A COURSE ON THE ISING MODEL

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Contents

Preface

These lecture notes are written progressively during the 2025 spring semester, as the course is taught at Sorbonne university in the M2 (second-year masters) programme. Its purpose is to give a broad introduction to the rigorous analysis of the Ising model. The main focus is on four techniques and their applications:

- The Peierls argument,
- The random-currents representation,
- The FKG inequality for the Ising spins,
- The FKG inequality for the random-cluster (FK) representation.

A basic understanding of analysis and probability theory is essential for following this course. Experience with other models in statistical mechanics (such as the Bernoulli percolation model) is a plus but by no means essential.

The appendices contain overviews of the main definitions, expansions, and inequalities in this text.

These notes are inspired by the lecture notes *Lectures on the Ising and Potts models on the hypercubic lattice* and the overview *100 Years of the (Critical) Ising Model on the Hypercubic Lattice*, both due to Hugo Duminil-Copin. The main text does not contain references at this stage; they will be added at a later time.

1. The Curie–Weiss model

At the end of the 19th century, Curie published his experimental results on *ferromagnetism*: the magnetic properties of metals. He made three striking observations.

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- The magnetic strength of a metal varies with the temperature. Increasing the temperature decreases the magnetic strength.
- Each metal has a certain temperature, specific to that metal, at which the magnetic properties disappear entirely. We call this temperature the *Curie temperature*.
- Around the Curie temperature, the magnetic strength drops continuously to zero. In other words, the magnetic strength does not "jump" to zero.

The first observation singles out the temperature as the driving parameter of the system. This is good news for us, since the temperature may be regarded informally as the amount of "randomness" or "entropy" in the system, justifying a probabilistic analysis of the situation. The second observation implies that there is a *phase transition*: there is a special temperature (in this case the Curie temperature) at which the system's behaviour undergoes a qualitative change. The third observation entails an important property of this phase transition.

The first mathematical explanation for Curie's experimental results came from Weiss. He proposed the following mathematical axioms for studying the magnetic properties of metals.

- The metal consists of n atoms.
- Each atom acts like a small magnet in itself. It is in one of two states, denoted $\pm$.
- The total strength of the metal is obtained by summing the states of all atoms.
- Each atom interacts with all other atoms. The atoms prefer to *align*, that is, to be in the same state. The temperature regulates the strength of the interaction.

Physically, it makes sense that the temperature regulates the interaction strength. When atoms move slowly, they will stabilise, oriented in alignment with the magnetic field imposed by the other atoms. When atoms move fast, they will not bother with the states of the other atoms, and simply align themselves randomly. It is thus natural to think of the interaction strength as the *inverse temperature*.

Definition 1.1 (Curie-Weiss model)**.** The Curie-Weiss model is the probability measure $\mathbb{P}_{n,\beta}^{\mathrm{CW}}$ on $\sigma \in \Omega := \{+,-\}^n$ defined via

$$\mathbb{P}(\sigma) := \mathbb{P}_{n,\beta}^{\mathrm{CW}}(\sigma) \propto e^{-H_{n,\beta}^{\mathrm{CW}}(\sigma)}; \qquad H(\sigma) := H_{n,\beta}^{\mathrm{CW}}(\sigma) := -\frac{\beta}{n}\sum_{i<j}\sigma_i\sigma_j,$$

where $n \in \mathbb{Z}_{\geq 1}$ and $\beta \in [0,\infty)$. The parameter β is called the *interaction strength* or *inverse temperature*. The function H is called the *Hamiltonian* and captures the *energy* in the system. The probability measure $\mathbb{P}$ is also called the *Boltzmann distribution*.

Let $n_+ = n_+(\sigma)$ denote the number of vertices with spin $+$ in a configuration $\sigma \in \Omega$. This is a random variable. Let us try to calculate the probability of the event $\{n_+ = k\}$, without worrying about the partition function (the normalising constant). One may easily check that the Hamiltonian satisfies

$$H(\sigma) = 2\frac{\beta}{n}n_+(n-n_+) + \mathrm{const}(n).$$

The distribution of n_+ can then be calculated as follows:

$$\mathbb{P}(\{n_+ = k\}) \propto \binom{n}{k} e^{-2\frac{\beta}{n}k(n-k)} \propto \frac{1}{k!(n-k)!}e^{-2\frac{\beta}{n}k(n-k)}. \tag{1}$$

Using Stirling's approximation for the factorials, we find that

$$\begin{gathered}\log \mathbb{P}(\{n_+ = k\}) \overset{\text{Stirling}}{\approx} -nf_\beta(k/n) + \mathrm{const}(n); \\ f_\beta : [0,1] \to \mathbb{R},\, x \mapsto x\log x + (1-x)\log(1-x) + 2\beta x(1-x).\end{gathered}$$

If we fix β and send n to infinity, then we discover a large deviations principle for the random variable n_+/n with rate function f_β and speed n. In particular, the random variable n_+/n concentrates around the minimisers of the function f_β.

Exercise 1.2 (The rate function of the Curie–Weiss model)**.**

(1) Show that for small β, the function f_β has a single minimum at $x = 1/2$, which means that the random variable n_+/n concentrates around the value $1/2$.
(2) Show that for large β, the function f_β has two minima at $(1 \pm m)/2$ for some $m > 0$, which means that the random variable n_+/n is concentrated around these minima. The value of m is called the *magnetisation*.
(3) Calculate the critical value for β. At this value, the second derivative of f_β vanishes at $x = 1/2$. What does this mean for the distribution of n_+/n? Estimate the order of magnitude of $\operatorname{Var} \frac{n_+}{n}$ as $n \to \infty$ for this value of β.

Remark 1.3 (Entropy versus energy in the Curie–Weiss model)**.** Reconsider Equation (1). In this equation, the competition between the two factors is extremely transparent.

- First, there is a combinatorial term or *entropy*, which favours values k for the random variable n_+ such that the cardinality of the set $\{n_+ = k\}$ is large. This means that values $k \approx n/2$ are preferred.
- Second, there is the *energy* term, which favours values such that the energy is minimised. This favours configurations where as many spins as possible align.

The interaction parameter β allows us to put more emphasis on the entropy term or on the energy term. In the $n \to \infty$ limit, there is a precise value for β where the behaviour of the random system undergoes a qualitative change: a rudimentary example of a *phase transition*.

2. Ising's model and basic notions

While the competition between entropy and energy is transparent in the Curie–Weiss model, the model does not encode any kind of geometry. Indeed, all atoms interact equally with all other atoms. It would perhaps be more realistic to place the atoms on a Euclidean grid, and let the interactions strength between two atoms depend on their distance. In the simplest case, we could simply let each atom interact only with the atoms closest to it. This is called the *nearest-neighbour interaction*. We mainly focus on this setup in these lecture notes.

Wilhelm Lenz challenged his doctoral student Ernst Ising to solve this nearest-neighbour model for magnetism on the one-dimensional line graph $\mathbb{Z}$. Lenz was not entirely precise when posing this question, and it was Ising who first formulated a definition for the model under consideration. The model is therefore called the *Ising model* in his honour. We shall later address Ising's analysis in an exercise (Exercise **??**). Add exercise

Definition 2.1 (Ising model)**.** The Ising model on a finite graph $G = (V, E)$ with *inverse temperature* $\beta \in [0, \infty)$ is defined as follows. Let $\Omega := \{\pm 1\}^V$ denote the set of spin configurations on the vertices of the graph; a typical element of Ω is denoted by $\sigma = (\sigma_u)_{u \in V}$. Elements $\sigma \in \Omega$ are called *spin configurations*; elements σ_u are called *spins*. The *energy* or *Hamiltonian* of a spin configuration σ is given by

$$H^{\mathrm{Ising}}_{G,\beta}(\sigma) := -\beta \sum_{uv \in E} \sigma_u \sigma_v.$$

We write $\mathbb{P}^{\mathrm{Ising}}_{G,\beta}$ for the associated *Boltzmann distribution* or *Gibbs measure*:

$$\mathbb{P}^{\mathrm{Ising}}_{G,\beta}(\sigma) := \frac{1}{Z^{\mathrm{Ising}}_{G,\beta}} e^{-H^{\mathrm{Ising}}_{G,\beta}(\sigma)},$$

where $Z^{\text{Ising}}_{G,\beta}$ is normalisation constant or *partition function* defined by

$$Z^{\text{Ising}}_{G,\beta} := \sum_{\sigma\in\Omega} e^{-H^{\text{Ising}}_{G,\beta}(\sigma)}.$$

We shall write $\langle\,\cdot\,\rangle^{\text{Ising}}_{G,\beta}$ for the expectation functional associated to this probability measure.

Remark. We shall often suppress subscripts and superscripts when they are clear from the context.

Remark. The mathematical community has widely adopted the terminology coming from the physics literature. We often prefer the symbol $\langle\,\cdot\,\rangle$ over $\mathbb{E}[\,\cdot\,]$ when taking expecations, expect when considering *conditional* expectations.

Exercise 2.2 (The edge graph)**.** Consider the Ising model on the complete graph on the two vertices $V := \{x, y\}$ at inverse temperature $\beta \in [0,\infty)$.

- Calculate $\langle\sigma_x\rangle_\beta$.
- Calculate $\langle\sigma_x\sigma_y\rangle_\beta$.

Definition 2.3 (Correlation functions)**.** Consider $\Omega := \{\pm\}^V$. Then for any finite subset $A \subset V$, we define $\sigma_A : \Omega \to \{\pm\}$, $\sigma \mapsto \prod_{x\in A}\sigma_x$. Its expectation $\langle\sigma_A\rangle$ in any probability measure $\langle\,\cdot\,\rangle$ on Ω is called a *correlation function.* If $|A| = n$ then $\langle\sigma_A\rangle$ is also called an *n-point correlation function.*

Remark (Flip-symmetry)**.** The Ising model is *flip-symmetric* in the sense that the distribution of the spins is invariant under the transformation $\sigma \mapsto -\sigma$. This is because the Hamiltonian is invariant under this transformation.

Exercise 2.4 (Flip-symmetry)**.** Consider the Ising model on a finite graph $G = (V, E)$.

- Prove that if $A \subset V$ contains an odd number of vertices, then $\langle\sigma_A\rangle = 0$.
- Prove that if $A \subset V$ contains an odd number of vertices and $x \in V$, then

$$\mathbb{E}[\sigma_A|\{\sigma_x = +\}] = \langle\sigma_A\sigma_x\rangle.$$

- Prove that if $A \subset V$ contains an even number of vertices and $x \in V$, then

$$\mathbb{E}[\sigma_A|\{\sigma_x = +\}] = \langle\sigma_A\rangle.$$

In practice, we are interested in the Ising model on finite portions of the square lattice $\mathbb{Z}^d$ endowed with nearest-neighbour connectivity. We now provide the definitions for this setup.

Definition 2.5 (Free boundary conditions)**.** Let $G = (V, E)$ denote a locally finite graph and $\Lambda \subset V$ a finite set. Write Λ^{f} for the subgraph of G induced by Λ. Write $\langle\,\cdot\,\rangle^{\mathrm{f}}_{\Lambda,\beta} := \langle\,\cdot\,\rangle_{\Lambda^{\mathrm{f}},\beta}$ for the *free-boundary Ising model* in Λ at inverse temperature $\beta \in [0,\infty)$.

Definition 2.6 (Fixed boundary conditions)**.** Let $G = (V, E)$ denote a locally finite graph and $\Lambda \subset V$ a finite set. Let $\partial\Lambda \subset V \setminus \Lambda$ denote the set of vertices adjacent to Λ. Write $\bar\Lambda$ for the graph defined by

$$V(\bar\Lambda) := \Lambda \cup \partial\Lambda; \qquad E(\bar\Lambda) := \{\{x, y\} \in E : \{x, y\} \cap \Lambda \neq \emptyset\}.$$

For any $\zeta \in \{\pm\}^{\partial\Lambda}$, we shall write $\langle\,\cdot\,\rangle^{\zeta}_{\Lambda,\beta}$ for the measure

$$\langle\,\cdot\,\rangle^{\zeta}_{\Lambda,\beta} := \mathbb{E}_{\bar\Lambda,\beta}[\,\cdot\,|\{\sigma|_{\partial\Lambda} = \zeta\}].$$

This is called the *fixed-boundary Ising model* with boundary conditions ζ. The boundary condition $\zeta \equiv \pm$ is of particular interest, and it is denoted $\langle\,\cdot\,\rangle^{\pm}_{\Lambda,\beta}$.

Exercise 2.7 (Markov property)**.** Consider the Ising model on some finite graph $G = (V, E)$ at inverse temperature β. Fix some $\Lambda \subset V$ and let $(\Lambda_i)_i$ denote the partition of Λ into connected components. Let $\zeta \in \{\pm\}^{\Lambda^c}$, and consider the conditional probability measure $\mathbb{P}[\,\cdot\,|\{\sigma|_{\Lambda^c} = \zeta\}]$.

- Prove that $(\sigma|_{\Lambda_i})_i$ is a family of independent random variables in this measure.
- Prove that the law of $\sigma|_{\Lambda_i}$ is $\langle\,\cdot\,\rangle_{\Lambda_i}^{\zeta|_{\partial\Lambda_i}}$.

Hint. Decompose the Hamiltonian according to $H(\sigma) = C + \sum_i H_i(\sigma)$, where each H_i is measurable in terms of $\zeta|_{\partial\Lambda_i}$ and $\sigma|_{\Lambda_i}$.

Ising proved that in one dimension, the Ising model exhibits exponential decay of correlations at all temperatures. In other words, there is no phase transition. We now state his result, without a proof. While the proof is quite straightforward even with elementary methods, its proof becomes entirely trivial after the introduction of more recent methods.

Theorem 2.8 (Ising, 1924)**.** *Consider the finite domains $\Lambda_n := \{-n, \ldots, n\}$ of the graph $\mathbb{Z}$. Then for any $\beta \in [0, \infty)$, there exists a constant $c = c_\beta > 0$ such that*

$$\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \leq \frac{1}{c} e^{-c_\beta \cdot n}.$$

Unfortunately, Ising wrongly conjectured that the same would be true in higher dimension. Disappointed with this prediction, he left academia.

3. Peierls' argument

Peierls disproved Ising's conjecture for the absence of phase transition in dimension $d \geq 2$.

Theorem 3.1 (Peierls, 1936)**.** *Consider the finite domains $\Lambda_n := \{-n, \ldots, n\}^2$ of the square lattice graph $\mathbb{Z}^2$. Then for sufficiently large $\beta \in [0, \infty)$, we have*

$$\inf_n \langle\sigma_{(0,0)}\rangle^+_{\Lambda_n,\beta} > 0.$$

Proof. Our objective is to prove that $\mathbb{P}^+_{\Lambda_n,\beta}[\{\sigma_{(0,0)} = -\}] \leq \frac{1}{3}$ for all n. Fix n.

Consider the set $\Omega' \subset \Omega$ of spin configurations on $\bar{\Lambda}_n$ which assign $+$ to $\partial\Lambda_n$. The two-dimensional square lattice graph $G = \mathbb{Z}^2$ is planar, and therefore we may consider its planar dual G^*. For any spin configuration $\sigma \in \Omega'$, we let $\mathcal{I}(\sigma) \subset E(G^*)$ denote its *interface*, that is, the set of dual edges separating two spins with a *distinct* value. Notice that:

- The map $\sigma \mapsto \mathcal{I}(\sigma)$ is injective,
- If $\sigma_{(0,0)} = -$, then $\mathcal{I}(\sigma)$ contains at least one self-avoiding loop around $(0, 0)$.

In particular, inclusion of events yields

$$\mathbb{P}^+_{\Lambda_n,\beta}[\{\sigma_{(0,0)} = -\}] \leq \mathbb{P}^+_{\Lambda_n,\beta}[\{\mathcal{I}(\sigma) \text{ contains a loop } \gamma \text{ around } (0, 0)\}]. \tag{2}$$

We would now like to make a competition between entropy and energy appear, as for the Curie-Weiss model. The entropy comes from the choice of the loop γ; the energy comes into play when upper bounding the probability that a particular loop belongs to $\mathcal{I}(\sigma)$. For large β, energy wins over entropy, yielding the desired bound. Let us start with the energy bound.

Claim. For any fixed loop γ, we may bound $\mathbb{P}^+_{\Lambda_n,\beta}[\{\gamma \subset \mathcal{I}(\sigma)\}] \leq e^{-2\beta|\gamma|}$.

Proof of the claim. We would like to define a *loop erasure map* $\mathcal{E} : \{\gamma \subset \mathcal{I}(\sigma)\} \to \Omega'$, which has the property that it removes the loop γ from the interface, that is,

$$\mathcal{I}(\mathcal{E}(\sigma)) = \mathcal{E}(\sigma) \setminus \gamma.$$

It is easy to realise such a map: we simply define $\mathcal{E}$ such that it flips the sign of every vertex of Λ_n which is surrounded by γ. Since $\mathcal{I}$ is injective, the map $\mathcal{E}$ is also injective, and we have

$$\mathbb{P}^+_{\Lambda_n,\beta}[\{\gamma\subset\mathcal{I}(\sigma)\}]=\frac{\sum_{\sigma\in\mathrm{Domain}(\mathcal{E})}e^{-H(\sigma)}}{\sum_{\sigma\in\Omega'}e^{-H(\sigma)}}\leq\frac{\sum_{\sigma\in\mathrm{Domain}(\mathcal{E})}e^{-H(\sigma)}}{\sum_{\sigma\in\mathrm{Image}(\mathcal{E})}e^{-H(\sigma)}}=e^{-2\beta|\gamma|}.$$

The last equality is easy, since for any $\sigma\in\{\gamma\subset\mathcal{I}(\sigma)\}$, we have

$$H(\mathcal{E}(\sigma))=H(\sigma)-2\beta|\gamma|,$$

since $\mathcal{E}$ removes precisely $|\gamma|$ disagreement edges from the interface. This proves the claim.

We use the energy bound to prove another interesting intermediate result.

Claim (Exponential decay of the loop length)**.** For any dual edge e, we have

$$\mathbb{P}^+_{\Lambda_n,\beta}[\{\mathcal{I}(\sigma)\text{ contains a loop of length at least }\ell\text{ through }e\}]\leq(3e^{-2\beta})^\ell\frac{1}{1-3e^{-2\beta}}$$

whenever $3e^{-2\beta}<1$.

Proof of the claim. Let $\mathcal{L}_k$ denote the set of self-avoiding loops through e of length k. Notice that $|\mathcal{L}_k|\leq 3^k$. A union bound yields

$$\begin{aligned}&\mathbb{P}^+_{\Lambda_n,\beta}[\{\mathcal{I}(\sigma)\text{ contains a loop of length at least }\ell\text{ through }e\}]\\&\qquad\leq\sum_{k=\ell}^\infty\sum_{\gamma\in\mathcal{L}_k}\mathbb{P}^+_{\Lambda_n,\beta}[\{\gamma\subset\mathcal{I}(\sigma)\}]\leq\sum_{k=\ell}^\infty|\mathcal{L}_k|\cdot e^{-2\beta k}\leq\sum_{k=\ell}^\infty 3^k\cdot e^{-2\beta k}\\&\qquad=(3e^{-2\beta})^\ell\frac{1}{1-3e^{-2\beta}}.\end{aligned}$$

This is the desired bound.

Return to Equation (2). If $\mathcal{I}(\sigma)$ contains a loop around $(0,0)$, then this loop must hit $(k-\frac12,0)$ for some $k\in\mathbb{Z}_{\geq1}$, and this loop must have at least k steps. Thus, another union bound yields

$$\mathbb{P}^+_{\Lambda_n,\beta}[\{\sigma_{(0,0)}=-\}]\leq\sum_{k=1}^\infty(3e^{-2\beta})^k\frac{1}{1-3e^{-2\beta}}=(3e^{-2\beta})\frac{1}{(1-3e^{-2\beta})^2}.$$

The right hand side is smaller than $\frac13$ when β is sufficiently large, independently of n. $\square$

Remark. Peierls' is robust, in the sense that it can be adapted to many other models in statistical mechanics.

Exercise 3.2 (The Peierls argument in higher dimensions)**.** Now consider the square lattice graph $\mathbb{Z}^d$ in dimension $d\geq3$. What is the structure of the interface in this case? Can we adapt Peierls' to prove magnetisation for sufficiently large β?

4. The high-temperature expansion

The previous section proved the Peierls argument. An essential ingredient was to view the Ising model in two dimensions through the *interfaces* of the spins. Such a transformation of the model may be viewed as a rudimentary version of an *expansion.* The interface perspective is sometimes called the *low-temperature expansion* because it works well in the low-temperature regime (when β is large). There are several useful expansions for the Ising model; each one of them is adapted to a different setting. In this section we discuss another expansion: the *high-temperature expansion.* As the name suggests, this expansion is well-adapted to situations where β is small, even though we can also use it to prove useful results in other regimes. Appendix ?? contains an overview of the expansions discussed in these notes, and may serve as a reference. Add Appendix and reference

For a streamlined presentation, we will henceforth present all our expansions for the Ising model on *finite graphs without boundary conditions*. This obviously includes the free boundary conditions. It is straightforward to see that fixed boundary conditions also fit into this framework, see Definition 4.5 and Lemma 4.6 below.

Consider the Ising model on a finite graph G. We are typically interested in the correlation functions, defined via

$$\langle \sigma_A \rangle = \frac{\sum_{\sigma\in\Omega} \sigma_A e^{-H(\sigma)}}{\sum_{\sigma\in\Omega} e^{-H(\sigma)}} = \frac{\sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} e^{\beta\sigma_x\sigma_y}}{\sum_{\sigma\in\Omega} \prod_{xy\in E} e^{\beta\sigma_x\sigma_y}}.$$

An *expansion* of the Ising model involves rewriting the sum

$$\sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} e^{\beta\sigma_x\sigma_y}.$$

A typical expansion comes down to rewriting the exponential, for example:

- We may write $e^{\beta\sigma_x\sigma_y} = \cosh\beta + \sigma_x\sigma_y \sinh\beta$,
- We may write $e^{\beta\sigma_x\sigma_y} = \sum_{k=0}^\infty (\beta\sigma_x\sigma_y)^k/k!$,
- We may write $e^{\beta\sigma_x\sigma_y} = e^{-2\beta} + 2\cdot \mathbb{1}(\sigma_x = \sigma_y)\sinh\beta$.

Every expansion comes with its own advantages and disadvantages. The high-temperature expansion derives from the first identity.

Definition 4.1 (High-temperature expansion)**.** Consider the Ising model on a finite graph $G = (V, E)$ at inverse temperature β. We consider *percolation configurations* $\omega \in \{0, 1\}^E$; each ω is also regarded a (random) set of edges. We write $\partial\omega \subset V$ for the set of vertices having *odd* degree in the graph (V, ω).

The *high-temperature expansion* is the measure $\mathbf{M}_{G,\beta}$ on $\omega \in \{0, 1\}^E$ defined by

$$\mathbf{M}[\omega] := \mathbf{M}_{G,\beta}[\omega] := (\cosh\beta)^{|E\setminus\omega|}(\sinh\beta)^{|\omega|}.$$

Theorem 4.2 (High-temperature expansion for correlation functions)**.** *Consider the Ising model on a finite graph $G = (V, E)$. Then for any $A \subset V$, we have*

$$Z\langle\sigma_A\rangle = 2^{|V|}\mathbf{M}[\{\partial\omega = A\}].$$

Proof. We claim that

$$\begin{aligned}
Z\langle\sigma_A\rangle &= \sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} e^{\beta\sigma_x\sigma_y} && (3)\\
&= \sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} (\cosh\beta + \sigma_x\sigma_y\sinh\beta) && (4)\\
&= \sum_{\sigma\in\Omega} \sigma_A \sum_{\omega\in\{0,1\}^E} \prod_{xy\in E} (\cosh\beta)^{1-\omega_{xy}}(\sigma_x\sigma_y\sinh\beta)^{\omega_{xy}} && (5)\\
&= \sum_{\omega\in\{0,1\}^E} (\cosh\beta)^{|E\setminus\omega|}(\sinh\beta)^{|\omega|} \sum_{\sigma\in\Omega} \sigma_A\sigma_{\partial\omega} && (6)\\
&= \sum_{\omega\in\{0,1\}^E} (\cosh\beta)^{|E\setminus\omega|}(\sinh\beta)^{|\omega|} 2^{|V|}\mathbb{1}(A = \partial\omega) && (7)\\
&= 2^{|V|}\mathbf{M}[\{\partial\omega = A\}].
\end{aligned}$$

Equations (3) and (4) come down to definitions and the identity for $e^{\beta\sigma_x\sigma_y}$. Swapping the sum and the product yields Equation (5). Equation (6) is a rearrangement of the terms, noting that $\prod_{xy}(\sigma_x\sigma_y)^{\omega_{xy}} = \sigma_{\partial\omega}$. Equation (7) is obtained by resolving the sum over σ. The final equation is the definition of $\mathbf{M}$. $\square$

We can use this theorem to state our first important *correlation inequality.*

Theorem 4.3 (First Griffiths inequality)**.** *Consider the Ising model on a finite graph $G = (V, E)$. Then for any $A \subset V$, we have $\langle \sigma_A \rangle \geq 0$.*

Proof. The previous theorem yields a nonnegative number for $Z\langle \sigma_A \rangle$. □

Theorem 4.4 (Exponential decay at high temperature)**.** *Consider the Ising model with $+$ boundary conditions on the graph $\mathbb{Z}^d$ for $d \in \mathbb{Z}_{\geq 1}$. Then for any $\beta \in [0, \infty)$ such that $(2d-1)\tanh\beta < 1$, there exists a constant $c = c_{d,\beta} > 0$ such that*

$$\langle \sigma_x \rangle^+_{\Lambda,\beta} \leq \tfrac{1}{c} e^{-c\,\mathrm{Distance}(x,\Lambda^c)}$$

for any $x \in \mathbb{Z}^d$ and any domain $\Lambda \subset \mathbb{Z}^d$.

We would like to use the high-temperature expansion, but for this we must first write $\langle \cdot \rangle^+_\Lambda$ as an Ising model on a finite graph without boundary condition.

Definition 4.5 (Ghost vertex)**.** Let $G = (V, E)$ denote a locally finite graph, and $\Lambda \subset V$ a finite domain. We already defined the graphs Λ^{f} and $\bar{\Lambda}$. Now define the graph $\Lambda^{\mathfrak{g}}$ as follows: it is obtained from the graph $\bar{\Lambda}$ by replacing all vertices in $\partial\Lambda$ by a single distinguished vertex $\mathfrak{g}$, called the *ghost vertex.* Its vertex set is given by $V(\Lambda^{\mathfrak{g}}) := \Lambda \cup \{\mathfrak{g}\}$, and there is a natural bijection from $E(\bar{\Lambda})$ to $E(\Lambda^{\mathfrak{g}})$.

Notice that $\Lambda^{\mathfrak{g}}$ is a multigraph when some $x \in \Lambda$ is connected to multiple vertices in $\partial\Lambda$ in the graph $\bar{\Lambda}$, but this does not really affect our setup.

It is easy to see that the following lemma holds true.

Lemma 4.6. *Let $G = (V, E)$ denote a locally finite graph, and $\Lambda \subset V$ a finite domain. Then the distribution of $\sigma|_\Lambda$ is the same in the following two measures:*

$$\langle \cdot \rangle^+_\Lambda \qquad \text{and} \qquad \mathbb{E}_{\Lambda^{\mathfrak{g}}}[\,\cdot\,|\sigma_{\mathfrak{g}} = +].$$

Correlation functions can thus be expressed in terms of correlation functions on finite graphs via Exercise 2.4.

Proof overview of Theorem 4.4. We have

$$\langle \sigma_x \rangle^+_\Lambda = \langle \sigma_x \sigma_{\mathfrak{g}} \rangle_{\Lambda^{\mathfrak{g}}} = \frac{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \{x, \mathfrak{g}\}\}]}{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \emptyset\}]}.$$

If $\partial\omega = \{x, \mathfrak{g}\}$, then ω contains a self-avoiding walk γ from x to $\mathfrak{g}$. A union bound yields

$$\langle \sigma_x \rangle^+_\Lambda \leq \sum_\gamma \frac{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \{x, \mathfrak{g}\}\} \cap \{\gamma \subset \omega\}]}{\mathbf{M}_{\Lambda^{\mathfrak{g}}}[\{\partial\omega = \emptyset\}]}.$$

The proof is now completed after performing the two steps of the Peierls argument:

- One bounds each term by $(\tanh\beta)^{|\gamma|}$,
- One bounds the number of walks γ of length n from x by $2d(2d-1)^{|\gamma|-1}$.

□

Exercise 4.7. Fill in the details of the previous proof overview.

Let us summarise what we have proved so far.

- Theorem 4.4 implies that there is exponential decay of correlations when β is sufficiently small.
- In dimension $d = 1$, Theorem 4.4 also implies Ising's result (Theorem 2.8), since there was no requirement on β when $d = 1$.
- In dimension $d \geq 2$, we proved that there is *magnetisation* via the Peierls argument (Theorem 3.1). Thus, in dimension $d \geq 2$, there must be a phase transition, and we aim to investigate further.

The high-temperature expansion is typically used to find upper bounds on correlation functions. However, it is also possible to use it to find lower bounds. Let $G = (V, E)$ denote a finite graph. For any fixed set $Q \subset E$ of edges, we define the *XOR map*

$$\Xi_Q : \{0,1\}^E \to \{0,1\}^E,\ \omega \mapsto \omega \Delta Q,$$

where Δ denotes the symmetric difference of two sets. This map is an involution. Moreover, for any $A \subset V$, it restricts to a bijection

$$\Xi_Q : \{\partial\omega = A\} \to \{\partial\omega = A\Delta\partial Q\}. \tag{8}$$

The measure $\mathbf{M}$ is not invariant under the involution Ξ_Q, but it is easy to see how the map affects the measure. More precisely, for any $\eta \in \{0,1\}^E$, we have

$$\mathbf{M}[\{\omega = \Xi_Q(\eta)\}] = (\tanh\beta)^{|Q\setminus\eta| - |Q\cap\eta|} \cdot \mathbf{M}[\{\omega = \eta\}].$$

The prefactor is upper bounded by $(\tanh\beta)^{-|Q|}$. Thus, writing Ξ_Q^* for the pushforward map, we obtain

$$\Xi_Q^*\mathbf{M} \le (\tanh\beta)^{-|Q|} \cdot \mathbf{M}.$$

For example, using the bijection in Equation (8), we obtain

$$\mathbf{M}[\{\partial\omega = A\}] \le (\tanh\beta)^{-|Q|} \cdot \mathbf{M}[\{\partial\omega = A\Delta\partial Q\}]. \tag{9}$$

We have now proved the following result.

Lemma 4.8. *Consider the Ising model on a finite graph $G = (V, E)$. Then for any $A \subset V$ and any $Q \subset E$, we have*

$$\langle \sigma_A \sigma_{\partial Q} \rangle \ge (\tanh\beta)^{|Q|} \cdot \langle \sigma_A \rangle.$$

In particular,

$$\langle \sigma_x \sigma_y \rangle \ge (\tanh\beta)^{\mathrm{Distance}(x,y)}.$$

Proof. The first inequality is Equation (9). For the second inequality, simply set $A = \emptyset$ and let Q denote a shortest path from x to y. □

This lemma complements Theorem 4.4 at high temperature, as the lemma asserts that the correlation functions cannot decay *faster* than exponentially at any temperature (except when $\beta = 0$).

5. Correlation inequalities

Peierls' argument is simple and robust, but also quite ad-hoc in the sense that it does not serve as a building block for further analysis. We now want to take a more systematic approach to the Ising model. At the centre of the modern study of the Ising model are *correlation functions* and *correlation inequalities.*

Let $\langle\,\cdot\,\rangle$ denote an Ising model (in a finite graph, or in a finite domain with boundary conditions). For any finite set $A \subset V$, we define

$$\sigma_A := \prod_{u\in A} \sigma_u.$$

Its expectation $\langle \sigma_A \rangle$ is called a *correlation function.*

Exercise 5.1. Consider an Ising model $\langle\,\cdot\,\rangle_{G,\beta}$ on a finite graph $G = (V, E)$. This is a probability measure on $\Omega = \{\pm 1\}^V$. Notice that the sample space Ω has the structure of a finite Abelian group. How is the Fourier transform of $\langle\,\cdot\,\rangle_{G,\beta}$ related to the family $(\langle \sigma_A \rangle_{G,\beta})_A$ of correlation functions?

Correlation functions are at the centre of the study of the Ising model. Inequalities between correlation functions are called *correlation inequalities.* We state some examples in the finite graph setting. These shall all be proved rigorously later.

- The *first Griffiths inequality*, which asserts that for any $A \subset V$,
$$\langle \sigma_A \rangle \geq 0.$$
- The *second Griffiths inequality*, which asserts that for any $A, B \subset V$,
$$\langle \sigma_A \sigma_B \rangle \geq \langle \sigma_A \rangle \langle \sigma_B \rangle.$$
- The *Fortuin–Kasteleyn–Ginibre (FKG) inequality*, which asserts that if $X, Y : \Omega \to \mathbb{R}$ are two non-decreasing functions on the partially ordered set Ω, then
$$\langle XY \rangle \geq \langle X \rangle \langle Y \rangle.$$

Such inequalities may be used to prove interesting properties about the Ising model.

Remark 5.2 (Infinite graphs with boundary conditions as finite graphs)**.** Until now, we always made a distinction between the Ising model on a finite graph and the Ising model on the infinite lattice with boundary conditions. But are they really different? Let us discuss the case of + boundary conditions. Let G denote an infinite graph and Λ a domain. Then we may consider the Ising model on the finite graph
$$G' := (V', E'); \qquad V' := \Lambda \cup \{\Lambda^c\}; \qquad E' := E(\Lambda).$$
This means that the vertices in Λ^c are collapsed into a single vertex. It is then easy to check that the distribution of $\sigma|_\Lambda$ is the same in the following two probability measures:
$$\mathbb{P}^+_{\Lambda,\beta} \qquad \text{and} \qquad \mathbb{P}_{G',\beta}(\,\cdot\,|\{\sigma_{\Lambda^c} = +\}).$$
This enables us to state all our inequalities in a unified way, namely on finite graphs.

Exercise 5.3. First consider the Ising model on a finite graph G at inverse temperature $\beta \in [0, \infty)$. Prove that $\langle \sigma_A \rangle_{G,\beta} = 0$ whenever $|A|$ is odd.

Now consider the Ising model on locally finite graph G at inverse temperature $\beta \in [0, \infty)$ with + boundary conditions outside the domain $\Lambda \subset V$, chosen such that $\Lambda \neq V$. Recall the construction in the previous remark. Prove that if $A \subset \Lambda$ contains an odd number of vertices, then
$$\langle \sigma_A \rangle^+_{\Lambda,\beta} = \mathbb{E}^+_{G',\beta}[\sigma_A | \{\sigma_{\Lambda^c} = +\}] = \langle \sigma_{A \cup \{\Lambda^c\}} \rangle_{G',\beta}.$$

6. Random currents

Definition 6.1 (Currents)**.** Let $G = (V, E)$ denote a graph. A *current* is a map $\mathbf{n} : E \to \mathbb{Z}_{\geq 0}$. We think of $(V, \mathbf{n})$ as a multigraph, where for each edge $uv \in E$ we have $\mathbf{n}_{uv}$ multi-edges between u and v. The set of *sources* $\partial \mathbf{n} \subset V$ of a current $\mathbf{n}$ is defined as the set of vertices $u \in V$ with an odd degree in the multigraph. If G is finite and $\beta \in [0, \infty)$, then the *weight* of a current is defined as
$$w_\beta(\mathbf{n}) := \prod_{xy \in E} \frac{\beta^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}.$$

Currents can be used to encode correlation functions of the Ising model.

Theorem 6.2 (Current representation of correlation functions)**.** *Consider the Ising model on a finite graph G at inverse temperature β. Let $A \subset V$ be a subset of vertices. Then*
$$Z_{G,\beta} \langle \sigma_A \rangle_{G,\beta} = 2^{|V|} \sum_{\mathbf{n} \colon \partial \mathbf{n} = A} w_\beta(\mathbf{n}).$$
In particular, the partition function is given by
$$Z_{G,\beta} = 2^{|V|} \sum_{\mathbf{n} \colon \partial \mathbf{n} = \emptyset} w_\beta(\mathbf{n}).$$

Proof. We claim that

$$
\begin{aligned}
Z_{G,\beta}\langle\sigma_A\rangle_{G,\beta} &= \sum_{\sigma\in\{\pm1\}^V}\sigma_A\prod_{uv\in E}e^{\beta\sigma_u\sigma_v} && (10)\\
&= \sum_{\sigma\in\{\pm1\}^V}\sigma_A\prod_{uv\in E}\sum_{\mathbf{n}=0}^{\infty}\frac{(\beta\sigma_u\sigma_v)^{\mathbf{n}}}{\mathbf{n}!} && (11)\\
&= \sum_{\mathbf{n}:E\to\mathbb{Z}_{\geq 0}}\sum_{\sigma\in\{\pm1\}^V}\sigma_A\prod_{uv\in E}\frac{(\beta\sigma_u\sigma_v)^{\mathbf{n}_{uv}}}{\mathbf{n}_{uv}!} && (12)\\
&= 2^{|V|}\sum_{\mathbf{n}:\,\partial\mathbf{n}=A}\prod_{uv\in E}\frac{\beta^{\mathbf{n}_{uv}}}{\mathbf{n}_{uv}!} && (13)\\
&= 2^{|V|}\sum_{\mathbf{n}:\,\partial\mathbf{n}=A}w_\beta(\mathbf{n}). && (14)
\end{aligned}
$$

Although this may come as a surprise, its justification is straightforward.

- Equation (10) is derived from the definition of the Ising model by pulling out the sum in the Hamiltonian in the exponential as a product.
- Equation (11) follows by expanding each exponential.
- Equation (12) is Fubini's theorem: we first swap the product and the sum (so that we have to perform one sum for each factor in the product), then we interchange the two sums. Absolute convergene comes from the factorial terms in the denominator.
- Equation (13) follows simply by noting that the sum over σ is zero unless all the factors of the form σ_u cancel.
- Equation (14) is the definition of the weight above.

This finishes the proof. □

Corollary 6.3 (First Griffiths inequality). *Consider the Ising model on a finite graph G at inverse temperature β. Then for any $A\subset V$, we have $\langle\sigma_A\rangle_{G,\beta}\geq 0$.*

Proof. The current expansion shows that the correlation function is a sum of non-negative terms. □

Exercise 6.4 (Correlation functions with odd sets).

- Consider the setting of the previous theorem. Show that if $|A|$ is odd, then $\langle\sigma_A\rangle_{G,\beta}=0$.
- Now consider an arbitrary graph G with some domain Λ. Show that if $|A|$ is odd, then $\langle\sigma_A\rangle^+_{G,\beta}\geq 0$. For this part, one needs Remark 5.2 and Exercise 5.3.

Definition 6.5 (Percolation of currents). Let G denote a graph and $\mathbf{n}$ a current. We associate $\mathbf{n}$ with the set $E(\mathbf{n}):=\{uv\in E:\mathbf{n}_{uv}>0\}\subset E$. Edges in $E(\mathbf{n})$ are called *open*, the other edges *closed*. We shall write $\{u\overset{\mathbf{n}}{\leftrightarrow}v\}$ for the event there is an open path from u to v (u and v may represent vertices or sets of vertices).

For fixed $S\subset V$, we shall also write $\mathcal{E}_S$ for the set

$$\{O\subset E:|C\cap S|\text{ is even for any connected component }C\subset V\text{ of }(V,O)\}.$$

Notice that if $\partial\mathbf{n}=S$, then $E(\mathbf{n})\in\mathcal{E}_S$.

We shall often simply write $\mathbf{n}$ for $E(\mathbf{n})$.

Exercise 6.6 (Currents and the Peierls argument). For the first two parts, consider the Ising model on a finite graph G at inverse temperature β.

(1) Let $\mathcal{L}\subset E$ denote a circuit through G (a closed path which traverses no edge more than once). Prove that

$$\frac{\sum_{\mathbf{n}:\,\partial\mathbf{n}=\emptyset,\,\mathcal{L}\subset E(\mathbf{n})}w_\beta(\mathbf{n})}{\sum_{\mathbf{n}:\,\partial\mathbf{n}=\emptyset}w_\beta(\mathbf{n})}\leq\beta^{|\mathcal{L}|}.$$

(2) Let $\mathcal{P} \subset E$ denote a path from u to v through G which traverses no edge more than once. Prove that

$$\frac{\sum_{\mathbf{n}\colon \partial\mathbf{n}=\{u,v\},\, \mathcal{P}\subset E(\mathbf{n})} w_\beta(\mathbf{n})}{\sum_{\mathbf{n}\colon \partial\mathbf{n}=\emptyset} w_\beta(\mathbf{n})} \leq \beta^{|\mathcal{P}|}.$$

(3) Let G denote an infinite graph of maximum degree $d \in \mathbb{Z}_{\geq 0}$. Let u denote a fixed vertex, and let Λ_n denote the set of vertices at distance at most $n-1$ from u. Prove that for any $\beta < 1/d$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$\langle \sigma_u \rangle^+_{\Lambda_n,\beta} \leq \frac{(d\beta)^n}{1-d\beta}.$$

For this part, one needs Remark 5.2 and Exercise 5.3.

Indeed, this is the high-temperature counterpart to the Peierls argument (Theorem **??**). We did not only prove that for small β, the magnetisation vanishes (tends to zero), but also that it tends to zero exponentially fast in the distance to the boundary.

In fact, the sum $\sum_{\mathbf{n}\colon \partial\mathbf{n}=A} w_\beta(\mathbf{n})$ can be calculated in a different way. We can first sum over the *parity* of each edge (which already determines $\partial\mathbf{n}$), then over all random currents consistent with that parity. Once the parity is fixed, the remaining sum decomposes as an independent product over the edges. More precisely, we get

$$\begin{aligned}\sum_{\mathbf{n}\colon \partial\mathbf{n}=A} w_\beta(\mathbf{n}) &= \sum_{\omega\subset E\colon \partial\mathbb{1}_\omega=A} \left(\sum\nolimits_{\mathbf{n}\colon \mathbf{n}-\mathbb{1}_\omega\in 2\mathbb{Z}^E} w_\beta(\mathbf{n})\right) \\ &= \sum_{\omega\subset E\colon \partial\mathbb{1}_\omega=A} (\sinh\beta)^{|\omega|}(\cosh\beta)^{|E\setminus\omega|}.\end{aligned}$$

The factors $\cosh\beta$ and $\sinh\beta$ appear when performing the sum $\sum_{k\geq 0}\beta^k/k!$ but restricting to even and odd values for k respectively. We have now proved the *high-temperature expansion* (see below for a formal statement). The high-temperature expansion is less versatile than the currents representation of the Ising model, but it is still somewhat useful.

Theorem 6.7 (High-temperature expansion)**.** *Consider the Ising model on a finite graph G at inverse temperature β. Let $A \subset V$ be a subset of vertices. Then*

$$Z_{G,\beta}\langle\sigma_A\rangle_{G,\beta} = 2^{|V|} \sum_{\omega\subset E\colon \partial\mathbb{1}_\omega=A} (\sinh\beta)^{|\omega|}(\cosh\beta)^{|E\setminus\omega|}.$$

In particular, the partition function is given by

$$Z_{G,\beta} = 2^{|V|} \sum_{\omega\subset E\colon \partial\mathbb{1}_\omega=\emptyset} (\sinh\beta)^{|\omega|}(\cosh\beta)^{|E\setminus\omega|}.$$

Exercise 6.8 (Log-Lipschitz property of correlation functions)**.** Let G denote any finite graph. We define the following metric on subsets of V: for any $A, B \subset V$, we set

$$d_{\text{Transport}}(A,B) := \min\{|\omega| : \omega\subset E \text{ and } A\Delta B = \partial\mathbb{1}_\omega\}.$$

Consider the Ising model on G at inverse temperature $\beta \in \mathbb{R}_{\geq 0}$. Use the high-temperature expansion to prove that

$$(\tanh\beta)^{d_{\text{Transport}}(A,B)}\langle\sigma_A\rangle \leq \langle\sigma_B\rangle.$$

This implies that for any $u \in V$, the function $v \mapsto -\log\langle\sigma_u\sigma_v\rangle$ is $(-\log\tanh\beta)$-Lipschitz.

Definition 6.9 (Replacing sums by measures)**.** Sums are measures. Even though there is no added mathematical value in replacing sums by measures, doing so will slightly change our perspective, and also shorten notations.

For an Ising model on some finite graph G with inverse temperature β, we shall write $\mathbb{M}_{G,\beta}$ for the measure on random currents $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^E$ such that

$$\mathbb{M}_{G,\beta}(\mathbf{n}) = w_\beta(\mathbf{n}).$$

For $A \subset V$, we shall also write $\mathbb{M}^A_{G,\beta}$ for the measure

$$\mathbb{M}_{G,\beta}[\,\cdot\,] := \mathbb{M}_{G,\beta}[(\,\cdot\,)\mathbb{1}_{\{\partial\mathbf{n}=A\}}].$$

Corollary 6.10. *The measure $\mathbb{M}_{G,\beta}$ is not a probability measure, but $e^{-\beta|E|}\mathbb{M}_{G,\beta}$ is a probability measure in which $(\mathbf{n}_{uv})_{uv\in E}$ is a family of independent random variables with distribution* Poisson(β)*.*

Corollary 6.11. *Theorem 6.2 says that*

$$Z_{G,\beta}\langle\sigma_A\rangle_{G,\beta} = 2^{|V|}\mathbb{M}_{G,\beta}[\{\partial\mathbf{n} = A\}] = 2^{|V|}\mathbb{M}^A_{G,\beta}[1].$$

7. Double random currents

The previous section explained how correlation functions are expressed in terms of random currents. We also proved a basic result, namely the existence of a demagnetised phase of the Ising model on graphs of bounded degree.

All results discussed so far concern the behaviour of the Ising model in the off-critical regime (very large values of β, very small values of β). Our main interest is however in the *critical regimes*: the values for β where the model undergoes a qualitative change, such as values in the topological boundary of the set

$$\{\beta \in [0,\infty) : \lim_{n\to\infty}\langle\sigma_u\rangle^+_{\Lambda_n,\beta} = 0\}$$

for a given infinite graph G with a reference point u (as per usual, Λ_n refers to the graph metric ball around u).

We shall now introduce a new tool to study correlation functions and random currents: the *switching lemma*. In recent years this tool has proved to be instrumental in the derivation of rigorous results on the critical behaviour of the Ising model, especially in graph dimensions 3 and 4.

Lemma 7.1 (Switching lemma)**.** *Consider the Ising model on a finite graph G at inverse temperature β. Let $A, B, S \subset V$. Then for any bounded function $F : (\mathbb{Z}_{\geq 0})^E \to \mathbb{C}$, the following identities hold true:*

$$\mathbb{M}^A \times \mathbb{M}^B[F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S)] = \mathbb{M}^{A\Delta S} \times \mathbb{M}^{B\Delta S}[F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S)],$$

where $A\Delta S$ denotes the symmetric difference of A and S.

In terms of weights, this is equivalent to

$$\sum_{\substack{\mathbf{n}:\,\partial\mathbf{n}=A\\ \mathbf{m}:\,\partial\mathbf{m}=B}} w_\beta(\mathbf{n})w_\beta(\mathbf{m})F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S)$$
$$= \sum_{\substack{\mathbf{n}:\,\partial\mathbf{n}=A\Delta S\\ \mathbf{m}:\,\partial\mathbf{m}=B\Delta S}} w_\beta(\mathbf{n})w_\beta(\mathbf{m})F(\mathbf{n}+\mathbf{m})\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_S).$$

Remark 7.2. While the switching lemma is an extremely powerful tool, its statement may appear daunting at first sight. Let us quickly see how switching may be applied to a simple example. Suppose that we record cars traversing a bridge on a road. Blue cars appear according to a Poisson process with rate λ. Let X_B denote the number of blue cars that passed after recording for one hour. Can we easily prove, without a calculation, that

$$\mathbb{P}[X_B \in 2\mathbb{Z}] \geq \mathbb{P}[X_B \in 2\mathbb{Z}+1]?$$

Suppose that there are also yellow cars, which arrive according to an independent Poisson process with rate λ. Let X_Y denote the number of yellow cars that passed. Suppose that, after waiting for one hour, $X_B + X_Y = N > 0$ cars passed. What is the *conditional* probability that X_B is even?

Well, we must have $\mathbb{P}[X_B \in 2\mathbb{Z} | \{X_B + X_Y = N\}] = 1/2$. Indeed, by the properties of the Poisson process, the distribution of the cars is invariant under *switching* the colour of the last car. If it was blue before, then we paint it yellow, and vice versa. This operation changes the parity of X_B but leaves the conditional distribution invariant: hence the symmetric probability $1/2$.

But we cannot always do the switch. If $X_B + X_Y = 0$ then there is no car to repaint, and also $X_B = 0$. Thus, we conclude that

$$\mathbb{P}[X_B \in 2\mathbb{Z}] - \mathbb{P}[X_B \in 2\mathbb{Z} + 1] = \mathbb{P}[\{X_B + X_Y = 0\}] \geq 0.$$

Notice that we originally asked a question about blue cars, but introducing yellow cars allowed us to answer it. This is the essence of the switching lemma.

The switching lemma is analogous to the above example:

- The product measure corresponds to the joint distribution of blue and yellow cars,
- The weights correspond to the rates of the Poisson processes,
- The function F corresponds to the conditioning event $\{X_B + X_Y = N\}$,
- The event $\{\mathbf{n} + \mathbf{m} \in \mathcal{E}_S\}$ corresponds to the event $\{X_B + X_Y > 0\}$.

Proof of the switching lemma. By linearity of expectation, it suffices to consider the case that $F(\mathbf{b}) := \mathbb{1}(\mathbf{b} = \mathbf{a})$ for some fixed $\mathbf{a} \in \mathcal{E}_S$. Our objective is then to derive the equality

$$\mathbb{M}^A \times \mathbb{M}^B[\{\mathbf{n} + \mathbf{m} = \mathbf{a}\}] = \mathbb{M}^{A\Delta S} \times \mathbb{M}^{B\Delta S}[\{\mathbf{n} + \mathbf{m} = \mathbf{a}\}]$$

or

$$\mathbb{M}^2[\{\partial\mathbf{n} = A,\, \partial\mathbf{m} = B,\, \mathbf{n} + \mathbf{m} = \mathbf{a}\}] = \mathbb{M}^2[\{\partial\mathbf{n} = A\Delta S,\, \partial\mathbf{m} = B\Delta S,\, \mathbf{n} + \mathbf{m} = \mathbf{a}\}].$$

Define the probability measure

$$\mathbb{P} :\propto \mathbb{M}^2[(\,\cdot\,)\mathbb{1}(\mathbf{n} + \mathbf{m} = \mathbf{a})].$$

It suffices to prove that

$$\mathbb{P}[\{\partial\mathbf{n} = A,\, \partial\mathbf{m} = B\}] = \mathbb{P}[\{\partial\mathbf{n} = A\Delta S,\, \partial\mathbf{m} = B\Delta S\}]. \tag{15}$$

By going back to the definition of $\mathbb{M}$ in terms of w_β, it is straightforward to see that the pair $(\mathbf{n}, \mathbf{m})$ has the following probability distribution under $\mathbb{P}$:

- The family $(\mathbf{n}_{uv})_{uv}$ is a family of independent random variables,
- The distribution of $\mathbf{n}_{uv}$ is $\mathrm{Binomial}(\mathbf{a}_{uv}, 1/2)$,
- We have $\mathbf{n} + \mathbf{m} = \mathbf{a}$ almost surely, which fixes the joint distribution of $(\mathbf{n}, \mathbf{m})$.

In fact, we may interpret $\mathbb{P}$ in a different way. Define the *multigraph*

$$\mathcal{M}_{\mathbf{a}} := \{(uv, k) \in E \times \mathbb{Z}_{\geq 0} : \mathbf{a}_{uv} < k\}$$

on the vertex set V. Then $\mathbb{P}$ is interpreted as follows:

- We let $\mathcal{K}$ denote a uniformly random subset of $\mathcal{M}_{\mathbf{a}}$,
- We let $\mathbf{n}_{uv}$ denote the number of multiedges in $\mathcal{K}$ between u and v,
- We let $\mathbf{m}_{uv}$ denote the number of multiedges in $\mathcal{M}_{\mathbf{a}} \setminus \mathcal{K}$ between u and v.

Indeed, this definition of $\mathbb{P}$ is consistent with our previous one.

Proving Equation (15) now comes down to proving that the number of submultigraphs $\mathcal{K} \subset \mathcal{M}_{\mathbf{a}}$ contributing to the events on the left and right, is the same. Let $E_S \subset E(\mathbf{a})$ denote an arbitrary subset such that $\partial E_S = S$, and write $E_{S,0} := E_S \times \{0\} \subset \mathcal{M}_{\mathbf{a}}$. The existence of the set E_S follows from the fact that $\mathbf{a} \in \mathcal{E}_S$. The reader may now verify that the map

$$\{\partial\mathbf{n} = A,\, \partial\mathbf{m} = B\} \to \{\partial\mathbf{n} = A\Delta S,\, \partial\mathbf{m} = B\Delta S\},\, \mathcal{K} \mapsto \mathcal{K}\Delta E_{S,0}$$

is a bijection. This proves that the two sets have the same cardinality, and thus the same probability under the measure $\mathbb{P}$. We have now established Equation (15) and therefore the lemma. □

Corollary 7.3 (Second Griffiths inequality)**.** *Consider the Ising model on a finite graph G at inverse temperature β. Then for any $A, B \subset V$, we have $\langle \sigma_{A\Delta B}\rangle_{G,\beta} - \langle \sigma_A\rangle_{G,\beta}\langle \sigma_B\rangle_{G,\beta} \geq 0$.*

Proof. We have $1 = \langle 1\rangle = \langle \sigma_\emptyset\rangle$. By the previous section (for example Corollary 6.11),

$$Z^2(\langle \sigma_{A\Delta B}\rangle\langle \sigma_\emptyset\rangle - \langle \sigma_A\rangle\langle \sigma_B\rangle) = 2^{2|V|}(\mathbb{M}^{A\Delta B} \times \mathbb{M}^\emptyset[1] - \mathbb{M}^A \times \mathbb{M}^B[1]).$$

Claim that the quantity on the right is nonnegative. Notice that if $\partial \mathbf{m} = B$, then $\mathbf{m} \in \mathcal{E}_S$, and therefore

$$\begin{aligned}\mathbb{M}^A \times \mathbb{M}^B[1] = \mathbb{M}^A \times \mathbb{M}^B[\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_B)] &\overset{\text{switch}}{=} M^{A\Delta B} \times \mathbb{M}^\emptyset[\mathbb{1}(\mathbf{n}+\mathbf{m} \in \mathcal{E}_B)] \\ &\leq M^{A\Delta B} \times \mathbb{M}^\emptyset[1].\end{aligned}$$

This inequality implies the claim, and therefore the second Griffiths inequality. □

Exercise 7.4 (Conditioning on equality increases the correlation functions)**.** Consider the Ising model on a finite graph G at inverse temperature β, and fix some subset $A \subset V$.

- Prove that for any two distinct vertices $u, v \in V$, we have
$$\mathbb{E}_{G,\beta}[\sigma_A|\{\sigma_u = \sigma_v\}] \geq \mathbb{E}_{G,\beta}[\sigma_A] = \langle \sigma_A\rangle_{G,\beta}.$$
- Prove for any $X \subset Y \subset V$, we have
$$\mathbb{E}_{G,\beta}[\sigma_A|\{\sigma \text{ is constant on } X\}] \leq \mathbb{E}_{G,\beta}[\sigma_A|\{\sigma \text{ is constant on } Y\}].$$

Exercise 7.5 (The two-point function as a metric)**.** Consider the Ising model on a finite graph G at inverse temperature $\beta > 0$. Prove that $V \times V \to [0,\infty]$, $(u,v) \mapsto -\log\langle \sigma_u\sigma_v\rangle_{G,\beta}$ defines a metric on V.

Definition 7.6 (Probability measures on currents)**.** Consider the Ising model on a finite graph G at inverse temperature β. For any $A \subset V$, define the probability measure

$$\mathbb{P}^A_{G,\beta} := \frac{2^{|V|}}{Z_{G,\beta}\langle \sigma_A\rangle_{G,\beta}}\mathbb{M}^A_{G,\beta}.$$

For any $A_1, \ldots, A_n$, write $\mathbb{P}^{A_1,\ldots,A_n} := \mathbb{P}^{A_1} \times \cdots \times \mathbb{P}^{A_n}$.

Exercise 7.7 (Correlation functions in terms of sourceless currents)**.** Consider the Ising model on a finite graph G at inverse temperature β. Prove that for any $A \subset V$,

$$\langle \sigma_A\rangle^2 = \mathbb{P}^{\emptyset,\emptyset}[\{\mathbf{n}+\mathbf{m} \in \mathcal{E}_A\}].$$

Observe that we can now express all correlation functions in terms of a single fixed probability measure on sourceless random currents.

8. Monotonicity in the temperature

Theorem 8.1 (Monotonicity in the temperature)**.** *Let G denote a finite graph and $A \subset V$ a finite set. Then the function $\beta \mapsto \langle \sigma_A\rangle_{G,\beta}$ is non-decreasing.*

Proof. We want to prove that

$$\frac{\partial}{\partial\beta}\langle \sigma_A\rangle_{G,\beta} = \frac{\partial}{\partial\beta}\left(\frac{\sum_\sigma \sigma_A \prod_{uv} e^{\beta\sigma_u\sigma_v}}{\sum_\sigma \prod_{uv} e^{\beta\sigma_u\sigma_v}}\right) \geq 0.$$

Since we are differentiating a fraction, it suffices to show that the numerator grows at a faster rate than the denominator, that is,

$$\frac{\frac{\partial}{\partial\beta}\sum_\sigma \sigma_A \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z\langle\sigma_A\rangle} \geq \frac{\frac{\partial}{\partial\beta}\sum_\sigma \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z}.$$

By multiplying either side by $\langle\sigma_A\rangle$ and differentiating each side, we see that this inequality is equivalent to

$$\sum_{xy} \frac{\sum_\sigma \sigma_x\sigma_y\sigma_A \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z} \geq \langle\sigma_A\rangle \sum_{xy} \frac{\sum_\sigma \sigma_x\sigma_y \prod_{uv} e^{\beta\sigma_u\sigma_v}}{Z}.$$

Each fraction may now be reinterpreted as a correlation function, so that the previous inequality is equivalent to

$$\sum_{xy}\langle\sigma_x\sigma_y\sigma_A\rangle \geq \langle\sigma_A\rangle \sum_{xy}\langle\sigma_x\sigma_y\rangle.$$

But this is just the second Griffiths inequality. □

Exercise 8.2. Prove that the function $[0,\infty) \to \mathbb{R}$, $\beta \mapsto \langle\sigma_A\rangle_{G,\beta}$ in the above context is an analytic function.

9. The thermodynamical limit. Wired boundary, existence

Consider the Ising model on the infinite graph $\mathbb{Z}^d$ for $d \geq 2$. Let $u = 0 \in \mathbb{Z}^d$ and let Λ_n denote the graph metric ball around u. We have already derived the following results.

- For large β, the Ising model exhibits magnetisation in the sense that
$$\inf_n \langle\sigma_0\rangle^+_{\Lambda_n,\beta} > 0.$$
This was proved via the Peierls argument, see Theorem 3.1 and Exercise 3.2.
- For small β, the Ising model *does not* exhibit magnetisation:
$$\lim_{n\to\infty} \langle\sigma_0\rangle^+_{\Lambda_n,\beta} = 0.$$
This was proved via a Peierls argument for random currents, see Exercise 6.6.

At the time moment of stating the Peierls argument (Section 3), we knew almost nothing about the Ising model. Our understanding is now advancing. We already used the first Griffiths inequality to show that $\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \geq 0$ (Corollary 6.3 and Exercise 6.4). Our first objective is now to prove the following result. To state it, we write

$$\lim_{\Lambda\uparrow V} f(\Lambda) \qquad \text{for} \qquad \lim_{n\to\infty} f(\Lambda_n),$$

where $(\Lambda_n)_n$ is any increasing sequence of domains with $\cup_n\Lambda_n = V$. This notation makes sense only when the limit is independent of the precise choice of the sequence $(\Lambda_n)_n$, and is called the *thermodynamical limit* or *infinite-volume limit.*

Lemma 9.1 (Correlation functions are monotone in the domain (wired boundary))**.** *Consider the Ising model on a locally finite graph G at inverse temperature β. Let $A \subset V$ denote any finite subset. Then the function*

$$\Lambda \mapsto \langle\sigma_A\rangle^+_{\Lambda,\beta}$$

is a nonincreasing function of the domain Λ.

In particular, we have well-definedness of the thermodynamical limit

$$\lim_{\Lambda\uparrow V}\langle\sigma_A\rangle^+_{\Lambda,\beta}.$$

Proof. Consider two domains $\Lambda \subset \bar{\Lambda}$. We want to show that
$$\langle \sigma_A \rangle^+_{\Lambda,\beta} \geq \langle \sigma_A \rangle^+_{\bar{\Lambda},\beta}.$$
Without loss of generality, $A \subset \bar{\Lambda}$ and $\bar{\Lambda} \setminus \Lambda = \{u\}$ for some vertex $u \in V$.

Let $G' = (\bar{\Lambda} \cup \{\bar{\Lambda}^c\}, E(\bar{\Lambda}))$ denote the graph obtained from $\bar{\Lambda}$ as in Remark 5.2 and Exercise 5.3. We refer to the Ising model on G' when subscripts are submitted from now on. Assume that $|A|$ is even for now. Then
$$\langle \sigma_A \rangle^+_{\bar{\Lambda},\beta} = \mathbb{E}[\sigma_A]; \qquad \langle \sigma_A \rangle^+_{\Lambda,\beta} = \mathbb{E}[\sigma_A | \{\sigma_u = \sigma_{\bar{\Lambda}^c}\}].$$
It suffices to show that the conditioning increases the expectation. This is just Exercise 7.4.

If $|A|$ is odd then we just need to replace the set A by $A' := A \cup \{\bar{\Lambda}^c\}$. More precisely, we have
$$\langle \sigma_A \rangle^+_{\bar{\Lambda},\beta} = \mathbb{E}[\sigma_{A'}]; \qquad \langle \sigma_A \rangle^+_{\Lambda,\beta} = \mathbb{E}[\sigma_{A'} | \{\sigma_u = \sigma_{\bar{\Lambda}^c}\}].$$
One may then simply apply Exercise 7.4 as for the even case. □

Perhaps we were wondering if $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$ was decreasing in n in the statement of the Peierls argument (Theorem **??**), but the result we proved just now is much stronger: we proved that the thermodynamical limit of any "local Fourier coefficient" is well-defined. Rather than taking a thermodynamical limit of observables, we would however like to make sense of the thermodynamical limit of the family of measures $\langle \cdot \rangle^+_{\Lambda,\beta}$. The previous lemma enables us to do this; we only need to set up the definitions to make formal sense of our limit.

Definition 9.2 (The compact "local convergence" topology)**.** Let G denote a locally finite graph. Recall that $(\Omega, \mathcal{F})$ is the measurable space $\Omega := \{\pm 1\}^V$ endowed with the product σ-algebra. For a domain Λ, we write $\mathcal{F}_\Lambda$ for the σ-algebra generated by spins in Λ. An observable $X : \Omega \to \mathbb{C}$ is called *local* if it is measurable with respect to $\mathcal{F}_\Lambda$ for some domain Λ.

Let $\mathcal{P}(\Omega, \mathcal{F})$ denote the set of all probability measures on this measurable space. We endow this set with the *local convergence topology*, which is defined as the topology making the map
$$\mathcal{P}(\Omega, \mathcal{F}) \to \mathbb{C},\ \mu \mapsto \mu[X]$$
continuous for any local observable X.

Remark 9.3. This topology is sometimes known under different names in the literature (such as the *weak topology*). I like the name *local convergence topology* because it captures the essence quite literally: if the statistics of the measures within a fixed domain Λ converge, then we have local convergence.

Exercise 9.4. Prove that $\mathcal{P}(\Omega, \mathcal{F})$ is a compact space in this topology.

Theorem 9.5 (Existence of the thermodynamical limit (wired boundary))**.** *Consider the Ising model on a locally finite graph G at inverse temperature β. Then there exists a unique probability measure $\langle \cdot \rangle^+_{G,\beta} \in \mathcal{P}(\Omega, \mathcal{F})$ such that*
$$\lim_{\Lambda \uparrow V} \langle X \rangle^+_{\Lambda,\beta} = \langle X \rangle^+_{G,\beta}$$
for any local observable $X : \Omega \to \mathbb{R}$. In other words,
$$\lim_{\Lambda \uparrow V} \langle \cdot \rangle^+_{\Lambda,\beta} =: \langle \cdot \rangle^+_{G,\beta}.$$
The measure $\langle \cdot \rangle^+_{G,\beta}$ is called the thermodynamical limit *or* infinite-volume limit *with + boundary conditions.*

Proof. Any local observable may be written as a finite linear conbination of observables of the form σ_A where A is a finite subset of V. The theorem then follows by compactness and Lemma 9.1. □

Exercise 9.6 (Continuity properties in β (wired boundary))**.** Consider the Ising model on a locally finite graph G. Prove all of the following statements.

- For any finite sets $A, \Lambda \subset V$, the function $[0,\infty) \to \mathbb{R}$, $\beta \mapsto \langle \sigma_A \rangle^+_{\Lambda,\beta}$ is non-decreasing and continuous.
- For any finite set $A \subset V$, the function $[0,\infty) \to \mathbb{R}$, $\beta \mapsto \langle \sigma_A \rangle^+_{G,\beta}$ is non-decreasing and right-continuous.
- The function $[0,\infty) \to \mathcal{P}(\Omega,\mathcal{F})$, $\beta \mapsto \langle \cdot \rangle^+_{G,\beta}$ is a right-continuous function.
- The points of discontinuity form a countable subset of $[0,\infty)$.

Definition 9.7 (Magnetisation and critical temperature)**.** Let G denote a locally finite graph and u some distinguished reference vertex. The function

$$m = m_G : [0,\infty) \to \mathbb{R},\ \beta \mapsto \langle \sigma_u \rangle^+_{G,\beta}$$

is called the *magnetisation*.

The *critical (inverse) temperature* is defined via

$$\beta_c := \inf\{\beta \in [0,\infty) : m(\beta) > 0\}.$$

Exercise 9.8. In general, the function m_G may depend on the choice of the reference vertex u. Show that if G is connected, then the definition of β_c does not depend on this choice. Hint: use Exercise 6.8.

Remark 9.9. The exercise above proves that m is a non-decreasing right-continuous function. We have already seen that:

- If G has max-degree d, then $m(\beta) = 0$ for $\beta < 1/d$ (Exercise 6.6),
- If G is the graph $\mathbb{Z}^d$ for $d \geq 2$, then $\lim_{\beta\to\infty} m(\beta) > 0$ (Exercise 3.2).

This implies in particular that on the square lattice graph $\mathbb{Z}^d$ in dimension $d \geq 2$, the critical inverse temperature β_c is a strictly positive real number. A key objective of our field is to understand the behaviour of the Ising model at $\beta = \beta_c$ and at $\beta \approx \beta_c$. As a very first question we can ask: is the function m continuous? We will see the answer in Theorem **??**.

Add ref to theorem

10. The thermodynamical limit: Wired boundary, demagnetisation

Our next goal is to prove the following theorem.

Theorem 10.1 ($+$ and $-$ boundary conditions coincide when the magnetisation vanishes)**.** *Let G denote a connected locally finite graph, endowed with some reference vertex u. Then*

$$\langle \cdot \rangle^+_{G,\beta} = \langle \cdot \rangle^-_{G,\beta} \qquad \iff \qquad m_G(\beta) = 0.$$

The theorem can be proved using the following bound.

Exercise 10.2 (Pairing bound, difficult)**.** Consider the Ising model on a finite graph G at inverse temperature β. Let $A \subset V$ denote any finite subset, and fix $u \in A$. Use the switching lemma to prove that

$$\langle \sigma_A \rangle \leq \sum_{v \in A\setminus\{u\}} \langle \sigma_u \sigma_v \rangle \langle \sigma_{A\setminus\{u,v\}} \rangle.$$

Hint: argue that

$$\mathbb{1}(\mathbf{n} \in \mathcal{E}_A) \leq \sum_{v \in A\setminus\{u\}} \mathbb{1}(\mathbf{n} \in \mathcal{E}_{\{u,v\}}) \mathbb{1}(\mathbf{n} \in \mathcal{E}_A).$$

Conclude that

$$\langle \sigma_A \rangle \leq \sum_{\pi} \prod_{\{u,v\}\in\pi} \langle \sigma_u \sigma_v \rangle$$

where π ranges over the *pairings* of A, that is, the set of partitions of A in which each member has two elements.

Proof of Theorem 10.1. Notice that $\langle\,\cdot\,\rangle^+_{G,\beta}$ and $\langle\,\cdot\,\rangle^-_{G,\beta}$ are related by a global spin flip (the pushforward map corresponding to $\sigma \mapsto -\sigma$). Therefore all of the following are equivalent:

- $\langle\,\cdot\,\rangle^+_{G,\beta} = \langle\,\cdot\,\rangle^-_{G,\beta}$,
- $\langle\,\cdot\,\rangle^+_{G,\beta}$ is invariant under the map $\sigma \mapsto -\sigma$,
- $\langle\sigma_A\rangle^+_{G,\beta} = 0$ whenever $A \subset V$ has odd cardinal.

The implication " $\implies$ " is now obvious, and we focus on " $\impliedby$ ". Suppose that $m(\beta) = 0$, that is, $\langle\sigma_u\rangle^+ = 0$ where u is the reference vertex. Fix $A \subset V$ with $|A|$ odd. It suffices to prove that $\langle\sigma_A\rangle^+ = 0$. We shall in fact give *two* proofs of this fact. In both proofs, we shall fix a sequence $(\Lambda_n)_n$ of increasing subsets of V with $\cup_n \Lambda_n = V$.

- *Proof 1, using the pairing bound.* For fixed n, we get
$$\begin{aligned}\langle\sigma_A\rangle^+_{\Lambda_n} = \langle\sigma_{A\cup\{\Lambda_n^c\}}\rangle_{G'_n} &\leq \sum_{v\in A}\langle\sigma_{\{v,\Lambda_n^c\}}\rangle_{G'_n}\langle\sigma_{A\setminus\{v\}}\rangle_{G'_n} = \sum_{v\in A}\langle\sigma_v\rangle^+_{\Lambda_n}\langle\sigma_{A\setminus\{v\}}\rangle^+_{\Lambda_n}\\ &\leq \sum_{v\in A}\langle\sigma_v\rangle^+_{\Lambda_n} \to_{n\to\infty} 0.\end{aligned}$$
The first inequality is the pairing bound, the second the generic bound $\langle\sigma_{A\setminus\{v\}}\rangle^+_{\Lambda_n} \in [0,1]$, and the convergence follows from Exercise 9.8.
- *Proof 2, using directly the high-temperature expansion.* We only consider $\beta > 0$, otherwise the spins are independent fair coin flips, and the result is automatic. Recall Exercise 6.8. For fixed n, we get
$$(\tanh\beta)^{d_{\mathrm{Transport}}(A,\{u\})}\cdot\langle\sigma_A\rangle^+_{\Lambda_n} \leq \langle\sigma_u\rangle^+_{\Lambda_n}.$$
On the left, the transport distance is calculated in the graph G'_n (the dependence on n is implicit). As $n \to \infty$, this transport distance stabilises at the finite transport distance in the infinite graph G. Since the right hand side tends to zero with n, we know that the left hand side also tends to zero. Since the prefactor remains uniformly positive, we must have $\langle\sigma_A\rangle^+ = 0$. $\square$

References

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