HW5

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17.3.25

$z^2 - \bar{z}^2 = 2$

$(x + yi)^2 + \overline{(x + yi)}^2 = 2$

$x^2 - y^2 + x^2 - y^2 = 2 \rightarrow x^2 - y^2 = 1$

The set of points simplifies to a hyperbola with its center at the origin.

17.4.24

$lim_{z \rightarrow 1} \frac{x+y-1}{z-1}$

For $y = 0,\ x \rightarrow 1 :\ lim_{x \rightarrow 1} \frac{x-1}{z-1} = 1$

For $y \rightarrow 0,\ x = 1 :\ lim_{x \rightarrow 1} \frac{y}{yi} = -i$

Since limit approaches different values, the limit does not exist.

17.4.37

$f(z) = \frac{z^3+z}{z^2+4}$

$z^2 + 4 = 0 \rightarrow z = \pm 2i$

Function will not be analytic at $z = \pm 2i$

17.5.18

$f(z) = 3x^2y^2 - 6x^2y^2i$

$u(x, y) = 3x^2y^2;\ v(x, y) = -6x^2y^2$

$\frac{\partial}{\partial x} u = 6xy^2$

$\frac{\partial}{\partial y} v = -12x^2y$

$6xy^2 = -12x^2y \rightarrow x = 0;\ y = 0;\ y = -2x$

$\frac{\partial}{\partial y} u = 6x^2y$

$\frac{\partial}{\partial x} v = -12xy^2$

$6x^2y = -12xy^2 \rightarrow x = 0;\ y = 0;\ x = 2y$

For both: $x = 0;\ y = 0$

If $x, y \neq 0 \rightarrow y = -4y \rightarrow y = 0$

$\therefore\ f(z)$ is not analytic but is differentiable along coordinate axes.

17.6.22a

$f(z) = e^{z^2} = e^{(x+iy)^2} = e^{(x^2-y^2)+2xyi} = e^{x^2-y^2}(cos2xy + isin2xy)$

$u(x, y) = e^{x^2-y^2}cos2xy;\ v(x, y) = e^{x^2-y^2}sin2xy$

$\frac{\partial}{\partial x}u = e^{x^2-y^2}(2xcos2xy - 2ysin2xy)$

$\frac{\partial}{\partial y}v = e^{x^2-y^2}(2xcos2xy - 2ysin2xy)$

$\frac{\partial}{\partial x}u = \frac{\partial}{\partial y}v$

$\frac{\partial}{\partial y}u = e^{x^2-y^2}(-2ycos2xy - 2xsin2xy)$

$\frac{\partial}{\partial x}v = e^{x^2-y^2}(2ycos2xy + 2xsin2xy)$

$\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}v$

Since both Cauchy-Riemann conditions are met, $f(z)$ is an entire function.

17.7.16

$cosz = -3i$

$\frac{e^{iz}+e^{-iz}}{2} = -3i$

$e^{iz} + e^{-iz} + 6i = 0 \rightarrow e^{2iz} + 1 + 6ie^{iz} = 0$

$e^{iz} = \frac{-61 \pm \sqrt{36i^2-4}}{2} = i(-3 \pm \sqrt{10})$

$iz = ln(i(-3 \pm \sqrt{10})) + \frac{\pi}{2}i + 2n\pi i$

$z = ln((-3 \pm \sqrt{10}))i \pm \frac{\pi}{2} + 2n\pi$

17.Review.38

$f(z) = x^3 + xy^2 - 4x + i(4y - y^3 - x^2y))$

$\frac{\partial}{\partial x}u = 3x^2 + y^2 - 4$

$\frac{\partial}{\partial y}v = 4 - 3y^2 - x^2$

Since: $\frac{\partial}{\partial x}u \neq \frac{\partial}{\partial y}v$, the function is not analytic but is differentiable given it is continuous.

18.1.11

$\int_C f(z)dz = \int_{C1} f(z)dz + \int_{C2} f(z)dz$

For line from $y = 0$ for $0 \leq x \leq 2$, $z(x) = x$. $f(z(x)) = e^x$

$\int_{C1} e^x dx = e^x|_0^2 = e^2 - 1$

For line from $z = 2$ to $z = 1 + i\pi$, for $1 \leq x \leq 2$, $y = -\pi(x - 2)$

$z(x) = x + i(2\pi - \pi x),\ z'(x) = 1 - \pi i$

$f(z(x)) = e^{x+i(2\pi-\pi x)}$

$(1 - \pi i)\int_2^1 e^{x+i(2\pi-\pi x)} dx = -e - e^2$

Finally:

$\int_C e^z dz = \int_{C1} e^z dz + \int_{C2} e^z dz = e^2 - 1 - e - e^2 = -e - 1$

18.2.13

$\int_C \frac{z}{z^2-\pi^2} dz;\ |z| = 3$

$\frac{z}{z^2-\pi^2} = \frac{A}{z-\pi} + \frac{B}{z+\pi} \rightarrow z = A(z + \pi) + B(z - \pi)$

For $z = \pi$, $A = \frac{1}{2}$

For $z = -\pi$, $B = \frac{1}{2}$

$\frac{z}{z^2-\pi^2} = \frac{\frac{1}{2}}{z-\pi} + \frac{\frac{1}{2}}{z+\pi}$

Since the function is not analytic at $z = \pm\pi$, $\int_C (\frac{\frac{1}{2}}{z-\pi} + \frac{\frac{1}{2}}{z+\pi})dz = 0$

18.3.17

$\int_C \frac{1}{z} dz;\ z = 4e^{it};\ -\pi/2 \leq t \leq \pi/2$

$f(z) = \frac{1}{z};\ \int f(z)dz = lnz + C$

For $t = -\pi/2$, $z(-\pi/2) = 4e^{-(\pi/2)i} = 4(cos(-\pi/2) + isin(-\pi/2)) = -4i$

For $t = \pi/2$, $z(\pi/2) = 4e^{(\pi/2)i} = 4(cos(\pi/2) + isin(\pi/2)) = 4i$

$\int_{-4i}^{4i} \frac{1}{z} dz = lnz|_{-4i}^{4i} = ln4 + i\pi/2 - (ln4 - i\pi/2) = i\pi$