# HW3

Patrick Gardocki

2023-10-18

## 1.1.28

Set $f(x) = \int_0^x e^{t^2} dt;\ f'(x) = \frac{d}{dx} \int_0^x e^{t^2} dt = e^{x^2}$

$\frac{dy}{dx} = \frac{d}{dx}[e^{-x^2}] + \frac{d}{dx}[e^{-x^2} f(x)] = -2xe^{-x^2} + dx[e^{-x^2}]f(x) + e^{-x^2} f'(x)$
$= -2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + e^{-x^2} \cdot e^{x^2}$
$= 1 - 2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt$

For $\frac{dy}{dx} + 2xy = 1$;
$1 - 2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + 2xe^{-x^2} + 2xe^{-x^2} \int_0^x e^{t^2} dt = 1$
$\therefore 1 = 1$

This is a solution.

## 1.2.22

$(1 + y^3)y' = x^2;\ f(x.y) = \frac{dy}{dx} = \frac{x^2}{1+y^3}$

$\frac{df}{dy} = -3x^2y^2 \left(1 + \frac{y^3}{3}\right)^2$

$\frac{dy}{dx}$ and $\frac{df}{dy}$ are both continuous when $y \neq -1$.

For $(x_0, y_0)$ such that $y \neq -1$, there is a unique solution.

## 2.2.25

Given $x^2 \frac{dy}{dx} = y - xy,\ y(-1) = -1$

$x^2 \frac{dy}{dx} = y(1 - x) \rightarrow \int \frac{dy}{y} = \int \frac{1-x}{x^2} dx \rightarrow ln|y| = -\frac{1}{x} - ln|x| + C$

If $y(-1) = -1, \rightarrow y = -1,\ x = -1$

Plug into previous equation to solve for C.

$C = -1$

$ln|y| = -\frac{1}{x} - ln|x| - 1$

Solve for $y$

$e^{ln|y|} = e^{-\frac{1}{x} - ln|x| - 1} \rightarrow y = \frac{1}{x} e^{-\frac{1}{x} - 1}$

## 2.3.13

Given: $x^2y' + x(x+2)y = e^x \rightarrow x^2y' + (x^2+2x)y = e^x$

$\frac{dy}{dx} + (1+\frac{2}{x})y = \frac{e^x}{x^2} \quad EQ1$

$P(x) = 1 + \frac{2}{x};\ f(x) = \frac{e^x}{x^2}$

$P(x)$ is continuous on $(-\infty, 0)$ and $f(x)$ is continuous on $(0, \infty)$.

$I(x) = e^{\int P(x)dx} = e^{\int 1+\frac{2}{x}dx} = e^x e^{2ln|x|} = x^2e^x$; I(x) is continuous on $(0, \infty)$

$EQ1 * I(x) = x^2e^x\frac{dy}{dx} + (x^2e^x + 2xe^2)y = e^{2x}$

$\frac{d}{dx}x^2e^xy = e^{2x} \rightarrow x^2e^xy = \int e^{2x}dx \rightarrow x^2e^xy = \frac{1}{2}e^{2x} + C$

Solve for y: General Solution: $y = \frac{1}{2x^2e^x}e^{2x} + \frac{C}{x^2e^x}$ on interval $(0, \infty)$

## 2.5.18

Given: $x\frac{dy}{dx} - (1+x)y = xy^2$

Substitution: $u = y^{-1};\ \frac{dy}{dx} = -u^{-2}\frac{du}{dx} \rightarrow\ \frac{du}{dx} + \frac{1+x}{x}u = -1$

Integrating Factor: $e^{\int \frac{1+x}{x}} = e^{lnx+x} = xe^x$

$xe^x\frac{du}{dx} + (e^x + xe^x)u = -xe^x;\ \frac{d}{dx}xe^xu = -xe^x \rightarrow xe^xu = \int -xe^xdx = -xe^x + e^x + c \rightarrow u = -1 + \frac{1}{x} + \frac{c}{x}e^{-x}$

$u = y^{-1} \rightarrow y = \frac{1}{-1+\frac{1}{x}+\frac{c}{x}e^{-x}}$

## 2.8.2

Using Equation: $P(t) = \frac{aP_0}{bP_0+(a-bP_0)e^{-at}}$

Given: $N(0) = N_0 = 500;\ N(1) = 1000;\ \lim_{\infty} = \frac{a}{b} = 50000$

$N(t) = \frac{500a}{500b+(a-500b)e^{-at}}$

$N(t) = \frac{500a/b}{500+(a/b-500)e^{-at}} = \frac{500(a/b)50000)}{500+(50000-500)e^{-at}} = \frac{50000}{1+99e^{-at}}$

Sub in $N(1) = 1000$, $a = -ln\frac{49}{99} \approx 0.7033; b \approx 0.00014$

$N(t) = \frac{5000}{1+99e^{-0.7033t}}$

## 3.1.26

For $f_1(x) = e^{x/2};\ f_2(x) = xe^{x/2}$

Linearly Independent if $W(f_1, f_2) \neq 0$

$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 \cdot f_2' - f_1' \cdot f_2$

$\begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & e^{x/2} + \frac{1}{2}xe^{x/2} \end{vmatrix} = e^x \neq 0$

$\therefore f_1 and f_2$ form a set of solutions on interval $(-\infty, \infty)$.

General Solution: $y = c_1e^{x/2} + c_2xe^{x/2}$

## 3.3.35

Given: $y''' + 12y'' + 36y' = 0;\ y(0) = 0;\ y'(0) = 1;\ y''(0) = -7$

$m^3 + 12m^2 + 36m = 0 = m(m^2 + 12m + 36) \rightarrow m_1 = 0;\ m^2 + 12m + 36 = 0 = (m+6)^2 = 0 \rightarrow m_2 = m_3 = -6$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 x e^{m_3 x} = c_1 + c_2 e^{-6x} + c_3 x e^{-6x}$

$y' = -6c_2 e^{-6x} + c_3(e^{-6x} - 6xe^{-6x})$

$y'' = 36c_2 e^{-6x} + c_3(-12e^{-6x} + 36xe^{-6x})$

Sub in $y(0) = 0$: $c_1 + c_2 = 0$

Sub in $y'(0) = 1$: $-6c_2 + c_3 = 1 \rightarrow -72c_2 + 12c_3 = 12$

Sub in $y''(0) = -7$: $36c_2 - 12c_3 = -7$

Add $y'$ and $y''$: $-36c_2 = 5 \rightarrow c_2 = \frac{-5}{36}, c_3 = \frac{1}{6}, c_1 = \frac{5}{36}$

$y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}$

## 3.4.15

Given: $y'' + y = 2xsinx$

For: $m^2 + 1 = 0;\ m^2 = -1 = \pm\sqrt{-1} = \pm i \rightarrow m_1 = i;\ m_2 = -i$

$\alpha = 0;\ \beta = 1 \rightarrow y_c = c_1 cosx + c_2 sinx$

Assume: $y_p = (Ax^2 + B)cosx + (Cx^2 + D)sinx$ $y_p' = (2Ax + B)cosx - (Ax^2 + bx)sinx + (2Cx + D)sinx + (Cx^2 + Dx)cosx$

$y_p'' = 2Acosx - (2Ax + B)sinx - (2Ax + B)sinx - (Ax^2 + bx)cosx + 2Csinx + (2Cx + D)cosx + (2Cx + D)cosx - (Cx^2 + Dx)sinx$

Subbing into DE:

$-4Axsinx = 2xsinx \rightarrow A = \frac{-1}{2}$

$4Cxcosx = 0 \rightarrow C = 0$

$(2A + 2D)cosx = 0 \rightarrow D = \frac{1}{2}$

$(2B + 2C)sinx = 0 \rightarrow B = 0$

$y_p = \frac{-1}{2}x^2 cosx + \frac{1}{2}xsinx$

$y = y_p + y_c = c_1 cosx + c_2 sinx - \frac{1}{2}x^2 cosx + \frac{1}{2}xsinx$

## 3.4.31

Given: $y'' + 4y' + 5y = 35e^{-4x},\ y(0) = -3,\ y'(0) = 1$

$m^2 + 4m + 5 = 0 \rightarrow m_1 = -2 + i,\ m_2 = -2 - i \therefore \alpha = -2,\ \beta = 1$

$y_c = e^{-2x}(c_1 cosx + c_2 sinx)$

Assume: $y_p = Ae^{-4x};\ y_p' = -4Ae^{-4x},\ y_p'' = 16Ae^{-4x}$

Subbing into DE:

$16Ae^{-4x} - 16Ae^{-4x} + 5Ae^{-4x} = 35e^{-4x} \rightarrow A = 7$

$\therefore y_p = 7e^{-4x},\ y = y_p + y_c$

$y = e^{-2x}(c_1 cosx + c_2 sinx) + 7e^{-4x};\ y' = -2e^{-2x}(c_1 cosx + c_2 sinx) + e^{-2x}(-c_1 sinx + c_2 cosx) - 28e^{-4x}$

For $y(0) = 0 \rightarrow c_1 = -10$

For $y'(0) = 1 \rightarrow c_2 = 9$

$y = e^{-2x}(-10cosx + 9sinx) + 7e^{-4x}$