

Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z) , which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

There are 18 basic types of geometric objects, which represent physical quantities of different types. We have inner-oriented scalars, inner-oriented 1-vectors up to 4-vectors, inner-oriented 1-covectors up to 4-covectors, and all their outer-oriented counterparts.

The exterior product ‘ \wedge ’ of vectors and covectors is usually represented just by juxtaposition, and a shortened notation is used for exterior products of several exterior derivatives ‘ d ’. For instance

$$d^2xy := dx \wedge dy \quad \partial_{xy}^2 := \partial_x \wedge \partial_y . \quad (1)$$

The associated bases for inner-oriented covector fields are

$$dt \quad dx \quad dy \quad dz \quad (2a)$$

$$d^2tx \quad d^2ty \quad d^2tz \quad d^2yz \quad d^2zx \quad d^2xy \quad (2b)$$

$$d^3xyz \quad -d^3tyz \quad -d^3tzx \quad -d^3txy \quad (2c)$$

$$d^4txyz \quad (2d)$$

and analogously for inner-oriented vector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it's only defined on a coordinate patch. It is idempotent: $\tilde{1}\tilde{1} = 1$.

A twisted or outer-oriented covector such as $d^3\tilde{x}yz$ has an associated *outer* direction, in this case positive t . We can adopt the following alternative notation for the outer-oriented versions of the bases above (analogous to the notation in Gotay & Marsden 1992 §2 p. 371); this notation emphasizes the actual orientation of the twisted covectors:

$$-dL_{xyz} \quad dL_{tyz} \quad dL_{tzz} \quad dL_{txy} \quad (3a)$$

$$d^2S_{yz} \quad d^2S_{zx} \quad d^2S_{xy} \quad d^2S_{tx} \quad d^2S_{ty} \quad d^2S_{tz} \quad (3b)$$

$$d^3V_t \quad d^3V_x \quad d^3V_y \quad d^3V_z \quad (3c)$$

$$d^4\Omega \quad (3d)$$

so that $-dL_{xyz} := d\tilde{f}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz := \partial_{\tilde{t}}$. Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} & n_{xyz} d^3\tilde{x}yz - n_{tyz} d^3\tilde{t}yz - n_{tzz} d^3\tilde{t}zx - n_{txy} d^3\tilde{t}xy \\ & \equiv n^t d^3V_t + n^x d^3V_x + n^y d^3V_y + n^z d^3V_z \end{aligned} \quad (4)$$

with $n^t \equiv n_{xyz}$, $n^x \equiv n_{tyz}$, and so on.

Contraction or dot-product of vectors and covectors is denoted by \cdot , and always contracts the maximal possible amount of contiguous elements. For instance

$$d^3xyz \cdot \partial_{yz}^2 = dx \quad - \partial_{txy}^3 \cdot dx = \partial_{ty}^2. \quad (5)$$

Contractions with the 4-vector ∂^4 and 4-covector $d^4\Omega$ establish a duality between outer n -covectors and inner $(4 - n)$ -vectors:

$$\left(\begin{array}{cccc} \partial^4 & & & \\ \partial_{xyz}^3 & \partial_{tyz}^3 & \partial_{tzx}^3 & \partial_{txy}^3 \\ & \partial_{tx}^2 & \partial_{ty}^2 & \partial_{tz}^2 \\ & \partial_{yz}^2 & \partial_{zx}^2 & \partial_{xy}^2 \\ & \partial_t & \partial_x & \partial_y & \partial_z \\ & & & & 1 \end{array} \right) \begin{array}{c} \xleftarrow{\partial^4 \cdot} \\ \xrightarrow{\cdot d^4\Omega} \end{array} \left(\begin{array}{cccc} \tilde{1} & & & \\ -dL_{xyz} & dL_{tyz} & dL_{tzx} & dL_{txy} \\ & d^2S_{yz} & d^2S_{zx} & d^2S_{xy} \\ & d^2S_{tx} & d^2S_{ty} & d^2S_{tz} \\ d^3V_t & d^3V_x & d^3V_y & d^3V_z \\ & & & d^4\Omega \end{array} \right) \quad (6)$$

These duals have special properties. For instance, for any 3-covector \mathcal{N} , we have

$$\mathcal{N} \cdot (\partial^4 \cdot \mathcal{N}) = (\partial^4 \cdot \mathcal{N}) \cdot \mathcal{N} = 0 \quad (7)$$

that is, the dual of \mathcal{N} is a vector belonging in the kernel of \mathcal{N} .

If $\frac{\sqrt{g}}{c}$ is a non-zero 4-covector and $\frac{c}{\sqrt{g}}$ the inverse 4-vector, that is, $\frac{\sqrt{g}}{c} \cdot \frac{c}{\sqrt{g}} = \frac{c}{\sqrt{g}} \cdot \frac{\sqrt{g}}{c} = 1$, and if \mathcal{N} is a 3-covector and ϕ a 1-covector, we have the useful identity

$$\mathcal{N} \wedge \phi = (\mathcal{N} \cdot \frac{c}{\sqrt{g}} \cdot \phi) \frac{\sqrt{g}}{c} \quad (8)$$

which also holds as long as the degrees of \mathcal{N} and ϕ sum up to 4.

We often consider vector-valued or covector-valued covectors, written as sums of tensor products. They can be outer- or inner-oriented. For instance

$$d^3V_t \otimes dx \equiv d^3\tilde{x}yz \otimes dx, \quad d^3V_x \otimes \partial_y. \quad (9)$$

The operation $\mathbf{\wedge}$ between a vector-valued covector and a covector-valued covector is the contraction of their vector- and covector-valued parts and the exterior product of their covector parts. For instance, if ϕ and ψ are covectors, ω is a covector, and u is a vector, then

$$(\phi \otimes \omega) \mathbf{\wedge} (\psi \otimes u) := (\phi \wedge \psi) \otimes (\omega \cdot u) \quad (10)$$

As another example,

$$(d^3V_t \otimes dx) \mathbf{\wedge} (dL_t \otimes \partial_x) = (d^3V_t \wedge dL_t)(dx \cdot \partial_x) = -d^4\Omega. \quad (11)$$

1.1 Moving area elements in spacetime

In 3D space we can consider a small area *which only exists at a given time t* , and with a crossing orientation; for instance a small area parallel to the y, z coordinates, with a positive- x crossing orientation. From a spacetime viewpoint we must also consider a temporal direction for this “crossing” orientation of the area. In a manner of speaking we have two choices: cross the surface in the positive- x direction, then “move forward” in time, in the positive- t direction; or move forward in time, then cross the surface. The first represents an xt orientation; the second a tx orientation.

If the area element in the example above has area a_x , expressed in units of the coordinates y and z , then the area element with the xt orientation is

$$a_x \partial^2 xt \equiv a_x \partial_{-\bar{y}z}^2 \equiv -a_x \partial_{\bar{y}z}^2 ,$$

whereas the one with the tx orientation is

$$a_x \partial^2 tx \equiv a_x \partial_{\bar{y}z}^2 .$$

An area element which *persists for a time lapse*, and which therefore has a motion with respect to the coordinate system (including rest as a special case), is a timelike 3D-volume from a spacetime viewpoint. For it we can for example choose a positive- x crossing direction; the temporal direction is irrelevant in this case. We can consider this timelike volume as produced by the area element displaced by a timelike vector.

Take the generic area element

$$\mathbf{A} := a_i \partial^2 x^i t \equiv a_x \partial^2 + a_y \partial^2 y t + a_z \partial^2 z t \quad (12)$$

and displace it by the timelike vector

$$\mathbf{V} := \delta_t (\partial_t + v^i \partial_{x^i}) \equiv \delta_t \partial_t + \delta_t v^x \partial_x + \delta_t v^y \partial_y + \delta_t v^z \partial_z \quad (13)$$

which represent a motion with coordinate velocity v^i for a time lapse δ_t . The resulting timelike volume is

$$\mathbf{A} \wedge \mathbf{V} = \delta_t (-a_i v^i \partial^3 t + a_i \partial^3 x^i) . \quad (14)$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta} du^\beta ,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$d\alpha' = \frac{\partial \alpha'}{\partial \alpha} d\alpha$$

- raising or lowering of indices with the metric, for example

$$dx^\alpha \mapsto g^{\alpha\beta} \partial_\beta .$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix¹. For the spaces of 3-vectors and 3-covectors we have simplified formulae²:

$$\begin{aligned} d^3 V_{\alpha'} &= \det\left(\frac{\partial \alpha'}{\partial \alpha}\right) \frac{\partial \alpha}{\partial \alpha'} d^3 V_{x^\alpha} \\ \partial^3 \alpha' &= \det\left(\frac{\partial \alpha}{\partial \alpha'}\right) \frac{\partial \alpha'}{\partial \alpha} \partial^3 \alpha \end{aligned} \tag{15}$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} d^3 V_{x^\alpha} &\mapsto \frac{1}{|g|} g_{\alpha\beta} \partial^3 \beta \\ \partial^3 \alpha &\mapsto |g| g^{\alpha\beta} d^3 V_\beta \end{aligned} \tag{16}$$

¹ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199. ² Gantmacher 2000 § I.4 eq. (33).

3 Metric

We take the metric \mathbf{g} to have signature $(-, +, +, +)$ and dimensions of area. The determinant and the negative of its square root are denoted shortly

$$g := \det \mathbf{g} \quad \sqrt{g} := \sqrt{-\det \mathbf{g}}. \quad (17)$$

The volume element induced by the metric \mathbf{g} has dimensions of volume-time and is denoted (note the boldface)

$$\frac{\sqrt{g}}{c} := \frac{\sqrt{g}}{c} d^4 \tilde{x} y z \equiv \frac{\sqrt{g}}{c} d^4 \Omega \quad (18)$$

and its corresponding twisted 4-vector:

$$\frac{c}{\sqrt{g}} := \frac{c}{\sqrt{g}} \partial^4. \quad (19)$$

Contraction with the volume element or its inverse establishes a “volume duality” between outer n -covectors and inner $(4 - n)$ -vectors:

$$\begin{pmatrix} \partial^4 \\ \partial_{xyz}^3 & \partial_{tyz}^3 & \partial_{tzz}^3 & \partial_{txy}^3 \\ \partial_{tx}^2 & \partial_{ty}^2 & \partial_{tz}^2 \\ \partial_{yz}^2 & \partial_{zx}^2 & \partial_{xy}^2 \\ \partial_t & \partial_x & \partial_y & \partial_z \\ 1 \end{pmatrix} \begin{matrix} \xleftarrow{\frac{\sqrt{g}}{c} \frac{c}{\sqrt{g}}} \\ \xrightarrow{\frac{\sqrt{g}}{c} \frac{c}{\sqrt{g}}} \end{matrix} \begin{pmatrix} \tilde{1} \\ dL_{xyz} & dL_{tyz} & dL_{tzz} & dL_{txy} \\ d^2 S_{yz} & d^2 S_{zx} & d^2 S_{xy} \\ d^2 S_{tx} & d^2 S_{ty} & d^2 S_{tz} \\ d^3 V_t & d^3 V_x & d^3 V_y & d^3 V_z \\ d^4 \Omega \end{pmatrix} \quad (20)$$

This is the reason why in older literature an outer-oriented n -covector is treated as a $(4 - n)$ -“vector density”, that is, a vector divided by the square root of the volume element.

The transpose of the map $\frac{c}{\sqrt{g}} \cdot$ depends on what this map is acting upon:

$$\blacksquare \cdot \frac{c}{\sqrt{g}} = \begin{cases} -\frac{c}{\sqrt{g}} \cdot \blacksquare & \text{on 1- or 3-covectors,} \\ \frac{c}{\sqrt{g}} \cdot \blacksquare & \text{on 0-, 2-, 4-covectors.} \end{cases} \quad (21)$$

Therefore we have the following inverse relationships:

$$\frac{\sqrt{g}}{c} \cdot \left(\frac{c}{\sqrt{g}} \cdot \blacksquare \right) = \begin{cases} -\blacksquare & \text{on 1- or 3-covectors,} \\ \blacksquare & \text{on 0-, 2-, 4-covectors.} \end{cases} \quad (22)$$

The metric on the space of 1-vectors also induces metrics on the spaces of 1-covectors, 2-vectors and 2-covectors, and so on. In particular, the metrics $\overset{3}{g}^{-1}$ on the space of 3-covectors and $\overset{3}{g}$ on the space of 3-vectors can be written in coordinates as

$$\overset{3}{g}^{-1} = \frac{g^{\mu\nu}}{g} \partial^3 x^\mu \otimes \partial^3 x^\nu \quad \text{with dimensions length}^{-6} \quad (23)$$

$$\overset{3}{g} = g g^{\mu\nu} d^3 V_\mu \otimes d^3 V_\nu \quad \text{with dimensions length}^6. \quad (24)$$

With these we can define squared norms $\|\cdot\|^2$ on all those spaces. In the case of 3-vectors, this norm represents the physical metric volume spanned by the 3-vector.

Note the following identity:

$$\|\frac{c}{\sqrt{g}} \cdot \mathcal{N}\|^2 = -c^2 \|\mathcal{N}\|^2 \quad \text{for every 3-covector } \mathcal{N}. \quad (25)$$

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (26)$$

and the volume element is simply $d^4\Omega$.

4 Matter current

The amount-of-matter current \mathcal{N} is an outer-oriented 3-covector

$$\mathcal{N} = N d^3 V_t + J^i d^3 V_{x^i} \quad (27)$$

of dimensions “amount of matter”, typically measured in moles, where

- N is the volumic amount of matter, measured per unit coordinate volume.
- J^i is the areic flux of amount of matter (equivalent to a molar flux) through a small area at rest in the coordinate system (t, x^i) , measured per unit coordinate area and unit coordinate time.

The current for every particular kind of matter satisfies the conservation law

$$d\mathbf{N} = 0 \quad \text{or} \quad \partial_t N + \partial_i J^i = 0 \quad (28)$$

independent of any metric.

The common contravariant form of the matter current, “ N^μ ”, is obtained by contracting the matter current with the inverse volume element:

$${}'N^\mu \triangleq \frac{c}{\sqrt{g}} \cdot \mathbf{N} = \frac{c}{\sqrt{g}} N \partial_t + \frac{c}{\sqrt{g}} J^i \partial_i. \quad (29)$$

If a metric is present, a four-velocity \mathbf{U} can be associated with the matter current \mathbf{N} , defined by the following properties and identity:

$$\mathbf{U} \cdot \mathbf{N} = 0 \quad \|\mathbf{U}\|^2 = -c^2 \quad (30)$$

$$\mathbf{U} = \frac{1}{\|\mathbf{N}\|} \frac{c}{\sqrt{g}} \cdot \mathbf{N} \quad (31)$$

which also implies (for normal matter)

$$\mathbf{N} = \|\mathbf{N}\| \mathbf{U} \cdot \frac{\sqrt{g}}{c} \quad (32)$$

For normal matter (as opposed to antimatter) $\|\mathbf{N}\|^2 \geq 0$.

The flux of matter through a small area moving with coordinate velocity v^i , represented by $\mathbf{A} \wedge \mathbf{V}$ (§ 1.1), is

$$\begin{aligned} \mathbf{N} \cdot (\mathbf{A} \wedge \mathbf{V}) &\equiv (N \, d^3 V_t + J^i \, d^3 V_{x^i}) \cdot \delta_t (-a_i v^i \, \partial^3 t + a_i \, \partial^3 x^i) \\ &= (-N a_i v^i + J^i a_i) \delta_t. \end{aligned} \quad (33)$$

The negative sign in front of the first summand makes sense. Take for instance a yz area with velocity in the positive- x direction. Then there's matter passing from its positive- x side of to its negative- x side, that is, a negative- x flux of matter.

5 Charge and magnetic flux

The electric-charge current \mathbf{Q} is an outer-oriented 3-covector

$$\mathbf{Q} = Q \, d^3 V_t + I^i \, d^3 V_{x^i} \quad (34)$$

of dimensions “charge”, typically measured in coulombs, where

- Q is the volumic charge, measured per unit coordinate volume.
- I^i is the areic flux of charge through a small area at rest in the coordinate system (t, x^i) , measured per unit coordinate area and unit coordinate time.

The electric-charge current satisfies the conservation law

$$d\mathbf{Q} = 0 \quad \text{or} \quad \partial_t Q + \partial_i I^i = 0 \quad (35)$$

independent of any metric.

Since \mathbf{Q} is a closed, outer-oriented 3-covector field, it can at least locally be expressed as the exterior differential of an outer-oriented 2-covector field \mathbf{G} :

$$\begin{aligned} \mathbf{G} &= H_x d^2\tilde{t}x + H_y d^2\tilde{t}y + H_z d^2\tilde{t}z + D_x d^2\tilde{y}z + D_y d^2\tilde{z}x + D_z d^2\tilde{x}y \\ &\equiv H^x d^2S_{yz} + H^y d^2S_{zx} + H^z d^2S_{xy} + \\ &\quad D^x d^2S_{tx} + D^y d^2S_{ty} + D^z d^2S_{tz} \end{aligned} \quad (36)$$

so that

$$d\mathbf{G} = \mathbf{Q} \quad \text{or} \quad \partial_i D^i = Q, \quad \partial_t D^i - \epsilon^{ijk} \partial_j H_k = -I^i. \quad (37)$$

The electromagnetic field, represented by the Faraday 2-covector, is typically decomposed as³

$$\begin{aligned} \mathbf{F} &= dt\mathbf{E} + \mathbf{B} \equiv -\mathbf{E}dt + \mathbf{B} \\ &\equiv -E_x d^2tx - E_y d^2ty - E_z d^2tz + B^x d^2yz + B^y d^2zx + B^z d^2xy. \end{aligned} \quad (38)$$

Where \mathbf{E} is a 1-covector and \mathbf{B} a 2-covector. Given a system of coordinates, this decomposition is unique. Neither the 2-covector \mathbf{B} or the 1-covector \mathbf{E} have any have dt components, that is, they don't have ∂_t in their kernels.

³ Frankel 1979 ch. 9.

The conservation of magnetic flux is expressed by

$$d\mathbf{F} = 0 \quad (39)$$

or equivalently

$$\begin{aligned} \partial_i B^i &= 0 & (d^3xyz \text{ component}) \\ \partial_t B^i + \epsilon^{ijk} \partial_j E_k &= 0 & (d^3tx^jx^k \text{ components}) \end{aligned} \quad (40)$$

It should be noted that all components of $d\mathbf{E}$ containing dt also disappear, because of the exterior product $\mathbf{E} dt$. The differential d therefore operates on \mathbf{E} as if this 1-covector belonged to a three-dimensional manifolds with coordinates (x, y, z) . Let's denote this operation by \mathbf{d} ; it is connected with the "curl" operator. Then the equations above can be written

$$d\mathbf{B} = 0 \quad \partial_t \mathbf{B} + d\mathbf{E} = 0. \quad (41)$$

The aether relations are constitutive relations between \mathbf{G} and \mathbf{F} ; they involve the metric:

$$\mathbf{G} = \frac{1}{\mu_0} \frac{\sqrt{g}}{c} : (g^{-1} \cdot \mathbf{F} \cdot g^{-1}) \quad (42)$$

or, at a point in inertial coordinates

$$D_i = \frac{1}{\mu_0 c^2} E_i, \quad H^i = \frac{1}{\mu_0} B^i. \quad (43)$$

6 Four-stress and its equations

6.1 Expressions

The energy-momentum tensor, or simply four-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\begin{aligned} \mathcal{T} &= \mathcal{T}^\mu{}_\nu d^3V_\mu \otimes dx^\nu \\ &= -e d^3V_t \otimes dt - q^i d^3V_i \otimes dt + p_j d^3V_t \otimes dx^j + \pi_j^i d^3V_i \otimes dx^j \end{aligned} \quad (44)$$

the indices i, j running over x, y, z , and where:

- The energy e is a density per unit *coordinate* volume xyz , and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this energy comprises “rest” energy, kinetic energy, potential gravitational energy, and centrifugal potential energy.
- The energy flux q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this flux term includes transport of the energy term above and also heat flux.
- The momentum p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i .
- The compressive three-stress π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j . If the point at which the four-stress is considered has a matter-current, then this stress comprises momentum transport and internal stresses.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} -|g_{tt}| & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \quad (45)$$

Note that if, for instance, x has dimensions of angle, then g_{xx} has dimensions of area per squared angle, and similarly for the other metric components.

The common contra-contra-variant form of the stress, “ $T^{\mu\nu}$ ”, is obtained by contracting the four-stress with the inverse volume element and the inverse metric:

$$\begin{aligned} {}'T^{\mu\nu} &\equiv \frac{c}{\sqrt{g}} \cdot \mathcal{T} \cdot g^{-1} = \frac{c|g^{tt}|}{\sqrt{g}} e \partial_t \otimes \partial_t + \frac{c|g^{tt}|}{\sqrt{g}} q^i \partial_i \otimes \partial_t \\ &\quad + \frac{c g^{ij}}{\sqrt{g}} p_i \partial_t \otimes \partial_j + \frac{c g^{kj}}{\sqrt{g}} \pi^i_k \partial_i \otimes \partial_j. \end{aligned} \quad (46)$$

6.2 Equations for the four-stress

If ϕ is a covector, ω a vector or covector, the *exterior covariant derivative* D is defined as

$$D(\phi \otimes \omega) := (d\phi) \otimes \omega + (-1)^{\deg \phi} \phi \wedge \nabla \omega \quad (47)$$

and is extended by linearity (with respect to constant coefficients).

As a consequence of the Einstein equations, the *total* four-stress satisfies two equations:

The differential equation

$$D\mathcal{T} = 0 \quad (48)$$

which is equivalent to the component equations

$$\partial_\mu \mathcal{T}^\mu{}_\alpha = \mathcal{T}^\mu{}_\nu \Gamma^\nu_{\mu\alpha} \quad (49)$$

or

$$\begin{aligned} \partial_t e + \partial_i q^i &= e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k \\ \partial_i p_j + \partial_i \pi_j^i &= -e \Gamma_{tj}^t - q^i \Gamma_{ij}^t + p_k \Gamma_{tj}^k + \pi_k^i \Gamma_{ij}^k. \end{aligned} \quad (50)$$

And the tensor equation

$$\left(\frac{c}{\sqrt{g}} \cdot \mathcal{T} \cdot g^{-1} \right)^\top = \frac{c}{\sqrt{g}} \cdot \mathcal{T} \cdot g^{-1} \quad (51)$$

which is equivalent to the component equations

$$\mathcal{T}^\nu{}_\alpha g^{\alpha\mu} = \mathcal{T}^\mu{}_\alpha g^{\alpha\nu}. \quad (52)$$

These two equations can be re-expressed in several ways, which we now discuss.

7 Balance laws from the four-stress

The four-stress determines an association between any 1-vector field \mathbf{V} and an outer-oriented 3-covector field, which can then be interpreted as a current:⁴

$$\mathbf{V} \mapsto \mathcal{T} \cdot \mathbf{V} \equiv (\mathcal{T}^\mu{}_\nu V^\nu) d^3 V_\mu \quad (53)$$

⁴ Choquet-Bruhat & DeWitt-Morette 2000 § II.7.III p. 87; Hawking & Ellis 1994 § 3.2 p. 62; Gotay & Marsden 1992; see also the discussion in van Dantzig 1934 part 4 § 1.

Taking the differential of the current $\mathcal{T} \cdot \mathbf{V}$ we have the following *identity*:

$$d(\mathcal{T} \cdot \mathbf{V}) = (D\mathcal{T}) \cdot \mathbf{V} - \mathcal{T} \mathbin{\wedge} \nabla \mathbf{V} \quad (54)$$

The identity above, combined with the four-stress equations (49) and (52), leads to a non-trivial balance law for the current $\mathcal{T} \cdot \mathbf{V}$ associated with any \mathbf{V} :⁵

$$\begin{aligned} d(\mathcal{T} \cdot \mathbf{V}) &= -\mathcal{T} \mathbin{\wedge} \frac{1}{2}(\nabla \mathbf{V} + \mathbf{g} \cdot \nabla \mathbf{V}^T \cdot \mathbf{g}^{-1}) \\ &= \text{tr} \left[\mathcal{T}^T \cdot \frac{c}{\sqrt{g}} \cdot \frac{1}{2}(\nabla \mathbf{V} + \mathbf{g} \cdot \nabla \mathbf{V}^T \cdot \mathbf{g}^{-1}) \right] \frac{\sqrt{g}}{c} \\ &\equiv \text{tr} \left[\mathcal{T}^T \cdot \frac{c}{\sqrt{g}} \cdot \frac{1}{2}(\mathbf{L}_V \mathbf{g}) \cdot \mathbf{g}^{-1} \right] \frac{\sqrt{g}}{c} . \end{aligned} \quad (55)$$

This is a *conservation* law if \mathbf{V} is a Killing vector, that is, if it satisfies

$$\nabla \mathbf{V} + \mathbf{g} \cdot \nabla \mathbf{V}^T \cdot \mathbf{g}^{-1} \equiv (\mathbf{L}_V \mathbf{g}) \cdot \mathbf{g}^{-1} = 0 . \quad (56)$$

Some examples. For $\mathbf{V} = \partial_\alpha$ the formula above becomes, thanks to equations (112) and (113),

$$\begin{aligned} d(\mathcal{T} \cdot \partial_\alpha) &= -\mathcal{T} \mathbin{\wedge} \nabla \partial_\alpha \iff d(\mathcal{T}^\mu{}_\alpha d^3 V_\mu) = \mathcal{T}^\mu{}_\nu \Gamma_{\mu\alpha}^\nu d^4 \Omega \iff \\ &\partial_\mu \mathcal{T}^\mu{}_\alpha = \mathcal{T}^\mu{}_\nu \Gamma_{\mu\alpha}^\nu \end{aligned} \quad (57)$$

which is the common expression of $D\mathcal{T} = 0$ in books where \mathcal{T} is interpreted as a tensor density.

For $\mathbf{V} = \mathbf{g}^{-1} \cdot dx^\alpha = g^{\mu\alpha} \partial_\mu$ we have

$$\begin{aligned} d(\mathcal{T} \cdot \mathbf{g}^{-1} \cdot dx^\alpha) &= -\mathcal{T} \mathbin{\wedge} \nabla(\mathbf{g}^{-1} \cdot dx^\alpha) \iff \\ d(\mathcal{T} \cdot \mathbf{g}^{-1} \cdot dx^\alpha) &= -\mathcal{T} \mathbin{\wedge} (\mathbf{g}^{-1} \cdot \nabla dx^\alpha) \iff \\ d(\mathcal{T}^{\mu\alpha} d^3 V_\mu) &= -\mathcal{T}^{\mu\nu} \Gamma_{\mu\nu}^\alpha d^4 \Omega \iff \\ \partial_\mu \mathcal{T}^{\mu\alpha} &= -\mathcal{T}^{\mu\nu} \Gamma_{\mu\nu}^\alpha , \end{aligned} \quad (58)$$

another common expression.

Equation (56) gives rise to an infinity of currents and related balance laws. Such balance laws are not independent; only ten of them are. The four-stress equations (49) and (52) can be re-expressed as a set of balance

⁵ Hawking & Ellis 1994 § 3.2 p. 62; Kopeikin et al. 2011 § 3.9.5.2 esp. eq. (3.424) and § 3.9.6.1 esp. eq. (3.458).

laws associated with ten different vector fields⁶. These balance laws are usually divided into a set of four for four-momentum, and a set of six for angular momentum, including “boost momentum”.

In the literature we find several inequivalent choices for these ten vector fields, leading to definitions of four-momentum and angular momentum that are different, even if they usually agree in a Newtonian approximation.

For the four-momentum, that is, energy and three-momentum, sometimes the vector fields

$$\mathbf{g}^{-1} \cdot dt, \quad \mathbf{g}^{-1} \cdot dx^i \quad (59)$$

are used⁷. Authors who make this choice can be recognized from their writing “ $T^{\mu\nu}_{;\mu}$ ”, or similar, for the covariant divergence of the energy-momentum tensor, with the non-contracted index up.

Sometimes the vector fields

$$\partial_t, \quad \partial_{x^i} \quad (60)$$

are used instead⁸. This choice is recognized by expressions such as “ $T^{\mu}_{\nu;\mu}$ ”, with the non-contracted index down.

In typical coordinate systems (for instance, locally free-falling), the vector $\mathbf{g}^{-1} \cdot dx^i$ is the one that gives the traditional momentum in the Newtonian limit, which is zero in a coordinate system where a matter element is at rest. The vector ∂_{x^i} instead yields additional terms related to the velocity of the frame with respect to another reference frame.

On the other hand, it is the vector $-c \partial_t / |||\partial_t|||$ (whose squared-norm is negative) which corresponds to the traditional total, “internal plus kinetic” energy in the Newtonian limit, where the “kinetic” part only involves the velocity of the matter element in the particular coordinates used.

For the angular-boost-momentum, one choice of vector fields, corresponding to (60) is

$$\mathbf{g}^{-1} \cdot (t dx^i - x^i dt), \quad \mathbf{g}^{-1} \cdot (x^i dx^j - x^j dx^i). \quad (61)$$

Another choice⁹ is

$$g_{t\mu} x^\mu \partial_{x^i} - g_{i\mu} x^\mu \partial_t, \quad g_{i\mu} x^\mu \partial_{x^j} - g_{j\mu} x^\mu \partial_i. \quad (62)$$

⁶ Hawking & Ellis 1994 § 3.2; see also Gotay & Marsden 1992. ⁷ for instance in Misner et al. 2017 § 5.3, ch. 20; Poisson & Will 2014 § 6.1.3. ⁸ for instance in Hawking & Ellis 1994 § 3.2 after eq. (3.2). ⁹ Kopeikin et al. 2011 § 3.9.6.3 before eq. (3.475).

See the brief remark in Dixon 1979 § 2.1.3 p. 160 for this particular, antisymmetrized, choice. Some texts¹⁰ also make the choice “ $x^\mu \partial_{x^\nu} - x^\nu \partial_{x^\mu}$ ”, but note that this in general may be dimensionally inconsistent.

Typically it is the first choice of vector that leads to the traditional angular momentum, in a similar way as discussed above for momentum.

With these vector fields it is possible to replace the four-stress equations (49) and (52) with a set of four plus six. For instance:

$$d(\mathcal{T} \cdot \partial_{x^\mu}) = -\mathcal{T} \wedge \nabla \partial_{x^\nu} \quad (63)$$

$$d[\mathcal{T} \cdot (x^\mu \mathbf{g}^{-1} \cdot dx^\nu - x^\nu \mathbf{g}^{-1} \cdot dx^\mu)] = -x^\mu \mathcal{T} \wedge \mathbf{g}^{-1} \cdot \nabla dx^\nu + x^\nu \mathcal{T} \wedge \mathbf{g}^{-1} \cdot \nabla dx^\mu \quad (64)$$

If Killing vectors are used, the corresponding supply terms vanish.

8 Four-stress for matter

Consider a region where there is a non-vanishing matter current \mathbf{N} with associated four-velocity \mathbf{U} , and define

$$\bar{\mathbf{U}} = -\frac{1}{c} \mathbf{g} \cdot \mathbf{U} \quad (65)$$

which satisfies

$$\bar{\mathbf{U}} \cdot \mathbf{U} = 1, \quad \nabla \mathbf{U} \cdot \bar{\mathbf{U}} = 0. \quad (66)$$

The last equality can be proved from $\nabla \mathbf{g} = 0$ and

$$0 = -\nabla(c^2) = \nabla(\mathbf{U} \cdot \mathbf{g} \cdot \mathbf{U}) = 2(\nabla \mathbf{U}) \cdot \mathbf{g} \cdot \mathbf{U}. \quad (67)$$

We can associate with the matter a four-stress \mathcal{T} which can be decomposed as follows:

$$\mathcal{T} = -e \mathbf{N} \otimes \bar{\mathbf{U}} + \mathbf{N} \otimes \mathbf{P} - (\bar{\mathbf{U}} \wedge \Phi) \otimes \bar{\mathbf{U}} + \bar{\mathbf{U}} \wedge \mathbf{S} \quad (68)$$

with $\mathbf{P} \cdot \mathbf{U} = 0 \quad \Phi \cdot \mathbf{U} = 0 \quad \mathbf{U} \cdot \mathbf{S} = 0 \quad \mathbf{S} \cdot \mathbf{U} = 0$

where

- e is a scalar, the molar energy density.
- \mathbf{P} is a 1-covector, the molar momentum density.

¹⁰ Burke 1987 §§ III.21, IX.62 p. 385; Hawking & Ellis 1994 § 3.2 after eq. (3.2).

- Φ is a 2-covector, the areic energy-flux density.
- S is a 1-covector-valued 2-covector, the three-stress related to the Cauchy stress.

Using the four-velocity \mathbf{U} associated with the matter current we obtain what could be called the “internal-energy current”:¹¹

$$\mathcal{T} \cdot \mathbf{U} \equiv -e \mathbf{N} - \bar{\mathbf{U}} \Phi \quad (69)$$

which, from eqs (56), (66), (35), satisfies the balance law

$$d(\mathcal{T} \cdot \mathbf{U}) = -\mathcal{T} \mathbin{\wedge} \nabla \mathbf{U} \quad (70)$$

or

$$d(-e \mathbf{N} - \bar{\mathbf{U}} \Phi) = (e \mathbf{N} \otimes \bar{\mathbf{U}}) \cdot \nabla \mathbf{U} - (\mathbf{N} \otimes \mathbf{P}) \cdot \nabla \mathbf{U} + (\bar{\mathbf{U}} \Phi \otimes \bar{\mathbf{U}}) \cdot \nabla \mathbf{U} - \bar{\mathbf{U}} S \cdot \nabla \mathbf{U} \quad (71)$$

or simply

$$\mathbf{N} de + \bar{\mathbf{U}} d\Phi - \Phi d\bar{\mathbf{U}} = -\mathbf{N} \nabla \mathbf{U} \cdot \mathbf{P} - (\bar{\mathbf{U}} S) \cdot \nabla \mathbf{U} . \quad (72)$$

The last balance equation corresponds to eq. (28) in Eckart (1940), except for the fact that here we are keeping the momentum \mathbf{P} and heat flux Φ distinct.

When \mathbf{P}, S, Φ are zero, one speaks of the four-stress of “dust”. It is noteworthy that in this case the four-stress is essentially proportional to the matter 3-covector, multiplied by a vector collinear with the matter four-velocity:

$$\mathcal{T} = -\mathbf{N} \otimes (e \bar{\mathbf{U}}) . \quad (73)$$

If we take the four-velocity itself as reference, given that $\bar{\mathbf{U}} \mathbf{U} = -c^2$, the energy 3-covector we obtain is proportional to the matter 3-covector:

$$-c^2 e \mathbf{N} \quad (74)$$

For this reason e is called the “proper internal energy”. If we take other vector fields as reference, then this energy will pick up further multiplicative terms consisting in the projection of the matter four-velocity \mathbf{U} onto the reference vector field. In particular taking ∂_t or ∂_i as reference we’ll have the projections of the four-velocity onto the coordinate covectors as multiplicative factors.

¹¹ Choquet-Bruhat & DeWitt-Morette 2000 § II.7.III p. 87; see also Gotay & Marsden 1992.

9 Explicit formulae for a GCRS-like coordinate system

9.1 Metric

Consider a coordinate system $(x^\mu) \equiv (t, x, y, z)$ like the Geocentric Celestial Reference System or the Barycentric Celestial Reference System¹².

Neglecting the timelike off-diagonal terms, the metric \mathbf{g} has components

$$(g_{\mu\nu}) = \begin{bmatrix} -c^2 + 2W + O(\frac{1}{c^2}) & O(\frac{1}{c^3}) & O(\frac{1}{c^3}) & O(\frac{1}{c^3}) \\ O(\frac{1}{c^3}) & 1 + \frac{2}{c^2}W + O(\frac{1}{c^4}) & 0 & 0 \\ O(\frac{1}{c^3}) & 0 & 1 + \frac{2}{c^2}W + O(\frac{1}{c^4}) & 0 \\ O(\frac{1}{c^3}) & 0 & 0 & 1 + \frac{2}{c^2}W + O(\frac{1}{c^4}) \end{bmatrix} \quad (75)$$

and its inverse \mathbf{g}^{-1} has components

$$(g^{\mu\nu}) = \begin{bmatrix} -\frac{1}{c^2} - \frac{2}{c^4}W + O(\frac{1}{c^6}) & O(\frac{1}{c^3}) & O(\frac{1}{c^3}) & O(\frac{1}{c^3}) \\ O(\frac{1}{c^3}) & 1 - \frac{2}{c^2}W + O(\frac{1}{c^4}) & 0 & 0 \\ O(\frac{1}{c^3}) & 0 & 1 - \frac{2}{c^2}W + O(\frac{1}{c^4}) & 0 \\ O(\frac{1}{c^3}) & 0 & 0 & 1 - \frac{2}{c^2}W + O(\frac{1}{c^4}) \end{bmatrix} \quad (76)$$

where the gravitational potential W generally depends on all coordinates.

Near Earth we have

$$W = \frac{GM}{r} \quad (77)$$

where M is Earth's mass and $r = \sqrt{x^2 + y^2 + z^2}$ the distance from its centre. Close to the surface, choosing a coordinate system in which z is vertical and pointing upward, we have $W \approx g(z_0 - z)$.

The volume element $\frac{\sqrt{g}}{c}$ and its inverse are

$$\frac{\sqrt{g}}{c} = \left[1 + \frac{2}{c^2}W + O(\frac{1}{c^4}) \right] d^4\Omega, \quad \frac{c}{\sqrt{g}} = \left[1 - \frac{2}{c^2}W + O(\frac{1}{c^4}) \right] \partial^4. \quad (78)$$

¹² Soffel et al. 2003; Petit & Wolf 2005; Poisson & Will 2014 § 8.1.1; see also Moyer 2000 § 2.3.1.1.

9.2 Matter current

Suppose the matter current in the geocentric coordinate system is given by

$$\mathbf{N} = N \, d^3V_t + J^i \, d^3V_{xi} . \quad (27)_r$$

The net flow of matter through a moving area element, that is, through a timelike volume, as the one discussed in § 1.1, is

$$\mathbf{N} \cdot (\mathbf{A} \wedge \mathbf{V}) = \delta_t (-N a_i v^i + J^i a_i) . \quad (79)$$

This flow is zero if the area element is moving with velocity $v^i = U^i := J^i/N$; this is what defines a “velocity of matter”. When represented as a normalized timelike 4-vector, this is the four-velocity \mathbf{U} discussed in § 4.

Note that the equations above are exact in full general relativity as well as in the Newtonian approximation. This happens because we’re working with tensor densities.

The representation of matter current as a 1-vector is

$$\left[N - \frac{2}{c^2} N W + O\left(\frac{1}{c^4}\right) , \quad J^i - \frac{2}{c^2} J^i W + O\left(\frac{1}{c^4}\right) \right] . \quad (80)$$

The four-velocity of matter normalized to $-c^2$ is given by

$$\mathbf{U} = \left[1 + \frac{1}{c^2} \left(\frac{1}{2} U^2 + W \right) + O\left(\frac{1}{c^4}\right) , \quad U^i + \frac{1}{c^2} U^i \left(\frac{1}{2} U^2 + W \right) + O\left(\frac{1}{c^4}\right) \right] . \quad (81)$$

9.3 Four-stress for matter

Using the decompositions (45) and (69), and defining $U_i := U^i$, the energy-momentum tensor for matter is¹³

$$\begin{aligned} \mathcal{T}^t_t &= -N\rho c^2 - N \left(\epsilon + \frac{1}{2} \rho U^2 - \rho W \right) + O\left(\frac{1}{c^2}\right) \\ \mathcal{T}^i_t &= -N\rho U^i c^2 - \left[N \left(\epsilon + \frac{1}{2} \rho U^2 - \rho W \right) U^i + q^i + \pi^i_k U^k \right] + O\left(\frac{1}{c^2}\right) \\ \mathcal{T}^t_j &= N\rho U^j + \frac{1}{c^2} \left[N \left(\epsilon + \frac{1}{2} \rho U^2 + 3\rho W \right) U_j + p_j + U_k \pi^k_j \right] + O\left(\frac{1}{c^4}\right) \\ \mathcal{T}^i_j &= \pi^i_j + N\rho U^i U_j + \\ &\quad \frac{1}{c^2} \left[N \left(\epsilon + \frac{1}{2} \rho U^2 + 3\rho W \right) U^i U_j + U^i p_j + q^i U_j \right] + O\left(\frac{1}{c^4}\right) \end{aligned} \quad (82)$$

with $p_i = q^i$ and $\pi^j_i = \pi^j_i$ when the symmetry equation (52) is enforced.

¹³ cf. Poisson & Will 2014 § 8.1.2, where it is assumed that there’s no heat flux and that the 3-stress is isotropic.

9.4 Energy, momentum, balance laws

Define the energy current with respect to the vector $-\partial_t$, as discussed in § 7. Then the volumic energy content, the areic energy flux across the moving area element $\mathbf{A} \wedge \mathbf{V}$, and the volumic energy supply are

$$\begin{aligned} e &= N\rho c^2 + N(\epsilon + \frac{1}{2}\rho U^2 - \rho W) + O(\frac{1}{c^2}) \\ \phi &= a_k(U^k - v^k)N\rho c^2 + \\ &\quad a_k(U^k - v^k)N(\epsilon + \frac{1}{2}\rho U^2 - \rho W) + a_k q^k + a_k \pi^k_j U^j + O(\frac{1}{c^2}) \\ r &= -N\rho \partial_t W + O(\frac{1}{c^2}). \end{aligned} \quad (83)$$

Define the x^a -momentum current with respect to the vector ∂_{x^a} . Then the volumic x^a -momentum content, the areic flux across the moving area element $\mathbf{A} \wedge \mathbf{V}$, and the volumic supply are

$$\begin{aligned} p_a &= N\rho U^a + \frac{1}{c^2}[q^a + \pi^a_j U^j + N(\epsilon + \frac{1}{2}\rho U^2 + 3\rho W)U^a] + O(\frac{1}{c^4}) \\ f_a &= a_k(U^k - v^k)U_a N\rho + a_j \pi^j_a + O(\frac{1}{c^2}) \\ g_a &= N\rho \partial_a W + O(\frac{1}{c^2}). \end{aligned} \quad (84)$$

Define the $x^a x^b$ -angular momentum current as in § 7. Then the volumic $x^a x^b$ -angular momentum content, the areic flux across the moving area element $\mathbf{A} \wedge \mathbf{V}$, and the volumic supply are

$$\begin{aligned} l^{ab} &= N\rho(x^a U^b - x^b U^a) + O(\frac{1}{c^2}) \\ &\equiv x^a p_b - x^b p_a + O(\frac{1}{c^2}) \\ \tau^{ab} &= a_k(U^k - v^k)(x^a U^b - x^b U^a)N\rho + x^a a_k \pi^k_b - x^b a_k \pi^k_a + O(\frac{1}{c^2}) \\ &\equiv x^a f_b - x^b f_a + O(\frac{1}{c^2}) \\ m^{ab} &= N\rho(x^a \partial_b W - x^b \partial_a W) + O(\frac{1}{c^2}) \\ &\equiv x^a g_b - x^b g_a + O(\frac{1}{c^2}). \end{aligned} \quad (85)$$

Finally define the x^a -boost momentum current as in § 7. Then the volumic x^a -boost momentum content, the areic flux across the moving

area element $\mathbf{A} \wedge \mathbf{V}$, and the volumic supply are

$$\begin{aligned}
 \text{content} &= N\rho (t U^a - x^a) + O\left(\frac{1}{c^2}\right) \\
 &\equiv t p_a - x^a e/c^2 + O\left(\frac{1}{c^2}\right) \\
 \text{flux} &= a_k (U^k - v^k) (t U^a - x^a) N\rho + t a_k \pi^k_a + O\left(\frac{1}{c^2}\right) \\
 &\equiv t f_a - x^a \phi/c^2 + O\left(\frac{1}{c^2}\right) \\
 \text{supply} &= N\rho t \partial_a W + O\left(\frac{1}{c^2}\right) \\
 &\equiv t g_a + O\left(\frac{1}{c^2}\right).
 \end{aligned} \tag{86}$$

Combining eqs (85) and (86) we obtain

$$\pi^a_b = \pi^b_a. \tag{87}$$

Combining eqs (84), (85), (87) we obtain

$$p_a = \phi^a/c^2. \tag{88}$$

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- \mathbf{u}^\bullet is a 1-vector, represented by the column-matrix \mathbf{u} .
- Similarly for \mathbf{v}^\bullet .
- $\boldsymbol{\omega}_\bullet$ is a 1-covector, represented by the row-matrix $\boldsymbol{\omega}$.
- Similarly for $\boldsymbol{\eta}_\bullet$.
- \mathbf{F}_\bullet is a 2-covector, represented by the matrix \mathbf{F} .
- \mathbf{G}_\bullet is a twisted 2-covector, represented by the matrix \mathbf{G} .
- \mathbf{g}_\bullet is a covector-valued covector, represented by the matrix \mathbf{g} . The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\mathbf{g}^{-1\bullet\bullet}$ is a vector-valued vector, inverse of \mathbf{g} , that is: $\mathbf{g} \cdot \mathbf{g}^1 = \mathbf{id}_\bullet$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\mathcal{N}_{|\dots|}$ is an twisted 3-covector, represented by the column-matrix \mathcal{N} .
- $\mathcal{T}_{|\dots|}$ is a covector-valued 3-covector, represented by the matrix \mathcal{T} . The rows represent the 3-covector components; the columns, the covector components.
- $\frac{\sqrt{g}}{c}_{|\dots|}$ is a 4-covector, represented by the number \sqrt{g}/c .
- $\frac{c}{\sqrt{g}}^{|\dots|}$ is a 4-vector, represented by the number c/\sqrt{g} .
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from “old” coordinates x to “new” coordinates x' is represented by the matrix \mathbf{J} . The rows correspond to the new coordinates x' ; the columns, to the old x .
- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x is represented by the matrix \mathbf{J}^{-1} . The rows correspond to the old coordinates x ; the columns, to the new x' .

Then – note that the order on the right side is important:

- Contractions, wedge product, index raising and lowering

$$\text{(object)} \qquad \qquad \text{(matrix repr)} \qquad \qquad (89)$$

$$\boldsymbol{\omega} \cdot \boldsymbol{u} \qquad \qquad \boldsymbol{\omega} \boldsymbol{u} \equiv \boldsymbol{u}^\top \boldsymbol{\omega}^\top \quad \text{(number)} \qquad (90)$$

$$\boldsymbol{v} \cdot \boldsymbol{g} \cdot \boldsymbol{u} \qquad \qquad \boldsymbol{v}^\top \boldsymbol{g} \boldsymbol{u} \quad \text{(number)} \qquad (91)$$

$$\boldsymbol{g} \cdot \boldsymbol{u} \qquad \qquad \boldsymbol{u}^\top \boldsymbol{g}^\top \quad \text{(row-matrix)} \qquad (92)$$

$$\boldsymbol{\omega} \wedge \boldsymbol{\eta} \qquad \qquad \boldsymbol{\omega}^\top \boldsymbol{\eta} - \boldsymbol{\eta}^\top \boldsymbol{\omega} \quad \text{(matrix)} \qquad (93)$$

$$\boldsymbol{u} \cdot \boldsymbol{F} \qquad \qquad \boldsymbol{u}^\top \boldsymbol{F} \quad \text{(row-matrix)} \qquad (94)$$

$$\boldsymbol{F} \cdot \boldsymbol{v} \qquad \qquad (\boldsymbol{F} \boldsymbol{v})^\top \quad \text{(row-matrix)} \qquad (95)$$

$$\boldsymbol{\omega} \cdot \boldsymbol{g}^{-1} \qquad \qquad \boldsymbol{g}^{-\top} \boldsymbol{\omega}^\top \quad \text{(column-matrix)} \qquad (96)$$

$$\frac{c}{\sqrt{g}} \cdot \boldsymbol{\mathcal{N}} \qquad \qquad \boldsymbol{\mathcal{N}} c / \sqrt{g} \quad \text{(column-matrix)} \qquad (97)$$

$$\boldsymbol{u} \cdot \boldsymbol{\mathcal{N}} \qquad \qquad \boldsymbol{u} \boldsymbol{\mathcal{N}}^\top - \boldsymbol{\mathcal{N}} \boldsymbol{u}^\top \quad \text{(matrix)} \qquad (98)$$

$$\boldsymbol{\omega} \wedge \boldsymbol{G} \qquad \qquad (\boldsymbol{\omega} \boldsymbol{G})^\top \quad \text{(column-matrix)} \qquad (99)$$

$$\frac{c}{\sqrt{g}} \cdot \boldsymbol{\mathcal{T}} \qquad \qquad \boldsymbol{\mathcal{T}} c / \sqrt{g} \quad \text{(matrix)} \qquad (100)$$

$$\boldsymbol{\mathcal{T}} \cdot \boldsymbol{u} \qquad \qquad \boldsymbol{\mathcal{T}} \boldsymbol{u} \quad \text{(column-matrix)} \qquad (101)$$

$$\boldsymbol{g} \cdot \left(\frac{c}{\sqrt{g}} \cdot \boldsymbol{\mathcal{T}} \right) \cdot \boldsymbol{u} \qquad \qquad \boldsymbol{g} \boldsymbol{\mathcal{T}} \boldsymbol{u} c / \sqrt{g} \quad \text{(matrix)} \qquad (102)$$

- Transformations

$$\text{(old coords)} \quad \mapsto \quad \text{(new coords)} \qquad (103)$$

$$\boldsymbol{u} \quad \mapsto \quad \boldsymbol{J} \boldsymbol{u} \qquad (104)$$

$$\boldsymbol{\omega} \quad \mapsto \quad \boldsymbol{\omega} \boldsymbol{J}^{-1} \qquad (105)$$

$$\boldsymbol{g} \quad \mapsto \quad \boldsymbol{J}^{-\top} \boldsymbol{g} \boldsymbol{J}^{-1} \qquad (106)$$

$$\boldsymbol{\mathcal{N}} \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}} \boldsymbol{J} \boldsymbol{\mathcal{N}} \qquad (107)$$

$$\sqrt{g}/c \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}} \sqrt{g}/c \qquad (108)$$

$$c/dg \quad \mapsto \quad \det \boldsymbol{J} c / \sqrt{g} \qquad (109)$$

$$\boldsymbol{\mathcal{T}} \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}} \boldsymbol{J} \boldsymbol{\mathcal{T}} \boldsymbol{J}^{-1} \qquad (110)$$

B Checks about optimal representation of four-stress

A lemma about the exterior derivative of some outer-oriented 3-covectors:

$$\begin{aligned}
 d(f \, d^3 V_t) &= d\left(f \, d^3 \tilde{x} y z\right) = \partial_t f \, dt \wedge d^3 \tilde{x} y z + 0 = \partial_t f \, d^4 \tilde{x} y z \\
 d(f \, d^3 V_x) &= -d\left(f \, d^3 \tilde{t} y z\right) = -\partial_x f \, dx \wedge d^3 \tilde{t} y z + 0 = \partial_x f \, d^4 \tilde{t} x y z \\
 d(f \, d^3 V_\alpha) &= \partial_\alpha f \, d^4 \tilde{t} x y z
 \end{aligned} \tag{111}$$

Another lemma about the exterior product of 3- and 1-covectors:

$$\begin{aligned}
 d^3 V_t \wedge dt &= -d^4 \Omega & d^3 V_x \wedge dx &= -d^4 \Omega & \dots \\
 d^3 V_\alpha \wedge dx^\beta &= -\delta_\alpha^\beta d^4 \Omega
 \end{aligned} \tag{112}$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned}
 \nabla(dt) &= -\Gamma_{tt}^t dt \otimes dt - \Gamma_{tj}^t dt \otimes dx^j - \Gamma_{it}^t dx^i \otimes dt - \Gamma_{ij}^t dx^i \otimes dx^j \\
 \nabla(dx^k) &= -\Gamma_{tt}^k dt \otimes dt - \Gamma_{tj}^k dt \otimes dx^j - \Gamma_{it}^k dx^i \otimes dt - \Gamma_{ij}^k dx^i \otimes dx^j
 \end{aligned} \tag{113}$$

If ϕ is a 3-covector, ω a 1-covector, and D the exterior covariant derivative, then

$$D(\phi \otimes \omega) = (d\phi) \otimes \omega - \phi \wedge \nabla \omega \tag{114}$$

and in particular

$$D(\phi \otimes dx^\alpha) = (d\phi) \otimes dx^\alpha + \Gamma_{\mu\nu}^\alpha (\phi \wedge dx^\mu) \otimes dx^\nu. \tag{115}$$

Let's also consider the contraction with a 1-vector u :

$$\begin{aligned}
 D(\phi \otimes \omega \cdot u) &\equiv d[\phi(\omega \cdot u)] \\
 &= (d\phi)(\omega \cdot u) - \phi \wedge d(\omega \cdot u) \\
 &= d\phi \otimes \omega \cdot u - \phi \wedge \nabla \omega \cdot u - \phi \wedge \nabla u \cdot \omega \\
 &\equiv (d\phi \otimes \omega - \phi \wedge \nabla \omega) \cdot u - \phi \wedge \nabla u \cdot \omega \\
 &\equiv D(\phi \otimes \omega) \cdot u - (\phi \otimes \omega) \wedge \nabla u.
 \end{aligned} \tag{116}$$

and in particular

$$D(\phi \otimes dx^\alpha \cdot u) = d\phi u^\alpha - \phi \wedge dx^\beta \partial_\beta u^\alpha \tag{117}$$

A balance equation with the exterior covariant derivative then becomes

$$D\mathcal{T} = 0 \quad \implies \quad d(\mathcal{T} \cdot u) = -\mathcal{T} \wedge \nabla u \tag{118}$$

Then

$$\begin{aligned}
 0 &= D\mathcal{T} \\
 &= D(-e \, d^3V_t \otimes dt - q^i \, d^3V_i \otimes dt + p_j \, d^3V_t \otimes dx^j + \pi^i_j \, d^3V_i \otimes dx^j) \\
 &= -\partial_t e \, d^4\Omega \otimes dt - \partial_i q^i \, d^4\Omega \otimes dt + \partial_t p_j \, d^4\Omega \otimes dx^j + \partial_i \pi^i_j \, d^4\Omega \otimes dx^j - \\
 &\quad \left[e \, \Gamma_{tt}^t \, d^3V_t \wedge dt \otimes dt + e \, \Gamma_{tj}^t \, d^3V_t \wedge dt \otimes dx^j + 0 \right. \\
 &\quad q^i \, \Gamma_{it}^t \, d^3V_i \wedge dx^i \otimes dt + q^i \, \Gamma_{ij}^t \, d^3V_i \wedge dx^i \otimes dx^j + 0 \\
 &\quad - p_j \, \Gamma_{tt}^j \, d^3V_t \wedge dt \otimes dt - p_k \, \Gamma_{tj}^k \, d^3V_t \wedge dt \otimes dx^j + 0 \\
 &\quad \left. - \pi^i_k \, \Gamma_{it}^k \, d^3V_i \wedge dx^i \otimes dt - \pi^i_k \, \Gamma_{ij}^k \, d^3V_i \wedge dx^i \otimes dx^j \right] \\
 &= d^4\Omega \otimes [\\
 &\quad -\partial_t e \, dt + e \, \Gamma_{tt}^t \, dt + e \, \Gamma_{tj}^t \, dx^j \\
 &\quad -\partial_i q^i \, dt + q^i \, \Gamma_{it}^t \, dt + q^i \, \Gamma_{ij}^t \, dx^j \\
 &\quad + \partial_t p_j \, dx^j - p_j \, \Gamma_{tt}^j \, dt - p_k \, \Gamma_{tj}^k \, dx^j \\
 &\quad + \partial_i \pi^i_j \, dx^j - \pi^i_k \, \Gamma_{it}^k \, dt - \pi^i_k \, \Gamma_{ij}^k \, dx^j \\
 &\quad] \\
 &\tag{119}
 \end{aligned}$$

The equation above corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e \, \Gamma_{tt}^t + q^i \, \Gamma_{it}^t - p_j \, \Gamma_{tt}^j - \pi^i_k \, \Gamma_{it}^k \tag{120}$$

$$\partial_t p_j + \partial_i \pi^i_j = -e \, \Gamma_{tj}^t - q^i \, \Gamma_{ij}^t + p_k \, \Gamma_{tj}^k + \pi^i_k \, \Gamma_{ij}^k \tag{121}$$

For $\mathcal{T} \cdot \mathbf{u}$ we find first

$$\mathcal{T} \cdot \mathbf{u} = (-e \, u^t + p_j \, u^j) \, d^3V_t + (-q^i \, u^t + \pi^i_j \, u^j) \, d^3V_i \tag{122}$$

$$\begin{aligned}
 \mathcal{T} \wedge \nabla \mathbf{u} &= (-e \, \partial_t u^t + p_j \, \partial_t u^j) \, d^3V_t \wedge dt + (-q^i \, \partial_i u^t + \pi^i_j \, \partial_i u^j) \, d^3V_i \wedge dx^i \\
 &\quad + \Gamma \text{ terms} \\
 &\tag{123}
 \end{aligned}$$

and therefore

$$\begin{aligned}
 \partial_t(-e u^t + p_j u^j) + \partial_i(-q^i u^t + \pi^i_j u^j) = \\
 -e \partial_t u^t + p_j \partial_t u^j - q^i \partial_i u^t + \pi^i_j \partial_i u^j \\
 - (e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi^i_k \Gamma_{it}^k) u^t \\
 - (e \Gamma_{tj}^t + q^i \Gamma_{ij}^t - p_k \Gamma_{tj}^k - \pi^i_k \Gamma_{ij}^k) u^j \quad (124)
 \end{aligned}$$

Radial case

$$\partial_t e + \partial_r q^r = e \Gamma_{tt}^t + q^r \Gamma_{rt}^t + p_r \Gamma_{tt}^r + \pi^r_r \Gamma_{rt}^r \quad (125)$$

$$\partial_t p_r + \partial_r \pi^r_r = e \Gamma_{tr}^t + q^r \Gamma_{rr}^t + p_r \Gamma_{tr}^r + \pi^r_r \Gamma_{rr}^r \quad (126)$$

$$\partial_t e + \partial_r q^r = q^r \frac{g}{c^2} + p_r g \quad (127)$$

$$\partial_t p_r + \partial_r \pi^r_r = e \frac{g}{c^2} \quad (128)$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have¹⁴

$$\Gamma_{jt}^t = \Gamma_{ij}^t = \frac{GM}{c^2} \frac{x^j}{r^3} \approx \frac{g}{c^2} \quad \Gamma_{tt}^j = GM \frac{x^j}{r^3} \approx g \quad (129)$$

where g is the standard acceleration, considered positive. Take also $p_j \approx mv_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

The balances above become

$$\partial_t e + \partial_i q^i = q^z \frac{g}{c^2} + p_z g \quad (130)$$

$$\partial_t p_z + \partial_i \pi^i_z = e \frac{g}{c^2} \quad (131)$$

Also,

¹⁴ Poisson & Will 2014 §5.2.3.

$$\begin{aligned}\Gamma_{tt}^t &\approx -2\frac{g}{c^2}v(t) & \Gamma_{jt}^t &= \Gamma_{tj}^t \approx \frac{g}{c^2} \\ \Gamma_{tt}^j &\approx g - 2\frac{g}{c^2}v(t)^2 - \dot{v}(t) & \Gamma_{jt}^j &= \Gamma_{tj}^j \approx \frac{g}{c^2}v(t)\end{aligned}\quad (132)$$

$$\partial_t e + \partial_i q^i = -e 2\frac{g}{c^2}v(t) + q^z \frac{g}{c^2} + p_j \left(g - 2\frac{g}{c^2}v(t)^2 - \dot{v}(t) \right) + \pi^z_z \frac{g}{c^2}v(t) \quad (133)$$

$$\partial_t p_z + \partial_i \pi^i_z = e \frac{g}{c^2} + p_z \frac{g}{c^2}v(t) \quad (134)$$

C Works with useful content

- Eq. (21) in Maugin 1974: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 inourgoulhon 2012
- Tensor-density expression of divergence law: Castrillón López et al. 2008 also Kopeikin et al. 2011; Soffel & Han 2019.
- For transformation or raising: Gantmacher 2000 § I.4 eq. (33).

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B} \gamma \cdot (A \wedge B) \quad (135)$$

with $\deg(B) = n - \deg(A)$ Barnabei et al. 1985 prop. 4.1

- Compound matrices: Choquet-Bruhat et al. 1996 § IV.A.1 p. 199, Problem 1 p. 270

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(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

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