A COURSE ON THE ISING MODEL

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Contents

Preface

These lecture notes are written progressively during the 2025 spring semester, as the course is taught at Sorbonne university in the M2 (second-year masters) programme. Its purpose is to give a broad introduction to the rigorous analysis of the Ising model. The main focus is on four techniques and their applications:

- The Peierls argument,
- The random-currents representation,
- The FKG inequality for the Ising spins,
- The FKG inequality for the random-cluster (FK) representation.

A basic understanding of analysis and probability theory is essential for following this course. Experience with other models in statistical mechanics (such as the Bernoulli percolation model) is a plus but by no means essential.

The appendices contain overviews of the main definitions, expansions, and inequalities in this text.

These notes are inspired by the lecture notes *Lectures on the Ising and Potts models on the hypercubic lattice* and the overview *100 Years of the (Critical) Ising Model on the Hypercubic Lattice*, both due to Hugo Duminil-Copin. The main text does not contain references at this stage; they will be added at a later time.

1. The Curie–Weiss model

At the end of the 19th century, Curie published his experimental results on *ferromagnetism*: the magnetic properties of metals. He made three striking observations.

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- The magnetic strength of a metal varies with the temperature. Increasing the temperature decreases the magnetic strength.
- Each metal has a certain temperature, specific to that metal, at which the magnetic properties disappear entirely. We call this temperature the *Curie temperature*.
- Around the Curie temperature, the magnetic strength drops continuously to zero. In other words, the magnetic strength does not "jump" to zero.

The first observation singles out the temperature as the driving parameter of the system. This is good news for us, since the temperature may be regarded informally as the amount of "randomness" or "entropy" in the system, justifying a probabilistic analysis of the situation. The second observation implies that there is a *phase transition*: there is a special temperature (in this case the Curie temperature) at which the system's behaviour undergoes a qualitative change. The third observation entails an important property of this phase transition.

The first mathematical explanation for Curie's experimental results came from Weiss. He proposed the following mathematical axioms for studying the magnetic properties of metals.

- The metal consists of n atoms.
- Each atom acts like a small magnet in itself. It is in one of two states, denoted $\pm$.
- The total strength of the metal is obtained by summing the states of all atoms.
- Each atom interacts with all other atoms. The atoms prefer to *align*, that is, to be in the same state. The temperature regulates the strength of the interaction.

Physically, it makes sense that the temperature regulates the interaction strength. When atoms move slowly, they will stabilise, oriented in alignment with the magnetic field imposed by the other atoms. When atoms move fast, they will not bother with the states of the other atoms, and simply align themselves randomly. It is thus natural to think of the interaction strength as the *inverse temperature*.

Definition 1.1 (Curie-Weiss model)**.** The Curie-Weiss model is the probability measure $\mathbb{P}_{n,\beta}^{\mathrm{CW}}$ on $\sigma \in \Omega := \{+,-\}^n$ defined via

$$\mathbb{P}(\sigma) := \mathbb{P}_{n,\beta}^{\mathrm{CW}}(\sigma) \propto e^{-H_{n,\beta}^{\mathrm{CW}}(\sigma)}; \qquad H(\sigma) := H_{n,\beta}^{\mathrm{CW}}(\sigma) := -\frac{\beta}{n}\sum_{i<j}\sigma_i\sigma_j,$$

where $n \in \mathbb{Z}_{\geq 1}$ and $\beta \in [0,\infty)$. The parameter β is called the *interaction strength* or *inverse temperature*. The function H is called the *Hamiltonian* and captures the *energy* in the system. The probability measure $\mathbb{P}$ is also called the *Boltzmann distribution*.

Let $n_+ = n_+(\sigma)$ denote the number of vertices with spin $+$ in a configuration $\sigma \in \Omega$. This is a random variable. Let us try to calculate the probability of the event $\{n_+ = k\}$, without worrying about the partition function (the normalising constant). One may easily check that the Hamiltonian satisfies

$$H(\sigma) = 2\frac{\beta}{n}n_+(n-n_+) + \mathrm{const}(n).$$

The distribution of n_+ can then be calculated as follows:

$$\mathbb{P}(\{n_+ = k\}) \propto \binom{n}{k} e^{-2\frac{\beta}{n}k(n-k)} \propto \frac{1}{k!(n-k)!}e^{-2\frac{\beta}{n}k(n-k)}. \tag{1}$$

Using Stirling's approximation for the factorials, we find that

$$\begin{aligned}\log \mathbb{P}(\{n_+ = k\}) &\overset{\text{Stirling}}{\approx} -nf_\beta(k/n) + \mathrm{const}(n);\\ f_\beta : [0,1] \to \mathbb{R},\ x &\mapsto x\log x + (1-x)\log(1-x) + 2\beta x(1-x).\end{aligned}$$

If we fix β and send n to infinity, then we discover a large deviations principle for the random variable n_+/n with rate function f_β and speed n. In particular, the random variable n_+/n concentrates around the minimisers of the function f_β.

Exercise 1.2 (The rate function of the Curie–Weiss model)**.**

(1) Show that for small β, the function f_β has a single minimum at $x = 1/2$, which means that the random variable n_+/n concentrates around the value $1/2$.
(2) Show that for large β, the function f_β has two minima at $(1 \pm m)/2$ for some $m > 0$, which means that the random variable n_+/n is concentrated around these minima. The value of m is called the *magnetisation*.
(3) Calculate the critical value for β. At this value, the second derivative of f_β vanishes at $x = 1/2$. What does this mean for the distribution of n_+/n? Estimate the order of magnitude of $\operatorname{Var} \frac{n_+}{n}$ as $n \to \infty$ for this value of β.

Remark 1.3 (Entropy versus energy in the Curie–Weiss model)**.** Reconsider Equation (1). In this equation, the competition between the two factors is extremely transparent.

- First, there is a combinatorial term or *entropy*, which favours values k for the random variable n_+ such that the cardinality of the set $\{n_+ = k\}$ is large. This means that values $k \approx n/2$ are preferred.
- Second, there is the *energy* term, which favours values such that the energy is minimised. This favours configurations where as many spins as possible align.

The interaction parameter β allows us to put more emphasis on the entropy term or on the energy term. In the $n \to \infty$ limit, there is a precise value for β where the behaviour of the random system undergoes a qualitative change: a rudimentary example of a *phase transition.*

2. Ising's model and basic notions

While the competition between entropy and energy is transparent in the Curie–Weiss model, the model does not encode any kind of geometry. Indeed, all atoms interact equally with all other atoms. It would perhaps be more realistic to place the atoms on a Euclidean grid, and let the interactions strength between two atoms depend on their distance. In the simplest case, we could simply let each atom interact only with the atoms closest to it. This is called the *nearest-neighbour interaction.* We mainly focus on this setup in these lecture notes.

Wilhelm Lenz challenged his doctoral student Ernst Ising to solve this nearest-neighbour model for magnetism on the one-dimensional line graph $\mathbb{Z}$. Lenz was not entirely precise when posing this question, and it was Ising who first formulated a definition for the model under consideration. The model is therefore called the *Ising model* in his honour. We shall later derive Ising's result from a broader theorem (Theorem 4.4).

Definition 2.1 (Ising model)**.** The Ising model on a finite graph $G = (V, E)$ with *inverse temperature* $\beta \in [0, \infty)$ is defined as follows. Let $\Omega := \{\pm 1\}^V$ denote the set of spin configurations on the vertices of the graph; a typical element of Ω is denoted by $\sigma = (\sigma_u)_{u \in V}$. Elements $\sigma \in \Omega$ are called *spin configurations*; elements σ_u are called *spins.* The *energy* or *Hamiltonian* of a spin configuration σ is given by

$$H^{\mathrm{Ising}}_{G,\beta}(\sigma) := -\beta \sum_{uv \in E} \sigma_u \sigma_v.$$

We write $\mathbb{P}^{\mathrm{Ising}}_{G,\beta}$ for the associated *Boltzmann distribution* or *Gibbs measure*:

$$\mathbb{P}^{\mathrm{Ising}}_{G,\beta}(\sigma) := \frac{1}{Z^{\mathrm{Ising}}_{G,\beta}} e^{-H^{\mathrm{Ising}}_{G,\beta}(\sigma)},$$

where $Z^{\text{Ising}}_{G,\beta}$ is normalisation constant or *partition function* defined by

$$Z^{\text{Ising}}_{G,\beta} := \sum_{\sigma\in\Omega} e^{-H^{\text{Ising}}_{G,\beta}(\sigma)}.$$

We shall write $\langle\,\cdot\,\rangle^{\text{Ising}}_{G,\beta}$ for the expectation functional associated to this probability measure.

Remark. We shall often suppress subscripts and superscripts when they are clear from the context.

Remark. The mathematical community has widely adopted the terminology coming from the physics literature. We often prefer the symbol $\langle\,\cdot\,\rangle$ over $\mathbb{E}[\,\cdot\,]$ when taking expecations, expect when considering *conditional* expectations.

Exercise 2.2 (The edge graph)**.** Consider the Ising model on the complete graph on the two vertices $V := \{x, y\}$ at inverse temperature $\beta \in [0, \infty)$.

- Calculate $\langle\sigma_x\rangle_\beta$.
- Calculate $\langle\sigma_x\sigma_y\rangle_\beta$.

Definition 2.3 (Correlation functions)**.** Consider $\Omega := \{\pm\}^V$. Then for any finite subset $A \subset V$, we define $\sigma_A : \Omega \to \{\pm\}$, $\sigma \mapsto \prod_{x\in A}\sigma_x$. Its expectation $\langle\sigma_A\rangle$ in any probability measure $\langle\,\cdot\,\rangle$ on Ω is called a *correlation function.* If $|A| = n$ then $\langle\sigma_A\rangle$ is also called an *n-point correlation function.*

Remark (Flip-symmetry)**.** The Ising model is *flip-symmetric* in the sense that the distribution of the spins is invariant under the transformation $\sigma \mapsto -\sigma$. This is because the Hamiltonian is invariant under this transformation.

Exercise 2.4 (Flip-symmetry)**.** Consider the Ising model on a finite graph $G = (V, E)$.

- Prove that if $A \subset V$ contains an odd number of vertices, then $\langle\sigma_A\rangle = 0$.
- Prove that if $A \subset V$ contains an odd number of vertices and $x \in V$, then
$$\mathbb{E}[\sigma_A|\{\sigma_x = +\}] = \langle\sigma_A\sigma_x\rangle.$$
- Prove that if $A \subset V$ contains an even number of vertices and $x \in V$, then
$$\mathbb{E}[\sigma_A|\{\sigma_x = +\}] = \langle\sigma_A\rangle.$$

In practice, we are interested in the Ising model on finite portions of the square lattice $\mathbb{Z}^d$ endowed with nearest-neighbour connectivity. We now provide the definitions for this setup.

Definition 2.5 (Free boundary conditions)**.** Let $G = (V, E)$ denote a locally finite graph and $\Lambda \subset V$ a finite set. Write Λ^{f} for the subgraph of G induced by Λ. Write $\langle\,\cdot\,\rangle^{\mathrm{f}}_{\Lambda,\beta} := \langle\,\cdot\,\rangle_{\Lambda^{\mathrm{f}},\beta}$ for the *free-boundary Ising model* in Λ at inverse temperature $\beta \in [0, \infty)$.

Definition 2.6 (Fixed boundary conditions)**.** Let $G = (V, E)$ denote a locally finite graph and $\Lambda \subset V$ a finite set. Let $\partial\Lambda \subset V \setminus \Lambda$ denote the set of vertices adjacent to Λ. Write $\bar\Lambda$ for the graph defined by

$$V(\bar\Lambda) := \Lambda \cup \partial\Lambda; \qquad E(\bar\Lambda) := \{\{x, y\} \in E : \{x, y\} \cap \Lambda \neq \emptyset\}.$$

For any $\zeta \in \{\pm\}^{\partial\Lambda}$, we shall write $\langle\,\cdot\,\rangle^\zeta_{\Lambda,\beta}$ for the measure

$$\langle\,\cdot\,\rangle^\zeta_{\Lambda,\beta} := \mathbb{E}_{\bar\Lambda,\beta}[\,\cdot\,|\{\sigma|_{\partial\Lambda} = \zeta\}].$$

This is called the *fixed-boundary Ising model* with boundary conditions ζ. The boundary condition $\zeta \equiv \pm$ is of particular interest, and it is denoted $\langle\,\cdot\,\rangle^\pm_{\Lambda,\beta}$.

Exercise 2.7 (Markov property)**.** Consider the Ising model on some finite graph $G = (V, E)$ at inverse temperature β. Fix some $\Lambda \subset V$ and let $(\Lambda_i)_i$ denote the partition of Λ into connected components. Let $\zeta \in \{\pm\}^{\Lambda^c}$, and consider the conditional probability measure $\mathbb{P}[\,\cdot\,|\{\sigma|_{\Lambda^c} = \zeta\}]$.

- Prove that $(\sigma|_{\Lambda_i})_i$ is a family of independent random variables in this measure.
- Prove that the law of $\sigma|_{\Lambda_i}$ is $\langle\,\cdot\,\rangle_{\Lambda_i}^{\zeta|_{\partial\Lambda_i}}$.

Hint. Decompose the Hamiltonian according to $H(\sigma) = C + \sum_i H_i(\sigma)$, where each H_i is measurable in terms of $\zeta|_{\partial\Lambda_i}$ and $\sigma|_{\Lambda_i}$.

Ising proved that in one dimension, the Ising model exhibits exponential decay of correlations at all temperatures. In other words, there is no phase transition. We now state his result, without a proof. While the proof is quite straightforward even with elementary methods, its proof becomes entirely trivial after the introduction of more recent methods.

Theorem 2.8 (Ising, 1924)**.** *Consider the finite domains $\Lambda_n := \{-n, \dots, n\}$ of the graph $\mathbb{Z}$. Then for any $\beta \in [0, \infty)$, there exists a constant $c = c_\beta > 0$ such that*

$$\langle\sigma_0\rangle_{\Lambda_n,\beta}^{+} \leq \frac{1}{c} e^{-c_\beta \cdot n}.$$

Unfortunately, Ising wrongly conjectured that the same would be true in higher dimension. Disappointed with this prediction, he left academia.

3. Peierls' argument

Peierls disproved Ising's conjecture for the absence of phase transition in dimension $d \geq 2$.

Theorem 3.1 (Peierls, 1936)**.** *Consider the finite domains $\Lambda_n := \{-n, \dots, n\}^2$ of the square lattice graph $\mathbb{Z}^2$. Then for sufficiently large $\beta \in [0, \infty)$, we have*

$$\inf_n \langle\sigma_{(0,0)}\rangle_{\Lambda_n,\beta}^{+} > 0.$$

Proof. Our objective is to prove that $\mathbb{P}_{\Lambda_n,\beta}^{+}[\{\sigma_{(0,0)} = -\}] \leq \frac{1}{3}$ for all n. Fix n.

Consider the set $\Omega' \subset \Omega$ of spin configurations on $\bar{\Lambda}_n$ which assign $+$ to $\partial\Lambda_n$. The two-dimensional square lattice graph $G = \mathbb{Z}^2$ is planar, and therefore we may consider its planar dual G^*. For any spin configuration $\sigma \in \Omega'$, we let $\mathcal{I}(\sigma) \subset E(G^*)$ denote its *interface*, that is, the set of dual edges separating two spins with a *distinct* value. Notice that:

- The map $\sigma \mapsto \mathcal{I}(\sigma)$ is injective,
- If $\sigma_{(0,0)} = -$, then $\mathcal{I}(\sigma)$ contains at least one self-avoiding loop around $(0, 0)$.

In particular, inclusion of events yields

$$\mathbb{P}_{\Lambda_n,\beta}^{+}[\{\sigma_{(0,0)} = -\}] \leq \mathbb{P}_{\Lambda_n,\beta}^{+}[\{\mathcal{I}(\sigma) \text{ contains a loop } \gamma \text{ around } (0, 0)\}]. \tag{2}$$

We would now like to make a competition between entropy and energy appear, as for the Curie-Weiss model. The entropy comes from the choice of the loop γ; the energy comes into play when upper bounding the probability that a particular loop belongs to $\mathcal{I}(\sigma)$. For large β, energy wins over entropy, yielding the desired bound. Let us start with the energy bound.

Claim. For any fixed loop γ, we may bound $\mathbb{P}_{\Lambda_n,\beta}^{+}[\{\gamma \subset \mathcal{I}(\sigma)\}] \leq e^{-2\beta|\gamma|}$.

Proof of the claim. We would like to define a *loop erasure map* $\mathcal{E} : \{\gamma \subset \mathcal{I}(\sigma)\} \to \Omega'$, which has the property that it removes the loop γ from the interface, that is,

$$\mathcal{I}(\mathcal{E}(\sigma)) = \mathcal{E}(\sigma) \setminus \gamma.$$

It is easy to realise such a map: we simply define $\mathcal{E}$ such that it flips the sign of every vertex of Λ_n which is surrounded by γ. Since $\mathcal{I}$ is injective, the map $\mathcal{E}$ is also injective, and we have

$$\mathbb{P}^+_{\Lambda_n,\beta}[\{\gamma \subset \mathcal{I}(\sigma)\}] = \frac{\sum_{\sigma \in \mathrm{Domain}(\mathcal{E})} e^{-H(\sigma)}}{\sum_{\sigma \in \Omega'} e^{-H(\sigma)}} \leq \frac{\sum_{\sigma \in \mathrm{Domain}(\mathcal{E})} e^{-H(\sigma)}}{\sum_{\sigma \in \mathrm{Image}(\mathcal{E})} e^{-H(\sigma)}} = e^{-2\beta|\gamma|}.$$

The last equality is easy, since for any $\sigma \in \{\gamma \subset \mathcal{I}(\sigma)\}$, we have

$$H(\mathcal{E}(\sigma)) = H(\sigma) - 2\beta|\gamma|,$$

since $\mathcal{E}$ removes precisely $|\gamma|$ disagreement edges from the interface. This proves the claim.

We use the energy bound to prove another interesting intermediate result.

Claim (Exponential decay of the loop length)**.** For any dual edge e, we have

$$\mathbb{P}^+_{\Lambda_n,\beta}[\{\mathcal{I}(\sigma) \text{ contains a loop of length at least } \ell \text{ through } e\}] \leq (3e^{-2\beta})^\ell \frac{1}{1-3e^{-2\beta}}$$

whenever $3e^{-2\beta} < 1$.

Proof of the claim. Let $\mathcal{L}_k$ denote the set of self-avoiding loops through e of length k. Notice that $|\mathcal{L}_k| \leq 3^k$. A union bound yields

$$\begin{aligned}
&\mathbb{P}^+_{\Lambda_n,\beta}[\{\mathcal{I}(\sigma) \text{ contains a loop of length at least } \ell \text{ through } e\}] \\
&\quad \leq \sum_{k=\ell}^{\infty} \sum_{\gamma \in \mathcal{L}_k} \mathbb{P}^+_{\Lambda_n,\beta}[\{\gamma \subset \mathcal{I}(\sigma)\}] \leq \sum_{k=\ell}^{\infty} |\mathcal{L}_k| \cdot e^{-2\beta k} \leq \sum_{k=\ell}^{\infty} 3^k \cdot e^{-2\beta k} \\
&\quad = (3e^{-2\beta})^\ell \frac{1}{1-3e^{-2\beta}}.
\end{aligned}$$

This is the desired bound.

Return to Equation (2). If $\mathcal{I}(\sigma)$ contains a loop around $(0,0)$, then this loop must hit $(k-\frac{1}{2}, 0)$ for some $k \in \mathbb{Z}_{\geq 1}$, and this loop must have at least k steps. Thus, another union bound yields

$$\mathbb{P}^+_{\Lambda_n,\beta}[\{\sigma_{(0,0)} = -\}] \leq \sum_{k=1}^{\infty} (3e^{-2\beta})^k \frac{1}{1-3e^{-2\beta}} = (3e^{-2\beta}) \frac{1}{(1-3e^{-2\beta})^2}.$$

The right hand side is smaller than $\frac{1}{3}$ when β is sufficiently large, independently of n. □

Remark. Peierls' is robust, in the sense that it can be adapted to many other models in statistical mechanics.

Exercise 3.2 (The Peierls argument in higher dimensions)**.** Now consider the square lattice graph $\mathbb{Z}^d$ in dimension $d \geq 3$. What is the structure of the interface in this case? Can we adapt Peierls' to prove magnetisation for sufficiently large β?

4. The high-temperature expansion

The previous section proved the Peierls argument. An essential ingredient was to view the Ising model in two dimensions through the *interfaces* of the spins. Such a transformation of the model may be viewed as a rudimentary version of an *expansion.* The interface perspective is sometimes called the *low-temperature expansion* because it works well in the low-temperature regime (when β is large). There are several useful expansions for the Ising model; each one of them is adapted to a different setting. In this section we discuss another expansion: the *high-temperature expansion.* As the name suggests, this expansion is well-adapted to situations where β is small, even though we can also use it to prove useful results in other regimes. Appendix ?? contains an overview of the expansions discussed in these notes, and may serve as a reference.

Add Appendix and reference

For a streamlined presentation, we will henceforth present all our expansions for the Ising model on *finite graphs without boundary conditions*. This obviously includes the free boundary conditions. It is straightforward to see that fixed boundary conditions also fit into this framework, see Definition 4.5 and Lemma 4.6 below.

Consider the Ising model on a finite graph G. We are typically interested in the correlation functions, defined via

$$\langle \sigma_A \rangle = \frac{\sum_{\sigma\in\Omega} \sigma_A e^{-H(\sigma)}}{\sum_{\sigma\in\Omega} e^{-H(\sigma)}} = \frac{\sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} e^{\beta\sigma_x\sigma_y}}{\sum_{\sigma\in\Omega} \prod_{xy\in E} e^{\beta\sigma_x\sigma_y}}.$$

An *expansion* of the Ising model involves rewriting the sum

$$\sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} e^{\beta\sigma_x\sigma_y}.$$

A typical expansion comes down to rewriting the exponential, for example:

- We may write $e^{\beta\sigma_x\sigma_y} = \cosh\beta + \sigma_x\sigma_y \sinh\beta$,
- We may write $e^{\beta\sigma_x\sigma_y} = \sum_{k=0}^\infty (\beta\sigma_x\sigma_y)^k/k!$,
- We may write $e^{\beta\sigma_x\sigma_y} = e^{-2\beta} + 2\cdot \mathbb{1}(\sigma_x = \sigma_y)\sinh\beta$.

Every expansion comes with its own advantages and disadvantages. The high-temperature expansion derives from the first identity.

Definition 4.1 (High-temperature expansion)**.** Consider the Ising model on a finite graph $G = (V, E)$ at inverse temperature β. We consider *percolation configurations* $\omega \in \{0, 1\}^E$; each ω is also regarded a (random) set of edges. We write $\partial\omega \subset V$ for the set of vertices having *odd* degree in the graph (V, ω).

The *high-temperature expansion* is the measure $\mathbf{M}_{G,\beta}$ on $\omega \in \{0, 1\}^E$ defined by

$$\mathbf{M}[\omega] := \mathbf{M}_{G,\beta}[\omega] := (\cosh\beta)^{|E\setminus\omega|}(\sinh\beta)^{|\omega|}.$$

Theorem 4.2 (High-temperature expansion for correlation functions)**.** *Consider the Ising model on a finite graph $G = (V, E)$. Then for any $A \subset V$, we have*

$$Z\langle\sigma_A\rangle = 2^{|V|}\mathbf{M}[\{\partial\omega = A\}].$$

In particular, $Z = 2^{|V|}\mathbf{M}[\{\partial\omega = \emptyset\}]$.

Proof. We claim that

$$\begin{aligned}
Z\langle\sigma_A\rangle &= \sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} e^{\beta\sigma_x\sigma_y} &(3)\\
&= \sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} (\cosh\beta + \sigma_x\sigma_y\sinh\beta) &(4)\\
&= \sum_{\sigma\in\Omega} \sigma_A \sum_{\omega\in\{0,1\}^E} \prod_{xy\in E} (\cosh\beta)^{1-\omega_{xy}}(\sigma_x\sigma_y\sinh\beta)^{\omega_{xy}} &(5)\\
&= \sum_{\omega\in\{0,1\}^E} (\cosh\beta)^{|E\setminus\omega|}(\sinh\beta)^{|\omega|} \sum_{\sigma\in\Omega} \sigma_A\sigma_{\partial\omega} &(6)\\
&= \sum_{\omega\in\{0,1\}^E} (\cosh\beta)^{|E\setminus\omega|}(\sinh\beta)^{|\omega|} 2^{|V|}\mathbb{1}(A = \partial\omega) &(7)\\
&= 2^{|V|}\mathbf{M}[\{\partial\omega = A\}].
\end{aligned}$$

Equations (3) and (4) come down to definitions and the identity for $e^{\beta\sigma_x\sigma_y}$. Swapping the sum and the product yields Equation (5). Equation (6) is a rearrangement of the terms, noting that $\prod_{xy}(\sigma_x\sigma_y)^{\omega_{xy}} = \sigma_{\partial\omega}$. Equation (7) is obtained by resolving the sum over σ. The final equation is the definition of $\mathbf{M}$. $\square$

We can use this theorem to state our first important *correlation inequality.*

Theorem 4.3 (First Griffiths inequality)**.** *Consider the Ising model on a finite graph $G = (V, E)$. Then for any $A \subset V$, we have $\langle\sigma_A\rangle \geq 0$.*

Proof. The previous theorem yields a nonnegative number for $Z\langle\sigma_A\rangle$. $\square$

One advantage of the high temperature expansion is that it yields a straightforward proof of exponential decay of the correlation functions at high temperature.

Theorem 4.4 (Exponential decay at high temperature)**.** *Consider the Ising model with $+$ boundary conditions on the graph $\mathbb{Z}^d$ for $d \in \mathbb{Z}_{\geq 1}$. Then for any $\beta \in [0,\infty)$ such that $(2d-1)\tanh\beta < 1$, there exists a constant $c = c_{d,\beta} > 0$ such that*

$$\langle\sigma_x\rangle^+_{\Lambda,\beta} \leq \tfrac{1}{c}e^{-c\,\mathrm{Distance}(x,\Lambda^c)}$$

for any $x \in \mathbb{Z}^d$ and any domain $\Lambda \subset \mathbb{Z}^d$.

In particular, in dimension $d = 1$, there is exponential decay at all temperatures.

We would like to use the high-temperature expansion, but for this we must first write $\langle\,\cdot\,\rangle^+_\Lambda$ as an Ising model on a finite graph without boundary condition.

Definition 4.5 (Ghost vertex)**.** Let $G = (V, E)$ denote a locally finite graph, and $\Lambda \subset V$ a finite domain. We already defined the graphs Λ^{f} and $\bar\Lambda$. Now define the graph $\Lambda^{\mathfrak{g}}$ as follows: it is obtained from the graph $\bar\Lambda$ by replacing all vertices in $\partial\Lambda$ by a single distinguished vertex $\mathfrak{g}$, called the *ghost vertex*. Its vertex set is given by $V(\Lambda^{\mathfrak{g}}) := \Lambda \cup \{\mathfrak{g}\}$, and there is a natural bijection from $E(\bar\Lambda)$ to $E(\Lambda^{\mathfrak{g}})$.

Notice that $\Lambda^{\mathfrak{g}}$ is a multigraph when some $x \in \Lambda$ is connected to multiple vertices in $\partial\Lambda$ in the graph $\bar\Lambda$, but this does not really affect our setup.

It is easy to see that the following lemma holds true.

Lemma 4.6. *Let $G = (V, E)$ denote a locally finite graph, and $\Lambda \subset V$ a finite domain. Then the distribution of $\sigma|_\Lambda$ is the same in the following two measures:*

$$\langle\,\cdot\,\rangle^+_\Lambda \qquad \text{and} \qquad \mathbb{E}_{\Lambda^{\mathfrak{g}}}[\,\cdot\,|\sigma_{\mathfrak{g}} = +].$$

Correlation functions can thus be expressed in terms of correlation functions on finite graphs via Exercise 2.4.

Proof overview of Theorem 4.4. We have

$$\langle\sigma_x\rangle^+_\Lambda = \langle\sigma_x\sigma_{\mathfrak{g}}\rangle_{\Lambda^{\mathfrak{g}}} = \frac{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \{x,\mathfrak{g}\}\}]}{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \emptyset\}]}.$$

If $\partial\omega = \{x,\mathfrak{g}\}$, then ω contains a self-avoiding walk γ from x to $\mathfrak{g}$. A union bound yields

$$\langle\sigma_x\rangle^+_\Lambda \leq \sum_\gamma \frac{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \{x,\mathfrak{g}\}\} \cap \{\gamma \subset \omega\}]}{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \emptyset\}]}.$$

The proof is now completed after performing the two steps of the Peierls argument:

- One bounds each term by $(\tanh\beta)^{|\gamma|}$,
- One bounds the number of walks γ of length n from x by $2d(2d-1)^{|\gamma|-1}$.

$\square$

Exercise 4.7. Fill in the details of the previous proof overview.

Let us summarise what we have proved so far.

- Theorem 4.4 implies that there is exponential decay of correlations when β is sufficiently small.
- In dimension $d = 1$, Theorem 4.4 also implies Ising's result (Theorem 2.8), since there was no requirement on β when $d = 1$.
- In dimension $d \geq 2$, we proved that there is *magnetisation* via the Peierls argument (Theorem 3.1). Thus, in dimension $d \geq 2$, there must be a phase transition, and we aim to investigate further.

The high-temperature expansion is typically used to find upper bounds on correlation functions. However, it is also possible to use it to find lower bounds. Let $G = (V, E)$ denote a finite graph. For any fixed set $Q \subset E$ of edges, we define the *XOR map*

$$\Xi_Q : \{0,1\}^E \to \{0,1\}^E,\ \omega \mapsto \omega \Delta Q,$$

where Δ denotes the symmetric difference of two sets. This map is an involution. Moreover, for any $A \subset V$, it restricts to a bijection

$$\Xi_Q : \{\partial\omega = A\} \to \{\partial\omega = A\Delta\partial Q\}. \tag{8}$$

The measure $\mathbf{M}$ is not invariant under the involution Ξ_Q, but it is easy to see how the map affects the measure. More precisely, for any $\eta \in \{0,1\}^E$, we have

$$\mathbf{M}[\{\omega = \Xi_Q(\eta)\}] = (\tanh\beta)^{|Q\setminus\eta| - |Q\cap\eta|} \cdot \mathbf{M}[\{\omega = \eta\}].$$

The prefactor is upper bounded by $(\tanh\beta)^{-|Q|}$. Thus, writing $(\Xi_Q)_*$ for the pushforward map, we obtain

$$(\Xi_Q)_*\mathbf{M} \leq (\tanh\beta)^{-|Q|} \cdot \mathbf{M}.$$

For example, using the bijection in Equation (8), we obtain

$$\mathbf{M}[\{\partial\omega = A\}] \leq (\tanh\beta)^{-|Q|} \cdot \mathbf{M}[\{\partial\omega = A\Delta\partial Q\}]. \tag{9}$$

We have now proved the following result.

Lemma 4.8. *Consider the Ising model on a finite graph $G = (V, E)$. Then for any $A \subset V$ and any $Q \subset E$, we have*

$$\langle\sigma_A\sigma_{\partial Q}\rangle \geq (\tanh\beta)^{|Q|} \cdot \langle\sigma_A\rangle.$$

In particular,

$$\langle\sigma_x\sigma_y\rangle \geq (\tanh\beta)^{\mathrm{Distance}(x,y)}.$$

Proof. The first inequality is Equation (9). For the second inequality, simply set $A = \emptyset$ and let Q denote a shortest path from x to y. ☐

This lemma complements Theorem 4.4 at high temperature, as the lemma asserts that the correlation functions cannot decay *faster* than exponentially at any finite temperature (that is, when $\beta > 0$).

5. The high-temperature expansion and switching

We already used the high-temperature expansion to prove one correlation inequality: the first Griffiths inequality, which asserts that $\langle\sigma_A\rangle \geq 0$. This was an immediate consequence of the fact that the high-temperature expansion is a sum of positive terms.

There are many other interested inequalities. Many of those are obtained via the *switching lemma*. The switching lemma is traditionally stated for the random-current expansion (which is a refinement of the high-temperature expansion introduced in the next section), but we shall first state it in the context of the high-temperature expansion because the setup is a little bit simpler. We can already use it to prove two interesting inequalities:

- The *pairing bound*, which relates multi-point and two-point correlation functions,
- The *Simon–Lieb inequality*, which yields a finite-size criterion for exponential decay.

We first prove the following switching lemma. We state it in its most general form, namely for overlapping graphs G and G'. In practice, we often care about the special case that $G = G'$, or the slightly more general case that $G \subset G'$.

Lemma 5.1 (Switching lemma for the high-temperature expansion)**.** *Let G and G' denote two finite graphs and fix $\beta \in [0,\infty)$. If $A \subset V$, $\eta \subset E \cup E'$, and $Q \subset \eta \cap E \cap E'$, then*

$$\mathbf{M}_{G,\beta} \times \mathbf{M}_{G',\beta}[\{\omega\Delta\omega' = \eta\} \cap \{\partial\omega = A\}] = \mathbf{M}_{G,\beta} \times \mathbf{M}_{G',\beta}[\{\omega\Delta\omega' = \eta\} \cap \{\partial\omega = A\Delta\partial Q\}].$$

Proof. First, assume simply that $G = G'$. Write $(\Xi_Q)^2$ for the map

$$(\Xi_Q)^2 : (\{0,1\}^E)^2 \to (\{0,1\}^E)^2,\ (\omega,\omega') \mapsto (\omega\Delta Q, \omega'\Delta Q).$$

We make two important observations:

- $(\Xi_Q)^2$ restricts to a involution on $\{\omega\Delta\omega' = \eta\}$,
- On $\{\omega\Delta\omega' = \eta\}$, the map $(\Xi_Q)^2$ does not modify the number $|\omega| + |\omega'|$ of edges.

Since the weight of each configuration (ω,ω') is a function of $|\omega|+|\omega'|$, the two observations imply that the measure

$$\mathbf{M}_{G,\beta} \times \mathbf{M}_{G',\beta}[\{\omega\Delta\omega' = \eta\} \cap (\,\cdot\,)]$$

is preserved by the involution $(\Xi_Q)^2$. The result follows since $(\Xi_Q)^2$ is is also a bijection from $\{\partial\omega = A\}$ to $\{\partial\omega = A\Delta\partial Q\}$.

If $G \neq G'$ then we simply view $(\Xi_Q)^2$ as an involution on $\{0,1\}^E \times \{0,1\}^{E'}$, and the rest of the proof works in the same way. □

To apply this switching lemma, it is useful to have some simple terminology for graphs and percolations.

Definition 5.2 (Percolation events)**.** Let $G = (V,E)$ denote a graph and $\omega \subset E$ a percolation configuration. Write

$$\{u \overset{\omega}{\leftrightarrow} v\}$$

for the event there is an open path from u to v (u and v may represent vertices or sets of vertices). For fixed $A \subset V$, we shall also write $\mathcal{E}_A$ for the set

$$\{\omega \subset E : |C \cap A| \text{ is even for any connected component } C \subset V \text{ of } (V,\omega)\}.$$

Exercise 5.3. Let G denote a graph and $x, y \in V$ distinct vertices. Prove that:

- If $A = \{x,y\}$, then $\{\omega \in \mathcal{E}_A\} = \{x \overset{\omega}{\leftrightarrow} y\}$,
- If $\omega \in \mathcal{E}_A$, then we may find a finite subset $\eta \subset \omega$ with $\partial\eta = A$,
- If G is a finite graph and $\partial\omega = A$, then $\omega \in \mathcal{E}_A$,
- For any $A \subset V$, the event $\{\omega \in \mathcal{E}_A\}$ is an increasing event of the percolation ω.

We can now prove some interesting bounds. The first bound is the pairing bound. This bound suggests that multi-correlation functions may be viewed as interacting paths which pair up the sources in our source set.

Theorem 5.4 (Pairing bound)**.** *Let $G = (V,E)$ denote a finite graph and $\beta \in [0,\infty)$. For any $x \in A \subset V$ we have*

$$\langle\sigma_A\rangle \le \sum_{y \in A\setminus\{x\}} \langle\sigma_x\sigma_y\rangle\langle\sigma_{A\setminus\{x,y\}}\rangle.$$

In particular, iterating yields

$$\langle\sigma_A\rangle \le \sum_{\pi} \prod_{xy\in\pi} \langle\sigma_x\sigma_y\rangle,$$

where π runs over all pairings *of A, that is, over all partitions of A into pairs.*

Proof. By the high-temperature expansion, we get

$$(2^{-|V|}Z)^2\langle\sigma_A\rangle = \mathbf{M}^2[\{\partial\omega = A,\, \partial\omega' = \emptyset\}].$$

But on this event we have $\partial(\omega\Delta\omega') = A$, which means that $\omega\Delta\omega'$ contains a path from x to at least one other vertex in A (see the exercise). In other words,

$$\mathbb{1}(\partial\omega = A,\, \partial\omega' = \emptyset) \le \sum\nolimits_{y\in A\setminus\{x\},\, \eta\in\{0,1\}^E,\, \{x\xleftrightarrow{\eta}y\}} \mathbb{1}(\omega\Delta\omega' = \eta,\, \partial\omega = A,\, \partial\omega' = \emptyset).$$

We now claim that

$$\begin{aligned}
&\mathbf{M}^2[\{\partial\omega = A,\, \partial\omega' = \emptyset\}] \\
&\le \sum\nolimits_{y\in A\setminus\{x\},\, \eta\in\{0,1\}^E,\, \{x\xleftrightarrow{\eta}y\}} \mathbf{M}^2[\{\omega\Delta\omega' = \eta,\, \partial\omega = A,\, \partial\omega' = \emptyset\}] \\
&= \sum\nolimits_{y\in A\setminus\{x\},\, \eta\in\{0,1\}^E,\, \{x\xleftrightarrow{\eta}y\}} \mathbf{M}^2[\{\omega\Delta\omega' = \eta,\, \partial\omega = A\setminus\{x,y\},\, \partial\omega' = \{x,y\}\}].
\end{aligned}$$

The inequality is the previous inequality, the equality is the switching lemma for the high-temperature expansion applied to each term (y,η), where Q is simply some path in η from x to y.

The final expression in the claim is equal to

$$\sum\nolimits_{y\in A\setminus\{x\}} \mathbf{M}^2[\{\partial\omega = A\setminus\{x,y\},\, \partial\omega' = \{x,y\}\}] = (2^{-|V|}Z)^2\langle\sigma_{A\setminus\{x,y\}}\rangle\langle\sigma_x\sigma_y\rangle,$$

which finishes the proof. □

Next, we focus on Simon's inequality. We already proved that there is exponential decay at high temperature. Simon's inequality says that *if* there is exponential decay of correlations, then it can be detected within a finite volume.

Recall that if $G = (V,E)$ is a graph and $\Lambda \subset V$ a subset, then $\partial\Lambda$ denotes the set of vertices in $V\setminus\Lambda$ which are adjacent to Λ. Write $\partial_\circ\Lambda$ for the *interior boundary*, that is, the set of vertices in Λ adjacent to $V\setminus\Lambda$. Write $\partial_e\Lambda$ for the *edge boundary*, that is, the set of edges connecting Λ and $V\setminus\Lambda$.

Theorem 5.5 (Simon's inequality)**.** *Let $G = (V,E)$ denote a finite graph, $\Lambda\subset V$ some domain, and let $\beta\in[0,\infty)$. Fix $x\in\Lambda$ and $y\in V\setminus\Lambda$.*

- *We have*
$$\langle\sigma_x\sigma_y\rangle_{G,\beta} \le \sum_{z\in\partial_\circ\Lambda} \langle\sigma_x\sigma_z\rangle^{\mathrm{f}}_{\Lambda,\beta}\langle\sigma_y\sigma_z\rangle_{G,\beta}.$$
- *We have*
$$\langle\sigma_x\sigma_y\rangle_{G,\beta} \le (\tanh\beta)\sum_{zz'\in\partial_e\Lambda} \langle\sigma_x\sigma_z\rangle^{\mathrm{f}}_{\Lambda,\beta}\langle\sigma_y\sigma_{z'}\rangle_{G,\beta}.$$

In fact, the converse inequalities are also true if we multiply the right hand sides by $1/|\partial_\circ\Lambda|$ and $1/|\partial_e\Lambda|$ respectively.

Proof. Focus on the first inequality. Let $\mathbf{M}' := \mathbf{M}_{G,\beta}\times\mathbf{M}_{\Lambda^{\mathrm{f}},\beta}$. Expanding the left hand side yields

$$\frac{Z_G Z_{\Lambda^{\mathrm{f}}}}{2^{|V|}2^{|\Lambda|}}\langle\sigma_x\sigma_y\rangle_{G,\beta} = \mathbf{M}'[\{\partial\omega = \{x,y\},\, \partial\omega' = \emptyset\}].$$

But on the event on the right, we have $\partial(\omega\Delta\omega') = \{x,y\}$, which means that $\omega\Delta\omega'$ contains a path which remains in Λ and which connects x to some vertex in $\partial_\circ\Lambda$. In other words,

$$\begin{aligned}
&\mathbb{1}(\partial\omega = \{x,y\},\, \partial\omega' = \emptyset) \\
&\qquad\le \sum\nolimits_{z\in\partial_\circ\Lambda,\, \eta\in\{0,1\}^E,\, x\xleftrightarrow{\eta\cap E(\Lambda^{\mathrm{f}})}z} \mathbb{1}(\omega\Delta\omega' = \eta,\, \partial\omega = \{x,y\},\, \partial\omega' = \emptyset). \qquad (10)
\end{aligned}$$

Using the switching lemma like for the pairing bound, we obtain

$$\begin{aligned}
&\mathbf{M}'[\{\partial\omega = \{x,y\},\, \partial\omega' = \emptyset\}] \\
&\leq \sum\nolimits_{z\in\partial_\circ\Lambda,\, \eta\in\{0,1\}^E,\, x\xleftrightarrow{\eta\cap E(\Lambda^{\mathrm{f}})} z} \mathbf{M}'[\{\omega\Delta\omega' = \eta,\, \partial\omega = \{x,y\},\, \partial\omega' = \emptyset\}] \\
&= \sum\nolimits_{z\in\partial_\circ\Lambda,\, \eta\in\{0,1\}^E,\, x\xleftrightarrow{\eta\cap E(\Lambda^{\mathrm{f}})} z} \mathbf{M}'[\{\omega\Delta\omega' = \eta,\, \partial\omega = \{y,z\},\, \partial\omega' = \{x,z\}\}] \\
&= \sum\nolimits_{z\in\partial_\circ\Lambda} \mathbf{M}'[\{\partial\omega = \{y,z\},\, \partial\omega' = \{x,z\}\}] \\
&= \frac{Z_G Z_{\Lambda^{\mathrm{f}}}}{2^{|V|}2^{|\Lambda|}} \sum\nolimits_{z\in\partial_\circ\Lambda} \langle\sigma_y\sigma_z\rangle_{G,\beta}\langle\sigma_x\sigma_z\rangle_{\Lambda^{\mathrm{f}},\beta}.
\end{aligned}$$

We use the switching lemma for the first equality; we choose Q to be a path from x to z in $\eta\cap E(\Lambda^{\mathrm{f}})$ to get an equality for each term. This proves the first inequality in the statement of the theorem.

The converse inequality is true simply because we may reverse the inequality in Equation (10) if we multiply the right hand side by $1/|\partial_\circ\Lambda|$.

For the second inequality we only give a proof outline. It is obtained in a similar fashion, noticing that if $\omega\Delta\omega'$ connects x and y, then there must be some edge $zz' \in \partial_e\Lambda$ such that $\omega\Delta\omega'$ contains a self-avoiding path from x to y, which passes through zz' and which does not leave Λ before using this edge. The switching lemma may then be applied in a similar fashion. By switching the edge zz', which only appears in the bigger graph, we make the extra factor $\tanh\beta$ appear. □

Corollary 5.6 (Finite size criterion)**.** *Consider the Ising model on $\mathbb{Z}^d$ at inverse temperature β. For any finite $\Lambda \subset \mathbb{Z}^d$ containing $0 \in \mathbb{Z}^d$, we define*

$$\varphi_\beta(\Lambda) := (\tanh\beta) \sum_{zz'\in\partial_e\Lambda} \langle\sigma_0\sigma_z\rangle^{\mathrm{f}}_{\Lambda,\beta}.$$

If $\varphi_\beta(\Lambda) < 1$ for some Λ, then there exists a constant $c = c_{d,\beta} > 0$ such that

$$\langle\sigma_x\rangle^+_{\Delta,\beta} \leq \tfrac{1}{c} e^{-c\,\mathrm{Distance}(x,\Delta^c)}$$

for any $x \in \mathbb{Z}^d$ and any domain $\Delta \subset \mathbb{Z}^d$.

Proof. Fix Λ with $\varphi_\beta(\Lambda) < 1$, and fix Δ. For any $x \in \mathbb{Z}^d$, define

$$a(x) := \left\lfloor \frac{\mathrm{Distance}(x,\Delta^c)}{\mathrm{Diameter}(\Lambda)+1} \right\rfloor.$$

It suffices to prove that for any $x \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$a(x) \geq n \qquad \Longrightarrow \qquad \langle\sigma_x\rangle^+_{\Delta,\beta} \leq \varphi_\beta(\Lambda)^n.$$

The induction basis $n = 0$ is obvious; we limit this proof to the induction step.

Using Simon's inequality, we get

$$\langle\sigma_x\rangle^+_{\Delta,\beta} = \langle\sigma_x\sigma_{\mathfrak{g}}\rangle_{\Delta^{\mathfrak{g}},\beta} \leq (\tanh\beta) \sum_{zz'\in\partial_e(\Lambda+x)} \langle\sigma_x\sigma_z\rangle^{\mathrm{f}}_{\Lambda+x,\beta}\langle\sigma_{\mathfrak{g}}\sigma_{z'}\rangle_{\Delta^{\mathfrak{g}},\beta}.$$

By induction, we may bound $\langle\sigma_{\mathfrak{g}}\sigma_{z'}\rangle_{\Delta^{\mathfrak{g}},\beta} = \langle\sigma_{z'}\rangle^+_{\Delta,\beta} \leq \varphi_\beta(\Lambda)^{a(x)-1}$, so that the previous line is upper bounded by

$$\langle\sigma_x\rangle^+_{\Delta,\beta} \leq \left((\tanh\beta)\sum\nolimits_{zz'\in\partial_e(\Lambda+x)}\langle\sigma_x\sigma_z\rangle^{\mathrm{f}}_{\Lambda+x,\beta}\right)\varphi_\beta(\Lambda)^{a(x)-1} = \varphi_\beta(\Lambda)^{a(x)}.$$

This is the desired bound. □

Remark 5.7. At this point, we have proved that

$$\inf_{\Lambda} \varphi_\beta(\Lambda) < 1 \qquad \Longrightarrow \qquad \text{exponential decay of correlations},$$

but not the converse implication.

It is easy to see that the converse implication is also true. Indeed, the converse version of the Simon inequality yields

$$\langle \sigma_0 \rangle^+_{\Lambda,\beta} \geq \frac{\tanh \beta}{|\partial_e \Lambda|} \sum_{zz' \in \partial_e \Lambda} \langle \sigma_0 \sigma_z \rangle^{\mathrm{f}}_{\Lambda,\beta} = \frac{\varphi_\beta(S)}{|\partial_e \Lambda|}.$$

If the left hand side decays exponentially fast in the distance from 0 to the boundary, then one may clearly choose Λ so large that $\langle \sigma_0 \rangle^+_{\Lambda,\beta} \leq 1/|\partial_e \Lambda|$, in which case $\varphi_\beta(\Lambda) < 1$.

6. The random-currents expansion

Next, we introduce random currents. Random currents are a refinement of the high-temperature expansion. Let us make precise what we mean, before turning to the details. A significant drawback of Lemma 5.1 is the fact that we can switch subsets of the *symmetric difference* $\omega \Delta \omega'$, but not of the *union* $\omega \cup \omega'$. Random currents carry slightly more information than high-temperature expansion, which enables this "upgrade". Switching over the union of the two percolations is important for several new correlation inequalities, and eventually the proof of continuity of the phase transition.

Definition 6.1 (Currents)**.** Let $G = (V, E)$ denote a graph. A *current* is a map $\mathbf{n} : E \to \mathbb{Z}_{\geq 0}$. We think of $(V, \mathbf{n})$ as a multigraph, where for each edge $uv \in E$ we have $\mathbf{n}_{uv}$ multi-edges between u and v. The set of *sources* $\partial \mathbf{n} \subset V$ of a current $\mathbf{n}$ is defined as the set of vertices $u \in V$ with an odd degree in the multigraph. We let $\hat{\mathbf{n}} := (\mathbf{n} \wedge 1) \in \{0,1\}^E$ denote the associated percolation, which simply contains the edges carrying at least one current.

If G is finite and $\beta \in [0, \infty)$, then the *weight* of a current is defined as

$$w(\mathbf{n}) := w_{G,\beta}(\mathbf{n}) := \prod_{xy \in E} \frac{\beta^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}.$$

The *random-currents measure* is the measure $\mathbb{M}_{G,\beta}$ on $(\mathbb{Z}_{\geq 0})^E$ defined by

$$\mathbb{M}[\mathbf{n}] := \mathbb{M}_{G,\beta}[\mathbf{n}] := w_{G,\beta}(\mathbf{n}).$$

Remark 6.2. Notice that $e^{-\beta|E|}\mathbb{M}_{G,\beta}$ is a probability measure in which $(\mathbf{n}_{xy})_{xy \in E}$ is a family of independent random variables with distribution Poisson(β).

The random-currents measure is a richer object than the high-temperature expansion. To see this, consider a single fixed edge $xy \in E$. Then we have the following correspondence between the high-temperature weights and the random-currents weights:

$$\cosh \beta = \sum_{\mathbf{n} \in 2\mathbb{Z}_{\geq 0}} \frac{\beta^{\mathbf{n}}}{\mathbf{n}!}; \qquad \sinh \beta = \sum_{\mathbf{n} \in 2\mathbb{Z}_{\geq 0}+1} \frac{\beta^{\mathbf{n}}}{\mathbf{n}!}.$$

We may thus interpret the relation between $e^{-\beta|E|}\mathbf{M}$ and $e^{-\beta|E|}\mathbb{M}$ as follows:

- $\mathbb{M}$ is a nonnormalised family of independent Poisson(β)-variables $(\mathbf{n}_{xy})_{xy \in E}$,
- $\mathbf{M}$ is obtained from $\mathbb{M}$ by writing ω_{xy} for the *parity* of $\mathbf{n}_{xy}$,
- In particular, $\partial \mathbf{n} \sim \mathbb{M}$ and $\partial \omega \sim \mathbf{M}$ have the same distribution,
- Moreover, the percolation $\hat{\mathbf{n}}$ may be viewed as

$$\hat{\mathbf{n}} = \omega \cup \{xy \in E : \mathbf{n}_{xy} \in \{2, 4, 8, \ldots\}\}.$$

In particular, the following distributions are the same:

$$\hat{\mathbf{n}} \text{ in the measure } \mathbb{M} \qquad \text{and} \qquad \omega \cup \eta \text{ in the measure } \mathbf{M} \times \mathbb{P}_p, \tag{11}$$

where $\eta \sim \mathbb{P}_p$ is an independent bond percolation on G with parameter

$$p = \frac{\cosh \beta - 1}{\cosh \beta} = \frac{\sum_{\mathbf{n} \in 2\mathbb{Z}_{\geq 1}} \frac{\beta^{\mathbf{n}}}{\mathbf{n}!}}{\sum_{\mathbf{n} \in 2\mathbb{Z}_{\geq 0}} \frac{\beta^{\mathbf{n}}}{\mathbf{n}!}}.$$

This is called *sprinkling*; $\hat{\mathbf{n}}$ is a *sprinkled* version of ω.

The last observation arises from the simple fact that edges carring an even current still have a probability p of carrying a strictly positive even current.

Theorem 4.2 translates to random currents as follows.

Theorem 6.3 (Current representation of correlation functions)**.** *Consider the Ising model on a finite graph G at inverse temperature β. Let $A \subset V$ be a subset of vertices. Then*

$$Z\langle \sigma_A \rangle = 2^{|V|} \sum_{\mathbf{n} \colon \partial \mathbf{n} = A} w(\mathbf{n}) = 2^{|V|} \mathbb{M}[\{\partial \mathbf{n} = A\}].$$

In particular, the partition function is given by

$$Z = 2^{|V|} \sum_{\mathbf{n} \colon \partial \mathbf{n} = \emptyset} w(\mathbf{n}) = 2^{|V|} \mathbb{M}[\{\partial \mathbf{n} = \emptyset\}].$$

We will now explain how random currents allow us to "upgrade" from switching over subsets of $\omega \Delta \omega'$ to subsets of $\hat{\mathbf{n}} \cup \hat{\mathbf{m}}$.

Exercise 6.4 (Poisson switching)**.** Suppose that we record cars traversing a bridge on a road. Blue cars pass according to a Poisson point process with rate 1 (per second). Let X_B denote the number of blue cars that pass after recording β seconds. Can we easily prove, without a calculation, that

$$\mathbb{P}[\{X_B \text{ is even}\}] \geq \mathbb{P}[\{X_B \text{ is odd}\}]?$$

Suppose that there are also yellow cars, which arrive according to an independent Poisson process with the same rate. Let X_Y denote the number of yellow cars that passed.

By considering the colour of the *last* car that passed the bridge, prove that:

(1) $\mathbb{P}[\{X_B \text{ is even}\} | \{X_B + X_Y = N\}] = \frac{1}{2}$ whenever $N > 0$,
(2) $\mathbb{P}[\{X_B \text{ is even}\} | \{X_B + X_Y = N\}] = 0$ whenever $N = 0$,
(3) $\mathbb{P}[\{X_B \text{ is even}\}] - \mathbb{P}[\{X_B \text{ is odd}\}] = \mathbb{P}[\{X_B + X_Y = 0\}] \geq 0$.

Lemma 6.5 (Switching lemma)**.** *Let G and G' denote two finite graphs and fix $\beta \in [0, \infty)$. If $A \subset V$, $\mathbf{s} \in (\mathbb{Z}_{\geq 0})^{E \cup E'}$, and $Q \subset \hat{\mathbf{s}} \cap E \cap E'$, then*

$$\mathbb{M}_{G,\beta} \times \mathbb{M}_{G',\beta}[\{\mathbf{n}+\mathbf{m} = \mathbf{s}\} \cap \{\partial \mathbf{n} = A\}] = \mathbb{M}_{G,\beta} \times \mathbb{M}_{G',\beta}[\{\mathbf{n}+\mathbf{m} = \mathbf{s}\} \cap \{\partial \mathbf{n} = A \Delta \partial Q\}].$$

Notice also that if $\mathbf{n} + \mathbf{m} = \mathbf{s}$, then $\partial \mathbf{s} = (\partial \mathbf{n}) \Delta (\partial \mathbf{m})$.

Proof. By induction, we may simply suppose that $|Q| = 1$, say $Q = \{xy\}$. Introduce the probability measure

$$\mathbb{P} :\propto \mathbb{M}_{G,\beta} \times \mathbb{M}_{G',\beta}[\{\mathbf{n} + \mathbf{m} = \mathbf{s}, \, \mathbf{n}|_{Q^c} = \mathbf{a}, \, \mathbf{m}|_{Q^c} = \mathbf{b}\} \cap (\,\cdot\,)];$$

for the claimed identity it suffices to derive, for fixed $\mathbf{a}$ and $\mathbf{b}$, the stronger equality

$$\mathbb{P}[\{\partial \mathbf{n} = A\}] = \mathbb{P}[\{\partial \mathbf{n} = A \Delta \partial Q\}].$$

But the only randomness that is left in the measure $\mathbb{P}$, are the values of $\mathbf{n}_{xy}$ and $\mathbf{m}_{xy}$, which are independent Poisson random variables conditioned to sum to $\mathbf{s}_{xy} > 0$. By the previous exercise, the parity of $\mathbf{n}_{xy}$ has the distribution of a fair coin flip, which proves the previous identity. □

An important corollary of the switching lemma is the *second Griffiths inequality.*

Lemma 6.6 (Second Griffiths inequality)**.** *Consider the Ising model on a finite graph G at inverse temperature β. Then for any $A, B \subset V$, we have $\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \geq 0$.*

The second Griffiths inequality is more subtle than the first, as it bounds a *difference* of correlation functions. This is typical for the switching lemma.

Proof. Claim that

$$\begin{aligned}
&\mathbb{M}^2[\{\partial\mathbf{n}=A\}\cap\{\partial\mathbf{m}=B\}]\\
&\quad=\mathbb{M}^2[\{\widehat{\mathbf{n}+\mathbf{m}}\in\mathcal{E}_B\}\cap\{\partial\mathbf{n}=A\}\cap\{\partial\mathbf{m}=B\}]\\
&\quad\overset{\text{switch}}{=}\mathbb{M}^2[\{\widehat{\mathbf{n}+\mathbf{m}}\in\mathcal{E}_B\}\cap\{\partial\mathbf{n}=A\Delta B\}\cap\{\partial\mathbf{m}=\emptyset\}]\\
&\quad\leq\mathbb{M}^2[\{\partial\mathbf{n}=A\Delta B\}\cap\{\partial\mathbf{m}=\emptyset\}].
\end{aligned}$$

For the first equality, we simply observe that $\{\partial\mathbf{m}=B\}\subset\{\widehat{\mathbf{n}+\mathbf{m}}\in\mathcal{E}_B\}$ (see Exercice 5.3). The switch is the switching lemma, where we switch over a subset of $\widehat{\mathbf{n}+\mathbf{m}}$ having B as source set. The inequality is inclusion of events.

By the random currents expansion of correlation functions (Theorem 6.3), the left- and rightmost expressions are given by

$$Z^2\langle\sigma_A\rangle\langle\sigma_B\rangle/4^{|V|}\leq Z^2\langle\sigma_A\sigma_B\rangle\langle\sigma_\emptyset\rangle/4^{|V|}.$$

Since $\langle\sigma_\emptyset\rangle=1$, this is the desired inequality. ☐

Exercise 6.7 (The two-point function as a metric)**.** Consider the Ising model on a finite graph G at inverse temperature $\beta>0$. Prove that $V\times V\to[0,\infty]$, $(u,v)\mapsto-\log\langle\sigma_u\sigma_v\rangle_{G,\beta}$ defines a metric on V. Use directly the switching lemma, and not the second Griffiths inequality. What does the percolation event $\mathcal{E}_S$ look like in this case?

Exercise 6.8 (Correlation functions in terms of sourceless currents)**.** Consider the Ising model on a finite graph G at inverse temperature β. Prove that for any $A\subset V$,

$$Z^2\langle\sigma_A\rangle^2/4^{|V|}=\mathbb{M}^2[\{\partial\mathbf{n}=\partial\mathbf{m}=\emptyset\}\cap\{\widehat{\mathbf{n}+\mathbf{m}}\in\mathcal{E}_A\}].$$

Observe that we can now express all correlation functions in terms of a single fixed probability measure $4^{|V|}\mathbb{M}^2[\{\partial\mathbf{n}=\partial\mathbf{m}=\emptyset\}\cap(\,\cdot\,)]/Z^2$ on sourceless random currents.

7. Monotonicity via the second Griffiths inequality

Theorem 7.1 (Monotonicity in the temperature)**.** *Let G denote a finite graph and $A\subset V$ a finite set. Then the function $\beta\mapsto\langle\sigma_A\rangle_{G,\beta}$ is non-decreasing.*

Proof. We want to prove that

$$\frac{\partial}{\partial\beta}\langle\sigma_A\rangle_{G,\beta}=\frac{\partial}{\partial\beta}\left(\frac{\sum_\sigma\sigma_A\prod_{uv}e^{\beta\sigma_u\sigma_v}}{\sum_\sigma\prod_{uv}e^{\beta\sigma_u\sigma_v}}\right)\geq 0.$$

Since we are differentiating a fraction, it suffices to show that the numerator grows at a faster rate than the denominator, that is,

$$\frac{\frac{\partial}{\partial\beta}\sum_\sigma\sigma_A\prod_{uv}e^{\beta\sigma_u\sigma_v}}{Z\langle\sigma_A\rangle}\geq\frac{\frac{\partial}{\partial\beta}\sum_\sigma\prod_{uv}e^{\beta\sigma_u\sigma_v}}{Z}.$$

We perform the differential and then multiply each side by $\langle\sigma_A\rangle$, to see that this inequality is equivalent to

$$\sum_{xy}\frac{\sum_\sigma\sigma_x\sigma_y\sigma_A\prod_{uv}e^{\beta\sigma_u\sigma_v}}{Z}\geq\langle\sigma_A\rangle\sum_{xy}\frac{\sum_\sigma\sigma_x\sigma_y\prod_{uv}e^{\beta\sigma_u\sigma_v}}{Z}.$$

Each fraction may now be reinterpreted as a correlation function, so that the previous inequality is equivalent to

$$\sum_{xy}\langle\sigma_x\sigma_y\sigma_A\rangle\geq\langle\sigma_A\rangle\sum_{xy}\langle\sigma_x\sigma_y\rangle.$$

But this is just the second Griffiths inequality. □

Exercise 7.2 (Regularity properties of the correlation functions in β)**.** Prove that the function $[0, \infty) \to \mathbb{R}$, $\beta \mapsto \langle \sigma_A \rangle_{G,\beta}$ in the above context is an analytic function.

Next, we want to prove monotonicity in domains. We first challenge the reader to prove the following exercise.

Exercise 7.3 (Conditioning on equality increases the correlation functions)**.** Consider the Ising model on a finite graph G at inverse temperature β, and fix some subset $A \subset V$.

- Prove that for any two distinct vertices $u, v \in V$, we have
$$\mathbb{E}_{G,\beta}[\sigma_A | \{\sigma_u = \sigma_v\}] \geq \mathbb{E}_{G,\beta}[\sigma_A] = \langle \sigma_A \rangle_{G,\beta}.$$
- Prove for any $X \subset Y \subset V$, we have
$$\mathbb{E}_{G,\beta}[\sigma_A | \{\sigma \text{ is constant on } X\}] \leq \mathbb{E}_{G,\beta}[\sigma_A | \{\sigma \text{ is constant on } Y\}].$$

Lemma 7.4 (Monotonicity in domains)**.** *Consider the Ising model on a locally finite graph $G = (V, E)$ at inverse temperature β. Consider two finite domains $\Lambda \subset \Lambda' \subset V$ and a subset $A \subset \Lambda$.*

- ***Free boundary.*** *We have $\langle \sigma_A \rangle^{\mathrm{f}}_{\Lambda,\beta} \leq \langle \sigma_A \rangle^{\mathrm{f}}_{\Lambda',\beta}$.*
- ***Wired boundary.*** *We have $\langle \sigma_A \rangle^{+}_{\Lambda,\beta} \geq \langle \sigma_A \rangle^{+}_{\Lambda',\beta}$.*

Proof for $\langle \cdot \rangle^{\mathrm{f}}$. We first prove the following claim: if G' and G'' are finite graphs on the same vertex set, and such that $E(G'') = E(G') \cup \{xy\}$, then
$$\langle \sigma_A \rangle_{G',\beta} \leq \langle \sigma_A \rangle_{G'',\beta}$$
for any $A \subset V(G')$. To prove the claim, we simply expand
$$\langle \sigma_A \rangle_{G'',\beta} = \frac{\langle e^{\beta \sigma_x \sigma_y} \sigma_A \rangle_{G'}}{\langle e^{\beta \sigma_x \sigma_y} \rangle_{G'}}.$$
Thus, we want to show that
$$\langle e^{\beta \sigma_x \sigma_y} \sigma_A \rangle_{G'} \geq \langle e^{\beta \sigma_x \sigma_y} \rangle_{G'} \langle \sigma_A \rangle_{G'}.$$
This follows from the second Griffiths inequality. We have now proved the claim.

Recall the definition of the finite graph Λ^{f}. Let $\tilde{\Lambda}^{\mathrm{f}} := ((\Lambda')^{\mathrm{f}}, E(\Lambda^{\mathrm{f}}))$; this is just the graph Λ^{f} supplemented with some isolated vertices $\Lambda' \setminus \Lambda$. The law of σ in $\langle \cdot \rangle_{\tilde{\Lambda}^{\mathrm{f}}}$ is just given by $\langle \cdot \rangle_{\Lambda^{\mathrm{f}}}$, with independent fair coin flips for the isolated vertices in $\Lambda' \setminus \Lambda$. Thus, it suffices to prove that
$$\langle \sigma_A \rangle^{\mathrm{f}}_{\Lambda} = \langle \sigma_A \rangle_{\tilde{\Lambda}^{\mathrm{f}}} \leq \langle \sigma_A \rangle_{(\Lambda')^{\mathrm{f}}} = \langle \sigma_A \rangle^{\mathrm{f}}_{\Lambda'}.$$
This follows from the claim. □

Proof for $\langle \cdot \rangle^{+}$. Without loss of generality, $\Lambda' \setminus \Lambda = \{u\}$ for some vertex $u \in V$. We make all calculations in the graph $(\Lambda')^{\mathfrak{g}}$ with the ghost vertex: we get
$$\langle \sigma_A \rangle^{+}_{\Lambda'} = \mathbb{E}_{(\Lambda')^{\mathfrak{g}}}[\sigma_A | \{\sigma_{\mathfrak{g}} = +\}]; \qquad \langle \sigma_A \rangle^{+}_{\Lambda} = \mathbb{E}_{(\Lambda')^{\mathfrak{g}}}[\sigma_A | \{\sigma_{\mathfrak{g}} = +\} \cap \{\sigma_u = \sigma_{\mathfrak{g}}\}].$$

Assume first that $|A|$ is even for now. Then
$$\langle \sigma_A \rangle^{+}_{\Lambda} = \mathbb{E}_{(\Lambda')^{\mathfrak{g}}}[\sigma_A | \{\sigma_u = \sigma_{\mathfrak{g}}\}] \geq \mathbb{E}_{(\Lambda')^{\mathfrak{g}}}[\sigma_A] = \langle \sigma_A \rangle^{+}_{\Lambda'},$$
due to Exercise 7.3.

If $|A|$ is odd, we just need to replace the set A by $A' := A \cup \{\mathfrak{g}\}$. More precisely,
$$\langle \sigma_A \rangle^{+}_{\Lambda} = \mathbb{E}_{(\Lambda')^{\mathfrak{g}}}[\sigma_{A'} | \{\sigma_u = \sigma_{\mathfrak{g}}\}] \geq \mathbb{E}_{(\Lambda')^{\mathfrak{g}}}[\sigma_{A'}] = \langle \sigma_A \rangle^{+}_{\Lambda'},$$
where the inequality uses the same exercise.

Those are the desired inequalities. □

8. Infinite-volume limits

Definition 8.1 (Infinite-volume limit)**.** Let $G = (V, E)$ denote a locally finite graph. Write

$$\lim_{\Lambda \uparrow V} f(\Lambda) \qquad \text{for} \qquad \lim_{n\to\infty} f(\Lambda_n),$$

where $(\Lambda_n)_n$ is any increasing sequence of finite domains with $\cup_n \Lambda_n = V$. This notation makes sense only when the limit is independent of the precise choice of the sequence $(\Lambda_n)_n$, and is called the *thermodynamical limit* or *infinite-volume limit*.

Let $(\Omega, \mathcal{F})$ denote the measurable space $\Omega := \{\pm 1\}^V$ endowed with the product σ-algebra. For a domain Λ, we write $\mathcal{F}_\Lambda$ for the σ-algebra generated by spins in Λ. An observable $X : \Omega \to \mathbb{C}$ is called *local* if it is measurable with respect to $\mathcal{F}_\Lambda$ for some domain Λ.

Let $\mathcal{P}(\Omega, \mathcal{F})$ denote the set of all probability measures on this measurable space. We endow this set with the *local convergence topology*, which is defined as the topology making the map

$$\mathcal{P}(\Omega, \mathcal{F}) \to \mathbb{C},\ \mu \mapsto \mu[X]$$

continuous for any local observable X.

Remark. This topology is sometimes known under different names in the literature, such as the *weak topology*. The name *local convergence topology* is quite explicit: if the statistics of the measures within a fixed domain Λ converge, then we have local convergence.

Exercise 8.2. Prove that $\mathcal{P}(\Omega, \mathcal{F})$ is a compact space in this topology.

Theorem 8.3 (Existence of the thermodynamical limit)**.** *Consider the Ising model on a locally finite graph G at inverse temperature β. Then there exists unique probability measures $\langle\cdot\rangle^{\mathrm{f}}_{G,\beta}, \langle\cdot\rangle^{+}_{G,\beta} \in \mathcal{P}(\Omega, \mathcal{F})$ such that*

$$\lim_{\Lambda\uparrow V} \langle X\rangle^*_{\Lambda,\beta} = \langle X\rangle^*_{G,\beta}$$

for $* \in \{\mathrm{f}, +\}$ and for any local observable $X : \Omega \to \mathbb{R}$. In other words,

$$\lim_{\Lambda\uparrow V} \langle\cdot\rangle^*_{\Lambda,\beta} =: \langle\cdot\rangle^*_{G,\beta}.$$

The measures $\langle\cdot\rangle^*_{G,\beta}$ are called the thermodynamical limits *or* infinite-volume limits*.*

Proof. Any local observable may be written as a finite linear combination of observables of the form σ_A where A is a finite subset of V. The theorem then follows by compactness and Lemma 7.4. □

Definition 8.4 (Shift operator)**.** Let $G = \mathbb{Z}^d$. Consider a measure $\langle\cdot\rangle \in \mathcal{P}(\Omega, \mathcal{F})$.

- For any $u \in \mathbb{Z}^d$, we define the *shift operator* $\tau_u : \Omega \to \Omega$ by

$$(\tau_u \sigma)_x = \sigma_{x-u}.$$

  An event A is *shift-invariant* if $\tau_u A := \{\tau_u\sigma : \sigma \in A\}$ for any $u \in \mathbb{Z}^d$.
- The measure is called *shift-invariant* if

$$\langle X \circ \tau_u\rangle = \langle X\rangle$$

  for any vertex $u \in \mathbb{Z}^d$ and for any bounded local observable X.

Theorem 8.5 (Shift-invariance)**.** *Let $G = \mathbb{Z}^d$. The measures $\langle\cdot\rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta}$ and $\langle\cdot\rangle^{+}_{\mathbb{Z}^d,\beta}$ are shift-invariant.*

Proof. The desired symmetry simply follows from the symmetry in the definitions. □

Exercise 8.6 (Continuity properties in β)**.** Consider the Ising model on a locally finite graph $G = (V, E)$. Fix $A \subset V$ finite.

- The function $\beta \mapsto \langle\sigma_A\rangle^*_{G,\beta}$ is non-decreasing for $* \in \{\mathrm{f}, +\}$.

- The function $\beta \mapsto \langle \sigma_A \rangle^{\mathrm{f}}_{G,\beta}$ is left continuous.
- The function $\beta \mapsto \langle \sigma_A \rangle^{+}_{G,\beta}$ is right continuous.

Hint. Argue that $\beta \mapsto \langle \sigma_A \rangle^{\mathrm{f}}_{G,\beta}$ is a limit of a non-decreasing sequence of non-decreasing functions.

Definition 8.7 (Magnetisation and critical temperature)**.** Let G be a vertex-transitive locally finite graph. The non-decreasing right-continuous function

$$m = m_G : [0, \infty) \to \mathbb{R},\ \beta \mapsto \langle \sigma_u \rangle^{+}_{G,\beta}$$

is called the *magnetisation* (u is an arbitrary reference vertex).

The *critical (inverse) temperature* is defined via

$$\beta_c := \beta_c(G) := \inf\{\beta \in [0, \infty) : m(\beta) > 0\}.$$

We have already proved that $\beta_c \in (0, \infty)$ for $G = \mathbb{Z}^d$ in dimension $d \geq 2$, and that $\beta_c = \infty$ for $G = \mathbb{Z}$.

It is easy to derive the following result when the $m(\beta) = 0$.

Theorem 8.8 ($+$ and $-$ boundary conditions coincide when the magnetisation vanishes)**.** *Let G denote a connected locally finite graph, endowed with some reference vertex u. Then*

$$\langle \cdot \rangle^{+}_{G,\beta} = \langle \cdot \rangle^{-}_{G,\beta} \qquad \iff \qquad m_G(\beta) = 0.$$

Proof. Notice that $\langle \cdot \rangle^{+}_{G,\beta}$ and $\langle \cdot \rangle^{-}_{G,\beta}$ are related by a global spin flip (the pushforward map corresponding to $\sigma \mapsto -\sigma$). Therefore all of the following are equivalent:

- $\langle \cdot \rangle^{+}_{G,\beta} = \langle \cdot \rangle^{-}_{G,\beta}$,
- $\langle \cdot \rangle^{+}_{G,\beta}$ is invariant under the map $\sigma \mapsto -\sigma$,
- $\langle \sigma_A \rangle^{+}_{G,\beta} = 0$ whenever $A \subset V$ has odd cardinal.

The implication " $\implies$ " is now obvious, and we focus on " $\impliedby$ ". Suppose that $m(\beta) = 0$, that is, $\langle \sigma_x \rangle^+ = 0$ for any $x \in V$. Fix $A \subset V$ with $|A|$ odd. It suffices to prove that $\langle \sigma_A \rangle^+ = 0$. But we simply observe that

$$\langle \sigma_A \rangle^+ = \lim_{\Lambda \uparrow V} \langle \sigma_A \sigma_{\mathfrak{g}} \rangle_{\Lambda^{\mathfrak{g}}} \leq \lim_{\Lambda \uparrow V} \sum_{x \in A} \langle \sigma_{A \setminus \{x\}} \rangle_{\Lambda^{\mathfrak{g}}} \langle \sigma_x \sigma_{\mathfrak{g}} \rangle_{\Lambda^{\mathfrak{g}}} \leq \sum_{x \in A} \langle \sigma_x \rangle^{+}_{\Lambda} = |A| \cdot m(\beta) = 0.$$

The first inequality is the pairing bound (Theorem 5.4). ☐

Add: Messager–Miracle-Solé

9. Continuity of the magnetisation in dimension $d \geq 3$

The objective of this section is to prove the following deep theorem.

Theorem 9.1 (Continuity in dimension $d \geq 3$)**.** *Consider the Ising model on the square lattice graph $G = \mathbb{Z}^d$ in dimension $d \in \mathbb{Z}_{\geq 3}$. Then $m(\beta_c) = 0$, that is, the magnetisation is continuous at $\beta = \beta_c$. Moreover, for $\beta \in [0, \beta_c]$, we have $\langle \cdot \rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta} = \langle \cdot \rangle^{+}_{\mathbb{Z}^d,\beta}$.*

The proof presented here works only in dimension $d \geq 3$, because we use an essential input called the *infrared bound*. The infrared bound is a classical tool in the analysis of spin systems. Unfortunately, its proof is beyond the scope of these lecture notes.

Theorem 9.2 (Infrared bound)**.** *Consider the Ising model on the square lattice graph $\mathbb{Z}^d$ for fixed $d \in \mathbb{Z}_{\geq 1}$. Then there exists a constant $C \in \mathbb{R}_{\geq 0}$ such that*

$$\langle \sigma_x \sigma_y \rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta} \leq C \frac{1}{\|y - x\|_2^{d-2}}$$

for any $\beta \in [0, \beta_c]$. In particular, if $d \geq 3$, then

$$\lim_{\|y-x\|_2 \to \infty} \langle \sigma_x \sigma_y \rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta} = 0. \tag{12}$$

Thus, we aim to prove that the infrared bound implies continuity (Theorem 9.1). In fact, once we proved that $m(\beta_c)=0$, it is quite easy to deduce the last part of Theorem 9.1. We focus on proving that $m(\beta_c)=0$ for now. Globally, the proof consists of the following two lemmas.

Lemma 9.3 (Continuity, Step 1). *Consider the Ising model on $\mathbb{Z}^d$ for $d\in\mathbb{Z}_{\geq 1}$ at $\beta\in[0,\infty)$. Then*

$$m(\beta)^2=\inf_{x,y}\langle\sigma_x\sigma_y\rangle^+_{\mathbb{Z}^d,\beta}.$$

Lemma 9.4 (Continuity, Step 2). *Consider the Ising model on $\mathbb{Z}^d$ for $d\in\mathbb{Z}_{\geq 3}$ at $\beta\in[0,\beta_c]$. Then*

$$\langle\sigma_x\sigma_y\rangle^+_{\mathbb{Z}^d,\beta}=\langle\sigma_x\sigma_y\rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta}.$$

for any $x,y\in\mathbb{Z}^d$. More generally, for any subset $A\subset\mathbb{Z}^d$ of even cardinal, we have

$$\langle\sigma_A\rangle^+_{\mathbb{Z}^d,\beta}=\langle\sigma_A\rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta}.$$

Proof that the two steps imply Theorem 9.1. Assume the two lemmas. The infrared bound then tells us that at β_c the two-point function tends to zero with the distance (for both free and wired boundary conditions, due to Step 2). Step 1 then tells us that the magnetisation vanishes. All odd correlation functions then vanish for $\langle\,\cdot\,\rangle^+$ by Theorem 8.8. The even correlation functions match those of $\langle\,\cdot\,\rangle^{\mathrm{f}}$ by the last part of Step 2. □

Step 2 is the hard step; we start with a proof of Step 1.

Proof of Continuity, Step 1. Fix $x,y\in\mathbb{Z}^d$. For any finite domain $\Lambda\ni x,y$, we have

$$\langle\sigma_x\rangle^+_{\Lambda,\beta}\langle\sigma_y\rangle^+_{\Lambda,\beta}=\langle\sigma_x\sigma_{\mathfrak{g}}\rangle_{\Lambda^{\mathfrak{g}},\beta}\langle\sigma_y\sigma_{\mathfrak{g}}\rangle_{\Lambda^{\mathfrak{g}},\beta}\leq\langle\sigma_x\sigma_y\rangle^+_{\Lambda,\beta}$$

by the second Griffiths inequality. Sending $\Lambda\uparrow\mathbb{Z}^d$ yields

$$m(\beta)^2\leq\langle\sigma_x\sigma_y\rangle^+_{\mathbb{Z}^d,\beta}.$$

It suffices to prove the other bound.

Fix $x=0\in\mathbb{Z}^d$, and let $\Lambda\ni x$ denote a large finite domain. For any $y\in\mathbb{Z}^d$, let $\Lambda_y:=\Lambda\cup(\Lambda+y)$. Then

$$\limsup_{\|y\|_2\to\infty}\langle\sigma_x\sigma_y\rangle^+_{\mathbb{Z}^d,\beta}\leq\limsup_{\|y\|_2\to\infty}\langle\sigma_x\sigma_y\rangle^+_{\Lambda_y,\beta}=(\langle\sigma_x\rangle^+_{\Lambda,\beta})^2\to_{\Lambda\uparrow\mathbb{Z}^d}m(\beta)^2.$$

The equality holds true because for $\|y\|_2$ sufficiently large, Λ and $\Lambda+y$ are no longer adjancent, and therefore the restrictions $\sigma|_\Lambda$ and $\sigma|_{\Lambda+y}$ behave like independent Ising models. □

We now turn to the proof of Step 2. Fix $A\subset\mathbb{Z}^d$ with $|A|$ even. We want to prove that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\left(\langle\sigma_A\rangle_{\Lambda^{\mathfrak{g}},\beta}-\langle\sigma_A\rangle_{\Lambda^{\mathrm{f}},\beta}\right)=0. \tag{13}$$

In order to state a useful upper bound, let us first introduce the probability measure

$$\mathbb{P}^A_G:=\frac{2^{|V|}}{Z_G\langle\sigma_A\rangle_G}\mathbb{M}_G[\{\partial\mathbf{n}=A\}\cap(\,\cdot\,)].$$

Lemma 9.5 (Continuity, Step 2a). *Fix $\Lambda\subset\mathbb{Z}^d$ finite, fix $\beta\in[0,\infty)$, and fix $A\subset\Lambda$ of even cardinal. Consider the random pair $(\mathbf{n},\mathbf{m})\sim\mathbb{P}^A_{\Lambda^{\mathfrak{g}},\beta}\times\mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}},\beta}$ Then*

$$\frac{\langle\sigma_A\rangle_{\Lambda^{\mathfrak{g}},\beta}-\langle\sigma_A\rangle_{\Lambda^{\mathrm{f}},\beta}}{\langle\sigma_A\rangle_{\Lambda^{\mathfrak{g}},\beta}}=\mathbb{P}^A_{\Lambda^{\mathfrak{g}}}\times\mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}[\{(\widehat{\mathbf{n}+\mathbf{m}}\cap E(\Lambda^{\mathrm{f}}))\notin\mathcal{E}_A\}].$$

The event on the right means that in order to pair up the vertices in A with edges in $\widehat{\mathbf{n}+\mathbf{m}}$, it is necessary to include a path through the ghost.

Proof. We have

$$\begin{aligned}Z_{\Lambda^{\mathfrak{g}}} Z_{\Lambda^{\mathrm{f}}} \langle \sigma_A \rangle_{\Lambda^{\mathrm{f}},\beta} &= 2^{2|\Lambda|+1} \mathbb{M}_{\Lambda^{\mathfrak{g}}} \times \mathbb{M}_{\Lambda^{\mathrm{f}}}[\{\partial \mathbf{n} = \emptyset,\, \partial \mathbf{m} = A\}] \\ &= 2^{2|\Lambda|+1} \mathbb{M}_{\Lambda^{\mathfrak{g}}} \times \mathbb{M}_{\Lambda^{\mathrm{f}}}[\{\partial \mathbf{n} = A,\, \partial \mathbf{m} = \emptyset\} \cap \{\widehat{\mathbf{n}+\mathbf{m}} \cap E(\Lambda^{\mathrm{f}}) \in \mathcal{E}_A\}],\end{aligned}$$

where we use switching for the last equality. In other words,

$$\langle \sigma_A \rangle_{\Lambda^{\mathrm{f}},\beta} = \langle \sigma_A \rangle_{\Lambda^{\mathfrak{g}},\beta} \cdot \mathbb{P}^A_{\Lambda^{\mathfrak{g}}} \times \mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}[\{(\widehat{\mathbf{n}+\mathbf{m}} \cap E(\Lambda^{\mathrm{f}})) \in \mathcal{E}_A\}].$$

This implies the desired identity. ☐

In order to prove Step 2 (or equivalently, Equation (13)), we now use the previous lemma to make a number of reductions.

- By the previous lemma, proving Equation (13) is equivalent to proving that
$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}^A_{\Lambda^{\mathfrak{g}}} \times \mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}[\{(\widehat{\mathbf{n}+\mathbf{m}} \cap E(\Lambda^{\mathrm{f}})) \not\in \mathcal{E}_A\}] = 0.$$
- Let $B_n = [-n,n]^d \cap \mathbb{Z}^d$ denote an extremely large box containing A. If the ghost is required to pair up the vertices in A, then B_n must be connected to the ghost. Therefore it suffices to prove that
$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}^A_{\Lambda^{\mathfrak{g}}} \times \mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}[\{B_n \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \mathfrak{g}\}] = 0. \tag{14}$$
- The percolation measure $\mathbb{P}^A_{\Lambda^{\mathfrak{g}}}$ may be viewed as a normalised version of
$$\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial \omega = A\} \cap (\,\cdot\,)] \times \mathbb{P}_p,$$
see Equation (11). Fix $Q \subset E(B_n^{\mathrm{f}})$ such that $\partial Q = A$. Equation (8) relates the measures $\mathbf{M}^A_{\Lambda^{\mathfrak{g}}}$ and $\mathbf{M}^{\emptyset}_{\Lambda^{\mathfrak{g}}}$: the map Ξ_Q acts as a pushforward map between the measures, up to a Radon–Nikodym derivative which is uniformly lower- and upper bounded. Since Ξ_Q only modifies the edges in $E(B_n^{\mathrm{f}})$, it leaves the event
$$\{B_n \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \mathfrak{g}\}$$
invariant. This implies that Equation (14) is equivalent to
$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}^{\emptyset}_{\Lambda^{\mathfrak{g}}} \times \mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}[\{B_n \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \mathfrak{g}\}] = 0.$$
- By inclusion of events, it suffices to show that
$$\lim_{N \to \infty} \lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}^{\emptyset}_{\Lambda^{\mathfrak{g}}} \times \mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}[\{B_n \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \partial B_N\}] = 0.$$
In fact, the event $\{B_n \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \partial B_N\}$ is a *local* event. We shall show below that the measures $\mathbb{P}^{\emptyset}_{\Lambda^{\mathfrak{g}}} \times \mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}$ converge in the local convergence topology as $\Lambda \uparrow \mathbb{Z}^d$. Thus, if we write $\mathbb{P}^{\emptyset,\emptyset}$ for its limit, it then suffices to show that
$$\lim_{N \to \infty} \mathbb{P}^{\emptyset,\emptyset}[\{B_n \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \partial B_N\}] = 0.$$
Since n is arbitrary, this is equivalent to showing that $\widehat{\mathbf{n}+\mathbf{m}}$ does not percolate $\mathbb{P}^{\emptyset,\emptyset}$-almost surely.

Thus, as explained above, it suffices to prove the following two steps.

Lemma 9.6 (Continuity, Step 2b)**.** *Consider the Ising model on $\mathbb{Z}^d$ with $d \in \mathbb{Z}_{\geq 1}$ and $\beta \in [0,\infty)$. Then the limit $\mathbb{P}^{\emptyset,\emptyset} := \lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}^{\emptyset}_{\Lambda^{\mathfrak{g}}} \times \mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}$ converges in the local convergence topology.*

Lemma 9.7 (Continuity, Step 2c)**.** *Consider the Ising model on $\mathbb{Z}^d$ with $d \in \mathbb{Z}_{\geq 3}$ and $\beta \in [0,\beta_c]$. Then $\widehat{\mathbf{n}+\mathbf{m}}$ does not percolate in the limit $\mathbb{P}^{\emptyset,\emptyset}$ constructed in Step 2b.*

Step 2b is straightforward, and follows from the convergence of the infinite-volume measures $\langle\,\cdot\,\rangle^{+}_{\mathbb{Z}^d}$ and $\langle\,\cdot\,\rangle^{\mathrm{f}}_{\mathbb{Z}^d}$. For Step 2c we combine a simple version of the Burton–Keane argument (not using ergodicity) with the infrared bound.

Exercise 9.8 (Proof of Continuity, Step 2b)**.** We want to prove convergence of $\mathbb{P}^{\emptyset}_{\Lambda^{\mathfrak{g}}}$ and $\mathbb{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}$; the proof of convergence is the same for the two families, and we focus on the second.

- Argue that the sprinkling relation implies that it suffices to prove that the probability measures $\mathbf{P}^{\emptyset}_{\Lambda^{\mathrm{f}}} :\propto \mathbf{M}_{\Lambda^{\mathrm{f}}}[\{\partial\mathbf{n} = \emptyset\} \cap (\,\cdot\,)]$ converge in the local convergence topology.
- Prove that for a fixed edge xy, we have
$$\mathbf{P}^{\emptyset}_{\Lambda^{\mathrm{f}}}[\{\omega_{xy} = a\}] \propto \langle(\sigma_x\sigma_y \tanh\beta)^a e^{-\beta\sigma_x\sigma_y}\rangle_{\Lambda^{\mathrm{f}},\beta}$$
as a ranges over $\{0,1\}$. Conclude that this value converges as $\Lambda \uparrow \mathbb{Z}^d$.
- Let $Q \subset E(\mathbb{Z}^d)$ finite. Find a local observable X_ζ for each $\zeta \in \{0,1\}^Q$ such that
$$\mathbf{P}_{\Lambda^{\mathrm{f}}}[\{\omega|_Q = \zeta\}] \propto \langle X_\zeta\rangle_{\Lambda^{\mathrm{f}}}.$$
Argue that the probabilities on the left converge as $\Lambda \uparrow \mathbb{Z}^d$.

Exercise 9.9 (Burton–Keane argument without ergodicity)**.** The measure $\mathbb{P}^{\emptyset,\emptyset}$ is insertion tolerant (due to sprinkling) and shift-invariant. Let N_∞ denote the number of infinite clusters of $\hat{\mathbf{n}}\cup\hat{\mathbf{m}}$. Prove that $\mathbb{P}^{\emptyset,\emptyset}[\{N_\infty > 2\}] = 0$ by appealing to trifurcation boxes. Why can we not rule out that $\mathbb{P}^{\emptyset,\emptyset}[\{N_\infty = 2\}] = 0$ at this stage?

Proof of Continuity, Step 2c. We want to prove that $p := \mathbb{P}^{\emptyset,\emptyset}[\{0 \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \infty\}] = 0$.

Define the random set $C_\infty := \{x : x \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} \infty\}$. Recall that $B_m := [-m,m]^d \cap \mathbb{Z}^d$. Shift-invariance implies that
$$\mathbb{E}^{\emptyset,\emptyset}[|C_\infty \cap B_m|^2] \geq \mathbb{E}^{\emptyset,\emptyset}[|C_\infty \cap B_m|]^2 = p^2|B_m|^2.$$
Since $N_\infty \leq 2$ almost surely, we have
$$\mathbb{E}^{\emptyset,\emptyset}[|\{(x,y) \in B_m \times B_m : x \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} y\}|] \geq \frac{1}{2}\mathbb{E}^{\emptyset,\emptyset}[|C_\infty \cap B_m|^2] \geq \frac{1}{2}p^2|B_m|^2.$$
By the switching lemma,
$$\langle\sigma_x\sigma_y\rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta} \geq \langle\sigma_x\sigma_y\rangle^{+}_{\mathbb{Z}^d,\beta}\langle\sigma_x\sigma_y\rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta} \geq \mathbb{P}^{\emptyset,\emptyset}[\{x \xleftrightarrow{\widehat{\mathbf{n}+\mathbf{m}}} y\}].$$
Thus, the inequality above yields
$$\sum_{x,y\in B_m}\langle\sigma_x\sigma_y\rangle^{\mathrm{f}}_{\mathbb{Z}^d,\beta} \geq \frac{1}{2}p^2|B_m|^2.$$
By the infrared bound (Equation (12)), the left side is of order $o(|B_m|^2)$ as $m \to \infty$. □

10. Coupling from the past

Consider the following practical objective: use a computer to sample a configuration of the Ising model on a finite graph. Computers generally have access to a source of i.i.d. randomness, but cannot immediately sample from complicated distributions. We must therefore transform the i.i.d. randomness into a sample from the Ising model.

Let us first consider *Glauber dynamics*, which is a standard strategy for reaching this objective. One uses the i.i.d. randomness to construct a Markov chain whose invariant distribution is the Ising model (for example, via the Metropolis–Hastings algorithm). To take a sample, one starts at a deterministic configuration, runs the algorithm for a deterministic number of steps (say N), and then takes the final state as the sample. This strategy has two drawbacks:

- The Markov chain approximates its invariant distribution; no sample is "perfect",
- The number N required for the desired precision is often hard to calculate.

Coupling from the past circumvents these problems. This section explains the basic algorithm, and the second section discusses theoretical implications (such as ergodicity, mixing, and uniqueness of Gibbs measures). Throughout this section, we consider the Ising model on a finite graph G at inverse temperature β.

Definition 10.1 (Treshold value at a single spin)**.** The distribution of σ_x conditional on $\{\sigma_{V\setminus\{x\}} = \zeta\}$ is given by

$$\mathbb{P}_{G,\beta}[\{\sigma_x = -\}|\{\sigma_{V\setminus\{x\}} = \zeta\}] = \tau_\beta(\textstyle\sum_{y\sim x} \zeta_x); \qquad \tau_\beta(a) := \dfrac{e^{-\beta a}}{2\cosh \beta a}.$$

The value τ_β is called the *treshold value.*

The conditional value of σ_x may thus be sampled as follows: first draw $U \sim U([0,1])$ (the uniform distribution on the unit interval), then set

$$\sigma_x := \begin{cases} - & \text{if } U \leq \tau_\beta(\sum_{y\sim x}\sigma_x), \\ + & \text{if } U > \tau_\beta(\sum_{y\sim x}\sigma_x). \end{cases}$$

In order to rigorously implement this conditional resampling into a Glauber dynamic, we first describe an explicit Glauber map.

Definition 10.2 (Glauber map)**.** Let G denote any locally finite graph. For any $x \in V$ and $U \in [0,1]$, define the *Glauber map*

$$R_{x,U} : \Omega \to \Omega,\, \sigma \mapsto \left(z \mapsto \begin{cases} \sigma_z & \text{if } z \neq x \\ - & \text{if } z = x \text{ and } U \leq \tau_\beta(\sum_{y\sim x}\sigma_y) \\ + & \text{if } z = x \text{ and } U > \tau_\beta(\sum_{y\sim x}\sigma_y) \end{cases} \right).$$

This map potentially flips the spin at x, while leaving all other spins invariant.

Let $\mathbb{P}_{\mathrm{iid}}$ denote a new probability measure in which $((x_k, U_k))_{k\in\mathbb{Z}}$ is an i.i.d. sequence of uniform random variables in $V \times [0,1]$. The idea is to use the i.i.d. randomness in this sequence to set up a Markov chain mixing to $\langle\,\cdot\,\rangle_{G,\beta}$ (as described above). In this probability space, we write $R^{k,\ell}$ for the random map

$$R^{k,\ell} := R_{x_{k+1},U_{k+1}} \circ R_{x_{k+2},U_{k+2}} \circ \cdots \circ R_{x_\ell,U_\ell}$$

for any $k, \ell \in \mathbb{Z}$ with $k \leq \ell$.

This setups is intrinsically linked to a Markov chain, described in the proposition below. We leave its proof to the reader.

Proposition 10.3 (Glauber dynamics)**.** *Let G denote a finite graph and let $\beta \in [0,\infty)$. Consider the measure $\mathbb{P}_{\mathrm{iid}}$ introduced above. Define the stochastic $|\Omega| \times |\Omega|$-matrix A via*

$$A_{\sigma',\sigma} := \mathbb{P}_{\mathrm{iid}}[\{R_{x_0,U_0}(\sigma) = \sigma'\}].$$

Then all of the following hold true.

- *The matrix A is the transition kernel of an irreducible aperiodic Markov chain.*
- *$\langle\,\cdot\,\rangle_{G,\beta}$ solves the detailed balance equations for A.*
- *For any $\sigma \in \Omega$ and $k \in \mathbb{Z}$, the random sequence $(R^{k,k+n}(\sigma))_{n\geq 0}$ in $\mathbb{P}_{\mathrm{iid}}$ has the law of the Markov chain with transition kernel A, started from σ.*

Coupling from the past relies on one crucial idea: one should not consider the asymptotic ($n \to \infty$) distribution of $R^{0,n}(\sigma)$, but rather of $R^{-n,0}(\sigma)$.

Theorem 10.4 (Coupling from the past)**.** *Let G denote a finite graph and let $\beta \in [0,\infty)$. Consider the measure $\mathbb{P}_{\mathrm{iid}}$ introduced above. Let $T \in \mathbb{Z}_{\geq 0}\cup\{\infty\}$ denote the random stopping time defined via*

$$T := \inf\{n \in \mathbb{Z}_{\geq 0} : \textit{the function } R^{-n,0} : \Omega \to \Omega \textit{ is constant as a function on } \Omega\}.$$

Suppose that $\mathbb{P}(\{T < \infty\}) = 1$. Then all of the following are true:

- *For any $S \geq T$, the function $R^{-S,0}$ is also constant on Ω, and $R^{-S,0} = R^{-T,0}$,*
- *The distribution of function $R^{-T,0}$ is $\langle\,\cdot\,\rangle_{G,\beta}$.*

Here we simply write $R^{-T,0}$ for $R^{-T,0}(\sigma)$ (with $\sigma \in \Omega$ arbitrary) whenever $R^{-T,0}$ is constant.

Proof. Fix a configuration $\sigma \in \Omega$ and a sequence $((x_k, U_k))_k$. If $R^{-n,0}$ is constant for some n, then

$$R^{-(n+1),0} = R^{-n,0} \circ R_{x_{-n},U_{-n}} = R^{-n,0}.$$

In that case, we simply have

$$\lim_{m\to\infty} R^{-m,0}(\sigma) = R^{-n,0}(\sigma) \equiv R^{-n,0}.$$

In particular, if $\mathbb{P}_{\mathrm{iid}}(\{T < \infty\}) = 1$, then $\mathbb{P}_{\mathrm{iid}}$-almost surely

$$\lim_{m\to\infty} R^{-m,0}(\sigma) = R^{-T,0}.$$

The distribution of $R^{-m,0}(\sigma)$ converges to $\langle\,\cdot\,\rangle_{G,\beta}$ as $m \to \infty$ (see the previous proposition), and therefore $R^{-T,0} \sim \langle\,\cdot\,\rangle_{G,\beta}$. □

It is clearly important that $\mathbb{P}_{\mathrm{iid}}(\{T < \infty\}) = 1$. We invite the reader to prove this in the following exercise. Another proof is provided later.

Exercise 10.5 (Coupling from the past: convergence (generic))**.** Show that the stopping time T in the theorem above is almost surely finite. Hint: observe that $\mathbb{P}_{\mathrm{iid}}$-almost surely, there are $|\Lambda|$ consecutive entries in the sequence $((x_k, U_k))_{k\in\mathbb{Z}_{\leq 0}}$ where x_k enumerates Λ and where $U_k \approx 0$.

Remark 10.6 (Algorithm description)**.** The algorithm can practically be implemented in a computer as follows.

(1) Fix a strictly increasing sequence $(n_k)_{k\geq 0}$ of integers with $n_0 = 0$.
(2) Set $k = 0$, and repeat the following procedure as long as $R^{-n_k,0}$ is *not* constant:
 (a) Add 1 to the counter k,
 (b) Draw $((x_i, U_i))_{-n_k < i \leq -n_{k-1}}$ from the independent source of randomness,
 (c) Calculate the map $R^{-n_k,0}$.
(3) Output the constant value $R^{-n_k,0}$ as our sample from $\langle\,\cdot\,\rangle_{G,\beta}$.

The algorithm has two drawbacks:

- It requires memory to record $R^{-n_k,0}$ or $((x_i, U_i))_{-n_k<i\leq 0}$ between iterations,
- It requires time to determine if $R^{-n_k,0}$ is constant on Ω (which has cardinal $2^{|V|}$).

We now describe an important ingredient not mentioned so far: *monotonicity.* With monotonicity, it is much easier to see that *coupling from the past* converges (more precisely, that $\mathbb{P}_{\mathrm{iid}}(\{T < \infty\}) = 1$). It is also much easier to check if the map $R^{-n,0}$ is constant or not. The monotonicity is relative to the partial ordering $\leq$ on the sample space $\Omega = \{\pm\}^V$; we say that $\sigma \leq \sigma'$ whenever $\sigma_x \leq \sigma'_x$ for all $x \in V$. Everything follows from the basic observation that each map R_{x_k,U_k} preserves $\leq$, in the sense that

$$\sigma \leq \sigma' \qquad \Longrightarrow \qquad R_{x_k,U_k}(\sigma) \leq R_{x_k,U_k}(\sigma').$$

Lemma 10.7 (Monotonicity of Glauber dynamics)**.** *Consider $((x_i, U_i))_{i\in\mathbb{Z}}$ fixed. Then:*

(1) Each map $R_{x_i,U_i} : \Omega \to \Omega$ preserves $\leq$,
(2) Each map $R^{i,j} : \Omega \to \Omega$ preserves $\leq$,
(3) $(R^{-n,0}(\sigma^{\max}))_n$ is decreasing in n, where $\sigma^{\max} \equiv + \in \Omega$ is the maximal element,
(4) $(R^{-n,0}(\sigma^{\min}))_n$ is increasing in n, where $\sigma^{\min} \equiv - \in \Omega$ is the minimal element,
(5) We have $R^{-\infty,0}(\sigma^{\min}) \leq R^{-\infty,0}(\sigma^{\max})$, where $R^{-\infty,0}(\cdots)$ denotes the $n \to \infty$ limit,
(6) If $R^{-n,0}(\sigma^{\min}) = R^{-n,0}(\sigma^{\max})$, then the map $R^{-n,0}$ is constant.

Proof. The first item can be proved by simply inspecting the definition of R_{x_i,U_i}. The second item then follows: indeed, a composition of increasing functions is increasing. For the third item (and similarly, the fourth item), observe that monotonicity of $R^{-n,0}$ implies:

$$R^{-(n+1),0}(\sigma^{\max}) = R^{-n,0}(R_{x_{-n},U_{-n}}(\sigma^{\max})) \leq R^{-n,0}(\sigma^{\max}).$$

The last two items follow from monotonicity of the map $R^{-n,0}$ (the second item). $\square$

Lemma 10.8 (Coupling from the past: convergence)**.** *Recall the context of Theorem 10.4. Then we have*

$$T = \inf\{n \in \mathbb{Z}_{\geq 0} : R^{-n,0}(\sigma^{\max}) = R^{-n,0}(\sigma^{\min})\},$$

and this random variable is $\mathbb{P}_{\mathrm{iid}}$-almost surely finite.

Proof. The reformulation of the stopping time is valid thanks to the previous lemma. It suffices to prove that it is almost surely finite. Notice that the distribution of both $R^{-\infty,0}(\sigma^{\max})$ and $R^{-\infty,0}(\sigma^{\min})$ is given by $\langle \cdot \rangle_{G,\beta}$. Since they have the same distribution and are almost surely $\leq$-ordered in $\mathbb{P}_{\mathrm{iid}}$, they must be almost surely equal. Since the state space Ω is discrete, this means that $\mathbb{P}_{\mathrm{iid}}$-almost surely there is some $n \in \mathbb{Z}_{\geq 0}$ such that

$$R^{-n,0}(\sigma^{\max}) = R^{-\infty,0}(\sigma^{\max}) \qquad \text{and} \qquad R^{-n,0}(\sigma^{\min}) = R^{-\infty,0}(\sigma^{\min}).$$

This finishes the proof. $\square$

References

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