,i)/choose(100,4)
  c2 = choose(60,4-2-i)*choose(30,2)*choose(10,i)/choose(100,4)
  c3 = choose(60,4-3-i)*choose(30,3)*choose(10,i)/choose(100,4)
  c4 = choose(60,4-4-i)*choose(30,4)*choose(10,i)/choose(100,4)
  out = c(c0,c1,c2,c3,c4)
  df = rbind(df,out)
}

colnames(df)<-c("x=0", "x=1", "x=2", "x=3", "x=4")
row.names(df) <- c("y=0", "y=1", "y=2", "y=3", "y=4")
```

```
kable(df,"latex",align="c", caption="PMF",label="5-6a")%>%
  kable_styling(latex_options = "hold_position")
```

Table 1: PMF

	x=0	x=1	x=2	x=3	x=4
y=0	0.1243578	0.2618059	0.1963545	0.0621234	0.0069889
y=1	0.0872686	0.1354169	0.0665608	0.0103539	0.0000000
y=2	0.0203125	0.0206568	0.0049921	0.0000000	0.0000000
y=3	0.0018362	0.0009181	0.0000000	0.0000000	0.0000000
y=4	0.0000536	0.0000000	0.0000000	0.0000000	0.0000000

b)

$f_x(x)$ is the sum of the columns in the PMF table.

$f_x(0) = 0.2338$; $f_x(1) = 0.4187$; $f_x(2) = 0.2679$; $f_x(3) = 0.0724$; $f_x(4) = 0.0069$

```
colSums(df[, na.rm=TRUE])
```

```
##           x=0           x=1           x=2           x=3           x=4
## 0.233828714 0.418797697 0.267907350 0.072477351 0.006988887
```

c)

$$E(X) = \sum_0^4 x_i * f(x_i) = 1.2001$$

```
0*0.2338 + 1*0.4188 + 2*0.2679 + 3*0.0725 + 4*0.0070
```

```
## [1] 1.2001
```

d)

$$f_{Y|3}(y) = \frac{f_{xy}(3,y)}{f_x(3)}$$

From 5-6 part b, using when $x=3$, and $f_x(3) = 0.0724$, the PMF

```
df1 = data.frame(y=c("0","1","2","3","4"),f=c(0.857,0.143,0,0,0))
names(df1) <- c("y", "$f_{Y|3}(y)$")

kable(df1,"latex",align="c", caption="PMF",label="5-6a",escape = FALSE)%>%
  kable_styling(latex_options = "hold_position")
```

Table 2: PMF

y	$f_{Y 3}(y)$
0	0.857
1	0.143
2	0.000
3	0.000
4	0.000

e)

$$E(Y|X=3) = \sum_0^4 y_i * f_{Y|3}(y)_i = 0 * 0.857 + 1 * 0.143 = 0.143$$

f)

$$V(Y|X=3) = \sum_0^4 (y_i^2 * f_{Y|3}(y)_i) - E(Y|X=3)^2 = 0^2 * 0.857 + 1^2 * 0.143 - 0.143^2 = 0.123$$

g)

X and Y are not independent.

$f_{xy}(x, y) \neq f_x(x) \times f_y(y)$ for all x and y

For example, $f_{xy}(1, 1) = 0.1354$; $f_x(1) = 0.4187$; $f_y(1) = 0.2994$

$0.1354 \neq 0.4187 \times 0.2994$

Q2

a)

The probability of a correct classification follows a binomial distribution. $P(X) = \binom{3}{x} \times 0.98^x \times 0.02^{3-x}$

```
df = data.frame()
library(kableExtra)
library(knitr)
c0 = dbinom(0:3,3,0.98)
for (x in 1:4){

  out = c(c0[x])
  df = rbind(df,out)
}

names(df) <- c("P(x)")
row.names(df) <- c("$f(0)$", "$f(1)$", "$f(2)$", "$f(3)$")

kable(df,"latex",align="c", caption="PMF",label="Q2",escape = FALSE)%>%
  kable_styling(latex_options = "hold_position")
```

Table 3: PMF

	P(x)
$f(0)$	0.000008
$f(1)$	0.001176
$f(2)$	0.057624
$f(3)$	0.941192

b)

```

c0 = pbinom(0:3,3,.98)
df = data.frame()
library(kableExtra)
library(knitr)

for (x in 1:4){

  out = c(c0[x])
  df = rbind(df,out)
}

names(df) <- c("P(x)")
row.names(df) <- c("$F(0)$", "$F(1)$", "$F(2)$", "$F(3)$")

kable(df,"latex",align="c", caption="CDF",label="Q2",escape = FALSE)%>%
  kable_styling(latex_options = "hold_position")

```

Table 4: CDF

	P(x)
$F(0)$	0.000008
$F(1)$	0.001184
$F(2)$	0.058808
$F(3)$	1.000000

$$F(x) = 0 \text{ for } x \leq 0$$

$$F(x) = 1 \text{ for } x \geq 3$$

c)

$$E(X) = \sum_{x=0}^3 x * f(x) = 2.94$$

$$V(X) = \sum_{x=0}^3 x^2 * f(x) - E(X)^2 = 0.0588$$

```
0*.000008 + 1*0.001176 + 2*0.057624 + 3*.941192
```

```
## [1] 2.94
```



```
0^2*.000008 + 1^2*0.001176 + 2^2*0.057624 + 3^2*.941192 - 2.94^2
```

```
## [1] 0.0588
```