Dimensional analysis on differential manifolds and in general relativity

P.G.L. Porta Mana
<pgl@portamana.org>

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Some notes on dimensional analysis on differential manifolds, with an eye on general relativity and the Einstein equation.

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1 Introduction

From the point of view of dimensional analysis, do all components of a tensor need to have the same dimension? What is the dimension of the metric and the curvature? And what is the dimension of the constant in the Einstein equations?

There seem to be insecurity and wrong ideas among some students and even some researchers in relativity, regarding the dimensions of tensors and of tensor components, the effect of tensor operators on their dimension, and the dimension of constants in field equations. I've met, for example, with the statement that the components of a tensor should all have the same dimension; and with calculations of the dimensions of curvature tensors starting from coordinates with dimensions of length. That statement is wrong, and that procedure is unnecessary.

Several factors probably cause or contribute to such difficulties. Modern texts in Lorentzian and general relativity commonly use geometrized units. They say that to find the dimension of some constant in a tensorial equations it's sufficient to compare the dimensions of the tensors in the equation. But this is not so immediate, because some tensors don't have universally agreed dimensions – prime example the metric tensor. Older texts often use coordinates with dimension of length. Often they even multiply a timelike coordinate or some tensorial components by powers of c, thus giving the impression that coordinates ought to always be lengths and that all components of a tensor ought to have the same dimension[1].

[1] e.g. Tolman 1949 § 37 eq. (37.8); Landau et al. 1996 § 32 eq. (32.15); Adler et al. 1975 § 10.1 eq. (10.15).

In this Note I want to clarify such misconceptions about dimensional analysis in general relativity and differential geometry, and to illustrate a simple way of reasoning for solving dimensional-analysis doubts and problems.

The most important step is the distinction between the *absolute dimension* of a tensor and the dimensions of that tensor's *components* in some coordinate system; this distinction is explained in § 4. We'll see in particular that *the components of a tensor need not have the same dimensions*.

The absolute dimension, introduced by Schouten and Dorgelo[2] and used in Truesdell et al.[3], is invariant with respect to the dimensions of the coordinates; it's an intrinsic property of the tensor. The dimensions of the components depend instead on the dimensions of the coordinate functions – which can be completely arbitrary, as discussed in § 3. The absolute dimension should thus be the primary focus in dimensional analysis in general relativity and in differential geometry.

The relation between absolute and component dimensions, and the dimensional result of tensor operations such as contraction, derivation, raising or lowering of indices, and so on, are straightforward to understand and compute using the usual intuitive rules of dimensional analysis: if two quantities are summed, then they must have the same dimension; the dimension of the product is the product of the dimensions; and so on. This intuitive way of reasoning relies on the coordinate-free, *intrinsic* view of tensors and other differential-geometrical objects. A brief reminder of it is given in § 2, with further references.

The results for the main tensor operations and operators are summarized in § 5. The results for the curvature tensors and other objects depending on a connection are presented in § 7. It is shown, in particular, that the absolute dimension of the Riemann and Ricci tensors is 1, that is, they are dimensionless. Some geometric objects or operators may be unfamiliar to researchers working in general relativity only; their discussion may simply be skipped by these readers.

The absolute dimension of the metric tensor is discussed in § 8. The literature on general relativity presents two standard choices for it. The absolute dimension of the stress-energy-momentum tensor is discussed in § 9, and the dimension of the constant in the Einstein equations is derived in § 10, where the two standard results from the literature are recovered.

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2 Dorgelo et al. 1946; Schouten 1989 ch. VI.

3 Truesdell et al. 1960 Appendix II.

The final § 11 gives a summary of the main points and results of the present Note.

Since the type of a tensor – that is, the amount and ordering of its covariant and contravariant 'slots'[4] – is important in dimensional analysis, I often denote it explicitly with the notation $\boldsymbol{A}_{\bullet}{}^{\bullet\bullet}$, for example, to indicate that $\boldsymbol{A}$ is covariant in its first slot and contravariant in its second and third slots. Its components would thus be $(A_i{}^{jk})$. For brevity I call this a 'co-contra-contravariant' tensor, with an obvious naming generalization for other types.

For the notation in dimensional analysis I use ISO conventions: $\dim(\boldsymbol{A})$ is the dimension of the quantity $\boldsymbol{A}$, and among the base quantities are mass M, length L, time T, temperature Θ. Note that I don't discuss *units* – here it doesn't matter if the unit for length is the metre or the centimetre, for example.

Finally, quoting Truesdell and Toupin[5], 'dimensional analysis remains a controversial and somewhat obscure subject. We do not attempt a complete presentation here.'

2 Intrinsic view of differential-geometric objects: brief reminder

From the intrinsic point of view, a tensor is defined by its geometric properties. For example, a vector field $\boldsymbol{v} \equiv \boldsymbol{v}(\cdot)$ is an object that operates on functions defined on the manifold, yielding new functions, with the properties $\boldsymbol{v}(af + bg) = a\boldsymbol{v}(f) + b\boldsymbol{v}(g)$ and $\boldsymbol{v}(fg) = \boldsymbol{v}(f)g + f\boldsymbol{v}(g)$ for all functions f, g and reals a, b. A covector field (1-form) $\boldsymbol{\omega}$ is an object that operates on vector fields, yielding functions ('duality'), with the property $\boldsymbol{\omega}(f\boldsymbol{u} + g\boldsymbol{v}) = f\boldsymbol{\omega}(\boldsymbol{u}) + g\boldsymbol{\omega}(\boldsymbol{v})$ for all vector fields $\boldsymbol{u}$, $\boldsymbol{v}$ and functions f, g. The sum of vector or covector fields, and their products by functions – let's call this 'linearity' – are defined in an obvious way. Tensors are constructed from these objects.

A system of coordinates (x^i) is just a set of linearly independent functions. This set gives rise to a set of vectors fields $\left(\frac{\partial}{\partial x^i}\right)$ and to a set of covector fields $(\mathrm{d}x^i)$ by the obvious requirements that $\frac{\partial}{\partial x^i}(x^j) = \delta_i{}^j$ and $\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j$. These two sets can be used as bases to express all other

[4] Misner et al. 1973 § 3.2. [5] Truesdell et al. 1960 Appendix § 7 footnote 4.

vectors and covectors as linear combinations. A vector field v can thus be written as

$$v \equiv \sum_i v^i \frac{\partial}{\partial x^i} \equiv v^i \frac{\partial}{\partial x^i}, \tag{1}$$

where the *functions* $v^i := v(x^i)$ are its components with respect to the basis $\left(\frac{\partial}{\partial x^i}\right)$. Analogously for a covector field.

For the full presentation of the intrinsic view I recommend the excellent texts by Choquet-Bruhat et al. (1996), Boothby (2003), Abraham et al.[6], Bossavit (1991), Burke (1987; 1980 ch. 2), and more on the general-relativity side Misner et al. (1973 ch. 9), Gourgoulhon (2012 ch. 2), Penrose et al. (2003 ch. 4).

3 Coordinates

From a physical point of view, a coordinate is just a function that associates a value of a physical quantity with every event in a region (the domain of the coordinate chart) of spacetime. Together with the other coordinates, such function allows us to uniquely identify every event within that region. Any physical quantity will do: the distance from something, the time elapsed since something, an angle, an energy density, the strength of a magnetic flux, a temperature, and so on. A coordinate can thus have any dimensions: length L, time T, angle 1, energy density $\mathsf{E} := \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}$, magnetic flux $\Phi := \mathsf{M}\mathsf{L}^{2}\mathsf{T}^{-2}\mathsf{I}^{-1}$, temperature Θ, and so on.

The dimensions of the coordinates don't matter, as we'll now see.

4 Tensors: absolute dimension and components' dimensions

Consider a system of coordinates (x^i) with dimensions (X_i), and the ensuing sets of covector fields (1-form) $\mathrm{d}x^i$ and of vector fields $\left(\frac{\partial}{\partial x^i}\right)$, bases for the cotangent and tangent spaces. Their tensor products are bases for the tangent spaces of higher tensor types.

The differential $\mathrm{d}x^i$ traditionally has the same dimension as x^i: $\dim(\mathrm{d}x^i) = \mathsf{X}_i$, and the operator (a vector) $\frac{\partial}{\partial x^i}$ traditionally has the inverse dimension: $\dim \frac{\partial}{\partial x^i} = {\mathsf{X}_i}^{-1}$. We'll see later that these conventions are self-consistent.

[6] Abraham et al. 1988.

For our discussion let's take a concrete example: a contra-co-tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$. The discussion generalizes to tensors of other types in an obvious way.

The tensor $\boldsymbol{A}$ can be expanded in terms of the basis vectors and covectors, as mentioned in § 2:

$$\boldsymbol{A} = A^i{}_j \, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \equiv A^0{}_0 \, \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^0 + A^0{}_1 \, \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^1 + \cdots . \tag{2}$$

Each function

$$A^i{}_j \coloneqq \boldsymbol{A}\Bigl(\mathrm{d}x^i, \frac{\partial}{\partial x^j}\Bigr) \tag{3}$$

is a component of the tensor in this coordinate system.

To make dimensional sense, all terms in the sum (2) must have the same dimension. This is possible only if the generic component $A^i{}_j$ has dimension

$$\dim(A^i{}_j) = \mathsf{A}\, \mathsf{X}_i\, {\mathsf{X}_j}^{-1}, \tag{4}$$

where A is common to all components. Suppose for example that we're using coordinates with dimensions

$$\dim(x^0) = \Theta, \quad \dim(x^1) = \mathsf{L}, \quad \dim(x^2) = \mathsf{L}, \quad \dim(x^3) = \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}; \tag{5}$$

then the components of $\boldsymbol{A}$ have dimensions

$$\bigl(\dim(A^i{}_j)\bigr) = \mathsf{A} \times \begin{pmatrix} 1 & \mathsf{L}^{-1}\Theta & \mathsf{L}^{-1}\Theta & \mathsf{M}^{-1}\mathsf{L}\mathsf{T}^2\Theta \\ \mathsf{L}\Theta^{-1} & 1 & 1 & \mathsf{M}^{-1}\mathsf{L}^2\mathsf{T}^2 \\ \mathsf{L}\Theta^{-1} & 1 & 1 & \mathsf{M}^{-1}\mathsf{L}^2\mathsf{T}^2 \\ \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}\Theta^{-1} & \mathsf{M}\mathsf{L}^{-2}\mathsf{T}^{-2} & \mathsf{M}\mathsf{L}^{-2}\mathsf{T}^{-2} & 1 \end{pmatrix}. \tag{6}$$

The dimension A, which is also the dimension of the sum (2), is called the *absolute dimension*[7] of the tensor $\boldsymbol{A}$, and we write

$$\dim(\boldsymbol{A}) = \mathsf{A}. \tag{7}$$

This is the intrinsic dimension of the tensor, independent of any coordinate system. It reflects the physical or operational[8] meaning of the tensor. We'll see an example of what this mean in § 8.

Different coordinate systems lead to different dimensions of the *components* of $\boldsymbol{A}$, but its absolute dimension remains the same. Formula (4)

[7] Dorgelo et al. 1946; Schouten 1989 ch. VI. § A.2; Truesdell et al. 1960 §§ A.3–4.

[8] Bridgman 1958; see also Synge 1960a

for the dimensions of the components is consistent under changes of coordinates. For example, in new coordinates (x'^k) with dimensions (X'_k), the new components of $\boldsymbol{A}$ are

$$A'^k{}_l = A^i{}_j \,\frac{\partial x'^k}{\partial x^i}\,\frac{\partial x^j}{\partial x'^l} \tag{8}$$

and a quick check shows that $\dim(A'^k{}_l) = \mathrm{A}\,\mathrm{X}'_k\,\mathrm{X}'_l{}^{-1}$, consistent with the general formula (4).

In the following I'll drop the adjective 'absolute' when it's clear from the context.

5 Tensor operations

By the reasoning of the previous section, which simply applies standard dimensional considerations to the basis expansion (2), it's easy to find out the resultant absolute dimension of various operations and operators on tensors and tensor fields.

The tensor product of $\boldsymbol{A}^{\bullet}{}_{\bullet}$ and $\boldsymbol{B}_{\bullet\bullet}{}^{\bullet}$, for example, can be written as the sum

$$\boldsymbol{A}\otimes\boldsymbol{B} = A^i{}_j\,B_{kl}{}^m\,\frac{\partial}{\partial x^i}\otimes \mathrm{d}x^j\otimes\mathrm{d}x^k\otimes\mathrm{d}x^l\otimes\frac{\partial}{\partial x^m} \tag{9}$$

from which it follows that

$$\dim(A^i{}_j\,B_{kl}{}^m) = \mathrm{A}\,\mathrm{B}\;\mathrm{X}_i\,\mathrm{X}_j{}^{-1}\,\mathrm{X}_k{}^{-1}\,\mathrm{X}_l{}^{-1}\,\mathrm{X}_m \tag{10}$$

with $\mathrm{A} = \dim(\boldsymbol{A})$ and $\mathrm{B} = \dim(\boldsymbol{B})$. The absolute dimension of $\boldsymbol{A}\otimes\boldsymbol{B}$ is therefore $\mathrm{AB} \equiv \dim(\boldsymbol{A})\,\dim(\boldsymbol{B})$.

Here is then a summary of the dimensional results of the main differential-geometric operations and operators, except for the covariant derivative and related tensors, discussed more in depth in § 7 below. In brackets I give the section of Choquet-Bruhat et al. (1996) where they are defined.

- *Tensor multiplication* [III.B.5] multiplies dimensions:

$$\dim(\boldsymbol{A}\otimes\boldsymbol{B}) = \dim(\boldsymbol{A})\,\dim(\boldsymbol{B}). \tag{11}$$

- The *contraction* [III.B.5] of the ith and jth slots (one covariant and one contravariant) of a tensor has the same dimension as the tensor:

$$\dim(\mathrm{tr}_{ij}\,\boldsymbol{A}) = \dim(\boldsymbol{A}). \tag{12}$$

Note that this only holds *without* raising or lowering indices.

• The *transposition* (swapping) of the ith and jth slots of a tensor has the same dimension as the tensor:

$$\dim(\boldsymbol{A}^{\intercal ij}) = \dim(\boldsymbol{A}). \tag{13}$$

• The *Lie bracket* [III.B.3] of two vectors has the product of their dimensions:

$$\dim([\boldsymbol{u}, \boldsymbol{v}]) = \dim(\boldsymbol{u}) \dim(\boldsymbol{v}). \tag{14}$$

• The *Lie derivative* [III.C.2] of a tensor with respect to a vector field has the product of the dimensions of the tensor and of the vector:

$$\dim(\mathrm{L}_{\boldsymbol{v}} \boldsymbol{A}) = \dim(\boldsymbol{v}) \dim(\boldsymbol{A}). \tag{15}$$

Regarding operations and operators on differential forms:

• The *exterior product* [IV.A.1] of two differential forms multiplies their dimensions:

$$\dim(\boldsymbol{\omega} \wedge \boldsymbol{\tau}) = \dim(\boldsymbol{\omega}) \dim(\boldsymbol{\tau}). \tag{16}$$

• The *interior product* [IV.A.4] of a vector and a form multiplies their dimensions:

$$\dim(\mathrm{i}_{\boldsymbol{v}} \boldsymbol{\omega}) = \dim(\boldsymbol{v}) \dim(\boldsymbol{\omega}). \tag{17}$$

• The *exterior derivative* [IV.A.2] of a form has the same dimension of the form:

$$\dim(\mathrm{d}\boldsymbol{\omega}) = \dim(\boldsymbol{\omega}). \tag{18}$$

This can be proven using the identity $\mathrm{d}\,\mathrm{i}_{\boldsymbol{v}} + \mathrm{i}_{\boldsymbol{v}}\,\mathrm{d} = \mathrm{L}_{\boldsymbol{v}}$ or similar identities[9] together with eqs (15) and (17).

• The *integral* [IV.B.1] of a form over a submanifold has the same dimension as the form:

$$\dim\Bigl(\int_c \boldsymbol{\omega}\Bigr) = \dim(\boldsymbol{\omega}). \tag{19}$$

The resultant absolute dimensions of other operators, for example the determinant[10], can be obtained by similar reasoning.

9 Curtis et al. 1985 ch. 9 p. 180 Theorem 9.78; Abraham et al. 1988 § 6.4 Theorem 6.4.8.
10 Abraham et al. 1988 § 6.2.

6 Curves and integral curves

Consider a curve into spacetime, $c\colon s \mapsto P(s)$, with the parameter s having dimension $\dim(s) = \mathrm{S}$.

If we consider the manifold as a dimensionless quantity (see §*** for what I mean by this), then the dimension of the tangent or velocity vector $\dot{C}$ to the curve is

$$\dim(\dot{C}) = \mathrm{S}^{-1}, \tag{20}$$

owing to the definition[11]

$$\dot{C} \coloneqq \frac{\partial x^i[C(s)]}{\partial s} \frac{\partial}{\partial x^i}. \tag{21}$$

This has a quirky but interesting consequence. Given a vector field $\boldsymbol{v}$ we say that C is an integral curve for it if

$$\boldsymbol{v} = \dot{C} \tag{22}$$

(or more precisely $\boldsymbol{v}_{C(s)} = \dot{C}_{C(s)}$ in usual differential-geometric notation[12]) at all events $C(s)$ in the image of the curve. From the point of view of dimensional analysis this definition can only be valid if $\boldsymbol{v}$ has dimension S^{-1}. If $\boldsymbol{v}$ and s^{-1} have different dimensions – a case which can happen for physical reasons – the condition (21) must be modified into $\boldsymbol{v} = k\dot{C}$, where k is a possibly dimensionful constant. This is equivalent to considering an affine and dimensional reparameterization of C.

7 Connection, covariant derivative, curvature tensors

Consider an arbitrary connection[13] with covariant derivative ∇. For the moment we don't assume the presence of any metric structure.

The covariant derivative of the product $f\boldsymbol{v}$ of a function and a vector satisfies[14]

$$\nabla(f\boldsymbol{v}) = \mathrm{d}f \otimes \boldsymbol{v} + f\nabla\boldsymbol{v}. \tag{23}$$

The first summand, from formulae (18) and (11), has dimension $\dim(f)\dim(\boldsymbol{v})$; for dimensional consistency this must also be the dimension of the second summand. Thus

$$\dim(\nabla\boldsymbol{v}) = \dim(\boldsymbol{v}). \tag{24}$$

[11] Choquet-Bruhat et al. 1996 § III.B.1; Boothby 2003 § IV.(1.9). [12] Choquet-Bruhat et al. 1996 § III.B.1. [13] Choquet-Bruhat et al. 1996 § V.B. [14] Choquet-Bruhat et al. 1996 § V.B.1.

It follows that the *directional* covariant derivative has dimension

$$\dim(\nabla_{\boldsymbol{u}} \boldsymbol{v}) = \dim(\boldsymbol{u}) \dim(\boldsymbol{v}), \tag{25}$$

and by its derivation properties[15] we see that formula (24) extends from vectors to tensors of arbitrary type.

In the coordinate system (x^i), the action of the covariant derivative is carried by the *connection coefficients* or Christoffel symbols $(\mathit{\Gamma}^i_{\ jk})$ defined by

$$\nabla \frac{\partial}{\partial x^k} = \mathit{\Gamma}^i_{\ jk}\ \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}. \tag{26}$$

From this equation and the previous ones it follows that these coefficients have dimensions

$$\dim(\mathit{\Gamma}^i_{\ jk}) = \mathrm{X}_i\, \mathrm{X}_j^{-1}\, \mathrm{X}_k^{-1}. \tag{27}$$

The *torsion* $\boldsymbol{\tau}^{\bullet}{}_{\bullet\bullet}$, *Riemann curvature* $\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}$, and *Ricci curvature* $\boldsymbol{Ric}_{\bullet\bullet}$ tensors are defined by[16]

$$\boldsymbol{\tau}(\boldsymbol{u}, \boldsymbol{v}) \coloneqq \nabla_{\boldsymbol{u}} \boldsymbol{v} - \nabla_{\boldsymbol{v}} \boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}], \tag{28}$$

$$\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{w}) \coloneqq \nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} \boldsymbol{w} - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}} \boldsymbol{w} - \nabla_{[\boldsymbol{u}, \boldsymbol{v}]} \boldsymbol{w}, \tag{29}$$

$$\boldsymbol{Ric}_{\bullet\bullet} \coloneqq \mathrm{tr}_{13}\, \boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}. \tag{30}$$

From these definitions and the results of § 5 we find the dimensional requirements

$$\dim(\boldsymbol{\tau}^{\bullet}{}_{\bullet\bullet}) \dim(\boldsymbol{u}) \dim(\boldsymbol{v}) = \dim(\boldsymbol{u}) \dim(\boldsymbol{v}), \tag{31}$$

$$\dim(\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}) \dim(\boldsymbol{u}) \dim(\boldsymbol{v}) \dim(\boldsymbol{w}) = \dim(\boldsymbol{u}) \dim(\boldsymbol{v}) \dim(\boldsymbol{w}), \tag{32}$$

$$\dim(\boldsymbol{Ric}_{\bullet\bullet}) = \dim(\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}), \tag{33}$$

which imply that *torsion, Riemann curvature, and Ricci curvature are dimensionless*:

$$\dim(\boldsymbol{\tau}^{\bullet}{}_{\bullet\bullet}) = \dim(\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}) = \dim(\boldsymbol{Ric}_{\bullet\bullet}) = 1. \tag{34}$$

The exact contra- and co-variant type used above for these tensors is very important in these equations. If we raise any of their indices using a metric, their dimensions will generally change.

Misner et al. (1973 pp. 35, 407) say that 'curvature', the Riemann or Einstein tensors (for which see § 8 below), has dimension L^{-2}, a statement

[15] Choquet-Bruhat et al. 1996 § V.B.1 p. 303.

[16] Choquet-Bruhat et al. 1996 § V.B.1.

seemingly at variance with the dimensionless results (34). But I believe that they refer to the *components* of those tensors, in specific coordinates of dimension L, and using geometrized units. In such specific coordinates every *component* $R^{i}{}_{jkl}$ does indeed have dimension L^{-2}, according to the general formula (4), if and only if the *absolute* dimension of $\boldsymbol{R}$ is unity, $\dim(\boldsymbol{R}) = 1$. So I believe that there's no real contradiction with that statement and the results (34). This possible misunderstanding shows that it's important to distinguish between absolute dimensions, which don't depend on any specific coordinate choice, and component dimensions, which do.

The formulae above are also valid if a metric is defined and the connection is compatible with it. The connection coefficients in this case are defined in terms of the metric tensor, but using the results of §*** it's easy to see that eqs (24), (25), (27), (34) still hold.

8 Metric tensor

Let's now consider a metric tensor $\boldsymbol{g}_{\bullet\bullet}$. What is its absolute dimension $\dim(\boldsymbol{g})$? There seem to be two or three choices in the literature; all three can be derived from an operational meaning of the metric.

Consider a (timelike) worldline $s \mapsto C(s)$, $s \in [a, b]$, between events $C(a)$ and $C(b)$. The metric tells us the *proper time* elapsed for an observer having that worldline, according to the formula

$$\triangle t = \int_a^b \mathrm{d}s \, \sqrt{\left|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]\right|}. \tag{35a}$$

From the results of § 5 this formula implies that $\mathsf{T} \equiv \dim(\triangle t) = \sqrt{\dim(\boldsymbol{g}_{\bullet\bullet})}$, (independently of the dimension of s) and therefore

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{T}^2. \tag{36a}$$

Many authors[17] prefer to include a dimensional factor $1/c$ in front of the integral (35a):

$$\triangle t = \frac{1}{c} \int_a^b \mathrm{d}s \, \sqrt{\left|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]\right|}, \tag{35b}$$

[17] e.g. Fock 1964 § V.62, eq. (62.02); Curtis et al. 1985 ch. 11 eq. (11.21); Rindler 1986 § 5.3 eq. (5.6); Hartle 2003 ch. 6 eq. (6.24).

thus obtaining

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{L}^2. \tag{36b}$$

The choice (36b) seems also supported by the traditional expression for the 'line element ds^2' as it appears in many works,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{37}$$

possibly with opposite signature[18]. If the coordinates (t, x, y, z) have the dimensions suggested by their symbols, this formula has dimension L^2, so that if we interpret 'ds^2' as $\boldsymbol{g}$ we find $\dim(\boldsymbol{g}) = \mathsf{L}^2$. The line-element expression above often has an ambiguous differential-geometric meaning, however, because it may represent *the metric applied to some unspecified vector*, that is, $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v})$, where $\boldsymbol{v}$ is left unspecified[19]. In this case we have

$$\mathsf{L}^2 = \dim(\boldsymbol{g}) \dim(\boldsymbol{v})^2 \tag{38}$$

and the dimension of $\boldsymbol{g}$ is ambiguous – possibly dimensionless if $\boldsymbol{v}$ has dimension of length, but we'll see in § 10 that a dimensionless $\boldsymbol{g}$ isn't quite compatible with the Einstein equation.

The standard choices for $\dim \boldsymbol{g}$ are thus T^2 or L^2. My favourite choice is the first, (36a), for reasons discussed by Synge and Bressan[20]. Synge gives a vivid summary:[21]

> We are now launched on the task of giving physical meaning to the Riemannian geometry [. . .]. It is indeed a Riemannian *chronometry* rather than *geometry*, and the word *geometry*, with its dangerous suggestion that we should go about measuring *lengths* with *yardsticks*, might well be abandoned altogether in the present connection

In fact, to measure the proper time $\triangle t$ defined above we only need to ensure that a clock has the worldline C, and then take the difference between its final and initial times. On the other hand, consider the case when the curve C is *spacelike*. Its proper length is still defined by the integral (35a) (apart from a dimensional constant). Its measurement, however, is more involved than the timelike case. It requires dividing the curve into very short pieces, and having specially-chosen observers (they must be orthogonal to the piece) measure each piece. But the measurement of each piece actually relies on the measurement of *proper*

18 for an exception, with dimension T^2, see Kilmister 1973 ch. II p. 25. **19** cf. Misner et al. 1973 Box 3.2 D, p. 77. **20** Synge 1960b §§ III.2–4; Bressan 1978 §§ 15, 18. **21** Synge 1960b § III.3 pp. 108–109.

time: each observer uses 'radar distance'[22], explained in appendix***. Even if rigid rods are used, their calibration still relies on a measurement of time – this is also reflected in the current definition of the standard metre[23].

The metric $\boldsymbol{g}$ can be considered as an operator mapping vectors to covectors, which we can compactly write as $\boldsymbol{\omega} = \boldsymbol{g}\boldsymbol{v}$, rather than $\boldsymbol{\omega} = \operatorname{tr}_{23}(\boldsymbol{g} \otimes \boldsymbol{v})$. The *inverse metric tensor* $\boldsymbol{g}^{-1\bullet\bullet}$ is then defined by the formula

$$\boldsymbol{g}^{-1}\boldsymbol{g} = \mathrm{id}^{\bullet}{}_{\bullet}, \qquad \boldsymbol{g}\boldsymbol{g}^{-1} = \mathrm{id}_{\bullet}{}^{\bullet}, \tag{39}$$

and obviously

$$\dim(\boldsymbol{g}^{-1}) = \dim(\boldsymbol{g})^{-1}. \tag{40}$$

The *metric volume element*[24] in spacetime is a 4-form $\boldsymbol{\gamma}$, equivalent to a completely antisymmetric tensor $\boldsymbol{\gamma}_{\bullet\bullet\bullet\bullet}$, such that $\boldsymbol{\gamma}(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = 1$ for every set of positively-oriented orthonormal vector fields $(\boldsymbol{e}_k)$, that is, such that $\boldsymbol{g}(\boldsymbol{e}_k, \boldsymbol{e}_l) = \pm\delta_{kl}$ (remember that the orientation is not determined by the metric). It has only one non-zero coordinate component, given by the square root of the determinant of the (positively ordered) components (g_{ij}) of the metric:

$$\boldsymbol{\gamma} = \sqrt{|\det(g_{ij})|}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{41}$$

From this expression and the results of § 5 it can be shown that, in spacetime,

$$\dim(\boldsymbol{\gamma}) = \dim(\boldsymbol{g})^2 \equiv \begin{cases} \mathsf{T}^4 \\ \mathsf{L}^4 \end{cases} \text{if } \dim(\boldsymbol{g}) \coloneqq \begin{cases} \mathsf{T}^2 \\ \mathsf{L}^2 \end{cases}. \tag{42}$$

(This is also the dimension of the *density* $|\boldsymbol{\gamma}|$, which, as opposed to the volume element, has the property that $|\boldsymbol{\gamma}|(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = 1$ for all sets of orthonormal vector fields, not only positively-oriented ones.)

The operation of *raising or lowering an index* of a tensor represents a contraction of the tensor product of that tensor with the metric or the

[22] Landau et al. 1996 § 84. [23] BIPM 1983 p. 98; Giacomo 1984 p. 25. [24] Abraham et al. 1988 § 6.2.

metric inverse, for example $\boldsymbol{A}_{\bullet\bullet} \equiv \operatorname{tr}_{13} \boldsymbol{A}^{\bullet}_{\bullet} \otimes \boldsymbol{g}_{\bullet\bullet}$ and similarly for tensors of other types. Therefore

$$\dim(\boldsymbol{A}_{\dots\bullet\dots}) = \dim(\boldsymbol{A}_{\dots}{}^{\bullet}{}_{\dots}) \dim(\boldsymbol{g}), \quad \dim(\boldsymbol{A}_{\dots}{}^{\bullet}{}_{\dots}) = \dim(\boldsymbol{A}_{\dots\bullet\dots}) \dim(\boldsymbol{g})^{-1}. \tag{43}$$

The formulae for the covariant derivative (24), connection coefficients (27), and curvature tensors (34) remain valid for a connection compatible with the metric. In this case the connection coefficients can be obtained from the metric by the formulae[25]

$$\Gamma^{i}_{\ jk} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} g_{jl} + \frac{\partial}{\partial x^j} g_{kl} - \frac{\partial}{\partial x^l} g_{jk}\right) g^{li}, \tag{44}$$

and it's easily verified that formula (27) still holds, and also (34) since the expression of the curvature tensors in terms of the connection coefficients is the same with or without a metric.

The *scalar curvature* ρ and the *Einstein tensor* $\boldsymbol{G}^{\bullet}_{\bullet}$,

$$\rho = \operatorname{tr} \boldsymbol{Ric}^{\bullet}_{\bullet} \equiv \operatorname{tr}_{23}(\boldsymbol{Ric} \otimes \boldsymbol{g}^{-1}), \qquad \boldsymbol{G}^{\bullet}_{\bullet} \coloneqq \boldsymbol{Ric}^{\bullet}_{\bullet} - \tfrac{1}{2}\rho \operatorname{id}^{\bullet}_{\bullet} \tag{45}$$

have therefore dimension

$$\dim(\rho) \coloneqq \dim(\boldsymbol{G}^{\bullet}_{\bullet}) = \dim(\boldsymbol{g})^{-1} \equiv \begin{cases} \mathsf{T}^{-2} \\ \mathsf{L}^{-2} \end{cases} \text{if } \dim(\boldsymbol{g}) \coloneqq \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases}. \tag{46}$$

9 Stress-energy-momentum tensor

To find the dimension of the stress-energy-momentum $\boldsymbol{T}$, or '4-stress' for short, let's start with the analysis of the (3-)stress in Newtonian mechanics. The stress $\boldsymbol{\sigma}$ is the projection of the 4-stress $\boldsymbol{T}$ onto a spacelike tangent plane with respect to some observer[26]. If we assume that such spatial projection preserve the absolute dimension, then the 4-stress and the stress have the same absolute dimension.

In Newtonian mechanics the stress $\boldsymbol{\sigma}$ is an object that, integrated over the boundary of a body, gives the total surface force acting on the body (such integration requires a flat connection). This means that it must be represented by a 'force-valued' 2-form. Force, in turn, can be interpreted

[25] Choquet-Bruhat et al. 1996 § V.B.2.

[26] Gourgoulhon 2012 § 3.4.1; Smarr et al. 1978; York 1979; Smarr et al. 1980; Wilson et al. 2007 § 1.3; the projection doesn't need to be orthogonal: Marsden et al. 1994 § 2.4; Hehl et al. 2003 § B.1.4.

as an object that, integrated over a (spacelike) trajectory, gives an energy – the work done by the force along the trajectory. It's therefore a 1-form. Putting these two requirements together we obtain a covector-valued 2-form, equivalent to a tensor $\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}$ antisymmetric in its last two indices. Integrated over a surface, and then over a trajectory, it yields an energy. From § 5, integration of a form does not change the dimension of the form. Therefore

$$\dim(\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}) = \mathsf{E} \equiv \mathsf{M}\mathsf{L}^2\mathsf{T}^{-2}. \tag{47}$$

Usually the stress is represented by a co-contravariant tensor $\boldsymbol{\sigma}_{\bullet}^{\bullet}$, however. This is done by contracting the last two slots of $\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}$ with the inverse of the volume element of the 3-metric – this is the duality[27] between k-vectors and $(n-k)$-covectors induced by the metric (and an orientation choice), where n is the geometric dimension of the manifold. If we assume the Newtonian 3-metric to have dimension L^2, it can be shown as done in § 8 that its volume element has dimension L^3, and the inverse volume element, L^{-3}. Thus we obtain

$$\dim(\boldsymbol{\sigma}_{\bullet}^{\bullet}) = \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}, \tag{48}$$

an energy density (or 'volumic energy' according to ISO[28]).

Since the stress $\boldsymbol{\sigma}_{\bullet}^{\bullet}$ is the projection of $\boldsymbol{T}_{\bullet}^{\bullet}$ and the projection preserves the absolute dimension, we finally find that $\boldsymbol{T}_{\bullet}^{\bullet}$ also has the dimension of an energy density:

$$\dim(\boldsymbol{T}_{\bullet}^{\bullet}) = \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}. \tag{49}$$

Note that other co- or contravariant versions of the 4-stress will have different absolute dimension, because they're obtained by lowering or raising indices. For example, $\dim(\boldsymbol{T}_{\bullet\bullet}) = \dim(\boldsymbol{T}_{\bullet}^{\bullet})\dim(\boldsymbol{g}) = \mathsf{M}\mathsf{L}^{-1}$ if $\dim(\boldsymbol{g}) \coloneqq \mathsf{T}^2$.

Even though in most texts the 4-stress is represented by a tensor of order 2, as above, its most fitting geometrical nature is still shrouded by mystery from a kinematic and dynamical point of view. There are indications that it could be more properly represented by a covector-valued 3-form (equivalent to a tensor $\boldsymbol{T}_{\bullet\,\bullet\bullet\bullet}$ antisymmetric in the last three slots), or by a 3-vector-valued 3-form (equivalent to a tensor $\boldsymbol{T}^{\bullet\bullet\bullet}{}_{\bullet\bullet\bullet}$ antisymmetric in the first three and last three slots), for reasons connected

[27] Bossavit 1991 § 4.1.2. [28] ISO 2009 item A.6.2.

with integration, similar to those mentioned above for the stress $\boldsymbol{\sigma}_{\bullet\,\bullet\bullet}$. See for example the discussion about '$^{*}\boldsymbol{T}$' by Misner et al. (1973 ch. 15), the works by Segev (2002; 1986; 1999; 2000a,b), the discussion by Burke (1987 § 41).

10 Einstein equation and Einstein's constant

We finally arrive at the Einstein equation,

$$\boldsymbol{G} = \kappa \boldsymbol{T} \tag{50}$$

where κ (sometimes seen with a minus[29] depending on the signature of the metric or on how the orientation of the stress is chosen) is Einstein's constant. For the dimension of κ we thus find

$$\dim(\kappa) = \dim(\boldsymbol{G}_{\bullet}{}^{\bullet}) \dim(\boldsymbol{T}_{\bullet}{}^{\bullet})^{-1} \equiv \begin{cases} \mathsf{M}^{-1}\mathsf{L} \\ \mathsf{M}^{-1}\mathsf{L}^{-1}\mathsf{T}^{2} \end{cases} \quad \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases} . \tag{51}$$

This constant can be obtained from the dimensions of Newton's gravitational constant $\dim(G) = \mathsf{M}^{-1}\mathsf{L}^{3}\mathsf{T}^{-2}$ and of the speed of light $\dim(c) = \mathsf{L}\mathsf{T}^{-1}$ only in this way, with an 8π factor coming from the Newtonian limit:

$$\kappa = \begin{cases} 8\pi G/c^{2} \\ 8\pi G/c^{4} \end{cases} \quad \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases} . \tag{52}$$

The second choice is by far the most common, consistently with the most common choice of $\dim(\boldsymbol{g}) = \mathsf{L}^{2}$ discussed before. The first choice appears for example in Fock (1964 § 55 eqs (55.15) and (52.06)) and Adler et al.[30].

11 Summary and conclusions

29 e.g. Tolman 1949 § 78 eq. (78.3); Fock 1964 § 52 eq. (52.06); Rindler 2006 § 14.2 eq. (14.8).
30 Adler et al. 1975 § 10.5 eq. (10.98).

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('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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