This pdf illustrates an adaptation of the method found in [1] for computing the density distribution of rotating stars. That method, which we will describe shortly, was specifically designed to combat two issues which occurred in earlier work on the same problem. The first is fast rotation speeds. Earlier methods by (refs) could handle low rotation speeds, but broke down at high speeds. The second issue was computational cost. Eriguchi and Mueller [2] developed a method which could handle rapid rotations, but which suffered from a high (for that time) computational cost. Hachisu's method avoids both of these problems. It appears, based on [3], that these algorithms are still the standard in this area.

We are not concerned with high rotation speeds, and the concerns about computational cost are not as relevant anymore. Here we are instead interested in adapting this method to allow for the inclusion of two charged densities instead of one. Due to the increase in number of equations involved, this adaptation is less straightforward than it first seems.

1 Hachisu's Method

Hachisu starts with

$$\int \rho^{-1} dP + \Phi - \int \Omega^2 \omega d\omega = C \tag{1}$$

where ω is the distance from the axis of rotation. He writes the algorithms for both polytropic and special relativistic pressure laws, but here I will only rewrite the polytropic ones. So he computes

$$\int \rho^{-1} dP = (1+N)P/\rho \equiv H \tag{2}$$

Like most of these computational methods, Hachisu expands the gravitational potential as a series in Legendre polynomials:

$$\Phi = -4\pi G \int_0^\infty \int_0^1 \sum_{n=0}^{\infty} f_{2n}(r', r) P_{2n}(\mu) P_{2n}(\mu') \rho(\mu', r') d\mu' dr' \tag{3}$$

where $\mu = \cos(\theta)$ and

$$f_{2n}(r', r) = \begin{cases} r'^{2n+1}/r^{2n+1} & r' < r \\ r^{2n}/r'^{2n-1} & r < r' \end{cases}$$

He works with more than just rigid rotation, but here we will stick to that. So he computes

$$-\int \Omega^2 \omega d\omega = -\Omega_0^2 \frac{\omega^2}{2} \tag{4}$$

where Ω_0 is the angular velocity. So he finally gets

$$H = C - \Phi - \Omega_0^2 \frac{\omega^2}{2} \tag{5}$$

Next, he introduces his iteration scheme. For his numerical procedure he fixes a single value, the axis ratio. He starts with some initial guess, ρ_0. From this, he uses (3) to compute Φ_0 (of course truncating appropriately). Since $H = 0$ on the surface of the star, he has from (5) the two equations

$$H_n(A) = 0 = C_n - \Phi_n(A) + \Omega_{0,n}^2 \frac{\omega(A)^2}{2} \tag{6}$$

and

$$H_n(B) = 0 = C_n - \Phi_n(B) + \Omega_{0,n}^2 \frac{\omega(B)^2}{2} \tag{7}$$

From these he can solve for C_n and $\Omega_{0,n}$. With those values, he then can use (5) again to solve for H_n everywhere. Finally, (2) can be used to calculate ρ_n, given some polytropic law. He then iterates this until the relative differences become less than a small value.

But he does not quite use these equations, he uses a dimensionless form of them. The basis he takes is ρ_{max}, the equatorial radius R_e, and G.

2 A two species version of Hachisu's scheme

Here I will detail a two species charged version of this scheme. This is more involved since we now have to consider electric and magnetic effects, but it is straightforward to write the equations down. The less certain part is how to generalize the numerical scheme, in particular, which variables do we fix, what do we take as our basis in writing down nondimensional equations, and how do we adapt the boundary conditions (6) and (7). Here we will again assume a polytropic pressure law and rigid rotation, but it is easy to see how to generalize as Hachisu does to his other studied cases. It should be pointed out immediately that Hachisu is working with mass densities while we will work with number densities since we have more than just gravitation.

First, let us write down the basic equations. We get two equations which are take the place of (5) (I will not use Hachisu's notation anymore):

$$H_p = \frac{5P_p}{3\rho_p} = \frac{5}{3}k_p\rho_p^{2/3} = -q\Phi_E + m_p\Phi_G - q\omega \times R_\perp \cdot \Phi_M + \frac{m_p}{2}|\omega|^2|R_\perp|^2 + \lambda_p \tag{8}$$

and

$$H_e = \frac{5P_e}{3\rho_e} = \frac{5}{3}k_e\rho_e^{2/3} = q\Phi_E + m_e\Phi_G + q\omega \times R_\perp \cdot \Phi_M + \frac{m_e}{2}|\omega|^2|R_\perp|^2 + \lambda_e \tag{9}$$

where now ω is the angular velocity, and

$$\Phi_E = Cq\int_{\mathbb{R}^3} \frac{(\rho_p - \rho_e)(y)}{|x-y|} d^3y \tag{10}$$

$$\Phi_G = G\int_{\mathbb{R}^3} \frac{(m_p\rho_p + m_e\rho_e)(y)}{|x-y|} d^3y \tag{11}$$

and

$$\Phi_M = \frac{q\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{(\rho_p - \rho_e)(y)\omega \times R_\perp}{|x-y|} d^3y \tag{12}$$

are the electric, gravitational, and magnetic potentials (in the Lorenz gauge), respectively. For the numerical scheme, we expand these in series just as Hachisu did. This gives us

$$\Phi_E = 2\pi Cq\int_0^\infty \int_0^\pi \sum_{k=0}^\infty f_k(r',r)P_k(\cos(\theta))P_k(\cos(\theta'))(\rho_p - \rho_e)(r',\theta')\sin(\theta')d\theta' dr' \tag{13}$$

$$\Phi_G = 2\pi G\int_0^\infty \int_0^\pi \sum_{k=0}^\infty f_k(r',r)P_k(\cos(\theta))P_k(\cos(\theta'))(m_p\rho_p + m_e\rho_e)(r',\theta')\sin(\theta')d\theta' dr' \tag{14}$$

$$\Phi_M = \frac{q\mu_0}{2}\int_0^\infty \int_0^\pi \sum_{k=0}^\infty f_k(r',r)P_k(\cos(\theta))P_k(\cos(\theta'))(\rho_p - \rho_e)(r',\theta')(\omega \times R_\perp)\sin(\theta')d\theta' dr' \tag{15}$$

Where Hachisu fixed one variable, we must fix two, but it is not apparent which to choose. It is not useful computationally to fix both B_p/A_p and B_e/A_e, for we would need to specify more information to determine on the grid we use in our computations where these values actually lie. Hachisu does not have this problem since he uses the equatorial length as a dimensional basis, so B is just one on his grid. But we still need to specify enough to solve for both λ_p and λ_e, so we choose to fix B_p, B_e, and ω.

Then we can follow essentially the same scheme as Hachisu:

1. Begin with a starting guess. Since we are only considering small rotation speeds, our starting guess will be the nonrotating solution, details of which can be found in [4]. In that work, the densities are found by solving a system of ODEs numerically (in the case of the quantum mechanical pressure law). This is much quicker than finding the potentials, but to facilitate the work with the rest of algorithm, we will solve the nondimensionalized nonrotating problem. These equations will be given below.

2. Compute Φ_E, Φ_G, and Φ_M using the series expansion.
3. Compute λ_e and λ_p using the boundary conditions

$$0 = -q\Phi_E(B_p) + m_p\Phi_G(B_p) - q\omega \times R_\perp(B_p) \cdot \Phi_M(B_p) + \frac{m_p}{2}|\omega|^2|R_\perp(B_p)|^2 + \lambda_p \tag{16}$$

and

$$0 = q\Phi_E(B_e) + m_e\Phi_G(B_e) + q\omega \times R_\perp(B_e) \cdot \Phi_M(B_e) + \frac{m_e}{2}|\omega|^2|R_\perp(B_e)|^2 + \lambda_e \tag{17}$$

4. Compute H_p and H_e.
5. Compute ρ_p and ρ_e.
6. Repeat until convergence.

Like Hachisu, we will use nondimensional quantities and equations. We write those here. For our basis we choose the central proton density $\rho_{p,c}$, the proton mass m_p, G, and the elementary charge q. Hachisu specifically notes that for his method, he uses $\rho_{\max}$ instead of ρ_c. This is for stability reasons since at high angular velocities, the configuration becomes 'hamburger' shaped, and eventually will become a ring. Since we are focusing on small angular velocities, it makes no difference: the maximum density will be found at the origin.

For our dimensionless variables, we then have

$$\hat{\rho}_p = \frac{\rho_p}{\rho_{p,c}} \tag{18}$$

$$\hat{\rho}_e = \frac{\rho_e}{\rho_{p,c}} \tag{19}$$

$$\hat{\Phi}_E = \Phi_E \frac{q}{G\rho_{p,c}^{1/3} m_p^2} \tag{20}$$

$$\hat{m}_e = \frac{m_e}{m_p} \tag{21}$$

$$\hat{R}_\perp = R_\perp \rho_{p,c}^{1/3} \tag{22}$$

$$\hat{\Phi}_G = \Phi_G \frac{1}{Gm_p\rho^{1/3}} \tag{23}$$

$$\hat{\omega} = \omega \frac{1}{G^{1/2} m_p^{1/2} \rho_{p,c}^{1/2}} \tag{24}$$

$$\hat{\Phi}_M = \Phi_M \frac{q}{G^{1/2} m_P^{3/2} \rho_{p,c}^{1/6}} \tag{25}$$

$$\hat{H}_p = H_p \frac{1}{Gm_p^2\rho_{p,c}^{1/3}} \tag{26}$$

$$\hat{H}_e = H_e \frac{1}{Gm_p^2\rho_{p,c}^{1/3}} \tag{27}$$

$$\hat{C} = C\frac{q^2}{Gm_p^2} \tag{28}$$

$$\hat{\mu}_0 = \mu_0 \frac{q^2\rho_{p,c}^{1/3}}{m_p} \tag{29}$$

For the equations themselves, we first get the new ODEs for the nonrotating system:

$$\frac{\partial^2 \hat{\rho}_p}{\partial \hat{r}^2} = -\frac{2}{\hat{r}}\frac{\partial \hat{\rho}_p}{\partial \hat{r}} + \frac{32\pi}{\hat{\hbar}^2}\left(\frac{\pi}{3}\right)^{2/3}\left[\hat{\rho}_p^{3/2}(\hat{C}-1) - \hat{\rho}_e^{3/2}(\hat{C}+\hat{m}_e)\right] \tag{30}$$

and

$$\frac{\partial^2 \hat{\rho}_e}{\partial \hat{r}^2} = -\frac{2}{\hat{r}}\frac{\partial \hat{\rho}_e}{\partial \hat{r}} + \frac{32\pi}{\hat{\hbar}^2}\left(\frac{\pi}{3}\right)^{2/3}\left[\hat{\rho}_e^{3/2}(\hat{C}-\hat{m}_e^2) - \hat{\rho}_p^{3/2}(\hat{C}+\hat{m}_e)\right] \tag{31}$$

$$\hat{H}_p = -\hat{\Phi}_E + \hat{\Phi}_G - \hat{\omega}\times\hat{R}_\perp\cdot\hat{\Phi}_M + \frac{\hat{R}_\perp^2}{2}\hat{\omega}^2 + \hat{\lambda}_p \tag{32}$$

$$\hat{H}_e = \hat{\Phi}_E + \hat{m}_e\hat{\Phi}_G + \hat{\omega}\times\hat{R}_\perp\cdot\hat{\Phi_M} + \frac{\hat{m}_e\hat{R}_\perp^2}{2}\hat{\omega}^2 + \hat{\lambda}_e \tag{33}$$

$$\hat{\rho}_p = \left(\frac{\hat{H}_p}{\hat{H}_{p,c}}\right)^{3/2} \tag{34}$$

$$\hat{\rho}_e = \left(\hat{m}_e\frac{\hat{H}_e}{\hat{H}_{p,c}}\right)^{3/2} \tag{35}$$

$$\hat{\Phi}_E = 4\pi\hat{C}\int_0^\infty\int_0^{\pi/2}\sin(\theta')\sum_{n=0}^{\infty} f_{2n}(\hat{r}',\hat{r})P_{2n}(\cos(\theta'))P_{2n}(\cos(\theta))(\hat{\rho}_p-\hat{\rho}_e)d\theta' d\hat{r}' \tag{36}$$

$$\hat{\Phi}_G = 4\pi\int_0^\infty\int_0^{\pi/2}\sin(\theta')\sum_{n=0}^{\infty} f_{2n}(\hat{r}',\hat{r})P_{2n}(\cos(\theta'))P_{2n}(\cos(\theta))(\hat{\rho}_p+\hat{m}_e\hat{\rho}_e)d\theta' d\hat{r}' \tag{37}$$

$$\hat{\Phi}_M = \hat{\mu}_0\int_0^\infty\int_0^{\pi/2}\sin(\theta')\sum_{n=0}^{\infty} f_{2n}(\hat{r}',\hat{r})P_{2n}(\cos(\theta'))P_{2n}(\cos(\theta))(\hat{\rho}_p-\hat{\rho}_e)\hat{\omega}\times\hat{R}_\perp d\theta' d\hat{r}' \tag{38}$$

To implement this numerically, we consider a box which will contain the support of the densities. This can be obtained from taking the radius of the star in the nonrotating case, and expanding it slightly, by multiplying it $\frac{16}{15}$. We then divide the box up into a square grid of some given size, using polar coordinates. So in each box of this grid, we can assign a value for the density of electrons and protons, using the values obtained from the nonrotating case.

With that, we estimate $\hat{\Phi}_E^{ji}$, $\hat{\Phi}_G^{ji}$, and $\hat{\Phi}_M^{ji}$ using a composite Simpson's rule. For $\hat{\Phi}_C^{ji}$ for example, we would have (j is the r coordinate and i is the polar coordinate, let g be the gridsize)

$$\hat{\Phi}_E^{ji} = 4\pi\hat{C}\sum_{k=0}^{\mathrm{LMAX}} D_{rG}^{k,j}P_{2k}\cos(\theta_i) \tag{39}$$

$$D_{rG}^{k,j} = \sum_{m=0,+2}^{g-2}\frac{1}{6}\frac{16}{15}r_{\max}\frac{2}{g}\left[f_{2k}^{m,j}D_{tG}^{m,k} + 4f_{2k}^{m+1,j}D_{tG}^{m+1,k} + f_{2k}^{m+2,j}D_{tG}^{m+2,k}\right] \tag{40}$$

$$D_{tG}^{m,k} = \sum_{n=0,+2}^{g-2}\frac{1}{6}\frac{\pi}{2}\frac{2}{g}\left[(\hat{\rho}_p^{m,n}-\hat{\rho}_e^{m,n})P_{2k}(\theta_n) + 4(\hat{\rho}_p^{m,n+1}-\hat{\rho}_e^{m,n+1})P_{2k}(\theta_{n+1}) + (\hat{\rho}_p^{m,n+2}-\hat{\rho}_e^{m,n+2})P_{2k}(\theta_{n+2})\right] \tag{41}$$

3 A revised two species version

The above scheme has not been successful, so I will here alter it slightly to make it even closer in spirit to Hachisu's code. What we want to do is to use the algebraic equations at the boundaries

$$H_p(B_p) = 0 = -q\Phi_E + m_p\Phi_G - q\omega \times R_\perp \cdot \Phi_M + \frac{m_p}{2}|\omega|^2|R_\perp|^2 + \lambda_p, \tag{42}$$

$$H_p(A_p) = 0 = -q\Phi_E + m_p\Phi_G + \lambda_p, \tag{43}$$

and

$$H_e(B_e) = q\Phi_E + m_e\Phi_G + q\omega \times R_\perp \cdot \Phi_M + \frac{m_e}{2}|\omega|^2|R_\perp|^2 + \lambda_e \tag{44}$$

to solve for λ_e, λ_p, and ω. I propose to fix A_p/B_p and B_e/B_p. In theory this is fine, but in practice this is more difficult because I actually have to put numbers into the computer. Hachisu knows his equatorial radius in the computer is just 1 since he uses this as a unit. He can only do this since he is using the mass density and not the number density. But we must use the number density, so we can only set the central number density to 1 or the proton radius to 1.

References

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