# A COURSE ON THE ISING MODEL

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## Contents



## Preface

These lecture notes are progressively written during the 2025 spring semester, as the course is taught at Sorbonne university in the M2 (second-year masters) programme. Its purpose is to give a broad introduction to the rigorous analysis of the Ising model. The main focus is on four techniques and their applications:

- The Peierls argument,
- The random-currents representation,
- The FKG inequality for the Ising spins,
- The FKG inequality for the random-cluster (FK) representation.

A basic understanding of analysis and probability theory is essential for following this course. Experience with other models in statistical mechanics (such as the Bernoulli percolation model) is a plus but by no means essential.

## 1. Introduction

The Ising model is the archetypal model for the study of phase transitions in mathematical physics. It was first introduced by Wilhelm Lenz in 1920 and later solved by Ernst Ising in 1924 in the one-dimensional case. The model consists of a lattice of spins, each of which can be in one of two states, up or down. Informally, one may think of these spins as the magnetic moments of atoms in a ferromagnetic material. The behaviour of this probabilistic model depends strongly on a few different parameters:

- The dimension $d$ of the lattice $\mathbb{Z}^d$ on which the spins are placed,
- The interaction strength $\beta$ between neighbouring spins,

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- The way that boundary conditions are imposed,
- The strength of external magnetic field $h$.

In fact, we shall start by defining the Ising model on arbitrary finite graphs. We shall now give a definition of the Ising model, although we keep boundary conditions and external magnetic fields for later.

**Definition 1.1** (Ising model on a finite graph)**.** The Ising model on a finite graph $G = (V, E)$ with *inverse temperature* $\beta \in [0, \infty)$ is defined as follows. Let $\Omega := \{\pm 1\}^V$ denote the set of spin configurations on the vertices of the graph; a typical element of $\Omega$ is denoted by $\sigma = (\sigma_u)_{u \in V}$. Elements $\sigma \in \Omega$ are called *spin configurations*; elements $\sigma_u$ are called *spins*. The *energy* or *Hamiltonian* of a spin configuration $\sigma$ is given by

$$H^{\mathrm{Ising}}_{G,\beta}(\sigma) := -\beta \sum_{uv \in E} \sigma_u \sigma_v.$$

We write $\mathbb{P}^{\mathrm{Ising}}_{G,\beta}$ for the associated *Boltzmann distribution* or *Gibbs measure*:

$$\mathbb{P}^{\mathrm{Ising}}_{G,\beta}(\sigma) := \frac{1}{Z^{\mathrm{Ising}}_{G,\beta}} e^{-H^{\mathrm{Ising}}_{G,\beta}(\sigma)},$$

where $Z^{\mathrm{Ising}}_{G,\beta}$ is normalisation constant or *partition function* defined by

$$Z^{\mathrm{Ising}}_{G,\beta} := \sum_{\sigma \in \Omega} e^{-H^{\mathrm{Ising}}_{G,\beta}(\sigma)}.$$

We shall write $\langle \cdot \rangle^{\mathrm{Ising}}_{G,\beta}$ for the expectation functional associated to this probability measure.

**Remark 1.2** (Flip-symmetry)**.** The Ising model is *flip-symmetric* in the sense that the distribution of the spins is invariant under the transformation $\sigma \mapsto -\sigma$. This is because the Hamiltonian is invariant under this transformation.

**Remark 1.3.** We shall often suppress subscripts and superscripts when they are clear from the context.

**Remark 1.4.** Adding a constant to the Hamiltonian does not change the distribution of the Ising model, even though it affects the partition function.

**Remark 1.5.** The mathematical community has widely adopted the terminology coming from the physics literature.

## 2. Early developments. 1907: The Curie–Weiss model

The Ising model is actually a generalisation of the Curie–Weiss model. Pierre-Ernest Weiss introduced this mathematical model in 1907 in order to find a theoretical explanation for the spontaneous magnetisation of ferromagnetic materials that had previously been studied by Pierre Curie. The Curie–Weiss model is a special case of the Ising model where the graph is a complete graph. This model can be solved with elementary methods and exhibits a phase transition. Moreover, the phase transition may be seen as a direct consequence of the competition between entropy and energy, which is why we quickly review it here.

**Definition 2.1** (Curie-Weiss model)**.** The Curie-Weiss model is the Ising model on the complete graph $G = K_n$ on $n$ vertices. We typically fix a parameter $\alpha \in [0, \infty)$ and then set the inverse temperature $\beta$ to $\beta = \alpha/n$. We shall write

$$\mathbb{P}^{\mathrm{CW}}_{n,\alpha} := \mathbb{P}^{\mathrm{Ising}}_{K_n, \alpha/n}$$

for the associated Boltzmann distribution. The Curie-Weiss model is often used to study mean-field behavior in statistical mechanics, as it captures the essential features of phase transitions and spontaneous magnetization in a simplified setting.

Let $n_+ = n_+(\sigma)$ denote the number of vertices with spin $+1$ in a configuration $\sigma \in \Omega$. This is a random variable. Let us try to calculate the probability of the event $\{n_+ = k\}$, without worrying about the partition function (the normalising constant). One may easily check that the Hamiltonian satisfies

$$H(\sigma) = 2\beta n_+(n - n_+) + \mathrm{const}(n).$$

The distribution of $n_+$ can then be calculated as follows:

$$\mathbb{P}^{\mathrm{Ising}}_{K_n,\beta}(\{n_+ = k\}) \propto \binom{n}{k} e^{-2\beta k(n-k)} \propto \frac{1}{k!(n-k)!} e^{-2\beta k(n-k)}. \tag{1}$$

Using Stirling's formula for the factorials, we find that

$$\log \mathbb{P}^{\mathrm{Ising}}_{K_n,\beta}(\{n_+ = k\}) \overset{\text{Stirling}}{\approx} -n f_{(n\beta)}(k/n) + \mathrm{const}(n);$$
$$f_{(\alpha)} : [0,1] \to \mathbb{R},\ x \mapsto x \log x + (1-x)\log(1-x) + 2\alpha x(1-x).$$

If we fix $\alpha$ and send $n$ to infinity, then we discover a large deviations principle for the random variable $n_+/n$ with rate function $f_{(\alpha)}$ and speed $n$. In particular, the random variable $n_+/n$ is concentrated around the minimisers of the function $f_{(\alpha)}$.

**Exercise 2.2.** (1) Formally verify that all of the above calculations are correct.

(2) Show that for small $\alpha$, the function $f_{(\alpha)}$ has a single minimum, which means that the random variable $n_+/n$ is concentrated around the value $1/2$.

(3) Show that for large enough $\alpha$, the function $f_{(\alpha)}$ has two minima at $(1 \pm m)/2$ for some $m > 0$, which means that the random variable $n_+/n$ is concentrated around these minima. The value of $m$ is called the *magnetisation*.

(4) Calculate the critical value for $\alpha$. At this value, the second derivative of $f_{(\alpha)}$ vanishes at $x = 1/2$. What does this mean for the distribution of $n_+/n$? More precisely, what is the order of magnitude of $\mathrm{Var}\,\frac{n_+}{n}$ as $n \to \infty$?

**Remark 2.3.** Reconsider Equation (1). In this equation, the competition between the two factors is extremely transparent.

- First, there is a combinatorial term or *entropy*, which favours values $k$ for the random variable $n_+$ such that the cardinality of the set $\{n_+ = k\}$ is large. This means that values $k \approx n/2$ are preferred.
- Second, there is the *energy* term, which favours values such that the Hamiltonian is minimised. This favours configurations where as many spins as possible align.

The interaction parameter $\beta$ allows us to put more emphasis on the entropy term or on the energy term. In the $n \to \infty$ limit, there is a precise value for $n\beta$ where the behaviour of the random system undergoes a qualitative change: a rudimentary example of a *phase transition.*

While the competition between entropy and energy is transparent in the case of the complete graph (the Curie–Weiss model), the situation is much more complicated when considering other graphs. We are particularly interested in graphs modelling Euclidean space, by taking large finite subgraphs of the square lattice graph $\mathbb{Z}^d$ in dimension $d \geq 1$. The geometry of the Ising model (absent in the case of Curie–Weiss) is an extremely rich and beautiful object, and most of this course is dedicated to understanding it.

## 3. Early developments. 1924: Ising's analysis

Wilhelm Lenz challenged his doctoral student Ernst Ising to solve the model of interest on the one-dimensional line graph $\mathbb{Z}$. Readers acquainted with percolation theory will suspect that such a simple model is unlikely to exhibit a phase transition. This suspicion is correct, but we stress that the percolation model had not yet been described at the time that Ising undertook his doctoral research.

We defined the Ising model in the previous section, but only on finite graphs. Let us extend this definition to infinite graphs.

**Definition 3.1** (The Ising model with boundary conditions)**.** Let $G = (V, E)$ denote a fixed locally finite graph. We consider the measurable space $(\Omega, \mathcal{F})$ where $\Omega = \{\pm 1\}^V$ and where $\mathcal{F}$ is the product-$\sigma$-algebra.

Let $\Lambda$ denote a *domain*, that is, a finite subset of $V$. The Ising model in the domain $\Lambda$ with inverse temperature $\beta \geq 0$ and boundary conditions $\zeta \in \{\pm 1\}^{\Lambda^c}$ is the probability measure $\mathbb{P}^{\mathrm{Ising},\zeta}_{\Lambda,\beta}$ on $(\Omega, \mathcal{F})$ defined via:

$$\mathbb{P}^{\mathrm{Ising},\zeta}_{\Lambda,\beta}(\sigma) = \frac{1}{Z^{\mathrm{Ising},\zeta}_{\Lambda,\beta}} \cdot \mathbb{1}(\sigma|_{\Lambda^c} = \zeta) \cdot e^{-H^{\mathrm{Ising}}_{\Lambda,\beta}(\sigma)},$$

where $H^{\mathrm{Ising}}_{\Lambda,\beta}(\sigma)$ is the Hamiltonian given by

$$H^{\mathrm{Ising}}_{\Lambda,\beta}(\sigma) = -\beta \sum_{uv \in E(\Lambda)} \sigma_u \sigma_v,$$

and where the partition function $Z^{\mathrm{Ising},\zeta}_{\Lambda,\beta}$ is given by

$$Z^{\mathrm{Ising},\zeta}_{\Lambda,\beta} = \sum_{\sigma \in \Omega} \mathbb{1}(\sigma|_{\Lambda^c} = \zeta) \cdot e^{-H^{\mathrm{Ising}}_{\Lambda,\beta}(\sigma)}.$$

The set $E(\Lambda) \subset E$ denotes the set of edges with at least one endpoint in $\Lambda$. Indeed, adding a constant to the Hamiltonian does not affect the measure, and edges which do not intersect $\Lambda$ contribute with a constant.

We write $+$ and $-$ for the boundary conditions $+1 \in \Omega$ and $-1 \in \Omega$ respectively.

**Theorem 3.2** (Ising, 1924)**.** *The one-dimensional Ising model is demagnetised at all temperatures. This means the following. Let $G = (V, E)$ denote the one-dimensional lattice $\mathbb{Z}$, and define $\Lambda_n := \{-n+1, \dots, n-1\}$. Then, for any $\beta \geq 0$, we have*

$$\lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} = 0.$$

*Proof.* Write $T$ for the matrix

$$T := \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

It is straightforward to work out that

$$Z^+_{\Lambda_n,\beta} \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} = \left( T^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^n \right)_{1,1};$$

$$Z^+_{\Lambda_n,\beta} = \left(T^{2n}\right)_{1,1},$$

see the exercise below. One may then conclude that the ratio of these two numbers tends to zero with $n \to \infty$ by simply diagonalising $T$. □

**Remark 3.3.** Although the intuition is reminiscent of the theory of Markov chains, we stress that the matrix $T$ above is *not* a stochastic matrix. This is why we need to consider the partition function (normalising constant) separately.

**Exercise 3.4.** Let $(f_k)_k$ denote a family of functions of the form $f_k : \{+1, -1\} \to \mathbb{R}$. For any $k$, define

$$M_k := \begin{pmatrix} f_k(+1) & 0 \\ 0 & f_k(-1) \end{pmatrix}.$$

Prove that for any $n$, we have

$$Z^+_{\Lambda_n,\beta} \langle \textstyle\prod_{k \in \Lambda_n} f_k(\sigma_k) \rangle^+_{\Lambda_n,\beta} = (T M_{n-1} T M_{n-2} T \cdots T M_{-n+1} T)_{1,1}.$$

**Remark 3.5.** Ising conjectured that the absence of magnetisation in the one-dimensional model would also hold in higher dimensions. This was later shown to be false: Peierls proved in 1936 that the two-dimensional model magnetises for sufficiently large $\beta$.

## 4. Early developments. 1936: Peierls' argument

**Theorem 4.1** (Peierls, 1936)**.** *The Ising model exhibits magnetisation in two dimensions.*

We shall discuss a slight variation of Peierls' original setup, so that we can fully focus the proof on the core idea. Let $\mathbb{T}$ denote the triangular lattice graph, comprised of vertices of the form

$$\mathbb{T} := \left\{ n + me^{\pi i/3} : n, m \in \mathbb{Z} \right\} \subset \mathbb{C},$$

and such that each vertex is connected to the six vertices at distance one. Let $\Lambda_n \subset \mathbb{T}$ denote the set of vertices at a graph distance at most $n-1$ from $0 \in \mathbb{T}$. We consider the Ising model on the infinite graph $\mathbb{T}$. We shall prove the following version of Peierls' result.

**Theorem 4.2** (Peierls, 1936)**.** *Consider the Ising model on the two-dimensional triangular lattice graph $\mathbb{T}$. For sufficiently large $\beta$, we have*

$$\inf_n \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} > 0.$$

Let $\mathbb{H} := \mathbb{T}^*$ denote the hexagonal lattice that is dual to the triangular lattice. For a fixed configuration $\sigma \in \Omega$, we let $\mathcal{I}(\sigma) \subset E(\mathbb{H})$ denote the set of hexagonal lattice edges separating hexagons with different spins. The set $\mathcal{I}(\sigma)$ is called the *interface* between the spins valued $+1$ and those valued $-1$. Notice that $\mathcal{I}(\sigma)$ has a partition into loops and bi-infinite paths. If only finitely many spins of $\sigma$ are valued $-1$, then there are no bi-infinite paths, and all connected components of $\mathcal{I}(\sigma)$ are loops. This happens almost surely when sampling from $\langle \cdot \rangle^+_{\Lambda_n,\beta}$.

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The core of Peierls' argument is the following lemma.

**Lemma 4.3** (Exponential decay of loop lengths)**.** *Consider the Ising model on the two-dimensional triangular lattice $\mathbb{T}$ at inverse temperature $\beta$. Suppose that $e^{-2\beta} < \frac{1}{2}$. Then for any hexagonal lattice edge $e \in E(\mathbb{H})$ and for any minimal loop length $\ell \in \mathbb{Z}_{\geq 1}$, we get*

$$\mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{I}(\sigma) \textit{ has a loop of length at least } \ell \textit{ through } e\}) \leq \frac{(2e^{-2\beta})^\ell}{1 - 2e^{-2\beta}},$$

*uniformly in $n$.*

*Proof.* Fix $\beta$, $e$, and $n$. Let $\mathcal{L}$ denote a loop through $e$, and consider the event $\{\mathcal{L} \subset \mathcal{I}\}$. We claim that

$$\mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{L} \subset \mathcal{I}\}) \leq e^{-2\beta|\mathcal{L}|}.$$

To prove the claim, we introduce the injective "loop erasure map"

$$\mathcal{E}_\mathcal{L} : \{\mathcal{L} \subset \mathcal{I}\} \to \Omega \setminus \{\mathcal{L} \subset \mathcal{I}\},$$

which is defined such that it flips all the spins inside the loop $\mathcal{L}$. As a consequence, $\mathcal{I}(\mathcal{E}_\mathcal{L}(\sigma)) = \mathcal{I}(\sigma) \setminus \mathcal{L}$. For any $\sigma \in \{\mathcal{L} \subset \mathcal{I}\}$, we have

$$\mathbb{P}^+_{\Lambda_n,\beta}(\sigma) = e^{-2\beta|\mathcal{L}|} \cdot \mathbb{P}^+_{\Lambda_n,\beta}(\mathcal{E}_\mathcal{L}(\sigma)).$$

We can write down this identity because we know that the loop erasure map decreases the Hamiltonian by $2\beta|\mathcal{L}|$. Since $\mathcal{E}_\mathcal{L}$ is injective, we get

$$\mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{L} \subset \mathcal{I}\}) = e^{-2\beta|\mathcal{L}|} \cdot \mathbb{P}^+_{\Lambda_n,\beta}(\mathrm{Image}(\mathcal{E}_\mathcal{L})) \leq e^{-2\beta|\mathcal{L}|},$$

which proves the claim.

To prove the lemma, observe simply that the number of loops of length $k$ through $e$ is bounded by $2^k$, so that

$$\begin{aligned}
&\mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{I}(\sigma) \text{ has a loop of length at least } \ell \text{ through } e\}) \\
&\quad = \sum\nolimits_{\mathcal{L} \text{ is a loop of length at least } \ell \text{ through } e} \mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{L} \subset \mathcal{I}\}) \\
&\quad \leq \sum\nolimits_{k \geq \ell} 2^k \cdot e^{-2\beta k}.
\end{aligned}$$

The final expression is a geometric series converging to the upper bound in the lemma. □

**Remark 4.4.** In the previous proof, the interplay between entropy and energy is quite transparent. The entropy in the argument comes from the number of loops of length $\ell$, which we upper bounded by $2^\ell$. Such a loop contributes a total of $2\beta\ell$ to the Hamiltonian. When $2e^{-2\beta} < 1$, the energy term dominates, forcing the loops to be small.

*Proof of Theorem 4.2.* For a fixed configuration $\sigma \in \Omega$ sampled from $\mathbb{P}^+_{\Lambda_n,\beta}$, we may express $\sigma_0$ as the parity of the number of loops in $\mathcal{I}(\sigma)$ surrounding 0. In particular, if no loop surrounds 0, then $\sigma_0 = +1$. Thus, for Theorem 4.2, it suffices to prove that

$$\mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{I}(\sigma) \text{ contains a loop surrounding } 0\}) < \frac{1}{2}, \tag{2}$$

for sufficiently large $\beta$, and uniformly in $n$.

Suppose given some loop $\mathcal{L} \subset E(\mathbb{H})$ surrounding 0. Then $\mathcal{L}$ must intersect the half-line $\mathbb{R}_{\geq 0} \subset \mathbb{C}$. More precisely, $\mathcal{L}$ must contain some edge $e$ whose midpoint lies precisely in the set of half-integers $-\frac{1}{2} + \mathbb{Z}_{\geq 1}$. If the endpoint of $e$ is $k - \frac{1}{2}$, then $|\mathcal{L}| \geq k$, otherwise it cannot surround 0. We are now ready to complete Peierls' argument, using exponential decay of the loop lengths (Lemma 4.3).

Let us perform a union bound over the intersection point, in order to obtain

$$\begin{aligned}
&\mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{I}(\sigma) \text{ contains a loop surrounding } 0\}) \\
&\quad \leq \sum_{k=1}^{\infty} \mathbb{P}^+_{\Lambda_n,\beta}(\{\mathcal{I}(\sigma) \text{ contains a loop surrounding } 0 \text{ and hitting } k - \tfrac{1}{2}\}) \\
&\quad \leq \sum_{k=1}^{\infty} \frac{(2e^{-2\beta})^k}{1 - 2e^{-2\beta}} = \frac{2e^{-2\beta}}{(1 - 2e^{-2\beta})^2}.
\end{aligned}$$

This upper bound is independent of $n$ and tends to 0 with $\beta \to \infty$, thus establishing Equation (2). □

**Exercise 4.5** (The Peierls argument on other graphs)**.**
1. Consider the Ising model on the two-dimensional square lattice graph $\mathbb{Z}^2$. In this case, the interface $\mathcal{I}(\sigma)$ does not consist of loops, but of even subgraphs of the dual lattice. How can Peierls' argument be adapted to this case?
2. Now consider the $d$-dimensional square lattice for $d \geq 3$. What is the structure of the interface in this case? Can we adapt Peierls' to prove magnetisation for sufficiently large $\beta$?

**Remark 4.6.** Peierls' is robust, in the sense that it can be adapted to many other models in statistical mechanics.

## 5. The Markov property

Recall the definition of the Ising model on a finite graph (Definition 1.1) and on general graphs with boundary conditions (Definition 3.1). The second definition includes the first, since we may simply choose our domain $\Lambda$ to be the full vertex set whenever the graph

$G = (V, E)$ is finite. That is why we state our results for general graphs with boundary conditions in this section.

One important property of the definition with boundary conditions is that it in fact encodes *conditional probability measures.*

**Lemma 5.1** (Boundary conditions as conditional measures)**.** *Let $G$ denote a locally finite graph and $\beta \in [0, \infty)$ an inverse temperature. Let $\Lambda \subset \Delta$ denote two finite domains and fix $\xi \in \{\pm 1\}^{\Delta^c}$ and $\zeta \in \{\pm 1\}^{\Delta \setminus \Lambda}$. Then*

$$\mathbb{P}^{\xi}_{\Delta,\beta}(\,\cdot\,|\{\sigma|_{\Delta\setminus\Lambda} = \zeta\}) = \mathbb{P}^{\xi\zeta}_{\Lambda,\beta}.$$

*Proof.* For the two measures, we get

$$\mathbb{P}^{\xi}_{\Delta,\beta}(\sigma|\{\sigma|_{\Delta\setminus\Lambda} = \zeta\}) \propto \mathbb{1}(\sigma|_{\Lambda^c} = \xi\zeta) \cdot e^{-H_{\Delta,\beta}(\sigma)};$$
$$\mathbb{P}^{\xi\zeta}_{\Lambda,\beta}(\sigma) \propto \mathbb{1}(\sigma|_{\Lambda^c} = \xi\zeta) \cdot e^{-H_{\Lambda,\beta}(\sigma)}.$$

But $H_{\Delta,\beta} - H_{\Lambda,\beta}$ is constant on the event $\{\sigma|_{\Lambda^c} = \xi\zeta\}$, which means that the two probability measures are the same. □

The Ising model is a *nearest-neighbour model*, meaning that the interactions are associated with the edges of the graph. A consequence of this is the so-called *Markov property.* There are several ways to state it. We shall first state and prove the following lemma. For any domain $\Lambda \subset V$, we let $\partial\Lambda \subset V$ denote the set of vertices at graph distance one from $\Lambda$. This is called the *boundary* of $\Lambda$.

**Lemma 5.2** (Markov property)**.** *Consider a locally finite graph $G$, an inverse temperature $\beta$, a domain $\Lambda$, and a boundary condition $\zeta$. Let $(\Lambda_i)_i$ denote the partition of $\Lambda$ into connected components. Then in the measure $\mathbb{P}^{\zeta}_{\Lambda,\beta}$, the family $(\sigma|_{\Lambda_i})_i$ is a family of independent random variables. Moreover, the distribution of $\sigma|_{\Lambda_i}$ only depends on $\zeta|_{\partial\Lambda_i}$.*

*Proof.* We have $\mathbb{P}^{\zeta}_{\Lambda,\beta}(\sigma) \propto \mathbb{1}(\sigma|_{\Lambda^c} = \zeta) \cdot e^{-H_{\Lambda,\beta}(\sigma)}$. The Hamiltonian may be written

$$H_{\Lambda,\beta}(\sigma) = \sum_i H_{\Lambda_i,\beta}(\sigma).$$

But each term $H_{\Lambda_i,\beta}(\sigma)$ is a function of $\sigma|_{\Lambda_i}$ and $\zeta|_{\partial\Lambda_i}$. This means that $e^{-H_{\Lambda,\beta}(\sigma)}$ may be written as a product of factors, where the factor corresponding to $\Lambda_i$ only depends on $\sigma|_{\Lambda_i}$ and $\zeta|_{\partial\Lambda_i}$. This implies the desired independence. □

The Markov property is often phrased in a slightly different fashion.

**Theorem 5.3** (Markov property)**.** *Consider a locally finite graph $G$, and inverse temperature $\beta$, and two domains $\Lambda \subset \Delta$. Let $\zeta \in \{\pm 1\}^{\Delta^c}$ denote a boundary condition, and fix $\xi \in \{\pm 1\}^{\partial\Lambda}$. If $\mathbb{P}^{\zeta}_{\Delta,\beta}(\{\sigma|_{\partial\Lambda = \xi}\}) > 0$, then in the conditional probability measure*

$$\mathbb{P}^{\zeta}_{\Delta,\beta}(\,\cdot\,|\{\sigma|_{\partial\Lambda=\xi}\}) = \mathbb{P}^{\zeta\xi}_{\Delta\cup\partial\Lambda,\beta},$$

*the random variables $\sigma|_{\Lambda}$ and $\sigma|_{\Lambda^c}$ are independent. Moreover, the distribution of $\sigma|_{\Lambda}$ only depends on $\xi$.*

*Proof.* The two measures in the display in this theorem are equal because of Lemma 5.1. The theorem is then a mere corollary of the previous lemma (Lemma 5.2). □

## 6. Correlation inequalities

Peierls' argument is simple and robust, but also quite ad-hoc in the sense that it does not serve as a building block for further analysis. We now want to take a more systematic approach to the Ising model. At the centre of the modern study of the Ising model are *correlation functions* and *correlation inequalities.*

Let $\langle\,\cdot\,\rangle$ denote an Ising model (in a finite graph, or in a finite domain with boundary conditions). For any finite set $A \subset V$, we define

$$\sigma_A := \prod_{u\in A} \sigma_u.$$

Its expectation $\langle\sigma_A\rangle$ is called a *correlation function.*

**Exercise 6.1.** Consider an Ising model $\langle\,\cdot\,\rangle_{G,\beta}$ on a finite graph $G = (V, E)$. This is a probability measure on $\Omega = \{\pm 1\}^V$. Notice that the sample space $\Omega$ has the structure of a finite Abelian group. How is the Fourier transform of $\langle\,\cdot\,\rangle_{G,\beta}$ related to the family $(\langle\sigma_A\rangle_{G,\beta})_A$ of correlation functions?

Correlation functions are at the centre of the study of the Ising model. Inequalities between correlation functions are called *correlation inequalities.* We state some examples in the finite graph setting. These shall all be proved rigorously later.

- The *first Griffiths inequality*, which asserts that for any $A \subset V$,
$$\langle\sigma_A\rangle \geq 0.$$
- The *second Griffiths inequality*, which asserts that for any $A, B \subset V$,
$$\langle\sigma_A\sigma_B\rangle \geq \langle\sigma_A\rangle\langle\sigma_B\rangle.$$
- The *Fortuin–Kasteleyn–Ginibre (FKG) inequality*, which asserts that if $X, Y : \Omega \to \mathbb{R}$ are two non-decreasing functions on the partially ordered set $\Omega$, then
$$\langle XY\rangle \geq \langle X\rangle\langle Y\rangle.$$

Such inequalities may be used to prove interesting properties about the Ising model.

**Remark 6.2** (Infinite graphs with boundary conditions as finite graphs)**.** Until now, we always made a distinction between the Ising model on a finite graph and the Ising model on the infinite lattice with boundary conditions. But are they really different? Let us discuss the case of + boundary conditions. Let $G$ denote an infinite graph and $\Lambda$ a domain. Then we may consider the Ising model on the finite graph

$$G' := (V', E'); \qquad V' := \Lambda \cup \{\Lambda^c\}; \qquad E' := E(\Lambda).$$

This means that the vertices in $\Lambda^c$ are collapsed into a single vertex. It is then easy to check that the distribution of $\sigma|_\Lambda$ is the same in the following two probability measures:

$$\mathbb{P}^+_{\Lambda,\beta} \qquad \text{and} \qquad \mathbb{P}_{G',\beta}(\,\cdot\,|\{\sigma_{\Lambda^c} = +\}).$$

This enables us to state all our inequalities in a unified way, namely on finite graphs.

**Exercise 6.3.** First consider the Ising model on a finite graph $G$ at inverse temperature $\beta \in [0, \infty)$. Prove that $\langle\sigma_A\rangle_{G,\beta} = 0$ whenever $|A|$ is odd.

Now consider the Ising model on locally finite graph $G$ at inverse temperature $\beta \in [0, \infty)$ with + boundary conditions outside the domain $\Lambda \subset V$, chosen such that $\Lambda \neq V$. Recall the construction in the previous remark. Prove that if $A \subset \Lambda$ contains an odd number of vertices, then

$$\langle\sigma_A\rangle^+_{\Lambda,\beta} = \mathbb{E}^+_{G',\beta}[\sigma_A|\{\sigma_{\Lambda^c} = +\}] = \langle\sigma_{A\cup\{\Lambda^c\}}\rangle_{G',\beta}.$$

## 7. Random currents

**Definition 7.1** (Currents)**.** Let $G = (V, E)$ denote a graph. A *current* is a map $\mathbf{n} : E \to \mathbb{Z}_{\geq 0}$. We think of $(V, \mathbf{n})$ as a multigraph, where for each edge $uv \in E$ we have $\mathbf{n}_{uv}$ multi-edges between $u$ and $v$. The set of *sources* $\partial\mathbf{n} \subset V$ of a current $\mathbf{n}$ is defined as the set of vertices $u \in V$ with an odd degree in the multigraph. If $G$ is finite and $\beta \in [0, \infty)$, then the *weight* of a current is defined as

$$w_\beta(\mathbf{n}) := \prod_{xy\in E} \frac{\beta^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}.$$

Currents can be used to encode correlation functions of the Ising model.

**Theorem 7.2** (Current representation of correlation functions)**.** *Consider the Ising model on a finite graph $G$ at inverse temperature $\beta$. Let $A \subset V$ be a subset of vertices. Then*

$$Z_{G,\beta}\langle\sigma_A\rangle_{G,\beta} = 2^{|V|} \sum_{\mathbf{n}\colon \partial\mathbf{n}=A} w_\beta(\mathbf{n}).$$

*In particular, the partition function is given by*

$$Z_{G,\beta} = 2^{|V|} \sum_{\mathbf{n}\colon \partial\mathbf{n}=\emptyset} w_\beta(\mathbf{n}).$$

*Proof.* We claim that

$$Z_{G,\beta}\langle\sigma_A\rangle_{G,\beta} = \sum_{\sigma\in\{\pm1\}^V} \sigma_A \prod_{uv\in E} e^{\beta\sigma_u\sigma_v} \tag{3}$$

$$= \sum_{\sigma\in\{\pm1\}^V} \sigma_A \prod_{uv\in E} \sum_{\mathbf{n}=0}^{\infty} \frac{(\beta\sigma_u\sigma_v)^{\mathbf{n}}}{\mathbf{n}!} \tag{4}$$

$$= \sum_{\mathbf{n}:E\to\mathbb{Z}_{\geq0}} \sum_{\sigma\in\{\pm1\}^V} \sigma_A \prod_{uv\in E} \frac{(\beta\sigma_u\sigma_v)^{\mathbf{n}_{uv}}}{\mathbf{n}_{uv}!} \tag{5}$$

$$= 2^{|V|} \sum_{\mathbf{n}\colon \partial\mathbf{n}=A} \prod_{uv\in E} \frac{\beta^{\mathbf{n}_{uv}}}{\mathbf{n}_{uv}!} \tag{6}$$

$$= 2^{|V|} \sum_{\mathbf{n}\colon \partial\mathbf{n}=A} w_\beta(\mathbf{n}). \tag{7}$$

Although this may come as a surprise, its justification is straightforward.

- Equation (3) is derived from the definition of the Ising model by pulling out the sum in the Hamiltonian in the exponential as a product.
- Equation (4) follows by expanding each exponential.
- Equation (5) is Fubini's theorem: we first swap the product and the sum (so that we have to perform one sum for each factor in the product), then we interchange the two sums. Absolute convergene comes from the factorial terms in the denominator.
- Equation (6) follows simply by noting that the sum over $\sigma$ is zero unless all the factors of the form $\sigma_u$ cancel.
- Equation (7) is the definition of the weight above.

This finishes the proof. □

**Corollary 7.3** (First Griffiths inequality)**.** *Consider the Ising model on a finite graph $G$ at inverse temperature $\beta$. Then for any $A \subset V$, we have $\langle\sigma_A\rangle_{G,\beta} \geq 0$.*

*Proof.* The current expansion shows that the correlation function is a sum of non-negative terms. □

**Exercise 7.4** (Correlation functions with odd sets).

- Consider the setting of the previous theorem. Show that if $|A|$ is odd, then $\langle \sigma_A \rangle_{G,\beta} = 0$.
- Now consider an arbitrary graph $G$ with some domain $\Lambda$. Show that if $|A|$ is odd, then $\langle \sigma_A \rangle^+_{G,\beta} \geq 0$. For this part, one needs Remark 6.2 and Exercise 6.3.

**Definition 7.5** (Percolation of currents). Let $G$ denote a graph and $\mathbf{n}$ a current. We associate $\mathbf{n}$ with the set $E(\mathbf{n}) := \{uv \in E : \mathbf{n}_{uv} > 0\} \subset E$. Edges in $E(\mathbf{n})$ are called *open*, the other edges *closed*. We shall write $\{u \overset{\mathbf{n}}{\leftrightarrow} v\}$ for the event there is an open path from $u$ to $v$ ($u$ and $v$ may represent vertices or sets of vertices).

For fixed $S \subset V$, we shall also write $\mathcal{E}_S$ for the set

$$\{O \subset E : |C \cap S| \text{ is even for any connected component } C \subset V \text{ of } (V, O)\}.$$

This definition will only become useful in the next section.

**Exercise 7.6** (Currents and the Peierls argument). For the first two parts, consider the Ising model on a finite graph $G$ at inverse temperature $\beta$.

(1) Let $\mathcal{L} \subset E$ denote a circuit through $G$ (a closed path which traverses no edge more than once). Prove that

$$\frac{\sum_{\mathbf{n}:\, \partial\mathbf{n}=\emptyset,\, \mathcal{L}\subset E(\mathbf{n})} w_\beta(\mathbf{n})}{\sum_{\mathbf{n}:\, \partial\mathbf{n}=\emptyset} w_\beta(\mathbf{n})} \leq \beta^{|\mathcal{L}|}.$$

(2) Let $\mathcal{P} \subset E$ denote a path from $u$ to $v$ through $G$ which traverses no edge more than once. Prove that

$$\frac{\sum_{\mathbf{n}:\, \partial\mathbf{n}=\{u,v\},\, \mathcal{P}\subset E(\mathbf{n})} w_\beta(\mathbf{n})}{\sum_{\mathbf{n}:\, \partial\mathbf{n}=\emptyset} w_\beta(\mathbf{n})} \leq \beta^{|\mathcal{P}|}.$$

(3) Let $G$ denote an infinite graph of maximum degree $d \in \mathbb{Z}_{\geq 0}$. Let $u$ denote a fixed vertex, and let $\Lambda_n$ denote the set of vertices at distance at most $n-1$ from $u$. Prove that for any $\beta < 1/d$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$\langle \sigma_u \rangle^+_{\Lambda_n,\beta} \leq \frac{(d\beta)^n}{1 - d\beta}.$$

For this part, one needs Remark 6.2 and Exercise 6.3.

Indeed, this is the high-temperature counterpart to the Peierls argument (Theorem 4.2). We did not only prove that for small $\beta$, the magnetisation vanishes (tends to zero), but also that it tends to zero exponentially fast in the distance to the boundary.

**Definition 7.7** (Replacing sums by measures). Sums are measures. Even though there is no added mathematical value in replacing sums by measures, doing so will slightly change our perspective, and also shorten notations.

For an Ising model on some finite graph $G$ with inverse temperature $\beta$, we shall write $\mathbb{M}_{G,\beta}$ for the measure on random currents $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^E$ such that

$$\mathbb{M}_{G,\beta}(\mathbf{n}) = w_\beta(\mathbf{n}).$$

For $A \subset V$, we shall also write $\mathbb{M}^A_{G,\beta}$ for the measure

$$\mathbb{M}_{G,\beta}[\,\cdot\,] := \mathbb{M}_{G,\beta}[(\,\cdot\,)\mathbb{1}_{\{\partial\mathbf{n}=A\}}].$$

**Corollary 7.8.** *The measure $\mathbb{M}_{G,\beta}$ is not a probability measure, but $e^{-\beta|E|}\mathbb{M}_{G,\beta}$ is a probability measure in which $(\mathbf{n}_{uv})_{uv\in E}$ is a family of independent random variables with distribution* Poisson$(\beta)$.

**Corollary 7.9.** *Theorem 7.2 says that*

$$Z_{G,\beta}\langle \sigma_A \rangle_{G,\beta} = 2^{|V|}\mathbb{M}_{G,\beta}[\{\partial\mathbf{n} = A\}] = 2^{|V|}\mathbb{M}^A_{G,\beta}[1].$$

## 8. Double random currents

The previous section explained how correlation functions are expressed in terms of random currents. We also proved a basic result, namely the existence of a demagnetised phase of the Ising model on graphs of bounded degree.

All results discussed so far concern the behaviour of the Ising model in the off-critical regime (very large values of $\beta$, very small values of $\beta$). Our main interest is however in the *critical regimes*: the values for $\beta$ where the model undergoes a qualitative change, such as values in the topological boundary of the set

$$\{\beta \in [0, \infty) : \lim_{n\to\infty} \langle \sigma_u \rangle^+_{\Lambda_n,\beta} = 0\}$$

for a given infinite graph $G$ with a reference point $u$ (as per usual, $\Lambda_n$ refers to the graph metric ball around $u$).

We shall now introduce a new tool to study correlation functions and random currents: the *switching lemma*. In recent years this tool has proved to be instrumental in the derivation of rigorous results on the critical behaviour of the Ising model, especially in graph dimensions 3 and 4.

**Lemma 8.1** (Switching lemma)**.** *Consider the Ising model on a finite graph $G$ at inverse temperature $\beta$. Let $A, B, S \subset V$. Then for any bounded function $F : (\mathbb{Z}_{\geq 0})^E \to \mathbb{C}$, the following identities hold true:*

$$\mathbb{M}^A \times \mathbb{M}^B[F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S)] = \mathbb{M}^{A\Delta S} \times \mathbb{M}^{B\Delta S}[F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S)],$$

*where $A\Delta S$ denotes the symmetric difference of $A$ and $S$.*

*In terms of weights, this is equivalent to*

$$\sum_{\substack{\mathbf{n}:\,\partial\mathbf{n}=A\\ \mathbf{m}:\,\partial\mathbf{m}=B}} w_\beta(\mathbf{n})w_\beta(\mathbf{m})F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S)$$

$$= \sum_{\substack{\mathbf{n}:\,\partial\mathbf{n}=A\Delta S\\ \mathbf{m}:\,\partial\mathbf{m}=B\Delta S}} w_\beta(\mathbf{n})w_\beta(\mathbf{m})F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S).$$

**Remark 8.2.** While the switching lemma is an extremely powerful tool, its statement may appear daunting at first sight. Let us quickly see how switching may be applied to a simple example. Suppose that we record cars traversing a bridge on a road. Blue cars appear according to a Poisson process with rate $\lambda$. Let $X_B$ denote the number of blue cars that passed after recording for one hour. Can we easily prove, without a calculation, that

$$\mathbb{P}[X_B \in 2\mathbb{Z}] \geq \mathbb{P}[X_B \in 2\mathbb{Z}+1]?$$

Suppose that there are also yellow cars, which arrive according to an independent Poisson process with rate $\lambda$. Let $X_Y$ denote the number of yellow cars that passed. Suppose that, after waiting for one hour, $X_B + X_Y = N > 0$ cars passed. What is the *conditional* probability that $X_B$ is even?

Well, we must have $\mathbb{P}[X_B \in 2\mathbb{Z} | \{X_B + X_Y = N\}] = 1/2$. Indeed, by the properties of the Poisson process, the distribution of the cars is invariant under *switching* the colour of the last car. If it was blue before, then we paint it yellow, and vice versa. This operation changes the parity of $X_B$ but leaves the conditional distribution invariant: hence the symmetric probability 1/2.

But we cannot always do the switch. If $X_B + X_Y = 0$ then there is no car to repaint, and also $X_B = 0$. Thus, we conclude that

$$\mathbb{P}[X_B \in 2\mathbb{Z}] - \mathbb{P}[X_B \in 2\mathbb{Z}+1] = \mathbb{P}[\{X_B + X_Y = 0\}] \geq 0.$$

Notice that we originally asked a question about blue cars, but introducing yellow cars allowed us to answer it. This is the essence of the switching lemma.

The switching lemma is analogous to the above example:

- The product measure corresponds to the joint distribution of blue and yellow cars,
- The weights correspond to the rates of the Poisson processes,
- The function $F$ corresponds to the conditioning event $\{X_B + X_Y = N\}$,
- The event $\{\mathbf{n}+\mathbf{m} \in \mathcal{E}_S\}$ corresponds to the event $\{X_B + X_Y > 0\}$.

*Proof of the switching lemma.* By linearity of expectation, it suffices to consider the case that $F(\mathbf{b}) := \mathbb{1}(\mathbf{b} = \mathbf{a})$ for some fixed $\mathbf{a} \in \mathcal{E}_S$. Our objective is then to derive the equality

$$\mathbb{M}^A \times \mathbb{M}^B[\{\mathbf{n}+\mathbf{m} = \mathbf{a}\}] = \mathbb{M}^{A\Delta S} \times \mathbb{M}^{B\Delta S}[\{\mathbf{n}+\mathbf{m} = \mathbf{a}\}]$$

or

$$\mathbb{M}^2[\{\partial\mathbf{n} = A,\, \partial\mathbf{m} = B,\, \mathbf{n}+\mathbf{m} = \mathbf{a}\}] = \mathbb{M}^2[\{\partial\mathbf{n} = A\Delta S,\, \partial\mathbf{m} = B\Delta S,\, \mathbf{n}+\mathbf{m} = \mathbf{a}\}].$$

Define the probability measure

$$\mathbb{P} :\propto \mathbb{M}^2[(\,\cdot\,)\mathbb{1}(\mathbf{n}+\mathbf{m} = \mathbf{a})].$$

It suffices to prove that

$$\mathbb{P}[\{\partial\mathbf{n} = A,\, \partial\mathbf{m} = B\}] = \mathbb{P}[\{\partial\mathbf{n} = A\Delta S,\, \partial\mathbf{m} = B\Delta S\}]. \tag{8}$$

By going back to the definition of $\mathbb{M}$ in terms of $w_\beta$, it is straightforward to see that the pair $(\mathbf{n}, \mathbf{m})$ has the following probability distribution under $\mathbb{P}$:

- The family $(\mathbf{n}_{uv})_{uv}$ is a family of independent random variables,
- The distribution of $\mathbf{n}_{uv}$ is $\mathrm{Binomial}(\mathbf{a}_{uv}, 1/2)$,
- We have $\mathbf{n}+\mathbf{m} = \mathbf{a}$ almost surely, which fixes the joint distribution of $(\mathbf{n}, \mathbf{m})$.

In fact, we may interpret $\mathbb{P}$ in a different way. Define the *multigraph*

$$\mathcal{M}_\mathbf{a} := \{(uv, k) \in E \times \mathbb{Z}_{\geq 0} : \mathbf{a}_{uv} < k\}$$

on the vertex set $V$. Then $\mathbb{P}$ is interpreted as follows:

- We let $\mathcal{K}$ denote a uniformly random subset of $\mathcal{M}_\mathbf{a}$,
- We let $\mathbf{n}_{uv}$ denote the number of multiedges in $\mathcal{K}$ between $u$ and $v$,
- We let $\mathbf{m}_{uv}$ denote the number of multiedges in $\mathcal{M}_\mathbf{a} \setminus \mathcal{K}$ between $u$ and $v$.

Indeed, this definition of $\mathbb{P}$ is consistent with our previous one.

Proving Equation (8) now comes down to proving that the number of submultigraphs $\mathcal{K} \subset \mathcal{M}_\mathbf{a}$ contributing to the events on the left and right, is the same. Let $E_S \subset E(\mathbf{a})$ denote an arbitrary subset such that $\partial E_S = S$, and write $E_{S,0} := E_S \times \{0\} \subset \mathcal{M}_\mathbf{a}$. The existence of the set $E_S$ follows from the fact that $\mathbf{a} \in \mathcal{E}_S$. The reader may now verify that the map

$$\{\partial\mathbf{n} = A,\, \partial\mathbf{m} = B\} \to \{\partial\mathbf{n} = A\Delta S,\, \partial\mathbf{m} = B\Delta S\},\, \mathcal{K} \mapsto \mathcal{K}\Delta E_{S,0}$$

is a bijection. This proves that the two sets have the same cardinality, and thus the same probability under the measure $\mathbb{P}$. We have now established Equation (8) and therefore the lemma. $\square$

**Corollary 8.3** (Second Griffiths inequality)**.** *Consider the Ising model on a finite graph $G$ at inverse temperature $\beta$. Then for any $A, B \subset V$, we have $\langle\sigma_{A\Delta B}\rangle_{G,\beta} - \langle\sigma_A\rangle_{G,\beta}\langle\sigma_B\rangle_{G,\beta} \geq 0$.*

*Proof.* We have $1 = \langle 1\rangle = \langle\sigma_\emptyset\rangle$. By the previous section (for example Corollary 7.9),

$$Z^2(\langle\sigma_{A\Delta B}\rangle\langle\sigma_\emptyset\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle) = 2^{2|V|}(\mathbb{M}^{A\Delta B} \times \mathbb{M}^\emptyset[1] - \mathbb{M}^A \times \mathbb{M}^B[1]).$$

Claim that the quantity on the right is nonnegative. Notice that if $\partial\mathbf{m} = B$, then $\mathbf{m} \in \mathcal{E}_S$, and therefore

$$\begin{aligned}\mathbb{M}^A \times \mathbb{M}^B[1] = \mathbb{M}^A \times \mathbb{M}^B[\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_B)] &\overset{\text{switch}}{=} M^{A\Delta B} \times \mathbb{M}^{\emptyset}[\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_B)] \\ &\le M^{A\Delta B} \times \mathbb{M}^{\emptyset}[1].\end{aligned}$$

This inequality implies the claim, and therefore the second Griffiths inequality. □

**Exercise 8.4** (Conditioning on equality increases the correlation functions)**.** Consider the Ising model on a finite graph $G$ at inverse temperature $\beta$, and fix some subset $A \subset V$.

- Prove that for any two distinct vertices $u, v \in V$, we have
$$\mathbb{E}_{G,\beta}[\sigma_A | \{\sigma_u = \sigma_v\}] \ge \mathbb{E}_{G,\beta}[\sigma_A] = \langle \sigma_A \rangle_{G,\beta}.$$
- Prove for any $X \subset Y \subset V$, we have
$$\mathbb{E}_{G,\beta}[\sigma_A | \{\sigma \text{ is constant on } X\}] \le \mathbb{E}_{G,\beta}[\sigma_A | \{\sigma \text{ is constant on } Y\}].$$

**Exercise 8.5** (The two-point function as a metric)**.** Consider the Ising model on a finite graph $G$ at inverse temperature $\beta > 0$. Prove that $V \times V \to [0,\infty]$, $(u,v) \mapsto -\log\langle \sigma_u \sigma_v \rangle_{G,\beta}$ defines a metric on $V$.

**Definition 8.6** (Probability measures on currents)**.** Consider the Ising model on a finite graph $G$ at inverse temperature $\beta$. For any $A \subset V$, define the probability measure

$$\mathbb{P}^A_{G,\beta} := \frac{2^{|V|}}{Z_{G,\beta}\langle \sigma_A \rangle_{G,\beta}} \mathbb{M}^A_{G,\beta}.$$

For any $A_1, \dots, A_n$, write $\mathbb{P}^{A_1,\dots,A_n} := \mathbb{P}^{A_1} \times \dots \times \mathbb{P}^{A_n}$.

**Exercise 8.7** (Correlation functions in terms of sourceless currents)**.** Consider the Ising model on a finite graph $G$ at inverse temperature $\beta$. Prove that for any $A \subset V$,

$$\langle \sigma_A \rangle^2 = \mathbb{P}^{\emptyset,\emptyset}[\{\mathbf{n}+\mathbf{m} \in \mathcal{E}_A\}].$$

Observe that we can now express all correlation functions in terms of a single fixed probability measure on sourceless random currents.

## 9. Monotonicity in the temperature

**Theorem 9.1** (Monotonicity in the temperature)**.** *Let $G$ denote a finite graph and $A \subset V$ a finite set. Then the function $\beta \mapsto \langle \sigma_A \rangle_{G,\beta}$ is non-decreasing.*

*Proof.* We want to prove that

$$\frac{\partial}{\partial \beta}\langle \sigma_A \rangle_{G,\beta} = \frac{\partial}{\partial\beta}\left(\frac{\sum_\sigma \sigma_A \prod_{uv} e^{\beta\sigma_u\sigma_v}}{\sum_\sigma \prod_{uv} e^{\beta\sigma_u\sigma_v}}\right) \ge 0.$$

Since we are differentiating a fraction, it suffices to show that the numerator grows at a faster rate than the denominator, that is,

$$\frac{\frac{\partial}{\partial\beta}\sum_\sigma \sigma_A \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z\langle\sigma_A\rangle} \ge \frac{\frac{\partial}{\partial\beta}\sum_\sigma \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z}.$$

By multiplying either side by $\langle \sigma_A \rangle$ and differentiating each side, we see that this inequality is equivalent to

$$\sum_{xy} \frac{\sum_\sigma \sigma_x\sigma_y\sigma_A \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z} \ge \langle\sigma_A\rangle \sum_{xy} \frac{\sum_\sigma \sigma_x\sigma_y \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z}.$$

Each fraction may now be reinterpreted as a correlation function, so that the previous inequality is equivalent to

$$\sum_{xy}\langle\sigma_x\sigma_y\sigma_A\rangle \geq \langle\sigma_A\rangle\sum_{xy}\langle\sigma_x\sigma_y\rangle.$$

But this is just the second Griffiths inequality. □

**Exercise 9.2.** Prove that the function $[0,\infty)\to\mathbb{R}$, $\beta\mapsto\langle\sigma_A\rangle_{G,\beta}$ in the above context is an analytic function.

## 10. The thermodynamical limit. Wired boundary

Consider the Ising model on the infinite graph $\mathbb{Z}^d$ for $d\geq 2$. Let $u=0\in\mathbb{Z}^d$ and let $\Lambda_n$ denote the graph metric ball around $u$. We have already derived the following results.

- For large $\beta$, the Ising model exhibits magnetisation in the sense that
$$\inf_n\langle\sigma_0\rangle^+_{\Lambda_n,\beta}>0.$$
This was proved via the Peierls argument, see Theorem 4.1 and Exercise 4.5.
- For small $\beta$, the Ising model *does not* exhibit magnetisation:
$$\lim_{n\to\infty}\langle\sigma_0\rangle^+_{\Lambda_n,\beta}=0.$$
This was proved via a Peierls argument for random currents, see Exercise 7.6.

At the time moment of stating the Peierls argument (Section 4), we knew almost nothing about the Ising model. Our understanding is now advancing. We already used the first Griffiths inequality to show that $\langle\sigma_0\rangle^+_{\Lambda_n,\beta}\geq 0$ (Corollary 7.3 and Exercise 7.4). Our first objective is now to prove the following result. To state it, we write

$$\lim_{\Lambda\uparrow V} f(\Lambda)\qquad\text{for}\qquad\lim_{n\to\infty}f(\Lambda_n),$$

where $(\Lambda_n)_n$ is any increasing sequence of domains with $\cup_n\Lambda_n=V$. This notation makes sense only when the limit is independent of the precise choice of the sequence $(\Lambda_n)_n$, and is called the *thermodynamical limit* or *infinite-volume limit*.

**Lemma 10.1** (Correlation functions are monotone in the domain (wired boundary))**.** *Consider the Ising model on a locally finite graph $G$ at inverse temperature $\beta$. Let $A\subset V$ denote any finite subset. Then the function*

$$\Lambda\mapsto\langle\sigma_A\rangle^+_{\Lambda,\beta}$$

*is a nonincreasing function of the domain $\Lambda$.*

*In particular, we have well-definedness of the thermodynamical limit*

$$\lim_{\Lambda\uparrow V}\langle\sigma_A\rangle^+_{\Lambda,\beta}.$$

*Proof.* Consider two domains $\Lambda\subset\bar\Lambda$. We want to show that

$$\langle\sigma_A\rangle^+_{\Lambda,\beta}\geq\langle\sigma_A\rangle^+_{\bar\Lambda,\beta}.$$

Without loss of generality, $A\subset\bar\Lambda$ and $\bar\Lambda\setminus\Lambda=\{u\}$ for some vertex $u\in V$.

Let $G'=(\bar\Lambda\cup\{\bar\Lambda^c\},E(\bar\Lambda))$ denote the graph obtained from $\bar\Lambda$ as in Remark 6.2 and Exercise 6.3. We refer to the Ising model on $G'$ when subscripts are submitted from now on. Assume that $|A|$ is even for now. Then

$$\langle\sigma_A\rangle^+_{\bar\Lambda,\beta}=\mathbb{E}[\sigma_A];\qquad\langle\sigma_A\rangle^+_{\Lambda,\beta}=\mathbb{E}[\sigma_A|\{\sigma_u=\sigma_{\bar\Lambda^c}\}].$$

It suffices to show that the conditioning increases the expectation. This is just Exercise 8.4.

If $|A|$ is odd then we just need to replace the set $A$ by $A' := A \cup \{\bar{\Lambda}^c\}$. More precisely, we have

$$\langle \sigma_A \rangle^+_{\Lambda,\beta} = \mathbb{E}[\sigma_{A'}]; \qquad \langle \sigma_A \rangle^+_{\Lambda,\beta} = \mathbb{E}[\sigma_{A'} | \{\sigma_u = \sigma_{\bar{\Lambda}^c}\}].$$

One may then simply apply Exercise 8.4 as for the even case. □

Perhaps we were wondering if $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$ was decreasing in $n$ in the statement of the Peierls argument (Theorem 4.2), but the result we proved just now is much stronger: we proved that the thermodynamical limit of any "local Fourier coefficient" is well-defined. Rather than taking a thermodynamical of observables, we would however like to make sense of the thermodynamical limit of the family of measures $\langle \cdot \rangle^+_{\Lambda,\beta}$. The previous lemma enables us to do this; we only need to set up the definitions to make formal sense of our limit.

**Definition 10.2** (The compact "local convergence" topology)**.** Let $G$ denote a locally finite graph. Recall that $(\Omega, \mathcal{F})$ is the measurable space $\Omega := \{\pm 1\}^V$ endowed with the product $\sigma$-algebra. For a domain $\Lambda$, we write $\mathcal{F}_\Lambda$ for the $\sigma$-algebra generated by spins in $\Lambda$. An observable $X : \Omega \to \mathbb{C}$ is called *local* if it measurable with respect to $\mathcal{F}_\Lambda$ for some domain $\Lambda$.

Let $\mathcal{P}(\Omega, \mathcal{F})$ denote the set of all probability measures on this measurable space. We endow this set with the *local convergence topology*, which is defined as the topology making the map

$$\mathcal{P}(\Omega, \mathcal{F}) \to \mathbb{C},\ \mu \mapsto \mu[X]$$

continuous for any local observable $X$.

**Remark 10.3.** This topology is sometimes known under different names in the literature (such as the *weak topology*). I like the name *local convergence topology* because it captures the essence quite literally: if the statistics of the measures within a fixed domain $\Lambda$ converge, then we have local convergence.

**Remark 10.4.** The local convergence topology turns $\mathcal{P}(\Omega, \mathcal{F})$ into a compact space. For a fixed domain $\Lambda$, the set of probability measures on $(\Omega, \mathcal{F}_\Lambda)$ is a compact simplex in some finite-dimensional real vector space. Sequences of probability measures of this type have converging subsequences by standard arguments. Convergence for arbitrary $\Lambda$ may be obtained by a standard diagonal argument.

**Theorem 10.5** (Existence of the thermodynamical limit (wired boundary))**.** *Consider the Ising model on a locally finite graph $G$ at inverse temperature $\beta$. Then there exists a unique probability measure $\langle \cdot \rangle^+_{G,\beta} \in \mathcal{P}(\Omega, \mathcal{F})$ such that*

$$\lim_{\Lambda \uparrow V} \langle X \rangle^+_{\Lambda,\beta} = \langle X \rangle^+_{G,\beta}$$

*for any local observable $X : \Omega \to \mathbb{R}$. In other words,*

$$\lim_{\Lambda \uparrow V} \langle \cdot \rangle^+_{\Lambda,\beta} = \langle \cdot \rangle^+_{G,\beta}.$$

*The measure $\langle \cdot \rangle^+_{G,\beta}$ is called the* thermodynamical limit *or* infinite-volume limit *with + boundary conditions.*

*Proof.* Any local observable may be written as a finite linear conbination of observables of the form $\sigma_A$ where $A$ is a finite subset of $V$. The theorem then follows by compactness and Lemma 10.1. □

**Exercise 10.6** (Right-continuity in $\beta$ for a wired boundary)**.** Consider the Ising model on a locally finite graph $G$. Prove all of the following statements.

- For any finite set $A \subset V$, the function $[0, \infty) \to \mathbb{R}$, $\beta \mapsto \langle \sigma_A \rangle^+_{G,\beta}$ is non-decreasing and right-continuous.
- The function $[0, \infty) \to \mathcal{P}(\Omega, \mathcal{F})$, $\beta \mapsto \langle \cdot \rangle^+_{G,\beta}$ is a right-continuous function.

- The points of discontinuity form a countable subset of $[0, \infty)$.

**Definition 10.7** (Magnetisation and critical temperature)**.** Let $G$ denote a locally finite graph and $u$ some distinguished reference vertex. The function

$$m = m_G : [0, \infty) \to \mathbb{R},\, \beta \mapsto \langle \sigma_u \rangle^+_{G,\beta}$$

is called the *magnetisation*.

The *critical (inverse) temperature* is defined via

$$\beta_c := \inf\{\beta \in [0, \infty) : m(\beta) > 0\}.$$

**Remark 10.8.** The exercise above proves that $m$ is a non-decreasing right-continuous function. We have already seen that:

- If $G$ has max-degree $d$, then $m(\beta) = 0$ for $\beta < 1/d$ (Exercise 7.6),
- If $G$ is the graph $\mathbb{Z}^d$ for $d \geq 2$, then $\lim_{\beta\to\infty} m(\beta) > 0$ (Exercise 4.5).

This implies in particular that on the square lattice graph $\mathbb{Z}^d$ in dimension $d \geq 2$, the critical inverse temperature $\beta_c$ is a strictly positive real number. A key objective of our field is to understand the behaviour of the Ising model at $\beta = \beta_c$ and at $\beta \approx \beta_c$. As a very first question we can ask: is the function $m$ continuous? We will see the answer in Theorem **??**. Add ref to theorem

## References

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