Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1   Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z), which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

There are 18 basic types of geometric objects, which represent physical quantities of different types. We have inner-oriented scalars, inner-oriented 1-vectors up to 4-vectors, inner-oriented 1-covectors up to 4-covectors, and all their outer-oriented counterparts.

The exterior product '∧' of vectors and covectors is usually represented just by juxtaposition, and a shortened notation is used for exterior products of several exterior derivatives 'd'. For instance

$$\mathrm{d}^2 xy := \mathrm{d}x \wedge \mathrm{d}y \qquad \partial_{xy}^2 := \partial_x \wedge \partial_y \ . \tag{1}$$

The associated bases for inner-oriented covector fields are

$$\mathrm{d}t \quad \mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z \tag{2}$$

$$\mathrm{d}^2 tx \quad \mathrm{d}^2 ty \quad \mathrm{d}^2 tz \quad \mathrm{d}^2 yz \quad \mathrm{d}^2 zx \quad \mathrm{d}^2 xy \tag{3}$$

$$\mathrm{d}^3 xyz \quad -\mathrm{d}^3 tyz \quad -\mathrm{d}^3 tzx \quad -\mathrm{d}^3 txy \tag{4}$$

$$\mathrm{d}^4 txyz \tag{5}$$

and analogously for inner-oriented vector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it's only defined on a coordinate patch. It is idempotent: $\tilde{1}\tilde{1} = 1$.

1

A twisted or outer-oriented 3-covector such as $\mathrm{d}^3\tilde{x}yz$ has an associated outer direction, in this case positive t. We adopt this shorter notation for the outer-oriented versions of the bases above (analogous to the notation in Gotay & Marsden 1992 § 2 p. 371):

$$-\mathrm{d}_{xyz} \quad \mathrm{d}_{tyz} \quad \mathrm{d}_{tzx} \quad \mathrm{d}_{txy} \tag{6a}$$

$$\mathrm{d}^2_{yz} \quad \mathrm{d}^2_{zx} \quad \mathrm{d}^2_{xy} \quad \mathrm{d}^2_{tx} \quad \mathrm{d}^2_{ty} \quad \mathrm{d}^2_{tz} \tag{6b}$$

$$\mathrm{d}^3_t \quad \mathrm{d}^3_x \quad \mathrm{d}^3_y \quad \mathrm{d}^3_z \tag{6c}$$

$$\mathrm{d}^4 \tag{6d}$$

so that $-\mathrm{d}_{xyz} := \mathrm{d}\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz := \partial_{\tilde{t}}$. Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$n_{xyz}\,\mathrm{d}^3\tilde{x}yz - n_{tyz}\,\mathrm{d}^3\tilde{t}yz - n_{tzx}\,\mathrm{d}^3\tilde{t}zx - n_{txy}\,\mathrm{d}^3\tilde{t}xy$$
$$\equiv n^t\,\mathrm{d}^3_t + n^x\,\mathrm{d}^3_x + n^y\,\mathrm{d}^3_y + n^z\,\mathrm{d}^3_z \tag{7}$$

with $n^t \equiv n_{xyz}$, $n^x \equiv n_{tyz}$, and so on.

Contraction or dot-product of vectors and covectors is denoted by '·', and always contracts the maximal possible amount of contiguous elements. For instance

$$\mathrm{d}^3xyz \cdot \partial^2_{yz} = \mathrm{d}x \qquad -\partial^3_{txy} \cdot \mathrm{d}x = \partial^2_{ty}\,. \tag{8}$$

Contractions with the 4-vector ∂^4 and 4-covector d^4 establish a duality between outer n-covectors and inner $(4-n)$-vectors:

$$
\left.
\begin{array}{c}
\partial^4 \\[2pt]
\partial^3_{xyz} \; \partial^3_{tyz} \; \partial^3_{tzx} \; \partial^3_{txy} \\[2pt]
\partial^2_{tx} \; \partial^2_{ty} \; \partial^2_{tz} \; \partial^2_{yz} \; \partial^2_{zx} \; \partial^2_{xy} \\[2pt]
\partial_t \; \partial_x \; \partial_y \; \partial_z \\[2pt]
1
\end{array}
\right)
\quad
\begin{array}{c}
\xleftarrow{\;\partial^4\cdot\;} \\[6pt]
\xrightarrow{\;\cdot\,\mathrm{d}^4\;}
\end{array}
\quad
\left(
\begin{array}{c}
\tilde{1} \\[2pt]
\mathrm{d}_{xyz} \; \mathrm{d}_{tyz} \; \mathrm{d}_{tzx} \; \mathrm{d}_{txy} \\[2pt]
\mathrm{d}^2_{yz} \; \mathrm{d}^2_{zx} \; \mathrm{d}^2_{xy} \; \mathrm{d}^2_{tx} \; \mathrm{d}^2_{ty} \; \mathrm{d}^2_{tz} \\[2pt]
\mathrm{d}^3_t \; \mathrm{d}^3_x \; \mathrm{d}^3_y \; \mathrm{d}^3_z \\[2pt]
\mathrm{d}^4
\end{array}
\right.
\tag{9}
$$

These duals have special properties. For instance, for any 3-covector $\boldsymbol{N}$, we have

$$\boldsymbol{N} \cdot (\partial^4 \cdot \boldsymbol{N}) \cdot \boldsymbol{N} = 0 \tag{10}$$

that is, the dual of $\boldsymbol{N}$ is a vector belonging in the kernel of $\boldsymbol{N}$.

If $\boldsymbol{\gamma}$ is a non-zero 4-covector and $\boldsymbol{\gamma}^{-1}$ the inverse 4-vector, that is, $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{\gamma} = 1$, and if $\boldsymbol{N}$ is a 3-covector and $\boldsymbol{\phi}$ a 1-covector, we have the useful identity

$$\boldsymbol{N} \wedge \boldsymbol{\phi} = (\boldsymbol{N} \cdot \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{\phi})\, \boldsymbol{\gamma} \,, \tag{11}$$

which also holds as long as the degrees of $\boldsymbol{N}$ and $\boldsymbol{\phi}$ sum up to 4.

We often consider vector- or covector-valued covectors, written as sums of tensor products. They can be outer- or inner-oriented. For instance

$$\mathrm{d}_t^3 \otimes \mathrm{d}x \equiv \mathrm{d}^3 \tilde{x} y z \otimes \mathrm{d}x \,, \qquad \mathrm{d}_x^3 \otimes \partial_y \,. \tag{12}$$

The operation $\curlywedge$ between a vector-valued covector and a covector-valued covector is the contraction of their vector- and covector-valued parts and the exterior product of their covector parts. For instance, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are covectors, $\boldsymbol{\omega}$ is a covector, and $\boldsymbol{u}$ is a vector, then

$$(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \curlywedge (\boldsymbol{\psi} \otimes \boldsymbol{u}) := (\boldsymbol{\phi} \wedge \boldsymbol{\psi}) \otimes (\boldsymbol{\omega} \cdot \boldsymbol{u}) \tag{13}$$

As another example,

$$(\mathrm{d}_t^3 \otimes \mathrm{d}x) \curlywedge (\mathrm{d}_t \otimes \partial_x) = (\mathrm{d}_t^3 \wedge \mathrm{d}_t)(\mathrm{d}x \cdot \partial_x) = -\mathrm{d}^4 \,. \tag{14}$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta}\, \mathrm{d}u^\beta \,,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$\mathrm{d}\alpha' = \frac{\partial \alpha'}{\partial \alpha}\, \mathrm{d}\alpha$$

- raising or lowering of indices with the metric, for example

$$\mathrm{d}\alpha \mapsto g^{\alpha\beta}\, \partial_\beta \,.$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix[1]. For the spaces of 3-vectors and 3-covectors we have simplified formulae[2]:

$$\mathrm{d}^3_{\alpha'} = \det\!\left(\tfrac{\partial \alpha'}{\partial \alpha}\right) \frac{\partial \alpha}{\partial \alpha'}\, \mathrm{d}^3_\alpha$$
$$\partial^3 \alpha' = \det\!\left(\tfrac{\partial \alpha}{\partial \alpha'}\right) \frac{\partial \alpha'}{\partial \alpha}\, \partial^3 \alpha \tag{15}$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\mathrm{d}^3_\alpha \mapsto \frac{1}{|g|}\, g_{\alpha\beta}\, \partial^3 \beta$$
$$\partial^3 \alpha \mapsto |g|\, g^{\alpha\beta}\, \mathrm{d}^3_\beta \tag{16}$$

3 Metric

We take the metric $\boldsymbol{g}$ to have signature $(-, +, +, +)$ and dimensions of area. The determinant and the negative of its square root are denoted shortly

$$g := \det \boldsymbol{g} \qquad \sqrt{g} := \sqrt{-\det \boldsymbol{g}}\,. \tag{17}$$

The volume element induced by the metric $\boldsymbol{g}$ has dimensions of volume-time and is denoted (note the boldface)

$$\boldsymbol{\gamma} := \frac{\sqrt{g}}{c}\mathrm{d}^4\tilde{t}xyz \equiv \frac{\sqrt{g}}{c}\,\mathrm{d}^4 \tag{18}$$

and its corresponding inverse, a twisted 4-vector:

$$\boldsymbol{\gamma}^{-1} := \frac{c}{\sqrt{g}}\,\partial^4\,. \tag{19}$$

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[1] Choquet-Bruhat et al. 1996 § IV.A.1 p. 199.   [2] Gantmacher 2000 § I.4 eq. (33).

Contraction with the volume element or its inverse establishes a "volume duality" between outer n-covectors and inner $(4 - n)$-vectors:

$$\left.\begin{array}{c} \partial^4 \\[4pt] \partial^3_{xyz}\ \partial^3_{tyz}\ \partial^3_{tzx}\ \partial^3_{txy} \\[4pt] \partial^2_{tx}\ \partial^2_{ty}\ \partial^2_{tz}\ \partial^2_{yz}\ \partial^2_{zx}\ \partial^2_{xy} \\[4pt] \partial_t\ \partial_x\ \partial_y\ \partial_z \\[4pt] 1 \end{array}\right) \quad \begin{array}{c} \xleftarrow{\frac{\sqrt{g}}{c}\,\boldsymbol{\gamma}^{-1}\,\cdot} \\[8pt] \xrightarrow{\ \cdot\,\boldsymbol{\gamma}\,\frac{c}{\sqrt{g}}\ } \end{array} \quad \left(\begin{array}{c} \tilde{1} \\[4pt] \mathrm{d}_{xyz}\ \mathrm{d}_{tyz}\ \mathrm{d}_{tzx}\ \mathrm{d}_{txy} \\[4pt] \mathrm{d}^2_{yz}\ \mathrm{d}^2_{zx}\ \mathrm{d}^2_{xy}\ \mathrm{d}^2_{tx}\ \mathrm{d}^2_{ty}\ \mathrm{d}^2_{tz} \\[4pt] \mathrm{d}^3_t\ \mathrm{d}^3_x\ \mathrm{d}^3_y\ \mathrm{d}^3_z \\[4pt] \mathrm{d}^4 \end{array}\right.$$

$$(20)$$

This is the reason why in older literature an outer-oriented n-covector is treated as a $(4 - n)$-"vector density", that is, a vector divided by the square root of the volume element.

The metric on the space of 1-vectors also induces metrics on the spaces of 1-covectors, 2-vectors and 2-covectors, and so on. In particular, the metrics $\overset{3}{\boldsymbol{g}}{}^{-1}$ on the space of 3-covectors and $\overset{3}{\boldsymbol{g}}$ on the space of 3-vectors can be written in coordinates as

$$\overset{3}{\boldsymbol{g}}{}^{-1} = \frac{g_{\mu\nu}}{g}\,\partial^3\mu \otimes \partial^3\nu \qquad \text{with dimensions length}^{-6} \tag{21}$$

$$\overset{3}{\boldsymbol{g}} = g\,g^{\mu\nu}\,\mathrm{d}^3_\mu \otimes \mathrm{d}^3_\nu \qquad \text{with dimensions length}^6\ . \tag{22}$$

With these we can define squared norms $\|\boldsymbol{.}\|^2$ on all those spaces. Note in particular the following identity:

$$\|\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{N}\|^2 = -c^2\,\|\boldsymbol{N}\| \qquad \text{for every 3-covector } \boldsymbol{N}. \tag{23}$$

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{24}$$

and the volume element is simply d^4.

4 Matter current

The amount-of-matter current $\boldsymbol{N}$ is an outer-oriented 3-covector

$$\boldsymbol{N} = N \, \mathrm{d}^3_t + J^i \, \mathrm{d}^3_i \tag{25}$$

of dimensions "amount of matter", typically measured in moles, where

- N is the volumic amount of matter, measured per unit coordinate volume.

- J^i is the aeric flux of amount of matter (equivalent to a molar flux), measured per unit coordinate area and unit coordinate time.

The current for every particular kind of matter satisfies the conservation law

$$\mathrm{d}\boldsymbol{N} = 0 \quad \text{or} \quad \partial_t N + \partial_i J^i = 0 \tag{26}$$

independent of any metric.

The common contravariant form of the matter current, "N^μ", is obtained by contracting the matter current with the inverse volume element:

$$‘N^\mu’ \triangleq \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{N} = \frac{c}{\sqrt{g}} N \, \partial_t + \frac{c}{\sqrt{g}} J^i \, \partial_t \ . \tag{27}$$

If a metric is present, a four-velocity $\boldsymbol{U}$ can be associated with the matter current $\boldsymbol{N}$, defined by the following properties and identity:

$$\boldsymbol{U} \cdot \boldsymbol{N} = 0 \qquad \|\boldsymbol{U}\|^2 = -c^2 \tag{28}$$

$$\boldsymbol{U} = \frac{1}{\|\|\boldsymbol{N}\|\|} \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{N} \tag{29}$$

which also implies (for normal matter)

$$\boldsymbol{N} = \|\boldsymbol{N}\| \, \boldsymbol{U} \cdot \boldsymbol{\gamma} \tag{30}$$

For normal matter (as opposed to antimatter) $\|\boldsymbol{N}\|^2 \geqslant 0$.

5 Four-stress

The stress-energy-momentum tensor, or simply four-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\begin{aligned}
\boldsymbol{T} &= T^{\mu}_{\ \nu}\ \mathrm{d}^3_{\mu} \otimes \mathrm{d}x^{\nu} \\
&= -\epsilon\, \mathrm{d}^3_t \otimes \mathrm{d}t - q^i\, \mathrm{d}^3_i \otimes \mathrm{d}t + p_j\, \mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\, \mathrm{d}^3_i \otimes \mathrm{d}x^j
\end{aligned} \tag{31}$$

the indices i, j running over x, y, z, and where:

- The energy ϵ is a density per unit *coordinate* volume xyz, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this energy comprises "rest" energy, kinetic energy, potential gravitational energy, and centrifugal potential energy.

- The energy flux q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this flux term includes transport of the energy term above and also heat flux.

- The momentum p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i.

- The compressive three-stress π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j. If the point at which the four-stress is considered has a matter-current, then this stress comprises momentum transport and internal stresses.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \tag{32}$$

The diagonal elements $g_{tt}, \ldots$ include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

The common contra-contra-variant form of the stress, "$T^{\mu\nu}$", is obtained by contracting the four-stress with the inverse volume element and the inverse metric:

$$'T^{\mu\nu}{}' \triangleq \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1} = -\frac{c\,g^{tt}}{\sqrt{g}}\,\epsilon\,\partial_t \otimes \partial_t - \frac{c\,g^{tt}}{\sqrt{g}}\,q^i\,\partial_i \otimes \partial_t$$

$$+ \sum_j \frac{c\,g^{jj}}{\sqrt{g}}\,p_j\,\partial_t \otimes \partial_j + \sum_j \frac{c\,g^{jj}}{\sqrt{g}}\,\pi_j^i\,\partial_i \otimes \partial_j \,. \tag{33}$$

One important detail in finding the Newtonian approximation of "energy density" is that *one takes different zeros of energy density in different coordinate systems*: the zero is taken as the molar mass times the molar density *in the current coordinate system*. By 'zero' I mean the arbitrary separation between "mass" and "energy".

The *total* four-stress satisfies the balance equation

$$\mathrm{D}\boldsymbol{T} = 0 \tag{34}$$

which is equivalent to the four balance equations

$$\partial_t e + \partial_i q^i = e\,\Gamma_{tt}^t + q^i\,\Gamma_{it}^t - p_j\,\Gamma_{tt}^j - \pi_k^i\,\Gamma_{it}^k$$

$$\partial_t p_j + \partial_i \pi_j^i = -e\,\Gamma_{tj}^t - q^i\,\Gamma_{ij}^t + p_k\,\Gamma_{tj}^k + \pi_k^i\,\Gamma_{ij}^k \tag{35}$$

In general relativity the *total* four-stress also satisfies

$$\left(\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1}\right)^{\mathsf{T}} - \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1} = 0 \,. \tag{36}$$

The four-stress determines an association between any 1-vector $\boldsymbol{V}$ field and an outer-oriented 3-covector field, interpreted as a current:

$$\boldsymbol{V} \mapsto \boldsymbol{T} \cdot \boldsymbol{V} \,. \tag{37}$$

This current satisfies the balance equation

$$\mathrm{d}(\boldsymbol{T} \cdot \boldsymbol{V}) = -\boldsymbol{T} \wedge \nabla\boldsymbol{V} = \mathrm{tr}(\boldsymbol{T}^{\mathsf{T}} \cdot \boldsymbol{\gamma}^{-1} \cdot \nabla\boldsymbol{V})\,\boldsymbol{\gamma} \tag{38}$$

which is a conservation law if $\boldsymbol{V}$ is a Killing vector.

For the special case $\boldsymbol{V} = \partial_\alpha$ the formula above becomes

$$\mathrm{d}(T^\mu{}_\alpha \, \mathrm{d}^3_\mu) = T^\mu{}_\nu \, \Gamma^\nu{}_{\mu\alpha} \, \mathrm{d}^4 \qquad \Longleftrightarrow \qquad \partial_\mu T^\mu{}_\alpha = T^\mu{}_\nu \, \Gamma^\nu{}_{\mu\alpha} \tag{39}$$

Consider a region where there is a non-vanishing matter current $\boldsymbol{N}$ with associated four-velocity $\boldsymbol{U}$, and define

$$\bar{\boldsymbol{U}} = -\frac{1}{c} \boldsymbol{g} \cdot \boldsymbol{U} \tag{40}$$

which statisfies

$$\bar{\boldsymbol{U}} \cdot \boldsymbol{U} = 1 \,, \qquad \nabla \boldsymbol{U} \cdot \bar{\boldsymbol{U}} = 0 \,. \tag{41}$$

The last equality can be proved from $\nabla \boldsymbol{g} = 0$ and

$$0 = -\nabla(c^2) = \nabla(\boldsymbol{U} \cdot \boldsymbol{g} \cdot \boldsymbol{U}) = 2\,(\nabla \boldsymbol{U}) \cdot \boldsymbol{g} \cdot \boldsymbol{U} \,. \tag{42}$$

We can associate with the matter a four-stress $\boldsymbol{T}$ which can be decomposed as follows:

$$\boldsymbol{T} = -\epsilon\,\boldsymbol{N} \otimes \bar{\boldsymbol{U}} + \boldsymbol{N} \otimes \boldsymbol{P} - (\bar{\boldsymbol{U}} \wedge \boldsymbol{Q}) \otimes \bar{\boldsymbol{U}} + \bar{\boldsymbol{U}} \wedge \boldsymbol{S}$$
$$\text{with} \quad \boldsymbol{P} \cdot \boldsymbol{U} = 0 \quad \boldsymbol{Q} \cdot \boldsymbol{U} = 0 \quad \boldsymbol{U} \cdot \boldsymbol{S} = 0 \quad \boldsymbol{S} \cdot \boldsymbol{U} = 0 \tag{43}$$

where

- ϵ is a scalar, the molar energy density.
- $\boldsymbol{P}$ is a 1-covector, the molar momentum density.
- $\boldsymbol{Q}$ is a 2-covector, the areic energy-flux density.
- $\boldsymbol{S}$ is a 1-covector-valued 2-covector, the three-stress related to the Cauchy stress.

Using the four-velocity $\boldsymbol{U}$ associated with the matter current we obtain what could be called the "internal-energy current":

$$\boldsymbol{T} \cdot \boldsymbol{U} \equiv -\epsilon\,\boldsymbol{N} - \bar{\boldsymbol{U}}\,\boldsymbol{Q} \tag{44}$$

which, from eqs (38), (40), (26), satisfies the balance law

$$\mathrm{d}(\boldsymbol{T} \cdot \boldsymbol{U}) = -\boldsymbol{T} \wedge \nabla \boldsymbol{U} \tag{45}$$

or

$$\mathrm{d}(-\epsilon\,\boldsymbol{N} - \bar{\boldsymbol{U}}\,\boldsymbol{Q}) = (\epsilon\,\boldsymbol{N} \otimes \bar{\boldsymbol{U}}) \cdot \nabla \boldsymbol{U} - (\boldsymbol{N} \otimes \boldsymbol{P}) \cdot \nabla \boldsymbol{U}$$
$$+ (\bar{\boldsymbol{U}}\,\boldsymbol{Q} \otimes \bar{\boldsymbol{U}}) \cdot \nabla \boldsymbol{U} - \bar{\boldsymbol{U}}\,\boldsymbol{S} \cdot \nabla \boldsymbol{U} \tag{46}$$

or simply

$$\boldsymbol{N}\mathrm{d}\epsilon + \bar{\boldsymbol{U}}\mathrm{d}\boldsymbol{Q} - \boldsymbol{Q}\,\mathrm{d}\bar{\boldsymbol{U}} = -\boldsymbol{N}\,\nabla\boldsymbol{U}\cdot\boldsymbol{P} - (\bar{\boldsymbol{U}}\,\boldsymbol{S})\cdot\nabla\boldsymbol{U}\ . \tag{47}$$

The last balance equation corresponds to eq. (28) in Eckart (1940), except for the fact that here we are keeping the momentum $\boldsymbol{P}$ and heat flux $\boldsymbol{Q}$ distinct.

When $\boldsymbol{P}$, $\boldsymbol{S}$, $\boldsymbol{Q}$ are zero, one speaks of the four-stress of "dust". It is noteworthy that in this case the four-stress is essentially proportional to the matter 3-covector, multiplied by a vector collinear with the matter four-velocity:

$$\boldsymbol{T} = -\boldsymbol{N}\otimes(\epsilon\,\bar{\boldsymbol{U}})\ . \tag{48}$$

If we take the four-velocity itself as reference, given that $\bar{\boldsymbol{U}}\boldsymbol{U} = -c^2$, the energy 3-covector we obtain is proportional to the matter 3-covector:

$$-c^2\epsilon\,\boldsymbol{N} \tag{49}$$

For this reason ϵ is called the "proper internal energy". If we take other vector fields as reference, then this energy will pick up further multiplicative terms consisting in the projection of the matter four-velocity $\boldsymbol{U}$ onto the reference vector field. In particular taking ∂_t or ∂_i as reference we'll have the projections of the four-velocity onto the coordinate covectors as multiplicative factors.

6   Electromagnetic field

The electromagnetic field, represented by the Faraday 2-covector, is typically decomposed as[3]

$$\begin{aligned}
\boldsymbol{F} &= \boldsymbol{E}\,\mathrm{d}t + \boldsymbol{B}\\
&\equiv E_x\,\mathrm{d}^2 xt + E_y\,\mathrm{d}^2 yt + E_z\,\mathrm{d}^2 zt + B^x\,\mathrm{d}^2 yz + B^y\,\mathrm{d}^2 zx + B^z\,\mathrm{d}^2 xy\ .
\end{aligned} \tag{50}$$

Given a system of coordinates, this decomposition is unique: the 2-covector $\boldsymbol{B}$ does not have $\mathrm{d}t$ components, that is, it has ∂_t in its kernel; and so does the 1-covector $\boldsymbol{E}$.

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[3] Frankel 1979 ch. 9.

The conservation of magnetic flux is expressed by

$$\mathrm{d}\boldsymbol{F} = 0 \tag{51}$$

or equivalently

$$
\begin{aligned}
&\partial_i B^i = 0 &&(\mathrm{d}^3 xyz \text{ component})\\
&\partial_t B^x + \partial_y E_z - \partial_z E_y = 0 &&(\mathrm{d}^3 tyz)\\
&\partial_t B^y + \partial_z E_x - \partial_x E_z = 0 &&(\mathrm{d}^3 tzx)\\
&\partial_t B^z + \partial_x E_y - \partial_y E_x = 0 &&(\mathrm{d}^3 txy)
\end{aligned}
\tag{52}
$$

It should be noted that all components of $\mathrm{d}\boldsymbol{E}$ containing $\mathrm{d}t$ also disappear, because of the exterior product $\boldsymbol{E}\,\mathrm{d}t$. The differential d therefore operates on $\boldsymbol{E}$ as if this 1-covector belonged to a three-dimensional manifolds with coordinates (x, y, z). Let's denote this operation by $\mathbf{d}$; it is connected with the "curl" operator. Then the equations above can be written

$$\mathbf{d}\boldsymbol{B} = 0 \qquad \partial_t \boldsymbol{B} + \mathbf{d}\boldsymbol{E} = 0 \;. \tag{53}$$

Appendices

A   Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- $\boldsymbol{u}^{\boldsymbol{\cdot}}$ is a 1-vector, represented by the column-matrix $\boldsymbol{u}$.

- Similarly for $\boldsymbol{v}^{\bullet}$.

- $\boldsymbol{\omega}_{\bullet}$ is a 1-covector, represented by the row-matrix $\boldsymbol{\omega}$.

- $\boldsymbol{g}_{\bullet\bullet}$ is a covector-valued covector, represented by the matrix $\boldsymbol{g}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.

- $\boldsymbol{g}^{-1\bullet\bullet}$ is a vector-valued vector, inverse of $\boldsymbol{g}$, that is: $\boldsymbol{g} \cdot \boldsymbol{g}^1 = \mathbf{id}_{\bullet}^{\bullet}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.

- $\boldsymbol{Q}_{|\bullet\bullet\bullet|}$ is a 3-covector, represented by the column-matrix $\boldsymbol{Q}$.

- $\boldsymbol{T}_{|\bullet\bullet\bullet|\bullet}$ is a covector-valued 3-covector, represented by the matrix $\boldsymbol{T}$. The rows represent the 3-covector components; the columns, the covector components.

- $\boldsymbol{\gamma}_{|\bullet\bullet\bullet\bullet|}$ is a 4-covector, represented by the number $\boldsymbol{\gamma}$.

- $\boldsymbol{\gamma}^{-1|\bullet\bullet\bullet\bullet|}$ is a 4-vector, represented by the number $\boldsymbol{\gamma}^{-1}$

- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from "old" coordinates x to "new" coordinates x' is represented by the matrix $\boldsymbol{J}$. The rows correspond to the new coordinates x'; the columns, to the old x.

- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x' is represented by the matrix $\boldsymbol{J}^{-1}$. The rows correspond to the old coordinates x; the columns, to the new x'.

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

(object)	(matrix repr)		
			(54)
$\boldsymbol{\omega} \cdot \boldsymbol{u}$	$\boldsymbol{\omega u} \equiv \boldsymbol{u}^{\mathsf{T}}\boldsymbol{\omega}^{\mathsf{T}}$	(number)	(55)
$\boldsymbol{v} \cdot \boldsymbol{g} \cdot \boldsymbol{u}$	$\boldsymbol{v}^{\mathsf{T}}\boldsymbol{gu}$	(number)	(56)
$\boldsymbol{g} \cdot \boldsymbol{u}$	$\boldsymbol{u}^{\mathsf{T}}\boldsymbol{g}^{\mathsf{T}}$	(row-matrix)	(57)
$\boldsymbol{\omega} \cdot \boldsymbol{g}^{-1}$	$\boldsymbol{g}^{-\mathsf{T}}\boldsymbol{\omega}^{\mathsf{T}}$	(column-matrix)	(58)
$\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{Q}$	$\boldsymbol{\gamma}^{-1}\boldsymbol{Q}$	(column-matrix)	(59)
$\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T}$	$\boldsymbol{\gamma}^{-1}\boldsymbol{T}$	(column-matrix)	(60)
$\boldsymbol{T} \cdot \boldsymbol{u}$	$\boldsymbol{Tu}$	(column-matrix)	(61)
$\boldsymbol{g} \cdot (\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T}) \cdot \boldsymbol{u}$	$\boldsymbol{\gamma}^{-1}\boldsymbol{gTu}$	(matrix)	(62)

- Transformations

$$\text{(old coords)} \quad \mapsto \quad \text{(new coords)} \tag{63}$$

$$\boldsymbol{u} \quad \mapsto \quad \boldsymbol{Ju} \tag{64}$$

$$\boldsymbol{\omega} \quad \mapsto \quad \boldsymbol{\omega J}^{-1} \tag{65}$$

$$\boldsymbol{g} \quad \mapsto \quad \boldsymbol{J}^{-\mathsf{T}}\boldsymbol{g J}^{-1} \tag{66}$$

$$\boldsymbol{Q} \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{JQ} \tag{67}$$

$$\boldsymbol{\gamma} \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{\gamma} \tag{68}$$

$$\boldsymbol{\gamma}^{-1} \quad \mapsto \quad \det \boldsymbol{J}\,\boldsymbol{\gamma}^{-1} \tag{69}$$

$$\boldsymbol{T} \quad \mapsto \quad \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{JTJ}^{-1} \tag{70}$$

B   Checks about optimal representation of four-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$
\begin{aligned}
\mathrm{d}\!\left(f\,\mathrm{d}_t^3\right) &= \mathrm{d}\!\left(f\,\mathrm{d}^3\tilde{x}yz\right) = \partial_t f\,\mathrm{d}t \wedge \mathrm{d}^3\tilde{x}yz + 0 = \partial_t f\,\mathrm{d}^4\tilde{t}xyz \\
\mathrm{d}\!\left(f\,\mathrm{d}_x^3\right) &= -\mathrm{d}\!\left(f\,\mathrm{d}^3\tilde{t}yz\right) = -\partial_x f\,\mathrm{d}x \wedge \mathrm{d}^3\tilde{t}yz + 0 = \partial_x f\,\mathrm{d}^4\tilde{t}xyz \\
\mathrm{d}\!\left(f\,\mathrm{d}_i^3\right) &= \partial_i f\,\mathrm{d}^4\tilde{t}xyz
\end{aligned}
\tag{71}
$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$
\begin{aligned}
\boldsymbol{\nabla}(\mathrm{d}t) &= -\Gamma^t_{tt}\,\mathrm{d}t \otimes \mathrm{d}t - \Gamma^t_{tj}\,\mathrm{d}t \otimes \mathrm{d}x^j - \Gamma^t_{it}\,\mathrm{d}x^i \otimes \mathrm{d}t - \Gamma^t_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \\
\boldsymbol{\nabla}\!\left(\mathrm{d}x^k\right) &= -\Gamma^k_{tt}\,\mathrm{d}t \otimes \mathrm{d}t - \Gamma^k_{tj}\,\mathrm{d}t \otimes \mathrm{d}x^j - \Gamma^k_{it}\,\mathrm{d}x^i \otimes \mathrm{d}t - \Gamma^k_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j
\end{aligned}
\tag{72}
$$

If $\boldsymbol{\phi}$ is a 3-covector, $\boldsymbol{\omega}$ a 1-covector, and D the exterior covariant derivative, then

$$\mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) = (\mathrm{d}\boldsymbol{\phi}) \otimes \boldsymbol{\omega} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega} \tag{73}$$

and in particular

$$\mathrm{D}\!\left(\boldsymbol{\phi} \otimes \mathrm{d}x^\alpha\right) = (\mathrm{d}\boldsymbol{\phi}) \otimes \mathrm{d}x^\alpha + \Gamma^\alpha_{\mu\nu}\left(\boldsymbol{\phi} \wedge \mathrm{d}x^\mu\right) \otimes \mathrm{d}x^\nu\,. \tag{74}$$

Let's also consider the contraction with a 1-vector $\boldsymbol{u}$:

$$\begin{aligned}
D(\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u}) &= D(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \boldsymbol{u} - (\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \curlywedge \nabla\boldsymbol{u} \\
&\equiv D(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega} \\
&= d\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega} \ .
\end{aligned} \tag{75}$$

and in particular

$$D(\boldsymbol{\phi} \otimes dx^\alpha \cdot \boldsymbol{u}) = d\boldsymbol{\phi} \, u^\alpha - \boldsymbol{\phi} \wedge dx^\beta \, \partial_\beta u^\alpha \tag{76}$$

A balance equation with the exterior covariant derivative then becomes

$$D\boldsymbol{T} = 0 \qquad \Longrightarrow \qquad d(\boldsymbol{T} \cdot \boldsymbol{u}) = -\boldsymbol{T} \curlywedge \nabla\boldsymbol{u} \tag{77}$$

Then

$$\begin{aligned}
0 = D\boldsymbol{T} \\
= D\big(&-e\, d_t^3 \otimes dt - q^i\, d_i^3 \otimes dt + p_j\, d_t^3 \otimes dx^j + \pi_j^i\, d_i^3 \otimes dx^j\big) \\
= &-\partial_t e\, d^4 \otimes dt - \partial_i q^i\, d^4 \otimes dt + \partial_t p_j\, d^4 \otimes dx^j + \partial_i \pi_j^i\, d^4 \otimes dx^j - \\
&\Big[e\, \Gamma_{tt}^t\, d_t^3 \wedge dt \otimes dt + e\, \Gamma_{tj}^t\, d_t^3 \wedge dt \otimes dx^j + 0 \\
&q^i\, \Gamma_{it}^t\, d_i^3 \wedge dx^i \otimes dt + q^i\, \Gamma_{ij}^t\, d_i^3 \wedge dx^i \otimes dx^j + 0 \\
&- p_j\, \Gamma_{tt}^j\, d_t^3 \wedge dt \otimes dt - p_k\, \Gamma_{tj}^k\, d_t^3 \wedge dt \otimes dx^j + 0 \\
&- \pi_k^i\, \Gamma_{it}^k\, d_i^3 \wedge dx^i \otimes dt - \pi_k^i\, \Gamma_{ij}^k\, d_i^3 \wedge dx^i \otimes dx^j\Big] \\
= d^4 \otimes \Big[\\
&- \partial_t e\, dt + e\, \Gamma_{tt}^t\, dt + e\, \Gamma_{tj}^t\, dx^j \\
&- \partial_i q^i\, dt + q^i\, \Gamma_{it}^t\, dt + q^i\, \Gamma_{ij}^t\, dx^j \\
&+ \partial_t p_j\, dx^j - p_j\, \Gamma_{tt}^j\, dt - p_k\, \Gamma_{tj}^k\, dx^j \\
&+ \partial_i \pi_j^i\, dx^j - \pi_k^i\, \Gamma_{it}^k\, dt - \pi_k^i\, \Gamma_{ij}^k\, dx^j \\
\Big] &
\end{aligned} \tag{78}$$

The equation above corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e\, \Gamma_{tt}^t + q^i\, \Gamma_{it}^t - p_j\, \Gamma_{tt}^j - \pi_k^i\, \Gamma_{it}^k \tag{79}$$

$$\partial_t p_j + \partial_i \pi_j^i = -e\, \Gamma_{tj}^t - q^i\, \Gamma_{ij}^t + p_k\, \Gamma_{tj}^k + \pi_k^i\, \Gamma_{ij}^k \tag{80}$$

For $\boldsymbol{T} \cdot \boldsymbol{u}$ we find first

$$\boldsymbol{T} \cdot \boldsymbol{u} = (-e\,u^t + p_j\,u^j)\,\mathrm{d}_t^3 + (-q^i\,u^t + \pi_j^i\,u^j)\,\mathrm{d}_i^3 \tag{81}$$

$$\boldsymbol{T} \wedge \nabla \boldsymbol{u} = (-e\,\partial_t u^t + p_j\,\partial_t u^j)\,\mathrm{d}_t^3 \wedge \mathrm{d}t + (-q^i\,\partial_i u^t + \pi_j^i\,\partial_i u^j)\,\mathrm{d}_i^3 \wedge \mathrm{d}x^i$$
$$+ \Gamma \text{ terms} \tag{82}$$

and therefore

$$\partial_t(-e\,u^t + p_j\,u^j) + \partial_i(-q^i\,u^t + \pi_j^i\,u^j) =$$
$$- e\,\partial_t u^t + p_j\,\partial_t u^j - q^i\,\partial_i u^t + \pi_j^i\,\partial_i u^j$$
$$- (e\,\Gamma_{tt}^t + q^i\,\Gamma_{it}^t - p_j\,\Gamma_{tt}^j - \pi_k^i\,\Gamma_{it}^k)\,u^t$$
$$- (e\,\Gamma_{tj}^t + q^i\,\Gamma_{ij}^t - p_k\,\Gamma_{tj}^k - \pi_k^i\,\Gamma_{ij}^k)\,u^j \tag{83}$$

Radial case

$$\partial_t e + \partial_r q^r = e\,\Gamma_{tt}^t + q^r\,\Gamma_{rt}^t + p_r\,\Gamma_{tt}^r + \pi_r^r\,\Gamma_{rt}^r \tag{84}$$

$$\partial_t p_r + \partial_r \pi_r^r = e\,\Gamma_{tr}^t + q^r\,\Gamma_{rr}^t + p_r\,\Gamma_{tr}^r + \pi_r^r\,\Gamma_{rr}^r \tag{85}$$

$$\partial_t e + \partial_r q^r = q^r\,\frac{g}{c^2} + p_r\,g \tag{86}$$

$$\partial_t p_r + \partial_r \pi_r^r = e\,\frac{g}{c^2} \tag{87}$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have[4]

$$\Gamma_{jt}^t = \Gamma_{tj}^t = \frac{GM}{c^2}\frac{x^j}{r^3} \approx \frac{g}{c^2} \qquad \Gamma_{tt}^j = GM\frac{x^j}{r^3} \approx g \tag{88}$$

where g is the standard acceleration, considered positive. Take also $p_j \approx m v_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

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[4] Poisson & Will 2014 § 5.2.3.

The balances above become

$$\partial_t e + \partial_i q^i = q^z \frac{g}{c^2} + p_z\, g \tag{89}$$

$$\partial_t p_z + \partial_i \pi_z^i = e\, \frac{g}{c^2} \tag{90}$$

Also,

$$\Gamma_{tt}^t \approx -2\frac{g}{c^2} v(t) \qquad \Gamma_{jt}^t = \Gamma_{tj}^t \approx \frac{g}{c^2}$$

$$\Gamma_{tt}^j \approx g - 2\frac{g}{c^2}\, v(t)^2 - \dot{v}(t) \qquad \Gamma_{jt}^j = \Gamma_{tj}^j \approx \frac{g}{c^2} v(t) \tag{91}$$

$$\partial_t e + \partial_i q^i = -e\, 2\frac{g}{c^2} v(t) + q^z\, \frac{g}{c^2} + p_j\left(g - 2\frac{g}{c^2}\, v(t)^2 - \dot{v}(t)\right) + \pi_z^z \frac{g}{c^2} v(t) \tag{92}$$

$$\partial_t p_z + \partial_i \pi_z^i = e\, \frac{g}{c^2} + p_z\, \frac{g}{c^2} v(t) \tag{93}$$

C Works with useful content

- Eq. (21) in[5]: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in[6]

For transformation or raising:[7]

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B} \gamma \cdot (A \wedge B) \tag{94}$$

with $\deg(B) = n - \deg(A)$[8]

Compound matrices:[9]

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("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

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