Fourier Series

Equipment

- Pasco WA-9307A Fourier synthesizer
- set of connecting leads (2)
- Fluke multimeter
- Oscilloscope

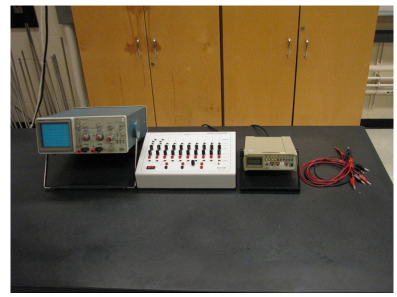


Figure 8.1: A photograph of the experimental setup.

Goals of the Experiment

- To become familiar with the properties of sinusoidal waves and superpositions of sinusoidal waves.
- To gain experience making phase measurements with an oscilloscope.
- To observe and measure beats.
- To investigate use of Fourier series for synthesizing new waveforms.

Theory

The everyday world is full of sound. All sounds are actually pressure waves transmitted through the air. Since they are waves they can be easily modeled using mathematics. For a pure tone, like a tuning fork, only one **frequency** is required. More complicated sounds, like that of a piano or the human voice, have many more frequencies. One method of describing these waves is to make use of a **Fourier series**. These series are composed of a multitude of sine and cosine functions of differing frequencies and amplitudes. Fourier series are used to deal with problems in such diverse fields as acoustics, geophysics, and astronomy. Although their validity was originally hotly debated, they are now a common and important tool. D'Alembert, in 1747, formulated the general equation of a vibrating string. D'Alembert felt that his equations only described smooth vibrations of the string. Euler, on the other hand, argued that the solutions could have sudden jumps and corners because strings can be plucked. Later, in 1753, Daniel Bernoulli formulated a solution of the vibrating string problem based on a series of sines and cosines. This started a debate whether smooth waves could be added up to produce waves with sharp jumps and corners. At first the smoothness problem seemed minor, but this same series began to appear elsewhere. In 1807, Fourier submitted a paper that explained how to use this series to solve problems in heat transfer. This paper was criticized, because researchers did not accept that smooth waves could sum to jagged waves. Later analysts proved that a series sum of smooth functions could indeed produce a non-smooth result. Today, these series of sines and cosines are used routinely and are named **Fourier series** in honour of Jean-Baptiste Fourier. Fourier series are based on the sine function. Sine is an example of a **periodic function**. It repeats at a regular interval called the **period**. As can be seen in Figure 8.2, the period, T, is the length of time for one cycle of the wave. The frequency, $f = 1/T$, is the number of cycles in one second. A sine wave has two other parameters that can be varied. The **amplitude**, A, determines the height of the wave. The **phase shift**, ϕ, is the amount the wave is shifted left or right. Sine waves are depicted mathematically by

$$f(t) = A \sin\left(\frac{2\pi}{T}t + \phi\right) \tag{8.1}$$

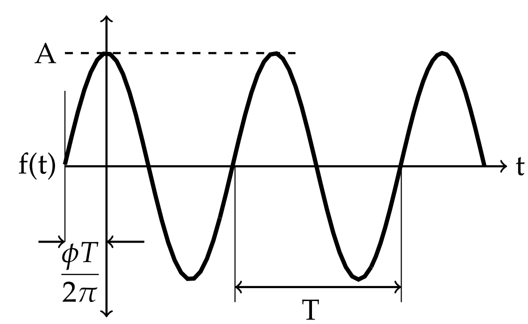


Figure 8.2: Phase shifting of a sine wave.

The quantity 2π/T=2πf is known as the **angular frequency**.

Sine waves are added together to form Fourier series. In this light, the addition properties of two sine waves is a special case of Fourier series. If the frequency and the phase are identical, then the amplitude of the sum is the sum of the amplitudes. The frequency and phase remain unchanged. This means that

$$A \sin\left(\frac{2\pi}{T}t + \theta\right) + B \sin\left(\frac{2\pi}{T}t + \theta\right) = \sin\left(\frac{2\pi}{T}t + \theta\right) \tag{8.2}$$

The addition of sines with the same frequency but **different** phases introduces new properties. Suppose the two signals are separated by a phase of $\pi/2$. Then

$$A \sin\left(\frac{2\pi}{T}t\right) + B \sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) = C \sin\left(\frac{2\pi}{T}t + \phi\right) \tag{8.3}$$

where

$$C = \sqrt{A^2 + B^2} \tag{8.4}$$

and

$$\phi = \tan^{-1}\left(\frac{B}{A}\right) \tag{8.5}$$

The amplitude and phase of the sum are now **functions** of the amplitude and phase of the addends and are given by Equations 8.4 and 8.5. However, the frequency remains unchanged and the sum is still a sine wave.

When sine waves of different frequencies are added, the sum is no longer a sine wave. For example, if sines of identical amplitude and unequal but close frequency are added, the result takes the form shown in Figure 8.3. This is the phenomenon known as **beats**. A lower frequency envelope is seen to be riding on the outside of a higher frequency signal, known as a **carrier**, because it "carries" the lower frequency beat signal along.

Mathematically, adding a sine wave of frequency n to a sine wave of frequency m yields the expression

$$sin(nx) + sin(mx) = 2sin\left[2\pi\left(\frac{m+n}{2}\right)x\right]cos\left[2\pi\left(\frac{m-n}{2}\right)x\right] \quad (8.6)$$

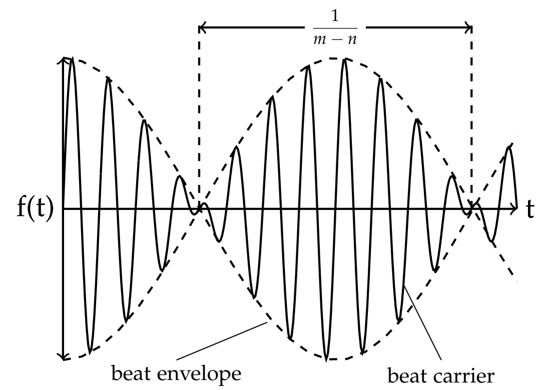


Figure 8.3: An illustration of beat frequency.

If m and n are close together, then Equation 8.6 can be thought of as a time varying amplitude applied to a sine wave of frequency $(m+n)/2$. Although this time varying amplitude has frequency $(m-n)/2$, Figure 8.3 shows that the frequency at which the minimum value appears is **twice** this amount. So the beat envelope is heard as the $m-n$.

The results of Equations 8.2 through 8.6 are found by use of standard trigonometric identities and result in waves that are repetitive in nature. However, the resulting sum of two sine waves **need not repeat itself**. An interesting fact about the addition of different frequencies, m and n, is that the resulting wave is periodic only if m and n are **commensurable**. Two numbers m and n are commensurable if there exists two rational numbers p and q such that

$$pm = qn \quad (8.7)$$

If the frequencies m and n are commensurable, then the frequency of the sum is given by the **greatest common divisor** of m and n.

A Fourier series generalizes the sum of two sine waves to the sum of an arbitrary number of sine and cosine waves. Each element of the series can have a different amplitude. However, the frequency of any element must be an integer multiple of one frequency known as the **fundamental**. The integer multiples of the fundamental are known as **harmonics**. Suppose the fundamental has a period T. Then a Fourier series has the form

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n cos\left(\frac{2nt\pi}{T}\right) + b_n sin\left(\frac{2nt\pi}{T}\right)\right) \quad (8.8)$$

The constants a_0, a_1, a_2... b_1, b_2... are called **Fourier coefficients**. The most significant property of Fourier series is that **any** function, $f(t)$, that repeats with a period T can be represented by such a series. This includes waves with sudden jumps and sharp corners.

A crucial problem is how to find the Fourier coefficients a_0, a_n and b_n for a given function $f(t)$. What Fourier found was a method for determining these coefficients. The Fourier coefficients are calculated by the formulae

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \tag{8.9}$$

$$a_n = \frac{2}{T}\int_0^T f(t)cos\left(\frac{2n\pi}{T}t\right)dt \tag{8.10}$$

and

$$b_n = \frac{2}{T}\int_0^T f(t)sin\left(\frac{2n\pi}{T}t\right)dt \tag{8.11}$$

It can be seen from Equations 8.9, 8.10, and 8.11 that the Fourier coefficients for a function f(t) are found by some form of integration over the period T. The reason for this is that certain integrals have the useful property of being zero for all terms of the Fourier series **except one**. They act as a **filter** that preserves one term of the Fourier series and eliminates all the others.

Equation 8.9 suggests that integrating the function and dividing by the period yields a_o. Since $f(t)$ has a Fourier series, integrating $f(t)$ gives

$$\int_0^T f(t) = \int_0^T \left[a_0 + a_1\cos\left(\frac{2\pi}{T}t\right) + a_2\cos\left(\frac{4\pi}{T}t\right) + ...b_1\sin\left(\frac{2\pi}{T}t\right) + ...\right]dt \tag{8.12}$$

Another useful property of integrals is that each element of a sum can be integrated separately. So,

$$\begin{aligned}\int_0^T f(t) = \int_0^T a_0 dt + \int_0^T a_1 cos\left(\frac{2\pi}{T}t\right)dt + \int_0^T a_2\cos\left(\frac{4\pi}{T}t\right)dt \\ + ...\int_0^T b_1\sin\left(\frac{2\pi}{T}t\right)dt + ...\end{aligned} \tag{8.13}$$

However, the net area under a full period of any sine or cosine is zero, so all the integrals on the right **vanish** except for a_o. This gives

$$\int_0^T f(t)dt = \int_0^T a_0 dt = Ta_0 \tag{8.14}$$

Therefore, a_o, the first Fourier coefficient, is found to be Equation 8.9.

The procedure for deducing a_n is nearly identical. Two additional integral properties are needed as filters. It turns out that the integral over a full period of $sin(mx)cos(nx)$ is always zero. Also, the integral over a full period of $cos(mx)cos(nx)$ is always zero **except** when $m = n$. This filters out all the a_n terms but one.

Using these facts, it can be seen that

$$\int_0^T f(t)\cos\left(\frac{2n\pi}{T}t\right)dt = \int_0^T a_n \cos^2\left(\frac{2n\pi}{T}t\right)dt \qquad (8.15)$$

This reduces to

$$\int_0^T f(t)\cos\left(\frac{2n\pi}{T}t\right)dt = \frac{a_n T}{2} \qquad (8.16)$$

which gives Equation 8.10.

The determination of b_n is identical except for the filtering property needed to isolate one b_n term. The integral over a full period of $sin(mx)sin(nx)$ as always zero except when $m = n$. This gives

$$\int_0^T f(t)\sin\left(\frac{2n\pi}{T}t\right)dt = \frac{b_n T}{2} \qquad (8.17)$$

Thus, for a given periodic function, f(t), Equations 8.9, 8.10 and 8.11 can be used to find the Fourier coefficients. Once these are known, the Fourier series can be used to represent or approximate $f(t)$. For example, Figure 8.4 shows a square wave. This signal is not smooth at all. Nevertheless, it has a Fourier series. It turns out that for this square wave $a_0 = 0$, $a_n = 0$ and $b_n = 0$ if n is even and $b_n = 1/n$ if n is odd. This implies that a square wave is composed of a fundamental sine wave and all of the odd harmonics. Figure 8.4 shows two other signals superimposed on the square wave. One signal is the fundamental, corresponding to the coefficient b_1. The other curve is the sum of the first four odd harmonics and the fundamental, namely, b_1, b_3, b_5, b_7, and b_9.

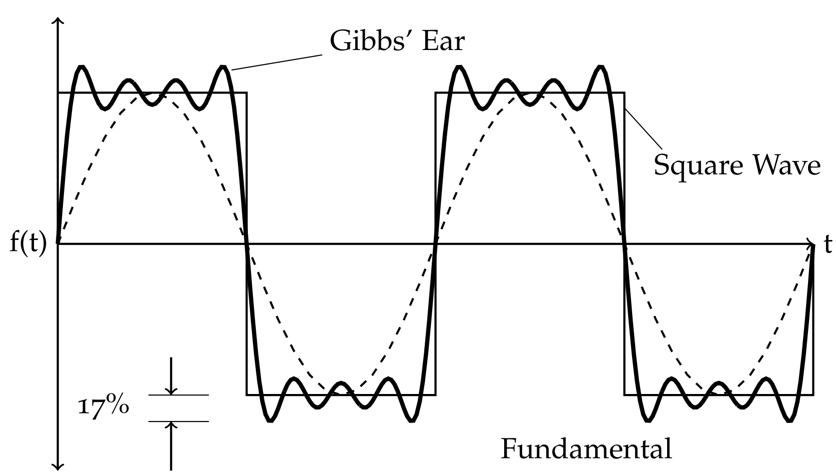


Figure 8.4: An illustration of Gibbs' phenomenon.

It can be seen that the shape is converging to a square wave. However, there seems to be a greater difficulty for the Fourier series to track at the discontinuity. There is a sudden jump or overshoot just before each rise and fall. This propensity for a Fourier series to overshoot at the edges is called **Gibb's phenomenon** or **Gibbs Ears**. It turns out that this behavior is a general property of all Fourier series, not just the square wave. In fact, as more terms are added, the ears get narrower but their height **may not decrease**. For the square wave, it can be shown that the height of the ears never falls below 17% above the square wave level.

This experiment examines the wave phenomena described here using an instrument called a **Fourier Synthesizer**. A Fourier synthesizer is an instrument that creates arbitrarily shaped periodic signals by use of Fourier series. The device generates two sine waves at a fundamental frequency of 440 Hz, as well as eight other harmonic sine waves at frequencies of 880 Hz, 1320 Hz,... up to 3960 Hz. The phase and amplitude of each of these ten signals can be independently adjusted. Any combination of them can be added together. In this way the synthesizer can generate the first nine terms of any Fourier series.

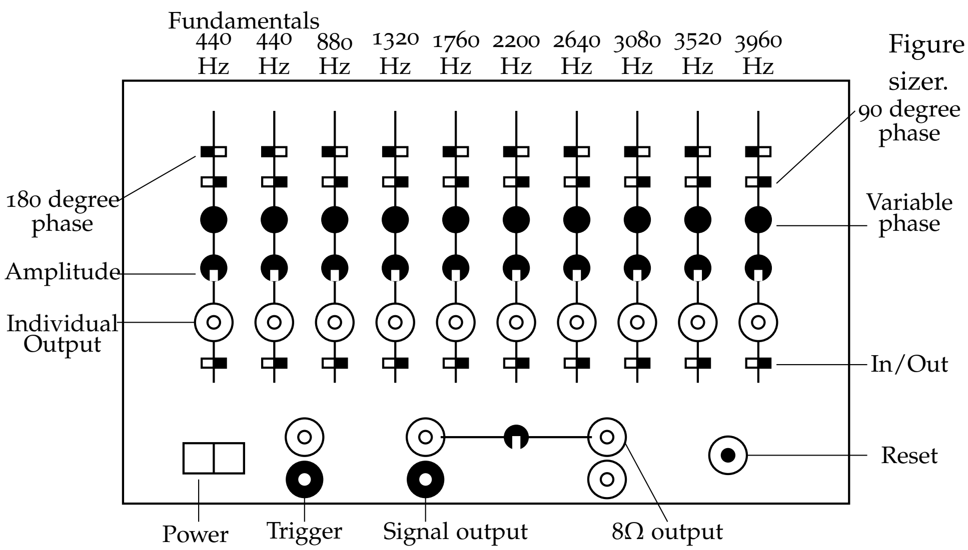


Figure 8.5: A Fourier synthesizer.

The Fourier synthesizer used in this experiment is shown in Figure 8.5. There are ten columns of controls. One column for each of the two fundamentals and eight harmonics. At the top of each column are the phase controls. Two switches select 90° and 180° phase shifts. The third phase control is continuously variable over a 90° range. Together, these three controls provide a full 360° phase adjustment range. Below the phase controls is an amplitude control to set the size of the signal.

An in/out switch selects whether the column gets added to the output or not. Each column also includes an output point where the signal generated by that column can be measured.

The three primary signal outputs generated by the synthesizer lie below the ten columns. All of the selected columns (those that are switched in) are **added** together and presented at the 10 kΩ output and the 8 Ω output. The 10 kΩ output is connected to an oscilloscope for viewing and measurement of the generated signal The 8 Ω output is used to drive a speaker for audible output.

The third output is called a **trigger**. This signal is a square wave locked in phase with the fundamental. The edges of this square wave represent a phase angle of 0°. This signal serves as a time marker (metronome) against which phases can be measured. Lastly, the reset button is used to periodically resynchronize all the harmonics back together.

The synthesizer can be used to experimentally examine the properties of the addition of sine waves. Equation 8.3 can be examined by adding a sine wave to another phase shifted sine wave of the identical frequency. The amplitude and phase of the sum would be expected to vary in the systematic manner suggested by Equations 8.4 and 8.5.

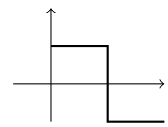
Equation 8.6 and Figure 8.3 can be tested by observing the resulting waveform when sines of identical amplitude but differing frequency are added together. For best results, the chosen frequencies should be relatively close together. This suggests adding together high harmonics such as 5, 6, 7, 8 and 9. The resulting frequencies of the beat envelope and the beat carrier can be measured with an oscilloscope and compared against Equation 8.6.

Similarly, Equation 8.7 can be examined by adding together two sinusoids of arbitrary amplitude, phase and frequency. The sum would be expected to have a frequency given by the greatest common divisor, independent of the amplitude and phase.

Most importantly, Equations 8.9, 8.10 and 8.11 can be used to find Fourier coefficients for various signals. The coefficients for a wide variety of common signals have been precomputed for implementation on the synthesizer. The selected signals are chosen from a variety of areas of electrical engineering and physics. It should be noted that the resulting shape is quite sensitive to the actual amplitude and phase of each harmonic. For best results, the amplitude of each harmonic should be set using the multimeter and the phase should be set using the oscilloscope.

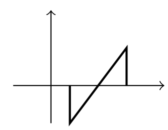
Experimental Procedure

1. Connect the Fourier synthesizer to the oscilloscope. One channel on the oscilloscope is best left connected to the trigger. The second channel on the oscilloscope is used to view the signal output or the individual output of each harmonic, whichever is most useful. The multimeter is connected to the individual output of whichever wave needs to be measured at the time. A common ground should be used between all the equipment. For best results, the oscilloscope should be AC coupled.

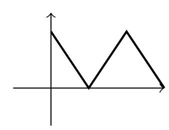


Square Wave

2. To test Equation 8.3, choose one fundamental to have an amplitude A and the other amplitude B. Set the phase of A to be 0° and the phase of B to be 90°. The phase is set using the oscilloscope, by observing the alignment of the sine wave against one edge of the trigger. A phase shift of 90° is $90^\circ/360^\circ = 1/4$ of a period. Since the period is $1/440$ s, the time shift for 90° is $0.25/440570$ s$\approx$ 570 μs. Fix the amplitude of A to some known value using the multimeter. Then, for at least eight different amplitudes of B, measure the amplitude of and phase of the sum. Amplitude is best measured with the AC voltmeter, and phase is best measured with the oscilloscope.



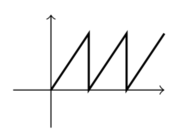
Ramp Function



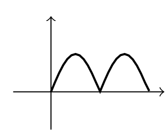
Triangle Wave

3. Choose any **one** of the following tests for the addition of sine waves of different frequency.

   (A) Examine beats as given by Equation 8.6 and Figure 8.3. Choose any two neighboring higher harmonics and set their amplitudes to be equal and the phases to be 0°. Measure the frequency of the beat envelope and beat carrier. Repeat for at least two other neighboring harmonics.

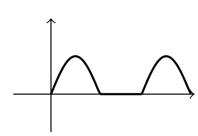


Sawtooth Wave



Full Wave

   (B) Examine the sum frequency predicted by Equation 8.7. Choose any two frequencies with arbitrary non zero amplitudes and arbitrary phase. Measure the period of their sum. The sum frequency can be compared to the expected value given by the greatest common divisor of the individual frequencies. Repeat for at least two other combinations.



Half Wave

4. The synthesizer can generate a 9 term Fourier series for the square wave, as seen in Figure 8.4. A recipe for synthesizing the square is given in Table 8.1. To synthesize a particular wave shape, each harmonic 1 through 9 must be set to a particular amplitude and phase. Each amplitude in the table is a percentage relative to a maximum amplitude within the first nine terms. Each phase angle is the number of degrees by which to shift that harmonic. Again, for best results, the AC voltmeter should be used to set the amplitude

and the oscilloscope used to set the phase. Synthesize the square wave.

Table 8.1:

	n	1	2	3	4	5	6	7	8	9
Square	% Ampl.	100	0.0	33.3	0.0	20.0	0.0	14.3	0.0	11.1
	Phase	90°	0°	90°	0°	90°	0°	90°	0°	90°

5. Gibbs Ears. Gibbs ears can be measured once the square wave has been synthesized. As each harmonic above number 1 is "switched in" record the height and width of Gibb's ears. This data can be used to check whether the width steadily decreases, while the height goes no lower than 17%.

6. The following signals are commonly found in sweep circuits, used in radar or television. Synthesize any **one** of the following waveforms. Record the result and note where it is different from the ideal shape. When synthesizing your signal **do not** trust the amplitude dials on the synthesizer. Instead set the amplitudes individually by measuring them with a multimeter.

Table 8.2:

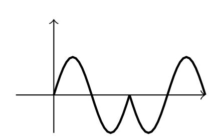
	n	1	2	3	4	5	6	7	8	9
Ramp	% Ampl.	100	78.5	11.1	39.3	4.0	26.2	2.0	19.6	1.2
	Phase	0°	270°	0°	270°	0°	270°	0°	270°	0°
Triangle	% Ampl.	100	0	11.1	0	4.0	0	2.0	0	1.2
	Phase	0°	0°	0°	0°	0°	0°	0°	0°	0°
Sawtooth	% Ampl.	100	50	33.3	25.0	20.0	16.7	14.3	12.5	11.1
	Phase	90°	90°	90°	90°	90°	90°	90°	90°	90°

7. The following are signals commonly found in electrical equipment like motors, generators, and power supplies. They are called **rectified sine waves**. Synthesize any one of the following waves. Record the result and note where there is deviation from the ideal desired shape.

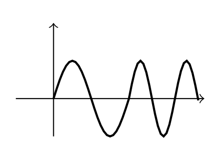
Table 8.3:

	n	1	2	3	4	5	6	7	8	9
Full	% Ampl.	0	100	0	20.0	0	8.6	0	4.8	0
	Phase	0°	180°	0°	180°	0°	180°	0°	180°	0°
Half	% Ampl.	100	42.4	0	8.5	0	3.6	0	2.0	0
	Phase	90°	180°	0°	180°	0°	180°	0°	180°	0°

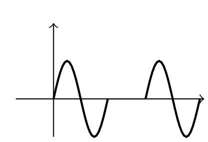
8. Signals of special form are frequently used in digital communication systems. In digital communications there are a number of standard methods for transmitting binary 'ones' and 'zeros.' Three of these are **Frequency shift keying** (FSK), **amplitude shift keying** (ASK) and **phase shift keying** (PSK). If ASK is used, one cycle of a sine wave represents a 'one' and a flat line is a 'zero.' Likewise, FSK represents a 'one' by one cycle of a sine wave and represents a 'zero' by one cycle at a different frequency sine wave. For PSK, a 'one' is designated by one cycle of a zero phase and a 'zero' is one cycle of a 180° phase shifted sine wave. Generate any **one** of ASK, FSK,



Pulse Shift Keying



Frequency Shift Keying



Amplitude Shift Keying

or PSK on the synthesizer using the recipes provided. Record the observed shape and note where there is significant deviation from the ideal desired shape.

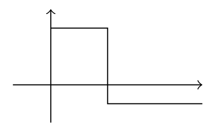
	n	1	2	3	4	5	6	7	8	9
PSK	% Ampl.	100	0	60.0	0	14.3	0	6.7	0	3.9
	Phase	0°	0°	180°	0°	180°	0°	180°	0°	180°
FSK	% Ampl.	56.8	100.0	41.6	22.0	8.4	0	2.9	2.1	0
	Phase	30°	150°	90°	30°	330°	0°	30°	330°	0°
ASK	% Ampl.	84.9	100	50.9	0	12.1	0	5.7	0	3.3
	Phase	0°	90°	180°	0°	180°	0°	180°	0°	180°

Table 8.4:

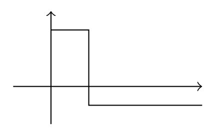
9. The following are **pulse waves** of various **duty cycles**. duty cycle is the proportion of time that the signal is at maximum. Again, generate **one** wave, record it, and note where it diverges from the ideal case.

Table 8.5:

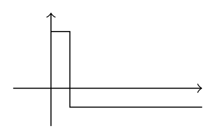
	n	1	2	3	4	5	6	7	8	9
25%	% Ampl.	100	70.7	33.3	0	20	23.6	14.3	0	11.1
	Phase	45°	90°	135°	0°	45°	90°	135°	0°	45°
33%	% Ampl.	100	50	0	25	20	0	14.3	12.5	0
	Phase	0°	330°	0°	90°	60°	0°	180°	150°	0°
8.3%	% Ampl.	100	95.1	87.3	76.9	64.7	51.3	37.4	23.8	11.1
	Phase	15°	30°	45°	60°	75°	90°	105°	120°	180°



25% wave



33% wave



8.3% wave

Error Analysis

Generally, the errors in an experiment arise from **systematic errors** in the apparatus and **measurement errors** in the instruments. In this experiment, the measurement errors are determined by the accuracy of the oscilloscope and the multimeter. The systematic errors arise from the behavior and performance of the Fourier synthesizer. The measurement errors of the oscilloscope and multimeter can be readily estimated.

However, there are many possible sources of systematic error lurking inside the synthesizer. The fundamental and harmonic sine waves generated by the synthesizer have some **distortion**. Also, the summation of all the components is not mathematically perfect. Furthermore, the synchronization of the harmonics to each other and to the fundamental is not exact. The magnitude of these and other internal errors is essentially unknown. For this reason error analysis is not required for this experiment.

To be handed in to laboratory instructor.

PreLab

Do any **two** of the following three questions.

1. The synthesizer can generate the first nine terms of a Fourier series. This means that once a1 through to a_9 and b_1 through to b_9 are known for a particular signal, that signal can be synthesized. The n'th term of a Fourier series has the form

$$a_n cos\left(\frac{2n\pi}{T}t\right) + b_n sin\left(\frac{2n\pi}{T}t\right) \tag{8.18}$$

However, the synthesizer does not generate sine and cosine waves. It generates a single wave with adjustable amplitude and phase.

Hence, the synthesizer actually generates a Fourier series n'th term of the form

$$A_n sin\left(\frac{2n\pi}{T}t + \phi_n\right) \tag{8.19}$$

Explain how to find the **amplitude coefficient**, A_n, and the **phase coefficient**, ϕ_n, from the Fourier sine and cosine coefficients a_n and b_n.

2. Derive Equation 8.10 or 8.11.

3. The synthesizer does not generate the a_0 term. How does this restrict the types of signals that can be synthesized? What does AC coupling the oscilloscope have to do with the a_0 term?

Data Requirements

4. A graph of C^2 versus B^2 from procedure step 2.

5. A graph of tan ϕ versus B from procedure step 2.

6. Results and observations from procedure step 3.

7. A sketch and observations of the synthesized square wave.

8. Data and observations of Gibbs' ears.

9. Sketches and relevant observations for the other synthesized signals.

Discussion

10. A discussion of the graphs from procedure step 2, the additions of sine waves of equal frequency. Is the synthesizer addition circuitry working correctly?

11. A discussion of the results from the addition of sine waves of different frequency. Is the synthesizer generating harmonic frequencies correctly?

12. Discussion and conclusions for the square wave and Gibb's ears.

13. Conclusions regarding the synthesis of signals by the use of Fourier series.

Fourier Series - Companion Guide

Equipment

- Pasco WA-9307A Fourier synthesizer
- set of connecting leads (2)
- Fluke multimeter
- Oscilloscope

Setup

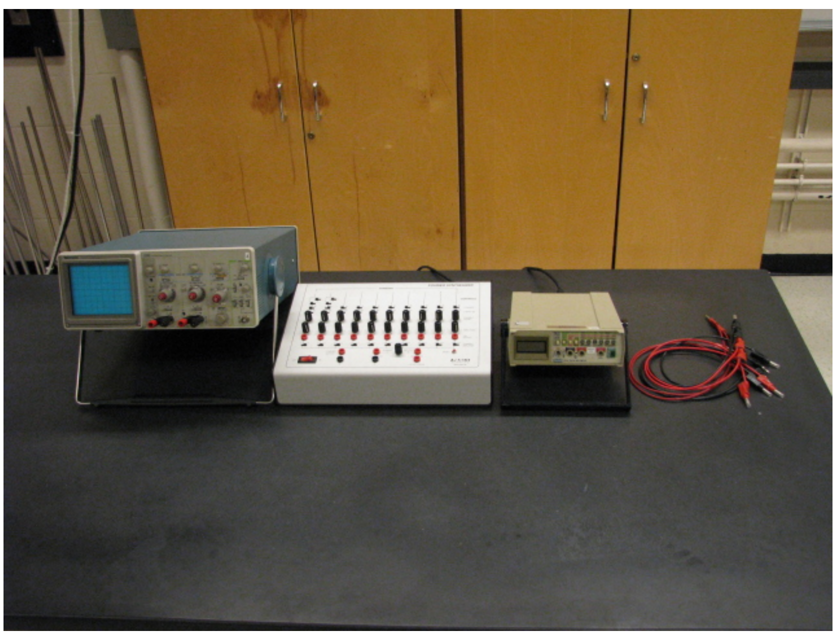


Figure 8.6: Equipment Setup

Set up bench as shown in Figure 8.6.

Maintenance

1.

2.

Critical Points of Failure

1. Make sure the students have the ground terminals on the Fourier synthesizer plugged into the black terminals on the oscilloscope. If they're plugged into the red terminals the oscilloscope won't properly display the signal.
2. It is essential for this lab that the oscilloscope be set to trigger on the proper channel. If the trigger output on the synthesizer is feeding into channel 2 on the oscilloscope then the oscilloscope must be set to trigger on channel 2.

Notes to the Instructor

1. When synthesizing pulse waves of various duty cycles in procedure step 9, students may need to select a 100% voltage for the fundamental frequency which is less than the maximum output on the Fourier synthesizer. This is because the upper harmonics of the synthesizer may not be able to output the required voltage percentage of the maximum output on the fundamental. For example, the maximum output of the fundamental is 0.715 V and harmonic 5 must be set to 64.7% which is 0.463 V but the maximum output of harmonic 5 is 0.420 V. It would then be best to set the fundamental to around 0.5 V so that harmonic 5 only needs to output 0.324 V.
2. Students should regularly check the oscilloscope trace to be sure it is oscillating around the x-axis by toggling the coupling switch on channel 1 from *AC* to *ground*, adjusting the vertical position of the trace so that the ground line is directly on top of the x-axis, and toggling back to AC. This makes it easier to accurately adjust the phase of each component when synthesizing a wave.
3. In step 5, students will measure the height and width of Gibbs' ears as higher harmonics are switched in. To obtain consistent measurements, channel 2 (or whatever channel the trigger is plugged into) may be switched to *DC* in order to create a straight and level reference line. It's important that the trigger channel be switched to *DC* to eliminate the inwardly directed slope of the line, as discussed in Data Requirement 7. The vertical and horizontal position of this reference line should be adjusted so that it mimics the ideal plateau of the square wave that the synthesized wave is oscillating around. Using this reference line, measurements of width are most easily taken to be where the Gibbs ears intersect and measurements of height can be taken to be the peak height of the ear with respect to the line.

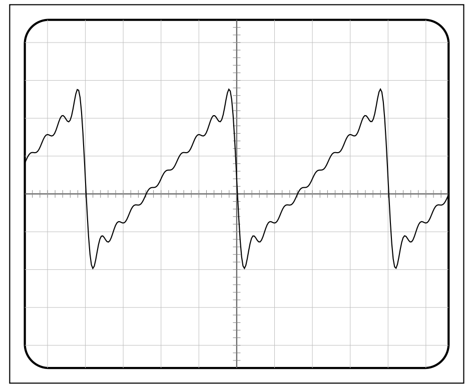


Figure 8.7: A sketch of the synthesized ramp function.

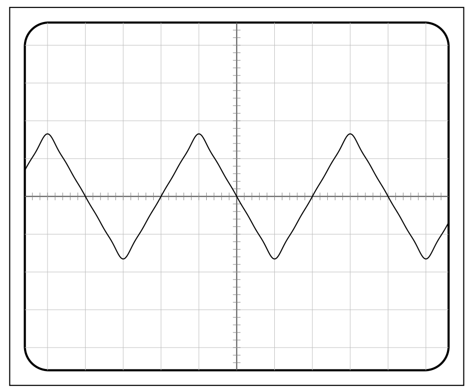


Figure 8.8: A sketch of the synthesized triangle wave.

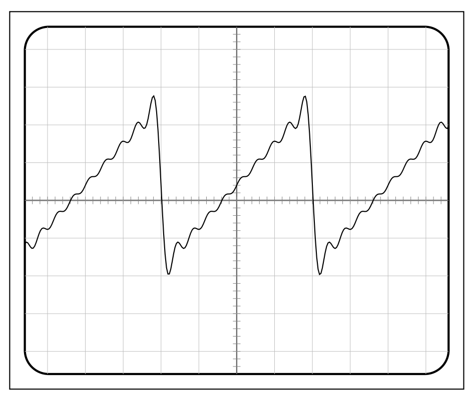


Figure 8.9: A sketch of the synthesized sawtooth wave.

Prelab

1. **Explain how to find the amplitude coefficient, A_n, and the phase coefficient, ϕ_n, from the Fourier sine and cosine coefficients, a_n and b_n.**

   We are given

$$A_n sin\left(\frac{2n\pi}{T}t + \phi_n\right) \tag{8.20}$$

   So we start with the following trigonometric identity,

$$Asin(x \pm y) = Asin(x)cos(y) \pm Acos(x)sin(y) \tag{8.21}$$

   Applying this identity to our original expression yields

$$A_n sin\left(\frac{2n\pi}{T}t\right) cos(\phi_n) + A_n cos\left(\frac{2n\pi}{T}t\right) sin(\phi_n) \tag{8.22}$$

   Rearranging gives us

$$A_n sin(\phi_n) cos\left(\frac{2n\pi}{T}t\right) + A_n cos(\phi_n) sin\left(\frac{2n\pi}{T}t\right) \tag{8.23}$$

   where

$$a_n = A_n sin(\phi_n) \qquad b_n = A_n cos(\phi_n) \tag{8.24}$$

   Solving for A_n and ϕ_n gives the resulting expressions

$$A_n = \sqrt{a_n^2 + b_n^2} \qquad \phi_n = tan^{-1}\left(\frac{a_n}{b_n}\right) \tag{8.25}$$

2. **Derive Equation 8.10 or 8.11.**

   A derivation of the equation for finding the coefficient a_n (Equation 8.10 in the lab manual) is included here. Deriving the equation for finding the coefficient b_n is a fairly trivial exercise after that, as will be seen.

   We start with Equation 8.8 from the lab manual,

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n cos\left(\frac{2n\pi}{T}t\right) + b_n sin\left(\frac{2n\pi}{T}t\right)\right] \tag{8.26}$$

   For concision we will make the substitution, $\omega_k = 2k\pi/T$, multiply both sides by $cos(\omega_k t)$ and integrate over a full period. If we wanted

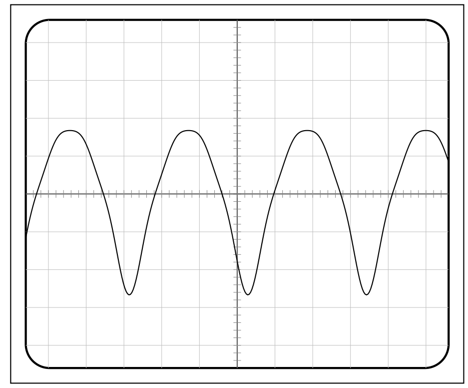


Figure 8.10: A sketch of the synthesized full rectified wave.

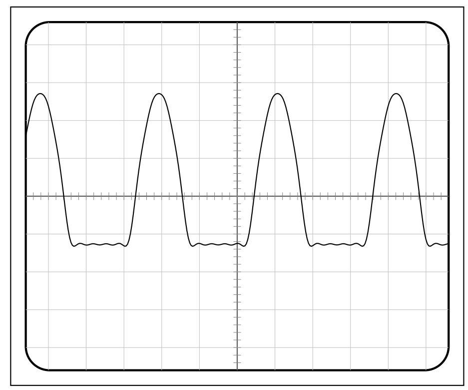


Figure 8.11: A sketch of the synthesized half rectified wave.

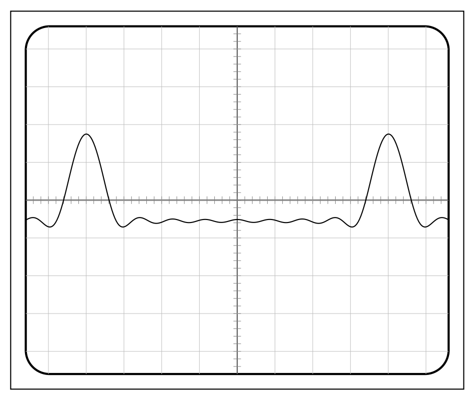


Figure 8.12: A sketch of the synthesized 25% duty cycle square wave.

to derive the equation for finding b_n we would instead multiply both sides by $sin(\omega_k t)$ and integrate over a full period.

$$\int_0^T f(t)cos(\omega_k t)dt = \int_0^T a_0 cos(\omega_k t)dt + \int_0^T \sum_{n=1}^{\infty} \left[a_n cos(\omega_n t)cos(\omega_k t) + b_n sin(\omega_n t)cos(\omega_k t)\right] dt \quad (8.27)$$

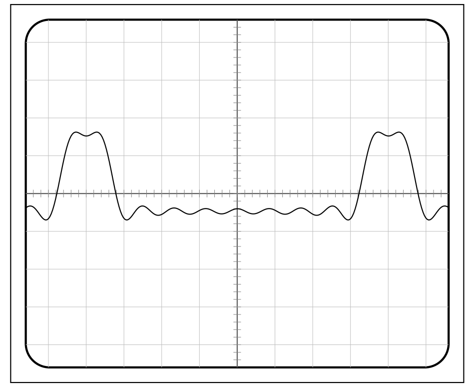


Figure 8.13: A sketch of the synthesized 33% duty cycle square wave.

Due to the orthogonality of the factors in each integrand, all terms in Equation 8.27 will become zero when integrated over a full period except for when $\omega_n = \omega_k$. We are then left with

$$\int_0^T f(t)cos(\omega_k t)dt = a_k \frac{T}{2} \quad (8.28)$$

After some rearranging we get,

$$a_k = \frac{2}{T} \int_0^T f(t)cos(\omega_k t)dt \quad (8.29)$$

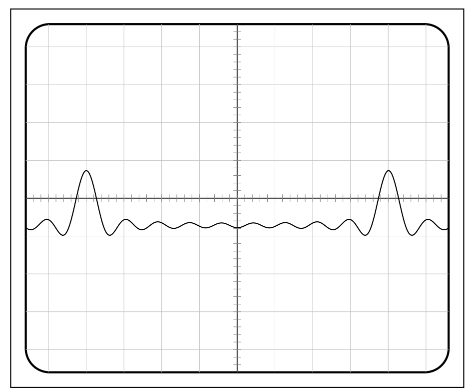


Figure 8.14: A sketch of the synthesized 8.3% duty cycle square wave.

3. **The synthesizer does not generate the a_0 term. How does this restrict the types of signals that can be synthesized? What does AC coupling the oscilloscope have to do with the a_0 term?**

The a_0 term represents a DC level in the signal. Because the synthesizer does not generate this term, AC coupling the oscilloscope should display the same signal as DC coupling. If we could generate the a_0 term then AC coupling would subtract it out and we would be left with a signal, displayed on the oscilloscope, that didn't have the a_0 term.

Data Requirements

4. **A graph of C^2 vs. B^2 from procedure step 2.**

   Here the y-intercept should be close to A^2, the voltage of the fixed frequency, which has been chosen to be A=0.727 V. So A^2=0.5285. The y-intercept can be read off the graph.

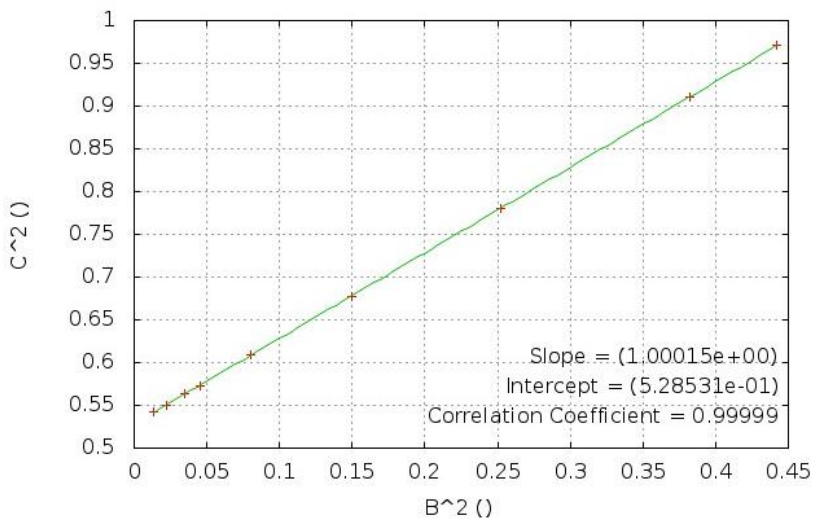


Figure 8.15: Graph for data requirement 4.

5. **A graph of tan(ϕ) vs. B from procedure step 2.**

   Here the slope of the graph should be close to 1/A. For this data set, 1/A=1.376 which closely matches the slope.

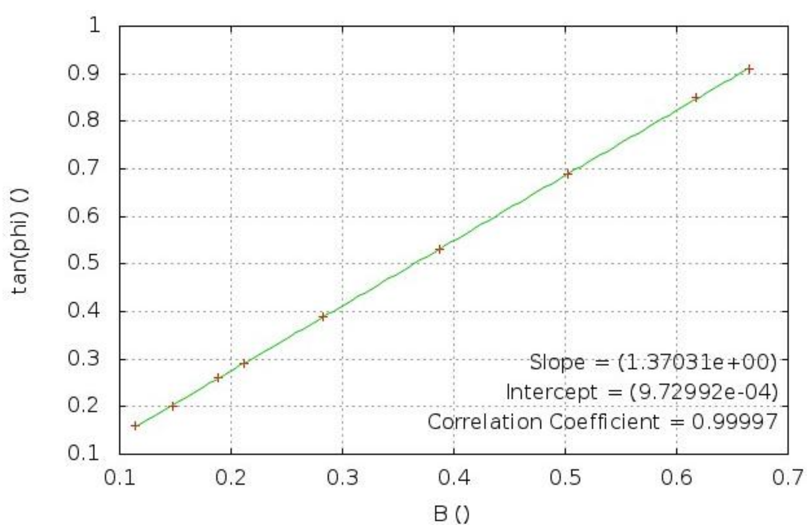


Figure 8.16: Graph for data requirement 5.

6. **Results and observations from procedure step 3.**

Students will choose one of the following two steps to perform.

(A) **Choose any two neighboring higher harmonics and set their amplitudes to be equal and the phases to be 0°. Measure the frequency of the beat envelope and beat carrier. Repeat for at least two other neighboring harmonics.**

For the following three harmonic pairs, Figure 8.17 will be referenced.

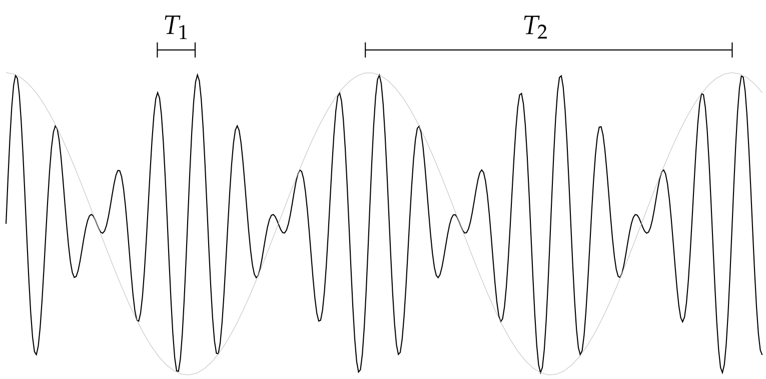


Figure 8.17: An illustration of a beat frequency with carrier period, T_1 and envelope period, T_2.

For harmonics 8 and 9, $f_1 = 1/T_1 = 1/(0.000265s) = 3.774$ kHz and $f_2 = 1/T_1 = 1/(0.0024s) = 0.4167$ kHz.

For harmonics 7 and 8, $f_1 = 1/T_1 = 1/(0.0003s) = 3.333$ kHz and $f_2 = 1/T_1 = 1/(0.0023s) = 0.4348$ kHz.

For harmonics 6 and 7, $f_1 = 1/T_1 = 1/(0.000325s) = 3.077$ kHz and $f_2 = 1/T_1 = 1/(0.0022s) = 0.4545$ kHz.

(B) **Choose any two frequencies with arbitrary non zero amplitudes and arbitrary phase. Measure the period of their sum. The sum frequency can be compared to the expected value given by the greatest common divisor of the individual frequencies. Repeat for at least two other combinations.**

For this problem, students will need to utilize the notion of commensurability, namely the commensurability of two waves of frequencies p and q, respectively.

$$mp = nq \tag{8.30}$$

So two waves of arbitrary frequency, amplitude, and phase will add together to form a periodic wave only if they differ by a rational constant. Additionally, the resulting wave will have a frequency, f, which is the greatest common divisor of the frequencies of its constituent waves, p and q.

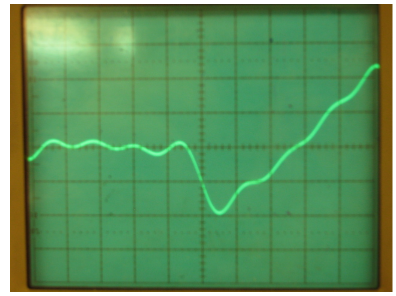


Figure 8.18: A photo of the synthesized ramp function.

For harmonics 4 and 6,

$$f = 888.9 \text{ Hz} \tag{8.31}$$
$$\times 2 = 1778 \text{ Hz} \approx 1760 \text{ Hz (4th Harmonic)} \tag{8.32}$$
$$\times 3 = 2667 \text{ Hz} \approx 2640 \text{ Hz (6th Harmonic)} \tag{8.33}$$

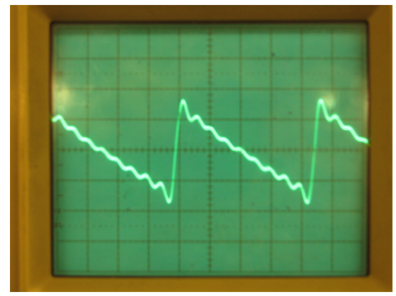


Figure 8.19: A photo of the synthesized sawtooth wave.

For harmonics 5 and 8,

$$f = 444 \text{ Hz} \tag{8.34}$$
$$\times 5 = 2220 \text{ Hz} \approx 2200 \text{ Hz (5th Harmonic)} \tag{8.35}$$
$$\times 8 = 3552 \text{ Hz} \approx 3520 \text{ Hz (8th Harmonic)} \tag{8.36}$$

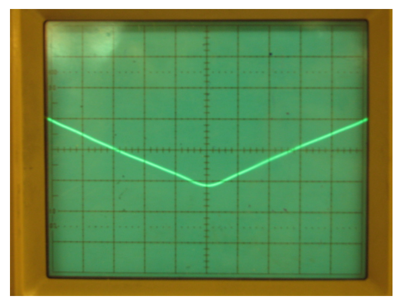


Figure 8.20: A photo of the synthesized triangle wave.

For harmonics 6 and 9,

$$f = 1333 \text{ Hz} \tag{8.37}$$
$$\times 2 = 2667 \text{ Hz} \approx 2640 \text{ Hz (6th Harmonic)} \tag{8.38}$$
$$\times 3 = 3999 \text{ Hz} \approx 3960 \text{ Hz (9th Harmonic)} \tag{8.39}$$

7. **A sketch and observations of the synthesized square wave.**

Figure 8.21 shows a 9 term Fourier series of a square wave, the same wave generated in step 4 of the lab. Students should see very close agreement with this shape with one exception; they may notice that the plateaus are not level but instead slope towards the central line in the direction of the sweep (left to right).

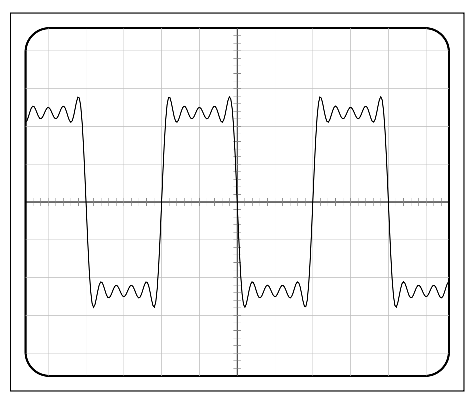


Figure 8.21: A sketch of the synthesized square wave.

8. **Data and observations of Gibbs' ears.**

Harmonic	Width (ms)	Height (V)	% Height
fundamental	0.50	0.20	25.6
+3	0.26	0.16	20.5
+5	0.17	0.16	20.5
+7	0.13	0.16	20.5
+9	0.11	0.18	23.1

Table 8.6: Data collected for Gibbs' ears.

9. **Sketches and relevant observations for the other synthesized signals.**

   Plots of 9 term Fourier series' have been included in the margins of this document for each synthesized wave as well as photos of the oscilloscope trace for comparison below.

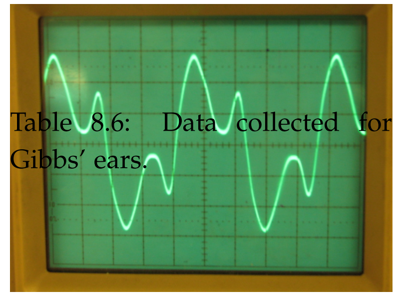


Figure 8.22: A photo of the synthesized PSK wave.

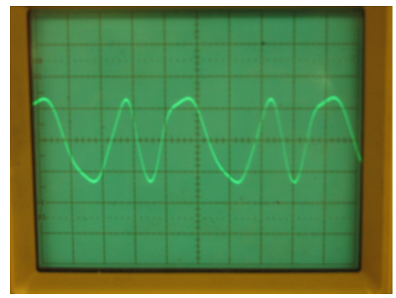


Figure 8.23: A photo of the synthesized FSK wave.

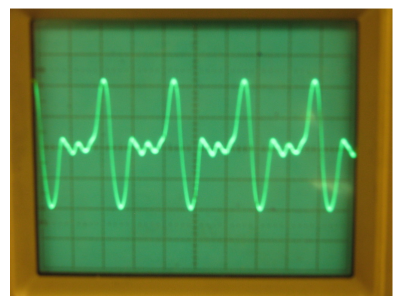


Figure 8.24: A photo of the synthesized ASK wave.

Discussion

10. Results of the addition of two waves of equal frequency agree with theory well. In both cases, data requirements 4 and 5, the experimental value for A, the fixed voltage, matched the expected value exactly to at least 3 significant digits. Given that the uncertainty in measurements made with the oscilloscope is plus or minus half the larger division on the voltage scale, i.e., ± 0.25 V for 0.5 V/div, the values obtained fall firmly within the expected range. The synthesizer appears to be adding functions together adequately.

11. If students complete part A then they will just be reporting the frequencies of the carrier wave and envelope with the associated uncertainties being those of the measurement of period from the oscilloscope screen, i.e., half the larger division. If students choose part B, they will be required to show that, upon addition of two waves of different frequencies, the resulting wave has a frequency that is the greatest common divisor of the constituent waves. The horizontal scale used here was 0.2 μs and so the uncertainty is ± 0.1 μs. Propagating this through the conversion from period to frequency gives an uncertainty in the frequency of ± 80 Hz. Results of the addition of waves of unequal frequencies agree well with Equation 8.30. Again, the synthesizer appears to be functioning correctly.

12. Gibbs's ears did not pass below 17% of the full amplitude, as the theory predicted though the width did decrease with each added harmonic. It should be noted that the square wave has an amplitude that decays along the direction of the oscilloscope sweep. This may be an error associated with either the synthesizer or the oscilloscope.

13. We conclude that a Fourier series is a useful method for approximating periodic functions with sinusoidally oscillating signals which are relatively easy to generate.