Dimensional analysis in general relativity and in differential geometry

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This note illustrates how to perform dimensional analysis in general relativity and differential geometry, and tries to revive Dorgelo & Schouten's notion of *absolute dimension* of a tensorial quantity. The absolute dimension is independent of the dimensions of the coordinate functions. The dimensional analysis of several important tensors and tensor operations is summarized. In particular it is shown that the components of a tensor need not have all the same dimension, and that the Riemann (once contravariant and trice covariant) and Ricci (fully covariant) tensors are dimensionless. The relation between dimension and operational meaning for the metric and stress-energy-momentum tensors is also discussed.

for Emma

1 Introduction

From the point of view of dimensional analysis, do all components of a tensor need to have the same dimension? What are the dimensions of the metric and of the curvature tensor? And what is the dimension of the constant in the Einstein equations?

Many students in relativity, but also some researchers, seem to be insecure when they have to discuss the dimensions of tensors and of tensor components, the effect of tensor operators on dimensions, and the dimensions of constants in differential-geometric and field equations. Some incorrect or unfounded notions about these topics are in circulation. For example, the notion that the components of a tensor should all have the same dimension. This is not true. Or the assumption that all coordinates have dimension of length. This is unnecessary. These points will be clarified in the next sections.

Several factors contribute to these insecurities and misconceptions. Modern texts in Lorentzian and general relativity commonly use geometrized units. They say that, for finding the dimension of some constant in a tensorial equation, it's sufficient to compare the dimensions of the tensors in the equation. But this is not so immediate, because some tensors don't have universally agreed dimensions – prime example the metric tensor. Older texts often use four coordinates with dimension of length, and base their dimensional analysis on that specific choice. They even multiply[1] coordinates or tensorial components having dimension of time by powers of c. A student thus gets the impression that coordinates ought to always be lengths, and that all components of a tensor ought to have the same dimension.

One purpose of the present note is to provide a short guide to the application of the usual rules of dimensional analysis in differential geometry. The usual rules are that if two quantities are summed, then they must have the same dimension; that the dimension of a product is the product of the dimensions; and so on. Any dimensional-analysis question in general relativity can thus be quickly and consistently settled; for example the dimension of the Riemann curvature tensor, or the dimensional results of contraction, of covariant derivative, of raising an index. A simple two-dimensional example is given in § 4. Sections 6–8 give a more general and systematic discussion and a synopsis of the main tensorial operations. It is shown, in particular, that *the components of a tensor need not have the same dimensions.* The dimension of the constant in the Einstein equations is derived in § 12.

The application of dimensional analysis in differential geometry pivots on the somewhat forgotten notion of *intrinsic dimension* of a tensor. The present note tries to revive this notion, introduced under the name of 'absolute dimension' by Schouten and Dorgelo[2] and used in Truesdell & Toupin[3]. The intrinsic or absolute dimension is an intrinsic property of a tensor, invariant with respect to the choice and dimensions of coordinates. It is distinct from the dimensions of the tensor's *components*, which instead depend on the dimensions of the coordinate functions. The intrinsic dimension should thus be the primary focus of dimensional analysis in general relativity and in differential geometry. This notion is explained in § 6. The intrinsic dimensions of the various curvature tensors, of the metric tensor, and of the stress-energy-momentum tensor are discussed in §§ 9–11, owing to their importance in general relativity. The (contravariant and thrice covariant) Riemann and (fully covariant) Ricci tensors are found to have intrinsic dimension 1, that is, to be dimensionless. Two standard choices for the intrinsic dimension of the metric tensor are discussed.

1 e.g. Tolman 1949 p. 72 eq. (37.8); Landau & Lifshitz 1996 p. 80 eq. (32.15); Adler et al. 1975 p. 332 eq. (10.15). **2** Dorgelo & Schouten 1946; Schouten 1989 ch. VI. **3** Truesdell & Toupin 1960 Appendix II.

Dimensional reasoning in differential geometry is clearest when we rely on the coordinate-free, intrinsic view of tensors and other differential-geometrical objects. A brief reminder of this view is given in § 3, with references. Special notation related to it is first explained in § 2.

This note assumes familiarity with basic tensor calculus and related notions, for example of co- and contra-variance, tensor product, contraction. Some passages assume familiarity with the exterior calculus of differential forms. Above all, familiarity with the intrinsic presentation of differential geometry mentioned above is assumed; but the general ideas should be understandable even without such familiarity.

Finally, quoting Truesdell & Toupin[4], "dimensional analysis remains a controversial and somewhat obscure subject. We do not attempt a complete presentation here." Some references about recent developments in this subject are given in the final § 13.

2 Notation

For the notation in dimensional analysis I use ISO conventions: $\dim(\boldsymbol{A})$ is the dimension of the quantity $\boldsymbol{A}$, and among the base quantities are mass M, length L, time T, temperature Θ, electric current I. Note that I don't discuss *units* – it doesn't matter here whether the unit for length is the metre or the centimetre, for example.

The number and ordering of a tensor's covariant and contravariant "slots"[5] will often be very important in our discussion. The traditional coordinate-free notation '$\boldsymbol{A}$' unfortunately omits this information. We thus need a coordinate-free notation that makes them explicit. Penrose & Rindler[6] propose an abstract-index notation, where '$A_i{}^{jk}$', for example, denotes a tensor covariant in its first slot and contravariant in its second and third slots. Every index in this notation is "simply as a label whose sole purpose is to keep track of the type of tensor under discussion"[7], so this notation doesn't stand for the set of *components* of the tensor. For the latter set, **bold** indices are used instead: '$A_{\boldsymbol{i}}{}^{\boldsymbol{jk}}$'. The difference between a tensor and its set of components is crucial to our discussion, however, and the abstract-index notation lends itself to conceptual and typographic misunderstanding.

I'll therefore use a notation such as '$\boldsymbol{A}_{\bullet}{}^{\bullet\bullet}$' to indicate that $\boldsymbol{A}$ is covariant in its first slot and contravariant in its second and third slots.

[4] Truesdell & Toupin 1960 Appendix § 7 footnote 4.
[5] Misner et al. 1973 § 3.2.
[6] Penrose & Rindler 2003 § 2.2.
[7] Penrose & Rindler 2003 p. 75.

Its components would thus be $(A_i{}^{jk})$. For brevity I'll call this a 'co-contra-contravariant' tensor, with an obvious naming generalization for other types.

The only weak points of this notation are the operations of contraction and transposition, which literal indices depict so well. Considering that contraction is a generalization of trace, and transposition a generalization of matrix transposition, I'll use the following notation:

- '$\operatorname{tr}_{\alpha\beta} \boldsymbol{A}$' is the contraction of the αth and βth slots (which must have opposite variant types)
- '$\boldsymbol{A}^{\intercal\alpha\beta}$' is the transposition (swapping) of the αth and βth slots.

In index notation these operations are the familiar

$$A_{\dots}{}^{\overset{\alpha\text{th slot}}{i}}{}_{\dots\underset{\beta\text{th slot}}{i}\dots} \qquad \text{and} \qquad A_{\dots}{}^{\overset{\alpha\text{th slot}}{i}}{}_{\dots\underset{\beta\text{th slot}}{j}\dots} \mapsto A_{\dots\underset{\alpha\text{th slot}}{j}\dots}{}^{\overset{\beta\text{th slot}}{i}}{}_{\dots}\;.$$

We'll consider contraction and transposition only sparsely, so I hope this notation won't be too uncomfortable.

3 Intrinsic view of differential-geometric objects: brief reminder

From the intrinsic point of view, a tensor is defined by its geometric properties. For example, a vector field $\boldsymbol{v}$ is an object that operates on functions defined on the (spacetime) manifold, yielding new functions, with the properties $\boldsymbol{v}(af+bg) = a\boldsymbol{v}(f)+b\boldsymbol{v}(g)$ and $\boldsymbol{v}(fg) = \boldsymbol{v}(f)g+f\boldsymbol{v}(g)$ for all functions f, g and reals a, b. A covector field (1-form) $\boldsymbol{\omega}$ is an object that operates on vector fields, yielding functions ('duality'), with the property $\boldsymbol{\omega}(f\boldsymbol{u}+g\boldsymbol{v}) = f\boldsymbol{\omega}(\boldsymbol{u})+g\boldsymbol{\omega}(\boldsymbol{v})$ for all vector fields $\boldsymbol{u}$, $\boldsymbol{v}$ and functions f, g. The sum of vector or covector fields, and their products by functions – let's call this 'linearity' – are defined in an obvious way. Tensors are constructed from these objects.

A system of coordinates (x^i) is just a set of linearly independent functions. This set gives rise to a set of vectors fields $\bigl(\frac{\partial}{\partial x^i}\bigr)$ and to a set of covector fields $(\mathrm{d}x^i)$ by the obvious requirements that $\frac{\partial}{\partial x^i}(x^j) = \delta_i{}^j$ and $\mathrm{d}x^i\bigl(\frac{\partial}{\partial x^j}\bigr) = \delta^i{}_j$. These two sets can be used as bases to express all other vectors and covectors as linear combinations. A vector field $\boldsymbol{v}$ can thus be written as

$$\boldsymbol{v} \equiv \sum_i v^i \frac{\partial}{\partial x^i} \equiv v^i \frac{\partial}{\partial x^i}\;, \tag{1}$$

where the *functions* $v^i := \boldsymbol{v}(x^i)$ are its components with respect to the basis $\bigl(\frac{\partial}{\partial x^i}\bigr)$. Analogously for a covector field.

For the presentation of the intrinsic view I recommend the excellent texts by Choquet-Bruhat et al. (1996), Boothby (2003), Abraham et al. (1988), Bossavit (1991), Burke (1987; 1980 ch. 2), and more on the general-relativity side Misner et al. (1973 ch. 9), Gourgoulhon (2012 ch. 2), Penrose & Rindler (2003 ch. 4).

4 An introductory two-dimensional example

Let me first present a simple example of dimensional analysis in a two-dimensional spacetime. I provide very little explanation, letting the analysis speaks for itself. The next sections will give a longer discussion of the general point of view, of the assumptions, and of cases with more complex geometrical objects.

In a region of a two-dimensional spacetime we use coordinates (x, y). These coordinates allow us to uniquely identify every event in the region (otherwise they wouldn't be coordinates). The coordinate x has dimension of temperature, and y of energy density: use entropy instead and cite Maugin, Eckart, and co.

$$\dim(x) = \Theta , \quad \dim(y) = \mathsf{E} := \mathsf{M}\mathsf{L}^{-2}\mathsf{T}^{-2} . \tag{2}$$

This may happen, for example, if the region is occupied by an electromagnetic material; its temperature is uniform over the slices of a specific spacelike foliation, but increasing across the slices; its internal energy is instead increasing across the 1-dimensional slice.

From these coordinates we construct two covector fields $(\mathrm{d}x, \mathrm{d}y)$, and two vector fields $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ that serve as bases for the spaces of tangent covectors, vectors, and tensors. Their dimensions are

$$\begin{aligned} \dim(\mathrm{d}x) &= \Theta & \qquad \dim(\mathrm{d}y) &= \mathsf{E} , \\ \dim\left(\frac{\partial}{\partial x}\right) &= \Theta^{-1} & \qquad \dim\left(\frac{\partial}{\partial y}\right) &= \mathsf{E}^{-1} . \end{aligned} \tag{3}$$

Consider a contra-co-variant tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$ in this region. Using the basis fields above it can be written as

$$\boldsymbol{A} = A^{x}{}_{x} \frac{\partial}{\partial x} \otimes \mathrm{d}x + A^{x}{}_{y} \frac{\partial}{\partial x} \otimes \mathrm{d}y + A^{y}{}_{x} \frac{\partial}{\partial y} \otimes \mathrm{d}x + A^{y}{}_{y} \frac{\partial}{\partial y} \otimes \mathrm{d}y , \tag{4}$$

where $A^{x}{}_{x} := \boldsymbol{A}\left(\mathrm{d}x, \frac{\partial}{\partial x}\right), \dots$ are the components of the tensor in the coordinate system (x, y).

The two sides of the expansion above must have the same dimension. The four summands of the right side must also have the same dimension. Denoting $\mathsf{A} := \dim(\boldsymbol{A})$, we have the four equations

$$\begin{aligned} \mathsf{A} &= \dim({A^x}_x) & \qquad \mathsf{A} &= \dim({A^x}_y)\,\Theta\,\mathsf{E}^{-1} \\ \mathsf{A} &= \dim({A^y}_x)\,\Theta^{-1}\,\mathsf{E} & \qquad \mathsf{A} &= \dim({A^y}_y)\ , \end{aligned}$$

or

$$\begin{aligned} \dim({A^x}_x) &= \mathsf{A} & \qquad \dim({A^x}_y) &= \mathsf{A}\,\Theta^{-1}\,\mathsf{E} \\ \dim({A^y}_x) &= \mathsf{A}\,\Theta\,\mathsf{E}^{-1} & \qquad \dim({A^y}_y) &= \mathsf{A}\ . \end{aligned} \tag{5}$$

The intrinsic dimension of the tensor $\boldsymbol{A}$ is A. The expansion (4) shows that this dimension is independent of the coordinate system, because the expansion could be done for any other coordinates but the left side would always be the same. The intrinsic dimension is determined by the physical or operational meaning of the tensor. Together with the dimensions of the coordinates it determines the dimensions of the components, eq. (5), which need not be all equal.

This simple example should have disclosed the main points of dimensional analysis on manifolds, which will now be discussed in more generality.

Note that in the example we silently adopted a couple of conventions; for example, that the tensor product behaves similarly to multiplication with regard to dimensions. Such conventions are briefly discussed in § 13.

5 Coordinates

From a physical point of view, a coordinate is just a function that associates one value of a physical quantity with every event in a region (the domain of the coordinate chart) of spacetime. Together with the other coordinates, such function allows us to uniquely identify every event within that region. Any physical quantity will do: the distance from something, the time elapsed since something, an angle, an energy density, the strength of a magnetic flux, a temperature, and so on. A coordinate can thus have any dimension: length L, time T, angle $\mathsf{1}$, temperature Θ, magnetic flux $\Phi := \mathsf{M}\mathsf{L}^2\mathsf{T}^{-2}\mathsf{I}^{-1}$, and so on.

The functional relation between two sets of coordinates must of course be dimensionally consistent. For example, if $\dim(x^0) = \mathsf{T}$ and $\dim(x^1) = \mathsf{L}$, and we introduce a coordinate $\xi(x^0, x^1)$ with dimension $\mathsf{1}$, additive in the previous two, then we must have $\xi = ax^0 + bx^1$ with $\dim(a) = \mathsf{T}^{-1}$ and $\dim(b) = \mathsf{L}^{-1}$.

6 Tensors: intrinsic dimension and components' dimensions

Consider a system of coordinates (x^i) with dimensions (X_i), and the ensuing sets of covector fields (1-forms) $\mathrm{d}x^i$ and of vector fields $\left(\frac{\partial}{\partial x^i}\right)$, bases for the cotangent and tangent spaces. Their tensor products are bases for the tangent spaces of higher tensor types.

The differential $\mathrm{d}x^i$ traditionally has the same dimension as x^i: $\dim(\mathrm{d}x^i) = \mathsf{X}_i$, and the vector $\frac{\partial}{\partial x^i}$ traditionally has the inverse dimension: $\dim \frac{\partial}{\partial x^i} = {\mathsf{X}_i}^{-1}$.

For our discussion let's take a concrete example: a contra-covariant tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$. The discussion generalizes to tensors of other types in an obvious way.

The tensor $\boldsymbol{A}$ can be expanded in terms of the basis vectors and covectors, as in § 3 and in the example of § 4:

$$\boldsymbol{A} = A^i{}_j \, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \equiv A^0{}_0 \, \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^0 + A^0{}_1 \, \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^1 + \cdots. \tag{6}$$

Each function

$$A^i{}_j \coloneqq \boldsymbol{A}\Bigl(\mathrm{d}x^i, \frac{\partial}{\partial x^j}\Bigr) \tag{7}$$

is a component of the tensor in this coordinate system.

To make dimensional sense, all terms in the sum (6) must have the same dimension. This is possible only if the generic component $A^i{}_j$ has dimension

$$\dim\bigl(A^i{}_j\bigr) = \mathsf{A}\,\mathsf{X}_i\,{\mathsf{X}_j}^{-1}, \tag{8}$$

where A is common to all components. In fact, the $\mathsf{X}_i\,{\mathsf{X}_j}^{-1}$ term cancels the ${\mathsf{X}_i}^{-1}\,\mathsf{X}_j$ term coming from $\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ in the sum (6), and each summand therefore has dimension A.

The generalization of the formula above to tensors of other types is obvious:

$$\dim\bigl(A^{ij\dots}{}_{kl\dots}\bigr) = \mathsf{A}\,\mathsf{X}_i\,\mathsf{X}_j \cdots {\mathsf{X}_k}^{-1}\,{\mathsf{X}_l}^{-1} \cdots, \quad \text{where the ordering of the indices doesn't matter.} \tag{9}$$

Clearly the components can have different dimensions. But this doesn't matter. What matters is that the sum (6) be dimensionally consistent. (Fokker[8], for example, uses a metric tensor with components having different dimensions.)

[8] Fokker 1965 § VII.1 p. 88.

The dimension A, which is also the dimension of the sum (6), I'll call the *intrinsic dimension* of the tensor $\boldsymbol{A}$, and we write

$$\dim(\boldsymbol{A}) = \mathsf{A}. \tag{10}$$

This dimension is independent of any coordinate system. It reflects the physical or operational[9] meaning of the tensor. We'll see an example of what this mean in § 10 with the metric tensor.

This notion was introduced by Dorgelo and Schouten[10] under the name ‘absolute dimension’. I find the adjective ‘intrinsic’ more congruous to modern terminology (and less prone to suggest spurious connections with absolute values).

Different coordinate systems lead to different dimensions of the *components* of a tensor $\boldsymbol{A}$, but the absolute dimension of the tensor remains the same. Formula (9) for the dimensions of the components is consistent under changes of coordinates. For example, in new coordinates $(\bar{x}^k)$ with dimensions $(\bar{\mathsf{X}}_k)$, the new components of $\boldsymbol{A}$ are

$$\bar{A}^k{}_l = A^i{}_j \, \frac{\partial \bar{x}^k}{\partial x^i} \, \frac{\partial x^j}{\partial \bar{x}^l} \tag{11}$$

and a quick check shows that $\dim(\bar{A}^k{}_l) = \mathsf{A}\,\bar{\mathsf{X}}_k\,\bar{\mathsf{X}}_l^{-1}$, consistently with the general formula (9).

In the following I'll drop the adjective ‘intrinsic’ when it's clear from the context.

7 Tensor operations

By the reasoning of the previous section, which simply applies standard dimensional considerations to the basis expansion (6), it's easy to find the resulting intrinsic dimension of various operations and operators on tensors and tensor fields.

Here is a summary of the dimensional rules for the main differential-geometric operations and operators, except for the covariant derivative, the metric, and related tensors, discussed more in depth in §§ 9–10 below. Some of these rules are actually definition or conventions, as briefly discussed in their description. The others can be proved; I only give a proof for one of them, leaving the other proofs as an exercise. For reference, in brackets I give the section of Choquet-Bruhat et al. (1996) where these operations are defined.

9 Bridgman 1958; see also Synge 1960a § A.2; Truesdell & Toupin 1960 §§ A.3–4.
10 Dorgelo & Schouten 1946; Schouten 1989 ch. VI.

• The *tensor product* [III.B.5] multiplies dimensions:

$$\dim(\boldsymbol{A} \otimes \boldsymbol{B}) = \dim(\boldsymbol{A}) \dim(\boldsymbol{B}). \tag{12}$$

This is actually a definition or convention. We tacitly used this rule already in the example of § 4 and in § 6 for the coordinate expansion (6). It is a natural definition, because for tensors of order 0 (functions) the tensor product is just the ordinary product, and the dimension of a product is the product of the dimensions. This definition doesn't lead to inconsistencies.

• The *contraction* [III.B.5] or trace of the αth and βth slots of a tensor has the same dimension as the tensor:

$$\dim(\operatorname{tr}_{\alpha\beta} \boldsymbol{A}) = \dim(\boldsymbol{A}). \tag{13}$$

Note that the formula above only holds *without raising or lowering indices*; see § 10 for those operations.

This operation can be traced back to the duality of vectors and covectors mentioned in § 3: a covector field $\boldsymbol{\omega}$ operates linearly on a vector field $\boldsymbol{v}$ to yield a function $f = \boldsymbol{\omega}(\boldsymbol{v})$. Also in this case we have that $\dim(f) = \dim(\boldsymbol{\omega}) \dim(\boldsymbol{v})$ by definition or convention, and the rule (13) follows from this convention. Also in this case this convention seems very natural, owing to the linearity properties of the trace, and doesn't lead to inconsistencies.

• The *transposition*[11] of the αth and βth slots of a tensor has the same dimension as the tensor:

$$\dim(\boldsymbol{A}^{\mathsf{T}_{\alpha\beta}}) = \dim(\boldsymbol{A}). \tag{14}$$

• The *Lie bracket* [III.B.3] of two vectors has the product of their dimensions:

$$\dim([\boldsymbol{u}, \boldsymbol{v}]) = \dim(\boldsymbol{u}) \dim(\boldsymbol{v}). \tag{15}$$

In fact, in coordinates (x^i) the bracket can be expressed as

$$[\boldsymbol{u}, \boldsymbol{v}] = \left(u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}, \tag{16}$$

and equating the dimensions of the left and right sides, considering that

$$\dim(u^i) = \dim(\boldsymbol{u})\,\mathsf{X}_i, \quad \dim(v^i) = \dim(\boldsymbol{v})\,\mathsf{X}_i, \tag{17}$$

we find again that all X terms cancel out, leaving the result (15).

11 called "building an isomer" by Schouten 1954 § I.3 p. 13; 1989 § II.4 p. 20.

• The *Lie derivative* [III.C.2] of a tensor with respect to a vector field has the product of the dimensions of the tensor and of the vector:

$$\dim(\mathrm{L}_{\boldsymbol{v}} \boldsymbol{A}) = \dim(\boldsymbol{v}) \dim(\boldsymbol{A}). \tag{18}$$

Regarding operations and operators on differential forms:

• The *exterior product* [IV.A.1] of two differential forms multiplies their dimensions:

$$\dim(\boldsymbol{\omega} \wedge \boldsymbol{\tau}) = \dim(\boldsymbol{\omega}) \dim(\boldsymbol{\tau}). \tag{19}$$

• The *interior product* [IV.A.4] of a vector and a form multiplies their dimensions:

$$\dim(\mathrm{i}_{\boldsymbol{v}} \boldsymbol{\omega}) = \dim(\boldsymbol{v}) \dim(\boldsymbol{\omega}). \tag{20}$$

• The *exterior derivative* [IV.A.2] of a form has the same dimension of the form:

$$\dim(\mathrm{d}\boldsymbol{\omega}) = \dim(\boldsymbol{\omega}). \tag{21}$$

This can be proven using the identity $\mathrm{d}\,\mathrm{i}_{\boldsymbol{v}} + \mathrm{i}_{\boldsymbol{v}}\,\mathrm{d} = \mathrm{L}_{\boldsymbol{v}}$ or similar identities[12] together with eqs (18) and (20).

• The *integral* [IV.B.1] of a form over a submanifold (or chain) M has the same dimension as the form:

$$\dim\bigl(\textstyle\int_M \boldsymbol{\omega}\bigr) = \dim(\boldsymbol{\omega}). \tag{22}$$

The reason is that the integral of a form over a submanifold or chain is ultimately based on the standard definition of integration on the real line[13], and the latter satisfies the dimensional rule above (possibly by convention).

note on inner-oriented forms

8 Curves and integral curves

Consider a curve into spacetime, $C\colon s \mapsto P(s)$, with the parameter s having dimension $\dim(s) = \mathsf{S}$.

[12] Curtis & Miller 1985 ch. 9 p. 180 Theorem 9.78; Abraham et al. 1988 § 6.4 Theorem 6.4.8. [13] e.g. Choquet-Bruhat et al. 1996 §§ IV.B.1–2; de Rham 1984 § 5 p. 21, § 6 p. 24.

If we consider the events of the spacetime manifold as dimensionless quantities, then the dimension of the tangent or velocity vector $\dot{C}$ to the curve is

$$\dim(\dot{C}) = \mathsf{S}^{-1}, \tag{23}$$

owing to the definition[14]

$$\dot{C} := \frac{\partial(x^i \circ C)}{\partial s}\,\frac{\partial}{\partial x^i}. \tag{24}$$

This has a quirky but interesting consequence. Given a vector field $\boldsymbol{v}$ we say that C is an integral curve for it if

$$\boldsymbol{v} = \dot{C} \tag{25}$$

at all events $C(s)$ in the image of the curve (or more precisely $\boldsymbol{v}_{C(s)} = \dot{C}_{C(s)}$ in usual differential-geometric notation[15]). From the point of view of dimensional analysis this definition can only be valid if $\boldsymbol{v}$ has dimension S^{-1}. If $\boldsymbol{v}$ and s^{-1} have different dimensions – a case which could happen for physical reasons – the condition (24) must be modified into $\boldsymbol{v} = k\dot{C}$, where k is a possibly dimensionful constant. This is equivalent to considering an affine and dimensional reparameterization of C.

9 Connection, covariant derivative, curvature tensors

Consider an arbitrary connection[16] with covariant derivative ∇. For the moment we don't assume the presence of any metric structure.

The covariant derivative of the product $f\boldsymbol{v}$ of a function and a vector satisfies[17]

$$\nabla(f\boldsymbol{v}) = \mathrm{d}f \otimes \boldsymbol{v} + f\nabla\boldsymbol{v}. \tag{26}$$

The first summand, from formulae (21) and (12), has dimension $\dim(f)\dim(\boldsymbol{v})$; for dimensional consistency this must also be the dimension of the second summand. Thus

$$\dim(\nabla\boldsymbol{v}) = \dim(\boldsymbol{v}). \tag{27}$$

It follows that the *directional* covariant derivative $\nabla_{\boldsymbol{u}}$ has dimension

$$\dim(\nabla_{\boldsymbol{u}}\boldsymbol{v}) = \dim(\boldsymbol{u})\dim(\boldsymbol{v}), \tag{28}$$

and by its derivation properties[18] we see that formula (27) extends from vectors to tensors of arbitrary type.

14 Choquet-Bruhat et al. 1996 § III.B.1; Boothby 2003 § IV.(1.9). **15** Choquet-Bruhat et al. 1996 § III.B.1. **16** Choquet-Bruhat et al. 1996 § V.B. **17** Choquet-Bruhat et al. 1996 § V.B.1. **18** Choquet-Bruhat et al. 1996 § V.B.1 p. 303.

In the coordinate system (x^i), the action of the covariant derivative is carried by the *connection coefficients* or Christoffel symbols $({\mathit{\Gamma}^i}_{jk})$ defined by

$$\nabla \frac{\partial}{\partial x^k} = {\mathit{\Gamma}^i}_{jk}\ \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}. \tag{29}$$

From this equation and the previous ones it follows that these coefficients have dimensions

$$\dim({\mathit{\Gamma}^i}_{jk}) = \mathsf{X}_i\, {\mathsf{X}_j}^{-1}\, {\mathsf{X}_k}^{-1}. \tag{30}$$

The *torsion* ${\boldsymbol{\tau}^{\bullet}}_{\bullet\bullet}$, *Riemann curvature* ${\boldsymbol{R}^{\bullet}}_{\bullet\bullet\bullet}$, and *Ricci curvature* $\boldsymbol{Ric}_{\bullet\bullet}$ tensors are defined by[19]

$$\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}) \coloneqq \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}], \tag{31}$$

$$\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{w}) \coloneqq \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}, \tag{32}$$

$$\boldsymbol{Ric}_{\bullet\bullet} \coloneqq \mathrm{tr}_{13}\, {\boldsymbol{R}^{\bullet}}_{\bullet\bullet\bullet}\ . \tag{33}$$

From these definitions and the results of § 7 we find the dimensional requirements

$$\dim({\boldsymbol{\tau}^{\bullet}}_{\bullet\bullet})\dim(\boldsymbol{u})\dim(\boldsymbol{v}) = \dim(\boldsymbol{u})\dim(\boldsymbol{v}), \tag{34}$$

$$\dim({\boldsymbol{R}^{\bullet}}_{\bullet\bullet\bullet})\dim(\boldsymbol{u})\dim(\boldsymbol{v})\dim(\boldsymbol{w}) = \dim(\boldsymbol{u})\dim(\boldsymbol{v})\dim(\boldsymbol{w}), \tag{35}$$

$$\dim(\boldsymbol{Ric}_{\bullet\bullet}) = \dim({\boldsymbol{R}^{\bullet}}_{\bullet\bullet\bullet}), \tag{36}$$

which imply that *torsion, Riemann curvature, and Ricci curvature are dimensionless*:

$$\dim({\boldsymbol{\tau}^{\bullet}}_{\bullet\bullet}) = \dim({\boldsymbol{R}^{\bullet}}_{\bullet\bullet\bullet}) = \dim(\boldsymbol{Ric}_{\bullet\bullet}) = 1. \tag{37}$$

The exact contra- and covariant type used above for these tensors is very important in these equations. If we raise any of their indices using a metric, their dimensions will generally change.

Misner et al. (1973 p. 35) say that "curvature", by which they seem to mean the Riemann tensor, has dimension L^{-2}. This statement is seemingly at variance with the dimensionless results (37). But I believe that Misner et al. refer to the *components* of the Riemann tensor, in specific coordinates of dimension L, and using geometrized units. In such specific coordinates every *component* ${R^i}_{jkl}$ does indeed have dimension L^{-2}, according to the general formula (9), if and only if the intrinsic dimension of $\boldsymbol{R}$ is unity, $\dim(\boldsymbol{R}) = 1$. So I believe that Mister et al.'s statement actually agrees with the results (37). This possible misunderstanding

[19] Choquet-Bruhat et al. 1996 § V.B.1.

shows that it's important to distinguish between intrinsic dimensions, which don't depend on any specific coordinate choice, and component dimensions, which do.

The formulae above are also valid if a metric is defined and the connection is compatible with it. The connection coefficients in this case are defined in terms of the metric tensor, but using the results of § 10 it's easy to see that eqs (27), (28), (30), (37) still hold.

10 Metric and related tensors and operations

Let's now consider a metric tensor $\boldsymbol{g}_{\bullet\bullet}$. What is its intrinsic dimension $\dim(\boldsymbol{g})$? There seem to be two choices in the literature; both can be derived from the operational meaning of the metric.

Consider a (timelike) worldline $s \mapsto C(s)$, $s \in [a, b]$, between events $C(a)$ and $C(b)$. The metric tells us the *proper time* $\triangle t$ elapsed for an observer having that worldline, according to the formula

$$\triangle t = \int_a^b \mathrm{d}s \, \sqrt{\left|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]\right|}. \tag{38}$$

From the results of § 7 this formula implies that $\mathsf{T} \equiv \dim(\triangle t) = \sqrt{\dim(\boldsymbol{g}_{\bullet\bullet})}$, (independently of the dimension of s) and therefore

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{T}^2. \tag{39}$$

Many authors[20], however, prefer to include a dimensional factor $1/c$ in front of the integral (38):

$$\triangle t = \frac{1}{c} \int_a^b \mathrm{d}s \, \sqrt{\left|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]\right|}, \tag{40}$$

thus obtaining

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{L}^2. \tag{41}$$

The choice (41) seems also supported by the traditional expression for the "line element ds^2" as it appears in many works (for an exception with dimension T^2 see Kilmister[21]),

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{42}$$

possibly with opposite signature. If the coordinates (t, x, y, z) have the dimensions suggested by their symbols, this formula has dimension L^2,

20 e.g. Fock 1964 § V.62 eq. (62.02); Curtis & Miller 1985 ch. 11 eq. (11.21); Rindler 1986 § 5.3 eq. (5.6); Hartle 2003 ch. 6 eq. (6.24). **21** Kilmister 1973 ch. II p. 25.

so that if we interpret "ds^2" as $\boldsymbol{g}$ we find $\dim(\boldsymbol{g}) = \mathsf{L}^2$. The line-element expression above often has an ambiguous differential-geometric meaning, however, because it may represent *the metric applied to some unspecified vector*, that is, $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v})$, where $\boldsymbol{v}$ is left unspecified[22]. In this case we have

$$\mathsf{L}^2 = \dim(\boldsymbol{g}) \dim(\boldsymbol{v})^2 \tag{43}$$

and the dimension of $\boldsymbol{g}$ is ambiguous.

The standard choices for $\dim \boldsymbol{g}$ are thus T^2 or L^2. My favourite choice is the first, (39), for reasons discussed by Synge and Bressan[23]. Synge gives a vivid summary:[24]

> We are now launched on the task of giving physical meaning to the Riemannian geometry [...]. It is indeed a Riemannian *chronometry* rather than *geometry*, and the word *geometry*, with its dangerous suggestion that we should go about measuring *lengths* with *yardsticks*, might well be abandoned altogether in the present connection

In fact, to measure the proper time $\triangle t$ defined above we only need to ensure that a clock has the worldline C, and then take the difference between the clock's final and initial times. On the other hand, consider the case when the curve C is *spacelike*. Its proper length is still defined by the integral (38) (apart from a dimensional constant). Its measurement, however, is more involved than the timelike case. It requires dividing the curve into very short pieces, and having specially-chosen observers (orthogonal to the pieces) measure each piece. But the measurement of each piece actually relies on the measurement of *proper time*: each observer uses 'radar distance'[25], sending a lightlike signal which bounces back at the end of the piece and measuring the time it takes to come back. Even if rigid rods are used, their calibration still relies on a measurement of time – this is also reflected in the current definition of the standard metre[26].

The metric $\boldsymbol{g}$ can be considered as an operator mapping vectors to covectors, which we can compactly write as $\boldsymbol{\omega} = \boldsymbol{g}\boldsymbol{v}$ (instead of the cumbersome $\boldsymbol{\omega} = \operatorname{tr}_{23}(\boldsymbol{g} \otimes \boldsymbol{v})$). The *inverse metric tensor* $\boldsymbol{g}^{-1\bullet\bullet}$ is then defined by the formula

$$\boldsymbol{g}^{-1}\boldsymbol{g} = \mathrm{id}^{\bullet}{}_{\bullet}, \qquad \boldsymbol{g}\boldsymbol{g}^{-1} = \mathrm{id}_{\bullet}{}^{\bullet} \quad , \tag{44}$$

so that, obviously,

$$\dim(\boldsymbol{g}^{-1}) = \dim(\boldsymbol{g})^{-1}. \tag{45}$$

22 cf. Misner et al. 1973 Box 3.2 D p. 77. **23** Synge 1960b §§ III.2–4; Bressan 1978 §§ 15, 18. **24** Synge 1960b § III.3 pp. 108–109. **25** Landau & Lifshitz 1996 § 84. **26** BIPM 1983 p. 98; Giacomo 1984 p. 25.

The *metric volume element*[27] in spacetime is a 4-form $\boldsymbol{\gamma}$, equivalent to a completely antisymmetric tensor $\boldsymbol{\gamma}_{\bullet\bullet\bullet\bullet}$, such that $\boldsymbol{\gamma}(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = 1$ for every set of positively-oriented orthonormal vector fields $(\boldsymbol{e}_k)$, that is, such that $\boldsymbol{g}(\boldsymbol{e}_k, \boldsymbol{e}_l) = \pm\delta_{kl}$ (remember that the orientation is not determined by the metric). It has only one non-zero component, given by the square root of the determinant of the (positively ordered) components (g_{ij}) of the metric:

$$\boldsymbol{\gamma} = \sqrt{|\det(g_{ij})|}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{46}$$

From this expression and the results of § 7 it can be shown that, in spacetime,

$$\dim(\boldsymbol{\gamma}) = \dim(\boldsymbol{g})^2 \equiv \begin{cases} \mathsf{T}^4 \\ \mathsf{L}^4 \end{cases} \quad \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{T}^2 \\ \mathsf{L}^2 \end{cases} . \tag{47}$$

(This is also the dimension of the *density* $|\boldsymbol{\gamma}|$, which, as opposed to the volume element, has the property that $|\boldsymbol{\gamma}|(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = 1$ for all sets of orthonormal vector fields, not only positively-oriented ones.)

The operation of *raising or lowering an index* of a tensor represents a contraction of the tensor product of that tensor with the metric or the metric inverse, for example $\boldsymbol{A}_{\bullet\bullet} \equiv \operatorname{tr}_{13} \boldsymbol{A}^{\bullet}{}_{\bullet} \otimes \boldsymbol{g}_{\bullet\bullet}$ and similarly for tensors of other types. Therefore

$$\dim(\boldsymbol{A}_{\dots\bullet\dots}) = \dim(\boldsymbol{A}_{\dots}{}^{\bullet}{}_{\dots}) \dim(\boldsymbol{g}), \quad \dim(\boldsymbol{A}_{\dots}{}^{\bullet}{}_{\dots}) = \dim(\boldsymbol{A}_{\dots\bullet\dots}) \dim(\boldsymbol{g})^{-1}. \tag{48}$$

The formulae for the covariant derivative (27), connection coefficients (30), and curvature tensors (37) remain valid for a connection compatible with the metric. In this case the connection coefficients can be obtained from the metric by the formulae[28]

$$\Gamma^i_{\ jk} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} g_{jl} + \frac{\partial}{\partial x^j} g_{kl} - \frac{\partial}{\partial x^l} g_{jk}\right) g^{li}, \tag{49}$$

and it's easily verified that the dimensions (30) of these coefficients still hold. Also the result for the curvature tensors (37) still holds, since their expressions in terms of the connection coefficients is the same with or without a metric.

[27] Abraham et al. 1988 § 6.2. [28] Choquet-Bruhat et al. 1996 § V.B.2.

The *scalar curvature* ρ and the *Einstein tensor* $\boldsymbol{G}_{\bullet}{}^{\bullet}$

$$\rho := \operatorname{tr} \boldsymbol{Ric}_{\bullet}{}^{\bullet} \equiv \operatorname{tr}_{23}(\boldsymbol{Ric} \otimes \boldsymbol{g}^{-1}), \qquad \boldsymbol{G}_{\bullet}{}^{\bullet} := \boldsymbol{Ric}_{\bullet}{}^{\bullet} - \tfrac{1}{2}\rho\, \mathrm{id}_{\bullet}{}^{\bullet} \tag{50}$$

have therefore dimension

$$\dim(\rho) = \dim(\boldsymbol{G}_{\bullet}{}^{\bullet}) = \dim(\boldsymbol{g})^{-1} \equiv \begin{cases} \mathsf{T}^{-2} \\ \mathsf{L}^{-2} \end{cases} \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases}. \tag{51}$$

11 Stress-energy-momentum tensor

To find the dimension of the stress-energy-momentum $\boldsymbol{T}$, or '4-stress' for short, let's start with the analysis of the (3-)stress $\boldsymbol{\sigma}$ in Newtonian mechanics. The stress $\boldsymbol{\sigma}$ is the projection of the 4-stress $\boldsymbol{T}$ onto a spacelike tangent plane with respect to some observer[29]. If we assume that such spatial projection preserves the intrinsic dimension, then the 4-stress and the stress have the same intrinsic dimension.

In Newtonian mechanics the stress $\boldsymbol{\sigma}$ is an object that, integrated over the boundary of a body, gives the total surface force acting on the body (such integration requires a flat connection). This means that it must be represented by a "force-valued" 2-form. Force, in turn, can be interpreted as an object that, integrated over a (spacelike) trajectory, gives an energy – the work done by the force along the trajectory. It's therefore a 1-form. Putting these two requirements together, the stress turns out to be a covector-valued 2-form, equivalent to a tensor $\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}$ antisymmetric in its last two indices. Integrated over a surface, and then over a trajectory, it yields an energy. From § 7, integration of a form does not change the dimension of the form. Therefore

$$\dim(\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}) = \mathsf{E} \equiv \mathsf{M}\mathsf{L}^{2}\mathsf{T}^{-2}. \tag{52}$$

But usually the stress is represented by a co-contravariant tensor $\boldsymbol{\sigma}_{\bullet}{}^{\bullet}$. This is obtained by contracting the last two slots of $\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}$ with the inverse of the volume element of the 3-metric – this is the duality[30] between k-vectors and $(n-k)$-covectors induced by the metric (and an orientation choice), where n is the geometric dimension of the manifold. If we assume the Newtonian 3-metric to have dimension L^2, it can be shown similarly

[29] Gourgoulhon 2012 § 3.4.1; Smarr & York 1978; York 1979; Smarr et al. 1980; Wilson & Mathews 2007 § 1.3; the projection doesn't need to be orthogonal: Marsden & Hughes 1994 § 2.4; Hehl & Obukhov 2003 § B.1.4. [30] Bossavit 1991 § 4.1.2.

to § 10 that its volume element has dimension L^3, and the inverse volume element has dimension L^{-3}. Thus we obtain

$$\dim(\boldsymbol{\sigma}_{\bullet}{}^{\bullet}) = \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}, \tag{53}$$

an energy density (or ‘volumic energy’ according to ISO[31]).

Since the stress $\boldsymbol{\sigma}_{\bullet}{}^{\bullet}$ is the projection of $\boldsymbol{T}_{\bullet}{}^{\bullet}$ and the projection preserves the intrinsic dimension, we finally find that $\boldsymbol{T}_{\bullet}{}^{\bullet}$ also has the dimension of an energy density:

$$\dim(\boldsymbol{T}_{\bullet}{}^{\bullet}) = \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}. \tag{54}$$

Note that other co- or contravariant versions of the 4-stress will have different intrinsic dimension, because they're obtained by lowering or raising indices. For example, $\dim(\boldsymbol{T}_{\bullet\bullet}) = \dim(\boldsymbol{T}_{\bullet}{}^{\bullet})\dim(\boldsymbol{g}) = \mathsf{M}\mathsf{L}^{-1}$ if $\dim(\boldsymbol{g}) \coloneqq \mathsf{T}^2$.

Let me add a passing remark. Even though in most texts the 4-stress is represented by a tensor of order 2, as above, its most fitting geometrical nature is still shrouded by mystery from a kinematic and dynamical point of view. There are indications that it could be more properly represented by a covector-valued 3-form (equivalent to a tensor $\boldsymbol{T}_{\bullet\,\bullet\bullet\bullet}$ antisymmetric in the last three slots), or by a 3-vector-valued 3-form (equivalent to a tensor $\boldsymbol{T}^{\bullet\bullet\bullet}{}_{\bullet\bullet\bullet}$ antisymmetric in the first three and last three slots), for reasons connected with integration, similar to those mentioned above for the stress $\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}$. See for example the discussion about “$^*\boldsymbol{T}$” by Misner et al.[32], the works by Segev[33], the discussion by Burke[34].

12 Einstein equation and Einstein's constant

We finally arrive at the Einstein equation,

$$\boldsymbol{G} = \kappa\boldsymbol{T} \tag{55}$$

where κ (sometimes seen with a minus[35] depending on the signature of the metric or on how the orientation of the stress is chosen) is Einstein's constant. For the dimension of κ we thus find

$$\dim(\kappa) = \dim(\boldsymbol{G}_{\bullet}{}^{\bullet})\dim(\boldsymbol{T}_{\bullet}{}^{\bullet})^{-1} \equiv \begin{cases}\mathsf{M}^{-1}\mathsf{L}\\ \mathsf{M}^{-1}\mathsf{L}^{-1}\mathsf{T}^2\end{cases} \quad \text{if } \dim(\boldsymbol{g}) \coloneqq \begin{cases}\mathsf{T}^2\\ \mathsf{L}^2\end{cases}. \tag{56}$$

This constant can be obtained from the dimensions of Newton's gravitational constant $\dim(G) = \mathsf{M}^{-1}\mathsf{L}^3\mathsf{T}^{-2}$ (this is not the Einstein tensor $\boldsymbol{G}$!)

31 ISO 2009 item A.6.2. 32 Misner et al. 1973 ch. 15. 33 Segev 2002; 1986; Segev & Rodnay 1999; Segev 2000a,b. 34 Burke 1987 § 41. 35 e.g. Tolman 1949 § 78 eq. (78.3); Fock 1964 § 52 eq. (52.06); Rindler 2006 § 14.2 eq. (14.8).

and of the speed of light $\dim(c) = \mathsf{LT}^{-1}$ only in the following ways, with an 8π factor coming from the Newtonian limit:

$$\kappa = \begin{cases} 8\pi G/c^2 \\ 8\pi G/c^4 \end{cases} \quad \text{if } \dim(\boldsymbol{g}) \coloneqq \begin{cases} \mathsf{T}^2 \\ \mathsf{L}^2 \end{cases} \quad . \tag{57}$$

The second choice is by far the most common, consistently with the most common choice of $\dim(\boldsymbol{g}) = \mathsf{L}^2$ discussed before. The first choice appears for example in Fock[36] and Adler et al.[37].

13 Summary and conclusions

We have seen that dimensional analysis in general relativity and differential geometry can be seamlessly done, following its usual standard rules, if we adopt a coordinate-free or coordinate-invariant view, typical of more modern texts. In this view, each tensor has an "intrinsic" dimension (Schouten & Dorgelo's terminology) that doesn't depend on the dimensions of the coordinates. It's the intrinsic dimension of the tensor. It would therefore be more profitable to focus on the intrinsic dimension of a tensor rather than on the dimensions of its components. The dimension of a specific component is easily found by formula (9): it's the product of the intrinsic dimension, times the dimension of the ith coordinate function for each contravariant index i, times the inverse of the dimension of the jth coordinate function for each covariant index j.

This generalized dimensional analysis rests mainly on two conventions: the tensor product and the action of covectors on vectors behave analogously to the usual multiplication for the purposes of dimensional analysis. add convention about integration

We found or re-derived some essential results for general relativity, in particular that the Riemann $\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}$ and Ricci $\boldsymbol{Ric}_{\bullet\bullet}$ tensors are dimensionless, and that the Einstein tensor $\boldsymbol{G}_{\bullet}{}^{\bullet}$ has the inverse dimension of the metric. Maybe these can be of importance for some current research involving scales and conformal factors[38].

Since the dimensions of the components are usually different from the intrinsic dimension and depend on the coordinates, I recommend to avoid statements such as "the tensor $A_i{}^{jk}$ has dimension X", because it leaves unclear whether "$A_i{}^{jk}$" is meant to represent the tensor in general (as in Penrose & Rindler's notation), or to represent its set of components, or to represent just a specific component.

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36 Fock 1964 § 55 eqs (55.15) and (52.06). **37** Adler et al. 1975 § 10.5 eq. (10.98).
38 e.g. Röhr & Uggla 2005; Cadoni & Tuveri 2019.

Finally, the above analysis, specific to general relativity, can be made along the same lines also for Newtonian mechanics.

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