Kinematics and dynamics from a modern perspective

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20 July 2022; updated 21 July 2022

The kinematics of mechanical, thermodynamic, and electromagnetic phenomena is developed in such a way as to be used for Newtonian, Lorentzian, and general relativity. The same is done, as much as possible, for their dynamics as well

1 Scribbles and memos

We can represent charge density and current density in one geometrical entity. We use a coordinate system (t, x, y, z); denote the wedge product by juxtaposition: $\mathrm{d}t \wedge \mathrm{d}x =: \mathrm{d}t\,\mathrm{d}x$ and so on; abbreviate $\mathrm{d}t\,\mathrm{d}x =: \mathrm{d}tx$ and so on; and denote twisted differential forms by $\tilde{\mathrm{d}}$. Then the charge-current density is

$$\rho\,\tilde{\mathrm{d}}xyz + j_x\,\tilde{\mathrm{d}}tyz + j_y\,\tilde{\mathrm{d}}tzx + j_z\,\tilde{\mathrm{d}}txy \equiv \rho\,\tilde{\mathrm{d}}xyz + \mathrm{d}t\,(j_x\,\tilde{\mathrm{d}}yz + j_y\,\tilde{\mathrm{d}}zx + j_z\,\tilde{\mathrm{d}}xy) \tag{1}$$

2 Twisted objects

Where an ordered set of coordinate functions (t, x, y, z) is chosen, the twisted unit $\tilde{1}$ is defined. It has unit magnitude and outer-orientation $txyz$, and the property $\tilde{1} \cdot \tilde{1} = 1$. In general it is only defined in chart domain, where coordinate functions can be defined, but not globally. We obtain twisted vectors and covectors by multiplying their non-twisted counterparts by the twisted unit. For a vector or covector ω we have that the orientation of its twisted counterpart is such that (see Burke 1983)

$$\big\{\{\tilde{\omega}\}, \{\omega\}\big\} = \big\{\tilde{1}\big\}\,. \tag{2}$$

Said otherwise, in the product $\tilde{1} \cdot \omega$ the *right* side of the orientation of $\tilde{1}$ cancels out with the orientation of ω. This rule must be respected even if we invert the product order, so $\tilde{1} \cdot \omega \equiv \omega \cdot \tilde{1}$.

Multiplying a twisted vector or covector by the twisted unit, we obtain its non-twisted counterpart. The resulting orientation is obtained by cancelling out the orientation of the twisted object and the *left* side of the orientation of $\tilde{1}$.

We have:

$$\{\tilde{1}\} = txyz\ ; \tag{3}$$

$$\{\tilde{\mathrm{d}}t\} = -xyz\ , \quad \{\tilde{\mathrm{d}}x\} = tyz\ , \quad \{\tilde{\mathrm{d}}y\} = tzx\ , \quad \{\tilde{\mathrm{d}}z\} = txy\ ; \tag{4}$$

$$\{\tilde{\mathrm{d}}tx\} = yz\ , \quad \{\tilde{\mathrm{d}}ty\} = zx\ , \quad \{\tilde{\mathrm{d}}tz\} = xy\ , \tag{5}$$

$$\{\tilde{\mathrm{d}}xy\} = tz\ , \quad \{\tilde{\mathrm{d}}yz\} = tx\ , \quad \{\tilde{\mathrm{d}}zx\} = ty\ ; \tag{6}$$

$$\{\tilde{\mathrm{d}}txy\} = -z\ , \quad \{\tilde{\mathrm{d}}tyz\} = -x\ , \quad \{\tilde{\mathrm{d}}tzx\} = -y\ , \quad \{\tilde{\mathrm{d}}xyz\} = t\ ; \tag{7}$$

$$\{\tilde{\mathrm{d}}txyz\} = +1\ . \tag{8}$$

The minus signs appear when we have t and an even number of other coordinates after the "$\tilde{\mathrm{d}}$".

Note that considering, say, the *function* t, we have

$$\{\tilde{t}\} = \begin{cases} \{txyz\} & \text{if } t > 0\ , \\ -\{txyz\} & \text{if } t < 0\ . \end{cases} \tag{9}$$

3 Orientation choices

Let's examine several conventions, keeping in mind the rules found in Burke 1983 eqs (5)–(11):

(i) (3+1)D, order (t, x, y, z):
$\{\tilde{\mathrm{d}}x\} = tyz$, $\{\tilde{\mathrm{d}}t\} = -xyz$, $\{\tilde{\mathrm{d}}tx\} = yz$, $\{\tilde{\mathrm{d}}xy\} = tz$, $\{\tilde{\mathrm{d}}xyz\} = t$.

(ii) (2+1)D, order (t, x, y):
$\{\tilde{\mathrm{d}}x\} = -ty$, $\{\tilde{\mathrm{d}}t\} = xy$, $\{\tilde{\mathrm{d}}tx\} = y$, $\{\tilde{\mathrm{d}}xy\} = t$.

(iii) (3+1)D, order (x, y, z, t):
$\{\tilde{\mathrm{d}}x\} = -yzt$, $\{\tilde{\mathrm{d}}t\} = xyz$, $\{\tilde{\mathrm{d}}xt\} = yz$, $\{\tilde{\mathrm{d}}xy\} = zt$, $\{\tilde{\mathrm{d}}xyz\} = -t$.

(iv) (2+1)D, order (x, y, t):
$\{\tilde{\mathrm{d}}x\} = yt$, $\{\tilde{\mathrm{d}}t\} = xy$, $\{\tilde{\mathrm{d}}xt\} = -y$, $\{\tilde{\mathrm{d}}xy\} = t$.

It is convenient to keep t factors always to either right or left, because this allows us to factor them in sums without worries about signs. One thing to be noted with this general choice is that we get either $\{\tilde{\mathrm{d}}t\} = -xyz$ or $\{\tilde{\mathrm{d}}xyz\} = -t$: one minus sign will appear in either case.

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3+1,\ txyz & 2+1,\ txy & 3+1,\ xyzt & 2+1,\ xyt \\
\{\tilde{\mathrm{d}}t\} = -xyz & \{\tilde{\mathrm{d}}t\} = xy & \{\tilde{\mathrm{d}}t\} = xyz & \{\tilde{\mathrm{d}}t\} = xy \\
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\{\partial_{xyz} \cdot \gamma\} = xyz & & \{\partial_{xyz} \cdot \gamma\} = xyz &
\end{array}
\tag{10}
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Bibliography

("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

Burke, W. L. (1983): *Manifestly parity invariant electromagnetic theory and twisted tensors*. J. Math. Phys. **24**[1], 65–69. DOI:10.1063/1.525603.