Notes on general-relativistic continuum electromagneto-thermo-mechanics

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20 July 2022; updated 24 November 2023 [draft]

Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z), which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

The associated bases for inner-oriented multicovector fields are

$$\mathrm{d}t \quad \mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z \tag{1}$$

$$\mathrm{d}^2tx \quad \mathrm{d}^2ty \quad \mathrm{d}^2tz \quad \mathrm{d}^2yz \quad \mathrm{d}^2zx \quad \mathrm{d}^2xy \tag{2}$$

$$\mathrm{d}^3xyz \quad -\mathrm{d}^3tyz \quad -\mathrm{d}^3tzx \quad -\mathrm{d}^3txy \tag{3}$$

$$\mathrm{d}^4txyz \tag{4}$$

and analogously for inner-oriented multivector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it's only defined on a coordinate patch. A twisted or outer 3-covector such as $\mathrm{d}^3\tilde{x}yz$ has an associated outer direction, in this case positive t. We adopt this shorter notation for the outer-oriented versions of the bases above:

$$-\mathrm{d}_{xyz} \quad \mathrm{d}_{tyz} \quad \mathrm{d}_{tzx} \quad \mathrm{d}_{txy} \tag{5}$$

$$\mathrm{d}^2_{yz} \quad \mathrm{d}^2_{zx} \quad \mathrm{d}^2_{xy} \quad \mathrm{d}^2_{tx} \quad \mathrm{d}^2_{ty} \quad \mathrm{d}^2_{tz} \tag{6}$$

$$\mathrm{d}^3_{t} \quad \mathrm{d}^3_{x} \quad \mathrm{d}^3_{y} \quad \mathrm{d}^3_{z} \tag{7}$$

$$\mathrm{d}^4 \tag{8}$$

so that $-\mathrm{d}_{xyz} \coloneqq \mathrm{d}\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz \coloneqq \partial_{\tilde{t}}$.

Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} & n_{xyz}\,\mathrm{d}^3\tilde{x}yz - n_{tyz}\,\mathrm{d}^3\tilde{t}yz - n_{tzx}\,\mathrm{d}^3\tilde{t}zx - n_{txy}\,\mathrm{d}^3\tilde{t}xy \\ \equiv {} & n^t\,\mathrm{d}^3_t + n^x\,\mathrm{d}^3_x + n^y\,\mathrm{d}^3_y + n^z\,\mathrm{d}^3_z \\ & \text{with } \; n^t \equiv n_{xyz}\;,\; n^x \equiv n_{tyz}\;,\; \text{ and so on.} \end{aligned} \tag{9}$$

2 Metric

We take the metric $\boldsymbol{g}$ to have signature $(-,+,+,+)$ and dimensions of area. The square root of its negative determinant is denoted shortly

$$\sqrt{g} := \sqrt{-\det \boldsymbol{g}}\;. \tag{10}$$

The volume element induced by the metric $\boldsymbol{g}$ has dimensions of volume-time and is denoted (note the boldface)

$$\boldsymbol{\gamma} := \frac{1}{c}\sqrt{g}\;\mathrm{d}^4\tilde{t}xyz \equiv \frac{1}{c}\sqrt{g}\;\mathrm{d}^4 \tag{11}$$

and its corresponding inverse, a twisted 4-vector:

$$\boldsymbol{\gamma}^{-1} := \frac{c}{\sqrt{g}}\,\partial^4 \tag{12}$$

Contraction with the volume element or its inverse establishes a "volume duality" between outer n-covectors and inner $(4-n)$-vectors:

$$\left.\begin{matrix} \mathrm{d}_{xyz}\;\; \mathrm{d}_{tyz}\;\; \mathrm{d}_{tzx}\;\; \mathrm{d}_{txy} \\ \mathrm{d}^2_{yz}\;\; \mathrm{d}^2_{zx}\;\; \mathrm{d}^2_{xy}\;\; \mathrm{d}^2_{tx}\;\; \mathrm{d}^2_{ty}\;\; \mathrm{d}^2_{tz} \\ \mathrm{d}^3_t\;\; \mathrm{d}^3_x\;\; \mathrm{d}^3_y\;\; \mathrm{d}^3_z \\ \mathrm{d}^4 \end{matrix}\right\} \quad \begin{matrix} \xrightarrow{\boldsymbol{\gamma}^{-1}\cdot} \\ \xleftarrow{\cdot\boldsymbol{\gamma}} \end{matrix} \quad \frac{c}{\sqrt{g}} \left\{\begin{matrix} \partial^3_{xyz}\;\; \partial^3_{tyz}\;\; \partial^3_{tzx}\;\; \partial^3_{txy} \\ \partial^2_{tx}\;\; \partial^2_{ty}\;\; \partial^2_{tz}\;\; \partial^2_{yz}\;\; \partial^2_{zx}\;\; \partial^2_{xy} \\ \partial_t\;\; \partial_x\;\; \partial_y\;\; \partial_z \\ \partial^4_{txyz} \end{matrix}\right. \tag{13}$$

This is the reason why in older literature an outer-oriented n-covector is treated as a $(4-n)$-"vector density", that is, a vector divided by the square root of the volume element.

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

its determinant is equal to c, and the volume element is simply d^4.

3 Four-stress

The stress-energy-momentum tensor, or simply 4-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\boldsymbol{T} = \epsilon\, \mathrm{d}^3_t \otimes \mathrm{d}t + q^i\, \mathrm{d}^3_i \otimes \mathrm{d}t + p_j\, \mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\, \mathrm{d}^3_i \otimes \mathrm{d}x^j \tag{15}$$

the indices i, j running over x, y, z, and where

$$\begin{aligned} \epsilon &= \text{volumic energy} & q^i &= \text{aeric energy flux} \\ p_i &= \text{volumic momentum} & \pi^i_j &= \text{3-stress} \end{aligned} \tag{16}$$

measured in the coordinate system $txyz$. The energy ϵ is a density per unit *coordinate* volume xyz, and possibly includes a conversion factor for the time unit. The component q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. The component p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i. The components π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} -c^2\, g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \tag{17}$$

The diagonal elements $g_{tt}, \dots$ include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

Contracting the 4-stress with the volume element and the inverse metric we obtain:

$$\gamma^{-1}\cdot\boldsymbol{T}\cdot\boldsymbol{g}^{-1} = -\frac{1}{c\,g_{tt}\sqrt{g}}\,\epsilon\,\partial_t\otimes\partial_t - \frac{1}{c\,g_{tt}\sqrt{g}}\,q^i\,\partial_i\otimes\partial_t + \sum_j \frac{c}{g_{jj}\sqrt{g}}\,p_j\,\partial_t\otimes\partial_j + \sum_j \frac{c}{g_{jj}\sqrt{g}}\,\pi^i_j\,\partial_i\otimes\partial_j \qquad (18)$$

Appendices

A Checks about optimal representation of 4-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$\begin{aligned}
\mathrm{d}\big(f\,\mathrm{d}^3_t\big) &= \mathrm{d}\Big(f\,\mathrm{d}^3\tilde{x}yz\Big) = \partial_t f\mathrm{d}t\wedge\mathrm{d}^3\tilde{x}yz + 0 = \partial_t f\ \mathrm{d}^4t\tilde{x}yz\\
\mathrm{d}\big(f\,\mathrm{d}^3_x\big) &= -\mathrm{d}\Big(f\,\mathrm{d}^3\tilde{t}yz\Big) = -\partial_x f\mathrm{d}x\wedge\mathrm{d}^3\tilde{t}yz + 0 = \partial_x f\ \mathrm{d}^4t\tilde{x}yz\\
\mathrm{d}\big(f\,\mathrm{d}^3_i\big) &= \partial_i f\ \mathrm{d}^4t\tilde{x}yz
\end{aligned} \qquad (19)$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned}
\boldsymbol{\nabla}(\mathrm{d}t) &= -\mathit{\Gamma}^t_{tt}\,\mathrm{d}t\otimes\mathrm{d}t - \mathit{\Gamma}^t_{tj}\,\mathrm{d}t\otimes\mathrm{d}x^j - \mathit{\Gamma}^t_{it}\,\mathrm{d}x^i\otimes\mathrm{d}t - \mathit{\Gamma}^t_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j\\
\boldsymbol{\nabla}\big(\mathrm{d}x^k\big) &= -\mathit{\Gamma}^k_{tt}\,\mathrm{d}t\otimes\mathrm{d}t - \mathit{\Gamma}^k_{tj}\,\mathrm{d}t\otimes\mathrm{d}x^j - \mathit{\Gamma}^k_{it}\,\mathrm{d}x^i\otimes\mathrm{d}t - \mathit{\Gamma}^k_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j
\end{aligned} \qquad (20)$$

Then

$$
\begin{aligned}
0 &= \mathrm{D}\boldsymbol{T} \\
&= \mathrm{D}\big(e\,\mathrm{d}^3_t \otimes \mathrm{d}t + q^i\,\mathrm{d}^3_i \otimes \mathrm{d}t + p_j\,\mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\,\mathrm{d}^3_i \otimes \mathrm{d}x^j\big) \\
&= \partial_t e\ \mathrm{d}^4 \otimes \mathrm{d}t + \partial_i q^i\ \mathrm{d}^4 \otimes \mathrm{d}t + \partial_t p_j\,\mathrm{d}^4 \otimes \mathrm{d}x^j + \partial_i \pi^i_j\,\mathrm{d}^4 \otimes \mathrm{d}x^j - {} \\
&\quad \Big[-e\,\Gamma^t_{tt}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t - e\,\Gamma^t_{tj}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0 + {} \\
&\quad - q^i\,\Gamma^t_{it}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t - q^i\,\Gamma^t_{ij}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j + 0 + {} \\
&\quad - p_j\,\Gamma^j_{tt}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t - p_k\,\Gamma^k_{tj}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0 + {} \\
&\quad - \pi^i_k\,\Gamma^k_{it}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t - \pi^i_k\,\Gamma^k_{ij}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j\Big] \\
&= \mathrm{d}^4 \otimes \big[\\
&\quad \partial_t e\ \mathrm{d}t - e\,\Gamma^t_{tt}\,\mathrm{d}t - e\,\Gamma^t_{tj}\,\mathrm{d}x^j + {} \\
&\quad \partial_i q^i\,\mathrm{d}t - q^i\,\Gamma^t_{it}\,\mathrm{d}t - q^i\,\Gamma^t_{ij}\,\mathrm{d}x^j + {} \\
&\quad \partial_t p_j\,\mathrm{d}x^j - p_j\,\Gamma^j_{tt}\,\mathrm{d}t - p_k\,\Gamma^k_{tj}\,\mathrm{d}x^j + {} \\
&\quad \partial_i \pi^i_j\,\mathrm{d}x^j - \pi^i_k\,\Gamma^k_{it}\,\mathrm{d}t - \pi^i_k\,\Gamma^k_{ij}\,\mathrm{d}x^j \\
&\quad \big]
\end{aligned}
\tag{21}
$$

This corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} + p_j\,\Gamma^j_{tt} + \pi^i_k\,\Gamma^k_{it} \tag{22}$$

$$\partial_t p_j + \partial_i \pi^i_j = e\,\Gamma^t_{tj} + q^i\,\Gamma^t_{ij} + p_k\,\Gamma^k_{tj} + \pi^i_k\,\Gamma^k_{ij} \tag{23}$$

Radial case

$$\partial_t e + \partial_r q^r = e\,\Gamma^t_{tt} + q^r\,\Gamma^t_{rt} + p_r\,\Gamma^r_{tt} + \pi^r_r\,\Gamma^r_{rt} \tag{24}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\Gamma^t_{tr} + q^r\,\Gamma^t_{rr} + p_r\,\Gamma^r_{tr} + \pi^r_r\,\Gamma^r_{rr} \tag{25}$$

$$\partial_t e + \partial_r q^r = q^r\,\frac{g}{c^2} + p_r\,g \tag{26}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\frac{g}{c^2} \tag{27}$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have[1]

$$\Gamma^t_{jt} = \Gamma^t_{tj} = \frac{GM}{c^2}\frac{x^j}{r^3} \approx \frac{g}{c^2} \qquad \Gamma^j_{tt} = GM\frac{x^j}{r^3} \approx g \tag{28}$$

where g is the standard acceleration, considered positive. Take also $p_j \approx mv_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

The balances above become

$$\partial_t e + \partial_i q^i = q^z\,\frac{g}{c^2} + p_z\,g \tag{29}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\,\frac{g}{c^2} \tag{30}$$

Also,

$$\begin{aligned} \Gamma^t_{tt} &\approx -2\frac{g}{c^2}v(t) \qquad & \Gamma^t_{jt} &= \Gamma^t_{tj} \approx \frac{g}{c^2} \\ \Gamma^j_{tt} &\approx g - 2\frac{g}{c^2}\,v(t)^2 - \dot{v}(t) \qquad & \Gamma^j_{jt} &= \Gamma^j_{tj} \approx \frac{g}{c^2}v(t) \end{aligned} \tag{31}$$

$$\partial_t e + \partial_i q^i = -e\,2\frac{g}{c^2}v(t) + q^z\,\frac{g}{c^2} + p_j\left(g - 2\frac{g}{c^2}\,v(t)^2 - \dot{v}(t)\right) + \pi^z_z\,\frac{g}{c^2}v(t) \tag{32}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\,\frac{g}{c^2} + p_z\,\frac{g}{c^2}v(t) \tag{33}$$

B References for useful mathematical identities

For transformation or raising:[2].

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B}\gamma \cdot (A \wedge B) \tag{34}$$

with $\deg(B) = n - \deg(A)$[3]

Compound matrices:[4]

[1] **poissonetal2014**. [2] **gantmacher1959_r2000**. [3] **barnabeietal1985**.
[4] **choquetbruhatetal1977_r1996**.