

Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z) , which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

There are 18 basic types of geometric objects, which represent physical quantities of different types. We have inner-oriented scalars, inner-oriented 1-vectors up to 4-vectors, inner-oriented 1-covectors up to 4-covectors, and all their outer-oriented counterparts.

The exterior product ‘ \wedge ’ of vectors and covectors is usually represented just by juxtaposition, and a shortened notation is used for exterior products of several exterior derivatives ‘ d ’. For instance

$$d^2xy := dx \wedge dy \quad \partial_{xy}^2 := \partial_x \wedge \partial_y . \quad (1)$$

The associated bases for inner-oriented covector fields are

$$dt \quad dx \quad dy \quad dz \quad (2)$$

$$d^2tx \quad d^2ty \quad d^2tz \quad d^2yz \quad d^2zx \quad d^2xy \quad (3)$$

$$d^3xyz \quad -d^3tyz \quad -d^3tzx \quad -d^3txy \quad (4)$$

$$d^4txyz \quad (5)$$

and analogously for inner-oriented vector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it’s only defined on a coordinate patch. It is idempotent: $\tilde{1}\tilde{1} = 1$.

A twisted or outer-oriented 3-covector such as $d^3\tilde{x}yz$ has an associated outer direction, in this case positive t . We adopt this shorter notation for the outer-oriented versions of the bases above (analogous to the notation in Gotay & Marsden 1992 §2 p. 371):

$$-d_{xyz} \quad d_{tyz} \quad d_{tzx} \quad d_{txy} \quad (6a)$$

$$d_{yz}^2 \quad d_{zx}^2 \quad d_{xy}^2 \quad d_{tx}^2 \quad d_{ty}^2 \quad d_{tz}^2 \quad (6b)$$

$$d_t^3 \quad d_x^3 \quad d_y^3 \quad d_z^3 \quad (6c)$$

$$d^4 \quad (6d)$$

so that $-d_{xyz} := d\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz := \partial_{\tilde{t}}$. Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} & n_{xyz} d^3\tilde{x}yz - n_{tyz} d^3\tilde{t}yz - n_{tzx} d^3\tilde{t}zx - n_{txy} d^3\tilde{t}xy \\ & \equiv n^t d_t^3 + n^x d_x^3 + n^y d_y^3 + n^z d_z^3 \end{aligned} \quad (7)$$

with $n^t \equiv n_{xyz}$, $n^x \equiv n_{tyz}$, and so on.

Contraction or dot-product of vectors and covectors is denoted by ‘ \cdot ’, and always contracts the maximal possible amount of contiguous elements. For instance

$$d^3xyz \cdot d_{yz}^2 = dx \quad - \partial_{txy}^3 \cdot dx = \partial_{ty}^2. \quad (8)$$

Contractions with the 4-vector ∂^4 and 4-covector d^4 establish a duality between outer n -covectors and inner $(4 - n)$ -vectors:

$$\begin{pmatrix} \partial^4 \\ \partial_{xyz}^3 \quad \partial_{tyz}^3 \quad \partial_{tzx}^3 \quad \partial_{txy}^3 \\ \partial_{tx}^2 \quad \partial_{ty}^2 \quad \partial_{tz}^2 \quad \partial_{yz}^2 \quad \partial_{zx}^2 \quad \partial_{xy}^2 \\ \partial_t \quad \partial_x \quad \partial_y \quad \partial_z \\ 1 \end{pmatrix} \begin{matrix} \xleftarrow{\partial^4 \cdot} \\ \xrightarrow{\cdot d^4} \end{matrix} \begin{pmatrix} \tilde{1} \\ d_{xyz} \quad d_{tyz} \quad d_{tzx} \quad d_{txy} \\ d_{yz}^2 \quad d_{zx}^2 \quad d_{xy}^2 \quad d_{tx}^2 \quad d_{ty}^2 \quad d_{tz}^2 \\ d_t^3 \quad d_x^3 \quad d_y^3 \quad d_z^3 \\ d^4 \end{pmatrix} \quad (9)$$

These duals have special properties. For instance, for any 3-covector N , we have

$$N \cdot (\partial^4 \cdot N) \cdot N = 0 \quad (10)$$

that is, the dual of N is a vector belonging in the kernel of N .

If γ is a non-zero 4-covector and γ^{-1} the inverse 4-vector, that is, $\gamma \cdot \gamma^{-1} = \gamma^{-1} \cdot \gamma = 1$, and if N is a 3-covector and ϕ a 1-covector, we have the useful identity

$$N \wedge \phi = (N \cdot \gamma^{-1} \cdot \phi) \gamma, \quad (11)$$

which also holds as long as the degrees of N and ϕ sum up to 4.

We often consider vector- or covector-valued covectors, written as sums of tensor products. They can be outer- or inner-oriented. For instance

$$d_t^3 \otimes dx \equiv d^3 \tilde{x} y z \otimes dx, \quad d_x^3 \otimes \partial_y. \quad (12)$$

The operation \wedge between a vector-valued covector and a covector-valued covector is the contraction of their vector- and covector-valued parts and the exterior product of their covector parts. For instance, if ϕ and ψ are covectors, ω is a covector, and u is a vector, then

$$(\phi \otimes \omega) \wedge (\psi \otimes u) := (\phi \wedge \psi) \otimes (\omega \cdot u) \quad (13)$$

As another example,

$$(d_t^3 \otimes dx) \wedge (d_t \otimes \partial_x) = (d_t^3 \wedge d_t) (dx \cdot \partial_x) = -d^4. \quad (14)$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta} du^\beta,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$d\alpha' = \frac{\partial \alpha'}{\partial \alpha} d\alpha$$

- raising or lowering of indices with the metric, for example

$$d\alpha \mapsto g^{\alpha\beta} \partial_\beta.$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix¹. For the spaces of 3-vectors and 3-covectors we have simplified formulae²:

$$\begin{aligned} d_{\alpha'}^3 &= \det\left(\frac{\partial \alpha'}{\partial \alpha}\right) \frac{\partial \alpha}{\partial \alpha'} d_{\alpha}^3 \\ \partial^3 \alpha' &= \det\left(\frac{\partial \alpha}{\partial \alpha'}\right) \frac{\partial \alpha'}{\partial \alpha} \partial^3 \alpha \end{aligned} \quad (15)$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} d_{\alpha}^3 &\mapsto \frac{1}{|g|} g_{\alpha\beta} \partial^3 \beta \\ \partial^3 \alpha &\mapsto |g| g^{\alpha\beta} d_{\beta}^3 \end{aligned} \quad (16)$$

3 Metric

We take the metric \mathbf{g} to have signature $(-, +, +, +)$ and dimensions of area. The determinant and the negative of its square root are denoted shortly

$$g := \det \mathbf{g} \quad \sqrt{g} := \sqrt{-\det \mathbf{g}}. \quad (17)$$

The volume element induced by the metric \mathbf{g} has dimensions of volume-time and is denoted (note the boldface)

$$\gamma := \frac{\sqrt{g}}{c} d^4 \tilde{t} x y z \equiv \frac{\sqrt{g}}{c} d^4 \quad (18)$$

and its corresponding inverse, a twisted 4-vector:

$$\gamma^{-1} := \frac{c}{\sqrt{g}} \partial^4. \quad (19)$$

¹ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199. ² Gantmacher 2000 § I.4 eq. (33).

Contraction with the volume element or its inverse establishes a “volume duality” between outer n -covectors and inner $(4 - n)$ -vectors:

$$\begin{pmatrix} \partial^4 \\ \partial_{xyz}^3 & \partial_{tyz}^3 & \partial_{tzx}^3 & \partial_{txy}^3 \\ \partial_{tx}^2 & \partial_{ty}^2 & \partial_{tz}^2 & \partial_{yz}^2 & \partial_{zx}^2 & \partial_{xy}^2 \\ \partial_t & \partial_x & \partial_y & \partial_z \\ 1 \end{pmatrix} \begin{matrix} \xleftarrow{\frac{\sqrt{g}}{c} \gamma^{-1} \cdot} \\ \xrightarrow{\cdot \gamma \frac{c}{\sqrt{g}}} \end{matrix} \begin{pmatrix} \tilde{1} \\ d_{xyz} & d_{tyz} & d_{tzx} & d_{txy} \\ d_{yz}^2 & d_{zx}^2 & d_{xy}^2 & d_{tx}^2 & d_{ty}^2 & d_{tz}^2 \\ d_t^3 & d_x^3 & d_y^3 & d_z^3 \\ d^4 \end{pmatrix} \quad (20)$$

This is the reason why in older literature an outer-oriented n -covector is treated as a $(4 - n)$ -“vector density”, that is, a vector divided by the square root of the volume element.

The metric on the space of 1-vectors also induces metrics on the spaces of 1-covectors, 2-vectors and 2-covectors, and so on. In particular, the metrics $\overset{3}{g}^{-1}$ on the space of 3-covectors and $\overset{3}{g}$ on the space of 3-vectors can be written in coordinates as

$$\overset{3}{g}^{-1} = \frac{g^{\mu\nu}}{g} \partial^3 \mu \otimes \partial^3 \nu \quad \text{with dimensions length}^{-6} \quad (21)$$

$$\overset{3}{g} = g g^{\mu\nu} d_\mu^3 \otimes d_\nu^3 \quad \text{with dimensions length}^6. \quad (22)$$

With these we can define squared norms $\|\cdot\|^2$ on all those spaces. Note in particular the following identity:

$$\|\gamma^{-1} \cdot N\|^2 = -c^2 \|N\| \quad \text{for every 3-covector } N. \quad (23)$$

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (24)$$

and the volume element is simply d^4 .

4 Matter current

The amount-of-matter current N is an outer-oriented 3-covector

$$N = N d_t^3 + J^i d_i^3 \quad (25)$$

of dimensions “amount of matter”, typically measured in moles, where

- N is the volumic amount of matter, measured per unit coordinate volume.
- J^i is the aeric flux of amount of matter (equivalent to a molar flux), measured per unit coordinate area and unit coordinate time.

The current for every particular kind of matter satisfies the conservation law

$$dN = 0 \quad \text{or} \quad \partial_t N + \partial_i J^i = 0 \quad (26)$$

independent of any metric.

The common contravariant form of the matter current, “ N^μ ”, is obtained by contracting the matter current with the inverse volume element:

$${}'N^\mu \triangleq \gamma^{-1} \cdot N = \frac{c}{\sqrt{g}} N \partial_t + \frac{c}{\sqrt{g}} J^i \partial_i. \quad (27)$$

If a metric is present, a four-velocity U can be associated with the matter current N , defined by the following properties and identity:

$$U \cdot N = 0 \quad \|U\|^2 = -c^2 \quad (28)$$

$$U = \frac{1}{\|N\|} \gamma^{-1} \cdot N \quad (29)$$

which also implies (for normal matter)

$$N = \|N\| U \cdot \gamma \quad (30)$$

For normal matter (as opposed to antimatter) $\|N\|^2 \geq 0$.

5 Four-stress

The stress-energy-momentum tensor, or simply four-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\begin{aligned} \mathbf{T} &= T^\mu{}_\nu \, d^3_\mu \otimes dx^\nu \\ &= -\epsilon \, d^3_t \otimes dt - q^i \, d^3_i \otimes dt + p_j \, d^3_i \otimes dx^j + \pi^i_j \, d^3_i \otimes dx^j \end{aligned} \quad (31)$$

the indices i, j running over x, y, z , and where:

- The energy ϵ is a density per unit *coordinate* volume xyz , and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this energy comprises “rest” energy, kinetic energy, potential gravitational energy, and centrifugal potential energy.
- The energy flux q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this flux term includes transport of the energy term above and also heat flux.
- The momentum p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i .
- The compressive three-stress π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j . If the point at which the four-stress is considered has a matter-current, then this stress comprises momentum transport and internal stresses.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \quad (32)$$

The diagonal elements g_{tt}, \dots include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

The common contra-contra-variant form of the stress, “ $T^{\mu\nu}$ ”, is obtained by contracting the four-stress with the inverse volume element and the inverse metric:

$$\begin{aligned} 'T^{\mu\nu}' \triangleq \gamma^{-1} \cdot \mathbf{T} \cdot \mathbf{g}^{-1} &= -\frac{c}{\sqrt{g}} \frac{g^{tt}}{\sqrt{g}} \epsilon \partial_t \otimes \partial_t - \frac{c}{\sqrt{g}} \frac{g^{tt}}{\sqrt{g}} q^i \partial_i \otimes \partial_t \\ &\quad + \sum_j \frac{c}{\sqrt{g}} \frac{g^{jj}}{\sqrt{g}} p_j \partial_t \otimes \partial_j + \sum_j \frac{c}{\sqrt{g}} \frac{g^{jj}}{\sqrt{g}} \pi_j^i \partial_i \otimes \partial_j . \end{aligned} \quad (33)$$

One important detail in finding the Newtonian approximation of “energy density” is that *one takes different zeros of energy density in different coordinate systems*: the zero is taken as the molar mass times the molar density in the current coordinate system. By ‘zero’ I mean the arbitrary separation between “mass” and “energy”.

The *total* four-stress satisfies the balance equation

$$D\mathbf{T} = 0 \quad (34)$$

which is equivalent to the four balance equations

$$\begin{aligned} \partial_t e + \partial_i q^i &= e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k \\ \partial_t p_j + \partial_i \pi_j^i &= -e \Gamma_{tj}^t - q^i \Gamma_{ij}^t + p_k \Gamma_{tj}^k + \pi_k^i \Gamma_{ij}^k \end{aligned} \quad (35)$$

In general relativity the *total* four-stress also satisfies

$$(\gamma^{-1} \cdot \mathbf{T} \cdot \mathbf{g}^{-1})^\top - \gamma^{-1} \cdot \mathbf{T} \cdot \mathbf{g}^{-1} = 0 . \quad (36)$$

The four-stress determines an association between any 1-vector V field and an outer-oriented 3-covector field, interpreted as a current:

$$V \mapsto \mathbf{T} \cdot V . \quad (37)$$

This current satisfies the balance equation

$$d(\mathbf{T} \cdot V) = -\mathbf{T} \wedge \nabla V = \text{tr}(\mathbf{T}^\top \cdot \gamma^{-1} \cdot \nabla V) \gamma \quad (38)$$

which is a conservation law if V is a Killing vector.

For the special case $V = \partial_\alpha$ the formula above becomes

$$d(T^\mu_\alpha d^3_\mu) = T^\mu_\nu \Gamma^\nu_{\mu\alpha} d^4 \iff \partial_\mu T^\mu_\alpha = T^\mu_\nu \Gamma^\nu_{\mu\alpha} \quad (39)$$

Consider a region where there is a non-vanishing matter current N with associated four-velocity U , and define

$$\bar{U} = -\frac{1}{c} \mathbf{g} \cdot U \quad (40)$$

which satisfies

$$\bar{U} \cdot U = 1, \quad \nabla U \cdot \bar{U} = 0. \quad (41)$$

The last equality can be proved from $\nabla \mathbf{g} = 0$ and

$$0 = -\nabla(c^2) = \nabla(U \cdot \mathbf{g} \cdot U) = 2(\nabla U) \cdot \mathbf{g} \cdot U. \quad (42)$$

We can associate with the matter a four-stress T which can be decomposed as follows:

$$T = -\epsilon N \otimes \bar{U} + N \otimes P - (\bar{U} \wedge Q) \otimes \bar{U} + \bar{U} \wedge S \quad (43)$$

with $P \cdot U = 0 \quad Q \cdot U = 0 \quad U \cdot S = 0 \quad S \cdot U = 0$

where

- ϵ is a scalar, the molar energy density.
- P is a 1-covector, the molar momentum density.
- Q is a 2-covector, the areic energy-flux density.
- S is a 1-covector-valued 2-covector, the three-stress related to the Cauchy stress.

Using the four-velocity U associated with the matter current we obtain what could be called the “internal-energy current”:

$$T \cdot U \equiv -\epsilon N - \bar{U} Q \quad (44)$$

which, from eqs (38), (40), (26), satisfies the balance law

$$d(T \cdot U) = -T \wedge \nabla U \quad (45)$$

or

$$d(-\epsilon N - \bar{U} Q) = (\epsilon N \otimes \bar{U}) \cdot \nabla U - (N \otimes P) \cdot \nabla U + (\bar{U} Q \otimes \bar{U}) \cdot \nabla U - \bar{U} S \cdot \nabla U \quad (46)$$

or simply

$$N d\epsilon + \bar{U} dQ - Q d\bar{U} = -N \nabla U \cdot P - (\bar{U} S) \cdot \nabla U. \quad (47)$$

The last balance equation corresponds to eq. (28) in Eckart (1940), except for the fact that here we are keeping the momentum P and heat flux Q distinct.

When P, S, Q are zero, one speaks of the four-stress of “dust”. It is noteworthy that in this case the four-stress is essentially proportional to the matter 3-covector, multiplied by a vector collinear with the matter four-velocity:

$$T = -N \otimes (\epsilon \bar{U}). \quad (48)$$

If we take the four-velocity itself as reference, given that $\bar{U}U = -c^2$, the energy 3-covector we obtain is proportional to the matter 3-covector:

$$-c^2 \epsilon N \quad (49)$$

For this reason ϵ is called the “proper internal energy”. If we take other vector fields as reference, then this energy will pick up further multiplicative terms consisting in the projection of the matter four-velocity U onto the reference vector field. In particular taking ∂_t or ∂_i as reference we’ll have the projections of the four-velocity onto the coordinate covectors as multiplicative factors.

6 Electromagnetic field

The electromagnetic field, represented by the Faraday 2-covector, is typically decomposed as³

$$\begin{aligned} F &= E dt + B \\ &\equiv E_x d^2xt + E_y d^2yt + E_z d^2zt + B^x d^2yz + B^y d^2zx + B^z d^2xy \end{aligned} \quad (50)$$

The conservation of magnetic flux is expressed by

$$dF = 0 \quad (51)$$

³ Frankel 1979 ch. 9.

or equivalently

$$\begin{aligned}
 \partial_i B^i &= 0 & (d^3xyz \text{ component}) \\
 \partial_t B^x + \partial_y E_z - \partial_z E_y &= 0 & (d^3tyz) \\
 \partial_t B^y + \partial_z E_x - \partial_x E_z &= 0 & (d^3txz) \\
 \partial_t B^z + \partial_x E_y - \partial_y E_x &= 0 & (d^3txy)
 \end{aligned} \tag{52}$$

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- u^\bullet is a 1-vector, represented by the column-matrix u .
- Similarly for v^\bullet .
- ω_\bullet is a 1-covector, represented by the row-matrix ω .
- $g_{\bullet\bullet}$ is a co-covector, represented by the matrix g . The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $g^{-1\bullet\bullet}$ is a contra-contravector, inverse of g , that is: $g \cdot g^1 = \text{id}_\bullet$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $Q_{|\dots|}$ is a 3-covector, represented by the column-matrix Q .

- $T_{|\dots|}$ is a 1-covector-valued 3-covector, represented by the matrix T . The rows represent the 3-covector components; the columns, the 1-covector components.
- $\gamma_{|\dots|}$ is a 4-covector, represented by the number γ .
- $\gamma^{-1|\dots|}$ is a 4-vector, represented by the number γ^{-1}
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from “old” coordinates x to “new” coordinates x' is represented by the matrix J . The rows correspond to the new coordinates x' ; the columns, to the old x .
- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x is represented by the matrix J^{-1} . The rows correspond to the old coordinates x ; the columns, to the new x' .

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

(object)	(matrix repr)	(53)
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$\omega \cdot u$	$\omega u \equiv u^\top \omega^\top$ (number)	(54)
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$v \cdot g \cdot u$	$v^\top g u$ (number)	(55)
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$g \cdot u$	$u^\top g^\top$ (row-matrix)	(56)
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$\omega \cdot g^{-1}$	$g^{-\top} \omega^\top$ (column-matrix)	(57)
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$\gamma^{-1} \cdot Q$	$\gamma^{-1} Q$ (column-matrix)	(58)
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$\gamma^{-1} \cdot T$	$\gamma^{-1} T$ (column-matrix)	(59)
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$T \cdot u$	$T u$ (column-matrix)	(60)
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$g \cdot (\gamma^{-1} \cdot T) \cdot u$	$\gamma^{-1} g T u$ (matrix)	(61)
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- Transformations

$$(\text{old coords}) \mapsto (\text{new coords}) \quad (62)$$

$$\mathbf{u} \mapsto \mathbf{J}\mathbf{u} \quad (63)$$

$$\boldsymbol{\omega} \mapsto \boldsymbol{\omega}\mathbf{J}^{-1} \quad (64)$$

$$\mathbf{g} \mapsto \mathbf{J}^{-\top}\mathbf{g}\mathbf{J}^{-1} \quad (65)$$

$$\mathbf{Q} \mapsto \frac{1}{\det \mathbf{J}} \mathbf{J}\mathbf{Q} \quad (66)$$

$$\boldsymbol{\gamma} \mapsto \frac{1}{\det \mathbf{J}} \boldsymbol{\gamma} \quad (67)$$

$$\boldsymbol{\gamma}^{-1} \mapsto \det \mathbf{J} \boldsymbol{\gamma}^{-1} \quad (68)$$

$$\mathbf{T} \mapsto \frac{1}{\det \mathbf{J}} \mathbf{J}\mathbf{T}\mathbf{J}^{-1} \quad (69)$$

B Checks about optimal representation of four-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$\begin{aligned} d(f d_t^3) &= d(f d^3 \tilde{x} y z) = \partial_t f dt \wedge d^3 \tilde{x} y z + 0 = \partial_t f d^4 \tilde{x} y z \\ d(f d_x^3) &= -d(f d^3 \tilde{t} y z) = -\partial_x f dx \wedge d^3 \tilde{t} y z + 0 = \partial_x f d^4 \tilde{t} x y z \\ d(f d_i^3) &= \partial_i f d^4 \tilde{t} x y z \end{aligned} \quad (70)$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned} \nabla(dt) &= -\Gamma_{tt}^t dt \otimes dt - \Gamma_{tj}^t dt \otimes dx^j - \Gamma_{it}^t dx^i \otimes dt - \Gamma_{ij}^t dx^i \otimes dx^j \\ \nabla(dx^k) &= -\Gamma_{tt}^k dt \otimes dt - \Gamma_{tj}^k dt \otimes dx^j - \Gamma_{it}^k dx^i \otimes dt - \Gamma_{ij}^k dx^i \otimes dx^j \end{aligned} \quad (71)$$

If $\boldsymbol{\omega}$ is a 3-covector, $\boldsymbol{\phi}$ a 1-covector, and \mathbf{D} the exterior covariant derivative, then

$$\mathbf{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) = (d\boldsymbol{\phi}) \otimes \boldsymbol{\omega} - \boldsymbol{\phi} \wedge \nabla \boldsymbol{\omega} \quad (72)$$

and in particular

$$\mathbf{D}(\boldsymbol{\phi} \otimes dx^\alpha) = (d\boldsymbol{\phi}) \otimes dx^\alpha + \Gamma_{\mu\nu}^\alpha (\boldsymbol{\phi} \wedge dx^\mu) \otimes dx^\nu. \quad (73)$$

Let's also consider the contraction with a 1-vector u :

$$\begin{aligned} D(\phi \otimes \omega \cdot u) &= D(\phi \otimes \omega) \cdot u - (\phi \otimes \omega) \wedge \nabla u \\ &\equiv D(\phi \otimes \omega) \cdot u - \phi \wedge \nabla u \cdot \omega \\ &= d\phi \otimes \omega \cdot u - \phi \wedge \nabla \omega \cdot u - \phi \wedge \nabla u \cdot \omega . \end{aligned} \quad (74)$$

and in particular

$$D(\phi \otimes dx^\alpha \cdot u) = d\phi u^\alpha - \phi \wedge dx^\beta \partial_\beta u^\alpha \quad (75)$$

A balance equation with the exterior covariant derivative then becomes

$$D\mathbf{T} = 0 \quad \implies \quad d(\mathbf{T} \cdot u) = -\mathbf{T} \wedge \nabla u \quad (76)$$

Then

$$\begin{aligned} 0 &= D\mathbf{T} \\ &= D(-e d^3 \otimes dt - q^i d_i^3 \otimes dt + p_j d_t^3 \otimes dx^j + \pi_j^i d_i^3 \otimes dx^j) \\ &= -\partial_t e d^4 \otimes dt - \partial_i q^i d^4 \otimes dt + \partial_t p_j d^4 \otimes dx^j + \partial_i \pi_j^i d^4 \otimes dx^j - \\ &\quad \left[e \Gamma_{tt}^t d_t^3 \wedge dt \otimes dt + e \Gamma_{tj}^t d_t^3 \wedge dt \otimes dx^j + 0 \right. \\ &\quad q^i \Gamma_{it}^t d_i^3 \wedge dx^i \otimes dt + q^i \Gamma_{ij}^t d_i^3 \wedge dx^i \otimes dx^j + 0 \\ &\quad - p_j \Gamma_{tt}^j d_t^3 \wedge dt \otimes dt - p_k \Gamma_{tj}^k d_t^3 \wedge dt \otimes dx^j + 0 \\ &\quad \left. - \pi_k^i \Gamma_{it}^k d_i^3 \wedge dx^i \otimes dt - \pi_k^i \Gamma_{ij}^k d_i^3 \wedge dx^i \otimes dx^j \right] \\ &= d^4 \otimes [\\ &\quad - \partial_t e dt + e \Gamma_{tt}^t dt + e \Gamma_{tj}^t dx^j \\ &\quad - \partial_i q^i dt + q^i \Gamma_{it}^t dt + q^i \Gamma_{ij}^t dx^j \\ &\quad + \partial_t p_j dx^j - p_j \Gamma_{tt}^j dt - p_k \Gamma_{tj}^k dx^j \\ &\quad + \partial_i \pi_j^i dx^j - \pi_k^i \Gamma_{it}^k dt - \pi_k^i \Gamma_{ij}^k dx^j \\ &\quad] \end{aligned} \quad (77)$$

The equation above corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k \quad (78)$$

$$\partial_t p_j + \partial_i \pi_j^i = -e \Gamma_{tj}^t - q^i \Gamma_{ij}^t + p_k \Gamma_{tj}^k + \pi_k^i \Gamma_{ij}^k \quad (79)$$

For $\mathbf{T} \cdot \mathbf{u}$ we find first

$$\mathbf{T} \cdot \mathbf{u} = (-e u^t + p_j u^j) d_t^3 + (-q^i u^t + \pi_j^i u^j) d_i^3 \quad (80)$$

$$\begin{aligned} \mathbf{T} \wedge \nabla \mathbf{u} &= (-e \partial_t u^t + p_j \partial_t u^j) d_t^3 \wedge dt + (-q^i \partial_i u^t + \pi_j^i \partial_i u^j) d_i^3 \wedge dx^i \\ &\quad + \Gamma \text{ terms} \end{aligned} \quad (81)$$

and therefore

$$\begin{aligned} \partial_t(-e u^t + p_j u^j) + \partial_i(-q^i u^t + \pi_j^i u^j) &= \\ &= -e \partial_t u^t + p_j \partial_t u^j - q^i \partial_i u^t + \pi_j^i \partial_i u^j \\ &\quad - (e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k) u^t \\ &\quad - (e \Gamma_{tj}^t + q^i \Gamma_{ij}^t - p_k \Gamma_{tj}^k - \pi_k^i \Gamma_{ij}^k) u^j \end{aligned} \quad (82)$$

Radial case

$$\partial_t e + \partial_r q^r = e \Gamma_{tt}^t + q^r \Gamma_{rt}^t + p_r \Gamma_{tt}^r + \pi_r^r \Gamma_{rt}^r \quad (83)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \Gamma_{tr}^t + q^r \Gamma_{rr}^t + p_r \Gamma_{tr}^r + \pi_r^r \Gamma_{rr}^r \quad (84)$$

$$\partial_t e + \partial_r q^r = q^r \frac{g}{c^2} + p_r g \quad (85)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \frac{g}{c^2} \quad (86)$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have⁴

$$\Gamma_{jt}^t = \Gamma_{ij}^t = \frac{GM}{c^2} \frac{x^j}{r^3} \approx \frac{g}{c^2} \quad \Gamma_{tt}^j = GM \frac{x^j}{r^3} \approx g \quad (87)$$

where g is the standard acceleration, considered positive. Take also $p_j \approx mv_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

⁴ Poisson & Will 2014 §5.2.3.

The balances above become

$$\partial_t e + \partial_i q^i = q^z \frac{g}{c^2} + p_z g \quad (88)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} \quad (89)$$

Also,

$$\begin{aligned} \Gamma_{tt}^t &\approx -2 \frac{g}{c^2} v(t) & \Gamma_{jt}^t &= \Gamma_{tj}^t \approx \frac{g}{c^2} \\ \Gamma_{tt}^j &\approx g - 2 \frac{g}{c^2} v(t)^2 - \dot{v}(t) & \Gamma_{jt}^j &= \Gamma_{tj}^j \approx \frac{g}{c^2} v(t) \end{aligned} \quad (90)$$

$$\partial_t e + \partial_i q^i = -e 2 \frac{g}{c^2} v(t) + q^z \frac{g}{c^2} + p_j \left(g - 2 \frac{g}{c^2} v(t)^2 - \dot{v}(t) \right) + \pi_z^z \frac{g}{c^2} v(t) \quad (91)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} + p_z \frac{g}{c^2} v(t) \quad (92)$$

C Works with useful content

- Eq. (21) in⁵: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in⁶

For transformation or raising:⁷.

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B} \gamma \cdot (A \wedge B) \quad (93)$$

with $\deg(B) = n - \deg(A)$ ⁸

Compound matrices:⁹

Bibliography

(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

Barnabei, M., Brini, A., Rota, G.-C. (1985): *On the exterior calculus of invariant theory*. J. Algebra **96**¹, 120–160. doi:10.1016/0021-8693(85)90043-2.

Choquet-Bruhat, Y., DeWitt-Morette, C., Dillard-Bleick, M. (1996): *Analysis, Manifolds and Physics. Part I: Basics*, rev. ed. (Elsevier, Amsterdam). First publ. 1977.

⁵ Maugin 1974. ⁶ Gourgoulhon 2012. ⁷ Gantmacher 2000 § I.4 eq. (33). ⁸ Barnabei et al. 1985 prop. 4.1. ⁹ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199, Problem 1 p. 270.

- Eckart, C. (1940): *The thermodynamics of irreversible processes. III. Relativistic theory of the simple fluid*. Phys. Rev. **58**¹⁰, 919–924. doi:10.1103/PhysRev.58.919.
- Frankel, T. (1979): *Gravitational Curvature: An Introduction to Einstein's Theory*. (W. H. Freeman and Company, San Francisco).
- Gantmacher, F. R. (2000): *The Theory of Matrices. Vol. 1*, repr. (American Mathematical Society, Providence, USA). <https://archive.org/details/gantmacher-the-theory-of-matrices-vol-1-1959>. First publ. in Russian 1959. Transl. by K. A. Hirsch.
- Gotay, M. J., Marsden, J. E. (1992): *Stress-energy-momentum tensors and the Belinfante-Rosenfeld formula*. Contemp. Math. **132**, 367–392. <https://www.cds.caltech.edu/~marsden/bib/1992/05-GoMa1992/>, doi:<https://doi.org/10.1090/conm/132>.
- Gourgoulhon, É. (2012): *3+1 Formalism in General Relativity: Bases of Numerical Relativity*. (Springer, Heidelberg). First publ. 2007 as arXiv doi:10.48550/arXiv.gr-qc/0703035. doi:10.1007/978-3-642-24525-1.
- Maugin, G. A. (1974): *Constitutive equations for heat conduction in general relativity*. J. Phys. A **7**⁴, 465–484. doi:10.1088/0305-4470/7/4/010.
- Poisson, E., Will, C. M. (2014): *Gravity: Newtonian, Post-Newtonian, Relativistic*. (Cambridge University Press, Cambridge). doi:10.1017/CB09781139507486.