Solutions: Exercise 2

July 17, 2014

1. **Several questions about the course:** (*0.5 points each*)

   (a) The Coriolis parameter f is defined as
   a) $f = \Omega \cos\varphi$
   b) $f = 2\Omega \cos\varphi$
   c) $f = 2\Omega \sin\varphi$
   d) $f = \beta y$
   Solution: C $f = 2\Omega \cos\varphi$

   (b) Please clarify: On the Northern Hemisphere, particles tend to go to the right or left relative to the direction of motion due to the Coriolis force?
   Solution: On the Northern hemisphere, the motion is deflected to the right.

   (c) Please write down the equation of state for the ocean and atmosphere!
   Solution: Atmosphere: $P = \rho RT$, Ocean: $\rho = \rho(s, T, P)$

   (d) What is the hydrostatic approximation in the momentum equations?
   Solution: $\frac{dP}{dz} = -g\rho$

2. **Short programming questions.** *(1 point each)*

   Write down and explain the output for the following R-commands:

   (a) `0:10`
   Solution displays numbers 0 to 10

   (b) `a<-c(0,5,3,4); mean(a)`
   Solution Sets up a vector with the assigned values, and displays the mean

   (c) `max(a)-min(a)`
   Solution Returns the difference between max and min.

   (d) `paste("The mean value of a is",mean(a),"for sure",sep="_")`
   Solution Displays the string and the mean

   (e) `a*2+c(1,1,1,0)`
   Solution Multiplies a by 2 and then sums up element-wise.

(f) `my.fun<-function(n){return(n*n+1)}`
`my.fun(10)-my.fun(1)`

Solution Defines a function which returns the square plus 1, and evaluates this for 10 minus 1.

3. **Evaluate the double vector product:** (*2 points*)

$$\Omega \times (\Omega \times r)$$

with vectors $\Omega = (0, 0, \omega)$ and $r = (x, y, z)$.

Solution

$$\begin{pmatrix} -x\omega^2 \\ -y\omega^2 \\ 0 \end{pmatrix}$$

4. **Given $f(x, y, z, t)$. What is the definition of partial derivatives for this variable? What is the definition of nabla, Laplacean, divergence, total (substantial) derivative, total differential?** (*2 points*)

Solution The partial derivative is the derivate of a function with respect to each variable at a time while considering all others constant.

The ∇ opeator represetns a vector of partial derivates for x, y, and z respectively.

The Laplacean ∇^2 represents the sum of the second partial derivates of a function. It is used to represent flux density of a gradient flow.

The divergence is the dot product of ∇ and the velocity vector.

The total derivative is the derivate of a function with respect to all variables that are not assumed constant.

The Total Differential is the sum over all of the independent variables of the partial derivative of the function with respect to a variables times the total differential of that variable.

5. **Population Dynamics** (1 point each)
Consider population dynamics with population $x > 0$ and reproduction (birth-death) r:

$$\frac{dx}{dt} = x \cdot r(x) \tag{1}$$

(a) Solve the differential equation for $r(x) = r_0 = const.$!
What happens for $t \to \infty$ when $r_0 > 0$ or $r_0 < 0$?

Solution

$$x = e^{r_o t}$$

for $r > 0, t \to \infty, x \to \infty$ for $r < 0, t \to \infty, x \to -\infty$

(b) Solve the differential equation for $r = r_0(1 - x)$! (limited growth)

Solution

$$x = -\frac{e^{r_o t}}{1 - e^{r_o t}}$$

What happens for $t \to \infty$?

Solution

$$t \to \infty : x = 1$$

(c) Consider the case $r = r_0(1 - x/K)$ with $K > 0$! Give a physical interpretation for K!

Solution
K (Carrying Capacity) gives the stable equilibrium point where the function stabilizes. Calculate the bifurcation with respect to parameter r; illustrate with a bifurcation diagram.
See lecture notes, a bifurcation diagram is presented

6. **Difference equations** (*1 point each*)
Consider the discretised form of (1) using the Euler scheme

$$\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{\Delta t}. \tag{2}$$

(a) Write down the iteration x_{n+1} as a function of x_n .

Solution $x_{n+1} = x_n + x_n \cdot r_0 \cdot \delta t$

(b) What is the solution of x_{n+1} as a function of x_0?
Consider the stability for the cases $r > 0$, $0 > \Delta t\, r > -1$, $-1 > \Delta t\, r > -2$, $-2 > \Delta t\, r$.
Do you have a graphical interpretation of the oscillation/decay?
Solution $x_{n+1} = x_0(1 + r_o \delta t)^{n+1}$

7. **Reynolds Number** (*3 points*)
Let us consider a flow problem where the fluid behaves according to the incompressible Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\, \vec{u} = -\dfrac{1}{\rho_0} \nabla p + \nu \nabla^2 \vec{u} \\ \nabla \cdot \vec{u} = 0 \end{cases}$$

where:

- $\vec{u}$ = flow velocity;
- t = time;

- ρ_0 = fluid density (constant);
- p = pressure;
- ν = kinematic viscosity (fluid property).

Using the reference density ρ_0 and further assuming that a reference lengthscale l_0 and a reference flow velocity U_0 have already been defined for the flow, transform these equations into a non-dimensional frame of reference. On how many parameters does the flow depend in the non-dimensional system? What is the role of the Reynolds number ($Re \equiv \frac{U_0 l_0}{\nu}$)?

Solution The Reynolds number $\left[\frac{v_o}{uL}\right]$ represents the ratio of the magnitudes of the inertial and viscous terms, if this is a small number, the larger the viscous terms are and thus the more turbulent the flow

In the nondimensional system, the flow depends on its velocity and the length of the motion.

Hint: Use U_0, ρ_0, l_0 to construct reference values for all other involved quantities (t, p) and operators $(\partial/\partial_t,\ \nabla)$, insert them into the equations and reduce as much as possible the terms which have constant factors.

Notes on submission of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. Please email your own answers to the questions to Paul.Gierz@awi.de.*