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Kinematics and dynamics from a modern perspective

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The kinematics of mechanical, thermodynamic, and electromagnetic phenomena is developed in such a way as to be used for Newtonian, Lorentzian, and general relativity. The same is done, as much as possible, for their dynamics as well

1 Vectors and covectors

1.1 Twisted objects

Denote the wedge product by juxtaposition: $\mathrm{d}t \wedge \mathrm{d}x =: \mathrm{d}t\,\mathrm{d}x$ and so on; abbreviate $\mathrm{d}t\,\mathrm{d}x =: \mathrm{d}tx$ and so on; and denote twisted differential forms by $\underset{\sim}{\mathrm{d}}$.

Where an ordered set of coordinate functions (t, x, y, z) is chosen, the twisted unit $\underset{\sim}{1}$ is defined. It has unit magnitude and outer-orientation $txyz$, and the property $\underset{\sim}{1} \cdot \underset{\sim}{1} = 1$. In general it is only defined in chart domain, where coordinate functions can be defined, but not globally. We obtain twisted vectors and covectors by multiplying their non-twisted counterparts by the twisted unit. For a vector or covector ω we have that the orientation of its twisted counterpart is such that (Burke 1987 eq. (28.1))

$$\big\{\{\underset{\sim}{\omega}\}, \{\omega\}\big\} = \big\{\underset{\sim}{1}\big\} \,. \tag{1}$$

Said otherwise, in the product $\underset{\sim}{1} \cdot \omega$ the *right* side of the orientation of $\underset{\sim}{1}$ cancels out with the orientation of ω. This rule must be respected even if we invert the product order, so $\underset{\sim}{1} \cdot \omega \equiv \omega \cdot \underset{\sim}{1}$.

Multiplying a twisted vector or covector by the twisted unit, we obtain its non-twisted counterpart. The resulting orientation is obtained by cancelling out the orientation of the twisted object and the *left* side of the orientation of $\underset{\sim}{1}$.

We have:

$$\{\underset{\sim}{1}\} = txyz \; ; \tag{2}$$

$$\{\underset{\sim}{\mathrm{d}}t\} = -xyz \,, \quad \{\underset{\sim}{\mathrm{d}}x\} = tyz \,, \quad \{\underset{\sim}{\mathrm{d}}y\} = tzx \,, \quad \{\underset{\sim}{\mathrm{d}}z\} = txy \; ; \tag{3}$$

$$\{\underset{\sim}{\mathrm{d}}tx\} = yz \,, \quad \{\underset{\sim}{\mathrm{d}}ty\} = zx \,, \quad \{\underset{\sim}{\mathrm{d}}tz\} = xy \,, \tag{4}$$

$$\{\underset{\sim}{\mathrm{d}}xy\} = tz \,, \quad \{\underset{\sim}{\mathrm{d}}yz\} = tx \,, \quad \{\underset{\sim}{\mathrm{d}}zx\} = ty \; ; \tag{5}$$

$$\{\underset{\sim}{\mathrm{d}}tyz\} = -x \,, \quad \{\underset{\sim}{\mathrm{d}}tzx\} = -y \,, \quad \{\underset{\sim}{\mathrm{d}}txy\} = -z \,, \quad \{\underset{\sim}{\mathrm{d}}xyz\} = t \; ; \tag{6}$$

$$\{\underset{\sim}{\mathrm{d}}txyz\} = +1 \,. \tag{7}$$

The minus signs appear in the odd ranks when we have t and an even number of other coordinates after the "$\underset{\sim}{\mathrm{d}}$". These minus signs flip if we keep t always to the right, with orientation $xyzt$.

Note that considering, say, the *function* x, we have

$$\{\underset{\sim}{x}\} = \begin{cases} \{txyz\} & \text{if } x > 0 \,, \\ -\{txyz\} & \text{if } x < 0 \,. \end{cases} \tag{8}$$

2 Charge-current density

2.1 Representation and integration

We can represent charge density and current density in one geometrical entity. Consider an ordered coordinate system (t, x, y, z). The charge-current density is

$$\begin{aligned} \boldsymbol{Q} &\coloneqq \rho \, \underset{\sim}{\mathrm{d}}xyz - j_x \, \underset{\sim}{\mathrm{d}}tyz - j_y \, \underset{\sim}{\mathrm{d}}tzx - j_z \, \underset{\sim}{\mathrm{d}}txy \\ &\equiv \rho \, \underset{\sim}{\mathrm{d}}xyz - \mathrm{d}t \, \left(j_x \, \underset{\sim}{\mathrm{d}}yz + j_y \, \underset{\sim}{\mathrm{d}}zx + j_z \, \underset{\sim}{\mathrm{d}}xy\right) \,. \end{aligned} \tag{9}$$

It has the dimensions of charge (current · time). The minus signs appear so that the x-component of the current, for example, is positive when $j_x > 0$ and so on.

This object automatically give us net volume charge when integrated over a three-dimensional region at constant time, or the net flux of charge when integrated over a two-dimensional surface – possibly even moving –

over a lapse of time. Consider for example a 3-volume V at constant time, having *outer* orientation in the positive t direction and parameterized by

$$(u, v, w) \mapsto (t_0, u, v, w)\ . \tag{10}$$

On it, the basis twisted 1-forms map to (Burke 1987 p. 192)

$$\begin{gathered}
\underset{-xyz}{\mathrm{d}}t\big|_V = 0\ , \quad \underset{tyz}{\mathrm{d}}x\big|_V = \underset{vw}{\mathrm{d}}u\ , \quad \underset{tzx}{\mathrm{d}}y\big|_V = \underset{wu}{\mathrm{d}}v\ , \quad \underset{txy}{\mathrm{d}}z\big|_V = \underset{uv}{\mathrm{d}}w\ , \\
\underset{t}{\mathrm{d}}xyz\big|_V = \underset{+}{\mathrm{d}}uvw\ .
\end{gathered} \tag{11}$$

Then we have

$$\big(\rho\, \underset{\sim}{\mathrm{d}}xyz - j_x\, \underset{\sim}{\mathrm{d}}tyz - j_y\, \underset{\sim}{\mathrm{d}}tzx - j_z\, \underset{\sim}{\mathrm{d}}txy\big)\big|_V = \rho\, \underset{+}{\mathrm{d}}uvw \tag{12}$$

and the current density gives no contribution.

2.2 Coordinate transformation

The charge-current density also correctly transform under coordinate changes. Consider for example

$$(t', x', y', z') = (t, x - vt, y, z)\ , \qquad (t, x, y, z) = (t', x' + vt', y', z')\ , \tag{13}$$

for which

$$\mathrm{d}t = \mathrm{d}t'\ , \quad \mathrm{d}y = \mathrm{d}y'\ , \quad \mathrm{d}z = \mathrm{d}z'\ , \quad \mathrm{d}x = \mathrm{d}x' + v\,\mathrm{d}t'\ ; \tag{14}$$

$$\begin{aligned}
\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= (\mathrm{d}x' + v\,\mathrm{d}t')\,\mathrm{d}y'\,\mathrm{d}z' = \mathrm{d}x'y'z' + v\,\mathrm{d}t'y'z'\ , \\
\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x &= \mathrm{d}t'\,\mathrm{d}z'\,(\mathrm{d}x' + v\,\mathrm{d}t') = \mathrm{d}t'\,\mathrm{d}z'\,\mathrm{d}x'\ , \\
\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y &= \mathrm{d}t'\,(\mathrm{d}x' + v\,\mathrm{d}t')\,\mathrm{d}y' = \mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}y'\ .
\end{aligned} \tag{15}$$

The charge-current density can then be rewritten as

$$\begin{aligned}
\boldsymbol{Q} &= \rho\, \underset{\sim}{\mathrm{d}}xyz - j_x\, \underset{\sim}{\mathrm{d}}tyz - j_y\, \underset{\sim}{\mathrm{d}}tzx - j_z\, \underset{\sim}{\mathrm{d}}txy \\
&= \rho\, \underset{\sim}{\mathrm{d}}x'y'z' - (j_x - \rho\, v)\, \underset{\sim}{\mathrm{d}}t'y'z' - j_y\, \underset{\sim}{\mathrm{d}}t'z'x' - j_z\, \underset{\sim}{\mathrm{d}}t'x'y'\ ,
\end{aligned} \tag{16}$$

which is indeed the correct transformation for the charge density and the x-component of the current density (Kovetz 2000 eq. (5.8)). It is important to note that we did not make any assumptions regarding spacetime symmetries and metric. The transformation (13) is a Galilei boost between Galileian inertial frames, if we assume Newtonian relativity, and a non-symmetry preserving coordinate transformation in Lorentzian or general relativity. So there is no contradiction with any of these theories. If we assume that Lorentzian relativity holds and (t, x, y, z) is a Lorentzian inertial frame, then a metric-preserving transformation would instead be

$$\begin{gathered}(t', x', y', z') = \Bigl((t - x\,v/c^2)/\gamma,\ (x - vt)/\gamma,\ y,\ z\Bigr)\,,\\ \gamma \coloneqq \sqrt{1 - v^2/c^2}\,,\end{gathered} \tag{17}$$

and a calculation similar to the previous one shows that the components of the charge-density would again transform as expected (Kovetz 2000 eqs (12.17)–(12.18)).

2.3 Charge-current balance

The law of charge conservation is simply expressed by

$$\mathrm{d}\boldsymbol{Q} = 0\,, \tag{18}$$

which leads to, considering permutations and antisymmetry,

$$\begin{aligned}0 = \mathrm{d}\boldsymbol{Q} &= \partial_t\rho\ \underset{\sim}{\mathrm{d}}txyz - \partial_x j_x\ \underset{\sim}{\mathrm{d}}xtyz - \partial_y j_y\ \underset{\sim}{\mathrm{d}}ytzx - \partial_x j_x\ \underset{\sim}{\mathrm{d}}ztxy\\ &= \bigl(\partial_t\rho + \partial_x j_x + \partial_y j_y + \partial_z j_z\bigr)\ \underset{\sim}{\mathrm{d}}txyz\,,\end{aligned} \tag{19}$$

implying the familiar (Kovetz 2000 eq. (1.14))

$$\partial_t\rho + \partial_x j_x + \partial_y j_y + \partial_z j_z = 0\,, \tag{20}$$

but now shown to be *valid in any coordinate system*.

All we have done in this section holds also for mass density and mass flux. It is important to keep mass flux and momentum separate, as they are not the same in Lorentzian and general relativity.

2.4 Charge-current potential

On a connected manifold the charge conservation $\partial\boldsymbol{Q} = 0$ implies that the charge can be written as the exterior derivative of a twisted 2-covector $\boldsymbol{f}$, the charge-current potential:

$$\boldsymbol{f} \coloneqq D_x\ \underset{\sim}{\mathrm{d}}yz + D_y\ \underset{\sim}{\mathrm{d}}zx + D_z\ \underset{\sim}{\mathrm{d}}xy + H_x\ \underset{\sim}{\mathrm{d}}tx + H_y\ \underset{\sim}{\mathrm{d}}ty + H_z\ \underset{\sim}{\mathrm{d}}tz\,. \tag{21}$$

In coordinates we find

$$
\begin{aligned}
\boldsymbol{Q} = \mathrm{d}\boldsymbol{f} &= (\partial_t D_x \,\underset{\sim}{\mathrm{d}}t + \partial_x D_x \,\underset{\sim}{\mathrm{d}}x)\,\underset{\sim}{\mathrm{d}}yz \\
&\quad + (\partial_t D_y \,\underset{\sim}{\mathrm{d}}t + \partial_y D_y \,\underset{\sim}{\mathrm{d}}y)\,\underset{\sim}{\mathrm{d}}zx \\
&\quad + (\partial_t D_z \,\underset{\sim}{\mathrm{d}}t + \partial_z D_z \,\underset{\sim}{\mathrm{d}}z)\,\underset{\sim}{\mathrm{d}}xy \\
&\quad + (\partial_y H_x \,\underset{\sim}{\mathrm{d}}y + \partial_z H_x \,\underset{\sim}{\mathrm{d}}z)\,\underset{\sim}{\mathrm{d}}tx \\
&\quad + (\partial_z H_y \,\underset{\sim}{\mathrm{d}}z + \partial_x H_y \,\underset{\sim}{\mathrm{d}}x)\,\underset{\sim}{\mathrm{d}}ty \\
&\quad + (\partial_x H_z \,\underset{\sim}{\mathrm{d}}x + \partial_y H_z \,\underset{\sim}{\mathrm{d}}y)\,\underset{\sim}{\mathrm{d}}tz \\
&\equiv (\partial_x D_x + \partial_y D_y + \partial_z D_z)\,\underset{\sim}{\mathrm{d}}xyz \\
&\quad + (\partial_t D_x - \partial_y H_z + \partial_z H_y)\,\underset{\sim}{\mathrm{d}}tyz \\
&\quad + (\partial_t D_y - \partial_z H_x + \partial_x H_z)\,\underset{\sim}{\mathrm{d}}tzx \\
&\quad + (\partial_t D_z - \partial_x H_y + \partial_y H_x)\,\underset{\sim}{\mathrm{d}}txy \;,
\end{aligned}
\tag{22}
$$

from which, by comparison with (9), we obtain the familiar equations

$$
\begin{aligned}
\partial_x D_x + \partial_y D_y + \partial_z D_z &= \rho \\
-\partial_t D_x + \partial_y H_z - \partial_z H_y &= j_x \\
-\partial_t D_y + \partial_z H_x - \partial_x H_z &= j_y \\
-\partial_t D_z + \partial_x H_y - \partial_y H_x &= j_z \;.
\end{aligned}
\tag{23}
$$

3 Electromagnetic flux

3.1 Representation and integration

We can represent the electric field and magnetic flux in one geometrical entity as well:

$$
\begin{aligned}
\boldsymbol{F} &\coloneqq B_x \,\mathrm{d}yz + B_y \,\mathrm{d}zx + B_z \,\mathrm{d}xy - E_x \,\mathrm{d}tx - E_y \,\mathrm{d}ty - E_z \,\mathrm{d}tz \\
&\equiv B_x \,\mathrm{d}yz + B_y \,\mathrm{d}zx + B_z \,\mathrm{d}xy - \mathrm{d}t \,\big(E_x \,\mathrm{d}x + E_y \,\mathrm{d}y + E_z \,\mathrm{d}z\big) \;.
\end{aligned}
\tag{24}
$$

This object automatically gives us the net magnetic flux, when integrated on a surface at a chosen time, or the time-integrated voltage, when integrated on a curve over a lapse of time.

3.2 Coordinate transformation

Under the coordinate transformation (13) we have

$$
\begin{gathered}
\mathrm{d}t\,\mathrm{d}x = \mathrm{d}t'\,\mathrm{d}x'\ , \quad \mathrm{d}t\,\mathrm{d}y = \mathrm{d}t'\,\mathrm{d}y'\ , \quad \mathrm{d}t\,\mathrm{d}z = \mathrm{d}t'\,\mathrm{d}z'\ , \\
\mathrm{d}y\,\mathrm{d}z = \mathrm{d}y'\,\mathrm{d}z'\ , \\
\mathrm{d}z\,\mathrm{d}x = \mathrm{d}z'\,\mathrm{d}x' - v\,\mathrm{d}t'\,\mathrm{d}z'\ , \quad \mathrm{d}x\,\mathrm{d}y = \mathrm{d}x'\,\mathrm{d}y' + v\,\mathrm{d}t'\,\mathrm{d}y'\ ,
\end{gathered}
\tag{25}
$$

and

$$
\begin{aligned}
\boldsymbol{F} &= B_x\ \mathrm{d}y'z' + B_y\ (\mathrm{d}z'x' - v\,\mathrm{d}t'z') + B_z\ (\mathrm{d}x'y' + v\,\mathrm{d}t'y') \\
&\qquad - E_x\ \mathrm{d}t'x' - E_y\ \mathrm{d}t'y' - E_z\ \mathrm{d}t'z' \\
&\equiv B_x\ \mathrm{d}y'z' + B_y\ \mathrm{d}z'x' + B_z\ \mathrm{d}x'y' \\
&\qquad - E_x\ \mathrm{d}t'x' - (E_y - v\,B_z)\ \mathrm{d}t'y' - (E_z + v\,B_y)\ \mathrm{d}t'z'\ .
\end{aligned}
\tag{26}
$$

which is again as expected in the Galileian case (Kovetz 2000 eq. (11.3)).

3.3 Electromagnetic-flux balance

The conservation of electromagnetic flux is simply expressed by

$$
\mathrm{d}\boldsymbol{F} = 0\ ,
\tag{27}
$$

which in coordinates becomes, keeping only terms that will not vanish owing to antisymmetry,

$$
\begin{aligned}
0 = \mathrm{d}\boldsymbol{F} &= (\partial_t B_x\ \mathrm{d}t + \partial_x B_x\ \mathrm{d}x)\ \mathrm{d}yz \\
&\quad + (\partial_t B_y\ \mathrm{d}t + \partial_y B_y\ \mathrm{d}y)\ \mathrm{d}zx \\
&\quad + (\partial_t B_z\ \mathrm{d}t + \partial_z B_z\ \mathrm{d}z)\ \mathrm{d}xy \\
&\quad - (\partial_y E_x\ \mathrm{d}y + \partial_z E_x\ \mathrm{d}z)\ \mathrm{d}tx \\
&\quad - (\partial_z E_y\ \mathrm{d}z + \partial_x E_y\ \mathrm{d}x)\ \mathrm{d}ty \\
&\quad - (\partial_x E_z\ \mathrm{d}x + \partial_y E_z\ \mathrm{d}y)\ \mathrm{d}tz \\
&\equiv (\partial_x B_x + \partial_y B_y + \partial_z B_z)\ \mathrm{d}xyz \\
&\quad + (\partial_t B_x + \partial_y E_z - \partial_z E_y)\ \mathrm{d}tyz \\
&\quad + (\partial_t B_y + \partial_z E_x - \partial_x E_z)\ \mathrm{d}tzx \\
&\quad + (\partial_t B_z + \partial_x E_y - \partial_y E_x)\ \mathrm{d}txy\ ,
\end{aligned}
\tag{28}
$$

where all four components must vanish, implying the familiar

$$\begin{aligned}
\partial_x B_x + \partial_y B_y + \partial_z B_z &= 0 \\
\partial_t B_x + \partial_y E_z - \partial_z E_y &= 0 \\
\partial_t B_y + \partial_z E_x - \partial_x E_z &= 0 \\
\partial_t B_z + \partial_x E_y - \partial_y E_x &= 0 \;.
\end{aligned} \tag{29}$$

Also these equations are *valid in any coordinate system*.

4 Scribbles and memos

Consider a coordinate system in which the metric tensor is written

$$\boldsymbol{g} = -c^2\, \mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \cdots \;. \tag{30}$$

The volume element (Porta Mana 2021 § 9.2) and its inverse are then

$$\gamma = \underset{\sim}{\mathrm{d}}txyz \;, \qquad \gamma^{-1} = \underset{\sim}{\partial}xyz \;. \tag{31}$$

The dot products (Truesdell & Toupin 1960 § 267 p. 661) of the inverse metric with the basis twisted covectors are

$$\boldsymbol{g} \cdot \mathrm{d}t = -\partial t / c^2 \;, \quad \boldsymbol{g} \cdot \mathrm{d}x = \partial x \;, \quad \text{and so on.} \tag{32}$$

We therefore have

$$\begin{aligned}
\boldsymbol{g} \cdot \boldsymbol{F} \cdot \boldsymbol{g} = {} & B_x\, \partial yz + B_y\, \partial zx + B_z\, \partial xy \\
& + E_x/c^2\, \partial tx + E_y/c^2\, \partial ty + E_z/c^2\, \partial tz \;.
\end{aligned} \tag{33}$$

Taking the dot product with the volume element on the left and dividing by μ_0

$$\begin{aligned}
\gamma \cdot (\boldsymbol{g} \cdot \boldsymbol{F} \cdot \boldsymbol{g})/\mu_0 = {} & B_x/\mu_0\, \underset{\sim}{\mathrm{d}}tx + B_y/\mu_0\, \underset{\sim}{\mathrm{d}}ty + B_z/\mu_0\, \underset{\sim}{\mathrm{d}}tz \\
& + \epsilon_0\, E_x\, \underset{\sim}{\mathrm{d}}yz + \epsilon_0\, E_y\, \underset{\sim}{\mathrm{d}}zx + \epsilon_0\, E_z\, \underset{\sim}{\mathrm{d}}xy \;.
\end{aligned} \tag{34}$$

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(“de *X*” is listed under D, “van *X*” under V, and so on, regardless of national conventions.)

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