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Aircraft Flight Path Optimization
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Abstract

Flight path optimization is designed for minimizing aircraft noise, fuel consumption and air pollution around airports. This paper gives theoretical considerations and algorithms solving the Hamilton-Jacobi-Bellman equation (HJB) of aircraft trajectory optimization. Comparisons with direct and indirect methods are carried out. The OCP problem is transformed into new equalities-constrained as a viscosity problem. This constitutes an original dynamical system extension where subsystems are linked to the original dynamics via algebraic coupling equations. A feedback control method using dynamic programming has been developed. Comparisons show its fast computing times. It provides the best optimized flight paths which could be more suitable for CDA approach applicability. A two-segment approach is provided by HJB method which also favors fuel consumption saving. This improved CDA approach could benefit both airlines and communities. Because of the processing speed and efficiency of the HJB method, it can be better interfaced with the in-flight management system respecting airspace system regulation constraints.

Subject Classification: xxxxxx
Keywords: Hamilton-Jacobi-Bellman equation, Dynamic programming, Aircraft, Flight path optimization, environment

1 Introduction

Due to the increase of air traffic, populations living near airports and the environment are impacted by commercial aircraft. This is considered to be one of the most environmental concerns affecting people and the physical environment [1, 2]. Technology development, airspace management, operational

improvement and system efficiency should be considered as an environmental innovation. There is no justification that air transport will not continue to progress without improving its environmental impacts [3]. This is because all types of procedures are not optimized but rather generic in nature, that new flight path development, associated to new aircraft design and engines, is a solution which should contribute to a decrease in aircraft annoyances. This development cannot be carried out without improvement of the scientific knowledge in this field, in particular the contribution of modeling. The latter consists in developing efficient data processing tools allowing in-flight diagnosis and control in real-time taking into account the FMS (flight management system) and the AMS (airspace management system) updates. In this paper, we have suggested a dynamic optimization method solving a model governed by an ODE system [4, 5, 6]. The cost function of this model describes aircraft noise and fuel consumption [7, 8, 9, 10, 11]. The ODE depends on the flight dynamics of the aircraft and considers flight safety and stability requirements (constraints and extreme conditions). Numerical methods solving control problems fall in many categories [12, 13, 14, 15, 16, 17, 18, 19, 20, 21] among which it is necessary to choose, improve or develop a new method. In this context, this paper gives theoretical considerations and algorithms solving Hamilton-Jacobi-Bellman equation "HJB" [22, 23] because the problem is transformed into new equalities-constrained as a viscosity problem. This is an original system extension where subsystems are linked to the dynamics via algebraic coupling equations [24, 25]. Among the existing methods, solving the HJB problem, a feedback control method using dynamic programming has been developed. The latter is a method used for solving complex problems by breaking them down into simpler subproblems. To solve the given problem, it solves subproblems, then combines solutions of them to reach a global solution. Because subproblems are generally the same, it seeks to solve each subproblem only once reducing the number of the total computations, in particular when the subproblem number is exponentially large. Comparisons have been performed between HJB, direct and indirect methods [26, 27, 28, 29] stressing the computing times with the aim of finding the best aircraft approach also favoring fuel consumption saving. Indeed, the direct approach has been used reducing the OCP to a finite-dimensional nonlinear program which is solved by a standard nonlinear programming solver. Algorithms are adapted and modified versus constraints, limits of flight dynamic parameters, and location points on the ground. For the indirect approach, optimality conditions given by Pontryagin's principle, have been discretized. An *AMPL* model (A Modeling Language for Mathematical Programming) [30] combined with *NLP* solver [31, 32, 33] has been performed for processing. In-depth details have been described in previous papers [34, 35, 36, 37, 38]. Technically, we analyze the processing speed and algorithm efficiency and their ability to be interfaced

with the in-flight management system respecting airspace system regulation constraints. This has to be fitted with the necessity to compensate both the growth in air traffic and the encroachment of airport-neighboring communities.

This paper presents in the first two sections an introduction and the optimal control problem of aircraft trajectory minimization (flight dynamics, constraints, and aircraft noise model). For comparison, the third section gives applied methods (indirect, direct, Hamilton-Jacobi-Bellman). The last two sections show numerical results followed by the conclusion.

2 Optimal Control Problem

We present in this section a summary of the optimal control problem that will be solved and methods compared [34, 35, 37, 38]. The system of differential equations commonly employed in aircraft trajectory analysis is the following six-dimension system derived at the center of mass of the aircraft [34, 35, 37, 39, 40, 41, 42]:

$$
(ED) \quad
\begin{cases}
\dot{V} &=& g\left(\dfrac{T\cos\alpha - D}{mg} - \sin\gamma\right) \\[2mm]
\dot{\gamma} &=& \dfrac{1}{mV}\left((T\sin\alpha + L)\cos\mu - mg\cos\gamma\right) \\[2mm]
\dot{\chi} &=& \dfrac{(T\sin\alpha + L)\sin\mu}{mV\cos\gamma} \\[2mm]
\dot{x} &=& V\cos\gamma\cos\chi \\[2mm]
\dot{y} &=& V\cos\gamma\sin\chi \\[2mm]
\dot{h} &=& V\sin\gamma
\end{cases}
$$

where V, γ, χ, α and μ are respectively the speed, the angle of descent, the yaw angle, the angle of attack and the roll angle. (x, y, h) is the position of the aircraft. The variables T, D, L, m and g are respectively the engine thrust, the drag force, the lift force, the aircraft mass and the aircraft weight acceleration given in previous paper [34, 35, 38, 39, 40, 41, 42, 43]. ED can be written in the following matrix form:

$$\dot{z}(t) = f(z(t), u(t))$$

where

$$
\begin{aligned}
z: \ & [t_0, t_f] \ & \longrightarrow \ & \mathbb{R}^6 \\
& t \ & \longrightarrow \ & z(t) = [V(t), \gamma(t), \chi(t), x(t), y(t), h(t)] \ \text{ are the state variables} \\
u: \ & [t_0, t_f] \ & \longrightarrow \ & \mathbb{R}^3 \\
& t \ & \longrightarrow \ & u(t) = [\alpha(t), \delta_x(t), \mu(t)] \ \text{ are the control variables}
\end{aligned}
$$

and t_0 and t_f are the initial and final times.

Along the trajectory, we have some safety requirements and comfort constraints. For that, we have to respect parameter limits related to the safety of flight and the operational modes of the aircraft.

Aircraft modeling continues to meet the increased demands associated with aviation and airport expansion. Aircraft noise footprints are commonly used for forecasting the impact of new developments, quantifying the noise trends around airports and evaluating new tools. Thus, aircraft models have become more sophisticated and their validation complex. A number of them are entirely based on empirical data [44]. Because of this complexity [45], such models are not characterized by a given analytical form describing noise at reception points on the ground. This paper uses the basic principles of aircraft noise modeling. The cost function may be chosen as any of the usual aircraft noise indices, which describes the effective noise level of the aircraft noise event [46, 47, 48, 49]. This study is limited to minimize the index $L_{eq,\Delta T}$ using a semi-empirical model of jet noise [7, 8, 10, 11, 50, 51, 52, 53, 54]. The cost function is expressed in the form:

$$
J : \mathcal{C}^1([t_0, t_f], \mathbb{R}^6) \times \mathcal{C}^1([t_0, t_f], \mathbb{R}^4) \qquad \longrightarrow \quad \mathbb{R}
$$
$$
J(X(t), U(t)) = \int_{t_0}^{t_f} (\ell(X(t), U(t)) + \Phi(X(t), t_f - t_0)) dt
$$

where J is the criterion which optimizes noise levels and fuel consumption. The cost function can be written in the following integral function form:

$$
J : \mathcal{C}^1([t_0, t_f], \mathbb{R}^6) \times \mathcal{C}^1([t_0, t_f], \mathbb{R}^3) \qquad \longrightarrow \quad \mathbb{R}
$$
$$
J(z(t), u(t)) = \int_{t_0}^{t_f} (\ell(z(t), u(t)) + \phi(z(t), (t_f - t_0))) dt
$$

where J is the criterion to be optimized. Finding an optimal trajectory can be stated as an optimal control problem as follows ($t_0 = 0$):

$$(OCP) \quad \begin{cases} \min \ J(z,u) = \int_0^{t_f} (\ell(z(t), u(t)) + \phi(z(t), t_f))dt \\[2ex] \dot{z}(t) = f(z(t), u(t)), \forall t \in [0, t_f] \\[2ex] z_{I_1}(0) = c_1, \ z_{I_2}(t_f) = c_2 \\[2ex] a \leq C(z(t), u(t)) \leq b \end{cases}$$

where $J : I\!R^{n+m} \to I\!R, f : I\!R^{n+m} \to I\!R^n$ and $C : I\!R^{n+m} \to I\!R^q$ correspond respectively to the cost function, the dynamic of the problem possessing a unique state trajectory and the constraints. The second equation giving the trajectory is a nonlinear system with states in $I\!R^n$. At t_f, terminal conditions are imposed. t_f can be fixed or unspecified. In this paper terminal conditions, describing the boundary conditions, are specified and their values are given. t_f is also fixed. In the next section, we present different methods solving the presented (OCP) (optimal control problem) problem. There is no practical theoretical limitations to using those methods that cannot be guaranteed to provide a global solution. We assume that the OCP has an optimal trajectory solution with the optimal cost. Subsequently, to reduce notations of l and ϕ, together, are called (replaced by) l.

3   Applied resolution methods

In this paper, we have applied three different approaches solving the OCP problems: direct, indirect, and dynamic approaches [12, 13, 18, 55, 56, 57, 58]. The direct method discretized the OCP for obtaining a finite-dimensional parameter optimization problem and solving the resulting nonlinear programming problem [59, 60, 61, 62]. It is well appropriated because of the domain of convergence and the efficient handling of constraints and the defined limits. It is opposed to the indirect approach based on Pontryagin's principle [63, 64, 65, 66, 67, 68] based on the assessment of variations requiring solutions of two-point boundary values problem. It provides a very fast computing times, in particular, in the vicinity of the optimal solution. Inequality constraints are carried out by Pontryagin's maximum principle. Another way can be suggested avoiding problems of constraints handling by transforming adequately the OCP in a new unconstrained OCP formulation that can be solved by a standard unconstrained numerical methods. Because of this change, a new unconstrained OCP is obtained having the same system dimension with new states and variables. The third method is based on the dynamic programming method [55, 56, 57, 58] than can be used to find the optimal state,

costate and the control variables which is focusing on the optimal function value [69]. Dynamic programming is a method for solving complex problems by breaking them down into simpler subproblems. To solve a given problem, it solves different parts of the problem called subproblems, then combines solutions of the subproblems to reach a global solution. Because of subproblems are generally the same, it seeks to solve each subproblem only once reducing the number of the total computations, in particular when subproblem number is exponentially large.

The first-order partial differential equation is derived using the Hamilton-Jacobi-Bellman equation which uses the principle of Optimality of bellman. The optimal value of the control vectors depending on the the date, the state and the parameters of the control problems is obtained. This way is called the feedback or the closed-loop form of the control. The Open-loop, where the form of the optimal control vector ca be obtained by the necessary conditions of optimal control theory can be used. The optimal value of the control vector can be obtained as a function of the independent variable time, the different used parameters and initial/final conditions of the cost function and the state vector. The optimal solution can be given for any period and for any possible state.

3.1   Indirect method

We set $\mathcal{H} : I\!R^n \times I\!R^m \times I\!R^n \times I\!R^q \times I\!R^q \longrightarrow I\!R$ the hamiltonian function of the problem (OCP):

$$\mathcal{H}(z, u, p, \lambda, \mu) = \ell(z, u) + \phi(z, t_f) + p^t f(z, u) + \lambda^t(C(z, u) - a) + \mu^t(b - C(z, u))$$

where λ, μ are the multiplicators associated to the constraints and p is the costate vector.
We describe now the optimality conditions (OC) for the (OCP) problem:

$$(OC) \quad \begin{cases} \dot{z}(t) & = f(z(t), u(t)) \\ \dot{p}(t) & = -\mathcal{H}_z(z(t), u(t), p(t), \lambda(t), \mu(t)) \\ u(t) & = Argmin_w \mathcal{H}(z(t), w, p(t), \lambda(t), \mu(t)) \\ 0 & = \lambda.(C(z(t), u(t) - a), \quad \lambda \geq 0 \\ 0 & = \mu.(b - C(z(t), u(t)), \quad \mu \leq 0 \end{cases}$$

In this paper, we have used the interior point method [14, 20, 21] discretizing the optimality conditions of the system. The method which solved the (OC) problem is described below. We explain the transformation of the (OC) problem into a sequence of problems. We also show that the solution of the optimality conditions is a solution of the (OCP) problem: discretization used

an Euler scheme and the resolution the Newton method [70].

By perturbing the last two equations (the complementary conditions) by a positive parameter ε we obtain the following system:

$$(OC_\varepsilon) \quad \begin{cases} \dot{z}(t) &= f(z(t), u(t)) \\ \dot{p}(t) &= -\mathcal{H}_z(z(t), u(t), p(t), \lambda(t), \mu(t)) \\ u(t) &= Argmin_w \mathcal{H}(z(t), w, p(t), \lambda(t), \mu(t)) \\ \mathbf{1}\varepsilon &= \lambda.(C(z(t), u(t)) - a), \quad \lambda \geq 0 \\ -\mathbf{1}\varepsilon &= \mu.(b - C(z(t), u(t))), \quad \mu \leq 0 \end{cases}$$

The previous system can be interpreted as the optimality conditions for the following problem:

$$(P_\varepsilon) \quad \begin{cases} \min \displaystyle\int_0^{t_f} (\ell_\varepsilon(z(t), u(t)) + \phi_\varepsilon(z(t), t_f)) dt \\[2ex] \dot{z}(t) = f(z(t), u(t)), \quad t \in [0, t_f] \end{cases}$$

where ℓ_ε is the barrier logarithmic of (P_ε), defined by:

$$\ell_\varepsilon(z, u) + \phi_\varepsilon(z, t_f) = \ell(z, u) + \phi(z, t_f) - \varepsilon \sum_i \left[\log(C_i(z, u) - a_i) + \log(b_i - C_i(z, u)) \right]$$
$$- \varepsilon D(z)$$

To solve (OC), we have to solve a sequence of problems (OC_ε) by tending ε to zero. When ε decreases to 0, the solution of optimal conditions (OC_ε) is a solution of (OC). To compute the solution of the continuous optimal conditions, we first discretized them. We obtained a set of non-linear equations, which has to be solved for the discretized control, state and costate vectors using a Newton method [21, 71, 72]. For the discretization, we have chosen an Euler schema [59, 70, 73, 74] providing for (OC_ε) the following system:

$$\begin{cases} z_{k+1} &= z_k + \mathrm{h}f(u_k, z_k), \quad k = 0, \ldots, N-1 \\ p_{k+1} &= p_k - \mathrm{h}\mathcal{H}_z(z_k, u_k, p_k, \lambda_k, \mu_k), \quad k = 0, \ldots, N-1 \\ 0 &= \mathcal{H}_u(u_k, z_k, p_k, \lambda_k, \mu_k), \quad k = 0, \ldots, N \\ \mathbf{1}\varepsilon &= \lambda_k.(C(z_k, u_k) - a), \quad \lambda_k \geq 0, \quad k = 0, \ldots, N \\ -\mathbf{1}\varepsilon &= \mu_k.(b - C(z_k, u_k)), \quad \mu_k \leq 0, \quad k = 0, \ldots, N \end{cases}$$

We have obtained a set of equations to be solved under the boundary constraints corresponding to the multiplicators:

$$(N_\varepsilon) \quad \begin{cases} F_\varepsilon(X) = 0 \\ \lambda_k \geq 0 \\ \mu_k \leq 0 \end{cases}$$

where F_ε is the set of optimal conditions, and $X = (z_k, u_k, p_k, \lambda_k, \mu_k)$ the variable vector. OCP is then successfully solved for decreasing ε with a non-growth of the cost.

3.2   Direct method

To solve (OC) problems, many methods exist in the open literature [75, 76, 77, 78]. In this section, we have used a direct optimal control technique. We discretize the control and the state for reducing the dimension of the optimal control problem. Then, we solve the resulting nonlinear programming problem using a standard NLP solver. The paragraph below gives discretization steps which used Euler scheme where the continuous set of the obtained equations is replaced by a discretized control problem which is solved thereafter.

To solve (OC), we have used in this section a direct optimal control technique. We discretize the control and the state for reducing the dimension of the optimal control problem. Then, we solve the resulting nonlinear programming problem using a standard NLP solver. We use an equidistant discretization of the time interval as:

$$t_k = t_0 + k\mathrm{h}, \quad k = 0, ..., N \quad \text{and} \quad \mathrm{h} = \frac{t_f - t_0}{N}$$

Then we consider that $u(.)$ is parameterized as a piecewise constant function:

$$u(t) := u_k \ \text{ for } \ t \in [t_{k-1}, t_k[$$

and we use an Euler scheme to discretize the dynamic:

$$z_{k+1} = z_k + \mathrm{h}f(z_k, u_k), \quad k = 0, \ldots, N-1$$

The new cost function can be written as:

$$\sum_{k=0}^{N}(\ell(z_k, u_k) + \phi(z_k))$$

The continuous set of equations is replaced by the following discretized control problem:

$$(NLP) \quad \begin{cases} \min_{(z_k, u_k)} \sum_{k=0}^{N}(\ell(z_k, u_k) + \phi(z_k)) \\ z_{k+1} = z_k + \mathrm{h}f(z_k, u_k), \quad k = 0, \ldots, N-1 \\ z_{0_{I_1}} = c_1, \ z_{N_{I_2}} = c_2 \\ a \le C(z_k, u_k) \le b, \quad k = 0, \ldots, N \end{cases}$$

3.3 Hamilton-Jacobi-Bellman method

The main idea behind this section is how to reduce an infinite-period optimization problem to a two-period or some-period optimization problem. Difficulties appear when the optimization problem is continuous. Two approaches to dynamic optimization: the Pontryagin approach which is Hamiltonian and the Bellman approach [55, 56, 57, 58]. Dynamic programming method solves a complex problem by dividing it into simpler subproblems solving different parts of the problem reducing the processing steps in particular for large dimensions. The global solution is reached by combining solutions of the subproblems. When used, the method is faster compared to other existing methods. Top-down or bottom-up dynamic programming exist. The first record the stages of calculation which are used thereafter, and the second reformulate the problem by a recursive series which calculations are where the processing are made easy. On the one hand, a candidate trajectory called the candidate solution is an element of a set of possible solutions for the given problem. Usually, a candidate solution could not be the best solution of the considered problem. It is the solution satisfying constraints. On the other hand the solution must belong the feasible region (solution space). The existence of the the trajectory solution is confirmed by Weierstrauss theorem which states that in a compact set the maximum and minimum values are reached for continuous or semi-continuous functions. Necessary conditions for optimality have been confirmed. Indeed, optima of the suggested inequality-constrained problem is instead found by the Lagrange multiplier method. A system of inequalities or the Karush-Kuhn-Tucker conditions, calculating the optimum, are met.

Sufficient conditions for optimality are met. First derivative tests identify the optima without differences between the minimum or the maximum. We assume that the cost function is twice differentiable. By controlling the second derivative or the Hessian matrix in the transformed unconstrained problem or the matrix of second derivatives of the cost function and the constraints / the bordered Hessian, we can easily distinguish maxima and minima from stationary points. It should be remembered that methods commonly used to evaluate Hessians (or approximate Hessians) are Newton's method or Sequential quadratic programming. The second is particulary used for small-medium scale constrained problems. Some versions exist and can handle large-dimensional problems. A bordered Hessian is performed for the second derivative test in flight path constrained optimization problem. This could be in relationship with the suggested further research described in the last section making possible the generalization method. If we consider m constraints, 0 of the above matrix is a m*m block of zeroes, m border rows at the top and m border columns at the left; positive definite and negative definite can not apply here since a bordered Hessian can not be definite, we have z'Hz=0 if the vector z

has non-zero as its first element followed by zeroes. The second derivative test consists of restrictions of the determinants of a set of n-m sub-matrices of the bordered Hessian. A constraint reduction of the problem by one with n-m free variables.

If we consider:

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f(x_1, x_2, \ldots, x_n) = (f_1, f_2, f_3, \ldots, f_n)$$

the array of the second partial derivatives is not a nxn matrix but a tensor of order 3 or a multi-dimensional array mxnxn which can be reduced to a usual Hessian matrix with m=1. A Riemannian manifold and its Levi-Civita connection could be used for considerations using a Hessian tensor.

Dynamic methods approximate continuous systems to discrete systems leading to recurrence relations making easier the processing. One approach that should be considered as alternative transforms the OCP system to a non-linear partial differential equations, often called the Hamilton-Jacobi-Bellman equation [79, 80, 81].

The following state equation can be written as:

$$\dot{z}(t) = f(z(t), u(t)), \forall t \in [0, t_f]$$

It minimizes:

$$J(z, u) = \int_0^{t_f} (\ell(z(t), u(t)) + \phi(z(t), t_f))dt$$

J(z,u) can be transformed as:

$$J = \kappa(z(t_f), t_f) + \int_0^{t_f} \ell(z(\tau), u(\tau))d\tau$$

where f and κ are given functions, and t_f is fixed. When we use imbedding principle, we can first give the following weak modification of the suggested problem at the moment t (less than t_f) at any acceptable state z(t) to make more large:

$$J(z(t), u(\tau)_{t \leq \tau \leq t_f}) = \kappa(z(t_f), t_f) + \int_t^{t_f} \ell(z(\tau), u(\tau))d\tau$$

Optimization process depends on numerical values of the state at t moments and the optimal control history in the considered time intervals. For a performed acceptable state for all $t \leq t_f$, we can assess controls minimizing the cost function J. This minimum can be written as:

$$J^*(z(t), t) = min_{[u(\tau), t \leq \tau \leq t_f]}\{\kappa(z(t_f), t_f) + \int_t^{t_f} \ell(z(\tau), u(\tau))d\tau\}$$

Dividing the main interval in infinitesimal intervals, we can write:

$$\begin{aligned}J^*(z(t),t) &= min_{[u(\tau),t\leq\tau\leq t_f]}\{\kappa(z(t_f),t_f) + \int_t^{t+\Delta t}\ell(z(\tau),u(\tau))d\tau \\ &+ \int_{t+\Delta t}^{t_f}\ell(z(\tau),u(\tau))d\tau\}\end{aligned}$$

Application of the optimality principle gives:

$$J^*(z(t),t) = min_{[u(\tau),t\leq\tau\leq t+\Delta t]}\{J^*(z(t+\Delta t),t+\Delta t),t)+\int_t^{t+\Delta t}\ell(z(\tau),u(\tau))d\tau\}$$

We expressed the cost function versus the minimum cost function for the interval $t + \Delta t \leq \tau \leq t_f$ where initial state is $z(t + \Delta t)$. Good mathematical conditions are filled, the second partial derivatives exist and are limited. The Taylor series development of J^* gives:

$$\begin{aligned}J^*(z(t),t) &= min_{[u(\tau),t\leq\tau\leq t+\Delta t]}\{\int_t^{t+\Delta t}\ell(z(\tau),u(\tau))d\tau \\ &+ J^*(z(t),t),t)+[\frac{\partial J^*}{\partial t}(z(t),t),t)]\Delta t \\ &+ [\frac{\partial J^*}{\partial t}(z(t),t),t)]^T[z(t+\Delta t)-z(t)] \\ &+ higher\ orders\}\end{aligned}$$

For infinitesimal Δt, as small as we can use, depending on the computer facilities during the processing step, we can write:

$$\begin{aligned}J^*(z(t),t) &= min_{u(t)}\{l(z(t),u(t)\Delta t) + J^*(z(t),t) \\ &+ J_t^*(z(t),t)\Delta t + J_z^{*T}(z(t),t)[f(z(t),u(t),t)]\Delta t \\ &+ o(\Delta t)\}\end{aligned}$$

where $o(\Delta t) \cong o(\Delta t)^p$ higher orders of $(\Delta t)^2$ following from integral approximations and the stop order of terms of Taylor series (dichotomy of Taylor series). Simplification f the last equation and tending $\Delta t \longrightarrow 0$, we obtain:

$$J_t^*(z(t),t) + min_{u(t)}l(z(t),u(t),t) + J_z^{*T}(z(t),t)[f(z(t),u(t),t)] = 0$$

$t = t_f$ provides the limit of this partial differential equation and then:

$$J^*(z(t_f),t_f) = \kappa(z(t_f),t_f)$$

Writing the Hamiltonian as:

$$\mathcal{H}(z(t),u(t),t,J_z^*(z(t),t)) \triangleq l(z(t),u(t),t) + J_z^{*T}(z(t),t)[f(z(t),u(t),t)]$$

and

$$\mathcal{H}(z(t), u^*(z(t), J_z^*, t), J_z^*, t) = min_{z(t)}[\mathcal{H}(z(t), u(t), t, J_z^*(z(t), t))]$$

We finally obtain the following Hamilton-Jacobi equation which can be called Hamilton-Jacobi-Bellman equation because it is based on the continuous time recurrence equation of Bellman [82, 83, 84, 79, 80, 81]:

$$J_t^*(z(t), t) + \mathcal{H}(z(t), u^*(z(t), J_z^*, t), J_z^*, t) = 0$$

The necessary optimality condition has then been filled because the cost function $J_t^*(z(t), t)$ satisfies the Hamilton-Jacobi-Bellman equation. We notice for the boundary conditions that some variables can be kept free. The Hamilton-Jacobi-Bellman equation will always function and can be solved. We can generally conclude that if the function satisfies the Hamilton-Jacobi-Bellman equation then it is the minimum cost function. This is proven and confirmed [85, 86]. Solutions are obtained using appropriate solvers. Discrete approximations of the given continuous OCP problem are established and solutions obtained using recurrence relations. Exact solutions of the discrete approximation of the the Hamilton-Jacobi-Bellman equation are obtained in the state-time space regions. The latter cannot be known since the beginning of the recurrence processing solving the Hamilton-Jacobi-Bellman equation. The question is to know if we obtain an exact solution of the discretized problem or an approximate solution of the exact optimization equation. The major assumption are: the state and the control variables are constrained, the final time t_f is fixed and $z(t_f)$ is free.

In practice, it is important to solve the equation numerically, if an analytical solution is not possible. The equation can be solved explicitly. In general, it is difficult to calculate the solution. Numerical method based on viscosity solutions to the Hamilton-Jacobi-Bellman equation is performed in this section [87]. First order of the Hamilton-Jacobi-Bellman equation is perturbed by an added diffusion term; a singular perturbation parameter is used. Time and variables are discretized.

There are many methods existing in the open literature solving the Hamilton-Jacobi-Bellman equation which can be considered as efficient. We have choose a feedback control method for the computation which solve the previous Hamilton-Jacobi-Bellman equation using the dynamic programming. The Hamilton-Jacobi-Bellman equation can be expressed as:

$$-\frac{\partial v}{\partial t} + sup_{u \in U}[-\nabla_z v.f(z, u, t) - l(z(t), u(t), t)] = 0$$

We consider a finite region Ξ for z included $\mathbb{R}^{n+m}$, U the control set, and the initial condition $v(t_f, z(t_f)) = \kappa(z(z(t_f), t_f)$ or the value function or the boundary condition, and $-\nabla_z v$ the gradient of v related to z. v and u are unknown. To assess gradients or approximate gradients (or even subgradients), the finite difference methods can be used: Quasi-Newton methods, Conjugate gradient methods, Interior point methods (a large class of methods for constrained optimization. Some interior-point methods use only (sub)gradient information, and others of which require the evaluation of Hessians), Gradient descent, Bundle method of descent, Ellipsoid method, Reduced gradient method (Frank-Wolfe), ... Finding a feedback control for the OCP is equal to solve the initial value problem. Many works are performed showing the existence and the unicity of the solution of this problem well know as a viscosity solution [87, 88, 89]. The extreme value theorem of Weierstrass or Weierstrauss theorem states that a continuous real-valued function on a compact set attains its maximum and minimum value. More generally, a lower semi-continuous function on a compact set attains its minimum; an upper semi-continuous function on a compact set attains its maximum.

Numerical approximations are need for initial value problem for solving the original OCP, which are difficult to find in the open literature in particular efficient algorithms. If we disturb the latter equation by a diffusion term $\varepsilon\nabla^2 v$ where $\varepsilon << 1$ a singular perturbating parameter, we can write the viscosity approximation of previous equation whose principal mathematical properties are shown by Zhou [89]:

$$\varepsilon\nabla^2 v - \frac{\partial v}{\partial t} + sup_{u \in U}[-\nabla_z v.f(z, u, t) - l(z(t), u(t), t)] = 0$$

This initial value problem can be solved explicitly by a time stepping schemes. t_f is fixed and no information is available on v. After domain extension of Ξ which is necessary for finding the exact solution, we can write:

$$\varepsilon\nabla^2 v - \frac{\partial v}{\partial t} - \nabla_z v.f(z, u^*, t) - l(z(t), u^*(t), t) = 0$$

with

$$u^* = arg \quad \sup_{u \in U}[-\nabla_z v.f(z, u, t) - l(z, u, t)]$$

t_f could not be exactly reach and the extension domain is a transitional region where the solution satisfies the artificial boundary conditions of the solutions of the Hamilton-Jacobi-Bellman equation. Finally, we obtain the following equations:

$$l(z(t), u^*(t), t) = \varepsilon\nabla^2 v - \{\frac{\partial v}{\partial t} + \nabla_z v.f(z, u^*, t)\}$$

$$u^* = arg \quad \sup_{u \in U}[-\nabla_z v.f(z,u,t) - l(z,u,t)]$$

To solve, by approximation, the previous equations (having the form of the convection-diffusion equation), we initially simplify the writing form of those equations:

$$l(z(t), u^*(z,t^n), t^n) = \varepsilon \nabla^2 v - \{\frac{v(\bar{x}, t^{n-1}) - v(x, t^n)}{\Delta t}$$

$$u^*(z, t^n) = arg \quad \sup_{u \in U}[-\nabla_z v.f(z, u(z, t^n), t^n) - l(z, u(z, t^n), t^n)]$$

the approximation of the solution at $t^n = 1 - n\Delta t$ ($\Delta t > 0$) with $\bar{x} = x + f(t, z, u(z, t^n))\Delta t$, and $v_t + f.\nabla v$ is an operator acting terms of the given equations which can express the differentiation in the characteristic direction $\zeta = \zeta(x)$ so that we can write:

$$l(z(t), u^*(t), t) = \varepsilon \nabla^2 v - \{\frac{\partial v}{\partial t} + \nabla_z v.f(z, u^*, t)\}$$

$$l(z(t), u^*(t), t) = \varepsilon \nabla^2 v - [1 + |f(z, u, t)|^2]^{\frac{1}{2}} \frac{\partial v}{\partial \zeta(z)}$$

with:

$$\frac{\partial}{\partial \zeta(x)} \cong [\frac{1}{[1 + |f(z, u, t)|^2]^{0.5}}][\frac{\partial}{\partial t} + f(z, u^*, t).\nabla]$$

The problem is the assessment of $v(\bar{x}, t^{n-1})$. This achieved by an extrapolation of $u(z, t^n)$ during processing. This discretization in time associated with the new form of those equations is called the continuous in space MMOC (the modified methods of characteristics) procedure of the original coupled and nonlinear system of equations. The idea behind this method [90, 91, 92] is to use a small extension of the domain. This method of characteristics usually solves partial differential equations. It is generally performed for first-order equations. The method can also be applied for any hyperbolic partial differential equation. Its power allows, inter alia, a reduction a partial differential equation to a family of ordinary differential equations where the solution can be obtained when initial data are given.

MMOC development by Huang et al. and by Cheng and Wang, provided the approximation of the characteristic derivative as:

$$[[1 + |f(z, u, t)|^2]^{0.5}]\frac{\partial v}{\partial \zeta(x)} \cong [[1 + |f(z, u, t)|^2]^{0.5}]\frac{v(\bar{x}, t^{n-1}) - v(x, t^n)}{[[\bar{x} - x|^2 + (\Delta t)^2]^{0.5}}$$

and

$$[[1 + |f(z, u, t)|^2]^{0.5}] \frac{v(\bar{x}, t^{n-1}) - v(x, t^n)}{[[|\bar{x} - x|^2 + (\Delta t)^2]^{0.5}} = \frac{v(\bar{x}, t^{n-1}) - v(x, t^n)}{\Delta t}$$

They suggested the use of the operator splitting technique providing calculation of variables of the first equation from the second one approximating ∇v and $\nabla^2 v$. For any (i, j, n):

$x_{i,j} = (-1 - a + i\kappa, -1 - a + j\kappa)$

For a function σ, $\sigma_{i,j}^n = \sigma(z_{i,j}, t^n)$

a. κ is given

b. $t^n = 1 - n\Delta t$

c. $\Delta t = \frac{1}{N}$

d. $\bar{z}_{i,j} = z + f_{i,j}^n \Delta t$

e. $\bar{w}_{i,j}^{n-1} = w(\bar{z}_{i,j})$

f. $w_{i,j}^n$ and $y_{i,j}^n$ are approximated values of the solution (v,u)

The definite discretization of the last Hamilton-Jacobi-Bellman equation given by the previous nonlinear system is:

$$l_{i,j}^n = \varepsilon \delta^2 w_{i,j}^n - \frac{\bar{w}_{i,j}^{n-1} - w_{i,j}^n}{\Delta t}$$

$$z_{i,j}^n = arg \quad \sup_{y^n \in U} [-\delta w_{i,j}^n . f_{i,j}^n - l_{i,j}^n]$$

Terms $l_{i,j}^n$ and $w_{i,j}^{n-1}$ made the system strongly coupled. It is dissociated by replacing $f_{i,j}^n$ by $f_{i,j}^{n-1}$ and $l_{i,j}^n$ by $l_{i,j}^{n-1}$. Now, we can write the dissociated terms, during numerical processing, using the following algorithm steps [12, 13, 18, 19]:

1. Give $w_{i,j}^0 = \kappa_{i,j}$.

2. Calculate $y_{i,j}^0 = arg \quad \sup_{y^0 \in U} [-\delta w_{i,j}^0 . f_{i,j}^0 - l_{i,j}^0)]$.

3. $n \in [1, N]$ for a given total processing number N, calculate $w_{i,j}^n$ from:

$$l_{i,j}^{n-1} = \varepsilon \delta^2 w_{i,j}^n - \frac{\bar{w}_{i,j}^{n-1} - w_{i,j}^n}{\Delta t}$$

4. Calculate $y_{i,j}^n = arg \quad \sup_{y^n \in U} [-\delta w_{i,j}^n . f_{i,j}^n - l_{i,j}^n)]$.

The processing stages are carried out led with an high calculation rate. The main remarks are given below.

* Discretization scheme given previously is coherent, consistent and stable. It is a copy of Euler's method.

* The matrix of the system is symmetric and positive with $\frac{1}{\Delta t} \gg \varepsilon$.

* High dimensions could be easily performed with those latter properties.

* The OCP dimension is then reduced to a finite-dimensional nonlinear program.
* The large nonlinear program is solved by a standard a robust *NLP* solver according to the discretized variables.
* To solve the obtained *NLP* problem, we developed an *AMPL* model.
* The viscosity coefficient $\varepsilon = 10^{-12}$ is used with this value in processing steps.
* $M = 81$, a and b can be choose in a random way. They are dimension-independent. Domain extension must gradually increase by steps of 10^{-3}.
* Δt=2.5 10^{-3} sec is the used time step. We have also performed calculation with a time step of $\Delta t = 0.5$ sec because radar data are updated every 0.5 sec. Our algorithms could be then interfaced with in flight and on the ground radars controlling aircraft flight paths.
* Asymptotic convergence has been reached because decoupling variables as described.

To conclude, the starting OCP problem is described by a Hamilton-Jacobi-Bellman equation which is transformed to being a convection-diffusion equation. The modified method of characteristics approximating the solution is used. It is then solved the problem in time and a finite-difference in the state space with a high accuracy.

In addition, Lie-Ovsyannikov infinitesimal approach applied for reduction of the corresponding Bellman equation has been described first by Garaev [97, 98] and the Noether theory of invariant variation problems. It could be suggested for the problem of optimum control [93, 94, 95, 96]. In the open literature, it is not usually considered. This approach is new in the area of optimal control problem related to aircraft annoyances minimization. The Bellman equation corollary can be obtained in the form of a linear partial differential equation. The use of the equation simplifies the construction of synthesizing controls.

4 Numerical results

We consider an aircraft landing by fixing initial and final flight conditions:

$$
\begin{aligned}
180\,m/s = V_{min} &\leq V &&\leq V_{max} = free \\
0.2 = \delta_{x_{min}} &\leq \delta_x &&\leq \delta_{x_{max}} = free \\
-10° = \gamma_{min} &\leq \gamma &&\leq \gamma_{max} = free \\
0° = \alpha_{min} &\leq \alpha &&\leq \alpha_{max} = +20° \\
-5° = \chi_{min} &\leq \chi &&\leq \chi_{max} = +5° \\
-5° = \mu_{min} &\leq \mu &&\leq \mu_{max} = +5° \\
-5° = \phi_{min} &\leq \phi &&\leq \phi_{max} = +5° \\
-60\,km = x_{min} &\leq x &&\leq x_{max} = 0\,km \\
-10\,km = y_{min} &\leq y &&\leq y_{max} = +10\,km \\
3500\,m = h_{min} &\leq h &&\leq h_{max} = free \\
t_{min} = 0 &\leq t &&\leq t_{max} = +10\,min
\end{aligned}
$$

V_{min} represents the aircraft velocity (stall velocity). As shown, some of these parameters are kept free. Once the processing steps and calculation efficiency are confirmed, their limit values are found and given. These inequalities are represented by:

$$
a \leq C(z(t), u(t)) \leq b
$$

where a and b are constant vectors. The used data in this optimization model are from an Airbus A300. The three-dimensional analysis is useful in enhancing the reliability of the optimization model applied in automatic detection of aircraft noise and in the aircraft noise features. We consider R the distance aircraft-observer:

$$
R = (x - x_{obs})^2 + (y - y_{obs})^2 + h^2
$$

where $(x_{obs}, y_{obs}, 0)$ is the coordinates of the observer on the ground. OCP is discretized along its state $z = (V, \gamma, \chi, x, y, h)$ and control $u = (\alpha, \delta_x, \mu)$ variables.

P1 to P12 are the considered observers on the ground for which noise levels have to be calculated:

Locations	$x_{obs}(m)$	$y_{obs}(m)$
$P1$	10000	2500
$P2$	10000	2000
$P3$	10000	3000
$P4$	5000	1250
$P5$	5000	1000
$P6$	5000	1500
$P7$	4000	1000
$P8$	4000	800
$P9$	4000	1200
$P10$	2000	500
$P11$	2000	400
$P12$	2000	600

Location points P2 and P3 are symmetrical compared to P1 and are regarded as side points. P5-P6, P8-P9 and P11-P12 are considered as side points compared respectively to P4, P7 and P10. P1, P4, P7 and P10 are under the flight path on the ground. We minimize noise levels in the cost function previously described. The problem to solve is written as follows :

$$(OCP)_3 \begin{cases} \min \vartheta \\ \vartheta >= J_{obs_j} \\ \dot{z}(t) = f(z(t), u(t)) \\ z_{I_1}(0) = c_1, \ z_{I_2}(t_f) = c_2 \\ a \leq C(z(t), u(t)) \leq b \end{cases} \tag{1}$$

where J_{obs_j} are noise levels corresponding to j fixed observers. For several observers, the method is applied; $SNOPT$ found a solution with a very high accuracy.

The discretization parameter is $N = 100$ points because the solution stability. Results will confirm this state. To solve the N_ε and NLP problems, we have used the $AMPL$ model [30] and a robust solver $SNOPT$ [35, 36, 31, 32, 33]. They have been chosen after numerous comparisons among other standard solvers available on the $NEOS$ optimization platform. We have used the call-by-need mechanism which memorized automatically the result of the cost function in order to speed up call-by-name evaluation ().

The (OCP) is transformed with the direct method into a NLP problem. The algorithm is adapted, it changes rules, and initializes points. The objective function has been minimized using 897 variables, 503 constraints, 500 non-linear equalities, and 3 inequalities. The number of nonzeros are respectively 3975 in Jacobian, and 80693 in Hessian. A locally optimal solution of the objective function has been found: the final objective value is 196.7 with a final

feasibility error (abs / rel) of 3.18 10-3 / 1.8 10-9, and a final optimality error (abs / rel) equal to 4.92 10-6 / 7.50 10-7) through 38 function evaluations, 39 gradients and 38 Hessian evaluations. The total program time is equal to 4.341 sec (4.344 CPU time / standard PC).

Concerning the indirect method, a sequence of N_ε problems (tending ε to zero). We initialize the problem N_ε by centering the state and the control. Then, we initialize the Lagrange multiplicators as follows:

$$\lambda = \varepsilon(C(z,u) - a)^{-1}, \quad \mu = \varepsilon(b - C(z,u))^{-1}$$

For the implementation of the penalty parameter ε and computation, we used the following strategy [99]:

$$\varepsilon_{k+1} = \varepsilon_k/a, \quad a > 1$$

Table below summarizes the obtained results for ε.

ε	Feasible error
1	$5.6e - 12$
0.2	$4.0e - 12$
0.04	$7.3e - 14$
0.008	$8.8e - 12$
0.0016	$1.1e - 11$
0.000032	$2.3e - 07$
$6.4e - 05$	$2.5e - 07$
$1.28e - 05$	$2.1e - 07$

Table 1 : The calculated feasibility errors versus ε

For each iteration of the interior point method, the algorithm found an optimal solution. First, it should be remembered that direct and indirect methods (DIM) provided the same optimal trajectory and the same throttle setting δ_x. The solution trajectory (optimal trajectory) and the control δ_x are shown in figure 1. The optimization processing found a constant throttle setting δ_x which corresponds to a stabilization flight or a constant flight level for the three applied methods (DIM and JHB). δ_x is bang-bang between its bounds, in particular for DIM methods where its increase is made in only once to 0.6 lasting slightly more with JHB that DIM. The altitude H, having a predominant role in the noise level behavior called the cost function, decreased with three soft slopes for DIM and two for JHB. These provided three and two constant flight segments in favor of the JHB method. One trajectory stage is observed for JBH before alignment on the runway with a slope of 3 degrees accompanied by a reduction of the power settings. Angles of descent are stable as recommended by ICAO and aircraft certification [100, 101, 102] in favor for JHB method because the continuous descent approach with one constant stage
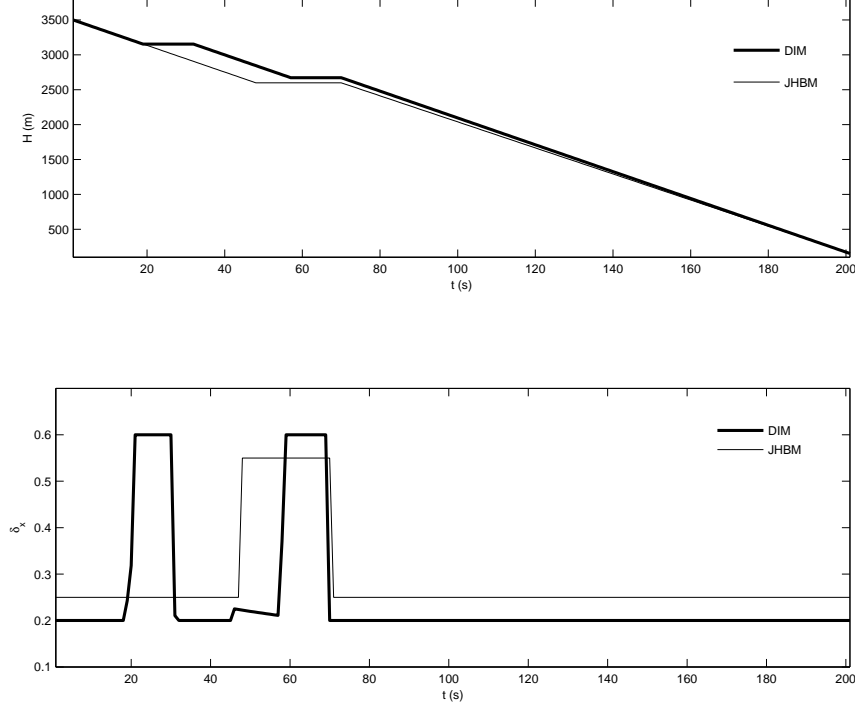
Figure 1: Obtained solution for 100 discretization points

showing the efficiency and performance of the aircraft approach. On the one hand, noise level decrease is confirmed. 6.5 to 9.3 dB reduction is obtained in favor of JHB method. On the other hand, when we compare the measured noise J_0 at a distance of 2 km under the flight path for a standard trajectory approach with the level J obtained with the optimal trajectory given by JHB, change varies with the altitude of approach between 4% to 11% of $\frac{J_0-J}{J_0}$. This is because optimization model, in particular the cost function, does not integrate all non-propulsive noise sources and because of optimization model makes noise reduction possible. The flight rate descent is varying between 896 and 1165 ft/mn which is close to the one recommended by ICAO and practiced by the airline companies (1000 ft/mn). The obtained JHB trajectory could be accepted into the airline community for a number of reasons. The soft JHB one-segment approach puts the aircraft in an appropriate envelope with margins for wind uncertainties and errors. There is no question of vortex separation and problems of intercepting a false glide-slope, given that it must be intercepted from above. With autopilot or flight director coupling, this approach would be acceptable for use in regular air carrier service.

Comparison between the described methods applied to our optimal control problem, confirms that the feasible errors are between $e-07$ and $e-09$ (table 2). Indeed, in terms of analysis of aircraft noise reduction because of the trajectory is optimal, the problem is more in favor of the case of several observers and the JHB method.

Feasible error (Direct method)	$7.39e-07$
Feasible error(Indirect method)	$5.6e-012 to 1.50e-07$
HJB computed errors	$2.3e-09$

Table 2 : Comparison of feasible errors

Although computing power has increased substantially making complex problems more practical for large projects, JHB optimization method offers a substantial advantage in detail over DIM methods with much less computer time and less discretization complexity. Optimization model is expected to replace empirical models for well-established applications such as predicting noise contours around airports and fuel saving. It is, now, practical for a wide range of situations where additional details are necessary. To conclude, the obtained results confirmed the good formulation of this problem of optimization and its effective resolution. They also provided good values, in particular for the flight parameters whose maximum values were maintained free during the processing. Further research is needed to include airframe noise sources, and air-brake systems. The cost function must integrate objectives like reduction of pollutant emissions linked to fuel consumption and air traffic constraints.

5 Conclusion

The objective of this paper, qualifying the best applied numerical method solving commercial aircraft trajectory optimization model taking into account noise sources, fuel consumption, constraints and extreme limits, has been reached. First, we described detailed theoretical considerations and algorithms solving the obtained Hamilton-Jacobi-Bellman HJB equation. Second, we carried out comparisons with direct and indirect methods. The OCP problem was transformed into new equalities-constrained as a viscosity problem constituting an original dynamic system extension. Among the existing methods solving the HJB equation a feedback control method using dynamic programming has been developed. Compared to the direct and indirect methods, we show that HJB dynamic method is characterized by its fast computing times and its efficiency. It provides the best optimized flight paths called the Shortest and Fastest Continuous Descent Approach (SF-CDA) which is able to reduce commercial

aircraft annoyances and fuel consumption. It is a two-segment approach confirmed as an optimized flight path reducing aircraft environmental impacts. Results show that the HJB method is well appropriated for aircraft trajectory optimization problem and could be implemented. Technically, because of its processing speed and algorithm efficiency, it can be better interfaced with the in-flight management system respecting airspace system regulation constraints. SF-CDA approach could benefit both airlines and communities. Further research is needed to consider non-propulsive sources and air traffic regulation.

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