Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z), which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

The associated bases for inner-oriented multicovector fields are

$$\mathrm{d}t \quad \mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z \tag{1}$$

$$\mathrm{d}^2tx \quad \mathrm{d}^2ty \quad \mathrm{d}^2tz \quad \mathrm{d}^2yz \quad \mathrm{d}^2zx \quad \mathrm{d}^2xy \tag{2}$$

$$\mathrm{d}^3xyz \quad -\mathrm{d}^3tyz \quad -\mathrm{d}^3tzx \quad -\mathrm{d}^3txy \tag{3}$$

$$\mathrm{d}^4txyz \tag{4}$$

and analogously for inner-oriented multivector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it's only defined on a coordinate patch. A twisted or outer 3-covector such as $\mathrm{d}^3\tilde{x}yz$ has an associated outer direction, in this case positive t. We adopt this shorter notation for the outer-oriented versions of the bases above:

$$-\mathrm{d}_{xyz} \quad \mathrm{d}_{tyz} \quad \mathrm{d}_{tzx} \quad \mathrm{d}_{txy} \tag{5}$$

$$\mathrm{d}^2_{yz} \quad \mathrm{d}^2_{zx} \quad \mathrm{d}^2_{xy} \quad \mathrm{d}^2_{tx} \quad \mathrm{d}^2_{ty} \quad \mathrm{d}^2_{tz} \tag{6}$$

$$\mathrm{d}^3_t \quad \mathrm{d}^3_x \quad \mathrm{d}^3_y \quad \mathrm{d}^3_z \tag{7}$$

$$\mathrm{d}^4 \tag{8}$$

so that $-\mathrm{d}_{xyz} \coloneqq \mathrm{d}\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz \coloneqq \partial_{\tilde{t}}$.

Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{gathered} n_{xyz}\,\mathrm{d}^3\tilde{x}yz - n_{tyz}\,\mathrm{d}^3\tilde{t}yz - n_{tzx}\,\mathrm{d}^3\tilde{t}zx - n_{txy}\,\mathrm{d}^3\tilde{t}xy \\ \equiv n^t\,\mathrm{d}^3_t + n^x\,\mathrm{d}^3_x + n^y\,\mathrm{d}^3_y + n^z\,\mathrm{d}^3_z \end{gathered} \tag{9}$$

$$\text{with } n^t \equiv n_{xyz}\ ,\ n^x \equiv n_{tyz}\ ,\ \text{ and so on.}$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta}\,\mathrm{d}u^\beta\ ,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$\mathrm{d}\alpha' = \frac{\partial\alpha'}{\partial\alpha}\,\mathrm{d}\alpha$$

- raising or lowering of indices with the metric, for example

$$\mathrm{d}\alpha \mapsto g^{\alpha\beta}\,\partial_\beta\ .$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix[1]. For the spaces of 3-vectors and 3-covectors we have simplified formulae[2]:

$$\begin{aligned} \mathrm{d}^3_{\alpha'} &= \det\!\left(\tfrac{\partial\alpha'}{\partial\alpha}\right)\frac{\partial\alpha}{\partial\alpha'}\,\mathrm{d}^3_\alpha \\ \partial^3\alpha' &= \det\!\left(\tfrac{\partial\alpha}{\partial\alpha'}\right)\frac{\partial\alpha'}{\partial\alpha}\,\partial^3\alpha \end{aligned} \tag{10}$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} \mathrm{d}^3_\alpha &\mapsto \frac{1}{|g|}\,g_{\alpha\beta}\,\partial^3\beta \\ \partial^3\alpha &\mapsto |g|\,g^{\alpha\beta}\,\mathrm{d}^3_\beta \end{aligned} \tag{11}$$

[1] Choquet-Bruhat et al. 1996 § IV.A.1 p. 199. [2] Gantmacher 2000 § I.4 eq. (33).

3 Metric

We take the metric $\boldsymbol{g}$ to have signature $(-,+,+,+)$ and dimensions of area. The square root of its negative determinant is denoted shortly

$$\sqrt{g} := \sqrt{-\det \boldsymbol{g}}\ . \tag{12}$$

The volume element induced by the metric $\boldsymbol{g}$ has dimensions of volume-time and is denoted (note the boldface)

$$\boldsymbol{\gamma} := \frac{1}{c}\sqrt{g}\ \mathrm{d}^4\tilde{t}xyz \equiv \frac{\sqrt{g}}{c}\ \mathrm{d}^4 \tag{13}$$

and its corresponding inverse, a twisted 4-vector:

$$\boldsymbol{\gamma}^{-1} := \frac{c}{\sqrt{g}}\ \partial^4 \tag{14}$$

Contraction with the volume element or its inverse establishes a "volume duality" between outer n-covectors and inner $(4-n)$-vectors:

$$\left.\begin{matrix} \partial^3_{xyz}\ \ \partial^3_{tyz}\ \ \partial^3_{tzx}\ \ \partial^3_{txy} \\ \partial^2_{tx}\ \ \partial^2_{ty}\ \ \partial^2_{tz}\ \ \partial^2_{yz}\ \ \partial^2_{zx}\ \ \partial^2_{xy} \\ \partial_t\ \ \partial_x\ \ \partial_y\ \ \partial_z \\ \partial^4_{txyz} \end{matrix}\right) \quad \begin{matrix} \xleftarrow{\ \frac{\sqrt{g}}{c}\,\boldsymbol{\gamma}^{-1}\,\cdot\ } \\ \xrightarrow{\ \cdot\,\boldsymbol{\gamma}\,\frac{c}{\sqrt{g}}\ } \end{matrix} \quad \left(\begin{matrix} \mathrm{d}_{xyz}\ \ \mathrm{d}_{tyz}\ \ \mathrm{d}_{tzx}\ \ \mathrm{d}_{txy} \\ \mathrm{d}^2_{yz}\ \ \mathrm{d}^2_{zx}\ \ \mathrm{d}^2_{xy}\ \ \mathrm{d}^2_{tx}\ \ \mathrm{d}^2_{ty}\ \ \mathrm{d}^2_{tz} \\ \mathrm{d}^3_t\ \ \mathrm{d}^3_x\ \ \mathrm{d}^3_y\ \ \mathrm{d}^3_z \\ \mathrm{d}^4 \end{matrix}\right. \tag{15}$$

This is the reason why in older literature an outer-oriented n-covector is treated as a $(4-n)$-"vector density", that is, a vector divided by the square root of the volume element.

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{16}$$

and the volume element is simply d^4.

4 Four-stress

The stress-energy-momentum tensor, or simply 4-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\boldsymbol{T} = -\epsilon\, \mathrm{d}^3_t \otimes \mathrm{d}t - q^i\, \mathrm{d}^3_i \otimes \mathrm{d}t + p_j\, \mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\, \mathrm{d}^3_i \otimes \mathrm{d}x^j \tag{17}$$

the indices i, j running over x, y, z, and where

$$\begin{aligned} \epsilon &= \text{volumic energy} & q^i &= \text{aeric energy flux} \\ p_i &= \text{volumic momentum} & \pi^i_j &= \text{3-stress} \end{aligned} \tag{18}$$

measured in the coordinate system $txyz$. The energy ϵ is a density per unit *coordinate* volume xyz, and possibly includes a conversion factor for the time unit. The component q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. The component p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i. The components π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} -c^2\, g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \tag{19}$$

The diagonal elements $g_{tt}, \dots$ include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

Contracting the 4-stress with the volume element and the inverse metric we obtain:

$$\begin{aligned} \gamma^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1} = {}& \frac{g^{tt}}{c\sqrt{g}}\, \epsilon\, \partial_t \otimes \partial_t + \frac{g^{tt}}{c\sqrt{g}}\, q^i\, \partial_i \otimes \partial_t \\ & + \sum_j \frac{g^{jj}}{c\sqrt{g}}\, p_j\, \partial_t \otimes \partial_j + \sum_j \frac{g^{jj}}{c\sqrt{g}}\, \pi^i_j\, \partial_i \otimes \partial_j \end{aligned} \tag{20}$$

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- $\boldsymbol{u}^{\bullet}$ is a 1-vector, represented by the column-matrix $\boldsymbol{u}$.
- Similarly for $\boldsymbol{v}^{\bullet}$.
- $\boldsymbol{\omega}_{\bullet}$ is a 1-covector, represented by the row-matrix $\boldsymbol{\omega}$.
- $\boldsymbol{g}_{\bullet\bullet}$ is a co-covector, represented by the matrix $\boldsymbol{g}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\boldsymbol{g}^{-1^{\bullet\bullet}}$ is a contra-contravector, inverse of $\boldsymbol{g}$, that is: $\boldsymbol{g} \cdot \boldsymbol{g}^{1} = \mathbf{id}_{\bullet}^{\bullet}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\boldsymbol{Q}_{|\bullet\bullet\bullet|}$ is a 3-covector, represented by the column-matrix $\boldsymbol{Q}$.
- $\boldsymbol{T}_{|\bullet\bullet\bullet|\bullet}$ is a 1-covector-valued 3-covector, represented by the matrix $\boldsymbol{T}$. The rows represent the 3-covector components; the columns, the 1-covector components.
- $\gamma_{|\bullet\bullet\bullet\bullet|}$ is a 4-covector, represented by the number γ.
- $\gamma^{-1^{|\bullet\bullet\bullet\bullet|}}$ is a 4-vector, represented by the number γ^{-1}
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from "old" coordinates x to "new" coordinates x' is represented by the matrix $\boldsymbol{J}$. The rows correspond to the new coordinates x'; the columns, to the old x.

- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x' is represented by the matrix $\boldsymbol{J}^{-1}$. The rows correspond to the old coordinates x; the columns, to the new x'.

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

$$\text{(object)} \qquad \text{(matrix repr)} \tag{21}$$

$$\boldsymbol{\omega} \cdot \boldsymbol{u} \qquad \boldsymbol{\omega}\boldsymbol{u} \equiv \boldsymbol{u}^{\intercal}\boldsymbol{\omega}^{\intercal} \quad \text{(number)} \tag{22}$$

$$\boldsymbol{v} \cdot \boldsymbol{g} \cdot \boldsymbol{u} \qquad \boldsymbol{v}^{\intercal}\boldsymbol{g}\boldsymbol{u} \quad \text{(number)} \tag{23}$$

$$\boldsymbol{g} \cdot \boldsymbol{u} \qquad \boldsymbol{u}^{\intercal}\boldsymbol{g}^{\intercal} \quad \text{(row-matrix)} \tag{24}$$

$$\boldsymbol{\omega} \cdot \boldsymbol{g}^{-1} \qquad \boldsymbol{g}^{-\intercal}\boldsymbol{\omega}^{\intercal} \quad \text{(column-matrix)} \tag{25}$$

$$\gamma^{-1} \cdot \boldsymbol{Q} \qquad \gamma^{-1}\boldsymbol{Q} \quad \text{(column-matrix)} \tag{26}$$

$$\gamma^{-1} \cdot \boldsymbol{T} \qquad \gamma^{-1}\boldsymbol{T} \quad \text{(column-matrix)} \tag{27}$$

$$\boldsymbol{T} \cdot \boldsymbol{u} \qquad \boldsymbol{T}\boldsymbol{u} \quad \text{(column-matrix)} \tag{28}$$

$$\boldsymbol{g} \cdot (\gamma^{-1} \cdot \boldsymbol{T}) \cdot \boldsymbol{u} \qquad \gamma^{-1}\,\boldsymbol{g}\boldsymbol{T}\boldsymbol{u} \quad \text{(matrix)} \tag{29}$$

- Transformations

$$\text{(old coords)} \quad \mapsto \quad \text{(new coords)} \tag{30}$$

$$\boldsymbol{u} \quad \mapsto \quad \boldsymbol{J}\boldsymbol{u} \tag{31}$$

$$\boldsymbol{\omega} \quad \mapsto \quad \boldsymbol{\omega}\boldsymbol{J}^{-1} \tag{32}$$

$$\boldsymbol{g} \quad \mapsto \quad \boldsymbol{J}^{-\intercal}\boldsymbol{g}\boldsymbol{J}^{-1} \tag{33}$$

$$\boldsymbol{Q} \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{J}\boldsymbol{Q} \tag{34}$$

$$\gamma \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}}\,\gamma \tag{35}$$

$$\gamma^{-1} \quad \mapsto \quad \det \boldsymbol{J}\,\gamma^{-1} \tag{36}$$

$$\boldsymbol{T} \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{J}\boldsymbol{T}\boldsymbol{J}^{-1} \tag{37}$$

B Checks about optimal representation of 4-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$
\begin{gathered}
\mathrm{d}\big(f\,\mathrm{d}^3_t\big) = \mathrm{d}\Big(f\,\mathrm{d}^3\tilde{x}yz\Big) = \partial_t f \mathrm{d}t \wedge \mathrm{d}^3\tilde{x}yz + 0 = \partial_t f\ \mathrm{d}^4\tilde{t}xyz \\
\mathrm{d}\big(f\,\mathrm{d}^3_x\big) = -\mathrm{d}\Big(f\,\mathrm{d}^3\tilde{t}yz\Big) = -\partial_x f \mathrm{d}x \wedge \mathrm{d}^3\tilde{t}yz + 0 = \partial_x f\ \mathrm{d}^4\tilde{t}xyz \\
\mathrm{d}\big(f\,\mathrm{d}^3_i\big) = \partial_i f\ \mathrm{d}^4\tilde{t}xyz
\end{gathered}
\tag{38}
$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$
\begin{aligned}
\boldsymbol{\nabla}(\mathrm{d}t) &= -\Gamma^t_{tt}\,\mathrm{d}t \otimes \mathrm{d}t - \Gamma^t_{tj}\,\mathrm{d}t \otimes \mathrm{d}x^j - \Gamma^t_{it}\,\mathrm{d}x^i \otimes \mathrm{d}t - \Gamma^t_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \\
\boldsymbol{\nabla}\big(\mathrm{d}x^k\big) &= -\Gamma^k_{tt}\,\mathrm{d}t \otimes \mathrm{d}t - \Gamma^k_{tj}\,\mathrm{d}t \otimes \mathrm{d}x^j - \Gamma^k_{it}\,\mathrm{d}x^i \otimes \mathrm{d}t - \Gamma^k_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j
\end{aligned}
\tag{39}
$$

If $\boldsymbol{\omega}$ is a 3-covector, $\boldsymbol{\phi}$ a 1-covector, and D the exterior covariant derivative, then

$$
\mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) = (\mathrm{d}\boldsymbol{\phi}) \otimes \boldsymbol{\omega} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega}
\tag{40}
$$

and in particular

$$
\mathrm{D}\big(\boldsymbol{\phi} \otimes \mathrm{d}x^\alpha\big) = (\mathrm{d}\boldsymbol{\phi}) \otimes \mathrm{d}x^\alpha + \Gamma^\alpha_{\mu\nu}\,\big(\boldsymbol{\phi} \wedge \mathrm{d}x^\mu\big) \otimes \mathrm{d}x^\nu\ .
\tag{41}
$$

Let's also consider the contraction with a 1-vector $\boldsymbol{u}$:

$$
\begin{aligned}
\mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u}) &= \mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \boldsymbol{u} - (\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \curlywedge \nabla\boldsymbol{u} \\
&\equiv \mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega} \\
&= \mathrm{d}\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega}\ .
\end{aligned}
\tag{42}
$$

and in particular

$$
\mathrm{D}(\boldsymbol{\phi} \otimes \mathrm{d}x^\alpha \cdot \boldsymbol{u}) = \mathrm{d}\boldsymbol{\phi}\, u^\alpha - \boldsymbol{\phi} \wedge \mathrm{d}x^\beta\, \partial_\beta u^\alpha
\tag{43}
$$

A balance equation with the exterior covariant derivative then becomes

$$
\mathrm{D}\boldsymbol{T} = 0 \qquad \Longrightarrow \qquad \mathrm{d}(\boldsymbol{T} \cdot \boldsymbol{u}) = -\boldsymbol{T} \curlywedge \nabla\boldsymbol{u}
\tag{44}
$$

Then

$$\begin{aligned}
0 &= \mathrm{D}\boldsymbol{T} \\
&= \mathrm{D}\big(-e\,\mathrm{d}^3_t \otimes \mathrm{d}t - q^i\,\mathrm{d}^3_i \otimes \mathrm{d}t + p_j\,\mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\,\mathrm{d}^3_i \otimes \mathrm{d}x^j\big) \\
&= -\partial_t e\ \mathrm{d}^4 \otimes \mathrm{d}t - \partial_i q^i\,\mathrm{d}^4 \otimes \mathrm{d}t + \partial_t p_j\,\mathrm{d}^4 \otimes \mathrm{d}x^j + \partial_i \pi^i_j\,\mathrm{d}^4 \otimes \mathrm{d}x^j - {} \\
&\quad \Big[e\,\Gamma^t_{tt}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t + e\,\Gamma^t_{tj}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0 \\
&\quad q^i\,\Gamma^t_{it}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t + q^i\,\Gamma^t_{ij}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j + 0 \\
&\quad - p_j\,\Gamma^j_{tt}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t - p_k\,\Gamma^k_{tj}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0 \\
&\quad - \pi^i_k\,\Gamma^k_{it}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t - \pi^i_k\,\Gamma^k_{ij}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j\Big] \\
&= \mathrm{d}^4 \otimes \Big[\\
&\quad - \partial_t e\ \mathrm{d}t + e\,\Gamma^t_{tt}\,\mathrm{d}t + e\,\Gamma^t_{tj}\,\mathrm{d}x^j \\
&\quad - \partial_i q^i\,\mathrm{d}t + q^i\,\Gamma^t_{it}\,\mathrm{d}t + q^i\,\Gamma^t_{ij}\,\mathrm{d}x^j \\
&\quad + \partial_t p_j\,\mathrm{d}x^j - p_j\,\Gamma^j_{tt}\,\mathrm{d}t - p_k\,\Gamma^k_{tj}\,\mathrm{d}x^j \\
&\quad + \partial_i \pi^i_j\,\mathrm{d}x^j - \pi^i_k\,\Gamma^k_{it}\,\mathrm{d}t - \pi^i_k\,\Gamma^k_{ij}\,\mathrm{d}x^j \\
&\quad \Big]
\end{aligned} \tag{45}$$

The equation above corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} - p_j\,\Gamma^j_{tt} - \pi^i_k\,\Gamma^k_{it} \tag{46}$$

$$\partial_t p_j + \partial_i \pi^i_j = -e\,\Gamma^t_{tj} - q^i\,\Gamma^t_{ij} + p_k\,\Gamma^k_{tj} + \pi^i_k\,\Gamma^k_{ij} \tag{47}$$

For $\boldsymbol{T} \cdot \boldsymbol{u}$ we find first

$$\boldsymbol{T} \cdot \boldsymbol{u} = (-e\,u^t + p_j\,u^j)\,\mathrm{d}^3_t + (-q^i\,u^t + \pi^i_j\,u^j)\,\mathrm{d}^3_i \tag{48}$$

$$\begin{aligned}
\boldsymbol{T} \curlywedge \nabla\boldsymbol{u} &= (-e\,\partial_t u^t + p_j\,\partial_t u^j)\,\mathrm{d}^3_t \wedge \mathrm{d}t + (-q^i\,\partial_i u^t + \pi^i_j\,\partial_i u^j)\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \\
&\qquad + \Gamma \text{ terms}
\end{aligned} \tag{49}$$

and therefore

$$\begin{aligned}
\partial_t(-e\,u^t + p_j\,u^j) + \partial_i(-q^i\,u^t + \pi^i_j\,u^j) ={}& \\
&-e\,\partial_t u^t + p_j\,\partial_t u^j - q^i\,\partial_i u^t + \pi^i_j\,\partial_i u^j \\
&-(e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} - p_j\,\Gamma^j_{tt} - \pi^i_k\,\Gamma^k_{it})\,u^t \\
&-(e\,\Gamma^t_{tj} + q^i\,\Gamma^t_{ij} - p_k\,\Gamma^k_{tj} - \pi^i_k\,\Gamma^k_{ij})\,u^j
\end{aligned} \tag{50}$$

Radial case

$$\partial_t e + \partial_r q^r = e\,\Gamma^t_{tt} + q^r\,\Gamma^t_{rt} + p_r\,\Gamma^r_{tt} + \pi^r_r\,\Gamma^r_{rt} \tag{51}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\Gamma^t_{tr} + q^r\,\Gamma^t_{rr} + p_r\,\Gamma^r_{tr} + \pi^r_r\,\Gamma^r_{rr} \tag{52}$$

$$\partial_t e + \partial_r q^r = q^r\,\frac{g}{c^2} + p_r\,g \tag{53}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\frac{g}{c^2} \tag{54}$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have[3]

$$\Gamma^t_{jt} = \Gamma^t_{tj} = \frac{GM}{c^2}\,\frac{x^j}{r^3} \approx \frac{g}{c^2} \qquad \Gamma^j_{tt} = GM\,\frac{x^j}{r^3} \approx g \tag{55}$$

where g is the standard acceleration, considered positive. Take also $p_j \approx m v_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

The balances above become

$$\partial_t e + \partial_i q^i = q^z\,\frac{g}{c^2} + p_z\,g \tag{56}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\,\frac{g}{c^2} \tag{57}$$

Also,

[3] Poisson & Will 2014 § 5.2.3.

$$\begin{aligned} \Gamma^t_{tt} &\approx -2\frac{g}{c^2}v(t) & \Gamma^t_{jt} &= \Gamma^t_{tj} \approx \frac{g}{c^2} \\ \Gamma^j_{tt} &\approx g - 2\frac{g}{c^2}\,v(t)^2 - \dot{v}(t) & \Gamma^j_{jt} &= \Gamma^j_{tj} \approx \frac{g}{c^2}v(t) \end{aligned} \tag{58}$$

$$\partial_t e + \partial_i q^i = -e\,2\frac{g}{c^2}v(t) + q^z\,\frac{g}{c^2} + p_j\left(g - 2\frac{g}{c^2}\,v(t)^2 - \dot{v}(t)\right) + \pi^z_z\,\frac{g}{c^2}v(t) \tag{59}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\,\frac{g}{c^2} + p_z\,\frac{g}{c^2}v(t) \tag{60}$$

C Works with useful content

- Eq. (21) in[4]: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in[5]

For transformation or raising:[6].

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B} \gamma \cdot (A \wedge B) \tag{61}$$

with $\deg(B) = n - \deg(A)$[7]

Compound matrices:[8]

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("de *X*" is listed under D, "van *X*" under V, and so on, regardless of national conventions.)

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