Minimal dominating sets in graph classes: combinatorial bounds and enumeration*

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Abstract. The maximum number of minimal dominating sets that a graph on n vertices can have is known to be at most 1.7159^n. This upper bound might not be tight, since no examples of graphs with 1.5705^n or more minimal dominating sets are known. For several classes of graphs, we substantially improve the upper bound on the maximum number of minimal dominating sets in graphs on n vertices. In some cases, we provide examples of graphs whose number of minimal dominating sets exactly matches the proved upper bound for that class, thereby showing that these bounds are tight. For all considered graph classes, the upper bound proofs are constructive and can easily be transformed into algorithms enumerating all minimal dominating sets of the input graph.

1 Introduction

Combinatorial questions of the type *"What is the maximum number of vertex subsets satisfying a given property in a graph?"* have found interest and applications in computer science, especially in exact exponential algorithms [5]. The question has been studied recently for minimal feedback vertex sets, minimal separators, potential maximal cliques, and for minimal feedback vertex sets in tournaments [3, 6, 7]. A famous classical example is the highly cited theorem of Moon and Moser [14], which states that the maximum number of maximal cliques and maximal independent sets, respectively, in any graph on n vertices is $3^{n/3}$. Although the original proof of the upper bound in [14] is by induction, it is not hard to transform it into a branching algorithm enumerating all maximal independent sets of a graph in time $O^*(3^{n/3})$, where the O^*-notation is used to suppress polynomial factors. These results were used by Lawler [13] to give an algorithm for graph coloring, which was the fastest algorithm for this purpose for over two decades. A faster algorithm for graph coloring was obtained by Eppstein [2] by improving the upper bound on the maximum number of maximal independent sets of small size.

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The number of papers on domination in graphs is in the thousands, and several well known surveys and books are dedicated to the topic (see, e.g., [9]). It is surprising that a first non-trivial answer to the Moon and Moser type question *"What is the maximum number of minimal dominating sets in a graph?"* was established only recently. Fomin, Grandoni, Pyatkin and Stepanov [4] showed that the maximum number of minimal dominating sets in a graph on n vertices is at most 1.7159^n. This result was used to derive an $O(2.8718^n)$ algorithm for the Domatic Number problem [4]. Although examples of graphs with 1.5704^n minimal dominating sets have been identified [4] (see Fig. 1), it is not known whether graphs with 1.5705^n or more minimal dominating sets exist.

Our interest in this combinatorial question was triggered by the large gap between the best known lower and upper bound for general graphs and the exact exponential algorithms background of the problem. We provide upper and lower bounds for the maximum number of minimal dominating sets in a variety of graph classes. Our upper bounds heavily rely on structural graph properties. Typically, either we have tight bounds, i.e., matching upper and lower bounds, that are proved using combinatorial arguments, or we have asymptotic bounds that are proved using branching algorithms. Our findings are summarized in Table 1.

Graph Class	Lower Bound	Upper Bound
general [4]	1.5704^n	1.7159^n
chordal	1.4422^n	1.6181^n
cobipartite	1.3195^n	1.5875^n
split	1.4422^n	1.4656^n
proper interval	1.4422^n	1.4656^n
cograph*	1.5704^n	1.5705^n
trivially perfect*	1.4422^n	1.4423^n
threshold*	$\omega(G)$	$\omega(G)$
chain*	$\lfloor n/2 \rfloor + m$	$\lfloor n/2 \rfloor + m$

Table 1. Lower and upper bounds on the maximum number of minimal dominating sets. The bounds for graph classes marked with an asterisk are tight; differences in the last digit are caused by rounding.

Very recently, Kanté et al. [11] showed that all minimal dominating sets in a split graph G can be enumerated in time polynomial in the number of minimal dominating sets of G. They did not provide an upper bound on the maximum number of such sets. As an important byproduct of our results, we obtain algorithms to enumerate all minimal dominating sets for graphs in each of the studied graph classes. In fact, all our upper bound proofs are constructive and the enumeration algorithms are easy consequences of them: simply check for all the generated candidate sets whether they are indeed minimal dominating sets. For each graph class with an exponential upper bound, say $O(c^n)$, the running time of the corresponding enumeration algorithm is $O^*(c^n)$. The enumeration algorithms for threshold graphs and chain graphs have polynomial running times.

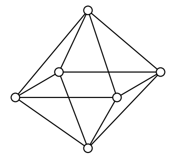

Fig. 1. The graph G^*, which has 6 vertices and 15 minimal dominating sets. The graph G^*_n on $n = 6k$ vertices, consisting of k disjoint copies of G^*, has $15^{n/6} \approx 1.5704^n$ mds.

We believe that our enumeration algorithms might have non-trivial algorithmic applications in domination type problems like Domatic Number.

Our paper is organized as follows. In Sections 3 and 4, we use branching algorithms to establish upper bounds on the number of minimal dominating sets in chordal graphs and split graphs, respectively. In Section 5, a combinatorial argument is applied to establish an upper bound for cobipartite graphs. In Sections 6 and 7, we determine *tight* upper bounds for cographs and chain graphs. The bounds for the remaining three graph classes are proved by similar techniques, and they are presented in a separate appendix.

2 Preliminaries

We work with simple undirected graphs. We denote such a graph by $G = (V, E)$, where V is the set of vertices and E is the set of edges of G. We adhere to the convention that $n = |V|$ and $m = |E|$. When the vertex set and the edge set of G are not specified, we use $V(G)$ and $E(G)$ to denote these, respectively. The set of *neighbors* of a vertex $v \in V$ is the set of vertices adjacent to v, and is denoted by $N_G(v)$. The *closed neighborhood* of v is $N_G[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, we define analogously $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$ and $N_G[S] = N_G(S) \cup S$. We will omit the subscript G when there is no ambiguity. A vertex v is *universal* if $N(v) = V$ and *isolated* if $N(v) = \emptyset$. The subgraph of G *induced* by S is denoted by $G[S]$. For ease of notation, we use $G-v$ to denote the graph $G[V \setminus \{v\}]$, and $G-S$ to denote the graph $G[V \setminus S]$. A graph is *connected* if there is a path between every pair of its vertices. A maximal connected subgraph of G is called a *connected component* of G. A set $S \subseteq V$ is called an *independent set* if $uv \notin E$ for every pair of vertices $u, v \in S$, and S is called a *clique* if $uv \in E$ for every pair of vertices $u, v \in S$. An independent set or a clique is *maximal* if no proper superset of it is an independent set or a clique, respectively.

A vertex set $S \subseteq V$ is a *dominating set* of G if $N[S] = V$. Every vertex v of a dominating set *dominates* the vertices in $N[v]$. A dominating set S is a *minimal dominating set* (*mds*) if no proper subset of S is a dominating set. It is an easy observation that, if S is a mds, then for every vertex $v \in S$, there is a vertex $x \in N[v]$ which is dominated only by v. We will call such a vertex x a *private neighbor* of v, since x is not adjacent to any vertex in $S \setminus \{v\}$. Note that a vertex in S might be its own private neighbor. The number of mds in a graph

G is denoted by $\mu(G)$. The following observation follows from the fact that every mds of G is the union of a mds of each connected component of G.

Observation 1 *Let G be a graph with connected components $G_1, G_2, \ldots, G_t$. Then $\mu(G) = \prod_{i=1}^{t} \mu(G_i)$.*

Each of the graph classes that we study will be defined in the section containing the results on that class. For the inclusion relationship between these graph classes, see Fig. 2 in the appendix. All of the graph classes mentioned in this paper can be recognized in linear time, and are closed under taking induced subgraphs [1, 8]. We now define two graph families that are useful for providing examples of lower bounds on the maximum number of mds. We write H_n to denote the graph on $n = 3k$ vertices which is the disjoint union of k triangles. We write S_n to denote the graph on $n = 3k$ vertices which consists of a clique C of size $2k$ and an independent set I of size k, such that each vertex of I has exactly two neighbors in C, and no two vertices in I have a common neighbor. It can be verified easily that $\mu(H_n) = \mu(S_n) = 3^{n/3} \approx 1.4422^n$. For several of our graph classes, the graph families H_n and C_n provide the best known lower bound on the maximum number of mds.

2.1 Preliminaries on branching

To prove the results given in the next two sections, we use branching algorithms to generate a collection of vertex subsets at the leaves of the corresponding search tree, containing *all* mds and possibly also subsets that are not mds. Consequently, the number of leaves of the search tree gives an upper bound on $\mu(G)$. By simply checking whether each generated vertex subset is indeed a mds, one can also obtain an enumeration algorithm for all mds of a graph G belonging to the studied graph class. Typically, every recursive call has input (G', D), where G' is an induced subgraph of the input graph G, and D is a subset of $V(G) \setminus V(G')$. The subset D contains vertices outside of G' that have been chosen to be in a possible minimal dominating set of G. Initially, no vertex has been chosen for a dominating set, so the algorithm starts with the call $(G, \emptyset)$. At every step, we make choices that result in new subproblems, in which the size of G' decreases and the size of D possibly increases. For every such set D, either there is a leaf of the search tree corresponding to a mds of G that contains D as a subset, or no mds of G contains D as a subset. Our algorithms always proceed in such a way that a vertex of G' is never needed to dominate a vertex outside of G'. As a consequence, no vertex of G' has a private neighbor outside of G'.

For the analysis of the number of leaves $T(n)$ in the search tree, we use standard terminology [5]. In particular, if at each step of the branching we make t new subproblems, where the size of the instance is decreased by $c_1, c_2, \ldots, c_t$ in each subproblem, respectively, we obtain a recurrence $T(n) \leq T(n - c_1) + T(n - c_2) + \ldots + T(n - c_t)$. Such a recurrence is said to have *branching vector* $(c_1, c_2, \ldots, c_t)$. The number of leaves in the search tree is upper bounded by $O^*(\alpha^n)$, where α is the largest real root of $x^n - x^{n-c_1} - \ldots - x^{n-c_t} = 0$ [5]. The number α is called the *branching number* of this branching vector. It is common to round α to the fourth digit after the decimal point. By rounding the last digit up, we can use O notation instead of O^* notation [5]. If different branching vectors are involved at different steps of an algorithm, then the branching vector with the highest branching number gives an upper bound on the number of leaves. In our results, we will not do the calculations of α explicitly, but just say, e.g., that branching vector $(2,2)$ has branching number 1.4143, which implies that the number of leaves in the search tree is bounded by $O(1.4143^n)$ in a branching algorithm where only branching vector $(2,2)$ occurs.

3 Chordal graphs

A *chord* of a cycle is an edge between two non consecutive vertices of the cycle. A graph is *chordal* if every cycle of length at least 4 has a chord. A vertex v is called *simplicial* if $N(v)$ is a clique. Every chordal graph has a simplicial vertex [8]. Observe that H_n and S_n are chordal, giving us examples of chordal graphs with $3^{n/3} \approx 1.4422^n$ mds.

Theorem 1. *A chordal graph has at most $O(1.6181^n)$ minimal dominating sets.*

Proof. Given an instance (G', D), we say that a vertex v of G' is *already dominated* if D contains a vertex of $N_G(v)$. Our branching algorithm picks a simplicial vertex x of G'. If x is isolated: if x is already dominated, then we do not add x to D, otherwise we add x to D. No branching is involved; we delete x from G' and continue with another simplicial vertex of G'. From now on, we assume that x has at least one neighbor in G'. We take action depending on whether or not x is already dominated, and on the number of neighbors x has in G'. Note that only one of the cases below applies, and will be executed by the algorithm.

Case 1: x is already dominated. We branch on the choice of either adding x to D or discarding x from inclusion in a possible mds containing D.

- *Add:* $x \in D$. Since x is already dominated, it needs a private neighbor in $N_{G'}(x)$. This means that no vertex of $N_{G'}(x)$ can appear in a mds containing D as a subset. Consequently, we can safely delete $N_{G'}[x]$, which results in the instance $(G' - N_{G'}[x], D \cup \{x\})$, and gives a decrease of at least 2 vertices.
- *Discard:* $x \notin D$. Since x is already dominated, it is safe to simply delete x from G'. This results in the instance $(G' - x, D)$, and gives a decrease of 1 vertex.

Case 2: x is not already dominated and has at least 2 neighbors in G'. Let y be any neighbor of x in G'. We branch on the choice of either adding y to D or discarding y with respect to D.

- *Add:* $y \in D$. When y is added to D, it dominates x. Then x will never be part of a mds containing D, since it would need a private neighbor, which does not exist since $N_{G'}(x) \subseteq N_{G'}(y)$ and every vertex in $G - V(G')$ is already dominated by D. We can therefore safely delete both x and y, which results in the instance $(G' - \{x, y\}, D \cup \{y\})$, and gives a decrease of 2.

– *Discard:* $y \notin D$. In this case, we simply delete y from G', which is safe for the following reason. Vertex x is not deleted and still needs to be dominated, and every neighbor of x in G' is also a neighbor of y in G', since x is simplicial. Hence, when x becomes dominated, then so will y. This might also happen by x being added to D at a later step. Hence we create a new instance $(G'-y, D)$, which gives a decrease of 1.

Case 3: x is not already dominated and has exactly one neighbor y in G'. Since x is not already dominated and is only adjacent to y in G', either x or y needs to be added to D to ensure that x is dominated. We branch on these possibilities.

– $x \in D$. In this case, y becomes dominated, and no mds containing D as a subset can contain y, since then x would not have a private neighbor. Consequently, we can safely delete x and y in this case. We get the instance $(G'-\{x, y\}, D \cup \{x\})$, and a decrease of 2.
– $y \in D$. Now x becomes dominated, and it can never become a member of a mds containing D, as x would then not have a private neighbor. Again, we can safely delete x and y, yielding the instance $(G'-\{x, y\}, D \cup \{y\})$, and a decrease of 2.

The branching vectors obtained in Cases 1, 2, and 3 are $(2, 1)$, $(2, 1)$, and $(2, 2)$, respectively. Branching vector $(2, 1)$ has the largest branching number, namely 1.6181, resulting in an upper bound of $O(1.6181^n)$. $\square$

For split graphs, which form a subset of chordal graphs, we are able to give a better upper bound in the next section.

4 Split graphs

A graph $G = (V, E)$ is a *split graph* if V can be partitioned into a clique C and an independent set I, where (C, I) is called a *split partition* of G. A split partition can be computed in linear time [8], and is not necessarily unique. In particular, if a vertex $c \in C$ has no neighbors in I, then $(C \setminus \{c\}, I \cup \{c\})$ is also a split partition of G. We will assume in the below that I is maximal, i.e., every vertex of C has a neighbor in I. Note that a mds of G cannot contain a vertex $u \in I$ together with a neighbor of u. Since S_n is a split graph, there are split graphs with $3^{n/3} \approx 1.4422^n$ mds.

Theorem 2. *A split graph has at most $O(1.4656^n)$ minimal dominating sets.*

Proof. As in the previous section, we use branching, but this time we use a two-phase branching algorithm. Our initial input is a split graph G and a split partition (C, I) of G, where I is maximal. At each step of the algorithm, the subproblem at hand is described by (G', C', I', D), where G' is an induced subgraph of G and (C', I') is a split partition of G' such that $C' \subseteq C$ and $I' \subseteq I$. Our algorithm proceeds in such a way that no vertex of I' is already dominated by a vertex in D. Consequently, if a vertex of I' has no neighbor in C', then it is added to D; this rule requires no branching. If no such isolated vertex exists in I', the algorithm chooses a vertex $c \in C'$ such that c has a maximum number of neighbors in I', as long as this maximum number is at least 2. Then it branches and recursively solves two subproblems: either it selects c to be added to D and removes c and all its neighbors in I' from the current graph, or it discards c from being added to D and removes c from the graph. Hence, the decrease in the size of the graph is at least 3 in the first subproblem, and 1 in the second subproblem. This implies the branching vector $(3, 1)$, and its branching number is 1.4656. This completes the description of the first phase of the algorithm.

Let us consider a leaf of the corresponding search tree, and the instance (G', C', I', D) at this leaf. Contrary to the analysis in the previous section, a leaf of the search tree may lead to more than one mds, since we stopped branching as soon as each vertex in C' had at most one neighbor in I', and thus $V(G')$ might be non-empty. We claim that there are at most $3^{t/3}$ mds of G that can be obtained from this instance, where t is the number of vertices of G'. To show this, we now describe the second phase of the algorithm. If there is a vertex c in C' that does not have any neighbor in I', then we know that c is already dominated. This is because c originally had at least one neighbor in I, and since the neighbors of c in I were deleted, while c itself was not deleted, a neighbor $x \in C$ of c must already have been placed in D, by the description of the algorithm. Hence, c is dominated by a vertex in $C \cap D$ and c has no private neighbor, which implies that c cannot be added to D. Consequently, as a first step, we remove all vertices of C' that do not have any neighbor in I'. Then, we choose a vertex u of I'. Note that u is not isolated. We branch into the following $|N_{G'}(u)| + 1$ subproblems: for every neighbor c of u, add c to D and remove u and all its neighbors from G', and, in the last subproblem, add u to D and remove u and all its neighbors. Since every vertex in C' has exactly one neighbor in I', exactly one vertex of $N_{G'}[u]$ belongs to a mds of G containing D, showing the correctness of the branching. The subproblems are solved recursively.

Assuming the vertex u has j neighbors, there are $j + 1$ subproblems, and for each one the decrease is $j + 1$. Simple analysis leads to the recurrences $T(t) = (j + 1) \cdot T(t - j - 1)$ for $j \geq 2$ for the number of leaves in the search tree. This is a well-known recurrence, and its solution is $T(t) = 3^{t/3}$ (see e.g. [5]). Thus, the number of leaves of the search tree for the second branching algorithm is at most $3^{t/3}$, and now each leaf contains at most one mds of G.

To establish an upper bound on $\mu(G)$, notice that the number of leaves of the first branching algorithm containing an instance with t vertices is at most $O(1.4656^{n-t})$, since those leaves correspond to a total decrease of $n - t$ of the size of the graph (from G to the graph G' of the leaf). Consequently, the number of leaves of the search tree of the first branching is at most $\sum_{t=0}^{n} O(1.4656^{n-t})$. Since each of those leaves leads to at most $3^{t/3} \approx 1.4423^t$ mds of G, we conclude that the maximum number of mds in a split graph G is at most $\sum_{t=0}^{n} O(1.4656^{n-t} \cdot 1.4423^t) \leq \sum_{t=0}^{n} O(1.4656^n) = O(1.4656^n)$. □

The next class of graphs are unrelated to chordal graphs, but related to split graphs in the sense that their vertex sets have a partition into two cliques.

5 Cobipartite graphs

A graph $G = (V, E)$ is *cobipartite* if V can be partitioned into two cliques. To obtain a lower bound, we define a graph family B_n as follows. For $n = 5k$, start with two disjoint cliques X and Y, where $|X| = 4k$ and $|Y| = k$. Make every vertex in Y adjacent to exactly four vertices in X, such that every vertex in X is adjacent to exactly one vertex in Y. The graph B_n has $4^{n/5} \approx 1.3195^n$ mds that are subsets of X, and $O(n^2)$ minimal dominating sets of the form $\{x, y\}$ with $x \in X$ and $y \in Y$.

Theorem 3. *A cobipartite graph has at most $O(1.5875^n)$ minimal dominating sets.*

Proof. Let $G = (V, E)$ be a cobipartite graph on n vertices, and let (X, Y) be a partition of V such that X and Y are cliques. Assume, without loss of generality, that $|X| = \alpha n$ with $0.5 \leq \alpha \leq 1$, and $|Y| = (1 - \alpha)n$. Let D be a mds of G. If $|D| = 1$, then $D = \{v\}$ for some universal vertex of G. Hence G has at most n mds of size 1. If $|D| \geq 2$ and $D \cap X \neq \emptyset$ and $D \cap Y \neq \emptyset$, then we must have $D = \{x, y\}$ for some vertices $x \in X$ and $y \in Y$, since every vertex in D needs a private neighbor. Hence there are at most $n^2/4$ mds of this type. Now assume that $|D| \geq 2$, and that we either have $D \subseteq X$ or $D \subseteq Y$. Clearly there are at most $2^{|Y|} \leq 2^{n/2}$ mds D of G with $D \subseteq Y$. It remains to find an upper bound on the number of mds D of G satisfying $D \subseteq X$. Let $|D| = \beta n$, where $\beta \leq \alpha$ and $2 \leq \beta n \leq |X|$. Every vertex of D must have a private neighbor; this can only be a vertex of Y, since D contains at least two vertices of X. This implies that $\beta \leq 1 - \alpha$. The number of subsets of X of size βn is $\binom{\alpha n}{\beta n}$. For every fixed α, the value of $\binom{\alpha n}{\beta n}$ is maximized for $\beta = \alpha/2$. To maximize the value of $\binom{\alpha n}{\beta n}$, note that $\beta = \alpha/2 \leq 1 - \alpha$ implies $\alpha \leq 2/3$. Hence the number of mds D, with $|D| \geq 2$ and $D \subseteq X$, is at most $\binom{2n/3}{n/3}$, which is less than or equal to $2^{2n/3}$, i.e., the number of all subsets of a set of size $2n/3$. In total, there are at most $n + n^2/4 + 2^{n/2} + 2^{2n/3} = O(2^{2n/3}) = O(1.5875^n)$ mds in G. □

We now move on to graph classes for which we have tight upper bounds.

6 Cographs

Cographs are of particular interest in the study of the maximum number of mds, as the only known examples of graphs with 1.5704^n mds are cographs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *disjoint union* of G_1 and G_2 is the graph $G_1 \uplus G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of G_1 and G_2 is the graph $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{v_1 v_2 \mid v_1 \in V_1, v_2 \in V_2\})$. A graph G is a *cograph* if it can be constructed from isolated vertices by disjoint union and join operations. The graph G^*, depicted in Fig. 1, is a cograph. It has 6 vertices and 15 mds. Any graph G_n^* on $n = 6k$ vertices consisting of k disjoint copies of G^* is a cograph containing $15^{n/6} \approx 1.5704^n$ mds. No example of a graph with 1.5705^n or more minimal dominating sets is known.

Theorem 4. *A cograph has at most $15^{n/6}$ minimal dominating sets, and there are cographs with $15^{n/6}$ minimal dominating sets.*

Proof. G_n^*, defined before the theorem, is a cograph with $15^{n/6}$ mds, so it remains to prove the upper bound. It can be verified exhaustively that the theorem holds for all cographs on at most 6 vertices. Let G be a cograph on $n \geq 7$ vertices. We prove the theorem by induction on the number of vertices. By the definition of a cograph, there exist two subgraphs G_1 and G_2 of G such that $G = G \uplus G_2$ or $G = G_1 \bowtie G_2$. Let $n_i = |V(G_i)|$ for $i = 1, 2$. If $G = G_1 \uplus G_2$, then by Observation 1, we have $\mu(G) = \mu(G_1) \cdot \mu(G_2) \leq 15^{n_1/6} \cdot 15^{n_2/6} = 15^{n/6}$. Suppose that $G = G_1 \bowtie G_2$. Then any mds of G_1 dominates every vertex in G_2, and vice versa. This means that any mds of G_1 is a mds of G, and the same holds for any mds of G_2. Since G is the complete join of G_1 and G_2, no mds of G contains more than one vertex from G_1 and more than one vertex from G_2. This means that any mds of G that is not a mds of G_1 or G_2 is of the form $\{v_1, v_2\}$, where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Hence $\mu(G) = \mu(G_1) + \mu(G_2) + n_1 n_2 \leq 15^{n_1/6} + 15^{n_2/6} + n_1 n_2 = 15^{n_1/6} + 15^{(n-n_1)/6} + n_1(n-n_1)$. Since we assumed that $1 \leq n_1 \leq n-1$ and $n \geq 7$, the function $15^{n_1/6} + 15^{(n-n_1)/6} + n_1(n-n_1)$ is maximal when $n_1 \in \{1, n-1\}$. In both cases, we get $\mu(G) \leq 15^{(n-1)/6} + 15^{1/6} + n - 1$, which is less than $15^{n/6}$ for any $n \geq 7$. $\square$

7 Chain graphs

In this section, we combine a study of structural properties of mds in chain graphs with combinatorial arguments to exactly determine the maximum number of mds in a chain graph on n vertices. A bipartite graph $G = (A, B, E)$ is a *chain graph* if there is an ordering $\sigma_A = \langle a_1, a_2, \ldots, a_k \rangle$ of the vertices of A such that $N(a_1) \subseteq N(a_2) \subseteq \cdots \subseteq N(a_k)$, as well as an ordering $\sigma_B = \langle b_1, b_2, \ldots, b_\ell \rangle$ of the vertices of B such that $N(b_1) \supseteq N(b_2) \supseteq \cdots \supseteq N(b_\ell)$ [15]. The orderings σ_A and σ_B together form a *chain ordering* of G. Note that if a chain graph G is disconnected, then at most one connected component of G contains edges.

In any graph, every maximal independent set is a minimal dominating set, and we start with the following result on chain graphs, which will help us prove the bound on mds, and which is also interesting on its own.

Lemma 1. *A chain graph has at most $\lfloor n/2 \rfloor + 1$ maximal independent sets, and there are chain graphs that have $\lfloor n/2 \rfloor + 1$ maximal independent sets.*

Proof. Let $G = (A, B, E)$ be a chain graph on n vertices and assume, without loss of generality, that $|A| \leq \lfloor n/2 \rfloor$. Since any isolated vertex must belong to every maximal independent set, we may assume that G is connected. Let $\sigma_A = \langle a_1, a_2, \ldots, a_k \rangle$ and $\sigma_B = \langle b_1, b_2, \ldots, b_\ell \rangle$ be a chain ordering of G. Observe that b_1 dominates A and a_k dominates B. Let $\nu_i(G)$ be the number of maximal independent sets in G containing a_i, but not containing any vertex a_j with $j > i$. Consider $\nu_i(G)$ for some $i \in \{1, \ldots, k\}$. If every maximal independent set in G contains a vertex a_j with $j > i$, then $\nu_i(G) = 0$. Suppose there exists a maximal independent set S containing a_i, but not containing any vertex a_j with $j > i$. Clearly, none of the neighbors of a_i can be in S. Since σ_A and σ_B form a chain ordering of G, $N(a_p) \subseteq N(a_i)$ for every $p < i$. This means in particular that there is no edge between any vertex in $\{a_1, \ldots, a_{i-1}\}$ and any vertex in $B \setminus N(a_i)$. Hence the set $\{a_1, \ldots, a_i\} \cup B \setminus N(a_i)$ is the unique maximal independent set in G containing a_i and not containing any a_j with $j > i$. As a result, $\nu_i(G) \leq 1$ for every $i \in \{1, \ldots, k\}$. Note that B forms the only maximal independent set in G containing none of the vertices of A, and that every other maximal independent set in G contains at least one vertex from A. Since $\nu_i(G) \leq 1$ for every $i \in \{1, \ldots, k\}$ and $k \leq \lfloor n/2 \rfloor$ by assumption, we conclude that G has at most $\lfloor n/2 \rfloor + 1$ maximal independent sets.

For every even $n \geq 2$, let G_n be the chain graph obtained from two independent sets $A = \{a_1, \ldots, a_{n/2}\}$ and $B = \{b_1, \ldots, b_{n/2}\}$ by making a_i adjacent to every vertex in $\{b_1, \ldots, b_i\}$, for $i = 1, \ldots, n/2$. For every even $n \geq 2$, the graph G_n contains exactly $\lfloor n/2 \rfloor + 1$ maximal independent sets. □

Lemma 2. *For every minimal dominating set S of a chain graph G, the graph $G[S]$ contains at most one edge.*

Proof. Let S be a mds of a chain graph $G = (A, B, E)$. We first show that every connected component of $G[S]$ contains at most two vertices. Suppose, for contradiction, that $G[S]$ contains a connected component on more than two vertices. Since G is bipartite, this means that $G[S]$ contains an induced path on three vertices. Without loss of generality, let $a', a'' \in A$ and $b \in B$ be three vertices such that $\{a', b, a''\}$ induces a path on three vertices in $G[S]$. Note that b dominates both a' and a''. Hence, in order for a' and a'' to have private neighbors, there must exist vertices b' and b'' such that a' is the only vertex of S dominating b', and a'' is the only vertex of S dominating b''. This contradicts that there is a chain ordering involving a and a', and hence that G is a chain graph.

Let $\sigma_A = \langle a_1, \ldots, a_k \rangle$ and $\sigma_B = \langle b_1, \ldots, b_\ell \rangle$ form a chain ordering of G. Suppose $G[S]$ contains at least one edge, and let $a_i b_j$ be an edge of $G[S]$. We already showed that $G[S]$ does not contain an induced path on three vertices, so none of the vertices of $N(a_i) \setminus \{b_j\}$ and $N(b_j) \setminus \{a_i\}$ is in S. This means that if $G[S]$ contains an edge other than $a_i b_j$, then this edge is of the form $a_p b_q$ with $p < i$ and $q > j$, where $a_p \notin N(b_j)$ and $b_q \notin N(a_i)$. But then $N(a_p) \nsubseteq N(a_i)$, contradicting the assumption that σ_A is an ordering of the vertices of A such that $N(a_1) \subseteq \cdots \subseteq N(a_q) \subseteq \cdots \subseteq N(a_i) \subseteq \cdots \subseteq N(a_k)$. We conclude that every connected component of $G[S]$, apart from the component containing a_i and b_j, contains exactly one vertex. □

The following lemma is an easy consequence of Lemma 2.

Lemma 3. *Let ab be an edge of a chain graph $G = (A, B, E)$ with $a \in A$ and $b \in B$. If a or b has degree 1, then there is no minimal dominating set in G containing both a and b. If both a and b have degree at least 2, then there is exactly one minimal dominating set in G containing both a and b.*

From Lemmas 1, 2 and 3, we can readily deduce that every chain graph has at most $\lfloor n/2 \rfloor + m + 1$ mds. This bound is tightened below.

Theorem 5. *A chain graph on at least 2 vertices has at most* $\lfloor n/2 \rfloor + m$ *minimal dominating sets, and there are chain graphs with* $\lfloor n/2 \rfloor + m$ *minimal dominating sets.*

Proof. Let $G = (A, B, E)$ be a chain graph on n vertices, and let $\sigma_A = \langle a_1, \ldots, a_k \rangle$ and $\sigma_B = \langle b_1, \ldots, b_\ell \rangle$ form a chain ordering of G. Without loss of generality, assume that $|A| \leq \lfloor n/2 \rfloor$. Since any isolated vertex must belong to every minimal dominating set, we may assume that G is connected. It is easy to check that G contains at most $\lfloor n/2 \rfloor + m$ mds in case $|A| = 1$ or $|B| = 1$. We therefore assume below that both A and B contain at least two vertices.

First suppose that at least one edge of G has an endpoint of degree 1. Then G has at most $m - 1$ mds S such that $G[S]$ contains an edge as a result of Lemmas 2 and 3. Every other mds in G must be a maximal independent set in G, and G has at most $\lfloor n/2 \rfloor + 1$ such sets by Lemma 1. Hence G has at most $\lfloor n/2 \rfloor + m$ mds in this case.

Now suppose that for every edge of G both endpoints have degree at least 2. In particular, b_ℓ has degree at least 2, which means that b_ℓ is adjacent to a_{k-1}, which in turn implies that a_{k-1} is adjacent to every vertex in B. Recall that a_k also dominates B. Let S be a maximal independent set in G containing a_{k-1}. Since a_{k-1} dominates B, S does not contain any vertex of B. Since S is a maximal independent set of G, we must have $S = A$. In particular, $a_k \in S$. Let $\nu_i(G)$ be the number of maximal independent sets in G containing a_i, but not containing any vertex a_j with $j > i$. Note that $\nu_{k-1}(G) = 0$. As shown in the proof of Lemma 1, $\nu_i(G) \leq 1$ for every $i \in \{1, \ldots, k\}$. The set B is the only maximal independent set in G containing no vertices of A. Every other maximal independent set contains at least one vertex of A. The assumption that $|A| \leq \lfloor n/2 \rfloor$, together with $\nu_{k-1}(G) = 0$ and $\nu_i(G) \leq 1$ for every $i \in \{1, \ldots, k\}$, implies that G has at most $\lfloor n/2 \rfloor$ maximal independent sets, each of which forms a mds in G. Due to Lemmas 2 and 3, G has exactly m mds S for which $G[S]$ contains an edge. Hence G contains at most $\lfloor n/2 \rfloor + m$ mds in total.

Recall the graph G_n that was defined in the proof of Lemma 1. For every even $n \geq 2$, let G'_n be the graph obtained from G_n by adding the edge $a_{k-1}b_\ell$. The graph G'_n contains exactly $\lfloor n/2 \rfloor + m$ mds: one for each of the $\lfloor n/2 \rfloor$ maximal independent sets, and one for each edge of G'_n, apart from the edge $a_1 b_1$. $\square$

8 Conclusions

We established new upper bounds for the number of mds in graphs on n vertices of various graph classes. All our bounds are significantly lower than the known upper bound for general graphs.

Could our enumeration algorithms be used to establish fast exact exponential algorithms solving Domatic Number or Connected Dominating Set on split and chordal graphs? We point out that both problems are NP-complete on split graphs, and thus also on chordal graphs [10, 12].

The maximum number of mds in a general graph on n vertices is still unknown. It is conjectured that $15^{n/6}$, the best known lower bound, is indeed the correct answer [4]. We have shown that a counterexample to this conjecture cannot belong to any of the graph classes studied in this paper, with the exception of cobipartite and chordal graphs. A known lower bound for bipartite graphs is $6^{n/4} \approx 1.5650^n$, which is the number of mds in the disjoint union of $n/4$ cycles of length 4. Is there an upper bound for bipartite graphs which is better than 1.7159^n? Finally, we conjecture that the maximum number of mds in split graphs and proper interval graphs is $3^{n/3}$.

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Appendix

In this appendix, after presenting a figure describing the inclusion relationship between the graph classes that we study, we give the full details of the remaining results that were mentioned in Table 1.

A. Inclusion relation of graph classes

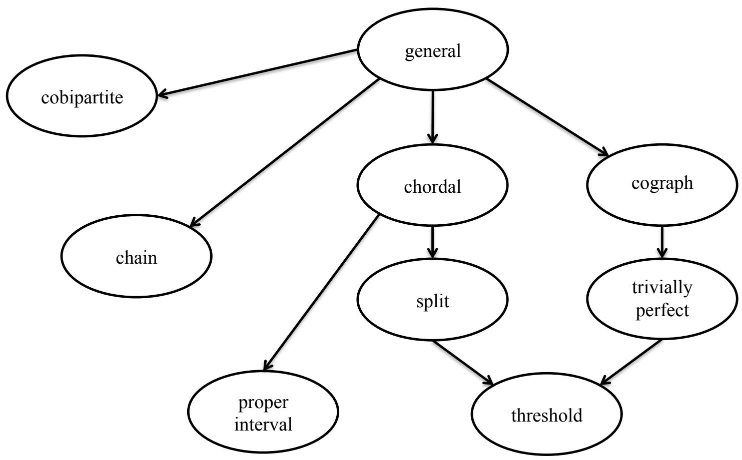

Fig. 2. The graph classes studied in this paper, where $\rightarrow$ represents the $\supset$ relation.

B. Proper interval graphs

A graph is a *proper* (or *unit*) *interval graph* if intervals of the real line of unit length can be assigned to its vertices, such that two vertices are adjacent if and only if their corresponding intervals intersect [19]. Proper interval graphs, also called *indifference graphs*, are chordal. A *proper interval ordering* $\langle v_1, v_2, \ldots, v_n \rangle$ of a graph satisfies the following property: if $v_i v_j$ is an edge with $i < j$, then the vertices $v_i, v_{i+1}, \ldots, v_j$ form a clique. A graph is a proper interval graph if and only if it has a proper interval ordering [17]. Observe that H_n is proper interval, giving us examples of proper interval graphs with $3^{n/3} \approx 1.4422^n$ mds.

Theorem 6. *A proper interval graph has at most $O(1.4656^n)$ minimal dominating sets.*

Proof. In our proof via a branching algorithm, we will apply the technique of Measure & Conquer [5], in which weights are assigned to the vertices in order to define the measure of an instance. Let (G', D) be an instance of a subproblem. We define three types of vertices. Any vertex of G' with a neighbor in D is *dominated*. A vertex of G' is *forbidden* if it is discarded from being added to D.

All other vertices, neither dominated nor forbidden, are said to be *free*. We define the weight of the vertices as follows: free and dominated vertices have weight 1, and forbidden vertices have weight 0. The measure of an instance (G', D) is the total weight of the vertices in G'. Initially, all vertices of the input graph G are free, and thus have weight 1. Hence the the total weight of the input instance $(G, \emptyset)$ is n. Observe that any vertex which is dominated and forbidden can be deleted from the graph, since such a vertex cannot be put into D and is already dominated. This gives a reduction rule, as a consequence of which, during the analysis below, we do not need to consider the cases that the vertex we are branching on is dominated and forbidden.

Let $\sigma = \langle v_1, v_2, \ldots, v_n \rangle$ be a proper interval ordering of the input graph G. In each instance (G', D), we will choose a vertex x of G' with the smallest index according to σ, and branch on this vertex. Observe that the relative ordering of the vertices of G' according to σ is a proper interval ordering of G'. Hence, the vertex x of G' with the smallest σ index is a simplicial vertex. Throughout the algorithm, the following two invariants will be true on every instance G':

1. *If a vertex v_j of G' is forbidden, then all vertices v_i of G' with $i < j$ are also forbidden.*
2. *If a vertex v_j of G' is dominated, then all vertices of v_i of G' with $i < j$ are also dominated.*

When we make new subproblems, whenever needed we will argue that the invariants are still true on the subproblems, assuming that they were true on the given instance. In the beginning of the algorithm, all vertices are free and the invariants are trivially true.

We now describe the branching algorithm. Let x be the vertex of G' with the smallest σ index. As we argued before, x is simplicial. We distinguish two main cases, depending on whether or not x is forbidden. Each case has subcases, depending on the number of neighbors of x in G'. Only one case applies and will be executed by the algorithm.

Case 1a: x is not forbidden and has at least 2 neighbors in G'. Notice that since x is not forbidden, by Invariant 1, none of its neighbors in G' is forbidden, since x is the vertex with the smallest σ index in G'. We branch on the possibilities of adding x to D or discarding x with respect to D.

- *Add:* $x \in D$. Vertex x dominates $N_{G'}(x)$ and needs a private neighbor in $N_{G'}(x)$. This means that no vertex of $N_{G'}(x)$ can appear in a mds containing D as a subset, since $N_{G'}(x)$ is a clique, and x would then not have a private neighbor. Consequently, we can safely delete $N_{G'}[x]$, which results in the instance $(G' - N_{G'}[x], D \cup \{x\})$. Since $N_{G'}(x)$ contains at least 2 vertices each of weight 1, and x has itself weight 1, this results in a decrease of at least 3 in weight.
- *Discard:* $x \notin D$. We simply forbid x. If x was already dominated, it can now be deleted. If not, the new instance is the same as before: (G', D), but the

weight of x has decreased from 1 to 0. We get a total weight decrease of 1. Invariant 1 is clearly preserved.

Case 1b: x is free (not forbidden and not dominated) and has exactly 1 neighbor in G'. Let y be the neighbor of x in G'. By Invariant 1, we know that y is not forbidden; otherwise x would also be forbidden. Since x is free, by Invariant 2, y is also free. Since x is not dominated and is only adjacent to y in G', either x or y needs to be added to D to ensure that x is dominated. We branch on these possibilities.

- $x \in D$. In this case, y becomes dominated, and no mds containing D as a subset can contain y, as x then would not have a private neighbor. Consequently, we can safely delete x and y in this case. We get the instance $(G' - \{x, y\}, D \cup \{x\})$, and a decrease of 2 in weight.
- $y \in D$. Now x becomes dominated, and it can never become a member of a mds containing D, as it would then not have a private neighbor. We can safely delete x and y, and mark all the remaining neighbors of y as dominated. Invariant 2 is preserved, and we get the instance $(G' - \{x, y\}, D \cup \{y\})$, with a weight decrease of 2.

Case 1c: x is not forbidden, is dominated, and has exactly 1 neighbor in G'. Let y be the neighbor of x in G'.

- $x \in D$. In this case x needs a private neighbor and the only possible one is y. Since y is the private neighbor of x, no other neighbor of y can be in D. Hence we mark all vertices in $N[y]$ as forbidden. This gives a total weight decrease of 3, since x and y are deleted, and the weight of at least one neighbor of y decreases by 1 when it becomes forbidden.
- $x \notin D$. In this case, x is already dominated and is now also forbidden. We therefore delete x, and the total weight decreases by 1.

Case 1d: x is not forbidden and is isolated G'. This case corresponds to a reduction rule and involves no branching. If x is not dominated, then we add it to D; otherwise, we simply delete x.

Case 2a: x is forbidden and has at least 2 non-forbidden neighbors in G'. Since x is forbidden and not yet dominated, one of the non-forbidden neighbors of x in G' must be added to D. Let $\langle y_1, y_2, \ldots, y_d \rangle$ be the non-forbidden vertices in $N_{G'}(x)$, ordered according to their ordering in σ. Note that $d \geq 2$. Observe that, since σ is a proper interval ordering, $N_{G'}[x] \subseteq N_{G'}[y_1] \subseteq N_{G'}[y_2] \subseteq \ldots \subseteq N_G[y_d]$. This means in particular that two vertices of $N_{G'}(x)$ cannot appear in a mds together, since the one with the smallest neighborhood then does not have a private neighbor. Hence, exactly one vertex of $N_{G'}(x)$ can and must be added to D. We branch on the choice of y_i to add to D. For each i between 1 and d, if y_i is added to D, then $N_{G'}[x]$ is dominated, and by the arguments above we can safely remove $N_{G'}[x]$ from G'. In addition, we mark the remaining vertices of $N_{G'}(y_i)$ as dominated. The invariants are preserved, by the neighborhood inclusion order of the y_i-vertices. This gives $d \geq 2$ new instances, each with a weight that is reduced by at least d, since there are at least d vertices of weight 1 in $N_{G'}(x)$. The branching vector for this case has length d, and is $(d, d, d, \ldots, d)$. The highest branching number is given by $d = 3$. (In particular, $(2, 2)$ and $(4, 4, 4, 4)$ have branching number 1.4143, $(3, 3, 3)$ has 1.4423, and $(5, 5, 5, 5, 5)$ has 1.3797.) Hence, we can take the branching vector to be $(3, 3, 3)$ for this case.

Case 2b: x is forbidden and has exactly 1 non-forbidden neighbor in G'. Let y be the only non-forbidden neighbor of x. Since x is not yet dominated, y must be added to D. We can safely delete x and y from G', and mark all remaining neighbors of y in G' as dominated. If any of these neighbors were forbidden, then can now be deleted. This is a reduction rule, involving no branching.

Case 2c: x is forbidden and has no non-forbidden neighbors in G'. In this case, D cannot be extended to a dominating set for G. Hence we stop recursing and simply discard D.

The branching vectors for Cases 1a, 1b, 1c, and 2a are $(3, 1)$, $(2, 2)$, $(3, 1)$, and $(3, 3, 3)$, respectively. Branching vector $(3, 1)$ gives the largest branching number, 1.4656, resulting in an upper bound of $O(1.4656^n)$. □

C. Trivially perfect graphs

Trivially perfect graphs form a subclass of cographs, and they have various characterizations [1, 8]. A graph G is *trivially perfect* if and only if each connected induced subgraph of G contains a universal vertex [20].

Theorem 7. *A trivially perfect graph has at most $3^{n/3}$ minimal dominating sets, and there are trivially perfect graphs with $3^{n/3}$ minimal dominating sets.*

Proof. Observe that H_n, defined in Section 2, is a trivially perfect graph. Hence we have examples of trivially perfect graphs with exactly $3^{n/3}$ mds. H_6 gives the base case for the upper bound, which we will prove by induction on the number of vertices. For smaller trivially perfect graphs, it can be verified easily that the theorem holds. Assume that the upper bound holds for all trivially perfect graphs on at most $n-1$ vertices, and let G be a trivially perfect graph on n vertices. If G is disconnected, where $n_1, \ldots, n_t$ are the sizes of its connected components, such that $n_1 + \ldots + n_t = n$ and $n_i < n$ for $1 \leq i \leq t$, then by our induction assumption and Observation 1, $\mu(G) \leq 3^{n_1/3} \cdot \ldots \cdot 3^{n_t/3} = 3^{(n_1+\ldots+n_t)/3} = 3^{n/3}$. Assume that G is connected. Then, by the definition of a trivially perfect graph, we know that G has a universal vertex u. The only mds of G that contains u is $\{u\}$. Furthermore, the set of mds of G that do not contain u is exactly the set of mds of $G{-}u$, since u is dominated by every vertex of $G{-}u$. Consequently, $\mu(G) = \mu(G{-}u) + 1 \leq 3^{(n-1)/3} + 1$. Since $3^{(n-1)/3} + 1 \leq 3^{n/3}$ for all values of $n \geq 3$, we conclude that $\mu(G) \leq 3^{n/3}$. □

D. Threshold graphs

Threshold graphs form a subclass of trivially perfect graphs. Threshold graphs have several characterizations [1, 8, 18]. In particular, a graph G is a *threshold graph* if and only if it is a split graph and, for any split partition (C, I) of G, there is an ordering $\langle x_1, x_2, \ldots, x_k \rangle$ of the vertices of C such that $N[x_1] \subseteq N[x_2] \subseteq \ldots \subseteq N[x_k]$, and there is an ordering $\langle y_1, y_2, \ldots, y_\ell \rangle$ of the vertices of I such that $N(y_1) \supseteq N(y_2) \supseteq \ldots \supseteq N(y_\ell)$ [18]. Threshold graphs are closely related to chain graphs, as a bipartite graph (A, B, E) is a *chain graph* if and only if turning A or B into a clique results in a threshold graph [18].

We write $\omega(G)$ to denote the size of a maximum clique in a graph G.

Theorem 8. *A threshold graph G has exactly $\omega(G)$ minimal dominating sets.*

Proof. Let G be a threshold graph with split partition (C, I). We can assume C to be a clique of size $\omega(G)$, since otherwise I contains a vertex that dominates C, and this vertex can be removed from I and added to C. Suppose G has a mds S that contains two vertices x_1 and x_2 of C. By definition, we either have $N[x_1] \subseteq N[x_2]$ or $N[x_2] \subseteq N[x_1]$; assume that $N[x_1] \subseteq N[x_2]$. Then $S \setminus \{x_1\}$ is a dominating set of G whose size is smaller than S, contradicting the minimality of S. This implies that any mds of G contains at most one vertex of C. Let $C' \subseteq C$ be the set of vertices of C that have no neighbor in I. Since G is threshold and C is a maximum clique, C' is not empty. This means that every mds of G contains at least, and therefore exactly, one vertex of C. For every vertex $x' \in C'$, the set $\{x'\} \cup I$ is a mds of G. For every vertex $x \in C \setminus C'$, the set $\{x\} \cup (V(G) \setminus N[x])$ is a mds of G. Hence G contains exactly $|C| = \omega(G)$ mds. $\square$

E. Additional references

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