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Kinematics, dynamics, force, inertia, metric in Newtonian relativity and in general relativity [draft]

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This note is an exploration of some notions around kinematics, dynamics, matter, inertia, and force in Newtonian- and general-relativistic continuum mechanics. The conceptual pivots of this exploration are these: (i) kinematics should not allow us to distinguish between Newtonian relativity and general relativity; (ii) in neither theory should there be any distinction between inertial and non-inertial motion, or between forced and unforced motion.
Note: Dear Reader & Peer, this manuscript is being peer-reviewed by you. Thank you.

This presentation is mainly discursive. For formulae see Porta Mana (2016).

1 Kinematics

1.1 Spacetime and matter

Our first primitive is the notion of spacetime, as the seats of all events. It's represented by a 4D differential manifold. In the following I misuse 'topological' as a shorthand for 'differential-geometric', to denote objects that only require a differential structure – so no connections, metrics, volume elements, or similar – for their definition. In spacetime there's no predefined notion of simultaneity nor of 'identity through time'.

Our second primitive is the notion of matter, of bodies. How to represent it? Matter is characterized by our ability to mark a small part of it – as when we draw a '○' sign on an body, or throw some leaves in flowing water – and then to follow it as it moves. It relates to the notions of identity and individuality. Two topological objects can represent these notions: an inner-oriented vector field or an outer-oriented 3-covector field[1] (or '3-form'). The first can be depicted as a collection of 1D curves, the second as a collection of tubes with 3D cross-sections[2]. Either has the property that, given one event in spacetime, a unique 1D curve through

[1] Schouten 1989. [2] **misneretal1970_r2003**; Schouten 1989.

it is selected. Either is in turn determined by such curves but only up to a multiplicative scalar function.

For the vector field, different functions determine different vectors tangent to the curves; they could represent a 4-velocity. But we know from Lorentzian relativity that such freedom is irrelevant: 4-velocities are by convention normalized to unity. For the 3-covector field, on the other hand, different scalar functions determine different cross-sections of the tubes.

The latter fact is connected with an additional feature of such a field: it can be integrated over an outer-oriented 3D surface (count the 3D tubes which cross that surface). Later on, such a surface could be characterized as spacelike or timelike. This 3-covector field thus allows us to speak of the 'quantity' of matter within a 3D region at some time, or the quantity of matter that has flowed through a 2D surface during a given time. It is not possible to decompose this flux into a 'density' and a 'velocity', though.

If such a field is closed (its exterior derivative, which is a topological operation, vanishes), then the matter it represents satisfies a conservation law: the total quantity of matter flowing through a closed 2D surface in a given time must be equal to the increase of matter within that surface during the same time. This property is useful in the theory of mixtures[3], where we model different kinds of matter that can transform into each other, their total amount being conserved. This situation can be represented by several non-closed 3-covector fields, their sum being closed. In general relativity, conservation of matter is more precisely conservation of baryonic-number[4]. In Newtonian relativity traditionally it's conservation of mass, but it's easy to show that this theory could be formulated with baryon conservation instead; a similar reformulation in fact takes place within the theory of mixtures mentioned above.

So we represent matter as an outer-oriented 3-covector field, called the 'matter field'.

A spacetime and one or more matter fields give us a topological kinematics. Such kinematics doesn't allow us to say that some parts of matter are at rest or in motion, or approaching or diverging. But it allows us to follow their trajectories through spacetime. It is remarkable that the conservation of matter turns out to be a kinematic and topological

[3] Samohýl 1975; Bowen 1976; Samohýl et al. 2014; Faria 2003. [4] Eckart 1940; Misner et al. 1973 § 22.2; Gourgoulhon 2012; Alcubierre 2008.

property. It is also remarkable that this is a true 'field' (an association with a geometric object to each point in spacetime) and yet it represents particles and their trajectories. This shows that there is no dualism between field and matter.

1.2 Simultaneity, rest, frames

The intuitive notion of simultaneity comes to us mainly from our sense of vision; our intuition would probably be different if it came from sound. I take the point of view, suggested by general relativity, that there is no natural spacetime-extended definition of simultaneity. We can make an arbitrary foliation of spacetime into 3D surfaces, which we can call 'instants', and even label each instant with a number, but this construction and labels need not have any physical meaning.

Even when instants are introduced, spacetime doesn't offer any canonical isomorphism among them. That is, there's no way of saying 'this event at this instant and that event at that instant occur at the *same* place'. It is possible to establish an arbitrary isomorphism among instants through a vector field not tangent to any of them. This vector field can also be chosen to have unit product with the 1-covector associated with the foliation. This construction needs not have any physical meaning. This vector field allows us to compare field quantities at different instants via the Lie derivative, which therefore acts as a 'time derivative at fixed position'.

The combination of a foliation and a vector field as described above is called a *frame* or an *observer*[5]. Once a frame is introduced we can speak of motion or rest of matter with respect to that frame, and even define a velocity and a space-density of matter[6]. But again, these are arbitrary constructions. A frame is a scaffolding that allows us to formulate kinematics in terms of 'rest' and topological versions of 'velocity' and 'density'. But it is clear that it doesn't add any physical feature. The rest of the development of the theory can be done without it. Frames may be useful for our intuition and to set up initial-value problems in numerical relativity.

It is possible to choose frames with particular physical meaning. For example we can choose the vector field of the frame to be tangent to the tubes of a matter field. That matter will then be at rest in that frame,

[5] Taub 1978; Smarr et al. 1978; York 1979; Smarr et al. 1980; Hehl et al. 2003 § B.1.4.
[6] Porta Mana 2016 § 3.2.

by decree. This frame is called *Lagrangean* or *comoving*[7]. Other choices become possible when a metric or a connection or a 4-stress tensor are introduced, and we have *Eulerian* frames, *inertial* or *free-fall* frames, and several others.

2 Metric

The kinematics just discussed is purely topological, even if we can introduce arbitrary notions of time-lapse and rest into it. In Newtonian relativity, kinematics traditionally includes actual chronometric and distance notions; that is, it includes a metric as part of its structure. This is possible because the metric is given and 'inert'.

[7] Taub 1978; Smarr et al. 1978; York 1979; Smarr et al. 1980.

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('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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