ME6406 Machine Vision

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Part 3A
Camera model and calibration

http://kmlee.gatech.edu/me6406

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Course Outline

- Introduction and low-level processing
  - Physics of digital images, histogram equalization, segmentation, edge detection, linear filters.
- Model-based Vision
  - Hough transform, pattern representation, matching
- **Geometric methods**
  - **Camera model, calibration, pose estimation**
- Neural network for machine vision
  - Basics, training algorithms, and applications
- Color images and selected topics
  - Physics, perception, processing and applications

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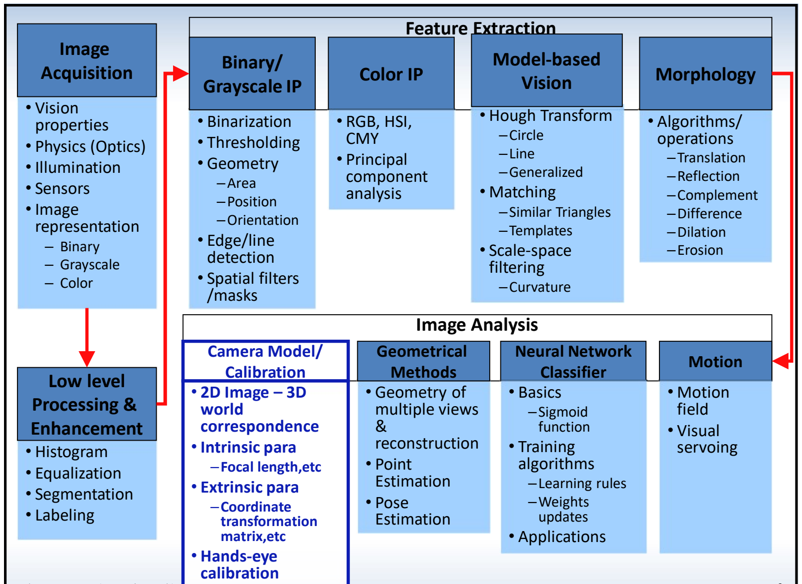

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Reading Materials:

5. Tsai, R. "A Versatile Camera Calibration Technique for High-accuracy 3D Machine Vision Metrology using Off-the-shelf TV Cameras and Lenses," IEEE Trans. on Robotics and Automation, Vol. 3, No.4, August 1987, pp: 323- 344

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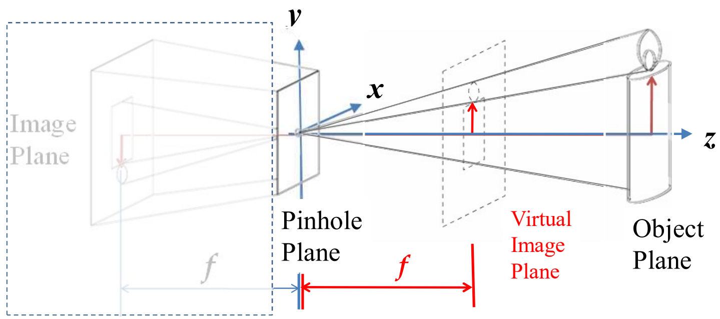
Pinhole Camera Model

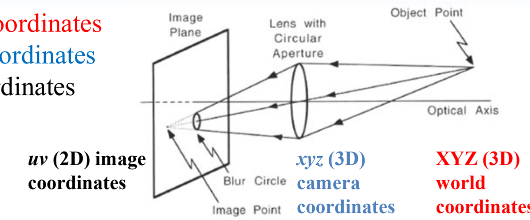


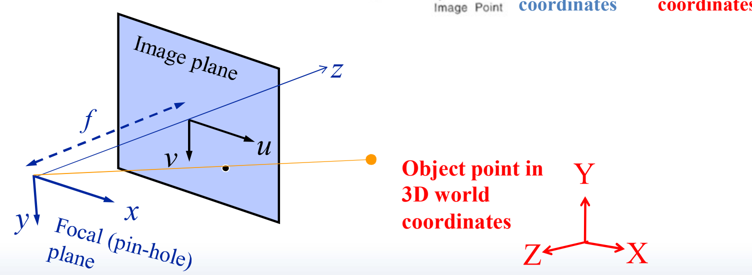
Camera Model

XYZ (3D) world coordinates

xyz (3D) camera coordinates

uv (2D) image coordinates





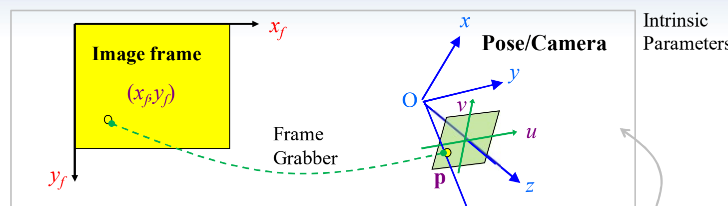
❑ Reasons for Camera Calibration

- Need to recover **3D quantitative measures** about the observed scene from **2D images**.
- **Model and predict** the performance or accuracy of any machine vision algorithms
- Determine the **camera location** relative to the calibration board (or the working plane)
- **Basis for other calibration**; robot kinematics, hand-eye relationship, geometric calibrations.

❑ Definition

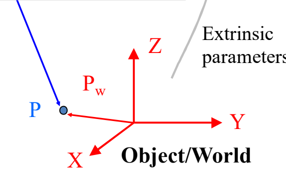
- The problem of determining the elements that govern the relationship or transformation between the 2D image that a camera sees and the 3D of the observed scene.
- Two kinds of parameters defining this 2D/3D relationship:
  - Intrinsic
  - Extrinsic

❑ Camera Model



- **Coordinate Systems** (**Right Hand Rule**)
  - Frame coordinates $(x_f\ y_f)$ pixels
  - Imaging sensor coordinates (u, v) in m (μm)
  - Camera coordinates (x, y, z) in m
  - World coordinates (X, Y, Z) in m

  Physical units



- **Camera Parameters**
  - Intrinsic Parameters (of the camera and the frame grabber): link the **frame coordinates** of an image point with its corresponding **camera coordinates.**
  - Extrinsic parameters: define the relative location and orientation of the **camera coordinate system** with the **world coordinate system.**

❑ Intrinsic parameters

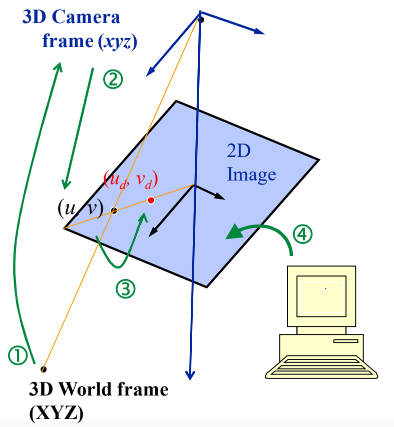
- Parameters that characterize the inherent geometric properties of the camera and optics:
  - Image center
  - Image X and Y scale factors
  - Lens principal distance (effective **focus length**)
  - **Lens distortion coefficients**

❑ Externsic parameters

- Parameters that indicates the position and orientation of the camera with respect to the world coordinate system:
  - Translation (T_x, T_y and T_z)
  - Rotation about X, Y and Z axes.

Tsai Camera Model

(Radial lens distortion assumption;)



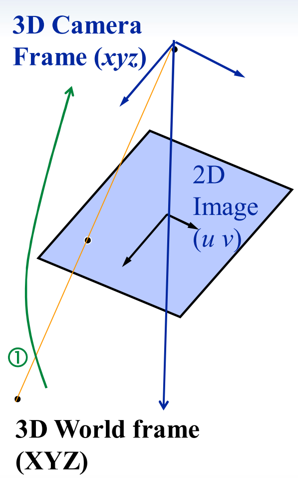
Step 1
Rigid body transformation from **world frame** to camera frame
Parameters to be calibrated: [R], T

Step 2
Perspective projection with pin hole geometry
Parameters to be calibrated: *f*

Step 3
Radial lens distortion correction
Parameters to be calibrated: k_1

Step 4
Relation between physical and computer coordinates
Parameters to be calibrated: scaling

Tsai Camera Model



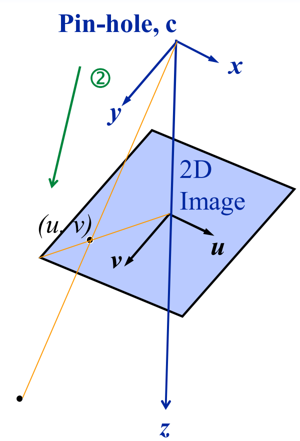
Step 1: Rigid body transformation from world frame to camera frame.

Intermediate Knowns

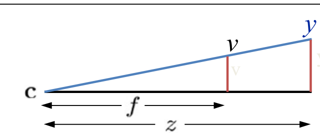
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \underbrace{\begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}}_{[R]}\begin{pmatrix}X\\Y\\Z\end{pmatrix}+\underbrace{\begin{pmatrix}T_x\\T_y\\T_z\end{pmatrix}}_{T}$$

Parameters to be calibrated: **[R], T**

Tsai Camera Model



Step 2: Perspective projection with pin-hole geometry



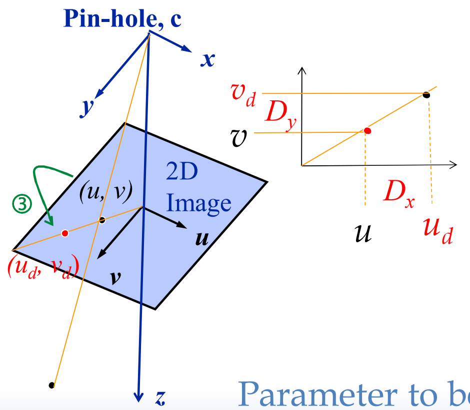
Similarly, Intermediate

$$u = f\frac{x}{z} \qquad v = f\frac{y}{z}$$

Parameter to be calibrated: ***f***

Tsai Camera Model

Step 3: Radial lens distortion correction



Assumption 1: Radial lens distortion

$$\begin{bmatrix} u_d \\ v_d \end{bmatrix} = L(r) \begin{bmatrix} u \\ v \end{bmatrix}$$

where $L = 1 + k_1 r^2 + \cdots$

Actual measurements

$$u_d = u(1 + k_1 r_d^2)$$
$$v_d = v(1 + k_1 r_d^2)$$

where $r_d^2 = u_d^2 + v_d^2$

Parameter to be calibrated: k_1

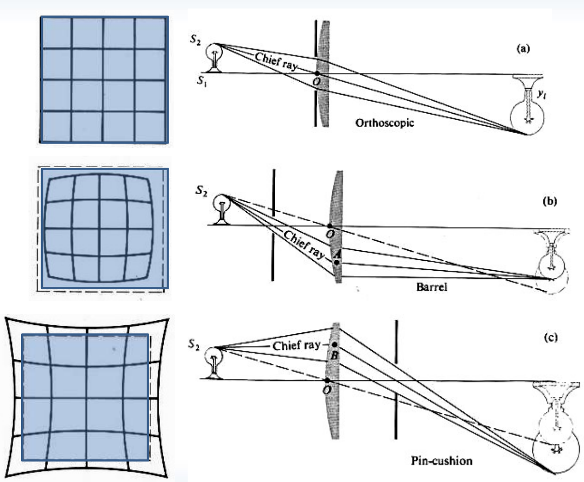
Radial Distortion (straight lines curve around the image)

Radial Distortion (straight lines curve around the image)

No distortion

Barrel

Pin cushion



Correcting Radial Distortion (example)

- Radial distortion of the image
  - Caused by imperfect lenses
  - Deviations are most noticeable for rays that pass through the edge of the lens

Distorted

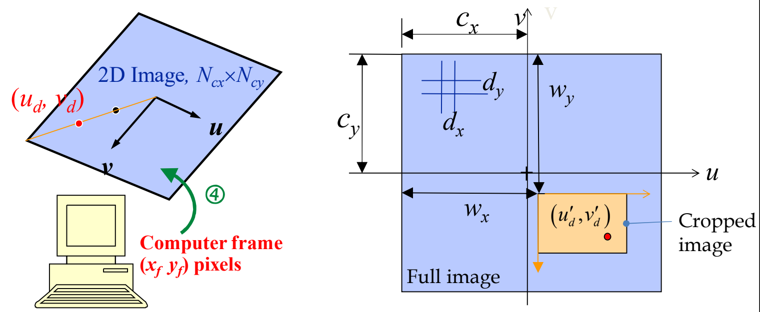
After correction

courtesy of Shawn Becker

Tsai's Camera Model

Step 4: Relation between physical and computer coordinates.

Frame $(x_f\ y_f)$ pixels
Image (u, v) in m (μm)
Camera (x, y, z) in m
World (X, Y, Z) in m



Given $\left(c_x, c_y\right), \left(w_x, w_y\right), \left(dx, dy\right)$, find $\left(u_d, v_d\right)$ from $\left(u'_d, v'_d\right)$.

$$\begin{bmatrix} u_d \\ v_d \end{bmatrix} = \begin{bmatrix} \left(w_x + u'_d - c_x\right) dx \\ -\left(w_y + v'_d - c_y\right) dy \end{bmatrix}$$

Tsai's Camera Model (Step 4 example)

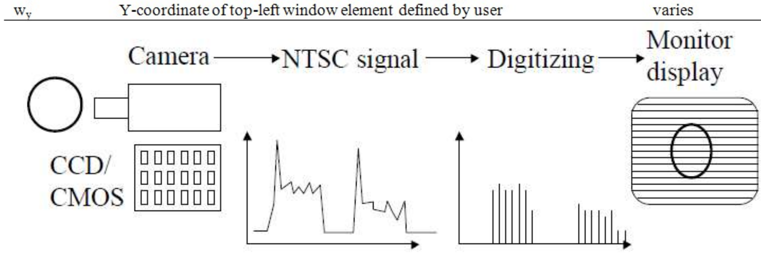
Table 1 PULNIX Camera Properties

Imager	2/3 inch progressive scanning interline transfer CCD
No. Pixels	768 (H) x 484 (V)
Cell Size	11.6 μm x 13.6 μm progressive scanning
Scanning	525 lines, 30 Hz or 60 Hz 2:1 interlace Internal/External autoswitch
Sync	HD/VD 4.0 ∀p-p impedance 4.7 kΩ VD=interlace/non-interlace, HD=15.734 kHz+/- 5%
Dataclock Output	14.31818 MHz
TV Resolution	470 (H) x 484 (V) analog 760(H) x 484 (V) digital sampling
S/N Ratio	50 dB min. (AGC=off)
Min. Illumination	10.0 lux, f=1.4 (no shutter) sensitivity 10 μV/e-
Size (WxHxL)	46 x 51 x 171.7 mm 1.81 x 2.0 x 6.766 inches
Weight	225 grams (4.3 oz)
Power Requirement	12 V DC 500 mA
Lens Mount	C Mount
Gamma	0.45 or 1.0 (0.45) std
Operating Temperature	-10° C to 50° C

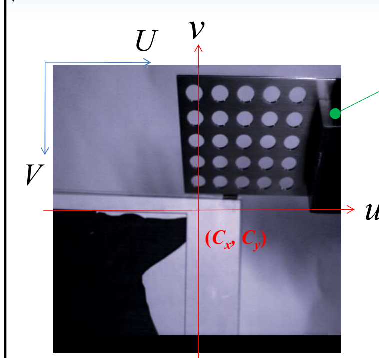
Tsai Camera Model (Step 4 example)

Table 2 Camera Sensor and Variables, Definitions, and Values

Variable	Definition	Value
dx	Center to center distance between adjacent sensor elements in the x scanline	11.6 μm
dy	Center to center distance between adjacent sensor elements in the y scanline	13.6 μm
N_{cx}	Number of sensor elements in the x direction	768 pixels
N_{cy}	Number of sensor elements in y direction	484 pixels
N_{fx}	Number of pixels in a line as sampled by the computer	512 pixels
N_{fy}	Number of rows (sensor elements plus blank rows) in y direction	512 pixels
C_x	Camera center x-coordinate taken to be the center of the camera sensor	768/2 = 384 pixels
C_y	Camera center y-coordinates taken to be the center of the camera sensor	484/2 = 242 pixels
w_x	X-coordinate of top-left window element defined by user	varies
w_y	Y-coordinate of top-left window element defined by user	varies



Example 1: **Use Camera Model Step 4 to convert from pixels to physical units**



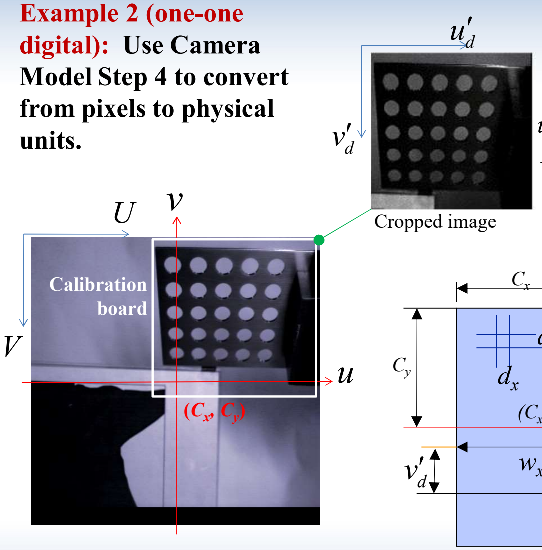
$$u_d = \left(U - C_x\right) dx'$$

$$v_d = \left(V - C_y\right) dy'$$

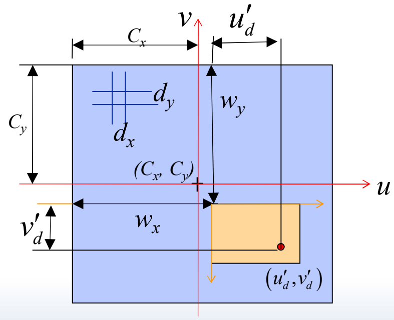
$$dx' = dx\frac{N_{cx}}{N_{fx}} = 11.6\left(\frac{768}{512}\right)\mu m$$

$$dy' = dy\frac{N_{cy}}{N_{fy}} = 13.6\left(\frac{484}{512}\right)\mu m$$

Example 2 (one-one digital): Use Camera Model Step 4 to convert from pixels to physical units.



$$u_d = \left(w_x + u'_d - C_x\right) dx$$
$$v_d = -\left(w_y + v'_d - C_y\right) dy$$



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Two-stage approach

Stage 1: Radial Alignment Constraint

1a) Solve for five parameters:

$r_{11}/T_y,\ r_{21}/T_y,\ r_{12}/T_y,\ r_{22}/T_y,\ T_x/T_y$

1b) Find T_y

1c) Solve for $[\mathbf{R}]$, T_x, T_y

Planar parameters

Out of plane parameters

Note: Only three of nine r_{ij} are independent.

$$[\mathbf{R}]^T = [\mathbf{R}]^{-1}$$

Stage 2: Perspective Constraint

Given $[\mathbf{R}]$, T_x, T_y solve for f, k, T_z

Depth related parameters

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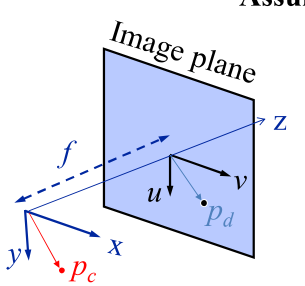
Two-stage approach (Stage 1a)

1a) Solve for five parameters:

$r_{11}/T_y,\ r_{21}/T_y,\ r_{12}/T_y,\ r_{22}/T_y,\ T_x/T_y$

Assumption 2: *xy* plane // *uv* plane,



$$\mathbf{p}_c \times \mathbf{p}_d = 0$$

$$\begin{bmatrix} i & j & k \\ x & y & 0 \\ u & v & 0 \end{bmatrix} = 0$$

$$xv - yu = 0 \qquad (1)$$

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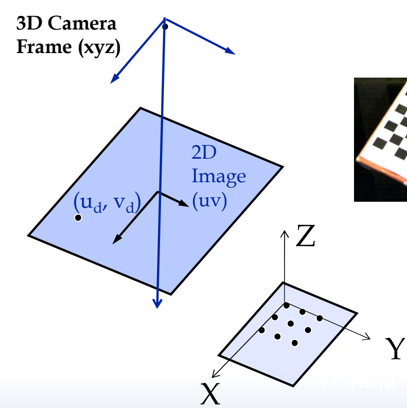
Two-stage approach (Stage 1a)

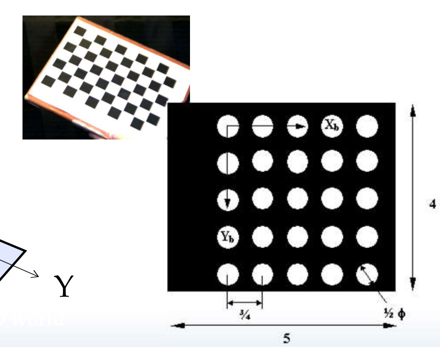
1a) Solve for five parameters:

$r_{11}/T_y,\ r_{21}/T_y,\ r_{12}/T_y,\ r_{22}/T_y,\ T_x/T_y$

Assumption 3:

2D Calibration board, $Z=0$





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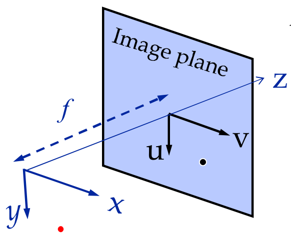
Two-stage approach (Stage 1a)

1a) Solve for five parameters:
$r_{11}/T_y,\ r_{21}/T_y,\ r_{12}/T_y,\ r_{22}/T_y,\ T_x/T_y$



Assumption 2: xy plane // uv plane,

$$xv_d - yu_d = 0 \qquad (1)$$

From Camera Model **Step 1**:

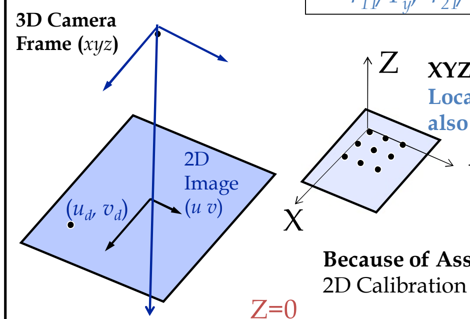
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} \qquad (2)$$

Substituting (2) into (1):

$$(r_{11}X + r_{12}Y + r_{13}Z + T_x)v_d - (r_{21}X + r_{22}Y + r_{23}Z + T_y)u_d = 0$$

Two-stage approach (Stage 1a)

1a) Solve for five parameters:
$r_{11}/T_y,\ r_{21}/T_y,\ r_{12}/T_y,\ r_{22}/T_y,\ T_x/T_y$



XYZ (world coordinate system)
Located on the calibration board; also called $(x_b\ y_b\ z_b = 0)$.

Because of Assumption 3:
2D Calibration board, Z=0

$$(r_{11}X_i + r_{12}Y_i + \cancel{r_{13}Z_i}^{\,Z=0} + T_x)v_{di} - (r_{21}X_i + r_{22}Y_i + \cancel{r_{23}Z_i}^{\,Z=0} + T_y)u_{di} = 0$$

6 unknowns: $r_{11},\ r_{12},\ r_{21},\ r_{22},\ T_x,\ T_y$

Two-stage approach (Stage 1a)

1a) Solve for five parameters:
$r_{11}/T_y,\ r_{21}/T_y,\ r_{12}/T_y,\ r_{22}/T_y,\ T_x/T_y$

$$(r_{11}X_i + r_{12}Y_i + T_x)v_{di} - (r_{21}X_i + r_{22}Y_i + T_y)u_{di} = 0$$

$$\underbrace{[\mathbf{A}]}_{n\times 6} \underbrace{\begin{bmatrix} r_{11} \\ r_{12} \\ r_{21} \\ r_{22} \\ T_x \\ T_y \end{bmatrix}}_{6\times 1} = 0$$

6 unknowns: $r_{11},\ r_{12},\ r_{21},\ r_{22},\ T_x,\ T_y$

Homogeneous equation: $[\mathbf{A}]\mathbf{x}^* = 0$

For calibration, over-determined system
$n \geq 6$ (trivial solutions, $\mathbf{x}^* = 0$)

Two-stage approach (Stage 1a)

1a) Solve for five parameters using least square (pseudo-inverse):
$\mu_1 = \mu_{11} = r_{11}/T_y;\ \mu_2 = \mu_{12} = r_{12}/T_y;\ \ \mu_3 = \mu_{21} = r_{21}/T_y;\ \mu_4 = \mu_{22} = r_{22}/T_y;$
$\mu_5 = T_x/T_y$

$$\left[X_i\left(\frac{r_{11}}{T_y}\right) + Y_i\left(\frac{r_{12}}{T_y}\right) + \left(\frac{T_x}{T_y}\right)\right]v_{di} - \left[X_i\left(\frac{r_{21}}{T_y}\right) + Y_i\left(\frac{r_{22}}{T_y}\right) + 1\right]u_{di} = 0$$

$$\underbrace{\begin{bmatrix} X_1 v_{d1} & Y_1 v_{d1} & -X_1 u_{d1} & -Y_1 u_{d1} & v_{d1} \\ X_2 v_{d2} & Y_2 v_{d2} & -X_2 u_{d2} & -Y_2 u_{d2} & v_{d2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_n v_{dn} & Y_2 v_{d2} & -X_n u_{dn} & -Y_n u_{dn} & v_{dn} \end{bmatrix}}_{\substack{[\mathbf{A}] \\ n\times 5}} \underbrace{\begin{bmatrix} r_{11}/T_y \\ r_{12}/T_y \\ r_{21}/T_y \\ r_{22}/T_y \\ T_x/T_y \end{bmatrix}}_{\substack{\boldsymbol{\mu} \\ 5\times 1}} = \underbrace{\begin{bmatrix} u_{d1} \\ u_{d2} \\ \vdots \\ u_{dn} \end{bmatrix}}_{\substack{\mathbf{b} \\ n\times 1}}$$

$$[\mathbf{A}]\boldsymbol{\mu} = \mathbf{b}$$

$$\boldsymbol{\mu} = \mathbf{A}^{+}\mathbf{b}$$

Two-stage approach (Stage 1b, 1c)

From pseudo-inverse solutions to Stage 1a): $\mu_1=\mu_{11y}$; $\mu_2=\mu_{12}$; $\mu_3=\mu_{21}$; $\mu_4=\mu_{22}$

Given $\mu_{ij} = {}^{r_{ij}}\!/_{T_y}$ $(i, j=1,2)$ and $\mu_5 = {}^{T_x}\!/_{T_y}$

$$[\mathbf{R}] = \begin{bmatrix} \boxed{\mu_{11}T_y \quad \mu_{12}T_y \quad r_{13}} \\ \boxed{\mu_{21}T_y \quad \mu_{22}T_y \quad r_{23}} \\ r_{31} \quad r_{32} \quad r_{33} \end{bmatrix} \begin{matrix} = \alpha^T \\ = \beta^T \\ \end{matrix} \qquad \boldsymbol{\alpha} = \begin{bmatrix} \mu_{11}T_y \\ \mu_{12}T_y \\ r_{13} \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mu_{21}T_y \\ \mu_{22}T_y \\ r_{23} \end{bmatrix}$$

Find T_y using the orthogonality of [R]

$$\boldsymbol{\alpha}\bullet\boldsymbol{\beta} = 0$$

$$\boldsymbol{\alpha}\bullet\boldsymbol{\alpha} = 1 \Rightarrow r_{13}^2 = 1 - T_y^2\left(\mu_{11}^2 + \mu_{12}^2\right)$$

$$\boldsymbol{\beta}\bullet\boldsymbol{\beta} = 1 \Rightarrow r_{23}^2 = 1 - T_y^2\left(\mu_{21}^2 + \mu_{22}^2\right)$$

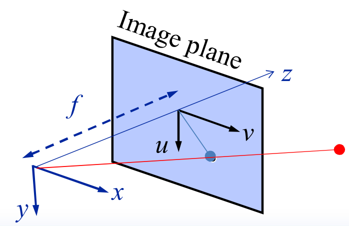
$$\begin{array}{ccccc} \boldsymbol{\gamma} & & \boldsymbol{\alpha} & & \boldsymbol{\beta} \\ \begin{bmatrix} r_{31} \\ r_{32} \\ r_{33} \end{bmatrix} & = & \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} & \times & \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} \end{array}$$

Two-stage approach (Stage 1(b, c)

Find T_y using the orthogonality of [R]

$$\boldsymbol{\alpha}\bullet\boldsymbol{\beta} = 0 \xrightarrow{\text{Step1}} \text{Solve for } T_y^2$$

$$\left.\begin{array}{l} \boldsymbol{\alpha}\bullet\boldsymbol{\alpha} = 1 \Rightarrow r_{13}^2 = 1 - T_y^2\left(\mu_{11}^2 + \mu_{12}^2\right) \\ \boldsymbol{\beta}\bullet\boldsymbol{\beta} = 1 \Rightarrow r_{23}^2 = 1 - T_y^2\left(\mu_{21}^2 + \mu_{22}^2\right) \end{array}\right\} \text{Solve for } r_{13}^2 \text{ and } r_{23}^2$$



Multiple "±" solutions exist !

T_y may be positive or negative but

1) the image and object are in the same quadrant; and
2) f and T_z are positive (based on the definition and the fact that object is in front of the camera.

Two-stage approach (Stage 1b, 1c)

Find T_y using the orthogonally of [R]

$$\boldsymbol{\alpha} = \begin{bmatrix} \mu_{11}T_y \\ \mu_{12}T_y \\ r_{13} \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mu_{21}T_y \\ \mu_{22}T_y \\ r_{23} \end{bmatrix}$$

$$\boldsymbol{\alpha}\bullet\boldsymbol{\beta} = 0$$

$$T_y^2\mu_{11}\mu_{21} + T_y^2\mu_{12}\mu_{22} + r_{13}r_{23} = 0$$

To avoid square roots,

$$T_y^4\left[\mu_{11}\mu_{21} + \mu_{12}\mu_{22}\right]^2 = \left[-r_{13}r_{23}\right]^2$$

$$\boldsymbol{\alpha}\bullet\boldsymbol{\alpha} = 1 \Rightarrow r_{13}^2 = 1 - T_y^2\left(\mu_{11}^2 + \mu_{12}^2\right)$$

$$\boldsymbol{\beta}\bullet\boldsymbol{\beta} = 1 \Rightarrow r_{23}^2 = 1 - T_y^2\left(\mu_{21}^2 + \mu_{22}^2\right)$$

$$T_y^4\left[\cancel{(\mu_{11}\mu_{21})^2} + 2(\mu_{11}\mu_{21})(\mu_{12}\mu_{22}) + \cancel{(\mu_{12}\mu_{22})^2}\right] = \underbrace{\left[1 - T_y^2\left(\mu_{11}^2+\mu_{12}^2\right)\right]\left[1 - T_y^2\left(\mu_{21}^2+\mu_{22}^2\right)\right]}$$

$$= 1 - T_y^2\left(\mu_{11}^2 + \mu_{12}^2 + \mu_{21}^2 + \mu_{22}^2\right) + T_y^4\left(\mu_{11}^2+\mu_{12}^2\right)\left(\mu_{21}^2+\mu_{22}^2\right)$$

$$= 1 - T_y^2 U + T_y^4\left(\mu_{11}^2+\mu_{12}^2\right)\left(\mu_{21}^2+\mu_{22}^2\right)$$

$$= 1 - T_y^2 U + T_y^4\left(\cancel{\mu_{11}^2\mu_{21}^2} + \mu_{11}^2\mu_{22}^2 + \mu_{12}^2\mu_{21}^2 + \cancel{\mu_{12}^2\mu_{22}^2}\right)$$

$$\boxed{\text{Let } U = \mu_{11}^2 + \mu_{12}^2 + \mu_{21}^2 + \mu_{22}^2 = \sum_{j=1}^{4}\mu_j^2}$$

Two-stage approach (Stage 1b)

Find T_y using the orthogonally of [R]

$$T_y^4\left[2(\mu_{11}\mu_{21})(\mu_{12}\mu_{22})\right] = 1 - T_y^2 U + T_y^4\left(\mu_{11}^2\mu_{22}^2 + \mu_{12}^2\mu_{21}^2\right)$$

$$T_y^4\underbrace{\left[\mu_{11}^2\mu_{22}^2 - 2(\mu_{11}\mu_{21})(\mu_{12}\mu_{22}) + \mu_{12}^2\mu_{21}^2\right]}_{(\mu_{11}\mu_{22} - \mu_{12}\mu_{21})^2} - UT_y^2 + 1 = 0$$

$$(\mu_1\mu_4 - \mu_2\mu_3)^2 T_y^4 - UT_y^2 + 1 = 0$$

$$T_y^2 = \begin{cases} \dfrac{1}{U} & \text{if } \mu_1\mu_4 = \mu_2\mu_3 \\ \dfrac{U - \left[U^2 - 4(\mu_1\mu_4 - \mu_2\mu_3)^2\right]^{1/2}}{2(\mu_1\mu_4 - \mu_2\mu_3)^2} & \text{if } \mu_1\mu_4 \neq \mu_2\mu_3 \end{cases}$$

where $U = \sum_{j=1}^{4}\mu_j^2$

From pseudo-inverse solutions to Stage 1a): $\mu_1=\mu_{11y}$; $\mu_2=\mu_{12}$; $\mu_3=\mu_{21}$; $\mu_4=\mu_{22}$

Only the negative sign is relevant. Proof is given in Tsai' paper (Appendix)

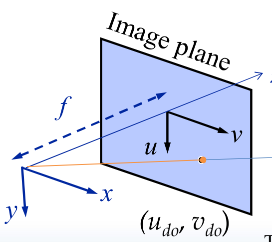
Two-stage approach (Stage 1b)

Let $T_y = \left(T_y^2\right)^{1/2}$ then $r_{ij} = \mu_{ij} T_y$ $(i, j = 1,2)$ and $T_x = \mu_5 T_y$

To determine the sign of T_y, select one object point P(X_0, Y_0, 0):

$$\begin{bmatrix} \xi_x \\ \xi_y \end{bmatrix} = \begin{bmatrix} r_{11} X_o + r_{12} Y_o + T_x \\ r_{21} X_o + r_{22} Y_o + T_y \end{bmatrix}$$

If (ξ_x, ξ_y) have the sign as (u_{do}, v_{do}), then T_y has the correct sign. Otherwise, negate it.



Select test point P(X_0, Y_0, 0)

The image and object are in the same quadrant;

Two-stage approach (Stage 1c)

1c) Solve for [**R**] T_x, T_y

The **unknown signs s_1 and s_2** are determined from the orthogonal property of [**R**].

$$[\mathbf{R}] = \begin{bmatrix} r_{11} & r_{12} & s_1\sqrt{1 - r_{11}^2 - r_{12}^2} \\ r_{21} & r_{22} & s_2\sqrt{1 - r_{21}^2 - r_{22}^2} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$(\boldsymbol{\alpha} \bullet \boldsymbol{\beta} =) \boldsymbol{\alpha}^T \boldsymbol{\beta} = 0$$

$$\Rightarrow \begin{bmatrix} r_{11} & r_{12} & r_{13} \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} = 0$$

$$\Rightarrow \underbrace{\left(r_{11} r_{21} + r_{12} r_{22}\right)}_{a} + s_1 s_2 \underbrace{\sqrt{1 - r_{11}^2 - r_{12}^2}\sqrt{1 - r_{21}^2 - r_{22}^2}}_{b} = 0$$

The two factors, a and b, must be equal and opposite.

Thus, choose $s_1 = s_2 = +1$, check $\operatorname{sgn}\left(r_{11} r_{21} + r_{12} r_{22}\right) \overset{?}{=}$ negative

If yes, keep the signs; otherwise $s_2 = -1$.

Summary of Stage 1

1a) Calculate T_y^2

1b) Let $T_y = \left(T_y^2\right)^{1/2}$ Determine the sign of T_y

$$\begin{bmatrix} \xi_x \\ \xi_y \end{bmatrix} = \begin{bmatrix} r_{11} X_o + r_{12} Y_o + T_x \\ r_{21} X_o + r_{22} Y_o + T_y \end{bmatrix}$$

If (ξ_x, ξ_y) have the sign as (u_{do}, v_{do}), then T_y has the correct sign.
Otherwise, negate it.

1c) Check $\operatorname{sgn}\left(r_{11} r_{21} + r_{12} r_{22}\right) \overset{?}{=}$ negative

Choose $s_1 = s_2 = +1$: If yes, keep the signs; otherwise $s_2 = -1$.

$$r_{13} = s_1\sqrt{1 - T_y^2\left(\mu_{11}^2 + \mu_{12}^2\right)} \qquad r_{23} = s_2\sqrt{1 - T_y^2\left(\mu_{21}^2 + \mu_{22}^2\right)}$$

Note: The signs of (s_1, s_2) and all associated signs may need to be adjusted after computing f and T_z, which must be positive; two other possibilities: $s_1 = s_2 = -1$ or $s_1 = -s_2 = -1$).

Calculate $\boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$ where $\boldsymbol{\gamma} = \begin{bmatrix} r_{31} & r_{32} & r_{33} \end{bmatrix}^T$

Two-stage approach (Stage 2)

Stage 2: Perspective Constraint
Given [**R**] T_x, T_y solve for f, k, T_z

Recall Camera Model Step 1 and Steps 2 and 3:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} \Rightarrow \frac{x_i}{z_i} = \frac{r_{11} X_i + r_{12} Y_i + T_x}{r_{31} X_i + r_{32} Y_i + T_z}$$

Z=0 (2D board)

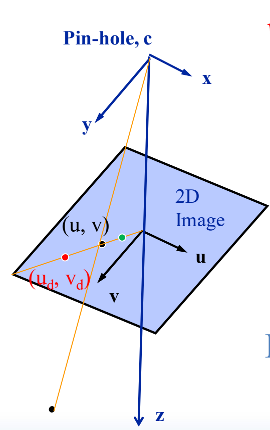
$$\left.\begin{aligned} u &= f\frac{x}{z} \\ u &= \frac{u_d}{\left(1 + k_1 r_d^{\,2}\right)} \end{aligned}\right\} \quad \frac{x}{z} = \frac{u_d}{f\left(1 + k_1 r_d^2\right)}$$

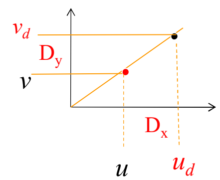
Eliminating u,

$$\boxed{u_d = f\left(1 + k_1 r_d^2\right)\frac{r_{11} X_i + r_{12} Y_i + T_x}{r_{31} X_i + r_{32} Y_i + T_z}}$$

Notes on Tsai's Camera Model

Recall Step 3: Radial lens distortion correction





$$u_d = u(1+k_1 r_d^2)$$

$$v_d = v(1+k_1 r_d^2)$$

where $r_d^2 = u_d^2 + v_d^2$

$$u = \frac{u_d}{\left(1+k_1 r_d^2\right)} \qquad \text{(Modified)}$$

Parameter to be calibrated: k_1

$$u = u_d\left(1+\kappa_1 r_d^2\right) \qquad \text{(Tsai's paper)}$$

Notes on Tsai's Radial lens distortion

If Tsai's model were used,

$$u_i = u_{di}\left(1+\kappa_1 r_{di}^{\,2}\right)$$

Recall Steps 1 and 2

$x_i = r_{11}X_i + r_{12}Y_i + T_x$=known

$z_i = r_{31}X_i + r_{32}Y_i + T_z$

$$u_i = f\frac{x_i}{z_i}$$

$$\frac{x_i}{z_i} = \frac{u_{di}}{f}\left(1+\kappa_1 r_{di}^{\,2}\right)$$

$$\frac{u_{di}}{f}\left(1+\kappa_1 r_{di}^2\right) = \frac{x_i}{r_{31}X_i + r_{32}Y_i + T_z}$$

$$u_{di}\left(r_{31}X_i + r_{32}Y_i + T_z\right)\left(1+\kappa_1 r_{di}^2\right) = f x_i$$

Tsai's model for solving T_z, k_1 and f are non-linear!

Notes on Tsai's Radial lens distortion

Recall Steps 1 and 2

$$\left.\begin{aligned} x_i &= r_{11}X_i + r_{12}Y_i + T_x = \text{known} \\ z_i &= r_{31}X_i + r_{32}Y_i + T_z \end{aligned}\right\}$$

$$\frac{u_i}{f} = \frac{x_i}{z_i} = \frac{x_i}{r_{31}X_i + r_{32}Y_i + T_z}$$

Modified: $u_i = \dfrac{u_{di}}{\left(1+k_1 r_{di}^2\right)}$

$$\frac{u_{di}}{f\left(1+\kappa_1 r_{di}^{\,2}\right)} = \frac{x_i}{r_{31}X_i + r_{32}Y_i + T_z}$$

$$\Rightarrow u_{di}\left(r_{31}X_i + r_{32}Y_i + T_z\right) = x_i f\left(1+\kappa_1 r_{di}^{\,2}\right)$$

$$x_i f + x_i r_{di}^{\,2} fk_1 - u_{di}T_z = \underbrace{u_{di}\left(r_{31}X_i + r_{32}Y_i\right)}_{\text{known}}$$

Two-stage approach *(Stage 2)*

Stage 2: Perspective Constraint
Given **[R]** T_x, T_y solve for f, k_1, T_z

$$\begin{bmatrix} x_1 & r_{d1}^2 x_1 & -u_{d1} \\ x_2 & r_{d2}^2 x_2 & -u_2 \\ \vdots & \vdots & \vdots \\ x_n & r_{dn}^2 x_n & -u_{dn} \end{bmatrix} \begin{bmatrix} f \\ fk_1 \\ T_z \end{bmatrix} = \begin{bmatrix} \left(r_{31}X_1 + r_{32}Y_1\right)u_{d1} \\ \left(r_{31}X_2 + r_{32}Y_2\right)u_{d2} \\ \vdots \\ \left(r_{31}X_n + r_{32}Y_n\right)u_{dn} \end{bmatrix}$$

where $x_i = r_{11}X_i + r_{12}Y_i + T_x$

$$[\mathbf{A}']\mathbf{x}' = \mathbf{b}' \qquad \mathbf{x}' = \mathbf{A}^{+}\mathbf{b}'$$

Note: The signs of (s_1, s_2) and all associated signs may need to be adjusted after computing f and T_z, which must be positive.

Numerical Example

- The following Table gives 5 point-correspondences input to the calibration system.
- The units for both the world coordinate system and the u-v image coordinate system are centimeters.
- Assume that this is no lens distortion, compute (f, **[R]**, **T**)

	Object points			Image points	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	–0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	–1.00

Numerical Example

points	Object			Image	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	–0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	–1.00

Stage 1 $[\mathbf{A}]\boldsymbol{\mu} = \mathbf{b}$

$$\boldsymbol{\mu} = \begin{bmatrix} \frac{r_{11}}{T_y} & \frac{r_{12}}{T_y} & \frac{r_{21}}{T_y} & \frac{r_{22}}{T_y} & \frac{T_x}{T_y} \end{bmatrix}^T$$

$$a_i = \begin{bmatrix} v_i X_i & v_i Y_i & -u_i X_i & -u_i Y_i & v_i \end{bmatrix} \qquad u_i$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 2.9 & 0 \\ 10 & 7.5 & -17.3 & -12.9 & 1 \\ 0 & 0 & -17.3 & -8.65 & 0 \\ 5 & 10 & 0 & 0 & 1 \\ -5 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -0.58 \\ 1.73 \\ 1.73 \\ 0 \\ 0 \end{bmatrix} \quad \boxed{\boldsymbol{\mu} = \mathbf{A}^{+}\mathbf{b} = \begin{bmatrix} -0.173 \\ 0 \\ 0 \\ -0.2 \\ 0.865 \end{bmatrix}}$$

Numerical Example (cont.)

Stage 1: Calculate T_y

points	Object			Image	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	-0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	-1.00

$$U = \sum_{j=1}^{4} \mu_j^2 = 0.0699$$

$$T_y^2 = \frac{U - \left[U^2 - 4(\mu_1\mu_4 - \mu_2\mu_3)^2\right]^{1/2}}{2(\mu_1\mu_4 - \mu_2\mu_3)^2} = 25$$

Try $T_y = +5$

$r_{11} = -0.865;\ r_{12} = r_{21} = 0;\ r_{22} = -1;\ T_x = 4.325$

Check Point 2: $\xi_x = r_{11}X + r_{12}Y + T_x = -4.325$

$\xi_y = r_{21}X + r_{22}Y + T_y = -2.5$

Wrong sign $\Rightarrow T_y = -5.$

$$\boxed{\boldsymbol{\mu} = \begin{bmatrix} r_{11}/T_y \\ r_{12}/T_y \\ r_{21}/T_y \\ r_{22}/T_y \\ T_x/T_y \end{bmatrix} = \begin{bmatrix} -0.173 \\ 0 \\ 0 \\ -0.2 \\ 0.865 \end{bmatrix}}$$

Numerical Example (cont.)

$T_y = -5$ Recalculate $r_{11} = 0.865;\ r_{12} = r_{21} = 0;\ r_{22} = 1;\ T_x = -4.325$

Try $s_1 = s_2 = +1$ $\quad r_{13} = s_1\sqrt{1 - r_{11}^2 - r_{12}^2} = 0.5018 \qquad r_{23} = s_2\sqrt{1 - r_{21}^2 - r_{22}^2} = 0$

$$sign(r_{11}r_{21} + r_{12}r_{22}) = 0 \quad \text{non-positve, OK}$$

$$\begin{array}{ccccc} \boldsymbol{\gamma} & & \boldsymbol{\alpha} & & \boldsymbol{\beta} \\ \begin{bmatrix} r_{31} \\ r_{32} \\ r_{33} \end{bmatrix} & = & \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} & \times & \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} \end{array} \qquad \begin{aligned} r_{31} &= r_{12}r_{23} - r_{13}r_{22} = -0.5018 \\ r_{32} &= r_{13}r_{21} - r_{11}r_{23} = 0 \\ r_{33} &= r_{11}r_{22} - r_{12}r_{21} = 0.8650 \end{aligned}$$

$$\Rightarrow \mathbf{R} = \begin{bmatrix} 0.865 & 0 & 0.5018 \\ 0 & 1 & 0 \\ -0.5018 & 0 & 0.865 \end{bmatrix}$$

Numerical Example (cont.)

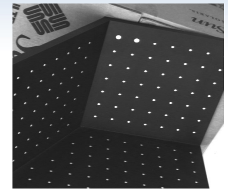
Stage 2

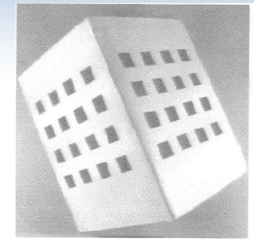
points	Object			Image	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	-0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	-1.00

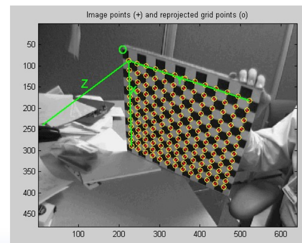
Solving for the unknown x', $\quad [\mathbf{A}']\underbrace{\begin{bmatrix} f \\ T_z \end{bmatrix}}_{\mathbf{x}'} = \mathbf{b}'$

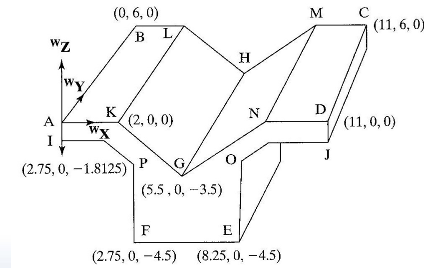
$\mathbf{x}' = \mathbf{A}^{+}\mathbf{b}'$ $\quad$ Hence, f =1.0123 $\quad \mathbf{T} = \begin{bmatrix} -4.325 \\ -5 \\ 7.5484 \end{bmatrix}$

Example calibration boards





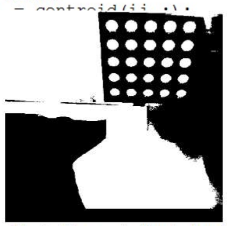


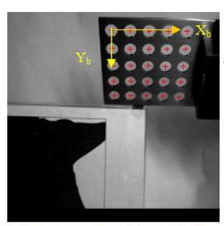


Example

Getting feature points from image (using regionprops):

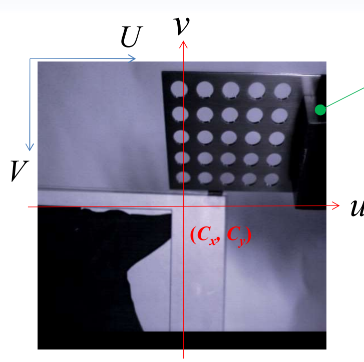
```
img_lab = bwlabel(img_bin);
props_struc = regionprops(img_lab, 'centroid','area'); %obtaining the
structure datatype with info
centroid = cat(1, centr_v.Centroid); %converting desired structured data
into numerical
count = 0;
%%using area to filter features
for ii = 1:length(areas)
    areav = areas(ii);
   if (areav > 200)  & (areav < 800)
        count = count+1;
        area_t(count) = areav;
        centr_t(count,:) = centroid(ii,:);
   end
end
```





Binarized image and calibration board image with 25 feature points labeled with '+'.

Use Camera Model Step 4 to convert from pixels to physical units



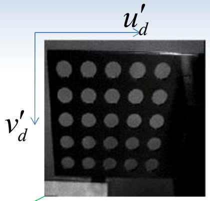
$$u_d = (U - c_x)dx'$$

$$v_d = (V - c_y)dy'$$

$$dx' = dx\frac{N_{cx}}{N_{fx}} = 11.6\left(\frac{768}{512}\right)\mu m$$

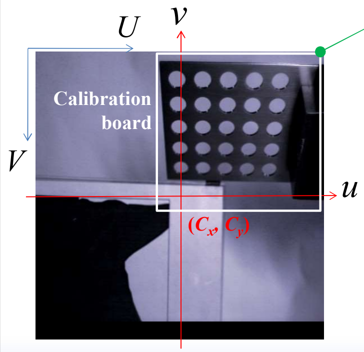
$$dy' = dy\frac{N_{cy}}{N_{fy}} = 13.6\left(\frac{484}{512}\right)\mu m$$

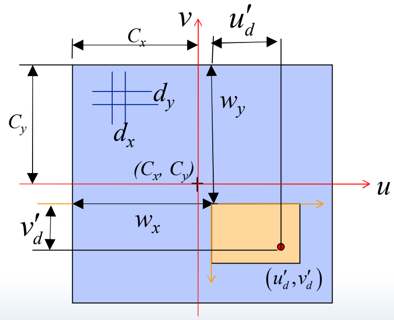
Use Camera Model Step 4 to convert from pixels to physical units



Cropped image

$$u_d = \left(w_x + u'_d - c_x\right) dx$$
$$v_d = \left(w_y + v'_d - c_y\right) dy$$





Example (cont., Results)

Camera Model Step 4:

```
%%%NOTE: IN THIS CASE, Cfx = 256, Cfy = Cfx = 256
Xd_phy =  (Xd_pix(:,1) - Cfx)*dxp; %use computer centers!!!
Yd_phy =  (Xd_pix(:,2) - Cfy)*dyp;  %use computer centers!!!
```

Stage I:

R - Rotation matrix			***T*** - Translation *mm (in)*	
0.9979	0.0412	0.0492	T_x	−1.8719 (−0.0737)
0.0147	0.5992	-0.8004	T_y	−63.5523 (−2.5011)
-0.0624	0.7995	0.5974		

Stage II:

f, κ_1, T_z – **Stage 2 parameters**

f = 14.1 *mm* (0. 5568 *in*) k_1 = 1228.8689 *mm*$^{-2}$ (0.7766 *in*$^{-2}$) T_z = 336.5 *mm* (13.2457 *in*)

Example (cont., Results)

Solving for parameters using distortion *model 1* for intrinsic parameters.

R - Rotation matrix			***T*** - Translation *mm (in)*	
0.9979	0.0412	0.0492	T_x	−1.8719 (−0.0737)
0.0147	0.5992	-0.8004	T_y	−63.5523 (−2.5011)
-0.0624	0.7995	0.5974	T_z	340.6396 (13.4057)

Intrinsic parameters

f = 14.3 *mm* (0.5622 *in*) k_1 = −879.7752 *mm*$^{-2}$ (-0.5560 *in*$^{-2}$)

In terms of Euler angles yaw θ, pitch ϕ, and tilt ψ for rotation,

$$R = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ -\sin\psi\cos\phi + \cos\psi\sin\theta\cos\phi & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\theta\sin\phi \\ \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi & \cos\theta\cos\phi \end{bmatrix}$$

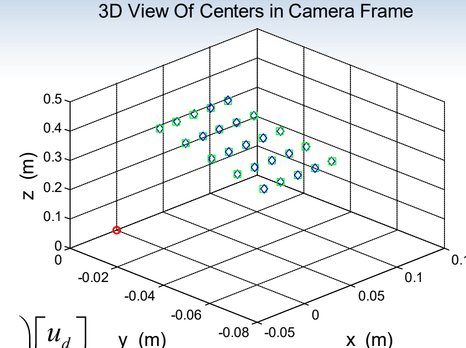
θ *(x-axis rotation)*	ϕ *(y-axis rotation)*	ψ *(z-axis rotation)*
$\theta = \tan^{-1}(r_{23}/r_{33})$	$\phi = -\sin^{-1}(r_{13})$	$\psi = \tan^{-1}(r_{12}/r_{11})$
-53°	-3°	2°

The results satisfy $[\mathbf{R}]^T = [\mathbf{R}]^{-1}$

Example

(validating results)

3D View Of Centers in Camera Frame



$$\boxed{\mathbf{x}_i = [\mathbf{R}]\mathbf{X}_i + \mathbf{T}}$$

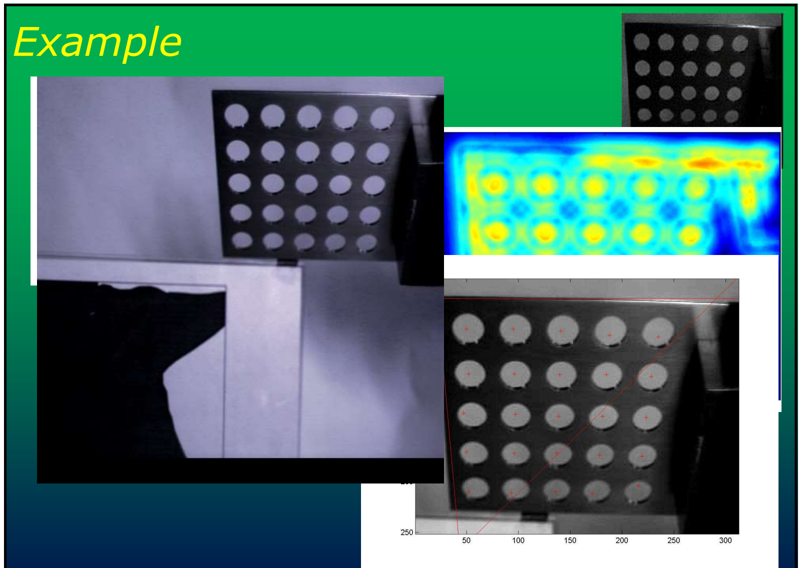
$$\text{RHS} = [\mathbf{R}]\mathbf{X}_i + \mathbf{T}$$

$$\text{LHS:} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \frac{z_i}{f}\left(\frac{1}{1+k_1 r_d^2}\right)\begin{bmatrix} u_d \\ v_d \end{bmatrix}$$

3 sample points from the figure above

***x* coordinates (*m*)**		***y* coordinates (*m*)**		***z* coordinates (*m*)**	
LHS	RHS	LHS	RHS	LHS	RHS
-0.0018444	-0.0018719	-0.0637924	-0.0635523	0.3388329	0.3388329
-0.0011130	-0.0010866	-0.0521947	-0.0521322	0.3540779	0.3540779
-0.0003378	-0.0003012	-0.0407762	-0.0407121	0.3693227	0.3693227

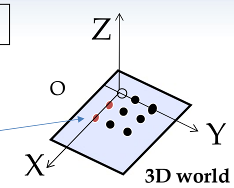
Example



Simplified for Stage 1a)

1a) Solve for two parameters: $r_{11}/T_y,\ r_{21}/T_y$

- Appropriately position the camera relative to the calibration such that $T_y > 0$ and $T_x/T_y = \rho$.
- Select points on the x-axis of the 2D calibration board, Y=Z=0.



$$(r_{11}X_i + \cancel{r_{12}Y_i} + \cancel{r_{13}Z_i} + T_x)v_{di} - (r_{21}X_i + \cancel{r_{22}Y_i} + \cancel{r_{23}Z_i} + T_y)u_{di} = 0$$

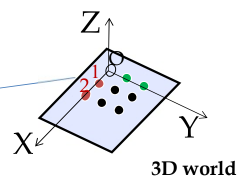
$$X_i v_{di}\underbrace{\left(\frac{r_{11}}{T_y}\right)}_{\mu_1} + v_{di}\underbrace{\left(\frac{T_x}{T_y}\right)}_{\text{known } \rho} - X_i u_{di}\underbrace{\left(\frac{r_{21}}{T_y}\right)}_{\mu_3} - u_{di} = 0$$

$$\begin{bmatrix} X_i v_{di} & -X_i u_{di} \end{bmatrix}\begin{bmatrix}\mu_1 \\ \mu_3\end{bmatrix} = u_{di} - \rho v_{di}$$

Simplified for Stage 1a)

1a) Similarly, solve for $r_{12}/T_y,\ r_{22}/T_y$

- Appropriately position the camera relative to the calibration such that $T_y > 0$ and $T_x/T_y = \rho$.
- Select points on the x-axis of the 2D calibration board, X=Z=0.



Let $a_{i1} = X_i v_{di};\ a_{i2} = X_i u_{di};$ and $b_i = u_{di} - \rho v_{di}$

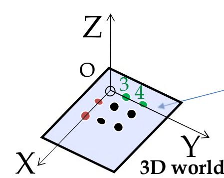
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\mu_1 \\ \mu_3\end{bmatrix} = \begin{bmatrix} b_1 \\ b_2\end{bmatrix} \Rightarrow \begin{bmatrix}\mu_1 \\ \mu_3\end{bmatrix} = \frac{1}{\det|\mathbf{A}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{bmatrix}\begin{bmatrix} b_1 \\ b_2\end{bmatrix}$$

where $\det|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$

$$\begin{bmatrix}\mu_1 \\ \mu_3\end{bmatrix} = \frac{1}{X_1 v_{d1} X_2 u_{d2} - X_1 u_{d1} X_2 v_{d2}}\begin{bmatrix} X_2 u_{d2} & -X_1 u_{d1} \\ -X_2 v_{d2} & X_1 v_{d1}\end{bmatrix}\begin{bmatrix} u_{d1} - \rho v_{d1} \\ u_{d2} - \rho v_{d2}\end{bmatrix}$$

Simplified for Stage 1a)

1a) Similarly, solve for $r_{12}/T_y,\ r_{22}/T_y$



- Appropriately position the camera relative to the calibration such that $T_y > 0$ and $T_x/T_y = \rho$.
- Select points on the x-axis of the 2D calibration board, X=Z=0.

$$(\cancel{r_{11}X_j} + r_{12}Y_j + \cancel{r_{13}Z_j} + T_x)v_{di} -$$
$$(\cancel{r_{21}X_j} + r_{22}Y_j + \cancel{r_{23}Z_j} + T_y)u_{dj} = 0$$

$$Y_j v_{dj}\underbrace{\left(\frac{r_{12}}{T_y}\right)}_{\mu_2} + v_{dj}\underbrace{\left(\frac{T_x}{T_y}\right)}_{\text{known } \rho} - Y_j u_{dj}\underbrace{\left(\frac{r_{22}}{T_y}\right)}_{\mu_4} - u_{dj} = 0 \qquad \begin{bmatrix} Y_j v_{dj} & -Y_j u_{dj} \end{bmatrix}\begin{bmatrix}\mu_2 \\ \mu_4\end{bmatrix} = u_{dj} - \rho v_{dj}$$

$$\begin{bmatrix}\mu_2 \\ \mu_4\end{bmatrix} = \frac{1}{Y_3 v_{d3} Y_4 u_{d4} - Y_3 u_{d3} Y_4 v_{d4}}\begin{bmatrix} Y_4 u_{d4} & -Y_3 u_{d3} \\ -Y_4 v_{d4} & Y_3 v_{d3}\end{bmatrix}\begin{bmatrix} u_{d3} - \rho v_{d3} \\ u_{d4} - \rho v_{d4}\end{bmatrix}$$