Elastic Curves

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Introduction

In 1691, James Bernoulli proposed a problem called the *elastica problem*: Assuming a lamina AB of uniform thickness and width and negligible weight of its own, supported on its lower perimeter at A, and with a weight hung from its top at B, the force from the weight along the line BC bends the lamina into a shape known as an elastic curve. What are the possible shapes?
Euler gave a detailed analysis on the problem in 1744 by solving a variational problem. His method gave birth to the method of variational calculus.
In this project, we solve the variational problem by minimising the bending energy of a thin inextensible wire, and from there to derive the equations of elasticae.

Background

We will deal primarily with curves in 2D space, i.e. planar curves. Let $I := [0, L] \subset \mathbb{R}^2$. A curve γ can be parametised as a function $\gamma : I \to \mathbb{R}^2$. The speed of the curve at t_0 is $v(t_0) = ||\gamma'(t_0)||$.
Given t_0, the tangent and normal vectors to the curve at the point $\gamma(t_0)$ is given by $T(t_0) := \frac{1}{v}\gamma'(t_0)$ and $N(t_0) := \frac{1}{||\gamma''(t_0)||}\gamma''(t_0)$ respectively. The Frenet-Seret Formulae gives us the following equations:

$$T' = v\kappa N$$
$$N' = v\kappa T$$

where $\kappa(t) = \frac{\gamma' \times \gamma''}{||\gamma'||^3}$ is the curvature of the curve.
The bending energy of this curve is given by

$$E = \int_0^L \kappa^2(s)ds.$$

To find the elastic curves, we need to minimise E. To do this, we define $\gamma : I \times \mathbb{R} \to \mathbb{R}^2$ where $\gamma(t, s)$ is a deformation of the original curve γ. Denote $\gamma(t, s) = \gamma_s(t)$ and $\frac{d}{dt}\gamma_s\big|_{s=0} = \dot{\gamma}$. Then we impose the conditions $\dot{\gamma}(0) = 0$ and $\dot{\gamma}(L) = 0$ on the variations so that they all have the same end-points.
Now γ is a free elastica if $\dot{E} = 0$ for all compactly supported variations.

Euler-Lagrange Equation

The problem will be solved using variational methods, and thus we first need to find the Euler-Lagrange equation for this problem. Assume $v(t, 0) = 1$, that is γ is a unit speed curve. Then

$$\begin{aligned}
\dot{E} &= \frac{d}{dt}|_{s=0}\int_0^L v(t,s)\kappa^2(t,s)dt \\
&= \int_0^L [\dot{v}(t,0)\kappa^2(t,0) + 2v(t,0)\kappa(t,0)\dot{\kappa}(t,0)]dt \\
&= \int_0^L [\dot{v}\kappa^2 + 2\ \kappa\dot{\kappa}]dt \\
&= \int_0^L [\kappa^2\langle\dot{\gamma}', T\rangle + 2\kappa\langle\dot{\gamma}'', N\rangle - 4\kappa^2\langle\dot{\gamma}', T\rangle]dt \\
&= \int_0^L [2\kappa\langle\dot{\gamma}'', N\rangle - 3\kappa^2\langle\dot{\gamma}', T\rangle]dt.
\end{aligned}$$

Continuing similarly, and using integration-by-parts as well as the Frenet-Seret formulae gives

$$\dot{E} = 0 \iff \kappa'' + \tfrac{1}{2}\kappa^3 = 0.$$

This is our Euler-Lagrange equation.
Multiplying across by κ' gives

$$\begin{aligned}
\kappa'\kappa'' + \tfrac{1}{2}\kappa'\kappa^3 &= (\tfrac{1}{2}(\kappa')^2 + \tfrac{1}{8}\kappa^4)' = 0. \\
&\implies (\kappa')^2 + \tfrac{1}{4}\kappa^4 - c_1 = 0
\end{aligned}$$

for some $c_1 \in \mathbb{R}_{>0}$.

Thus

$$\begin{aligned}
\frac{d\kappa}{dt} &= \tfrac{1}{2}\sqrt{4c_1 - \kappa^4} \\
\implies dt &= 2\frac{d\kappa}{\sqrt{4c_1 - \kappa^4}} \\
\implies t + c_2 &= 2\int \frac{d\kappa}{\sqrt{2\sqrt{c_1} - \kappa^2}\sqrt{2\sqrt{c_1} + \kappa^2}}
\end{aligned}$$

We now make the substitution $\kappa = \sqrt{2\sqrt{c_1}}\,sin\phi$ to finally get

$$t + c_2 = \frac{\sqrt{2}}{c_1^{\frac{1}{2}}}\int \frac{d\phi}{\sqrt{1 - (-1)sin^2\phi}}.$$

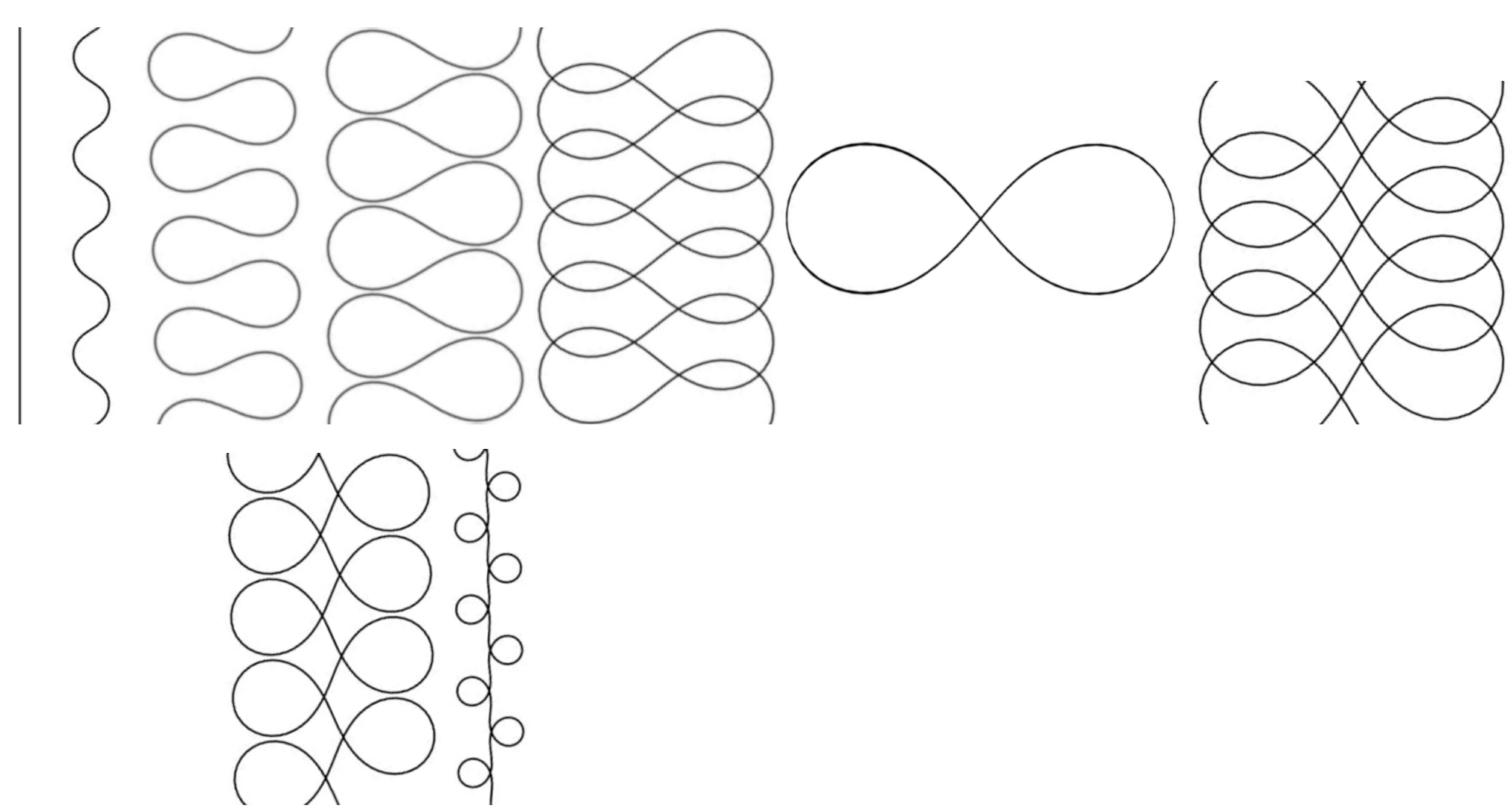
The right-hand-side is the elliptic integral of the first kind and has the associated inverse

$$\kappa(t) = \sqrt{2\sqrt{c_1}}\,sn(\tfrac{c_1^{\frac{1}{4}}(t + c_2)}{\sqrt{2}}| - 1).$$

Now that we have an explicit equation for κ, we now invoke another result: the planar curve γ with curvature κ with $\gamma(0) = 0$ and $\gamma'(0) = (1, 0)$ is given by

$$\gamma(t) = \int_0^t exp(i\int_0^s \kappa(u)du)ds.$$

The latter condition is already satisfied, and the former can be easily made so by way of translation.
Now we note that the curvature depends on two constants. Varying these constants result in different curves, as can be seen in the images below.



Closed Curves

Let $\kappa : \mathbb{R} \mapsto \mathbb{R}$ be a smooth periodic function of perion ρ_κ and $\gamma_\kappa(t)$ be the curve with curvature $\kappa(t)$. Then $\gamma_\kappa(t)$ has period ρ such that $\rho = n\rho_\kappa$ for some natural number $n > 1$. Then $\gamma_\kappa(t)$ is closed if there exists an interger m such that

$$\frac{1}{2\pi}\int_0^{\rho_\kappa}\kappa(s)ds = \frac{m}{n} \in \mathbb{Q} - \mathbb{Z}.$$

Here m is the rotation index of the associate curve.

References

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