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Dimensional analysis on differential manifolds [draft]

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Some notes on dimensional analysis on differential manifolds, with an eye on general relativity and the Einstein equation.
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1 ***

'Dimensional analysis remains a controversial and somewhat obscure subject. We do not attempt a complete presentation here.'[1]

Let's start from more general facts about dimensional analysis on differential manifolds.

For dimensional analysis I use ISO conventions and notation. I sometimes use notation such as $\boldsymbol{T}_{\bullet}{}^{\bullet}$ to indicate that the tensor $\boldsymbol{T}$ is covariant in its first slot and contravariant in its second; I call this a "co-contra-variant tensor".[2]

2 Coordinates

From a physical point of view, a coordinate is just a function that associates a value of a physical quantity with every event in a region (the domain of the coordinate chart) of spacetime. Together with the other coordinates, such function allows us to uniquely identify every event within that region. Any physical quantity will do: the distance from something, the time elapsed since something, an angle, an energy density, the strength of a magnetic flux, a temperature, and so on. A coordinate can thus have any dimensions: length L, time T, angle 1, energy density $\mathrm{E} = \mathrm{L}^{-1}\mathrm{M}\mathrm{T}^{-2}$, magnetic flux $\Phi = \mathrm{L}^{2}\mathrm{M}\mathrm{T}^{-2}\mathrm{I}^{-1}$, temperature Θ, and so on.

The dimensions of the coordinates don't matter, as we'll now see.

[1] Truesdell et al. 1960 Appendix § 7 footnote 4. [2] Aldersley 1977.

3 Tensors

Consider a system of coordinates (x^i) with dimensions (X_i). Each coordinate x^i begets a covector field (1-form) $\mathrm{d}x^i$, and the coordinates together beget a set of dual vector fields $\left(\frac{\partial}{\partial x^i}\right)$. These covectors and vectors can be used as bases for the cotangent and tangent spaces, and their tensor products as bases for tensor tangent spaces of higher type.

The differential $\mathrm{d}x^i$ traditionally has the same dimension as x^i: $\dim(\mathrm{d}x^i) = \mathrm{X}_i$, and the operator $\frac{\partial}{\partial x^i}$ traditionally has the inverse dimension: $\dim \frac{\partial}{\partial x^i} = {\mathrm{X}_i}^{-1}$. We'll see later that these conventions are self-consistent.

For our discussion let's now take a contra-co-tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$; the discussion generalizes to tensors of other types in an obvious way.

The tensor $\boldsymbol{A}$ can be expanded in terms of the basis vectors and covectors:

$$\boldsymbol{A} = A^i{}_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \equiv A^0{}_0 \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^0 + A^0{}_1 \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^1 + \cdots, \tag{1}$$

where each

$$A^i{}_j \coloneqq \boldsymbol{A}\left(\mathrm{d}x^i, \frac{\partial}{\partial x^j}\right) \tag{2}$$

is a component of the tensor in this coordinate system.

To make sense dimensionally, every term the sum (1) must have the same dimension. This is possible only if the generic component $A^i{}_j$ has dimension

$$\dim(A^i{}_j) = \mathrm{A}\, \mathrm{X}_i\, {\mathrm{X}_j}^{-1}, \tag{3}$$

where A is common to all components and is also the dimension of the sum (1). For example, suppose we're using coordinates with dimensions

$$\dim(x^0) = \Theta, \quad \dim(x^1) = \mathrm{L}, \quad \dim(x^2) = \mathrm{L}, \quad \dim(x^3) = \mathrm{L}^{-1}\mathrm{M}\mathrm{T}^{-2}; \tag{4}$$

then the components of $\boldsymbol{A}$ have dimensions

$$\left(\dim(A^i{}_j)\right) = \mathrm{A} \begin{pmatrix} 1 & \mathrm{L}^{-1}\Theta & \mathrm{L}^{-1}\Theta & \mathrm{L}\mathrm{M}^{-1}\mathrm{T}^{2}\Theta \\ \mathrm{L}\Theta^{-1} & 1 & 1 & \mathrm{L}^{2}\mathrm{M}^{-1}\mathrm{T}^{2} \\ \mathrm{L}\Theta^{-1} & 1 & 1 & \mathrm{L}^{2}\mathrm{M}^{-1}\mathrm{T}^{2} \\ \mathrm{L}^{-1}\mathrm{M}\mathrm{T}^{-2}\Theta^{-1} & \mathrm{L}^{-2}\mathrm{M}\mathrm{T}^{-2} & \mathrm{L}^{-2}\mathrm{M}\mathrm{T}^{-2} & 1 \end{pmatrix}. \tag{5}$$

The dimension A is called the *absolute dimension*[3] of the tensor $\boldsymbol{A}$. This is the intrinsic dimension of the tensor, independently of any coordinate system, so we write

$$\dim(\boldsymbol{A}) = \mathrm{A}. \tag{6}$$

Different coordinate systems simply lead to different dimensions of the components of $\boldsymbol{A}$. Formula (3) for the dimensions of the components is consistent under changes of coordinates. For example, in coordinates (x'^k) with dimensions (X'_k), the components of $\boldsymbol{A}$ are

$$A'^k{}_l = A^i{}_j \frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} \tag{7}$$

and a quick check shows that $\dim(A'^k{}_l) = \mathrm{A}\,\mathrm{X}'_k\,\mathrm{X}'^{-1}_l$, consistent with (3).

4 Tensor operations

By the reasoning of the previous section it's easy to see how various tensor and tensor-field operations affect the absolute dimension of their arguments. For example, the tensor product of $\boldsymbol{A}^{\bullet}{}_{\bullet}$ and $\boldsymbol{B}_{\bullet\bullet}{}^{\bullet}$ can be written as the sum

$$\boldsymbol{A} \otimes \boldsymbol{B} = A^i{}_j\, B_{kl}{}^m\, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l \otimes \frac{\partial}{\partial x^m} \tag{8}$$

from which it follows that

$$\dim(A^i{}_j\, B_{kl}{}^m) = \mathrm{A}\,\mathrm{B}\,\mathrm{X}_i\,\mathrm{X}_j{}^{-1}\,\mathrm{X}_k{}^{-1}\,\mathrm{X}_l{}^{-1}\,\mathrm{X}_m \tag{9}$$

with $\mathrm{A} = \dim(\boldsymbol{A})$ and $\mathrm{B} = \dim(\boldsymbol{B})$. The absolute dimension of $\boldsymbol{A} \otimes \boldsymbol{B}$ is therefore $\mathrm{AB} \equiv \dim(\boldsymbol{A})\,\dim(\boldsymbol{B})$.

I'll drop the adjective 'absolute' when it's clear from the context.

- *Tensor multiplication* multiplies dimensions:

$$\dim(\boldsymbol{A} \otimes \boldsymbol{B}) = \dim(\boldsymbol{A})\,\dim(\boldsymbol{B}). \tag{10}$$

- The *contraction* of the ith and jth slots (one covariant and one contravariant) of a tensor has the same dimension as the tensor (but without raising or lowering indices; see below):

$$\dim(\operatorname{tr}_{ij} \boldsymbol{A}) = \dim(\boldsymbol{A}). \tag{11}$$

[3] Dorgelo et al. 1946; Schouten 1989 ch. VI.

- The *Lie bracket* of two vectors has the product of their dimensions:

$$\dim([\boldsymbol{u}, \boldsymbol{v}]) = \dim(\boldsymbol{u}) \dim(\boldsymbol{v}). \tag{12}$$

- The *Lie derivative* of a tensor with respect to a vector field has the product of the dimensions of the tensor and of the vector:

$$\dim(\mathrm{L}_{\boldsymbol{v}} \boldsymbol{A}) = \dim(\boldsymbol{v}) \dim(\boldsymbol{A}). \tag{13}$$

Regarding operations with differential forms:

- The *exterior product* of two differential forms multiplies their dimensions:

$$\dim(\boldsymbol{\omega} \wedge \boldsymbol{\tau}) = \dim(\boldsymbol{\omega}) \dim(\boldsymbol{\tau}). \tag{14}$$

- The *interior product* of a vector and a form multiplies their dimensions:

$$\dim(\mathrm{i}_{\boldsymbol{v}} \boldsymbol{\omega}) = \dim(\boldsymbol{v}) \dim(\boldsymbol{\omega}). \tag{15}$$

- The *exterior derivative* of a form has the same dimension of the form:

$$\dim(\mathrm{d}\boldsymbol{\omega}) = \dim(\boldsymbol{\omega}). \tag{16}$$

This can be proven using Cartan's identity with eqs (13) and (15).

- The *integral* of a form over a submanifold has the same dimension as the form:

$$\dim\Bigl(\int_c \boldsymbol{\omega}\Bigr) = \dim(\boldsymbol{\omega}). \tag{17}$$

5 Connection, covariant derivative, curvature tensors

Consider an arbitrary connection with covariant derivative ∇. For the moment we don't assume the presence of any metric structure.

The *covariant derivative* of the product $f\boldsymbol{v}$ of a function and a vector satisfies[4]

$$\nabla(f\boldsymbol{v}) = \mathrm{d}f \otimes \boldsymbol{v} + f\nabla\boldsymbol{v}. \tag{18}$$

The first summand has dimension $\dim(f)\dim(\boldsymbol{v})$, which must also be the dimension of the second summand. Thus we see that

$$\dim(\nabla\boldsymbol{v}) = \dim(\boldsymbol{v}). \tag{19}$$

It follows that the *directional covariant derivative* has dimension

$$\dim(\nabla_{\boldsymbol{u}}\boldsymbol{v}) = \dim(\boldsymbol{u})\dim(\boldsymbol{v}), \tag{20}$$

and by its derivation properties[5] we see that formula (19) extends from vectors to tensors of arbitrary type.

In the coordinate system (x^i), the action of the covariant derivative is carried by the *connection coefficients* or Christoffel symbols $(\Gamma^i_{\ jk})$ defined by

$$\nabla\frac{\partial}{\partial x^k} = \Gamma^i_{\ jk}\ \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}. \tag{21}$$

From this equation and the previous ones it follows that the coefficients have dimensions

$$\dim(\Gamma^i_{\ jk}) = \mathrm{X}_i\,\mathrm{X}_j{}^{-1}\,\mathrm{X}_k{}^{-1}. \tag{22}$$

The *torsion* $\boldsymbol{T}^{\bullet}{}_{\bullet\bullet}$, *Riemann curvature* $\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}$, and *Ricci curvature* $\boldsymbol{R}_{\bullet\bullet}$ tensors are defined by

$$\boldsymbol{T}(\boldsymbol{u},\boldsymbol{v}) \coloneqq \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}], \tag{23}$$

$$\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\cdot\boldsymbol{w} \coloneqq \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}, \tag{24}$$

$$\boldsymbol{R}_{\bullet\bullet} \coloneqq \mathrm{tr}_{13}\,\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}. \tag{25}$$

From these definitions, formula (19), and the dimensional effects of tensor product (10) and contraction (11) we see that

$$\dim(\boldsymbol{T}^{\bullet}{}_{\bullet\bullet}) = \dim(\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}) = \dim(\boldsymbol{R}_{\bullet\bullet}) = 1. \tag{26}$$

The exact contra- and co-variant type used above for these tensors is very important in these equations. If we raise any of their indices using a metric, their dimensions will generally change.

4 Choquet-Bruhat et al. 1996 § V.B.1 p. 300. **5** Choquet-Bruhat et al. 1996 § V.B.1 p. 303.

6 Curves and integral curves

Consider a curve to the manifold, $c\colon s \mapsto P$, where the parameter s has dimension [S]. If we consider the manifold as "adimensional" (if this makes sense), then the dimensions of the tangent vector $\dot{c}$ to the curve are $\dim(\dot{c}) = [\mathrm{S}^{-1}]$. This follows either from $\dot{c} := \partial x^i[c(s)]/\partial s \; \partial_{x^i}$, or considering that $\dot{c}$ can be interpreted as the push-forward of ∂_s, that is, $c_*(\partial_s)$.

This has an interesting, quirky implication. Given a vector field $P \mapsto \boldsymbol{v}(P)$ we say that c is an integral curve for it if

$$\boldsymbol{v}[c(s)] = \dot{c}(s).$$

But this equation is only valid if $\boldsymbol{v}$ has dimensions $[\mathrm{S}^{-1}]$. For the general case a constant dimensional factor needs to be introduced in the equation above.

** Metric tensor **

From the above discussion we see that the component g_{ij} of the metric $\boldsymbol{g}$ has dimensions $[\mathrm{Z}\,\mathrm{X}_i\,{\mathrm{X}_j}^{-1}\,{\mathrm{X}_k}^{-1}]$, where [Z] are the absolute dimensions of the metric. What are these absolute dimensions?

The answer probably depends on how you see the operational meaning of the metric. Here I offer my personal point of view. We can use the metric to measure the "length" of (timelike or spacelike) paths in spacetime. The "length" of a path $c(s)$ with $s \in [a, b]$ is

$$\int_a^b \mathrm{d}s \; \sqrt{\left|g_{ij}[c(s)]\; \dot{c}^i(s)\, \dot{c}^j(s)\right|}.$$

We see that this "length" has dimensions $[\mathrm{Z}^{1/2}]$ and not unexpectedly it doesn't depend on the dimensions of the curve parameter s.

If the path is timelike, this "length" can be measured by a clock having that path as worldline – it's its proper time. Thus, for me $[\mathrm{Z}^{1/2}] = [\mathrm{T}]$, a time, and therefore the absolute dimensions of the metric tensor are time squared:

$$\dim(\boldsymbol{g}) = [\mathrm{T}^2].$$

I believe that these dimensions also make sense for spacelike paths: in this case we would have to measure the "length" by dividing it in very small pieces and using radar coordinates on each piece. So we're measuring the "length" by checking clocks, to see how long it takes for the light to bounce back: time [T], again.

By our usual argument it's possible to see that the Riemann curvature tensor $\boldsymbol{R}^{\bullet}{}_{\bullet\bullet\bullet}$, the Ricci tensor $\boldsymbol{R}_{\bullet\bullet}$, and the Einstein tensor $\boldsymbol{G}_{\bullet\bullet}$ are adimensional – [1] – and the scalar curvature has dimensions $[\mathrm{T}^{-2}]$. Note that the Riemann and Ricci tensors (with the contra/co-variant type specified above) do not require a metric for their definition, but an affine connection. They are adimensional no matter what dimensions we give the metric. By construction the (fully co-variant) Einstein tensor is always adimensional, too.

An important operation done with the metric:

- "lowering an index" of a tensor multiplies its dimensions by $[\mathrm{T}^2]$, and "rising an index" multiplies them by $[\mathrm{T}^{-2}]$ (if you agree with my discussion above).

** Stress-energy-momentum tensor **

What are the absolute dimensions of the co-contra-variant stress-energy-momentum tensor $\boldsymbol{T}_{\bullet}{}^{\bullet}$? We must look for an operational meaning here too. I'll try to sketch an informal argument that reflects my point of view. The argument can be made more rigorous but that would take too long to do here.

The dynamics equation $\nabla \cdot \boldsymbol{T} = 0$ holds in general-relativistic (thermo)mechanics, and also in Newtonian (thermo)mechanics when no body forces and no body heating are present. In Newtonian mechanics it's the formal combination of the balances of momentum density and energy density – which incidentally have the same dimensions $[\mathrm{M}\,\mathrm{L}^{-1}\,\mathrm{T}^{-3}]$, energy/(volume × time).

The divergence of the stress-energy-momentum gives us a 4-force density, just like the 3-divergence of the stress gives us a force density. Please check Misner &al (1973), chap. 14, for a very interesting discussion of these matters, and also Eckart (1940) and Burke (1980, 1987).

Further, the 4-force is an object that, integrated over a path, gives us an energy density (cf Milne 1951 chap. IV, and Burke again). The integral of a force in Newtonian mechanics is the work done by the force. In general-relativistic mechanics, the timelike component of the 4-force additionally gives us the increase in energy owing to heating (Eckart 1940).

So $\nabla \cdot \boldsymbol{T} \equiv T_i{}^j{}_{;j}\,\mathrm{d}x^i$ has the dimensions of energy density, $[\mathrm{M}\,\mathrm{L}^{-1}\mathrm{T}^{-2}]$. The *co-contra-variant* stress-energy-momentum $\boldsymbol{T}_{\bullet}{}^{\bullet}$ has therefore the same dimensions. But the *co-co-variant* tensor, obtained by contraction with the metric, $\boldsymbol{T}_{\bullet\bullet} \equiv \boldsymbol{T} \cdot \boldsymbol{g}$, has dimensions of energy density times squared time: $[\mathrm{M}\,\mathrm{L}^{-1}]$, a mass over length.

Einstein's constant κ therefore relates a dimensionless quantity and a mass over length:

$$\boldsymbol{G}_{\bullet\bullet} = \kappa \boldsymbol{T}_{\bullet\bullet} \ .$$

Its dimension must be $[\mathrm{M}^{-1}\,\mathrm{L}]$, and it's easily seen that these are the dimensions of G/c^2. So I'm one of those people (like Fock 1964 p. 199) who define

$$\kappa = 8\pi G/c^2.$$

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('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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