Math 6701 HW2

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8.2.33

$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 4 & -1 & -1 \end{bmatrix}$; $B = \begin{bmatrix} -12 \\ 1 \\ 10 \end{bmatrix}$; $X = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ $AX = B \rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \\ 10 \end{bmatrix}$ If $c_1 = 0,\ AX = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \\ 10 \end{bmatrix}$

If X is solution for $AX = 0$, then X is a solution for any constant, c_1.

$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 4 & -1 & -1 \end{bmatrix} \xrightarrow[\text{R2-R1}]{\text{R1/2}} \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 5/2 & -3/2 \\ 4 & -1 & -1 \end{bmatrix} \xrightarrow[\text{R1+R2*3/2}]{\text{R2*2/5}} \begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & -3/5 \\ 0 & -5 & -3 \end{bmatrix} \xrightarrow{\text{R3-5R2}} \begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & -3/5 \\ 0 & 0 & 0 \end{bmatrix} = RREF(A)$

$Rank(A) = 2$
Nullity(A) = # of Columns - Rank $\therefore$ $Nullity(A) = 3 - 2 = 1$

8.3.5

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{bmatrix} \xrightarrow{\text{-(-R1 + R2)}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{R3-3R2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{bmatrix} \xrightarrow{\text{R3/9}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{R1}-R2-R3]{\text{R2+3R3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = RREF(A)$

Rank = number of non-zero rows in RREF = 3

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The null space is a zero vector, therefore the nullity is 0.

8.4.24

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2+x & 3+y & 4+z \end{vmatrix} = 1 \cdot \begin{vmatrix} y & z \\ 3+y & 4+z \end{vmatrix} - 1 \cdot \begin{vmatrix} x & z \\ 2+x & 4+z \end{vmatrix} + 1 \cdot \begin{vmatrix} x & y \\ 2+x & 3+y \end{vmatrix}$$

$$\begin{vmatrix} y & z \\ 3+y & 4+z \end{vmatrix} = 4y + yz - 3z - yz$$

$$\begin{vmatrix} x & z \\ 2+x & 4+z \end{vmatrix} = 4x + xz - 2z - xz$$

$$\begin{vmatrix} x & y \\ 2+x & 3+y \end{vmatrix} = 3x + xy - 2y - xy$$

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2+x & 3+y & 4+z \end{vmatrix} = -x + 2y - z$$

8.5.26

$det(AB) = detA * detB = detB * detA = det(BA)$

8.6.31

If $AA = I$, A is its own inverse. $\begin{bmatrix} 4 & -3 \\ x & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ x & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 16-3x & 0 \\ 0 & 16-3x \end{bmatrix} \rightarrow 16 - 3x = 1 \rightarrow x = 5$
Since $AA = I$, A is its own inverse.

8.6.36

If either A or B is singular, then $det(A) = 0 \therefore det(AB) = det(A) * det(B) = 0$ Thus AB is singular if A or B is singular

8.12.36

$S = PDP^T$

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{8}{3} & \frac{4}{3} & \frac{-1}{3} \\ \frac{4}{3} & \frac{11}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{4}{3} & \frac{8}{3} \end{bmatrix}$$

8.9.5

$A = \begin{bmatrix} 8-\lambda & 5 \\ 4 & -\lambda \end{bmatrix} \rightarrow \lambda^2 - 8\lambda - 20 = 0 \rightarrow \lambda = -2,\ 10\ (-2)^m = c_o - 2c_1$
$10^m = c_o + 10c_1$ $c_o = \frac{5(-2)^m + 10^m}{6}$
$c_1 = \frac{-(-2)^m + 10^m}{12}$

$A^m = c_o I + c_1 A = \begin{bmatrix} \frac{(-2)^m + 2^m 5^{m+1}}{6} & \frac{5(-(-2)^m + 10^m)}{12} \\ \frac{-(-2)^m + 10^m}{3} & \frac{5(-2)^m + 10^m}{6} \end{bmatrix}$

$m = 5;\ A^5 = \begin{bmatrix} 83328 & 41680 \\ 33344 & 16640 \end{bmatrix}$

8.10.16

$$\begin{bmatrix}0&1&1\\1&1&1\\1&1&0\end{bmatrix}\rightarrow\begin{bmatrix}0-\lambda&1&1\\1&1-\lambda&1\\1&1&0-\lambda\end{bmatrix}=0;\ \lambda=-1,\ 1\pm\sqrt{2}$$

Gauss-Jordan Elimination $\lambda=-1$; $\begin{bmatrix}1&1&1\\1&2&1\\1&1&1\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1\\0&1&0\\0&0&0\end{bmatrix}\therefore K_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\lambda=1+\sqrt{2}$; $\begin{bmatrix}-1+\sqrt{2}&1&1\\1&\sqrt{2}&1\\1&1&-1+\sqrt{2}\end{bmatrix}\rightarrow\begin{bmatrix}1&\sqrt{2}&1\\0&1&\sqrt{2}\\0&0&0\end{bmatrix}\therefore K_2=\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix}$

$\lambda=1-\sqrt{2}$; $\begin{bmatrix}-1-\sqrt{2}&1&1\\1&-\sqrt{2}&1\\1&1&-1-\sqrt{2}\end{bmatrix}\rightarrow\begin{bmatrix}1&-\sqrt{2}&1\\0&1&-\sqrt{2}\\0&0&0\end{bmatrix}\therefore K_3=\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix}$

Norm of K_1 is $\sqrt{2}$, K_2 and K_3 is 2,

Orthogonal Matrix: $P=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{2}&\frac{1}{2}\\0&\frac{-\sqrt{2}}{\sqrt{2}}&\frac{\sqrt{2}}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}$

8.10.21

a and b

$$AK_1=\begin{bmatrix}0&2&2\\2&0&2\\2&2&0\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}-2\\2\\0\end{bmatrix}=-2\begin{bmatrix}1\\-1\\0\end{bmatrix}=\lambda K_1;\ \lambda_1=-2$$

$$AK_2=\begin{bmatrix}0&2&2\\2&0&2\\2&2&0\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}-2\\0\\2\end{bmatrix}=-2\begin{bmatrix}1\\0\\-1\end{bmatrix}=\lambda K_2;\ \lambda_2=-2$$

$$AK_2=\begin{bmatrix}0&2&2\\2&0&2\\2&2&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}4\\4\\4\end{bmatrix}=4\begin{bmatrix}1\\4\\1\end{bmatrix}=\lambda K_2;\ \lambda_3=4$$

c

$K_1^TK_2=1(1)-1(0)+0(-1)=1\neq 0\therefore$ K_1K_2 are not orthogonal. $K_2^TK_2=1(1)+0(1)-1(1)=0\therefore$ *orthogonal* $K_1^TK_3=1(1)-1(1)+0(1)=0\therefore$ *orthogonal*

Gram-Schmidt Process: $V_1=K_1$

$V_2=K_2-\frac{K_2^TV_1}{V_1^TV_1}V_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\end{bmatrix}$

Orthogonal set of eigenvectors: (V_1,V_2,K_3)

Norms:

$||V_1||=\sqrt{2}$ $||V_2||=\frac{\sqrt{3}}{\sqrt{2}}$ $||K_3||=\sqrt{3}$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 2 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$