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Notes on energy and momentum

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1 Questions about momentum and energy

The notions of momentum and energy in Newtonian physics present many peculiarities, at least to the eye of a student who's learning about them. Consider for instance the following questions, some of which have indeed been asked by students, including me:

(a) How should *total energy* be defined? Is it just the sum of internal and kinetic energies? or should it also include gravitational potential energy?

(b) Why is *internal energy* the same in two reference frames, whereas *kinetic energy* differs?

(c) Why does force, which is related to change of momentum, appear in the laws for the change of energy (in the formula for "work")? Why doesn't change of energy appear, vice versa, in the law for change of momentum?

(d) What is the *law of transformation for momentum* between reference frames? For instance, if we know that the components of momentum in an inertial frame are $[0, 1, 2]\,\mathrm{N\,s}$, then what are the components in another inertial frame with constant velocity $\boldsymbol{v}$ with respect to the first?

Tentative answers to some of these questions may lead to embarrassing further questions.

Take question (a) for example. A possible answer is that the definition of "total energy" is arbitrary. For an object near Earth's surface, we may avoid speaking of gravitational potential energy if we include the work done by gravitational forces in the balance of energy; or vice versa we may include a gravitational potential energy in the total energy, avoiding the inclusion of work by gravitational forces in the energy balance. To this explanation the student may ask why we have such definition freedom for energy, but not an analogous one for momentum.

Or take question (d). A possible answer is that in order to know the momentum in another frame we need to know both the mass and the velocity of an object. At this answer the student may have the following questions: What in the case of an electromagnetic field, where there's no mass or velocity? And why do we need such extra information for the transformation of momentum, when we don't need extra information for the transformation of velocity or of mass?

There are different perspectives from which one can try to answer questions like these. One can take a historical, rather than physical, point of view. Or one may say that it's just a matter of definitions. The literature also offers more physical answers for some of these questions, based for instance on symmetry, or on variational principles, or on the Newton-Cartan theory. As an example, some work of Šilhavý[1], derives the notions of mass, momentum, and kinetic energy from Galileian invariance.

Here we show how these questions receive answers from the point of view of general relativity.

2 Energy-momentum tensor

Energy, momentum, flux of energy, and force, or more generally flux of momentum, are all components of a single object, the *energy-momentum tensor*, also called 'energy-momentum-stress tensor', with permutations, or 'four-stress'. It can be represented by a 4-by-4 matrix. Roughly speaking they can be identified as the following entries in the matrix:

$$\begin{bmatrix} \text{energy} & \text{momentum} \\ \text{energy flux} & \begin{matrix}\text{force} \\ \text{(momentum flux)}\end{matrix} \end{bmatrix} . \tag{1}$$

The "roughly speaking" will be made more precise later.

[1] Šilhavý 1989; 1992; see also Serrin 1998.

The fact that energy, momentum, and their fluxes are components of a simple object, already gives a partial answer to question (c). We know already in the simpler examples of vectors that their components get mixed in a change of frame. This makes it plausible that forces, for instance, should appear in the energy flux upon a change of frame. Also question (d) starts to receive an answer. Generally all components of an object are needed to find the new ones in another frame. The three components of momentum are not really a vector by themselves, but are partial components of something larger; this is why they are not enough, by themselves, for finding the momentum components in a new frame. This also hints at the fact that *mass* or something related to it should be one of the additional components.

There are several ways to represent the energy-momentum tensor

Bibliography

(“de *X*” is listed under D, “van *X*” under V, and so on, regardless of national conventions.)

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