Forbidden Induced Subgraphs and the Price of Connectivity for Feedback Vertex Set

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Abstract. Let $\mathrm{fvs}(G)$ and $\mathrm{cfvs}(G)$ denote the cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph G, respectively. For a graph class $\mathcal{G}$, the price of connectivity for feedback vertex set (poc-fvs) for $\mathcal{G}$ is defined as the maximum ratio $\mathrm{cfvs}(G)/\mathrm{fvs}(G)$ over all connected graphs G in $\mathcal{G}$. The poc-fvs for general graphs is unbounded, as the ratio $\mathrm{cfvs}(G)/\mathrm{fvs}(G)$ can be arbitrarily large. We study the poc-fvs for graph classes defined by a finite family $\mathcal{H}$ of forbidden induced subgraphs. We characterize exactly those finite families $\mathcal{H}$ for which the poc-fvs for $\mathcal{H}$-free graphs is bounded by a constant. Prior to our work, such a result was only known for the case where $|\mathcal{H}| = 1$.

1 Introduction

A *feedback vertex set* of a graph is a subset of its vertices whose removal yields an acyclic graph, and a feedback vertex set is connected if it induces a connected graph. We write $\mathrm{fvs}(G)$ and $\mathrm{cfvs}(G)$ to denote the cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph G, respectively. Let $\mathcal{G}$ be a class of graphs. The *price of connectivity for feedback vertex set* (poc-fvs) for $\mathcal{G}$ is defined to be the maximum ratio $\mathrm{cfvs}(G)/\mathrm{fvs}(G)$ over all connected graphs G in $\mathcal{G}$. Graphs consisting of two disjoint cycles that are connected to each other by an arbitrarily long path show that the poc-fvs for general graphs is not upper bounded by a constant, and the same clearly holds for planar graphs. Interestingly, Grigoriev and Sitters [6] showed that the poc-fvs for planar graphs of minimum degree at least 3 is at most 11. Schweitzer and Schweitzer [7] later improved this upper bound from 11 to 5, and showed the upper bound of 5 to be tight.

In a previous paper [1], we studied the poc-fvs for graph classes characterized by a single forbidden induced subgraph. We proved that the poc-fvs for H-free graphs is bounded by a constant c_H if and only if H is a linear forest, i.e., a disjoint union of

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paths. In fact, we obtained a more refined tetrachotomy result that determines, for every graph H, which of the following cases holds: (i) $\mathrm{cfvs}(G) = \mathrm{fvs}(G)$ for every connected H-free graph G; (ii) there exists a constant c_H such that $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G)+c_H$ for every connected H-free graph G; (iii) there exists a constant c_H such that $\mathrm{cfvs}(G) \leq c_H \cdot \mathrm{fvs}(G)$ for every connected H-free graph G; (iv) there does not exist a constant c_H such that $\mathrm{cfvs}(G) \leq c_H \cdot \mathrm{fvs}(G)$ for every connected H-free graph G.

The concept of "price of connectivity", introduced by Cardinal and Levy [4], has been studied for other parameters as well. One such parameter is the vertex cover number of a graph. Let $\tau(G)$ and $\tau_c(G)$ denote the cardinalities of a minimum vertex cover and a minimum connected vertex cover of a graphs G, respectively. For a graph class $\mathcal{G}$, the price of connectivity for vertex cover for $\mathcal{G}$ is defined as the worst-case ratio $\tau_c(G)/\tau(G)$ over all connected graphs G in $\mathcal{G}$. It is known that for general graphs, the price of connectivity for vertex cover is upper bounded by 2, and this bound is sharp [2]. Cardinal and Levy [4] showed that for n-vertex graphs with average degree ϵn, this bound can be improved to $2/(1+\epsilon)$. Camby et al. [2] provided forbidden induced subgraph characterizations of graph classes for which the price of connectivity for vertex cover is upper bounded by 1, 4/3, and 3/2, respectively.

The price of connectivity for dominating set (poc-ds) for a graph class $\mathcal{G}$ is defined as the maximum ration $\gamma_c(G)/\gamma(G)$ over all connected graphs G in $\mathcal{G}$, where $\gamma_c(G)$ and $\gamma(G)$ denote the domination number and the connected domination number of G, respectively. It is easy to prove that the poc-ds for general graphs is upper bounded by 3 [5]. Motivated by the work of Zverovich [8], Camby and Schaudt [3] studied the poc-ds for (P_k, C_k)-free graphs for several values of k. Their results show that the poc-ds for (P_8, P_9)-free graphs is upper bounded by 2, while the general upper bound of 3 is asymptotically sharp for (P_9, C_9)-free graphs.

Our contribution. We continue the line of research on the price of connectivity for feedback vertex set we initiated in [1]. For a family of graphs $\mathcal{H}$, a graph G is called *$\mathcal{H}$-free* if G does not contain an induced subgraph isomorphic to any graph $H \in \mathcal{H}$. The vast majority of well-studied graph classes have forbidden induced subgraphs characterizations, and such characterizations can often be exploited when proving structural or algorithmic properties of these graph classes. In fact, for every hereditary graph class $\mathcal{G}$, that is, for every graph class $\mathcal{G}$ that is closed under taking induced subgraphs, there exists a family $\mathcal{H}$ of graphs such that $\mathcal{G}$ is exactly the class of $\mathcal{H}$-free graphs. Notable examples of graphs classes that can be characterized using a *finite* family of forbidden induced subgraphs include claw-free graphs, line graphs, proper interval graphs, split graphs and cographs.

Our main result establishes a dichotomy between the finite families $\mathcal{H}$ for which the price of connectivity for feedback vertex set for $\mathcal{H}$-free graphs is upper bounded by a constant $c_{\mathcal{H}}$ and the families $\mathcal{H}$ for which such a constant $c_{\mathcal{H}}$ does not exist. This can be seen as an extension of the case (iii) from [1] (mentioned above) from monogenic to finitely defined classes of graphs. In order to formally state our main result, we need to introduce some terminology.

For two graphs H_1 and H_2, we write $H_1 + H_2$ to denote the disjoint union of H_1 and H_2. We write sH to denote the disjoint union of s copies of H. For any $r \geq 3$, we write C_r to denote the cycle on r vertices. For any three integers i, j, k with $i, j \geq 3$ and $k \geq 1$, we define $B_{i,j,k}$ to be the graph obtained from $C_i + C_j$ by choosing a vertex x in

C_i and a vertex y in C_j, and adding a path of length k between x and y; see Figure 1 for a picture of the graph $B_{5,9,4}$. We call a graph of the form $B_{i,j,k}$ a *butterfly*.

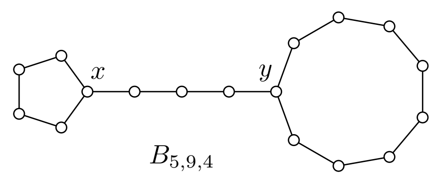

Fig. 1: The butterfly $B_{5,9,4}$.

It is clear that the price of connectivity for feedback vertex set for the class of all butterflies is not bounded by a constant, since $\mathrm{fvs}(B_{i,j,k}) = 2$ and $\mathrm{cfvs}(B_{i,j,k}) = k+1$ for every $i, j \geq 3$ and $k \geq 1$. Roughly speaking, our main result states that the price of connectivity for feedback vertex set for the class of $\mathcal{H}$-free graphs is bounded by a constant $c_{\mathcal{H}}$ if and only if the forbidden induced subgraphs in $\mathcal{H}$ prevent arbitrarily large butterflies from appearing as induced subgraphs. To make this statement more concrete, we use the following definition.

Definition 1. *Let $i, j \geq 3$ be two integers, let $\mathcal{H}$ be a family of graphs, and let $N = 2 \cdot \max_{H \in \mathcal{H}} |V(H)| + 1$. The family $\mathcal{H}$* covers *the pair (i, j) if $\mathcal{H}$ contains an induced subgraph of $B_{i,j,N}$. A graph H covers the pair (i, j) if $\{H\}$ covers (i, j).*

Theorem 1. *Let $\mathcal{H}$ be a finite family of graphs. Then the poc-fvs for $\mathcal{H}$-free graphs is upper bounded by a constant $c_{\mathcal{H}}$ if and only if $\mathcal{H}$ covers the pair (i, j) for every $i, j \geq 3$.*

Section 2 is devoted to the proof of Theorem 1. In Section 3, we prove a sequence of lemmata that show exactly which graphs H cover which pairs (i, j). In Section 4, we present some applications of the results in Sections 2 and 3. In particular, we describe a procedure that, given a positive integer k, yields an explicit description of all the minimal graph families $\mathcal{H}$ with $|\mathcal{H}| = k$ for which the poc-fvs for $\mathcal{H}$-free graphs is upper bounded by a constant. For $k = 1$, this immediately yields the aforementioned result from [1], stating that the poc-fvs for H-free graphs is upper bounded by a constant if and only if H is a linear forest (Corollary 1). We also demonstrate the procedure for the case $k = 2$, and obtain an explicit description of exactly those families $\{H_1, H_2\}$ for which the poc-fvs for $\{H_1, H_2\}$-free graphs is upper bounded by a constant (Corollary 2). Section 5 contains some concluding remarks.

We end this section by defining some additional terminology that will be used throughout the paper. For any $k, p, q \geq 1$, let P_k denote the path on k vertices, and let $T_k^{p,q}$ denote the graph obtained from $P_k + P_p + P_q$ by making a new vertex adjacent to one end-vertex of each path. For any $k \geq 0$ and $r \geq 3$, let D_k^r denote the graph obtained from $P_k + C_r$ by adding an edge between a vertex of the cycle and an end-vertex of the path; in particular, D_0^r is isomorphic to C_r.

2 Proof of Theorem 1

In this section, we prove the dichotomy result given in Theorem 1. We will several times make use of the following two observations.

Observation 1 *Let i, j, k, ℓ be integers such that $i, j \geq 3$ and $\ell \geq k \geq 1$. A graph on at most k vertices is an induced subgraph of $B_{i,j,k}$ if and only if it is an induced subgraph of $B_{i,j,\ell}$.*

Observation 2 *Let G be a connected graph that is not a cycle. Then G has a minimum feedback vertex set F such that each vertex in F lies on a cycle and has degree at least 3 in G.*

We are now ready to present the proof of Theorem 1.

Proof (of Theorem 1). First suppose there exists a pair (i, j) with $i, j \geq 3$ such that $\mathcal{H}$ does not cover (i, j). For contradiction, suppose there exists a constant $c_{\mathcal{H}}$ as in the statement of the theorem. By Definition 1, $\mathcal{H}$ does not contain an induced subgraph of $B_{i,j,N}$, and hence $B_{i,j,N}$ is $\mathcal{H}$-free. As a result of Observation 1, $B_{i,j,k}$ is $\mathcal{H}$-free for every $k \geq N$. In particular, the graph $B_{i,j,N+2c_{\mathcal{H}}}$ is $\mathcal{H}$-free. Note that $\text{fvs}(B_{i,j,N+2c_{\mathcal{H}}}) = 2$ and $\text{cfvs}(B_{i,j,N+2c_{\mathcal{H}}}) = N+2c_{\mathcal{H}}+1$. This implies that $\text{cfvs}(B_{i,j,N+2c_{\mathcal{H}}}) > c_{\mathcal{H}} \cdot \text{fvs}(B_{i,j,N+2c_{\mathcal{H}}})$, yielding the desired contradiction.

For the converse direction, suppose $\mathcal{H}$ covers the pair (i, j) for every $i, j \geq 3$. Let G be a connected $\mathcal{H}$-free graph. Observe that $\text{cfvs}(G) = \text{fvs}(G)$ if G is a cycle or a tree, so we assume that G is neither a cycle nor a tree. Let F be a minimum feedback vertex set of G such that each vertex in F lies on a cycle and has degree at least 3 in G; the existence of such a feedback vertex set is guaranteed by Observation 2. Below, we will prove that the distance in G between any two vertices of F is at most $5N$. To see why this suffices to prove the theorem, observe that we can transform F into a connected feedback vertex set of G of size at most $5N \cdot |F| = 5N \cdot \text{fvs}(G)$ by choosing an arbitrary vertex $x \in F$ and adding, for each $y \in F \setminus \{x\}$, all the internal vertices of a shortest path between x and y.

Let $x, y \in F$, and let P be a shortest path from x to y. For contradiction, suppose P has length at least $5N + 1$. Recall that by the definition of F, there exist cycles C_x and C_y that contain x and y, respectively; assume, without loss of generality, that C_x and C_y are induced cycles in G. Let $X = \{v \in V(C_x) \mid d_{G[V(C_x)]}(v, x) \leq N\}$. Note that X induces the cycle C_x in case $|V(C_x)| \leq 2N$, and X induces a path of length at most $2N$ otherwise. We also define $Y = \{v \in V(C_y) \mid d_{G[V(C_y)]}(v, y) \leq N\}$. We partition the vertex set of P into three sets: $L = \{v \in V(P) \mid d_G(v, x) \leq 2N + 1\}$, $M = \{v \in V(P) \mid d_G(v, x) \geq 2N + 2 \text{ and } d_G(v, y) \geq 2N + 2\}$, and $R = \{v \in V(P) \mid d_G(v, y) \leq 2N + 1\}$. For any two distinct vertices u and v on the path P, we say that u is to the left of v (and, equivalently, v is to the right of u) if the subpath of P from x to u does not contain v.

Claim 1. $G[X \cup L]$ contains a graph in $\{D_N^i \mid i \geq 3\} \cup \{T_N^{N,N}\}$ as an induced subgraph.

Let x' be the vertex of P closest to y that has a neighbor $x_1 \in X \setminus \{x\}$; possibly $x' = x$. Let P' be the subpath of P from x to x'. By the definition of X, the distance between x_1 and x is at most N, implying that $d_G(x, x') \leq N+1$. Since P is a shortest path from

x to y, we find that the length of P' is at most $N+1$. Let x'' be the unique vertex of P such that x'' is to the right of x' and $d_G(x'',x') = N$, and let P'' be the subpath of P from x' to x''. Since $|L| = 2N+2$, path P' has length at most $N+1$, and path P'' has length N, it follows that $V(P'') \subseteq L$. Observe that x' is the only vertex of P'' that has a neighbor in $X \setminus \{x\}$.

Suppose $x = x'$. Then $X \cap V(P) = \{x\}$, and hence $G[X \cup V(P'')]$ is isomorphic to either $D_N^{|V(C_x)|}$ or $T_N^{N,N}$, implying that the claim holds in this case. From now on, we assume that $x' \neq x$. We distinguish two cases, depending on how many neighbors x' has in X.

If x' has at least two neighbors in X, then x' has two neighbors x_1, x_2 in X such that there is a path in X from x_1 to x_2 whose internal vertices are not adjacent to x'. This path, together with the edges x_1x' and x_2x', forms an induced cycle C in G. Then $G[V(C) \cup V(P'')]$ is isomorphic to $D_N^{|V(C)|}$, so the claim holds.

Now suppose x' has exactly one neighbor $x_1 \in X$. If X induces a cycle in G, then the cycle $G[X]$, the path P'', and the edge $x'x_1$ together form a graph that is isomorphic to $D_N^{|X|}$, so the claim holds. Suppose X induces a path in G; recall that this path has exactly $2N+1$ vertices, and x is the middle vertex of this path. If $x_1 = x$, then $G[X \cup V(P'')]$ is isomorphic to $T_N^{N,N}$. Suppose $x_1 \neq x$. Let P_X be the unique path in $G[X]$ from x_1 to x. Then the graph $G[V(P_X) \cup V(P')]$ contains an induced cycle C such that x' lies on C, and the graph $G[V(C) \cup V(P'')]$ is isomorphic to $D_N^{|V(C)|}$. This completes the proof of Claim 1.

Let G_x be an induced subgraph of $G[X \cup L]$ that is isomorphic to a graph in $\{D_N^i \mid i \geq 3\} \cup \{T_N^{N,N}\}$ and that is constructed from the cycle C_x in the way described in the proof of Claim 1. In particular, let x'' be the vertex of G_x that is closest to y in G. Recall that x'' is a vertex of P and has degree 1 in G_x. It is clear from the construction of G_x that every vertex in G_x has distance at most $2N+1$ to x. By symmetry, we can define an induced subgraph G_y of $G[Y \cup R]$ and a vertex y'' in G_y in an analogous way, that is, G_y is isomorphic to a graph in $\{D_N^i \mid i \geq 3\} \cup \{T_N^{N,N}\}$, and y'' is the vertex of G_y that is closest to x in G.

Let P^* be the subpath of P from x'' to y''. The fact that P is a shortest path from x to y implies that x'' and y'' are the only two vertices of G_x and G_y that are adjacent to internal vertices of P^*. Moreover, there are no edges between G_x and G_y, as otherwise there would be a path from x to y of length at most $4N+2$, contradicting the fact that P is a shortest path from x to y. Let G^* denote the induced subgraph of G obtained from $G_x + G_y$ by connecting x'' and y'' using the path P^*. We distinguish four cases, and obtain a contradiction in each case. We will repeatedly use the fact that in each case, G^* can be obtained from a "large" butterfly by deleting at most two vertices.

Case 1. G_x is isomorphic to D_N^i and G_y is isomorphic to D_N^j for some $i,j \geq 3$.

In this case, G^* is isomorphic to $B_{i,j,k}$ for some $k \geq 2N$. Since $\mathcal{H}$ covers the pair (i,j), there exists a graph $H \in \mathcal{H}$ such that H is an induced subgraph of $B_{i,j,N}$ by Definition 1. Due to Observation 1, H is also an induced subgraph of G^* and hence also of G. This contradicts the assumption that G is $\mathcal{H}$-free.

Case 2. G_x is isomorphic to D_N^i for some $i \geq 3$ and G_y is isomorphic to $T_N^{N,N}$.

Since $\mathcal{H}$ covers the pair $(i, 2N)$, there exists a graph $H \in \mathcal{H}$ such that H is an induced subgraph of $B_{i,2N,N}$. Since $|V(H)| \leq N$, the graph H contains at most one cycle, and this cycle, if it exists, is of length i. Hence it is clear that H is also an induced subgraph of G^*. This contradicts the assumption that G and thus G^* is $\mathcal{H}$-free.

Case 3. G_x is isomorphic to $T_N^{N,N}$ and G_y is isomorphic to D_N^i for some $i \geq 3$.

By symmetry, we obtain a contradiction in the same way as in Case 2.

Case 4. Both G_x and G_y are isomorphic to $T_N^{N,N}$.

Since $\mathcal{H}$ covers the pair $(2N, 2N)$, there exists a graph $H \in \mathcal{H}$ such that H is an induced subgraph of $B_{2N,2N,N}$. This graph H has at most N vertices, which implies that H has no cycle. But then H is an induced subgraph of G^*, again yielding the desired contradiction. This completes the proof of Theorem 1. $\square$

3 Which Graphs H Cover Which Pairs (i, j)?

Recall that by Definition 1, a graph H covers a pair (i, j) if and only if H is an induced subgraph of $B_{i,j,N}$, where $N = 2 \cdot |V(H)| + 1$. In particular, if a graph H is not an induced subgraph of a butterfly, then H does not cover any pair (i, j). However, it is important to note that some induced subgraphs of $B_{i,j,N}$ cover more pairs than others. For example, as we will see in Lemma 6, a linear forest covers all pairs (i, j) with $i, j \geq 3$, but this is not the case for any induced subgraph of $B_{i,j,N}$ that is not a linear forest.

In this section, we will prove exactly which pairs (i, j) are covered by which graphs H. For convenience, we first describe all the possible induced subgraphs of $B_{i,j,N}$ in the following observation.

Observation 3 *Let H be a graph, let $N = 2 \cdot |V(H)| + 1$, and let $i, j \geq 3$ be two integers. Then H is an induced subgraph of $B_{i,j,N}$ if and only if H is isomorphic to the disjoint union of a linear forest (possibly on zero vertices) and at most one of the following graphs:*

- *(i) D_ℓ^i for some $\ell \geq 0$;*
- *(ii) D_ℓ^j for some $\ell \geq 0$;*
- *(iii) $D_\ell^i + D_{\ell'}^j$ for some $\ell, \ell' \geq 0$;*
- *(iv) $T_k^{p,q}$ for some $k, p, q \geq 1$ such that $p + q + 2 \leq \max\{i, j\}$;*
- *(v) $T_k^{p,q} + T_{k'}^{p',q'}$ for some $k, p, q, k', p', q' \geq 1$ such that $p+q+2 \leq i$ and $p'+q'+2 \leq j$;*
- *(vi) $D_\ell^i + T_k^{p,q}$ for some $\ell \geq 0$ and $k, p, q \geq 1$ such that $p + q + 2 \leq j$;*
- *(vii) $D_\ell^j + T_k^{p,q}$ for some $\ell \geq 0$ and $k, p, q \geq 1$ such that $p + q + 2 \leq i$.*

The lemmata below show, for each of the induced subgraphs described in Observation 3, exactly which pairs (i, j) they cover. At the end of the proof of each of the lemmata, we refer to a table in which the set of covered pairs is depicted. This will be helpful in the applications presented in Section 4.

Lemma 1. *Let H be a graph, let $p \geq 3$, and let $\mathcal{X}$ be the set consisting of the pairs (i, j) with $i, j \geq 3$ and $p \in \{i, j\}$.*

- *(i) If H is an induced subgraph of D_k^p for some $k \geq 0$, then H covers all the pairs in $\mathcal{X}$.*

(ii) If D_k^p is an induced subgraph of H for some $k \geq 0$, then H does not cover any pair that does not belong to $\mathcal{X}$.

Proof. Let $N = 2 \cdot |V(H)| + 1$. Suppose H is an induced subgraph of D_k^p for some $k \geq 0$. Then H is also an induced subgraph of $B_{i,j,N}$ for every $i, j \geq 3$ such that $p \in \{i, j\}$. Hence, by Definition 1, H covers the pairs (p, j) and (i, p) for every $i, j \geq 3$.

Now suppose D_k^p is an induced subgraph of H for some $k \geq 0$. Then H contains a cycle of length p. Hence it is clear that if H is an induced subgraph of a butterfly $B_{i,j,N}$, then we must have $p \in \{i, j\}$. This shows that H cannot cover any pair that does not belong to $\mathcal{X}$.

See Table 1 for an illustration of the pairs in $\mathcal{X}$. □

Observe that H covers exactly these pairs and no other pairs if H is isomorphic to D_k^p for some $k \geq 0$, as hence both conditions (i) and (ii) are satisfied. In fact, the same holds when H is the disjoint union of a linear forest and D_k^p, since then H is an induced subgraph of $D_{k'}^p$ for some $k' \geq k$, and hence conditions (i) and (ii) are also satisfied in this case.

Lemma 2. *Let H be a graph, let $p, q \geq 3$, and let $\mathcal{X} = \{(p, q), (q, p)\}$.*

(i) If H is an induced subgraph of $D_k^p + D_k^q$ for some $k \geq 0$, then H covers all the pairs in $\mathcal{X}$.

(ii) If $D_k^p + D_k^q$ is an induced subgraph of H for some $k \geq 0$, then H does not cover any pair that does not belong to $\mathcal{X}$.

Proof. Let $N = 2 \cdot |V(H)| + 1$. First suppose H is an induced subgraph of $D_k^p + D_k^q$ for some $k \geq 0$. Then H is also an induced subgraph of $B_{p,q,N}$. Hence, by Definition 1, graph H covers the pairs (p, q) and (q, p).

To prove (ii), suppose $D_k^p + D_k^q$ is an induced subgraph of H for some $k \geq 0$. Then H contains a cycle of length p and a cycle of length q. Hence, from the definition of butterflies and by Definition 1, it is clear that H is not an induced subgraph of $B_{i,j,N}$ for any $i, j \geq 3$ such that $\{p, q\} \neq \{i, j\}$. This proves claim (ii).

See Table 2 for an illustration of the pairs in $\mathcal{X}$. □

Lemma 3. *Let H be a graph, let $p, q \geq 1$, and let $\mathcal{X}$ be the set consisting of the pairs (i, j) with $i, j \geq 3$ and $\max\{i, j\} \geq p + q + 2$.*

(i) If H is an induced subgraph of $T_r^{p,q}$ for some $r \geq 1$, then H covers all the pairs in $\mathcal{X}$.

(ii) If $T_r^{p,q}$ is an induced subgraph of H for some $r \geq 1$, then H does not cover any pair that does not belong to $\mathcal{X}$.

Proof. Let $N = 2 \cdot |V(H)| + 1$. Suppose H is an induced subgraph of $T_r^{p,q}$ for some $r \geq 1$. Then H is also an induced subgraph of the butterfly $B_{i,j,N}$ for every $i, j \geq 3$ such that $\max\{i, j\} \geq p + q + 2$. Hence H covers all the pairs in $\mathcal{X}$ due to Definition 1.

For the converse direction, suppose $T_r^{p,q}$ is an induced subgraph of H for some $r \geq 1$. Then H is not an induced subgraph of $B_{i,j,N}$ whenever $\max\{i, j\} < p + q + 2$. This shows that H does not cover any pair that does not belong to $\mathcal{X}$.

See Table 3 for an illustration of the pairs in $\mathcal{X}$. □

$i \backslash j$	3	$\cdots$	$\cdots$	p	$\cdots$	$\cdots$
3				✓		
$\vdots$				✓		
$\vdots$				✓		
p	✓	✓	✓	✓	✓	✓
$\vdots$				✓		
$\vdots$				✓		

Table 1. Pairs (i, j) covered by H when H is isomorphic to D_k^p.

$i \backslash j$	3	$\cdots$	$\cdots$	p	$\cdots$	q	$\cdots$
3							
$\vdots$							
p						✓	
$\vdots$							
q				✓			
$\vdots$							

Table 2. Pairs (i, j) covered by H when H is isomorphic to $D_k^p + D_k^q$, with $p \leq q$.

Lemma 4. *Let H be a graph, let $p, q, p', q' \geq 1$ be such that $p + q \leq p' + q'$, and let $\mathcal{X}$ consist of all the pairs (i, j) with $\min\{i, j\} \geq p + q + 2$ and $\max\{i, j\} \geq p' + q' + 2$.*

(i) If H is an induced subgraph of $T_r^{p,q} + T_r^{p',q'}$ for some $r \geq 1$, then H covers all the pairs in $\mathcal{X}$.
(ii) If $T_r^{p,q} + T_r^{p',q'}$ is an induced subgraph of H for some $r \geq 1$, then H does not cover any pair that does not belong to $\mathcal{X}$.

Proof. Let $N = 2 \cdot |V(H)| + 1$. Suppose H is an induced subgraph of $T_r^{p,q} + T_r^{p',q'}$ for some $r \geq 1$. Then H is also an induced subgraph of $B_{i,j,N}$ for any i, j with $\min\{i, j\} \geq p + q + 2$ and $\max\{i, j\} \geq p' + q' + 2$. Hence H covers all the pairs in $\mathcal{X}$.

To prove (ii), suppose $T_r^{p,q} + T_r^{p',q'}$ is an induced subgraph of H for some $r \geq 1$. Then H is not an induced subgraph of $B_{i,j,N}$ whenever $\min\{i, j\} < p + q + 2$ or $\max\{i, j\} < p' + q' + 2$. Hence, by Definition 1, H cannot cover any pair that does not belong to $\mathcal{X}$.

See Table 4 for an illustration of the pairs in $\mathcal{X}$. □

Lemma 5. *Let H be a graph, let $p \geq 3$ and $p', q' \geq 1$, and let $\mathcal{X}$ be the set consisting of the pairs (i, j) with either $p = i$ and $j \geq p' + q' + 2$ or $p = j$ and $i \geq p' + q' + 2$.*

(i) If H is an induced subgraph of $D_k^p + T_k^{p',q'}$ for some $r \geq 1$, then H covers all the pairs in $\mathcal{X}$.
(ii) If $D_k^p + T_k^{p',q'}$ is an induced subgraph of H for some $r \geq 1$, then H does not cover any pair that does not belong to $\mathcal{X}$.

Proof. Let $N = 2 \cdot |V(H)| + 1$. Suppose H is an induced subgraph of $D_k^p + T_k^{p',q'}$ for some $r \geq 1$. Then H is an induced subgraph of $B_{i,j,N}$ for every i, j with either $p = i$ and $j \geq p' + q' + 2$ or $p = j$ and $i \geq p' + q' + 2$. Hence, by Definition 1, H covers all the pairs in $\mathcal{X}$.

To prove (ii), suppose $D_k^p + T_k^{p',q'}$ is an induced subgraph of H for some $r \geq 1$. Suppose H covers a pair (i, j). By Definition 1, H is an induced subgraph of $B_{i,j,N}$. Since H contains a cycle of length p due to the presence of D_k^p as induced subgraph, it holds that $p \in \{i, j\}$. Suppose $p = i$. Since H contains $T_k^{p',q'}$ as an induced subgraph,

i \ j	3	$\cdots$	$\cdots$	$p+q+2$	$\cdots$	$\cdots$
3				✓	✓	✓
$\vdots$				✓	✓	✓
$\vdots$				✓	✓	✓
$p+q+2$	✓	✓	✓	✓	✓	✓
$\vdots$	✓	✓	✓	✓	✓	✓
$\vdots$	✓	✓	✓	✓	✓	✓

Table 3. Pairs (i, j) covered by H when H is isomorphic to $T_k^{p,q}$.

i \ j	3	$\cdots$	$p+q+2$	$\cdots$	$p'+q'+2$	$\cdots$
3						
$\vdots$						
$p+q+2$					✓	✓
$\vdots$					✓	✓
$p'+q'+2$			✓	✓	✓	✓
$\vdots$			✓	✓	✓	✓

Table 4. Pairs (i, j) covered by H when H is isomorphic to $T_k^{p,q} + T_k^{p',q'}$, with $p + q \leq p' + q'$.

we must have that $j \geq p' + q' + 2$. Similarly, if $p = j$, then it holds that $i \geq p' + q' + 2$. We conclude that $(i, j) \in \mathcal{X}$, which suffices to prove (ii).

See Tables 5, 6, and 7 for an illustration of the pairs in $\mathcal{X}$ in the cases where $p < p' + q' + 2$, $p > p' + q' + 2$, and $p = p' + q' + 2$, respectively. □

i \ j	3	$\cdots$	p	$\cdots$	$p'+q'+2$	$\cdots$
3						
$\vdots$						
p					✓	✓
$\vdots$						
$p'+q'+2$			✓			
$\vdots$			✓			

Table 5. Pairs (i, j) covered by H when H is isomorphic to $D_k^p + T_k^{p',q'}$ when $p \leq p' + q' + 2$.

i \ j	3	$\cdots$	$p'+q'+2$	$\cdots$	p	$\cdots$
3						
$\vdots$						
$p'+q'+2$					✓	
$\vdots$					✓	
p			✓	✓	✓	✓
$\vdots$					✓	

Table 6. Pairs (i, j) covered by H when H is isomorphic to $D_k^p + T_k^{p',q'}$ when $p \geq p' + q' + 2$.

Lemma 6. *A graph H covers every pair (i, j) with $i, j \geq 3$ if and only if H is a linear forest.*

Proof. If H is a linear forest, then H is an induced subgraph of a path on $2 \cdot |V(H)|$ vertices. Hence H is also an induced subgraph of $B_{i,j,N}$ for every $i, j \geq 3$, where $N = 2 \cdot |V(H)| + 1$. By Definition 1, H covers every pair (i, j) with $i, j \geq 3$.

For the reverse direction, suppose H covers every pair (i, j) with $i, j \geq 3$. For contradiction, suppose H is not a linear forest. Then, as a result of Definition 1 and

$i \backslash j$	3	$\cdots$	$\cdots$	p	$\cdots$	$\cdots$
3						
$\vdots$						
$\vdots$						
p				✓	✓	✓
$\vdots$				✓		
$\vdots$				✓		

Table 7. Pairs (i,j) covered by H when H is isomorphic to $D_k^p + T_k^{p',q'}$ when $p = p' + q' + 2$.

Observation 3, either H contains $T_k^{p,q}$ as an induced subgraph for some $p, q, k \geq 1$, or H contains D_ℓ^r as an induced subgraph for some $r \geq 3$ and $\ell \geq 0$. In the first case, it follows from Lemma 3(ii) that H does not cover the pair $(3,3)$. In the second case, it follows from Lemma 1(ii) that H does not cover any pair (i,j) with $r \notin \{i,j\}$. In both cases, we obtain the desired contradiction. $\square$

4 Applications of Our Results

In this section, we show how we can apply Theorem 1 and the lemmata from Section 3 in order to obtain some concrete characterizations. Let us first remark that the following result, previously obtained in [1], immediately follows from Theorem 1 and Lemma 6.

Corollary 1 ([1]). *Let H be a graph. Then the poc-fvs for H-free graphs is upper bounded by a constant c_H if and only if H is a linear forest.*

Obtaining similar characterizations for finite families $\mathcal{H}$ with $|\mathcal{H}| \geq 2$ is slightly more involved, but can be done using the finite procedure we informally describe below. We then illustrate the procedure in Corollary 2 below for the case where $|\mathcal{H}| = 2$.

Let $k \geq 2$. Suppose we want to characterize the families of graphs $\mathcal{H}$ with $|\mathcal{H}| = k$ for which the poc-fvs for $\mathcal{H}$-free graphs is upper bounded by a constant. It follows from Theorem 1 and Lemma 6 that the poc-fvs for $\mathcal{H}$-free graphs is bounded whenever $\mathcal{H}$ contains a linear forest. What about families $\mathcal{H}$ that do not contain a linear forest?

Consider the infinite table containing all the pairs (i,j) with $i, j \geq 3$. From Lemmata 3–5 and Tables 1–7, we can observe two important things. First, the only graphs H that cover the pair $(3,3)$ are induced subgraphs of $2D_\ell^3$ for some $\ell \geq 0$. Second, the only graphs H that cover infinitely many rows and columns of this table are induced subgraphs of $T_r^{p,q} + T_r^{p',q'}$ for some $r, p, q, p', q' \geq 1$. Hence, any finite family $\mathcal{H}$ that covers all pairs (i,j) must contain at least one graph of both types. Formally, we have the following observation (observe that every linear forest is an induced subgraph of $2D_\ell^p$ for some $\ell \geq 0$ and of $T_r^{p,q} + T_r^{p',q'}$ for some $r, p, q, p', q' \geq 1$):

Observation 4 *Let $\mathcal{H}$ be a finite family of graphs such that $|\mathcal{H}| \geq 2$. If the poc-fvs for $\mathcal{H}$-free graphs is upper bounded by a constant $c_{\mathcal{H}}$, then $\mathcal{H}$ contains an induced subgraph of $2D_\ell^3$ for some $\ell \geq 0$ and an induced subgraph of $T_r^{p,q} + T_r^{p',q'}$ for some $r, p, q, p', q' \geq 1$.*

Suppose $\mathcal{H}$ is a family of k graphs such that the poc-fvs for $\mathcal{H}$-free graphs is bounded by a constant. By Observation 4, $\mathcal{H}$ contains a graph H_1 that is an induced subgraph of $T_r^{p,q} + T_r^{p',q'}$ for some $r, p, q, p', q' \geq 1$.

If H_1 is also an induced subgraph of $T_r^{p,q}$ for some $r, p, q \geq 1$, or if $\mathcal{H}$ contains another graph that is of this form, then Lemma 3 and Table 3 show that there are only finitely many pairs (i, j) that are not covered by H_1. These cells need to be covered by the remaining graphs in $\mathcal{H}$. Using Lemmata 3–5, we can determine exactly which combination of graphs covers exactly those remaining pairs.

Suppose $\mathcal{H}$ does not contain induced subgraph of $T_r^{p,q}$ for any $r, p, q \geq 1$. Then Lemma 4 and Table 4 imply that there are finitely many rows and columns in which no pair is covered by H_1. In particular, since $p, q, p', q' \geq 1$, the pairs $(i, 3)$ and $(3, j)$ are not covered for any $i, j \geq 3$. From the lemmata in Section 3 and the corresponding tables, it it clear that the only graphs H that cover infinitely many pairs of this type are induced subgraphs of $T_r^{p,q}$ for some $r, p, q \geq 1$ or of $D_{r'}^3 + T_r^{p,q}$ for some $r' \geq 0$ and $p, q \geq 1$. Hence, $\mathcal{H}$ must contain a graph H_2 that is isomorphic to such an induced subgraph. Similarly, if the pairs $(i, 4)$ and $(4, j)$ are not covered for any $i, j \geq 3$, then $\mathcal{H}$ must contain an induced subgraph of $T_r^{p,q}$ for some $r, p, q \geq 1$ or of $D_{r'}^4 + T_r^{p,q}$ for some $r' \geq 0$ and $p, q \geq 1$, etcetera. Once all rows and columns contain only finitely many pairs that are not covered yet, we can determine all possible combinations of graphs that cover those last pairs.

To illustrate the above procedure, we now give an explicit description of exactly those families $\{H_1, H_2\}$ for which the poc-fvs for $\{H_1, H_2\}$-free graphs is upper bounded by a constant.

Corollary 2. *Let H_1 and H_2 be two graphs, and let $\mathcal{H} = \{H_1, H_2\}$. Then the poc-fvs for $\mathcal{H}$-free graphs is upper bounded by a constant $c_{\mathcal{H}}$ if only if there exist integers $\ell \geq 0$ and $r \geq 1$ such that one of the following conditions holds:*

- *H_1 or H_2 is a linear forest;*
- *H_1 and H_2 are induced subgraphs of D_ℓ^3 and $2T_r^{1,1}$, respectively;*
- *H_1 and H_2 are induced subgraphs of $2D_\ell^3$ and $T_r^{1,1}$, respectively.*

Proof. First suppose that the price of connectivity for feedback vertex set for $\mathcal{H}$-free graphs is bounded by some constant $c_{\mathcal{H}}$, and suppose that neither H_1 nor H_2 is a linear forest. Due to Observation 4, we may without loss of generality assume that H_1 is an induced subgraph of $2D_\ell^3$ for some $\ell \geq 0$ and H_2 is an induced subgraph of $T_r^{p,q} + T_r^{p',q'}$ for some $r, p, q, p', q' \geq 1$. From Lemmata 1 and 2 and the assumption that H_1 is not a linear forest, it follows that H_1 does not cover the pair $(4, 4)$. Hence H_2 must cover this pair. This, together with Lemma 4, implies that $p = q = p' = q' = 1$, i.e., H_2 is an induced subgraph of $2T_r^{1,1}$ for some $r \geq 1$.

If H_1 is an induced subgraph of $D_{\ell'}^3$ for some $\ell' \geq 0$, then the second condition holds and we are done. Suppose this is not the case. Then H_1 covers only the pair $(3, 3)$ due to Lemma 2. This means that all the pairs (i, j) with $i, j \geq 3$ and $3 \in \{i, j\}$, apart from $(3, 3)$, must be covered by H_2. From Lemma 3 and 4 it is clear that this only holds if H_2 is an induced subgraph of $T_{r'}^{1,1}$ for some $r' \geq 1$. Hence the third condition holds.

The converse direction follows by combining Theorem 1 with Lemma 6, Lemmata 1 and 4, and Lemmata 2 and 3, respectively. $\square$

Observe that Corollary 2 generalizes Corollary 1, as the class of H-free graphs is equivalent to the class of $\{H, H\}$-free graphs. We point out that any graph H that is an induced subgraph of both D_ℓ^3 for some $\ell \geq 0$ and of $2T_r^{1,1}$ for some $r \geq 1$ is a linear forest.

5 Conclusion

Recall that in [1], we proved for every graph H which of the following cases holds: (i) $\mathrm{cfvs}(G) = \mathrm{fvs}(G)$ for every connected H-free graph G; (ii) there exists a constant c_H such that $\mathrm{cfvs}(G) \leq \mathrm{fvs}(G) + c_H$ for every connected H-free graph G; (iii) there exists a constant c_H such that $\mathrm{cfvs}(G) \leq c_H \cdot \mathrm{fvs}(G)$ for every connected H-free graph G; (iv) there does not exist a constant c_H such that $\mathrm{cfvs}(G) \leq c_H \cdot \mathrm{fvs}(G)$ for every connected H-free graph G. Theorem 1 extends the case of (iii) to all finite families $\mathcal{H}$. A natural question to ask is to characterize all finite families $\mathcal{H}$ for (i) and (ii) as well.

Another natural question to ask is whether Theorem 1 can be extended to families $\mathcal{H}$ which are not finite, i.e., to all hereditary classes of graphs. Definition 1 and Theorem 1 show that for any finite family $\mathcal{H}$, the poc-fvs for $\mathcal{H}$-free graphs is bounded essentially when the graphs in this class do not contain arbitrarily large induced butterflies. The following example shows that when $\mathcal{H}$ is infinite, it is no longer only butterflies that can cause the poc-fvs to be unbounded. Let G be a graph obtained from K_3 by first duplicating every edge once, and then subdividing every edge arbitrarily many times. Let $\mathcal{G}$ be the class of all graphs that can be constructed this way. In order to make $\mathcal{G}$ hereditary, we take its closure under the induced subgraph relation. Let $\mathcal{G}'$ be the resulting graph class. Observe that graphs in this class have arbitrarily large minimum connected feedback vertex sets, while $\mathrm{fvs}(G) \leq 2$ for every graph $G \in \mathcal{G}'$. Hence, the poc-fvs for $\mathcal{G}'$ is not bounded. However, no graph in this family contains a butterfly as an induced subgraph.

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