IOP Publishing European Journal of Physics
Eur. J. Phys. **42** (2021) 045601 (18pp) https://doi.org/10.1088/1361-6404/aba90b

Dimensional analysis in relativity and in differential geometry

P G L Porta Mana

Department of Computer Science, Electrical Engineering, and Mathematical Sciences, Western Norway University of Applied Sciences, Bergen, Norway

E-mail: pgl@portamana.org

Received 2 May 2020, revised 4 July 2020
Accepted for publication 24 July 2020
Published 20 April 2021

Abstract
This note provides a short guide to dimensional analysis in Lorentzian and general relativity and in differential geometry. It tries to revive Dorgelo and Schouten's notion of 'intrinsic' or 'absolute' dimension of a tensorial quantity. The intrinsic dimension is independent of the dimensions of the coordinates and expresses the physical and operational meaning of a tensor. The dimensional analysis of several important tensors and tensor operations is summarized. In particular it is shown that the components of a tensor need not have all the same dimension, and that the Riemann (once contravariant and thrice covariant) and Ricci (fully covariant) curvature tensors are dimensionless. The relation between dimension and operational meaning for the metric and stress–energy–momentum tensors is discussed; and the main conventions for the dimensions of these two tensors and of Einstein's constant are reviewed. A more thorough and updated analysis is available as a preprint.

Keywords: dimensional analysis, relativity, differential geometry, units, Einstein equations

Dedicated to little Emma

1. Introduction

From the point of view of dimensional analysis, do all components of a tensor need to have the same dimension? What happens to these components if we choose coordinates that do not have dimensions of length or time? And if the components of a tensor have different dimensions, then does it make sense to speak of 'the dimension of the tensor'? What are the dimensions of the metric and of the curvature tensors? What is the dimension of the constant in the Einstein equations?

A sense of insecurity often gets hold of many students (and possibly of some researchers) in relativity, when they have to discuss and answer questions like the above. This is evident in many question and answer websites and wiki pages, where some incorrect or unfounded statements about dimensional analysis in relativity are in circulation. For example, the notion that the components of a tensor or the coordinate functions should all have the same dimension. As we will see shortly, this notion is false.

Several factors contribute to these misconceptions and insecurity. Modern texts in Lorentzian and general relativity commonly use geometrized units. They say that, for finding the dimension of some constant in a tensorial equation, it is sufficient to compare the dimensions of the terms in the equation. But the application of this procedure is sometimes not so immediate, because some tensors do not have universally agreed dimensions—prime example the metric tensor. Older texts often use four coordinates with dimension of length, and base their dimensional analyses on that specific choice. They even multiply[1] coordinates or tensorial components having dimension of time by powers of c. A student thus gets the impression that coordinates ought to always be lengths, and that all components of a tensor ought to have the same dimension[2].

Dimensional analysis may thus not be as self-evident in relativity and in differential geometry as authors in these subjects take it to be. The present note wants to provide a short but exhaustive guide to it. Some important dimensional-analysis questions in general relativity are also consistently settled in this note; for example the dimension of the Riemann curvature tensor, or the effect of the covariant or Lie derivatives on dimensions.

The application of dimensional analysis in relativity seems to me most straightforward and self-evident if we rely on the coordinate-free or intrinsic approach to differential geometry, briefly recalled below, and if we adopt the perhaps overlooked notion of *intrinsic dimension* of a tensor.

The intrinsic dimension of a tensor was introduced under the name of 'absolute dimension' by Schouten and Dorgelo[3] and used in Truesdell and Toupin[4]. As its name implies, this dimension is independent of the choice and dimensions of coordinate functions. It is distinct from the dimensions of the tensor's *components*, which instead depend on the dimensions of the coordinates. The intrinsic dimension of a tensor is determined by the latter's physical and operational[5] meaning. It is therefore a natural notion for dimensional analysis in relativity.

A brief reminder of the intrinsic approach to differential geometry, with references, is given in the next section, together with some special notation necessary to our discussion. An introductory example of the basic way of reasoning about dimensional analysis in differential geometry is then presented in section 3. Sections 4–7 offer a more systematic discussion and a synopsis of dimensional analysis for the main tensorial operations. The notion of intrinsic dimension is explained in section 5. The intrinsic dimensions of the various curvature tensors, of the metric tensor, and of the stress–energy–momentum tensor are separately discussed in sections 8–10. The (contravariant and thrice covariant) Riemann and (fully covariant) Ricci tensors, in particular, are found to have intrinsic dimension $\mathsf{1}$, that is, to be dimensionless. The operational motivation of the two standard choices for the dimension of the metric tensor are also discussed. The dimension of the constant in the Einstein equations is finally derived in section 11. This note obviously assumes familiarity with basic tensor calculus and related notions, for example of co- and contra-variance, tensor product, contraction. Some passages assume familiarity with the exterior calculus of differential forms. The general ideas, however, should be understandable even without such familiarity.

[1] e.g. Tolman (1949) p 71 equation (37.1); Landau and Lifshitz (1996) p 80 equation (32.15); Adler *et al* (1975) p 332 equation (10.15).

[2] A recent work explicitly stating, if only in passing, that this needs not be the case is Kitano (2013) section X.

[3] Dorgelo and Schouten (1946); Schouten (1989) ch VI.

[4] Truesdell and Toupin (1960) appendix II.

[5] Bridgman (1958).

Finally, quoting Truesdell and Toupin[6], 'dimensional analysis remains a controversial and somewhat obscure subject. We do not attempt a complete presentation here'. References about recent developments in this subject are given in the final summary, section 12.

An updated version of the analysis, results, and bibliography of the present work is given in Porta Mana (2020).

2. Intrinsic view of differential-geometric objects: brief reminder and notation

From the intrinsic point of view, a tensor is defined by its geometric properties. For example, a vector field $\boldsymbol{v}$ is an object that operates on functions defined on the (spacetime) manifold, yielding new functions, with the properties $\boldsymbol{v}(af + bg) = a\boldsymbol{v}(f) + b\boldsymbol{v}(g)$ and $\boldsymbol{v}(fg) = \boldsymbol{v}(f)g + f\boldsymbol{v}(g)$ for all functions f, g and reals a, b. A covector field (1-form) $\boldsymbol{\omega}$ is an object that operates on vector fields, yielding functions ('duality'), with the property $\boldsymbol{\omega}(f\boldsymbol{u} + g\boldsymbol{v}) = f\boldsymbol{\omega}(\boldsymbol{u}) + g\boldsymbol{\omega}(\boldsymbol{v})$ for all vector fields $\boldsymbol{u}$, $\boldsymbol{v}$ and functions f, g. The sum of vector or covector fields, and their products by functions—let us call this 'linearity'—are defined in an obvious way. Tensors are constructed from these objects.

A system of coordinates (x^i) is just a set of linearly independent functions. This set gives rise to a set of vectors fields $\left(\frac{\partial}{\partial x^i}\right)$ and to a set of covector fields $(\mathrm{d}x^i)$ by the obvious requirements that $\frac{\partial}{\partial x^i}(x^j) = \delta_i{}^j$ and $\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j$. These two sets can be used as bases to express all other vectors and covectors as linear combinations. A vector field $\boldsymbol{v}$ can thus be written as

$$\boldsymbol{v} \equiv \sum_i v^i \frac{\partial}{\partial x^i} \equiv v^i \frac{\partial}{\partial x^i}, \tag{1}$$

where the *functions* $v^i := \boldsymbol{v}(x^i)$ are its components with respect to the basis $\left(\frac{\partial}{\partial x^i}\right)$. Analogously for a covector field.

For the presentation of the intrinsic view I recommend the excellent texts by Choquet-Bruhat *et al* (1996), Boothby (2003), Abraham *et al* (1988), Bossavit (1991), Burke (1987), (1980) (ch 2), and more on the general-relativity side Misner *et al* (1973) (ch 9), Gourgoulhon (2012) (ch 2), Penrose and Rindler (2003) (ch 4). If you find this you can claim a postcard from me.

For the notation in dimensional analysis I use International Organization for Standardization conventions:[*] dim($\boldsymbol{A}$) is the dimension of the quantity $\boldsymbol{A}$, and among the base quantities are mass M, length L, time T, temperature Θ, electric current I. Note that I do not discuss units—it does not matter here whether the unit for length is the metre or the centimetre, for example.

The number and ordering of a tensor's covariant and contravariant 'slots'[7] will often be important in our discussion. The traditional coordinate-free notation '$\boldsymbol{A}$' unfortunately omits this information. We thus need a coordinate-free notation that makes it explicit. Penrose and Rindler[8] propose an abstract-index notation where '$A_i{}^{jk}$', for example, denotes a tensor covariant in its first slot and contravariant in its second and third slots. Every index in this notation is 'a *label*' whose sole purpose is to keep track of the type of tensor under discussion[9]. So this notation does not stand for the set of *components* of the tensor. For the latter set, **bold** indices are used instead: '$A_{\boldsymbol{i}}{}^{\boldsymbol{jk}}$'. In our discussion, where the difference between a tensor and its set of components will be crucial, this abstract-index notation unfortunately lends itself to conceptual and typographic misunderstanding.

[6] Truesdell and Toupin (1960) appendix section 7 footnote 4.

[*] ISO 2009 section 5.

[7] Misner *et al* (1973) section 3.2.

[8] Penrose and Rindler (2003) section 2.2.

[9] Penrose and Rindler (2003) p 75.

I shall therefore use a notation such as $\boldsymbol{A}_{\bullet}{}^{\bullet\bullet}$ to indicate that $\boldsymbol{A}$ is covariant in its first slot and contravariant in its second and third slots. Its components would thus be $(A_i{}^{jk})$. For brevity I will call this a 'co-contra–contra-variant' tensor, with an obvious naming generalization for other tensor types.

The only weak points of this notation are the operations of transposition and contraction, which literal indices depict so well instead. Considering that transposition is a generalization of matrix transposition, and contraction a generalization of trace, I will use the following notation:

- $\boldsymbol{A}^{\intercal\alpha\beta}$ is the transposition (swapping) of the αth and βth slots. Its coordinate-free definition is

$$(\boldsymbol{A}^{\intercal\alpha\beta})(\ldots,\overset{\alpha\text{th slot}}{\boldsymbol{\zeta}},\ldots,\underset{\beta\text{th slot}}{\boldsymbol{\eta}},\ldots):=\boldsymbol{A}(\ldots,\overset{\alpha\text{th slot}}{\boldsymbol{\eta}},\ldots,\underset{\beta\text{th slot}}{\boldsymbol{\zeta}},\ldots) \tag{2}$$

  for all $\boldsymbol{\zeta},\boldsymbol{\eta}$ of appropriate variance type.
- $\mathrm{tr}_{\alpha\beta}\boldsymbol{A}$ is the contraction of the αth and βth slots (which must have opposite variant types). Its coordinate-free definition is

$$(\mathrm{tr}_{\alpha\beta}\boldsymbol{A})(\ldots,\ldots,\ldots):=\sum_i \boldsymbol{A}(\ldots,\overset{\alpha\text{th slot}}{\boldsymbol{u}_i},\ldots,\underset{\beta\text{th slot}}{\boldsymbol{\omega}^i},\ldots) \tag{3}$$

  for any arbitrary complete and linearly independent sets $\{\boldsymbol{u}_i\}$, $\{\boldsymbol{\omega}^j\}$ such that $\boldsymbol{\omega}^j(\boldsymbol{u}_i)=\delta^j{}_i$. An analogous definition holds if the αth slot is covariant and the βth contravariant.

In index notation these operations are the familiar

$$A_{\dots}{}^{\overset{\alpha\text{th slot}}{i}}{}_{\dots\underset{\beta\text{th slot}}{j}\dots}\mapsto A_{\dots\underset{\alpha\text{th slot}}{j}\dots}{}^{\overset{\beta\text{th slot}}{i}}{}_{\dots}\qquad\text{and}\qquad A_{\dots}{}^{\overset{\alpha\text{th slot}}{i}}{}_{\dots\underset{\beta\text{th slot}}{i}\dots}\,.$$

Contraction and transposition will be discussed only sparsely, so I hope you will not find the notation above too uncomfortable.

It is possible to build the tensor-product architecture not on vectors and covectors, but on multi-vectors and multi-covectors, with their straight and twisted (or 'even' and 'odd') orientations. This possibility is examined in Porta Mana (2020).

3. An introductory two-dimensional example

Let me first present a simple example of dimensional analysis in a two-dimensional spacetime. I provide very little explanation, letting the analysis speak for itself. The next sections will give a longer discussion of the general point of view, of the assumptions, and of cases with more elaborate geometric objects.

In a region of a two-dimensional spacetime we use coordinates (x,y). These coordinates allow us to uniquely label every event in the region (otherwise they would not be coordinates). Let us say that coordinate x has dimension of temperature, and y of specific entropy:

$$\dim(x)=\Theta\,,\quad \dim(y)=\mathsf{s}:=\mathsf{L}^2\mathsf{T}^{-2}\Theta^{-1}. \tag{4}$$

This choice may be possible for several reasons. For example, the region could be occupied by a heat-conducting material; in a specific spacetime foliation, its temperature increases along each one-dimensional spacelike slice, and its entropy density is uniform on each slice but increases from slice to slice.[10] Owing to this kind of monotonic behaviour for these quantities, if we are given a pair of temperature and specific-entropy values we can identify a unique event associated to them in this spacetime region. They can thus be used as a coordinate system. The point here is that coordinates can have any dimensions owing to physical reasons. In atmospheric and ocean dynamics, for example, pressure or mass density are sometimes used as coordinates for depth[11].

From these coordinates we construct two covector fields $(\mathrm{d}x, \mathrm{d}y)$, and two vector fields $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ that serve as bases for the spaces of tangent covectors, vectors, and tensors. Their dimensions are

$$\begin{aligned} \dim(\mathrm{d}x) &= \Theta & \dim(\mathrm{d}y) &= \mathsf{s}, \\ \dim\left(\frac{\partial}{\partial x}\right) &= \Theta^{-1} & \dim\left(\frac{\partial}{\partial y}\right) &= \mathsf{s}^{-1}. \end{aligned} \tag{5}$$

Consider a contra-co-variant tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$ in this region. Using the basis fields above it can be written as

$$\boldsymbol{A} = A^x{}_x \frac{\partial}{\partial x} \otimes \mathrm{d}x + A^x{}_y \frac{\partial}{\partial x} \otimes \mathrm{d}y + A^y{}_x \frac{\partial}{\partial y} \otimes \mathrm{d}x + A^y{}_y \frac{\partial}{\partial y} \otimes \mathrm{d}y, \tag{6}$$

where $A^x{}_x := \boldsymbol{A}\left(\mathrm{d}x, \frac{\partial}{\partial x}\right)$ and so on are the components of the tensor in the coordinate system (x, y).

By the rules of dimensional analysis, the two sides of the expansion above must have the same dimension. The same holds for the four summands on the right side. Denoting $\mathsf{A} := \dim(\boldsymbol{A})$, we thus have the four equations

$$\begin{aligned} \mathsf{A} &= \dim(A^x{}_x) & \mathsf{A} &= \dim(A^x{}_y)\,\Theta^{-1}\,\mathsf{s} \\ \mathsf{A} &= \dim(A^y{}_x)\,\Theta\,\mathsf{s}^{-1} & \mathsf{A} &= \dim(A^y{}_y), \end{aligned}$$

or

$$\begin{aligned} \dim(A^x{}_x) &= \mathsf{A} & \dim(A^x{}_y) &= \mathsf{A}\,\Theta\,\mathsf{s}^{-1} \equiv \mathsf{A}\,\mathsf{L}^{-2}\mathsf{T}^2\Theta^2 \\ \dim(A^y{}_x) &= \mathsf{A}\,\Theta^{-1}\,\mathsf{s} & \dim(A^y{}_y) &= \mathsf{A}. \end{aligned} \tag{7}$$

The intrinsic dimension of the tensor $\boldsymbol{A}$ is A. The expansion (6) shows that this dimension is independent of the coordinate system, by construction—such expansion could be done in any other coordinate system, and the left side would be the same. The effect of coordinate transformations is examined more in detail in section 5. The intrinsic dimension A is determined by the physical and operational meaning of the tensor; see sections 9 and 10 for concrete examples. Together with the dimensions of the coordinates it determines the dimensions of the components, equation (7), which need not be all equal.

This simple example should have disclosed the main points of dimensional analysis on manifolds, which will now be discussed in more generality. In the derivation above we silently adopted a couple of natural conventions; for example, that the tensor product behaves similarly to multiplication with regard to dimensions. Such conventions are briefly discussed in section 12.

[10] For general-relativistic thermomechanics see e.g. Eckart (1940); Maugin (1974); (1978a), (b), (c), (d); Muschik and von Borzeszkowski (2014).

[11] Griffies (2004) ch 6; Vallis (2006) section 2.6.2.

4. Coordinates

From a physical point of view, a coordinate is just a function that associates values of some physical quantity with the events in a region (the domain of the coordinate chart) of spacetime. Together with the other coordinates, such function allows us to uniquely identify every event within that region. Any physical quantity will do: the distance from something, the time elapsed since something, an angle, an energy density, the strength of a magnetic flux, a temperature, and so on. A coordinate can thus have any dimension: length L, time T, angle $\mathsf{1}$, temperature Θ, magnetic flux $\Phi := \mathsf{ML}^2\mathsf{T}^{-2}\mathsf{I}^{-1}$, and so on.

The functional relation between two sets of coordinates must of course be dimensionally consistent. For example, if $\dim(x^0) = \mathsf{T}$ and $\dim(x^1) = \mathsf{L}$, and we introduce a coordinate $\xi(x^0, x^1)$ with dimension $\mathsf{1}$, additive in the previous two, then we must have $\xi = ax^0 + bx^1$ with $\dim(a) = \mathsf{T}^{-1}$ and $\dim(b) = \mathsf{L}^{-1}$.

5. Tensors: intrinsic dimension and components' dimensions

Consider a system of coordinates (x^i) with dimensions (X_i), and the ensuing sets of covector fields (1-forms) $\mathrm{d}x^i$ and of vector fields $\left(\frac{\partial}{\partial x^i}\right)$, bases for the cotangent and tangent spaces. Their tensor products are bases for the tangent spaces of higher tensor types.

The differential $\mathrm{d}x^i$ traditionally has the same dimension as x^i: $\dim(\mathrm{d}x^i) = \mathsf{X}_i$, and the vector $\frac{\partial}{\partial x^i}$ traditionally has the inverse dimension: $\dim \frac{\partial}{\partial x^i} = {\mathsf{X}_i}^{-1}$.

For our discussion let us take a concrete example: a contra-co-variant tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$. The discussion generalizes to tensors of other types in an obvious way.

The tensor $\boldsymbol{A}$ can be expanded in terms of the basis vectors and covectors, as in section 2 and in the example of section 3:

$$\boldsymbol{A} = A^i{}_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \equiv A^0{}_0 \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^0 + A^0{}_1 \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^1 + \cdots. \tag{8}$$

Each function

$$A^i{}_j := \boldsymbol{A}\left(\mathrm{d}x^i, \frac{\partial}{\partial x^j}\right) \tag{9}$$

is a component of the tensor in this coordinate system.

To make dimensional sense, all terms in the sum (8) must have the same dimension. This is possible only if the generic component $A^i{}_j$ has dimension

$$\dim\left(A^i{}_j\right) = \mathsf{A}\, \mathsf{X}_i \, {\mathsf{X}_j}^{-1}, \tag{10}$$

where A is common to all components. In fact, the $\mathsf{X}_i \, {\mathsf{X}_j}^{-1}$ term cancels the ${\mathsf{X}_i}^{-1} \mathsf{X}_j$ term coming from $\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ in the sum (8), and each summand therefore has dimension A.

The generalization of the formula above to tensors of other types is obvious:

$$\dim\left(A^{ij\ldots}{}_{kl\ldots}\right) = \mathsf{A}\, \mathsf{X}_i \, \mathsf{X}_j \cdots {\mathsf{X}_k}^{-1} {\mathsf{X}_l}^{-1} \cdots \tag{11}$$

where the ordering of the indices doesn't matter.

Clearly the components can have different dimensions. But this does not matter. What matters is that the sum (8) be dimensionally consistent. (Fokker[12], for example, uses a metric tensor with components having different dimensions.)

The dimension A, which is also the dimension of the sum (8), I will call the *intrinsic dimension* of the tensor $\boldsymbol{A}$, and we write

$$\dim(\boldsymbol{A}) = \mathsf{A}. \tag{12}$$

This dimension is independent of any coordinate system. It reflects the physical or operational[13] meaning of the tensor. We shall see an example of such an operational analysis in sections 9 and 10 for the metric and stress–energy–momentum tensors.

The notion of intrinsic dimension was introduced by Dorgelo and Schouten[14] under the name 'absolute dimension'. I find the adjective 'intrinsic' more congruous to modern terminology (and less prone to suggest spurious connections with absolute values).

Different coordinate systems lead to different dimensions of the *components* of a tensor $\boldsymbol{A}$, but the absolute dimension of the tensor remains the same. Formula (11) for the dimensions of the components is consistent under changes of coordinates. For example, in new coordinates $(\bar{x}^k)$ with dimensions $(\bar{\mathsf{X}}_k)$, the new components of $\boldsymbol{A}$ are

$$\bar{A}^k{}_l = A^i{}_j \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^l} \tag{13}$$

and a quick check shows that $\dim(\bar{A}^k{}_l) = \mathsf{A}\, \bar{\mathsf{X}}_k\, \bar{\mathsf{X}}_l{}^{-1}$, consistently with the general formula (11).

In the following I will drop the adjective 'intrinsic' when it is clear from the context.

6. Tensor operations

By the reasoning of the previous section, which simply applies standard dimensional considerations to the basis expansion (8), it is easy to find the resulting intrinsic dimension of various operations and operators on tensors and tensor fields.

Here is a summary of the dimensional rules for the main differential-geometric operations and operators, except for the covariant derivative, the metric, and related tensors, discussed more in depth in sections 8 and 9 below. Some of these rules are actually definition or conventions, as briefly discussed in their description. The others can be proved; I only give a proof for one of them, leaving the other proofs as an exercise. For reference, in brackets I give the section of Choquet-Bruhat *et al* (1996) where these operations are defined.

- The *tensor product* [III.B.5] multiplies dimensions:

$$\dim(\boldsymbol{A} \otimes \boldsymbol{B}) = \dim(\boldsymbol{A})\ \dim(\boldsymbol{B}). \tag{14}$$

  This is actually a definition or convention. We tacitly used this rule already in the example of section 3 and in section 5 for the coordinate expansion (8). It is a natural definition, because for tensors of order 0 (functions) the tensor product is just the ordinary product, and the dimension of a product is the product of the dimensions. This definition does not lead to inconsistencies.

[12] Fokker (1965) section VII.1 p 88.
[13] Bridgman (1958); see also Synge (1960a) section A.2; Truesdell and Toupin (1960) sections A.3–4.
[14] Dorgelo and Schouten (1946); Schouten (1989) ch VI.

- The *contraction* [III.B.5] or trace of the αth and βth slots of a tensor has the same dimension as the tensor:

$$\dim(\mathrm{tr}_{\alpha\beta}\,\boldsymbol{A}) = \dim(\boldsymbol{A}). \tag{15}$$

  Note that the formula above only holds *without raising or lowering indices*; see section 9 for those operations.

  This operation can be traced back to the duality of vectors and covectors mentioned in section 2: a covector field $\boldsymbol{\omega}$ operates linearly on a vector field $\boldsymbol{v}$ to yield a function $f = \boldsymbol{\omega}(\boldsymbol{v})$. Also in this case we have that $\dim(f) = \dim(\boldsymbol{\omega})\ \dim(\boldsymbol{v})$ by definition or convention, and the rule (15) follows from this convention. Also in this case this convention seems very natural, owing to the linearity properties of the trace, and does not lead to inconsistencies.
- The *transposition*[15] of the αth and βth slots of a tensor has the same dimension as the tensor:

$$\dim(\boldsymbol{A}^{\intercal_{\alpha\beta}}) = \dim(\boldsymbol{A}). \tag{16}$$

- The *Lie bracket* [III.B.3] of two vectors has the product of their dimensions:

$$\dim([\boldsymbol{u}, \boldsymbol{v}]) = \dim(\boldsymbol{u})\ \ \dim(\boldsymbol{v}). \tag{17}$$

  In fact, in coordinates (x^i) the bracket can be expressed as

$$[\boldsymbol{u}, \boldsymbol{v}] = \left(u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j}\right)\ \frac{\partial}{\partial x^i}, \tag{18}$$

  and equating the dimensions of the left and right sides, considering that

$$\dim(u^i) = \dim(\boldsymbol{u})\,\mathsf{X}_i, \quad \dim(v^i) = \dim(\boldsymbol{v})\,\mathsf{X}_i, \tag{19}$$

  we find again that all X terms cancel out, leaving the result (17).
- The *pull-back* [III.A.2], *tangent map* [III.B.1] and *push-forward* of a map F between manifolds do not change the dimensions of the tensors they map. The reason, evident from their definitions, is that they all rest on the pull-back of any function $F^*(f) := F \circ f$, which being a composition does not alter the dimension of the function.
- The *Lie derivative* [III.C.2] of a tensor with respect to a vector field has the product of the dimensions of the tensor and of the vector:

$$\dim(\mathrm{L}_{\boldsymbol{v}}\boldsymbol{A}) = \dim(\boldsymbol{v})\ \ \dim(\boldsymbol{A}). \tag{20}$$

Regarding operations and operators on differential forms:

- The *exterior product* [IV.A.1] of two differential forms multiplies their dimensions:

$$\dim(\boldsymbol{\omega} \wedge \boldsymbol{\tau}) = \dim(\boldsymbol{\omega})\ \ \dim(\boldsymbol{\tau}). \tag{21}$$

- The *interior product* [IV.A.4] of a vector and a form multiplies their dimensions:

$$\dim(\mathrm{i}_{\boldsymbol{v}}\boldsymbol{\omega}) = \dim(\boldsymbol{v})\ \ \dim(\boldsymbol{\omega}). \tag{22}$$

[15] called ‘building an isomer’ by Schouten (1954) section I.3 p 13; 1989 section II.4 p 20.

- The *exterior derivative* [IV.A.2] of a form has the same dimension of the form:

$$\dim(\mathrm{d}\boldsymbol{\omega}) = \dim(\boldsymbol{\omega}). \tag{23}$$

This can be proven using the identity $\mathrm{d}\,\mathrm{i}_{\boldsymbol{v}} + \mathrm{i}_{\boldsymbol{v}}\,\mathrm{d} = \mathrm{L}_{\boldsymbol{v}}$ or similar identities[16] together with equations (20) and (22).

- The *integral* [IV.B.1] of a form over a submanifold (or more generally a chain) M has the same dimension as the form:

$$\dim\left(\int_M \boldsymbol{\omega}\right) = \dim(\boldsymbol{\omega}). \tag{24}$$

The reason is that the integral of a form over a submanifold or chain ultimately rests on the standard definition of integration on the real line[17], which satisfies the dimensional rule above. In fact, the integral is invariant with respect to reparameterizations of the chain; it depends only on its image (some texts[18] even define chains as equivalence classes determined by their image).

All rules above extend in obvious ways to inner-oriented forms[19] (also called 'odd'[20] or 'twisted'[21] forms) and to tensor densities.

7. Curves and integral curves

Consider a curve into spacetime, $C : s \mapsto P(s)$, with the parameter s having dimension $\dim(s) = \mathsf{S}$.

If we consider the events of the spacetime manifold as dimensionless quantities, then the dimension of the tangent or velocity vector $\dot{C}$ to the curve is

$$\dim(\dot{C}) = \mathsf{S}^{-1}, \tag{25}$$

owing to the definition[22]

$$\dot{C} := \frac{\partial(x^i \circ C)}{\partial s}\,\frac{\partial}{\partial x^i}. \tag{26}$$

This has a quirky interesting consequence. Given a vector field $\boldsymbol{v}$ we say that C is an integral curve for it if

$$\boldsymbol{v} = \dot{C} \tag{27}$$

at all events $C(s)$ in the image of the curve (or more precisely $\boldsymbol{v}_{C(s)} = \dot{C}_{C(s)}$ in usual differential-geometric notation[23]). From the point of view of dimensional analysis this definition can only be valid if $\boldsymbol{v}$ has dimension S^{-1}. If $\boldsymbol{v}$ and s^{-1} have different dimensions—a case which could happen for physical reasons—the condition (26) must be modified into $\boldsymbol{v} = k\dot{C}$, where k is a

[16] Curtis and Miller 1985 ch 9 p 180 Theorem 9.78; Abraham *et al* (1988) section 6.4 theorem 6.4.8.
[17] e.g. Choquet-Bruhat *et al* (1996) sections 4.B.1–2; de Rham (1984) section 5 p 21, section 6 p 24; Abraham *et al* (1988) section 7.1; Boothby (2003) section VI.2.
[18] e.g. Martin (2004) section 10.4 p 297; Fecko (2006) section 7.3.
[19] Schouten (1989) ch II.
[20] De Rham (1984) ch II.
[21] Burke (1983); (1995); Bossavit (1991) ch 3.
[22] Choquet-Bruhat *et al* (1996) section III.B.1; Boothby 2003 section IV(1.9).
[23] Choquet-Bruhat *et al* (1996) section III.B.1.

possibly dimensionful constant. This is equivalent to considering an affine and dimensional reparameterization of C.

8. Connection, covariant derivative, curvature tensors

Consider an arbitrary connection[24] with covariant derivative ∇. For the moment we do not assume the presence of any metric structure.

The covariant derivative of the product $f\boldsymbol{v}$ of a function and a vector satisfies[25]

$$\nabla(f\,\boldsymbol{v}) = \mathrm{d}f \otimes \boldsymbol{v} + f\,\nabla\boldsymbol{v}. \tag{28}$$

The first summand, from formulae (23) and (14), has dimension $\dim(f)\ \dim(\boldsymbol{v})$; for dimensional consistency this must also be the dimension of the second summand. Thus

$$\dim(\nabla\boldsymbol{v}) = \dim(\boldsymbol{v}). \tag{29}$$

It follows that the *directional* covariant derivative $\nabla_{\boldsymbol{u}}$ has dimension

$$\dim(\nabla_{\boldsymbol{u}}\boldsymbol{v}) = \dim(\boldsymbol{u})\ \dim(\boldsymbol{v}), \tag{30}$$

and by its derivation properties[26] we see that formula (29) extends from vectors to tensors of arbitrary type.

In the coordinate system (x^i), the action of the covariant derivative is carried by the *connection coefficients* or Christoffel symbols $(\varGamma^i{}_{jk})$ defined by

$$\nabla\frac{\partial}{\partial x^k} = \varGamma^i{}_{jk}\,\mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}. \tag{31}$$

From this equation and the previous ones it follows that these coefficients have dimensions

$$\dim(\varGamma^i{}_{jk}) = \mathsf{X}_i\,\mathsf{X}_j{}^{-1}\,\mathsf{X}_k{}^{-1}. \tag{32}$$

The *torsion* $\boldsymbol{\tau}^{\bullet}{}_{\bullet\bullet}$, *Riemann curvature* $\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}$, and *Ricci curvature* $\boldsymbol{Ric}_{\bullet\bullet}$ tensors are defined by[27]

$$\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}) := \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}], \tag{33}$$

$$\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{w}) := \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}, \tag{34}$$

$$\boldsymbol{Ric}_{\bullet\bullet} := \mathrm{tr}_{13}\ \boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}. \tag{35}$$

From these definitions and the results of section 6 we find the dimensional requirements

$$\dim(\boldsymbol{\tau}^{\bullet}{}_{\bullet\bullet})\dim(\boldsymbol{u})\ \dim(\boldsymbol{v}) = \dim(\boldsymbol{u})\ \dim(\boldsymbol{v}), \tag{36}$$

$$\dim(\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet})\dim(\boldsymbol{u})\ \dim(\boldsymbol{v})\ \dim(\boldsymbol{w}) = \dim(\boldsymbol{u})\ \dim(\boldsymbol{v})\ \dim(\boldsymbol{w}), \tag{37}$$

$$\dim(\boldsymbol{Ric}_{\bullet\bullet}) = \dim(\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}), \tag{38}$$

[24] Choquet-Bruhat *et al* (1996) section V.B.
[25] Choquet-Bruhat *et al* (1996) section V.B.1.
[26] Choquet-Bruhat *et al* (1996) section V.B.1 p 303.
[27] Choquet-Bruhat *et al* (1996) section V.B.1.

which imply that *torsion, Riemann curvature, and Ricci curvature tensors are dimensionless*:

$$\dim(\boldsymbol{\tau}^{\bullet}{}_{\bullet\bullet}) = \dim(\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}) = \dim(\boldsymbol{Ric}_{\bullet\bullet}) = \mathsf{1}. \tag{39}$$

The exact contra- and co-variant type of these tensors is very important in the equations above. If a metric tensor is also introduced and used to raise or lower any indices of these tensors, the resulting tensors will have different dimensions; see next section, especially equation (50).

Misner *et al*[28] say that 'curvature', by which they seem to mean the Riemann tensor, has dimension L^{-2}. This statement is seemingly at variance with the dimensionless results (39). But I believe that Misner *et al* refer to the *components* of the Riemann tensor, in specific coordinates of dimension L, and using geometrized units. In such specific coordinates every *component* $R^{i}{}_{jkl}$ does indeed have dimension L^{-2}, according to the general formula (11), if and only if the intrinsic dimension of $\boldsymbol{R}$ is unity, $\dim(\boldsymbol{R}) = \mathsf{1}$. So I believe that Misner *et al*'s statement actually agrees with the results (39). This possible misunderstanding shows the importance of distinguishing between the intrinsic dimension, which does not depend on any specific coordinate choice, and component dimensions, which do.

The formulae above are also valid if a metric is defined and the connection is compatible with it. The connection coefficients in this case are defined in terms of the metric tensor, but using the results of section 9 it is easy to see that equations (29), (30), (32) and (39) still hold.

9. Metric and related tensors and operations

Let us now consider a metric tensor $\boldsymbol{g}_{\bullet\bullet}$. What is its intrinsic dimension $\dim(\boldsymbol{g})$? There seem to be two choices in the literature; both can be derived from the operational meaning of the metric.

Consider a (timelike) worldline $s \mapsto C(s)$, $s \in [a, b]$, between events $C(a)$ and $C(b)$. The metric tells us the *proper time* Δt elapsed for an observer having that worldline, according to the formula

$$\triangle t = \int_a^b \sqrt{|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]|}\; \mathrm{d}s. \tag{40}$$

From the results of section 6 this formula implies that $\mathsf{T} \equiv \dim(\triangle t) = \sqrt{\dim(\boldsymbol{g}_{\bullet\bullet})}$, (independently of the dimension of s) and therefore

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{T}^2. \tag{41}$$

Many authors[29], however, prefer to include a dimensional factor $1/c$ in front of the integral (40):

$$\triangle t = \frac{1}{c}\int_a^b \sqrt{|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]|}\; \mathrm{d}s, \tag{42}$$

thus obtaining

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{L}^2. \tag{43}$$

[28] Misner *et al* (1973) p 35.

[29] e.g. Fock (1964) section V.62 equation (62.02); Curtis and Miller (1985) ch 11 equation (11.21); Rindler (1986) section 5.3 equation (5.6); Hartle (2003) ch 6 equation (6.24).

The choice (43) seems also supported by the traditional expression for the 'line element ds^2' as it appears in many works:

$$ds^2 = c^2\ dt^2 - dx^2 - dy^2 - dz^2, \tag{44}$$

possibly with opposite signature (for an exception with dimension T^2 see Kilmister[30]). If the coordinates (t, x, y, z) have the dimensions suggested by their symbols, this formula has dimension L^2, so that if we interpret 'ds^2' as $\boldsymbol{g}$ we find $\dim(\boldsymbol{g}) = \mathsf{L}^2$. The line-element expression above often has an ambiguous differential-geometric meaning, however, because it may also represent the metric applied to some *unspecified* vector, that is, $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v})$, where $\boldsymbol{v}$ is left unspecified[31]. In this case we have

$$\mathsf{L}^2 = \dim(\boldsymbol{g})\ \dim(\boldsymbol{v})^2 \tag{45}$$

and the dimension of $\boldsymbol{g}$ is ambiguous or undefined, because the vector $\boldsymbol{v}$ could have any dimension.

The standard choices for dim $\boldsymbol{g}$ are thus T^2 or L^2. My favourite choice is the first, for reasons discussed by Synge and Bressan[32]. Synge gives a vivid summary:[33]

> We are now launched on the task of giving physical meaning to the Riemannian geometry [...]. It is indeed a Riemannian *chronometry* rather than *geometry*, and the word *geometry*, with its dangerous suggestion that we should go about measuring *lengths* with *yardsticks*, might well be abandoned altogether in the present connection

In fact, to measure the proper time Δt defined above we only need to ensure that a clock has the worldline C, and then take the difference between the clock's final and initial times. On the other hand, consider the case when the curve C is *spacelike*. Its proper length is still defined by the integral (40) apart from a dimensional constant. Its measurement, however, is more involved than in the timelike case. It requires dividing the curve into very short pieces, and having specially-chosen observers (orthogonal to the pieces) measure each piece. But the measurement of each piece actually relies on the measurement of *proper time*: each observer uses radar distance[34], sending a lightlike signal which bounces back at the end of the piece, and measuring the time it takes to come back. Even if rigid rods are used, their calibration still relies on a measurement of time—this is also reflected in the current definition of the standard metre[35].

The metric $\boldsymbol{g}$ can be considered as an operator mapping vectors to covectors, which we can compactly write as $\boldsymbol{\omega} = \boldsymbol{g}\boldsymbol{v}$ (instead of the cumbersome $\boldsymbol{\omega} = \mathrm{tr}_{23}(\boldsymbol{g} \otimes \boldsymbol{v})$). The *inverse metric tensor* $\boldsymbol{g}^{-1\bullet\bullet}$ is then defined by the formula

$$\boldsymbol{g}^{-1}\boldsymbol{g} = \mathbf{id}^{\bullet}{}_{\bullet}, \tag{46}$$

where $\mathbf{id}^{\bullet}{}_{\bullet} : \boldsymbol{v} \mapsto \boldsymbol{v}$ is the dimensionless identity operator (tensor) on the tangent space. Hence

$$\dim(\boldsymbol{g}^{-1}) = \dim(\boldsymbol{g})^{-1}. \tag{47}$$

[30] Kilmister (1973) ch II p 25.
[31] cf. Misner *et al* (1973) Box 3.2 D p 77.
[32] Synge (1960b) sections III.2–4; Bressan (1978) section 15, 18.
[33] Synge (1960b) section III.3 pp 108–109.
[34] Landau and Lifshitz (1996) section 84.
[35] BIPM (1983) p 98; Giacomo (1984) p 25.

The operation of *raising or lowering an index* of a tensor represents a contraction of the tensor product of that tensor with the metric or the metric inverse, for example $\boldsymbol{A}_{\bullet\bullet} \equiv \mathrm{tr}_{13}(\boldsymbol{A}^{\bullet}{}_{\bullet} \otimes \boldsymbol{g}_{\bullet\bullet})$ and similarly for tensors of other types. Therefore

$$\begin{aligned} \dim(\boldsymbol{A}_{\dots\bullet\dots}) &= \dim(\boldsymbol{A}_{\dots}{}^{\bullet}{}_{\dots})\ \dim(\boldsymbol{g}) \\ \dim(\boldsymbol{A}_{\dots}{}^{\bullet}{}_{\dots}) &= \dim(\boldsymbol{A}_{\dots\bullet\dots})\ \dim(\boldsymbol{g})^{-1}\ . \end{aligned} \tag{50}$$

The formulae for the covariant derivative (29), connection coefficients (32), and curvature tensors (39) remain valid for a connection compatible with the metric. In this case the connection coefficients can be obtained from the metric by the formulae[37]

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\left(\frac{\partial}{\partial x^{k}} g_{jl} + \frac{\partial}{\partial x^{j}} g_{kl} - \frac{\partial}{\partial x^{l}} g_{jk}\right) g^{li}, \tag{51}$$

and it is easily verified that the dimensions of these coefficients given in equation (32) still hold. Also the results for the curvature tensors (39) still hold, since their expressions in terms of the connection coefficients is the same with or without a metric.

The *scalar curvature* ρ and the *Einstein tensor* $\boldsymbol{G}_{\bullet}{}^{\bullet}$

$$\rho := \mathrm{tr}\, \boldsymbol{Ric}_{\bullet}{}^{\bullet} \equiv \mathrm{tr}_{23}(\boldsymbol{Ric} \otimes \boldsymbol{g}^{-1}), \quad \boldsymbol{G}_{\bullet}{}^{\bullet} := \boldsymbol{Ric}_{\bullet}{}^{\bullet} - \frac{1}{2}\rho\ \mathbf{id}_{\bullet}{}^{\bullet} \tag{52}$$

have therefore dimension

$$\dim(\rho) = \dim(\boldsymbol{G}_{\bullet}{}^{\bullet}) = \dim(\boldsymbol{g})^{-1} \equiv \begin{cases} \mathsf{T}^{-2} \\ \mathsf{L}^{-2} \end{cases} \qquad \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases}\ . \tag{53}$$

The analysis of the metric volume element, of its inverse, and of the four-velocity is given in Porta Mana (2020).

10. Stress–energy–momentum tensor

This section offer a heuristic analysis of the dimension of the stress-energy-momentum tensor. For a more in-depth discussion, involving also variational principles and other dimensional choices that appear in the literature, see Porta Mana (2020). To find the dimension of the stress–energy–momentum $\boldsymbol{T}$, or '4-stress' for short, let us start with the analysis of the 3-stress $\boldsymbol{\sigma}$ in Newtonian mechanics. The stress $\boldsymbol{\sigma}$ is the projection of the 4-stress $\boldsymbol{T}$ onto a spacelike tangent plane with respect to some observer[38]. If we assume that such spatial projection preserves the intrinsic dimension, then the 4-stress and the stress have the same intrinsic dimension.

In Newtonian mechanics the stress $\boldsymbol{\sigma}$ is an object that, integrated over the boundary of a body, gives the total surface force acting on the body[39] (such integration requires a flat connection). This means that it must be represented by a 'force-valued' 2-form. Force, in turn, can be interpreted as an object that, integrated over a (spacelike) trajectory, gives an energy—the work done by the force along the trajectory. It is therefore a 1-form. Putting these two requirements together, the stress turns out to be a covector-valued 2-form, equivalent to a tensor $\boldsymbol{\sigma}_{\bullet\bullet\bullet}$

[37] Choquet-Bruhat *et al* (1996) section V.B.2.

[38] Gourgoulhon 2012 section 3.4.1; Smarr and York (1978); York (1979); Smarr *et al* (1980); Wilson and Mathews (2007) section 1.3; the projection does not need to be orthogonal: Marsden and Hughes (1994) section 2.4; Hehl and Obukhov (2003) section B.1.4.

[39] Truesdell (1991) ch III.

antisymmetric in its last two indices. Integrated over a surface, and then over a trajectory, it yields an energy. From section 6, integration of a form does not change the dimension of the form. Therefore

$$\dim(\boldsymbol{\sigma}_{\bullet\ \bullet\bullet}) = \mathsf{E} \equiv \mathsf{ML}^2\mathsf{T}^{-2}. \tag{54}$$

But usually the stress is represented by a co-contra-variant tensor $\boldsymbol{\sigma}_{\bullet}{}^{\bullet}$. The latter is obtained by contracting the last two slots of $\boldsymbol{\sigma}_{\bullet\bullet\bullet}$ with the inverse of the volume element of the 3-metric—this is the duality[40] between k-vectors and $(n-k)$-covectors induced by the metric (and an orientation choice), where n is the geometric dimension of the manifold. If we assume the Newtonian 3-metric to have dimension L^2, it can be shown similarly to section 9 that its volume element has dimension L^3, and the inverse volume element has dimension L^{-3}. Thus we obtain

$$\dim(\boldsymbol{\sigma}_{\bullet}{}^{\bullet}) = \mathsf{EL}^{-3} \equiv \mathsf{ML}^{-1}\mathsf{T}^{-2}, \tag{55}$$

an energy density (or 'volumic energy' according to ISO[41]).

Since the stress $\boldsymbol{\sigma}_{\bullet}{}^{\bullet}$ is the projection of $\boldsymbol{T}_{\bullet}{}^{\bullet}$ and the projection preserves the intrinsic dimension, we finally find that $\boldsymbol{T}_{\bullet}{}^{\bullet}$ also has the dimension of an energy density:

$$\dim(\boldsymbol{T}_{\bullet}{}^{\bullet}) = \mathsf{EL}^{-3} \equiv \mathsf{ML}^{-1}\mathsf{T}^{-2}. \tag{56}$$

Note that other co- or contra-variant versions of the 4-stress have different intrinsic dimension, because they are obtained by lowering or raising indices. For example, $\dim(\boldsymbol{T}_{\bullet\bullet}) = \dim(\boldsymbol{T}_{\bullet}{}^{\bullet})\dim(\boldsymbol{g}) = \mathsf{ML}^{-1}$ if $\dim(\boldsymbol{g}) := \mathsf{T}^2$.

Let me add a passing remark. Even though in most texts the 4-stress is represented by a tensor of order 2, as above, its most fitting geometrical nature is still shrouded in mystery from the kinematic and the dynamical points of view. There are indications that it could be more properly represented by a covector-valued 3-form (equivalent to a tensor $\boldsymbol{T}_{\bullet\bullet\bullet\bullet}$ antisymmetric in the last three slots), or by a 3-vector-valued 3-form (equivalent to a tensor $\boldsymbol{T}^{\bullet\bullet\bullet}{}_{\bullet\bullet\bullet}$ antisymmetric in the first three and last three slots), for reasons connected with integration, similar to those mentioned above for the stress $\boldsymbol{\sigma}_{\bullet\bullet\bullet}$. See for example the discussion about '${}^*\boldsymbol{T}$' by Misner *et al*[42], the works by Segev[43], the discussion by Burke[44].

11. The constant in the Einstein equations

We finally arrive at the Einstein equations,

$$\boldsymbol{G} = \kappa\boldsymbol{T} \tag{57}$$

where κ (sometimes seen with a minus[45] depending on the signature of the metric or on the orientation of the stress) is Einstein's constant. For the dimension of κ we thus find

[40] Bossavit (1991) section 4.1.2.

[41] ISO (2009) item A.6.2.

[42] Misner *et al* (1973) ch 15.

[43] Segev (2002); (1986); Segev and Rodnay (1999); Segev (2000a),(b).

[44] Burke (1987) section 41.

[45] e.g. Tolman (1949) section 78 equation (78.3); Fock (1964) section 52 equation (52.06); Rindler (2006) section 14.2 equation (14.8).

$$\dim(\kappa) = \dim(\boldsymbol{G}_{\bullet}{}^{\bullet})\ \dim(\boldsymbol{T}_{\bullet}{}^{\bullet})^{-1} \equiv \begin{cases} \mathsf{M}^{-1}\mathsf{L} \\ \mathsf{M}^{-1}\mathsf{L}^{-1}\mathsf{T}^{2} \end{cases} \qquad \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases} \tag{58}$$

This constant can be obtained from the dimensions of Newton's gravitational constant $\dim(G) = \mathsf{M}^{-1}\mathsf{L}^{3}\mathsf{T}^{-2}$ (this is not the Einstein tensor $\boldsymbol{G}$!) and of the speed of light $\dim(c) = \mathsf{L}\mathsf{T}^{-1}$ only in the following ways, with an 8π factor coming from the Newtonian limit:

$$\kappa = \begin{cases} 8\pi G/c^{2} \\ 8\pi G/c^{4} \end{cases} \qquad \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases} \qquad . \tag{59}$$

The second choice is by far the most common, consistently with the most common choice of $\dim(\boldsymbol{g}) = \mathsf{L}^{2}$ discussed before. The first choice appears for example in Fock[46] and Adler *et al*[47]. Another, quite curious convention for the dimension of the Einstein constant is also possible: $\kappa = 8\pi$ G, without c factors at all. Such convention comes from attributing the dimension MT to the stress-energy-momentum tensor, a choice that appears among some authors. This possibility is discussed in Porta Mana (2020).

12. Summary and conclusions

We have seen that dimensional analysis, with its familiar rules, can be seamlessly performed in Lorentzian and general relativity and in differential geometry if we adopt the coordinate-free approach typical of modern texts. In this approach each tensor has an *intrinsic* dimension (a notion introduced by Schouten and Dorgelo). This dimension does not depend on the dimensions of the coordinates, and is determined by the physical and operational meaning of the tensor. It is therefore generally more profitable to focus on the intrinsic dimension of a tensor rather than on the dimensions of its components. The dimension of each specific component is easily found by formula (11): it is the product of the intrinsic dimension by the dimension of the ith coordinate function for each contravariant index i, by the inverse of the dimension of the jth coordinate function for each covariant index j.

Dimensional analysis in differential geometry seems to rest on two main conventions: the tensor product and the action of covectors on vectors behave analogously to usual multiplication for the purposes of dimensional analysis. Alternative, equivalent sets of conventions can also be considered, for example involving the exterior derivative.

We found or re-derived some essential results for general relativity, in particular that the Riemann $\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}$ and Ricci $\boldsymbol{Ric}_{\bullet\bullet}$ curvature tensors are dimensionless, and that the Einstein tensor $\boldsymbol{G}_{\bullet}{}^{\bullet}$ has the inverse dimension of the metric tensor. Maybe these results can be of importance for some current research involving scales and conformal factors[48]. We also discussed the operational reasons behind two common choices of dimension for the metric tensor.

Since the dimensions of the components are usually different from the intrinsic dimension and depend on the coordinates, I recommend to avoid statements such as 'the tensor $A_i{}^{jk}$ has dimension X', which leave it unclear whether '$A_i{}^{jk}$' is meant to represent the tensor in general

[46] Fock (1964) section 55 equations (55.15) and (52.06).

[47] Adler *et al* (1975) section 10.5 equation (10.98).

[48] e.g. Röhr and Uggla (2005); Cadoni and Tuveri (2019).

(as in Penrose and Rindler's notation), or to represent its set of components, or to represent just a specific component.

Dimensional analysis remains a controversial, obscure, but fascinating subject still today, 60 years from Truesdell and Toupin's remark quoted in the Introduction. For an overview of some recent and creative approaches to it, going beyond Bridgman's text* (whose point of view is in many respects at variance with modern developments: see the following references), I recommend for example the works by Mari *et al*[49], Domotor and Batitsky[50], Kitano[51], the extensive analysis by Dybkaer[52], the historical review by de Boer[53], and references therein.

Acknowledgements

Thanks to (chronologically) Mariano Cadoni, Ingemar Bengtsson, Iván Davidovich, Claudia Battistin for valuable comments on previous drafts. To the staff of the NTNU library for their always prompt support. To Mari, Miri, Emma for continuous encouragement and affection, and to Buster Keaton and Saitama for filling life with awe and inspiration. To the developers and maintainers of LATEX, Emacs, AUCTEX, Open Science Framework, R, Python, Inkscape, Sci-Hub for making a free and impartial scientific exchange possible. This work is financially supported partly by the Kavli Foundation and the Centre of Excellence scheme of the Research Council of Norway (Roudi group), and partly by the Trond Mohn Research Foundation, grant number BFS2018TMT07.

ORCID iDs

P G L Porta Mana https://orcid.org/0000-0002-6070-0784

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* Bridgman 1963.

[49] Mari and Giordani (2012); Frigerio *et al* (2010).

[50] Domotor (2017); Domotor and Batitsky (2016); Domotor (2012).

[51] Kitano (2013).

[52] Dybkær (2010).

[53] De Boer (1995).

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