G021 Microeconomics
Lecture notes

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1 Consumption set and budget set

The consumption set X is the set of all conceivable consumption bundles $\mathbf{q}$, usually identified with $\mathbb{R}^n_+$

The budget set $B \subset X$ is the set of affordable bundles

In standard model individuals can purchase unlimited quantities at constant prices $\mathbf{p}$ subject to total budget y. The budget set is the *Walrasian*, *competitive* or *linear* budget set:

$$B = \{\mathbf{q} \in \mathbb{R}^n_+ | \mathbf{p}'\mathbf{q} \leq y\}$$

Notice this is a convex, closed and bounded set with linear boundary $\mathbf{p}'\mathbf{q} = y$. Maximum affordable quantity of any commodity is y/p_i and slope $\mathrm{d}q_i/\mathrm{d}q_j|_B = -p_j/p_i$ is constant and independent of total budget.

In practical applications budget constraints are frequently kinked or discontinuous as a consequence for example of taxation or non-linear pricing.

2 Marshallian demands, elasticities and types of good

The consumer chooses bundles $\mathbf{f}(y, \mathbf{p}) \in B$ known as *Marshallian*, *uncompensated*, *competitive* or *market* demands. In general the consumer may be prepared to choose more than one bundle in which case $\mathbf{f}(y, \mathbf{p})$ is a demand correspondence but typically a single bundle is chosen and $\mathbf{f}(y, \mathbf{p})$ is a demand function.

We wish to understand the effects of changes in y and $\mathbf{p}$ on demand for, say, the ith good:

- total budget y
  - the path traced out by demands in q-space as y increases is called the *income expansion path* whereas the graph of $f_i(y, \mathbf{p})$ as a function of y is called the *Engel curve*
  - for differentiable demands we can summarise dependence in the total budget elasticity

$$\epsilon_i = \frac{y}{q_i}\frac{\partial q_i}{\partial y} = \frac{\partial \ln q_i}{\partial \ln y}$$

  - if demand for a good rises with total budget, $\epsilon_i > 0$, then we say it is a *normal* good and if it falls, $\epsilon_i < 0$, we say it is an *inferior* good
  - if budget share of a good, $w_i = p_i q_i / y$, rises with total budget, $\epsilon_i > 1$, then we say it is a *luxury* or *income elastic* and if it falls, $\epsilon_i < 1$, we say it is a *necessity* or *income inelastic*

- own price p_i
  - the path traced out by demands in q-space as p_i increases is called the *offer curve* whereas the graph of $f_i(y, \mathbf{p})$ as a function of p_i is called the *demand curve*
  - for differentiable demands we can summarise dependence in the (uncompensated) own price elasticity

$$\eta_{ii} = \frac{p_i}{q_i}\frac{\partial q_i}{\partial p_i} = \frac{\partial \ln q_i}{\partial \ln p_i}$$

  - if uncompensated demand for a good rises with own price, $\eta_{ii} > 0$, then we say it is a *Giffen* good
  - if budget share of a good rises with price, $\eta_{ii} > -1$, then we say it is *price inelastic* and if it falls, $\eta_{ii} < -1$, we say it is *price elastic*

- other price $p_j,\ j \neq i$
  - for differentiable demands we can summarise dependence in the (uncompensated) cross price elasticity

$$\eta_{ij} = \frac{p_j}{q_i}\frac{\partial q_i}{\partial p_j} = \frac{\partial \ln q_i}{\partial \ln p_j}$$

  - if uncompensated demand for a good rises with the price of another, $\eta_{ij} > 0$, then we can say it is an (uncompensated) *substitute* whereas if it falls with the price of another, $\eta_{ij} < 0$, then we can say it is an (uncompensated) *complement*. These are not the best definitions of complementarity and substitutability however since they may not be symmetric ie q_i could be a substitute for q_j while q_j is a complement for q_j. A better definition, guaranteed to be symmetric, is one based on the concept of compensated demand to be introduced below.

3 Properties of demands

3.1 Adding up

We know that demands must lie within the budget set: $\mathbf{p}'\mathbf{f}(y, \mathbf{p}) \leq y$. If consumer spending exhausts the total budget then this holds as an equality, $\mathbf{p}'\mathbf{f}(y, \mathbf{p}) = y$, which is known as *adding up*, *Walras' law* or *budget balancedness*.

If we differentiate wrt y then we get a property known as *Engel aggregation*

$$\sum_i p_i \frac{\partial f_i}{\partial y} = \sum_i w_i \epsilon_i = 1$$

- It is clear from this that not all goods can be inferior ($\epsilon_i < 0$), not all goods can be luxuries ($\epsilon_i > 1$) and not all goods can be necessities ($\epsilon_i < 1$)
- Also certain specifications are ruled out for demand systems . It is not possible, for example, for all goods to have constant income elasticities unless these elasticities are all 1. Otherwise $p_i q_i = A_i y^{\alpha_i}$ and $1 = \sum_i A_i \alpha_i y^{\alpha_i - 1}$ for all y and for some $A_i > 0,\ \alpha_i \neq 1,\ i = 1, \ldots, n$ which is impossible.

If we differentiate wrt an arbitrary price p_j then we get a property known as *Cournot aggregation*

$$f_j + \sum_i p_i \frac{\partial f_i}{\partial p_j} = 0 \quad \Rightarrow w_j \quad + \sum_i w_i \eta_{ij} = 0$$

- From this, no good can be a Giffen good unless it has strong complements

3.2 Homogeneity

If we assume that demands depend on y and $\mathbf{p}$ only insofar as these determine the budget set B then values of y and $\mathbf{p}$ giving the same budget set should give the same demands. Hence, since scaling y and $\mathbf{p}$ simultaneously by the same factor does not affect B, demands should be *homogeneous* of degree zero

$$\mathbf{f}(\lambda y, \lambda \mathbf{p}) = \mathbf{f}(y, \mathbf{p}) \text{ forany } \lambda > 0$$

Differentiating wrt λ and setting $\lambda = 1$

$$y \frac{\partial \mathbf{f}}{\partial y} + \sum_j p_j \frac{\partial \mathbf{f}}{\partial p_j} = 0 \quad \Rightarrow \quad \epsilon_i + \sum_j \eta_{ij} = 0$$

which is just an application of *Euler's theorem*.

3.3 Negativity

The *Weak Axiom of Revealed Preference* or *WARP*, stated for the most general case, says that if $\mathbf{q}^0$ is chosen from a budget set B^0 which also contains $\mathbf{q}^1$ then there should exist no budget set B^1 containing $\mathbf{q}^0$ and $\mathbf{q}^1$ from which $\mathbf{q}^1$ is chosen and not $\mathbf{q}^0$. It is a statement of consistency in choice behaviour.

For the case of linear budget constraints, WARP says that if $\mathbf{q}^0 \neq \mathbf{q}^1$ and $\mathbf{q}^0$ is chosen at prices $\mathbf{p}^0$ when $\mathbf{p}^{0\prime}\mathbf{q}^0 \geq \mathbf{p}^{0\prime}\mathbf{q}^1$ then $\mathbf{q}^1$ should never be chosen at prices $\mathbf{p}^1$ when $\mathbf{p}^{1\prime}\mathbf{q}^0 \leq \mathbf{p}^{1\prime}\mathbf{q}^1$

We say that $\mathbf{q}^0$ is (directly) revealed preferred to $\mathbf{q}^1$, written $\mathbf{q}^0 R \mathbf{q}^1$, if $\mathbf{q}^0$ is chosen at prices $\mathbf{p}^0$ when $\mathbf{p}^{0\prime}\mathbf{q}^0 \geq \mathbf{p}^{0\prime}\mathbf{q}^1$. Hence WARP says that we should never find different bundles $\mathbf{q}^0$ and $\mathbf{q}^1$ such that $\mathbf{q}^0 R \mathbf{q}^1$ and $\mathbf{q}^1 R \mathbf{q}^0$

Consider increasing the price of the first good p_1 (by an amount Δp_1) at the same time as increasing total budget by exactly enough to keep the initial choice affordable. This is called *Slutsky compensation* and the extra budget required is easily calculated as $q_1 \Delta p_1$. Any alternative choice within the new budget set which involves a greater quantity of q_1 must previously have been affordable and the consumer cannot now make that choice without violating WARP since the initial choice is also still in the budget set. The consumer must therefore decrease demand for the first good. Slutsky compensated own price effects are necessarily negative.

Since Slutsky compensation was positive the uncompensated own price effect must be even more negative if the good is normal. Hence the *Law of Demand* states that demand curves slope down for normal goods.

We can generalise this to changes in the price of any number of goods. Consider a Slutsky compensated change in the price vector from $\mathbf{p}^0$ to $\mathbf{p}^1 = \mathbf{p}^0 + \Delta\mathbf{p}$ inducing a change in demand from $\mathbf{q}^0$ to $\mathbf{q}^1 = \mathbf{q}^0 + \Delta\mathbf{q}$. By Slutsky compensation both $\mathbf{q}^0$ and $\mathbf{q}^1$ are affordable after the price change: $\mathbf{p}^{1\prime}\mathbf{q}^0 = \mathbf{p}^{1\prime}\mathbf{q}^1$. By WARP, $\mathbf{q}^1$ could not have been affordable before the price change: $\mathbf{p}^{0\prime}\mathbf{q}^0 > \mathbf{p}^{0\prime}\mathbf{q}^1$. By subtraction, therefore, we get the general statement of *negativity*: $\Delta\mathbf{p}'\Delta\mathbf{q} < 0$.

3.4 The Slutsky equation

Slutsky compensated demands $\mathbf{h}(\mathbf{q}^0, \mathbf{p})$ are functions of an initial bundle $\mathbf{q}^0$ and prices $\mathbf{p}$ and are given by Marshallian demands at a budget which maintains affordability of $\mathbf{q}^0$ ie $\mathbf{h}(\mathbf{q}^0, \mathbf{p}) = \mathbf{f}(\mathbf{p}'\mathbf{q}^0, \mathbf{p})$. Differentiating provides a link between the price derivatives of Marshallian and Slutsky-compensated demands

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial f_i}{\partial p_j} + \frac{\partial f_i}{\partial y} q_j^0$$

known as the *Slutsky equation*. Since all terms on the right hand side are observable from market demand responses we can calculate Slutsky compensated price effects and check for negativity more precisely than simply checking to see whether the law of demand is satisfied.

Let $\mathbf{S}$, the *Slutsky matrix*, be the matrix with elements given by the Slutsky compensated price terms $\partial h_i / \partial p_j$. Consider a price change $\Delta\mathbf{p} = \lambda\mathbf{d}$ where $\lambda > 0$ and $\mathbf{d}$ is some arbitrary vector. As $\lambda \to 0$, $\Delta\mathbf{p}'\Delta\mathbf{q} \to \lambda^2 \mathbf{d}'\mathbf{S}\mathbf{d}$ hence negativity requires $\mathbf{d}'\mathbf{S}\mathbf{d} \leq 0$ for any $\mathbf{d}$ which is to say the Slutsky matrix $\mathbf{S}$ must be negative semidefinite. Note how weak have been the assumptions needed to get this result.

4 Preferences

We write $\mathbf{q}^0 \succsim \mathbf{q}^1$ to mean $\mathbf{q}^0$ is at least as good as $\mathbf{q}^1$. For the purpose of constructing a theory of consumer choice behaviour we need only construe this as a statement about willingness to choose $\mathbf{q}^0$ over $\mathbf{q}^1$. For welfare analysis we need to read in a link to consumer wellbeing.

From this basic preference relation we can pull out a symmetric part $\mathbf{q}^0 \sim \mathbf{q}^1$ meaning that $\mathbf{q}^0 \succsim \mathbf{q}^1$ and $\mathbf{q}^1 \succsim \mathbf{q}^0$ and capturing the notion of indifference. We can also pull out an antisymmetric part $\mathbf{q}^0 \succ \mathbf{q}^1$ meaning that $\mathbf{q}^0 \succsim \mathbf{q}^1$ and $\mathbf{q}^0 \nsim \mathbf{q}^1$ capturing the notion of strict preference.

4.1 Rationality

We want the preference relation to provide a basis to consistently identify a set of most preferred elements in any possible budget set. A minimal set of properties comprises:

- *Completeness*: for any $\mathbf{q}^0$ and $\mathbf{q}^1$ either $\mathbf{q}^0 \succsim \mathbf{q}^1$ or $\mathbf{q}^1 \succsim \mathbf{q}^0$
- *Transitivity*: for any $\mathbf{q}^0$, $\mathbf{q}^1$ and $\mathbf{q}^2$, if $\mathbf{q}^0 \succsim \mathbf{q}^1$ and $\mathbf{q}^1 \succsim \mathbf{q}^2$ then $\mathbf{q}^0 \succsim \mathbf{q}^2$

Completeness ensures that choice is possible in any budget set and transitivity ensures that there are no *cycles* in preferences within any budget set. Together they ensure that the preference relation is a *preference ordering*.

4.2 Continuity and utility functions

We can use the preference ordering to define several sets for any bundle $\mathbf{q}^0$:

- the *weakly preferred set*, *upper contour set* or *at least as good as* set is the set $R(\mathbf{q}^0) = \{\mathbf{q}^1 | \mathbf{q}^1 \succsim \mathbf{q}^0\}$
- the *indifferent set* is the set $I(\mathbf{q}^0) = \{\mathbf{q}^1 | \mathbf{q}^1 \sim \mathbf{q}^0\}$
- the *lower contour set* is the set $L(\mathbf{q}^0) = \{\mathbf{q}^1 | \mathbf{q}^0 \succsim \mathbf{q}^1\}$

Plainly $I(\mathbf{q}^0) = R(\mathbf{q}^0) \cap L(\mathbf{q}^0)$ but no assumptions made so far ensure that $R(\mathbf{q}^0)$ or $L(\mathbf{q}^0)$ contain their boundaries and therefore that $I(\mathbf{q}^0)$ can be identified with the boundaries of either. The following assumption guarantees this:

Continuity: Both $R(\mathbf{q}^0)$ and $L(\mathbf{q}^0)$ are closed sets. Equivalently, for any sequences of bundles $\mathbf{q}^i$ and $\mathbf{r}^i$ such that $\mathbf{q}^i \succsim \mathbf{r}^i$ for all i, $\lim \mathbf{q}^i \succsim \lim \mathbf{r}^i$.

If preferences satisfy continuity then there exists a continuous function $u : X \to \mathbb{R}$ such that $u(\mathbf{q}^0) \geq u(\mathbf{q}^1)$ whenever $\mathbf{q}^0 \succsim \mathbf{q}^1$. Such a function is called a *utility function* representing the preferences. The utility function is not unique: if $u(\cdot)$ represents preferences then so does any function $\phi(u(\cdot))$ where $\phi(.)$ is increasing. All that matters for describing choice is the ordering over bundles induced by the utility function and it is therefore said to be an *ordinal* function.

Any continuous function attains a maximum on a closed and bounded set so continuity ensures that the linear budget set has a well identified set of most preferred elements.

If the consumer chooses those most preferred elements then their behaviour satisfies WARP. If there are only two goods then such behaviour is equivalent to

WARP. If there are more goods then such behaviour is equivalent to the *Strong Axiom of Revealed Preference* or *SARP* which says that there should never exist a sequence of bundles $\mathbf{q}^i,\ i = 1, \ldots, n$ such that $\mathbf{q}^0 R \mathbf{q}^1,\ \mathbf{q}^1 R \mathbf{q}^2,\ \ldots,\ \mathbf{q}^{n-1} R \mathbf{q}^n$ but $\mathbf{q}^n R \mathbf{q}^0$.

4.3 Nonsatiation and monotonicity

Nonsatiation says that consumers are never fully satisfied:

Nonsatiation: For any bundle $\mathbf{q}^0$ and any $\epsilon > 0$ there exists another bundle $\mathbf{q}^1 \in X$ where $|\mathbf{q}^0 - \mathbf{q}^1| < \epsilon$ and $\mathbf{q}^1 \succ \mathbf{q}^0$

This, with continuity, ensures that indifferent sets are indifference curves - they cannot have any "thick" regions to them

Monotonicity strengthens nonsatiation to specify the direction in which preferences are increasing:

Monotonicity: If $\mathbf{q}^1 \gg \mathbf{q}^0$ ie $q_i^1 > q_i^0$ for all i, then $\mathbf{q}^1 \succsim \mathbf{q}^0$

Strong monotonicity: If $q_i^1 > q_i^0$ for some i and $q_i^1 < q_i^0$ for no i, then $\mathbf{q}^1 \succsim \mathbf{q}^0$

Monotonicity ensures that indifference curves slope down and that further out indifference curves represent higher utility. The slope of the indifference curve is called the *marginal rate of substitution* or *MRS*.

Any utility function representing (strongly) monotonic preferences has the property that utility is increasing in all arguments. If the utility function is differentiable then $\partial u / \partial q_i > 0$ for all i and

$$MRS = \left.\frac{\mathrm{d}q_j}{\mathrm{d}q_i}\right|_u = -\frac{\partial u / \partial q_i}{\partial u / \partial q_j}$$

The implied marginal rates of substitution are features of the utility function which are invariant to monotonic transformation.

4.4 Convexity

Convexity captures the notion that consumers prefer variety:

Convexity: If $\mathbf{q}^0 \sim \mathbf{q}^1$ then $\lambda \mathbf{q}^0 + (1 - \lambda)\mathbf{q}^1 \succsim \mathbf{q}^0$

Upper contour sets are convex sets and the MRS is diminishing (in magnitude): $\left.\mathrm{d}^2 q_j / \mathrm{d}q_i^2\right|_u > 0$. The corresponding property of the utility function is known as *quasiconcavity*: $u(\lambda \mathbf{q}^0 + (1 - \lambda)\mathbf{q}^1) \geq \min(u(\mathbf{q}^0), u(\mathbf{q}^1))$.

4.5 Homotheticity and quasilinearity

Preferences are *homothetic* if indifference is invariant to scaling up consumption bundles: $\mathbf{q}^0 \sim \mathbf{q}^1$ implies $\lambda \mathbf{q}^0 \sim \lambda \mathbf{q}^1$ for any $\lambda > 0$. This imposes no restriction on the shape of any one indifference curve considered in isolation but implies that all indifference curves have the same shape in the sense that those further out are magnified versions from the origin of those further in. As a consequence, marginal rates of substitution are constant along rays through the origin.

Homotheticity clearly holds if the utility function is homogeneous of degree one: $u(\lambda \mathbf{q}) = \lambda u(\mathbf{q})$ for $\lambda > 0$. In fact, up to increasing transformation, this is the only class of utility functions which give homothetic preferences ie preferences are homothetic iff $u(\mathbf{q}) = \phi(\upsilon(\mathbf{q}))$ where $\upsilon(\lambda\mathbf{q}) = \lambda\upsilon(\mathbf{q})$ for $\lambda > 0$.

Quasilinearity is a somewhat similar idea in that it requires indifference curves all to have the same shape, but in the sense of being translated versions of each other. In this case indifference is invariant to adding quantities to a particular good: preferences are quasilinear wrt the ith good if $\mathbf{q}^0 \sim \mathbf{q}^1$ implies $\mathbf{q}^0 + \lambda\mathbf{e}_i \sim \mathbf{q}^1 + \lambda\mathbf{e}_i$ for any $\lambda > 0$ and $\mathbf{e}_i$ is the n-vector with zeroes in all places except the ith.

In terms of the utility function, preferences are quasilinear iff $u(\mathbf{q}) = \phi(\upsilon(\mathbf{q}))$ where $\upsilon(\mathbf{q} + \lambda\mathbf{e}_i) = \upsilon(\mathbf{q}) + \lambda$ for $\lambda > 0$.

5 Choice

An individual chooses $\mathbf{q}^0$ if $\mathbf{q}^0 \in B$ and there is no other $\mathbf{q}^1 \in B$ where $\mathbf{q}^1 \succ \mathbf{q}^0$. If preferences are continuous and the budget constraint is linear then there exists a utility function $u(\mathbf{q})$ to represent preferences and the choice solves the consumer problem

$$\max u(\mathbf{q}) \ s.t. \ \mathbf{p}'\mathbf{q} \leq y$$

.

The demands solving such a problem

- satisfy homogeneity
- satisfy WARP
- satisfy adding up if preferences are nonsatiated (otherwise there would exist a preferred bundle within the budget set which was not chosen)
- are unique if preferences are convex

The solution is at a point where an indifference curve just touches the boundary of the budget set. If utility is differentiable at that point then the MRS between any two goods consumed in positive quantities equals the ratio of their prices

$$\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = \frac{p_i}{p_j}$$

.

This could be deduced from the first order conditions for solving the consumer problem:

$$\frac{\partial u}{\partial q_i} = \lambda p_i \qquad i = 1, \ldots, n$$

where λ is the Lagrange multiplier on the budget constraint.

5.1 Income expansion paths

As y is increased the budget set expands but the slope of its boundary is unchanged. The points of tangency trace out a path along which the MRS between goods are constant and this characterises the income expansion path.

- For homothetic preferences such paths are rays through the origin and ratios between chosen quantities are independent of y given $\mathbf{p}$, as also therefore are budget shares.
- For quasilinear preferences such paths are straight lines parallel to the ith axis and quantities of all goods except the ith good are independent of y given $\mathbf{p}$, provided that the fixed quantities of these goods in question remain affordable

6 Duality

6.1 Hicksian demands

Just as upper contour sets can be ordered by utility, budget sets can be ordered (given $\mathbf{p}$) by total budget y. Just as Marshallian demands maximise utility given total budget y and prices $\mathbf{p}$ so the same quantities minimise the expenditure necessary to each given utility u given prices $\mathbf{p}$.

Consider the *dual* problem

$$\min \mathbf{p}'\mathbf{q} \; s.t. \; u(\mathbf{q}) \geq u$$

to be contrasted with the *primal* problem above. The quantities solving this problem can be written as functions of utility u and prices $\mathbf{p}$ and are called the *Hicksian* or *compensated* demands, which we write as $\mathbf{g}(u, \mathbf{p})$.

First order conditions for this problem are clearly similar to those for solution of the primal problem

$$p_i = \mu \frac{\partial u}{\partial q_i} \qquad i = 1, \ldots, n$$

where μ is the Lagrange multiplier on the utility constraint

The demands solving such a problem

- satisfy homogeneity in prices, $g(u, \lambda p) = g(u, p)$
- satisfy WARP
- satisfy the utility constraint with equality if preferences are nonsatiated, $u(g(u, p)) = u$
- are unique if preferences are convex

6.2 Indirect utility function and expenditure function

We can define functions giving the values of the primal and dual problems. these are known as the indirect utility function

$$v(y, \mathbf{p}) = \max u(\mathbf{q}) \ s.t. \ \mathbf{p}'\mathbf{q} \leq y$$

and the expenditure function

$$e(u, \mathbf{p}) = \min \mathbf{p}'\mathbf{q} \ s.t. \ u(\mathbf{q}) \geq u.$$

These functions can be derived from the corresponding demands by evaluating the objective functions at those demands ie

$$v(y, \mathbf{p}) = u(\mathbf{f}(y, \mathbf{p})) \qquad e(u, \mathbf{p}) = \mathbf{p}'\mathbf{g}(u, \mathbf{p}).$$

The duality between the two problems can be expressed by noting the equality of the quantities solving the two problems

$$\mathbf{f}(e(u, \mathbf{p}), \mathbf{p}) = \mathbf{g}(u, \mathbf{p}) \qquad \mathbf{f}(y, \mathbf{p}) = \mathbf{g}(v(y, \mathbf{p}), \mathbf{p})$$

or noting that $v(y, \mathbf{p})$ and $e(u, \mathbf{p})$ are inverses of each other in their first arguments

$$v(e(u, \mathbf{p}), \mathbf{p}) = u \qquad e(v(y, \mathbf{p}), \mathbf{p}) = y.$$

The expenditure function has the properties that

- it is homogeneous of degree one in prices $\mathbf{p}$, $e(u, \lambda\mathbf{p}) = \lambda e(u, \mathbf{p})$. The Hicksian demands are homogeneous of degree zero so the total cost of purchasing them must be homogeneous of degree one

$$e(u, \lambda\mathbf{p}) = \lambda\mathbf{p}'\mathbf{g}(u, \lambda\mathbf{p}) = \lambda\mathbf{p}'\mathbf{g}(u, \mathbf{p}) = \lambda e(u, \mathbf{p})$$

- it is increasing in $\mathbf{p}$ and u.
- it is concave in prices

$$\begin{aligned} e(u, \lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^0 &= \lambda\mathbf{p}^{1\prime}\mathbf{g}(u, \lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^0) \\ &\qquad +(1-\lambda)\mathbf{p}^{0\prime}\mathbf{g}(u, \lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^0) \\ &\geq \lambda e(u, \mathbf{p}^1) + (1-\lambda)e(u, \mathbf{p}^0) \end{aligned}$$

since $\mathbf{p}^{1\prime}\mathbf{g}(u, \lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^0) \geq e(u, \mathbf{p}^1)$ and $\mathbf{p}^{0\prime}\mathbf{g}(u, \lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^0) \geq e(u, \mathbf{p}^0)$

These are all of the properties that an expenditure function must have.

The properties of the indirect utility function follow immediately from those of the expenditure function given the inverse relationship between them

- it is homogeneous of degree zero in total budget y and prices $\mathbf{p}$, $v(\lambda y, \lambda\mathbf{p}) = v(y, \mathbf{p})$. This should be apparent also from the homogeneity properties of Marshallian demands
- it is decreasing in $\mathbf{p}$ and increasing in y.
- it is quasiconvex in prices

$$v(y, \lambda\mathbf{p}^1 + (1-\lambda)\mathbf{p}^0 \quad \leq \quad \max(v(y, \mathbf{p}^1), v(y, \mathbf{p}^0))$$

6.3 Shephard's lemma and Roy's identity

Since $e(u,\mathbf{p}) = \mathbf{p}'\mathbf{g}(u,\mathbf{p})$

$$\begin{aligned}
\frac{\partial e(u,\mathbf{p})}{\partial p_i} &= g_i(u,\mathbf{p}) + \mathbf{p}'\frac{\partial \mathbf{g}(u,\mathbf{p})}{\partial p_i} \\
&= g_i(u,\mathbf{p}) + \mu \sum_j \frac{\partial u}{\partial q_j}\frac{\partial g_j(u,\mathbf{p})}{\partial p_i} \\
&= g_i(u,\mathbf{p})
\end{aligned}$$

using the first order condition and the fact that utility u is held constant. This is *Shephard's Lemma* Its importance is that it allows compensated demands to be deduced simply from the expenditure function by differentiation.

Since $v(e(u,\mathbf{p}),\mathbf{p}) = u$

$$\begin{aligned}
\frac{\partial v(y,\mathbf{p})}{\partial p_i} + \frac{\partial v(y,\mathbf{p})}{\partial y}\frac{\partial e(u,\mathbf{p})}{\partial p_i} &= 0 \\
\Rightarrow -\frac{\partial v(y,\mathbf{p})/\partial p_i}{\partial v(y,\mathbf{p})/\partial y} &= g_i(v(y,\mathbf{p}),\mathbf{p}) \\
&= f_i(y,\mathbf{p})
\end{aligned}$$

using Shephard's Lemma

This is *Roy's identity*. Its importance is that it allows uncompensated demands to be deduced simply from the indirect utility function, again solely by differentiation

In many ways it is therefore easier to derive a demand system by beginning with $v(y,\mathbf{p})$ or $e(u,\mathbf{p})$ than by solving the consumer problem directly given $u(\mathbf{q})$

6.4 The Slutsky equation

Since $\mathbf{g}(u,p) = \mathbf{f}(e(u,\mathbf{p}),\mathbf{p})$

$$\begin{aligned}
\frac{\partial g_i(u,\mathbf{p})}{\partial p_j} &= \frac{\partial f_i(y,\mathbf{p})}{\partial p_j} + \frac{\partial f_i(y,\mathbf{p})}{\partial y}\frac{\partial e(u,\mathbf{p})}{\partial p_j} \\
&= \frac{\partial f_i(y,\mathbf{p})}{\partial p_j} + \frac{\partial f_i(y,\mathbf{p})}{\partial y} f_j(y,\mathbf{p})
\end{aligned}$$

The equation relating price derivatives of Hicks-compensated to Marshallian demands has the same form as that relating Slutsky-compensated to Marshallian demands. Hicks-compensated price derivatives are the same as Slutsky-compensated price derivatives since the two notions of compensation coincide at the margin.

The Slutsky matrix can therefore be defined using either notion of compensation and Hicksian demands therefore also satisfy negativity at the margin.

6.5 Slutsky symmetry

From Shephard's Lemma

$$\frac{\partial g_i(u, \mathbf{p})}{\partial p_j} = \frac{\partial^2 e(u, \mathbf{p})}{\partial p_i p_j} = \frac{\partial g_j(u, \mathbf{p})}{\partial p_i}$$

The Slutsky matrix of compensated price derivatives is not only negative definite but also *symmetric*.

Note the implication that notions of complementarity and substitutability are consistent between demand equations if using compensated demands. This is *not* true of uncompensated demands because income effects are not symmetric and it is therefore preferable to base definitions of complements and substitutes on compensated demands.

6.6 Integrability

The properties of *adding up*, *homogeneity*, *negativity* and *symmetry* are not only necessary but sufficient for consumer optimisation. If Marshallian demands satisfy these restrictions then there is a utility function $u(\mathbf{q})$ which they maximise subject to the budget constraint. We say in such a case that demands are *integrable*. The implied Hicksian demands define the expenditure function through a soluble set of differential equations by Shephard's Lemma

We know a system of demands is integrable if any of the following hold

- They were derived as solutions to the dual or primal problem given a well specified direct utility function
- They were derived by Shephard's Lemma from a cost function satisfying the appropriate requirements or they were derived by Roy's identity from an indirect utility function satisfying the appropriate requirements
- They satisfy adding up, homogeneity, symmetry and negativity

6.7 Homotheticity

We know that homothetic preferences lead to constant budget shares given prices. Therefore $\mathbf{f}(y, \mathbf{p}) = y\alpha(\mathbf{p})$ for some $\alpha(\mathbf{p})$ and, if we were to take the homogeneous representation of utility, which necessarily exists, then doubling y doubles demands and would double utility. Thus, more generally, $v(y, \mathbf{p}) = \phi(y/a(\mathbf{p}))$ for some $a(\mathbf{p})$ and some increasing $\phi(\cdot)$ and $e(u, \mathbf{p}) = \phi^{-1}(u)a(\mathbf{p})$. The function $a(\mathbf{p})$ should be interpreted as a price index which is independent of utility u.

6.8 Summary

$$\begin{array}{ccc}
\textit{Utility maximisation problem} & & \textit{Cost minimisation problem} \\
\max u(\mathbf{q}) \quad s.t. \quad \mathbf{p}'\mathbf{q} \leq y & \longleftrightarrow & \min \mathbf{p}'\mathbf{q} \quad s.t. \quad u(\mathbf{q}) \geq u \\
\downarrow & & \downarrow \\
\textit{Uncompensated demands} & & \textit{Compensated demands} \\
\mathbf{q} = \mathbf{f}(y, \mathbf{p}) & \longleftrightarrow & \mathbf{q} = \mathbf{g}(u, \mathbf{p}) \\
\updownarrow & & \updownarrow \\
\textit{Indirect utility function} & & \textit{Expenditure function} \\
v(y, \mathbf{p}) & \longleftrightarrow & e(u, \mathbf{p})
\end{array}$$

7 Grouping and Separability

The need to simplify often requires that we consider grouping goods and analysing their demand in isolation from the rest of the consumer problem. This could be justified either by assumptions about price movements or by assumptions about preferences. Suppose that we partition the consumption bundle $\mathbf{q}$ into groups $\mathbf{q}_\mathcal{I}$, $\mathbf{q}_\mathcal{J}$, $\mathbf{q}_\mathcal{K}$, ... and so on.

7.1 Price movements

7.1.1 Hicks aggregation

Suppose prices for goods within one group move together in the sense that $\mathbf{p}_\mathcal{I} = \theta \mathbf{p}^0_\mathcal{I}$ for some fixed vector of relativities $\mathbf{p}^0_I$. This sort of assumption might be justified by features of the technology producing these goods or by arbitrage possibilities. Then we can think of θ as a sort of scalar group price. The expenditure function takes the form $e(u, \mathbf{p}) = e(u, \theta\mathbf{p}^0_\mathcal{I}, \mathbf{p}_\mathcal{J}, \dots) = e^*(u, \theta, \mathbf{p}_\mathcal{J}, \dots)$ for some function $e^*(\cdot)$. Note that $\partial e^*/\partial\theta = \mathbf{p}^{0\prime}_\mathcal{I} \partial e/\partial \mathbf{p}_\mathcal{I} = \mathbf{p}^{0\prime}_\mathcal{I}\mathbf{q}_\mathcal{I}$. Hence Shephard's lemma holds if we regard θ as the price of a *composite commodity* formed by adding up the within-group quantities weighted by the fixed relative prices $\mathbf{p}^0_\mathcal{I}$. This sort of grouping is called *Hicks aggregation* or *Hicksian separability.*

7.2 Preference structure

7.2.1 Weak separability

We say that the group $\mathcal{I}$ is a *weakly separable* group in preferences if preferences over $\mathbf{q}_\mathcal{I}$ are independent of the quantities in other groups. Precisely, weak separability requires that

$$(\mathbf{q}^0_\mathcal{I}, \mathbf{q}_\mathcal{J}, \dots) \succsim (\mathbf{q}^1_\mathcal{I}, \mathbf{q}_\mathcal{J}, \dots)$$

only if

$$(\mathbf{q}^0_\mathcal{I}, \mathbf{q}_\mathcal{J}', \dots) \succsim (\mathbf{q}^1_\mathcal{I}, \mathbf{q}_\mathcal{J}', \dots)$$

for all $\mathbf{q}_\mathcal{J}', \dots$ If there exists a utility function then weak separability implies it has the form

$$u = v(\phi_\mathcal{I}(\mathbf{q}_\mathcal{I}), \mathbf{q}_\mathcal{J}, \mathbf{q}_\mathcal{K}, \dots)$$

for some within-group utility function $\phi_{\mathcal{I}}(\cdot)$. Note that this implies the MRS between goods in group $\mathcal{I}$ are independent of quantities of goods outside the group:

$$\left.\frac{\mathrm{d}q_{\mathcal{I}i}}{\mathrm{d}q_{\mathcal{I}j}}\right|_u = \frac{\partial\phi_{\mathcal{I}}/\partial q_{\mathcal{I}j}}{\partial\phi_{\mathcal{I}}/\partial q_{\mathcal{I}i}}.$$

If we consider the problem of choosing $\mathbf{q}_{\mathcal{I}}$ conditional on quantities of other goods

$$\max_{\mathbf{q}_{\mathcal{I}}} v(\phi_{\mathcal{I}}(\mathbf{q}_{\mathcal{I}}), \mathbf{q}_{\mathcal{J}}, \mathbf{q}_{\mathcal{K}}, \dots)\ s.t.\ y \geq \mathbf{p}_{\mathcal{I}}{}'\mathbf{q}_{\mathcal{I}} + \mathbf{p}_{\mathcal{J}}{}'\mathbf{q}_{\mathcal{J}} + \mathbf{p}_{\mathcal{K}}{}'\mathbf{q}_{\mathcal{K}} + \dots$$

we see that the solution reduces to solving

$$\max_{\mathbf{q}_{\mathcal{I}}} \phi_{\mathcal{I}}(\mathbf{q}_{\mathcal{I}})\ s.t.\ \mathbf{p}_{\mathcal{I}}{}'\mathbf{q}_{\mathcal{I}} \leq y - \mathbf{p}_{\mathcal{J}}{}'\mathbf{q}_{\mathcal{J}} - \mathbf{p}_{\mathcal{K}}{}'\mathbf{q}_{\mathcal{K}} - \dots \equiv y_{\mathcal{I}}.$$

The only information needed to solve the problem is the amount left to spend on the group $y_{\mathcal{I}}$ and prices within the group $\mathbf{p}_{\mathcal{I}}$. Within group demands can therefore be written $\mathbf{q}_{\mathcal{I}} = \mathbf{f}_{\mathcal{I}}(y_{\mathcal{I}}, \mathbf{p}_{\mathcal{I}})$ so lower stage demands depend only on total within group spending and within group prices. This sort of preference structure justifies common empirical simplifications such as

- analysing spending on one period's demands as a function of that period's budget and prices ignoring patterns of spending in other periods
- analysing spending on commodity demands as a function of total commodity spending and prices ignoring decisions on allocation of time
- analysing spending on nondurable goods as a function of nondurable spending and prices ignoring patterns of spending on durable goods

We can substitute the within group demands into the within group utility to get a within group indirect utility function

$$v_{\mathcal{I}}(y_{\mathcal{I}}, \mathbf{p}_{\mathcal{I}}) = \phi_I(\mathbf{f}_{\mathcal{I}}(y_{\mathcal{I}}, \mathbf{p}_{\mathcal{I}}))$$

and invert it to get a well-defined within group expenditure function

$$e_{\mathcal{I}}(u_{\mathcal{I}}, \mathbf{p}_{\mathcal{I}}) = \min_{\mathbf{q}_{\mathcal{I}}} \mathbf{p}_{\mathcal{I}}{}'\mathbf{q}_{\mathcal{I}}\ s.t.\ \phi(\mathbf{q}_{\mathcal{I}}) \geq u_{\mathcal{I}}.$$

Standard duality results such as Roy's identity and Shephard's lemma apply to these within group functions.

Weak separability justifies treating the lower stage of the individual's consumer choices separately from decisions about how much budget to allocate to the group. However the upper stage decision about how much to allocate to the group cannot in general be made without simultaneously considering the lower stage decision. If lower stage preferences are *homothetic* then we can write $v_{\mathcal{I}}(y_{\mathcal{I}}, \mathbf{p}_{\mathcal{I}}) = y_{\mathcal{I}}/a_{\mathcal{I}}(\mathbf{p}_{\mathcal{I}})$ for some lower stage price index $a_{\mathcal{I}}(\mathbf{p}_{\mathcal{I}})$ and the top stage decision can then be written

$$\max_{u_{\mathcal{I}}, \mathbf{q}_{\mathcal{J}}, \dots} v(u_{\mathcal{I}}, \mathbf{q}_{\mathcal{J}}, \dots)\ s.t.\ y \geq a_{\mathcal{I}}(\mathbf{p}_{\mathcal{I}})u_{\mathcal{I}} + \mathbf{p}_{\mathcal{J}}{}'\mathbf{q}_{\mathcal{J}} + \mathbf{p}_{\mathcal{K}}{}'\mathbf{q}_{\mathcal{K}} + \dots$$

and solved without considering the lower stage decision.

If all groups are weakly separable then we say simply that preferences are *weakly separable*

$$u = \upsilon(\phi_{\mathcal{I}}(\mathbf{q}_{\mathcal{I}}), \phi_{\mathcal{J}}(\mathbf{q}_{\mathcal{J}}), \phi_{\mathcal{K}}(\mathbf{q}_{\mathcal{K}}), \dots)$$

and if preferences within all groups are homothetic then we have *homothetic weak separability* and complete *two stage budgeting* with a top stage:

$$\max_{u_{\mathcal{I}}, u_{\mathcal{J}}, \dots} \upsilon(u_{\mathcal{I}}, u_{\mathcal{J}}, \dots) \; s.t. \; y \geq a_{\mathcal{I}}(\mathbf{p}_{\mathcal{I}})u_{\mathcal{I}} + a_{\mathcal{J}}(\mathbf{p}_{\mathcal{J}})u_{\mathcal{J}} + \dots$$

7.2.2 Strong separability

Now suppose that preferences can be written in an *additive* form

$$u = \phi_{\mathcal{I}}(\mathbf{q}_{\mathcal{I}}) + \phi_{\mathcal{J}}(\mathbf{q}_{\mathcal{J}}) + \phi_{\mathcal{K}}(\mathbf{q}_{\mathcal{K}}) + \dots$$

Then not only is the MRS within any group independent of quantities in other groups but the marginal utilities of quantities within the group are independent of quantities in other groups: $\partial u/\partial q_{\mathcal{I}i} = \partial \phi_{\mathcal{I}}/\partial q_{\mathcal{I}i}$. Hence the MRS between goods in any two groups (being the ratio of these marginal utilities) is independent of quantities in any other group. In other words, groups can be arbitrarily combined to give further separable groups - for example,

$$(\mathbf{q}^0_{\mathcal{I}}, \mathbf{q}^0_{\mathcal{J}}, \mathbf{q}_{\mathcal{K}}, \dots) \succsim (\mathbf{q}^1_{\mathcal{I}}, \mathbf{q}^1_{\mathcal{J}}, \mathbf{q}_{\mathcal{K}}, \dots)$$

only if

$$(\mathbf{q}^0_{\mathcal{I}}, \mathbf{q}^0_{\mathcal{J}}, \mathbf{q}_{\mathcal{K}}', \dots) \succsim (\mathbf{q}^1_{\mathcal{I}}, \mathbf{q}^1_{\mathcal{J}}, \mathbf{q}_{\mathcal{K}}', \dots)$$

for all $\mathbf{q}_{\mathcal{K}}', \dots$ Gorman proved that the possibility of writing utility in this additive form is the only preference structure which allows this. It is known as *strong separability*, *additive separability* or *additivity*.

8 Demand with Endowments

Until now we have assumed consumers' budgets are in the form of endowed nominal income and that their values do not therefore change as prices change. If consumers have endowments of goods then this is no longer true. The exchangeable values of these endowments change with prices. Suppose the endowment is ω. The budget constraint becomes $y + \mathbf{p}'\omega \geq \mathbf{p}'\mathbf{q}$ or equivalently $y \geq \mathbf{p}'\mathbf{z}$ where $\mathbf{z} = \mathbf{q} - \omega$ denotes the *net demands*.

Quantities consumed will be $\mathbf{q} = \mathbf{f}(y + \mathbf{p}'\omega, \mathbf{p})$. Differentiating and using the Slutsky equation

$$\begin{aligned} \left.\frac{\partial q_i}{\partial p_j}\right|_{y,\omega} &= \frac{\partial f_i}{\partial p_j} + \frac{\partial f_i}{\partial y}\omega_j \\ &= \frac{\partial g_i}{\partial p_j} - \frac{\partial f_i}{\partial y}[q_j - \omega_j] \end{aligned}$$

The income effect now depends upon whether the individual is a net buyer or seller. If the individual is a net seller then the uncompensated demand curve need not slope down for a normal good - in fact income and substitution effects will be opposing.

9 Labour supply

Labour supply choice is a key example of demand with endowments. We treat this as a two good choice problem where the consumer has an endowment of one of the goods, time.

Combining all commodities into a single good c with a price p can be justified by invoking either

- Hicks aggregation under the assumption that all consumers under consideration face the same commodity prices or
- homothetic weak separability of goods from leisure.

The consumer has an endowment T of time of which l is consumed (sometimes referred to as *leisure*) and $h = T - l$ is sold to the market at a price of w. The consumer possibly also has an endowment of unearned income y so the budget constraint is $y + wT \geq wl + pc$. The value of total available resources $y + wT$ is sometimes called *full income.*

Uncompensated demand for time is $l = f(y + wT, w, p)$ and compensated demand is $l = g(u, w, p)$. The Slutsky equation has the form

$$\begin{aligned} \left.\frac{\partial l}{\partial w}\right|_y &= \frac{\partial f}{\partial w} + \frac{\partial f}{\partial y}T \\ &= \frac{\partial g}{\partial w} + \frac{\partial f}{\partial y}(T - l) \end{aligned}$$

Note that the two terms have opposite sign if time is a normal good since the consumer must be a net seller of time, $T - l \geq 0$. The corresponding Slutsky equation for labour supply takes the form

$$\left.\frac{\partial h}{\partial w}\right|_y = \left.\frac{\partial h}{\partial w}\right|_u + \left.\frac{\partial h}{\partial y}\right|_w h.$$

10 Intertemporal choice

The intertemporal nature of consumer choice can be dealt with by treating the vectors of commodities consumed in different time periods, $t = 0, 1, , \ldots, T$, as different goods, $\mathbf{q}_t$. An intertemporal budget constraint then requires that the discounted present value of the lifetime spending stream equals the discounted present value of the lifetime income stream, $y_t,\ t = 0, 1, , \ldots, T$:

$$\sum_t^T \mathbf{p}_t' \mathbf{q}_t/(1+r)^t \leq \sum_t^T y_t/(1+r)^t = Y$$

where r is the market discount rate.

It is very natural in such a context to assume some sort of separability of preferences across periods since goods in different periods are not consumed at the same time. Any such assumption restricts the role of memory and anticipation in determining preferences, ruling out habits in consumption, for example. Preferences are often assumed strongly intertemporally separable

$$u = \sum_t^T \phi_t(\mathbf{q}_t)$$

for some concave functions $\phi_t(\cdot)$. Thus the MRS between goods consumed in any two periods is independent of the quantities consumed in any third period.

If, furthermore, within period preferences are homothetic then the problem of allocating spending across periods can be written

$$\max \sum_t^T \psi_t(c_t) \ s.t. \ \sum_t^T c_t a_t(\mathbf{p}_t/(1+r)^t) \leq \sum_t^T y_t/(1+r)^t$$

for some concave functions $\psi_t(\cdot)$ and time-specific price indices $a_t(\cdot)$. The intertemporal choice problem now has the form of a problem of demand with endowments where the goods are total consumptions in each period and endowments are incomes in each period. Effects of interest rate changes therefore depend upon whether the consumer is a net seller or net buyer of these goods or, in other words, whether they are a saver or a borrower.

Frequently $\psi_t(c_t) = \upsilon(c_t)/(1+\delta)^t$ for some concave function $\upsilon(\cdot)$. The parameter $\delta > 0$ is a *subjective discount rate* reflecting downweighting of future utility relative to the present and therefore capturing impatience. Assuming prices $\mathbf{p}_t$ are constant, for simplicity, then first order conditions for intertemporal choice require

$$\frac{\upsilon'(c_t)}{\upsilon'(c_s)} = \left(\frac{1+\delta}{1+r}\right)^{|t-s|}.$$

If $\delta = r$ then $c_t = c_s$ given concavity of $\upsilon(\cdot)$ so concavity can be seen as capturing the desire to smooth the consumption stream. If $r > \delta$ then chosen consumption will follow a rising path with a steepness determined by the degree of concavity in $\upsilon(\cdot)$.

11 Uncertainty

Choice under uncertainty can be fitted into the standard consumer framework by treating commodities consumed in different possible states of the world, $s = 0, 1, 2, \ldots, S$, as different goods, $\mathbf{q}_s$. In fact, for most of what is said below about preferences we can interpret $\mathbf{q}_s$ more generally as a description of any pertinent feature of s, not simply quantities of commodities consumed, and we refer to $\mathbf{q} = (\mathbf{q}_0, \mathbf{q}_1, \ldots)$ as the vector of *outcomes*.

The budget constraint linking what can be consumed in different states of the world is typically determined by means for transferring wealth between uncertain states of the world such as insurance, gambling, risky investment and so on. The relative prices on consumption in different states of the world are then set by the premia in insurance contracts, betting odds and so on.

11.1 Expected utility

What is distinctive about the case of uncertainty is the probabilistic framework. The probabilities of states occurring, $\pi_s,\ s = 0, 1, 2, \ldots, S$, enter not the budget constraint but preferences. To bring these in we define preferences over combinations of vectors of outcomes and probabilities known as *lotteries*, *gambles* or *prospects*. A *simple* lottery $L = (\mathbf{q}; \pi)$ is a list of outcomes $\mathbf{q}$ and associated probabilities π.

This is a context in which separability assumptions are often regarded as extremely persuasive. Consider the choice between two lotteries

$$L^0 = (\mathbf{q}_0^0, \mathbf{q}_1^0, \mathbf{q}_2, \mathbf{q}_3, \ldots; \pi) \qquad \text{and} \qquad L^1 = (\mathbf{q}_0^1, \mathbf{q}_1^1, \mathbf{q}_2, \mathbf{q}_3, \ldots; \pi)$$

. What happens in states of the world other than $s = 0, 1$ are the same in both cases and since these states of the world will not have occurred if either of the first two states transpire then it is felt that they should not matter to the choice. In other words if $L^0 \succsim L^1$ then this should be so whatever the outcomes $\mathbf{q}_2, \mathbf{q}_3, \ldots$. This is referred to as the *sure thing principle* (because it says choice should ignore outcomes in states of the world where the outcome is a "sure thing") and it is plainly recognisable as a strong separability assumption. If it is true then, given π, preferences have an additive utility representation

$$u(L) = \sum_s \upsilon_s(\mathbf{q}_s, \pi).$$

The sure thing principle implies the MRS between outcomes in two different states of the world should be independent of outcomes in any third state of the world. What is being ruled out is any influence on preferences between any two states from what merely might happen in a third state as might be felt to be the case if choice were influenced by considerations, say, of potential for regret. The assumption is incompatible with behaviour in certain well known examples such as the *Allais paradox*.

We can put further structure on preferences by thinking about the way that probabilities enter preferences. It is usual to assume firstly that the preference relation is continuous in the probabilities. Secondly we can make assumptions about preferences over combinations of lotteries known as *compound lotteries*. Let $\pi \circ L^0 + (1-\pi) \circ L^1$ denote the lottery which gives a chance π of entering lottery L^0 and a chance $(1-\pi)$ of entering lottery L^1. The *betweenness* assumption then says that that if $L^0 \succsim L^1$ then $L^0 \succsim \pi \circ L^0 + (1-\pi) \circ L^1 \succsim L^1$ so that any compound lottery mixing the two lies between them in the consumer's preference ordering. This already implies linearity of indifference curves in probability space since if $L^0 \sim L^1$ then

$$u(L^0) = u(L^1) = \pi u(L^0) + (1-\pi)u(L^1) = u(\pi \circ L^0 + (1-\pi) \circ L^1).$$

If we combine this assumption with the sure thing principle[1] then we get the *strong independence axiom*: $L^0 \succsim L^1$ if and only if $\pi \circ L^0 + (1-\pi) \circ L^2 \succsim \pi \circ L^1 + (1-\pi) \circ L^2$ for any third lottery L^2. Given strong independence, preferences are both additive across states and linear in probabilities - in fact they take the *expected utility form*

$$u(L) = \sum_s \pi_s \upsilon_s(\mathbf{q}_s).$$

Such a utility function $u(L)$ is called a *(von Neumann-Morgenstern) expected utility function*

If we add the assumption that the description of the state of the world is irrelevant to the utility gained from the outcome, in the sense that, say, $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \ldots; \pi_0, \pi_1, \pi_2, \ldots) \sim (\mathbf{q}_1, \mathbf{q}_0, \mathbf{q}_2, \ldots; \pi_1, \pi_0, \pi_2, \ldots)$ then we can drop the distinction between within period utility functions and write simply

$$u(L) = \sum_s \pi_s \upsilon(\mathbf{q}_s).$$

11.2 Risk aversion

Suppose the outcomes are monetary amounts, y_s, so that

$$u(L) = \sum_s \pi_s \upsilon(y_s).$$

It is through the properties of $\upsilon(\cdot)$, the state specific utility function sometimes called the *Bernoulli utility function*, that properties such as aversion to risk are captured. We say that someone is *risk averse* if they would always prefer to take the expected monetary value of a gamble with certainty to participating in the gamble. Thus a person is risk averse if $E\upsilon(y_s) \leq \upsilon(Ey_s)$ for any monetary lottery. Taking a case with two outcomes

$$\pi\upsilon(y_0) + (1-\pi)\upsilon(y_1) \leq \upsilon(\pi y_0 + (1-\pi)y_1)$$

for $0 \leq \pi \leq 1$. Since this is a statement of concavity of the Bernoulli utility function $\upsilon(\cdot)$ we identify risk aversion with concavity.

Concavity is not a property preserved under arbitrary increasing transformations of $\upsilon(\cdot)$ and representations of preferences under uncertainty are not preserved by increasing transformations of the Bernoulli utility function. If two alternative expected utility functions represent the same preferences then

[1] If $L^0 \succsim L^1$ then $\pi \circ L^0 + (1-\pi) \circ L^0 \sim L^0 \succsim \pi \circ L^1 + (1-\pi) \circ L^0$ and the sure thing principle allows us to replace the second occurrence of L^0 in these compound lotteries with any lottery L^2.

it must be that the Bernoulli utility functions are affine transformations of each other.

We can evaluate the degree of someone's risk aversion by asking how much they would be prepared to pay to avoid a gamble. The *certainty equivalent* to a monetary gamble, M, is the amount which if received with certainty would give the same expected utility as the actual gamble: $\upsilon(M) = E\upsilon(y)$. The difference between the expected monetary value of the gamble and the certainty equivalent is what the individual would pay to avoid the gamble and is known as the risk premium, $m = Ey - M$. Taking Taylor expansions:

$$\begin{aligned}
\upsilon(Ey - m) &\simeq \upsilon(Ey) - m\upsilon'(Ey) \\
E\upsilon(y) &\simeq \upsilon(Ey) + E(y - Ey)\upsilon'(Ey) + \frac{1}{2}E(y - Ey)^2\upsilon''(Ey) \\
&= \upsilon(Ey) + \frac{1}{2}Var(y)\upsilon''(Ey) \\
\Rightarrow m &\simeq -\frac{1}{2}\frac{\upsilon''(Ey)}{\upsilon'(Ey)}Var(y)
\end{aligned}$$

The ratio $R(y) = -\upsilon''(y)/\upsilon'(y)$ is known as the *Arrow-Pratt coefficient of absolute risk aversion*. These ideas for comparing risk aversion come together in three equivalent definitions of what it might mean for one individual to be more risk averse than another:

- the more risk averse individual has a Bernoulli utility function which is an increasing concave transformation of that of the other
- the more risk averse individual has a higher risk premium for any gamble
- the more risk averse individual has a higher coefficient of absolute risk aversion at all y

Multiplying $R(y)$ by y gives the *coefficient of relative risk aversion*, $r(y)$. Just as $m \simeq \frac{1}{2}R(Ey)Var(y)$, $m/Ey \simeq \frac{1}{2}r(Ey)Var(y)/(Ey)^2$.

11.3 An example: Insurance

Suppose someone has initial wealth A but would lose all of it in an event occurring with probability π. They can purchase insurance K for a premium of γK where γ is the rate at which the insurance premium is charged. The individual's levels of wealth in good and bad states are therefore $y_0 = A - \gamma K$ and $y_1 = K - \gamma K$. The budget constraint linking them is $(1-\gamma)y_0 + \gamma y_1 = A(1-\gamma)$. The relative price of wealth in the two states y_0 and y_1 is set by the rate of premium γ and the structure of the choice problem is like a demand problem where the individual has an endowment A of y_0 of which they are a net seller.

The consumer problem is

$$\max_K [(1 - \pi)\upsilon(A - \gamma K) \ + \ \pi\upsilon(K(1 - \gamma))]$$

and the first order condition for an interior solution is

$$-(1-\pi)\gamma v'(A-\gamma K)+(1-\gamma)\pi v'(K(1-\gamma))=0.$$

If $\pi=\gamma$ then insurance is *actuarially fair* in the sense that the expected monetary return from taking out an insurance contract is zero and, given concavity $v''(\cdot)<0$, the solution to the first order condition implies *full insurance*, $A=K$. All points on the budget constraint imply the same expected wealth and the risk averse individual will choose the unique point at which risk is eliminated.

If $\gamma>\pi$ then insurance is less than actuarially fair but a risk averse consumer may still be willing to pay to reduce some but not all of the risk they would face in the uninsured state. Taking a Taylor expansion of the first order condition around $K=A$

$$\begin{aligned} 0 &\simeq [(1-\pi)\gamma-\pi(1-\gamma)]v'(A(1-\gamma)) \\ &\qquad -[\gamma^2(1-\pi)+(1-\gamma)^2\pi]v''(A(1-\gamma))[K-A] \\ \Rightarrow A-K &\simeq \frac{[(1-\pi)\gamma-\pi(1-\gamma)]}{[\gamma^2(1-\pi)+(1-\gamma)^2\pi]\ R(A(1-\gamma))} \end{aligned}$$

so that the degree of underinsurance depends on the departure from actuarial fairness $(1-\pi)\gamma-\pi(1-\gamma)$ and the coefficient of absolute risk aversion at full insurance $R(A(1-\gamma))$ (and the degree of underinsurance relative to initial wealth $1-K/A$ depends upon the departure from actuarial fairness and the coefficient of relative risk aversion). A more risk averse individual will therefore underinsure less heavily.

12 Production

12.1 Technology

Let a firm's activities be described by a *production plan*, *production vector* or *net output vector*, $\mathbf{y}$, in which negative quantities correspond to goods consumed (*inputs*) and positive quantities to goods produced (*outputs*). The technological constraints facing the firm are captured in the *production set*, Y, which includes all technologically feasible production plans.

The production set Y is usually assumed closed and nonempty. In particular, assuming Y to contain at least $\mathbf{y}=0$ means that shutdown is feasible. On the other hand, assuming Y to contain no $\mathbf{y}\gg 0$ rules out the possibility of producing positive output without consuming any inputs. *Monotonicity* or *free disposal* requires that it is always possible to reduce the net output of any good so if $\mathbf{y}\in Y$ then so also is any $\mathbf{y}'\ll y$.

The *transformation function* $F(\mathbf{y})$ is an equivalent means of describing production possibilities, defined to take nonpositive values $F(\mathbf{y})\leq 0$ if and only if the production plan is feasible, $\mathbf{y}\in Y$. Moving along the boundary of Y, known as the *transformation frontier*, keeping $F(\mathbf{y})=0$, the rate at which any one net output can be transformed into another is known as the *marginal rate of transformation*,

$$MRT_{ij} = \mathrm{d}y_i/\mathrm{d}y_j|_{F(\mathbf{y})=0} = -\frac{\partial F/\partial y_j}{\partial F/\partial y_j}.$$

Where there is a single output q produced with several inputs $\mathbf{z}$ we define a production function, $f(\mathbf{z}) = \max q\ s.t.\ (q, -\mathbf{z}) \in Y$, giving the maximum output feasible with any combination of inputs. The production set can be defined by a transformation function $F(y) = q - f(z)$. For any given output level q the *input requirement set*, $V(q) = \{\mathbf{z}|(q, -\mathbf{z}) \in Y\}$, gives all input bundles capable of producing q. The set of input combinations producing exactly the same output forms the boundary to this set, known as an *isoquant*, and its slope is known as the *marginal rate of technical substitution*

$$MRTS_{ij} = \mathrm{d}z_i/\mathrm{d}z_j|_{f(\mathbf{z})=q} = -\frac{\partial f/\partial z_j}{\partial f/\partial z_j}.$$

(Note the similarity of the input requirement set to the weakly preferred set in consumer theory and of an isoquant to an indifference curve).

Returns to scale are concerned with the feasibility of scaling up and down production plans.

- If any feasible production plan can always be scaled down then we say that there are *nonincreasing returns to scale*, $\mathbf{y} \in Y$ implies $\alpha\mathbf{y} \in Y$ for $0 \leq \alpha \leq 1$. If there is only one output and one input then this is equivalent to concavity of the production function.
- If any feasible production plan can always be scaled up then we say that there are *nondecreasing returns to scale*, $\mathbf{y} \in Y$ implies $\alpha\mathbf{y} \in Y$ for $\alpha \geq 1$. If there is only one output and one input then this is equivalent to convexity of the production function.
- If production shows both nonincreasing and nondecreasing returns to scale then we say that there are *constant returns to scale* (CRS), $\mathbf{y} \in Y$ implies $\alpha\mathbf{y} \in Y$ for $\alpha \geq 0$. Mathematically, this says that the production set is a *cone*. If there is only one output then this is equivalent to homogeneity of degree one of the production function, $f(\alpha\mathbf{z}) = \alpha f(\mathbf{z})$ and if there is also only one input then production is linear.

Convexity of the input requirement set $V(q)$ means that convex combinations of input vectors $\mathbf{z}$ and $\mathbf{z}'$ which produce the same output q will also produce at least q. In terms of the slope of isoquants, convexity implies a decreasing MRTS.

Convexity of the production set Y is a stronger assumption, implying not only convexity of input requirement sets $V(q)$ but also nonincreasing returns to scale, neither of which implies the other. If there is a single output then this sort of convexity is equivalent to concavity of the production function $f(\mathbf{z})$. If Y is convex and there are constant returns to scale then Y is a *convex cone*.

12.2 Profit maximisation

Supposing goods prices are $\mathbf{p}$, then firm profits are $\mathbf{p}'\mathbf{y}$ and we assume the firm chooses its production plan so as to earn the highest possible profits

$$\max_{\mathbf{y}} \mathbf{p}'\mathbf{y} \; s.t. \; \mathbf{y} \in Y.$$

We consider here only the case of competitive behaviour in which the firm takes prices $\mathbf{p}$ to be fixed. Assuming a solution exists with finite $\mathbf{y} \neq 0$ and characterised by first order conditions

$$p_i = \lambda \partial F / \partial y_i, \; i = 1, 2, \ldots$$

where λ is a Lagrange multiplier on the technology constraint and therefore

$$MRT_{ij} = -p_i/p_j.$$

The marginal rate of transformation between any two outputs, between any two inputs or between any input and output is equal to their price ratio.

If there is a single output at price p produced with inputs priced at $\mathbf{w}$ then the problem reduces to

$$\max_{\mathbf{z}} pf(\mathbf{z}) - \mathbf{w}'\mathbf{z}$$

for which first order conditions require $p\partial f/\partial z_i - w_i = 0$ so that each input is used up to the point that the value of its marginal product $\partial f/\partial z_i$ equals its input price w_i.

The value of the profit maximising net supplies $\mathbf{y}(\mathbf{p})$ gives the *profit function*

$$\pi(\mathbf{p}) = \mathbf{p}'\mathbf{y}(\mathbf{p}) = \max_{\mathbf{y}} \mathbf{p}'\mathbf{y} \; s.t. \; \mathbf{y} \in Y.$$

It has the properties that

- it is homogeneous of degree one: $\pi(\lambda\mathbf{p}) = \lambda\pi(\mathbf{p})$
- it is convex:

$$\begin{aligned}
\pi(\lambda\mathbf{p}^0 + (1-\lambda)\mathbf{p}^1) &= \lambda{\mathbf{p}^0}'\mathbf{y}(\lambda\mathbf{p}^0 + (1-\lambda)\mathbf{p}^1) + (1-\lambda){\mathbf{p}^1}'\mathbf{y}(\lambda\mathbf{p}^0 + (1-\lambda)\mathbf{p}^1) \\
&\leq \lambda{\mathbf{p}^0}'\mathbf{y}(\mathbf{p}^0) + (1-\lambda){\mathbf{p}^1}'\mathbf{y}(\mathbf{p}^1) \\
&= \lambda\pi(\mathbf{p}^0) + (1-\lambda)\pi(\mathbf{p}^1)
\end{aligned}$$

Differentiating the identity $\pi(\mathbf{p}) = \mathbf{p}'\mathbf{y}(\mathbf{p})$:

$$\begin{aligned}
\frac{\partial \pi}{\partial p_i} &= y_i(\mathbf{p}) + \sum_j p_j \frac{\partial y_j}{\partial p_i} \\
&= y_i(\mathbf{p}) + \lambda \sum_j \frac{\partial F}{\partial y_i}\frac{\partial y_j}{\partial p_i} \; = \; y_i(\mathbf{p})
\end{aligned}$$

using the first order condition for profit maximisation and noting that $F(\mathbf{y})$ is held constant as $\mathbf{p}$ changes. This is *Hotelling's Lemma*: the supply functions are the derivatives of the profit function.

The properties of the net supply functions themselves follow from the properties of the profit function given Hotelling's lemma. They are homogeneous of degree zero: $\mathbf{y}(\lambda\mathbf{p}) = \mathbf{y}(\mathbf{p})$. Since $\partial y_i/\partial p_j = \partial^2\pi/\partial p_i \partial p_j$, the matrix of price derivatives of the supply functions, known as the *supply substitution matrix* is necessarily symmetric and, given convexity of $\pi(\mathbf{p})$, positive semidefinite. As a particular implication of this, $\partial y_i/\partial p_i \geq 0$ so supply functions must slope up in own prices - the *law of supply*. An increase in price of an output increases its supply and an increase in an input price reduces its demand. (Note that this result needs no qualification regarding compensation since there is no budget constraint associated with the producer problem).

This last result could have been established by appeal to the *Weak Axiom of Profit Maximisation*: if a producer chooses $\mathbf{y}^0$ at prices $\mathbf{p}^0$ then that output choice must make greater profit at those prices than any output choice $\mathbf{y}^1$ made under any other prices $\mathbf{p}^1$. Hence $\mathbf{p}^{0\prime}\mathbf{y}^0 \geq \mathbf{p}^{0\prime}\mathbf{y}^1$ and $\mathbf{p}^{1\prime}\mathbf{y}^1 \geq \mathbf{p}^{1\prime}\mathbf{y}^0$. By subtraction $\Delta\mathbf{p}'\Delta\mathbf{y} \geq 0$ where $\Delta\mathbf{p} = \mathbf{p}^1 - \mathbf{p}^0$ and $\Delta\mathbf{y} = \mathbf{y}^1 - \mathbf{y}^0$.

Existence of a single valued solution to the consumer problem with finite y is by no means trivial. Suppose there are nondecreasing returns to scale. Then either

- there are no production plans earning positive profits, $\pi(\mathbf{p}) = 0$ and if any $\mathbf{y} \neq 0$ earns zero profits so do infinitely many any other production plans $\alpha\mathbf{y},\ \alpha \geq 1$
- there is some production plan $\mathbf{y}$ earning positive profit but this can be scaled up indefinitely earning higher profits without limit and $\pi(\mathbf{p}) = +\infty$.

The possibility of indefinitely increasing profits without limit will be avoided if either

- at sufficiently high output levels there are decreasing returns to scale
- at sufficiently high output levels the assumption of price-taking behaviour does not hold
- there are economic forces (such as free entry) which drive prices to the point where positive profits can not be earned

12.3 Cost minimisation

For firms producing a single output, an implication of profit maximisation is that the chosen output level is produced at least cost. That is to say, choice of inputs $\mathbf{z}$ solves

$$\min_{\mathbf{z}} \mathbf{w}'\mathbf{z} \ s.t.\ \mathbf{z} \in V(q).$$

This is useful because it is true even where firms are not output price-takers and even where decisions about scale of production may be difficult to model.

First order conditions require $\lambda \partial f/\partial z_i = w_i$ where λ is the Lagrange multiplier on the output constraint, and therefore imply equality between the marginal rate of technical substitution and input price ratio, $MRTS_{ij} = w_i/w_j$. This is a particular case of the MRT condition derived earlier. The cost minimising input quantities are known as the *conditional factor demands*, $\mathbf{z}(q, \mathbf{w})$ and the function giving minimum cost given q and $\mathbf{z}$ is known as the *cost function*, $c(q, \mathbf{w}) = \mathbf{w}'\mathbf{z}(q, \mathbf{w})$.

The cost minimisation problem is formally identical to the consumer's problem of minimising the expenditure required to reach a given utility (if inputs are identified with commodities consumed, output with utility and input prices with commodity prices). Results derived for that case can therefore be adapted to the current context. The cost function is nondecreasing in q, increasing, homogeneous of degree one and concave in $\mathbf{w}$. Furthermore, *Shephard's lemma* requires that its derivatives equal the conditional factor demands $\partial c(q, \mathbf{w})/\partial w_i = z_i(q, \mathbf{w})$. The conditional factor demands are therefore homogeneous of degree zero in $\mathbf{w}$ and obey symmetry and negativity conditions.

12.4 Cost functions

Having defined the cost function we can use it to rewrite the problem of output choice as

$$\max_q pq - c(q, \mathbf{w}).$$

The first order condition for this problem requires that price be equated to marginal cost, $p = \partial c/\partial q$.

In the case of constant returns to scale, the cost function and conditional factor demands are proportional to q, $c(q, \mathbf{w}) = q\kappa(\mathbf{w})$. Marginal and average cost are equal at all q to the unit cost function $\kappa(\mathbf{w})$. For concave $f(\mathbf{z})$ the marginal cost is nondecreasing. Other technologies can give various shapes to average and marginal cost curves. Simple calculus establishes the useful result that the average cost curve is flat where average cost equals marginal cost since

$$\frac{\mathrm{d}(c/q)}{\mathrm{d}q} = \frac{1}{q}\left(\frac{\mathrm{d}c}{\mathrm{d}q} - \frac{c}{q}\right).$$

The lowest quantity q at which average cost reaches a minimum is known as the *minimum efficient scale*.

13 Equilibrium

13.1 Exchange economies

Consider a population of N individuals with endowments of goods ω^i, $i = 1, \ldots, N$. An *allocation* of consumption bundles $\mathbf{q}^i$, $i = 1, \ldots, N$ is said to be *feasible* if the aggregate endowments are sufficient to cover the total consumption for each good, $\sum_i \mathbf{q}^i \leq \sum_i \omega^i$. The initial endowments trivially constitute a feasible allocation.

Assume that agents trade competitively at prices $\mathbf{p}$ - that is to say they regard themselves as individually unable to affect the given prices. Gross demands are $\mathbf{f}^i(\mathbf{p}'\omega^i, \mathbf{p})$ and net demands are therefore $\mathbf{z}^i(\mathbf{p}) = \mathbf{f}^i(\mathbf{p}'\omega^i, \mathbf{p}) - \omega^i$. We say that a *Walrasian equilibrium*, *competitive equilibrium*, *general equilibrium*, *market equilibrium* or *price-taking equilibrium* exists at prices $\mathbf{p}$ if the gross demands at those prices constitute a feasible allocation, $\mathbf{z}(\mathbf{p}) = \sum_i \mathbf{z}^i(\mathbf{p}) \leq 0$.

For the case of two goods and two persons the *Edgeworth box* provides an illuminating graphical representation of feasible allocations. An arbitrary price vector will not lead to mutually compatible desired trades from arbitrary endowments but by plotting offer curves we can identify competitive equilibria with crossings of offer curves and see that such equilibria, if they exist, will involve mutual tangencies between the two consumers' indifference curves.

13.2 Walras' Law

Because individual demands are homogeneous, $\mathbf{z}(\mathbf{p}) = \mathbf{z}(\lambda\mathbf{p})$ and if any price vector $\mathbf{p}$ constitutes a Walrasian equilibrium then so does any multiple $\lambda\mathbf{p}$, $\lambda > 0$. Finding an equilibrium therefore involves finding a vector of $n-1$ relative prices which ensures satisfaction of the n conditions, $\mathbf{z}_j(\mathbf{p}) \leq 0,\ j = 1, \ldots, n$.

At first it might seem that this gives inadequate free prices to solve the required number of inequalities but this is not so because of *Walras' Law*. Because each individual is on their individual budget constraint the value of their excess demand is zero, $\mathbf{p}'\mathbf{z}^i(\mathbf{p}) = 0$. But then the value of aggregate excess demand must also be zero, $\mathbf{p}'\mathbf{z}(\mathbf{p}) = \sum_i \mathbf{p}'\mathbf{z}^i(\mathbf{p}) = 0$, which is Walras' Law. Hence if prices can be found to clear only $n-1$ of the markets then the clearing of the remaining market is guaranteed: $z_j(\mathbf{p}) = 0,\ j = 1, \ldots, n-1 \ \Rightarrow\ z_n(\mathbf{p}) = 0$.

Walras' Law holds both in and out of equilibrium. However in equilibrium it carries further consequences. Since all prices are nonnegative, $p_j \geq 0$, and all aggregate net demands are nonpositive in equilibrium, $z_j(\mathbf{p}) \leq 0$, the value of excess demand must be nonnegative, $p_j z_j(\mathbf{p}) \leq 0$. Walras' Law, $\sum_j p_j z_j(\mathbf{p}) = 0$ cannot therefore be satisfied unless $p_j = 0$ whenever $z_j(\mathbf{p}) < 0$, ie any good in excess supply must be free. We would of course expect free goods to be in excess demand not excess supply. If that is so then Walras' Law implies that demand must actually equal supply on all markets, $z_j(\mathbf{p}) = 0,\ j = 1, \ldots, n$.

13.3 Existence

At least one Walrasian equilibrium exists if the aggregate net demand functions $\mathbf{z}(\mathbf{p})$ are continuous. Since strict convexity of preferences guarantees uniqueness and continuity of individual net demands it is therefore sufficient for existence of an equilibrium in an exchange economy.

To prove this we typically make use of a *fixed point theorem*. For example, *Brouwer's* fixed point theorem states that every continuous mapping $F(\cdot)$ from the unit simplex $S = \{\mathbf{x} | \sum_j x_j = 1\}$ to itself has a fixed point, or in other words there exists a $\xi \in S$ such that $\xi = F(\xi)$. Given this theorem we know that an equilibrium exists if we can find a particular mapping from the unit simplex to itself which has a fixed point only if there exists a Walrasian equilibrium.

Given homogeneity we can always scale prices so that the price vector $\mathbf{p}$ lies in the unit simplex (just divide by $\sum_j p_j$) without affecting demands. We can therefore concentrate attention on finding a $\mathbf{p} \in S$ which sustains a feasible set of demands.

Define a vector function $\mathbf{F} : S \to S$ as having elements

$$F_j(\mathbf{p}) = \frac{p_j + \max(0, z_j)}{1 + \sum_k \max(0, z_k)}.$$

This function, as required, provides a continuous mapping from the unit simplex to itself, provided that aggregate net demands are continuous, since $\sum_j F_j(\mathbf{p}) = 1$ if $\sum_j p_j = 1$.

Now suppose we have a fixed point so that $F_j(\mathbf{p}) = p_j$.

$$\begin{aligned} p_j &= \frac{p_j + \max(0, z_j)}{1 + \sum_k \max(0, z_k)} \\ \Rightarrow \max(0, z_j) &= p_j \sum_k \max(0, z_k) \\ \Rightarrow z_j \max(0, z_j) &= p_j z_j \sum_k \max(0, z_k) \\ \Rightarrow \sum_j z_j \max(0, z_j) &= \sum_j p_j z_j \sum_k \max(0, z_k) \end{aligned}$$

But $\sum_j p_j z_j = 0$ by Walras' Law so $\sum_j z_j \max(0, z_j) = 0$. Every term in this sum is nonnegative so all terms must be zero and therefore $z_j = 0,\ j = 1, \ldots, n$. The price vector is therefore a Walrasian equilibrium and, by Brouwer's fixed point theorem such a point exists. This completes the proof.

13.4 Uniqueness

In general there is no guarantee that equilibrium is unique. It is plain that one can, for instance, draw an Edgeworth box with multiply crossing offer curves.

If we want to establish uniqueness then it is necessary to make restrictions on preferences. An example of such a restriction would be that all goods are gross substitutes in the sense that $\partial z_i / \partial p_j > 0$ whenever $i \neq j$. It is simple to prove that there cannot be more than one equilibrium price vector in S given such an assumption. Suppose that there were two such vectors $\mathbf{p}^0$ and $\mathbf{p}^1$ and let λ denote the maximum ratio between prices in the two vectors $\max_j(p_j^1/p_j^0)$. By homogeneity if $\mathbf{p}^0$ sustains an equilibrium then so does $\lambda\mathbf{p}^0$. To get from $\lambda\mathbf{p}^0$ to $\mathbf{p}^1$ no price has to be raised and at least one has to be reduced. But then excess demand for any good which has the same price in $\lambda\mathbf{p}^0$ and $\mathbf{p}^1$ would be reduced below 0 and $\mathbf{p}^1$ could not be an equilibrium.

13.5 Dynamics

To develop a theory of how prices change out of equilibrium we need further theory. If prices are not in equilibrium then desired trades are not feasible, some consumers must be rationed and their demands for other goods will be affected. Without a theory of how rationing is implemented and how consumer demands respond it is difficult to reach any conclusion.

Walras is responsible for introducing the fictional notion of a *tâtonnement* process. Suppose the market is overseen by an auctioneer who announces candidate prices, collects declarations of demand from agents, checks for disequilibrium, revises prices according to some rule and repeats the process until, if ever, equilibrium is found. The ability of such a process to find equilibrium cannot be guaranteed except under specific restrictions on preferences and on the nature of the rule for updating prices but there certainly are assumptions which will ensure movement towards equilibrium.

It is sensible to think that prices should rise where there is excess demand so suppose $\mathrm{d}p_j/\mathrm{d}t = \alpha_j z_j$ for some positive constants $\alpha_j,\ j = 1, \ldots, n$. Suppose that there exists an equilibrium $\mathbf{p}^*$ and measure distance from equilibrium by $D(\mathbf{p}) = \sum_j (p_j - p_j^*)^2/2\alpha_i > 0$. Then $\mathrm{d}D(\mathbf{p})/\mathrm{d}t = \sum_j (p_j - p_j^*)z_j = -\sum_j p_j^* z_j$ (using Walras' law). Hence disequilibrium falls continuously if $\sum_j p_j^* z_j > 0$. This can be seen as an application of the Weak Axiom to aggregate demands in comparisons involving equilibrium prices $\mathbf{p}^*$. Unfortunately satisfaction of the Weak Axiom by individuals does not generally guarantee that it holds in the aggregate but it does do so for certain demands, for example any where goods are gross substitutes.

13.6 Welfare

Simply looking at the representation of equilibrium in an Edgeworth box makes plain that neither individual can be made better off without making the other worse off. In other words equilibrium is *Pareto optimal* or *Pareto efficient*. This is a general fact about Walrasian equilibria captured in the *First Fundamental theorem of Welfare Economics*: Walrasian equilibria are Pareto efficient.

The proof is simple. Suppose a feasible allocation $\mathbf{r}$ is *Pareto superior* to a Walrasian equilibrium $\mathbf{q}$. In other words it makes someone better off and noone worse off. Since $\mathbf{q}$ is chosen at equilibrium prices $\mathbf{p}$, the allocations in $\mathbf{r}$ cannot be affordable at $\mathbf{p}$ by anyone who is better off: $\mathbf{pr}^i > \mathbf{pq}^i$ if $\mathbf{r}^i \succ^i \mathbf{q}^i$ and $\mathbf{pr}^i \geq \mathbf{pq}^i$ if $\mathbf{r}^i \succsim^i \mathbf{q}^i$. But then $\mathbf{p}' \sum_i \mathbf{r}^i > \mathbf{p}' \sum_i \mathbf{q}^i$. But since $\mathbf{r}$ and $\mathbf{q}$ are both affordable $\mathbf{p}' \sum_i \mathbf{r}^i = \mathbf{p}' \sum_i \omega^i = \sum_i \mathbf{q}^i$ which is a contradiction.

Is the reverse true? Suppose $\mathbf{q}$ is Pareto efficient and suppose an equilibrium exists with initial endowments $\omega = \mathbf{q}$. Call that equilibrium $\mathbf{r}$. Then $\mathbf{r}^i \succsim^i \mathbf{q}^i$ for all households since $\mathbf{q}^i$ is in each household's budget constraint. But $\mathbf{q}$ is Pareto efficient so it must be that $\mathbf{r}^i \sim^i \mathbf{q}^i$ for all households. But in that case $\mathbf{q}$ is itself an equilibrium. Hence any Pareto efficient allocation can be sustained as a Walrasian equilibrium provided an equilibrium exists with that allocation as the initial endowment point. We know that strict convexity of preferences is sufficient to guarantee that. This is the *Second Fundamental Theorem of Welfare Economics*.

13.7 Production

Production can be introduced into the economy by allowing for the existence of M firms each trying to maximise profits within its production set, $\max \mathbf{p}'\mathbf{y}^j$ *s.t.* $\mathbf{y}^j \in Y^j$. Inputs to production come from consumers' net supply of endowments (and particularly labour supply). Net outputs of firms supplement consumers' endowments as sources of resources for consumption. Firms are owned by consumers and profits are returned to them according to some matrix $\Theta = \{\theta_{ij}\}$ of profit shares reflecting ownership of firms. Budget constraints are therefore

$$\mathbf{p}'\mathbf{q}^i \leq \mathbf{p}'\omega^i + \sum_j \theta_{ij}\mathbf{p}'\mathbf{y}^j$$

with $\sum_i \theta_{ij} = 1$.

To analyse such an economy redefine aggregate excess demand as the excess of aggregate consumption over endowments and production: $\mathbf{z}(\mathbf{p}) = \sum_i(\mathbf{q}^i - \omega^i) - \sum_j \mathbf{y}^j$. Walras' Law still holds provided $\sum_i \theta_{ij} = 1$.

In place of the Edgeworth box we can base intuition on diagrams representing the *Robinson Crusoe* economy where one individual trades with himself acting as consumer on one side and producer on the other.

Existence of an equilibrium is guaranteed if aggregate net demands are continuous, which in such an economy is assured by strict convexity of preferences and production possibilities.

The First Fundamental Theorem still holds: Walrasian equilibrium is still Pareto efficient. The proof is only a little more complicated. Suppose a feasible consumption allocation $\mathbf{r}$ and production plan $\mathbf{x}$ Pareto dominates Walrasian equilibrium allocation $\mathbf{q}$ and production plan $\mathbf{y}$. As argued above $\mathbf{r}^i$ must be unaffordable at equilibrium prices $\mathbf{p}$ by anyone better off under it given $\mathbf{y}$: $\mathbf{p}'r^i > \mathbf{p}'\omega^i + \sum_j \theta_{ij}\mathbf{y}^j$ if $\mathbf{r}^i \succ_i \mathbf{q}_i$ and $\mathbf{p}'r^i \geq \mathbf{p}'\omega^i + \sum_j \theta_{ij}\mathbf{y}^j$ if $\mathbf{r}^i \succsim_i \mathbf{q}_i$. But then $\mathbf{p}'\sum_i r^i > \mathbf{p}'[\sum_i \omega^i + \sum_j \mathbf{y}^j]$ given $\sum_i \theta_{ij} = 1$. By feasibility, $\sum_i \mathbf{r}^i = \sum_i \omega^i + \sum_j \mathbf{x}^j$, therefore $\mathbf{p}'\sum_j \mathbf{x}^j > \mathbf{p}'\sum_j y^j$ and profits can not be being maximised in the Walrasian equilibrium.

The Second Fundamental Theorem can also be appropriately extended.

13.8 Optimality conditions

At a Pareto optimum, the utility of each individual is maximised given the utilities of everyone else and given firms' production sets. This involves solving:

$$\begin{aligned}
\max u^1(\mathbf{q}^1) & & & \\
s.t. \quad u^i(\mathbf{q}^i) &\geq u^i & & i = 2, \ldots, N \\
F^j(\mathbf{y}^j) &\leq 0 & & j = 1, \ldots, M \\
\sum_i \mathbf{q}^i - \sum_j \mathbf{y}^j - \sum_i \omega^i &\leq 0 & &
\end{aligned}$$

First order conditions require

$$\begin{aligned}
\mu^A \partial u^A / \partial q_i^A - \lambda_i &= 0 \\
\mu^B \partial u^B / \partial q_i^B - \lambda_i &= 0 \\
\mu^A \partial u^A / \partial q_j^A - \lambda_j &= 0 \\
\mu^B \partial u^B / \partial q_j^B - \lambda_j &= 0 \\
\kappa^a \partial F^a / \partial y_i^a - \lambda_i &= 0 \\
\kappa^b \partial F^b / \partial y_i^b - \lambda_i &= 0 \\
\kappa^a \partial F^a / \partial y_j^a - \lambda_j &= 0 \\
\kappa^b \partial F^b / \partial y_j^a - \lambda_j &= 0
\end{aligned}$$

for Lagrange multipliers $\lambda_i,\ \lambda_j,\ i, j = 1, \ldots, n,\ \mu^A,\ \mu^B,\ A, B = 1, \ldots, N$ and $\kappa^a,\ \kappa^b,\ a, b = 1, \ldots, M$.

Thus

$$\frac{\partial u^A / \partial q_i^A}{\partial u^A / \partial q_j^A} = \frac{\partial u^B / \partial q_i^B}{\partial u^B / \partial q_j^B} = \frac{\partial F^a / \partial y_i^a}{\partial F^a / \partial y_j^a} = \frac{\partial F^b / \partial y_i^b}{\partial F^b / \partial y_j^b}.$$

Pareto optimality therefore requires that

- all consumers have the same MRS_{ij}
- all producers have the same MRT_{ij}
- these are equal: $MRS_{ij} = MRT_{ij}$

All of these conditions are ensured by the working of the common price mechanism in Walrasian equilibrium.