Notes on multivector algebra on differential manifolds

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1 Multivector tensor algebra

The idea is to build tensors not from the two spaces of vectors and covectors, but from the $2N^2$ spaces of multivectors and multicovectors with their possible straight or twisted orientations.

2 Comparison of exterior-product definitions

Antisymmetrizer A (a projection):

$$AT \coloneqq \frac{1}{(\deg T)!} \sum_{\pi} \operatorname{sgn}(\pi)\, T \circ \pi \tag{1}$$

Abraham et al. (1988), Choquet-Bruhat et al. (1996), Bossavit (1991):

$$\begin{aligned} \alpha \wedge \beta &\coloneqq \frac{(\deg \alpha + \deg \beta)!}{(\deg \alpha)!\,(\deg \beta)!}\, A(\alpha \otimes \beta) \\ &= \frac{1}{(\deg \alpha)!\,(\deg \beta)!} \sum_{\pi} \operatorname{sgn}(\pi)\, (\alpha \otimes \beta) \circ \pi \end{aligned} \tag{2}$$

Note that for 1-covectors $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$.

The fact that the definition of multi(co)vector is not unique shows that the multivector algebra is a separate entity from the tensor algebra, not a subspace of it[1].

3 Inner or dual or dot product

[1] cf. Deschamps 1970; 1981.

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Bibliography

("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

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