

Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z) , which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

There are 18 basic types of geometric objects, which represent physical quantities of different types. We have inner-oriented scalars, inner-oriented 1-vectors up to 4-vectors, inner-oriented 1-covectors up to 4-covectors, and all their outer-oriented counterparts.

The exterior product ‘ \wedge ’ of vectors and covectors is usually represented just by juxtaposition, and a shortened notation is used for exterior products of several exterior derivatives ‘ d ’. For instance

$$d^2xy := dx \wedge dy \quad \partial_{xy}^2 := \partial_x \wedge \partial_y . \quad (1)$$

The associated bases for inner-oriented covector fields are

$$dt \quad dx \quad dy \quad dz \quad (2)$$

$$d^2tx \quad d^2ty \quad d^2tz \quad d^2yz \quad d^2zx \quad d^2xy \quad (3)$$

$$d^3xyz \quad -d^3tyz \quad -d^3tzx \quad -d^3txy \quad (4)$$

$$d^4txyz \quad (5)$$

and analogously for inner-oriented vector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it’s only defined on a coordinate patch. It is idempotent: $\tilde{1}\tilde{1} = 1$.

A twisted or outer-oriented 3-covector such as $d^3\tilde{x}yz$ has an associated outer direction, in this case positive t . We adopt this shorter notation for the outer-oriented versions of the bases above:

$$-d_{xyz} \quad d_{tyz} \quad d_{tzx} \quad d_{txy} \quad (6)$$

$$d_{yz}^2 \quad d_{zx}^2 \quad d_{xy}^2 \quad d_{tx}^2 \quad d_{ty}^2 \quad d_{tz}^2 \quad (7)$$

$$d_t^3 \quad d_x^3 \quad d_y^3 \quad d_z^3 \quad (8)$$

$$d^4 \quad (9)$$

so that $-d_{xyz} := d\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz := \partial_{\tilde{t}}$. Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} & n_{xyz} d^3\tilde{x}yz - n_{tyz} d^3\tilde{t}yz - n_{tzx} d^3\tilde{t}zx - n_{txy} d^3\tilde{t}xy \\ & \equiv n^t d_t^3 + n^x d_x^3 + n^y d_y^3 + n^z d_z^3 \end{aligned} \quad (10)$$

with $n^t \equiv n_{xyz}$, $n^x \equiv n_{tyz}$, and so on.

Contraction or dot-product of vectors and covectors is denoted by ‘ \cdot ’, and always contracts the maximal possible amount of contiguous elements. For instance

$$d^3xyz \cdot \partial_{yz}^2 = dx \quad - \partial_{txy}^3 \cdot dx = \partial_{ty}^2. \quad (11)$$

We often consider vector- or covector-valued covectors, written as sums of tensor products. They can be outer- or inner-oriented. For instance

$$d_t^3 \otimes dx \equiv d^3\tilde{x}yz \otimes dx, \quad d_x^3 \otimes \partial_y. \quad (12)$$

The dot-product of a vector-valued covector and a covector-valued covector represents the contraction of their vector- and covector-valued parts and the exterior product of their covector parts. For instance, if ϕ and ψ are covectors, ω is a covector, and u is a vector, then

$$(\phi \otimes \omega) \cdot (\psi \otimes u) := (\phi \wedge \psi) \otimes (\omega \cdot u) \quad (13)$$

It is usually clear from the context which parts go into the exterior product and which parts need to be contracted. As a concrete example,

$$(d_t^3 \otimes dx) \cdot (d_t \otimes \partial_x) = (d_t^3 \wedge d_t)(dx \cdot \partial_x) = -d^4. \quad (14)$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta} du^\beta ,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$d\alpha' = \frac{\partial \alpha'}{\partial \alpha} d\alpha$$

- raising or lowering of indices with the metric, for example

$$d\alpha \mapsto g^{\alpha\beta} \partial_\beta .$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix¹. For the spaces of 3-vectors and 3-covectors we have simplified formulae²:

$$\begin{aligned} d_{\alpha'}^3 &= \det\left(\frac{\partial \alpha'}{\partial \alpha}\right) \frac{\partial \alpha}{\partial \alpha'} d_\alpha^3 \\ \partial^3 \alpha' &= \det\left(\frac{\partial \alpha}{\partial \alpha'}\right) \frac{\partial \alpha'}{\partial \alpha} \partial^3 \alpha \end{aligned} \tag{15}$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} d_\alpha^3 &\mapsto \frac{1}{|g|} g_{\alpha\beta} \partial^3 \beta \\ \partial^3 \alpha &\mapsto |g| g^{\alpha\beta} d_\beta^3 \end{aligned} \tag{16}$$

¹ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199.

² Gantmacher 2000 § I.4 eq. (33).

3 Metric

We take the metric g to have signature $(-, +, +, +)$ and dimensions of area. The determinant and the negative of its square root are denoted shortly

$$g := \det g \quad \sqrt{g} := \sqrt{-\det g} . \quad (17)$$

The volume element induced by the metric g has dimensions of volume-time and is denoted (note the boldface)

$$\gamma := \frac{\sqrt{g}}{c} d^4 \tilde{t} x y z \equiv \frac{\sqrt{g}}{c} d^4 \quad (18)$$

and its corresponding inverse, a twisted 4-vector:

$$\gamma^{-1} := \frac{c}{\sqrt{g}} \partial^4 . \quad (19)$$

Contraction with the volume element or its inverse establishes a “volume duality” between outer n -covectors and inner $(4 - n)$ -vectors:

$$\left(\begin{array}{cccc} \partial_{xyz}^3 & \partial_{tyz}^3 & \partial_{tzx}^3 & \partial_{txy}^3 \\ \partial_{tx}^2 & \partial_{ty}^2 & \partial_{tz}^2 & \partial_{yz}^2 & \partial_{zx}^2 & \partial_{xy}^2 \\ \partial_t & \partial_x & \partial_y & \partial_z \\ \partial_{txyz}^4 \end{array} \right) \begin{array}{c} \xleftarrow{\frac{\sqrt{g}}{c} \gamma^{-1}} \\ \xrightarrow{\gamma \frac{c}{\sqrt{g}}} \end{array} \left(\begin{array}{cccc} d_{xyz} & d_{tyz} & d_{tzx} & d_{txy} \\ d_{yz}^2 & d_{zx}^2 & d_{xy}^2 & d_{tx}^2 & d_{ty}^2 & d_{tz}^2 \\ d_t^3 & d_x^3 & d_y^3 & d_z^3 \\ d^4 \end{array} \right) \quad (20)$$

This is the reason why in older literature an outer-oriented n -covector is treated as a $(4 - n)$ -“vector density”, that is, a vector divided by the square root of the volume element.

The metric on the space of 1-vectors also induces metrics on the spaces of 1-covectors, 2-vectors and 2-covectors, and so on. In particular, the metrics $\overset{3}{g}^{-1}$ on the space of 3-covectors and $\overset{3}{g}$ on the space of 3-vectors can be written in coordinates as

$$\overset{3}{g}^{-1} = \frac{g^{\mu\nu}}{g} \partial^3 \mu \otimes \partial^3 \nu \quad \text{with dimensions length}^{-6} \quad (21)$$

$$\overset{3}{g} = g g^{\mu\nu} d_\mu^3 \otimes d_\nu^3 \quad \text{with dimensions length}^6 . \quad (22)$$

With these we can define squared norms $\|\cdot\|^2$ on all those spaces. Note in particular the following identity:

$$\|\gamma^{-1} \cdot N\|^2 = -c^2 \|N\| \quad \text{for every 3-covector } N. \quad (23)$$

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (24)$$

and the volume element is simply d^4 .

4 Matter current

The amount-of-matter current N is an outer-oriented 3-covector

$$N = N d_t^3 + J^i d_i^3 \quad (25)$$

of dimensions “amount of matter”, typically measured in moles, where

- N is the volumic amount of matter, measured per unit coordinate volume.
- J^i is the aeric flux of amount of matter (equivalent to a molar flux), measured per unit coordinate area and unit coordinate time.

The current for every particular kind of matter satisfies the conservation law

$$dN = 0 \quad \text{or} \quad \partial_t N + \partial_i J^i = 0 \quad (26)$$

independent of any metric.

The common contravariant form of the matter current, “ N^μ ”, is obtained by contracting the matter current with the inverse volume element:

$${}'N^\mu \triangleq \gamma^{-1} \cdot N = \frac{c}{\sqrt{g}} N \partial_t + \frac{c}{\sqrt{g}} J^i \partial_i. \quad (27)$$

If a metric is present, a four-velocity U can be associated with the matter current N , defined by the following properties and identity:

$$U \cdot N = 0 \quad \|U\|^2 = -c^2 \quad (28)$$

$$U = \frac{1}{\|N\|} \gamma^{-1} \cdot N \quad (29)$$

which also implies (for normal matter)

$$N = \|N\| U \cdot \gamma \quad (30)$$

For normal matter (as opposed to antimatter) $\|N\|^2 \geq 0$.

5 Four-stress

The stress-energy-momentum tensor, or simply four-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\mathbf{T} = -\epsilon \, d_t^3 \otimes dt - q^i \, d_t^3 \otimes dt + p_j \, d_t^3 \otimes dx^j + \pi_j^i \, d_t^3 \otimes dx^j \quad (31)$$

the indices i, j running over x, y, z , and where:

- The energy ϵ is a density per unit *coordinate* volume xyz , and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this energy comprises “rest” energy, kinetic energy, potential gravitational energy, and centrifugal potential energy.
- The energy flux q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this flux term includes transport of the energy term above and also heat flux.
- The momentum p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i .
- The compressive three-stress π_j^i are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j . If the point at which the four-stress is considered has a matter-current, then this stress comprises momentum transport and internal stresses.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \quad (32)$$

The diagonal elements g_{tt}, \dots include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

The common contra-contra-variant form of the stress, “ $T^{\mu\nu}$ ”, is obtained by contracting the four-stress with the inverse volume element and the inverse metric:

$$\begin{aligned} 'T^{\mu\nu}' \triangleq \gamma^{-1} \cdot \mathbf{T} \cdot \mathbf{g}^{-1} = & -\frac{c}{\sqrt{g}} \frac{g^{tt}}{\sqrt{g}} \epsilon \partial_t \otimes \partial_t - \frac{c}{\sqrt{g}} \frac{g^{tt}}{\sqrt{g}} q^i \partial_i \otimes \partial_t \\ & + \sum_j \frac{c}{\sqrt{g}} \frac{g^{jj}}{\sqrt{g}} p_j \partial_t \otimes \partial_j + \sum_j \frac{c}{\sqrt{g}} \frac{g^{jj}}{\sqrt{g}} \pi_j^i \partial_i \otimes \partial_j . \end{aligned} \quad (33)$$

The *total* four-stress satisfies the balance equation

$$D\mathbf{T} = 0 \quad (34)$$

which is equivalent to the four balance equations

$$\begin{aligned} \partial_t e + \partial_i q^i &= e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k \\ \partial_t p_j + \partial_i \pi_j^i &= -e \Gamma_{tj}^t - q^i \Gamma_{ij}^t + p_k \Gamma_{tj}^k + \pi_k^i \Gamma_{ij}^k \end{aligned} \quad (35)$$

In general relativity the *total* four-stress also satisfies

$$(\gamma^{-1} \cdot \mathbf{T} \cdot \mathbf{g}^{-1})^\top - \gamma^{-1} \cdot \mathbf{T} \cdot \mathbf{g}^{-1} = 0 . \quad (36)$$

The four-stress determines an association between any 1-vector V field and an outer-oriented 3-covector field, interpreted as a current:

$$V \mapsto \mathbf{T} \cdot V . \quad (37)$$

This current satisfies the balance equation

$$d(\mathbf{T} \cdot V) = -\mathbf{T} \wedge \nabla V \quad (38)$$

which is a conservation law if V is a Killing vector.

Consider a region where there is a non-vanishing matter current N with associated four-velocity \mathbf{U} , and define

$$\bar{\mathbf{U}} = -\frac{1}{c} \mathbf{g} \cdot \mathbf{U} \quad (39)$$

which statisfies

$$\bar{\mathbf{U}} \cdot \mathbf{U} = 1 , \quad \nabla \mathbf{U} \cdot \bar{\mathbf{U}} = 0 . \quad (40)$$

The last equality can be proved from $\nabla g = 0$ and

$$0 = -\nabla(c^2) = \nabla(\mathbf{U} \cdot \mathbf{g} \cdot \mathbf{U}) = 2(\nabla \mathbf{U}) \cdot \mathbf{g} \cdot \mathbf{U} . \quad (41)$$

We can associate with the matter a four-stress \mathbf{T} which can be decomposed as follows:

$$\begin{aligned} \mathbf{T} &= -\epsilon \mathbf{N} \otimes \bar{\mathbf{U}} + \mathbf{N} \otimes \mathbf{P} - (\bar{\mathbf{U}} \wedge \mathbf{Q}) \otimes \bar{\mathbf{U}} + \bar{\mathbf{U}} \wedge \mathbf{S} \\ \text{with } \mathbf{P} \cdot \mathbf{U} &= 0 \quad \mathbf{Q} \cdot \mathbf{U} = 0 \quad \mathbf{U} \cdot \mathbf{S} = 0 \quad \mathbf{S} \cdot \mathbf{U} = 0 \end{aligned} \quad (42)$$

where

- ϵ is a scalar, the molar energy density.
- \mathbf{P} is a 1-covector, the molar momentum density.
- \mathbf{Q} is a 2-covector, the areic energy-flux density.
- \mathbf{S} is a 1-covector-valued 2-covector, the three-stress related to the Cauchy stress.

Using the four-velocity \mathbf{U} associated with the matter current we obtain what could be called the “internal-energy current”:

$$\mathbf{T} \cdot \mathbf{U} \equiv -\epsilon \mathbf{N} - \bar{\mathbf{U}} \mathbf{Q} \quad (43)$$

which, from eqs (38), (39), (26), satisfies the balance law

$$d(\mathbf{T} \cdot \mathbf{U}) = -\mathbf{T} \cdot \nabla \mathbf{U} \quad (44)$$

or

$$\begin{aligned} d(-\epsilon \mathbf{N} - \bar{\mathbf{U}} \mathbf{Q}) &= (\epsilon \mathbf{N} \otimes \bar{\mathbf{U}}) \cdot \nabla \mathbf{U} - (\mathbf{N} \otimes \mathbf{P}) \cdot \nabla \mathbf{U} \\ &\quad + (\bar{\mathbf{U}} \mathbf{Q} \otimes \bar{\mathbf{U}}) \cdot \nabla \mathbf{U} - \bar{\mathbf{U}} \mathbf{S} \cdot \nabla \mathbf{U} \end{aligned} \quad (45)$$

or simply

$$N d\epsilon + \bar{\mathbf{U}} d\mathbf{Q} - \mathbf{Q} d\bar{\mathbf{U}} = -N \nabla \mathbf{U} \cdot \mathbf{P} - (\bar{\mathbf{U}} \mathbf{S}) \cdot \nabla \mathbf{U} . \quad (46)$$

The last balance equation corresponds to eq. (28) in Eckart (1940), except for the fact that here we are keeping the momentum \mathbf{P} and heat flux \mathbf{Q} distinct.

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- u^\bullet is a 1-vector, represented by the column-matrix u .
- Similarly for v^\bullet .
- ω_\bullet is a 1-covector, represented by the row-matrix ω .
- $g_{\bullet\bullet}$ is a co-covector, represented by the matrix g . The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $g^{-1\bullet\bullet}$ is a contra-contravector, inverse of g , that is: $g \cdot g^1 = \text{id}_\bullet^\bullet$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $Q_{|...|}$ is a 3-covector, represented by the column-matrix Q .
- $T_{|...|}$ is a 1-covector-valued 3-covector, represented by the matrix T . The rows represent the 3-covector components; the columns, the 1-covector components.
- $\gamma_{|...|}$ is a 4-covector, represented by the number γ .
- $\gamma^{-1|...|}$ is a 4-vector, represented by the number γ^{-1} .
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from “old” coordinates x to “new” coordinates x' is represented by the matrix J . The rows correspond to the new coordinates x' ; the columns, to the old x .

- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x is represented by the matrix J^{-1} . The rows correspond to the old coordinates x ; the columns, to the new x' .

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

$$\text{(object)} \qquad \qquad \text{(matrix repr)} \qquad \qquad (47)$$

$$\omega \cdot u \qquad \qquad \omega u \equiv u^\top \omega^\top \quad \text{(number)} \qquad (48)$$

$$v \cdot g \cdot u \qquad \qquad v^\top g u \quad \text{(number)} \qquad (49)$$

$$g \cdot u \qquad \qquad u^\top g^\top \quad \text{(row-matrix)} \qquad (50)$$

$$\omega \cdot g^{-1} \qquad \qquad g^{-\top} \omega^\top \quad \text{(column-matrix)} \qquad (51)$$

$$\gamma^{-1} \cdot Q \qquad \qquad \gamma^{-1} Q \quad \text{(column-matrix)} \qquad (52)$$

$$\gamma^{-1} \cdot T \qquad \qquad \gamma^{-1} T \quad \text{(column-matrix)} \qquad (53)$$

$$T \cdot u \qquad \qquad T u \quad \text{(column-matrix)} \qquad (54)$$

$$g \cdot (\gamma^{-1} \cdot T) \cdot u \qquad \qquad \gamma^{-1} g T u \quad \text{(matrix)} \qquad (55)$$

- Transformations

$$\text{(old coords)} \quad \mapsto \quad \text{(new coords)} \qquad (56)$$

$$u \mapsto J u \qquad (57)$$

$$\omega \mapsto \omega J^{-1} \qquad (58)$$

$$g \mapsto J^{-\top} g J^{-1} \qquad (59)$$

$$Q \mapsto \frac{1}{\det J} J Q \qquad (60)$$

$$\gamma \mapsto \frac{1}{\det J} \gamma \qquad (61)$$

$$\gamma^{-1} \mapsto \det J \gamma^{-1} \qquad (62)$$

$$T \mapsto \frac{1}{\det J} J T J^{-1} \qquad (63)$$

B Checks about optimal representation of four-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$\begin{aligned} d(f d_t^3) &= d(f d^3 \tilde{x} y z) = \partial_t f dt \wedge d^3 \tilde{x} y z + 0 = \partial_t f d^4 \tilde{x} y z \\ d(f d_x^3) &= -d(f d^3 \tilde{t} y z) = -\partial_x f dx \wedge d^3 \tilde{t} y z + 0 = \partial_x f d^4 \tilde{t} x y z \\ d(f d_i^3) &= \partial_i f d^4 \tilde{t} x y z \end{aligned} \quad (64)$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned} \nabla(dt) &= -\Gamma_{tt}^t dt \otimes dt - \Gamma_{tj}^t dt \otimes dx^j - \Gamma_{it}^t dx^i \otimes dt - \Gamma_{ij}^t dx^i \otimes dx^j \\ \nabla(dx^k) &= -\Gamma_{tt}^k dt \otimes dt - \Gamma_{tj}^k dt \otimes dx^j - \Gamma_{it}^k dx^i \otimes dt - \Gamma_{ij}^k dx^i \otimes dx^j \end{aligned} \quad (65)$$

If ω is a 3-covector, ϕ a 1-covector, and D the exterior covariant derivative, then

$$D(\phi \otimes \omega) = (d\phi) \otimes \omega - \phi \wedge \nabla \omega \quad (66)$$

and in particular

$$D(\phi \otimes dx^\alpha) = (d\phi) \otimes dx^\alpha + \Gamma_{\mu\nu}^\alpha (\phi \wedge dx^\mu) \otimes dx^\nu. \quad (67)$$

Let's also consider the contraction with a 1-vector u :

$$\begin{aligned} D(\phi \otimes \omega \cdot u) &= D(\phi \otimes \omega) \cdot u - (\phi \otimes \omega) \wedge \nabla u \\ &\equiv D(\phi \otimes \omega) \cdot u - \phi \wedge \nabla u \cdot \omega \\ &= d\phi \otimes \omega \cdot u - \phi \wedge \nabla \omega \cdot u - \phi \wedge \nabla u \cdot \omega. \end{aligned} \quad (68)$$

and in particular

$$D(\phi \otimes dx^\alpha \cdot u) = d\phi u^\alpha - \phi \wedge dx^\beta \partial_\beta u^\alpha \quad (69)$$

A balance equation with the exterior covariant derivative then becomes

$$DT = 0 \quad \implies \quad d(T \cdot u) = -T \wedge \nabla u \quad (70)$$

Then

$$0 = D\mathbf{T}$$

$$\begin{aligned}
 &= D(-e \, d_t^3 \otimes dt - q^i \, d_i^3 \otimes dt + p_j \, d_t^3 \otimes dx^j + \pi_j^i \, d_i^3 \otimes dx^j) \\
 &= -\partial_t e \, d^4 \otimes dt - \partial_i q^i \, d^4 \otimes dt + \partial_t p_j \, d^4 \otimes dx^j + \partial_i \pi_j^i \, d^4 \otimes dx^j - \\
 &\quad \left[e \, \Gamma_{tt}^t \, d_t^3 \wedge dt \otimes dt + e \, \Gamma_{tj}^t \, d_t^3 \wedge dt \otimes dx^j + 0 \right. \\
 &\quad q^i \, \Gamma_{it}^t \, d_i^3 \wedge dx^i \otimes dt + q^i \, \Gamma_{ij}^t \, d_i^3 \wedge dx^i \otimes dx^j + 0 \\
 &\quad - p_j \, \Gamma_{tt}^j \, d_t^3 \wedge dt \otimes dt - p_k \, \Gamma_{tj}^k \, d_t^3 \wedge dt \otimes dx^j + 0 \\
 &\quad \left. - \pi_k^i \, \Gamma_{it}^k \, d_i^3 \wedge dx^i \otimes dt - \pi_k^i \, \Gamma_{ij}^k \, d_i^3 \wedge dx^i \otimes dx^j \right] \\
 &= d^4 \otimes [\\
 &\quad - \partial_t e \, dt + e \, \Gamma_{tt}^t \, dt + e \, \Gamma_{tj}^t \, dx^j \\
 &\quad - \partial_i q^i \, dt + q^i \, \Gamma_{it}^t \, dt + q^i \, \Gamma_{ij}^t \, dx^j \\
 &\quad + \partial_t p_j \, dx^j - p_j \, \Gamma_{tt}^j \, dt - p_k \, \Gamma_{tj}^k \, dx^j \\
 &\quad + \partial_i \pi_j^i \, dx^j - \pi_k^i \, \Gamma_{it}^k \, dt - \pi_k^i \, \Gamma_{ij}^k \, dx^j \\
 &\quad] \\
 &\tag{71}
 \end{aligned}$$

The equation above corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e \, \Gamma_{tt}^t + q^i \, \Gamma_{it}^t - p_j \, \Gamma_{tt}^j - \pi_k^i \, \Gamma_{it}^k \tag{72}$$

$$\partial_t p_j + \partial_i \pi_j^i = -e \, \Gamma_{tj}^t - q^i \, \Gamma_{ij}^t + p_k \, \Gamma_{tj}^k + \pi_k^i \, \Gamma_{ij}^k \tag{73}$$

For $\mathbf{T} \cdot \mathbf{u}$ we find first

$$\mathbf{T} \cdot \mathbf{u} = (-e \, u^t + p_j \, u^j) \, d_t^3 + (-q^i \, u^t + \pi_j^i \, u^j) \, d_i^3 \tag{74}$$

$$\begin{aligned}
 \mathbf{T} \wedge \nabla \mathbf{u} &= (-e \, \partial_t u^t + p_j \, \partial_t u^j) \, d_t^3 \wedge dt + (-q^i \, \partial_i u^t + \pi_j^i \, \partial_i u^j) \, d_i^3 \wedge dx^i \\
 &\quad + \Gamma \text{ terms} \\
 &\tag{75}
 \end{aligned}$$

and therefore

$$\begin{aligned}
 \partial_t(-e u^t + p_j u^j) + \partial_i(-q^i u^t + \pi_j^i u^j) = \\
 -e \partial_t u^t + p_j \partial_t u^j - q^i \partial_i u^t + \pi_j^i \partial_i u^j \\
 - (e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k) u^t \\
 - (e \Gamma_{tj}^t + q^i \Gamma_{ij}^t - p_k \Gamma_{tj}^k - \pi_k^i \Gamma_{ij}^k) u^j \quad (76)
 \end{aligned}$$

Radial case

$$\partial_t e + \partial_r q^r = e \Gamma_{tt}^t + q^r \Gamma_{rt}^t + p_r \Gamma_{tt}^r + \pi_r^r \Gamma_{rt}^r \quad (77)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \Gamma_{tr}^t + q^r \Gamma_{rr}^t + p_r \Gamma_{tr}^r + \pi_r^r \Gamma_{rr}^r \quad (78)$$

$$\partial_t e + \partial_r q^r = q^r \frac{g}{c^2} + p_r g \quad (79)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \frac{g}{c^2} \quad (80)$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have³

$$\Gamma_{jt}^t = \Gamma_{tj}^t = \frac{GM}{c^2} \frac{x^j}{r^3} \approx \frac{g}{c^2} \quad \Gamma_{tt}^j = GM \frac{x^j}{r^3} \approx g \quad (81)$$

where g is the standard acceleration, considered positive. Take also $p_j \approx mv_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

The balances above become

$$\partial_t e + \partial_i q^i = q^z \frac{g}{c^2} + p_z g \quad (82)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} \quad (83)$$

Also,

³ Poisson & Will 2014 §5.2.3.

$$\begin{aligned}\Gamma_{tt}^t &\approx -2\frac{g}{c^2}v(t) & \Gamma_{jt}^t &= \Gamma_{tj}^t \approx \frac{g}{c^2} \\ \Gamma_{tt}^j &\approx g - 2\frac{g}{c^2}v(t)^2 - \dot{v}(t) & \Gamma_{jt}^j &= \Gamma_{tj}^j \approx \frac{g}{c^2}v(t)\end{aligned}\quad (84)$$

$$\partial_t e + \partial_i q^i = -e 2\frac{g}{c^2}v(t) + q^z \frac{g}{c^2} + p_j \left(g - 2\frac{g}{c^2}v(t)^2 - \dot{v}(t) \right) + \pi_z^z \frac{g}{c^2}v(t) \quad (85)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} + p_z \frac{g}{c^2}v(t) \quad (86)$$

C Works with useful content

- Eq. (21) in⁴: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in⁵

For transformation or raising:⁶.

$$\gamma' \cdot (B \wedge A) = (-1)^{\deg A \deg B} \gamma' \cdot (A \wedge B) \quad (87)$$

with $\deg(B) = n - \deg(A)$ ⁷

Compound matrices:⁸

Bibliography

(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

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⁴ Maugin 1974. ⁵ Gourgoulhon 2012. ⁶ Gantmacher 2000 § I.4 eq. (33). ⁷ Barnabei et al. 1985 prop. 4.1. ⁸ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199, Problem 1 p. 270.

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