

Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z) , which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

There are 18 basic types of geometric objects, which represent physical quantities of different types. We have inner-oriented scalars, inner-oriented 1-vectors up to 4-vectors, inner-oriented 1-covectors up to 4-covectors, and all their outer-oriented counterparts.

The exterior product ‘ \wedge ’ of vectors and covectors is usually represented just by juxtaposition, and a shortened notation is used for exterior products of several exterior derivatives ‘ d ’. For instance

$$d^2xy := dx \wedge dy \quad \partial_{xy}^2 := \partial_x \wedge \partial_y . \quad (1)$$

The associated bases for inner-oriented covector fields are

$$dt \quad dx \quad dy \quad dz \quad (2)$$

$$d^2tx \quad d^2ty \quad d^2tz \quad d^2yz \quad d^2zx \quad d^2xy \quad (3)$$

$$d^3xyz \quad -d^3tyz \quad -d^3tzx \quad -d^3txy \quad (4)$$

$$d^4txyz \quad (5)$$

and analogously for inner-oriented vector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it’s only defined on a coordinate patch. It is idempotent: $\tilde{1}\tilde{1} = 1$.

A twisted or outer-oriented 3-covector such as $d^3\tilde{x}yz$ has an associated outer direction, in this case positive t . We adopt this shorter notation for the outer-oriented versions of the bases above (analogous to the notation in Gotay & Marsden 1992 §2 p. 371):

$$-d_{xyz} \quad d_{tyz} \quad d_{tzx} \quad d_{txy} \quad (6a)$$

$$d_{yz}^2 \quad d_{zx}^2 \quad d_{xy}^2 \quad d_{tx}^2 \quad d_{ty}^2 \quad d_{tz}^2 \quad (6b)$$

$$d_t^3 \quad d_x^3 \quad d_y^3 \quad d_z^3 \quad (6c)$$

$$d^4 \quad (6d)$$

so that $-d_{xyz} := d\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz := \partial_{\tilde{t}}$. Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} & n_{xyz} d^3\tilde{x}yz - n_{tyz} d^3\tilde{t}yz - n_{tzx} d^3\tilde{t}zx - n_{txy} d^3\tilde{t}xy \\ & \equiv n^t d_t^3 + n^x d_x^3 + n^y d_y^3 + n^z d_z^3 \end{aligned} \quad (7)$$

with $n^t \equiv n_{xyz}$, $n^x \equiv n_{tyz}$, and so on.

Contraction or dot-product of vectors and covectors is denoted by ‘ \cdot ’, and always contracts the maximal possible amount of contiguous elements. For instance

$$d^3xyz \cdot \partial_{yz}^2 = dx \quad - \partial_{txy}^3 \cdot dx = \partial_{ty}^2. \quad (8)$$

Contractions with the 4-vector ∂^4 and 4-covector d^4 establish a duality between outer n -covectors and inner $(4 - n)$ -vectors:

$$\begin{pmatrix} \partial^4 \\ \partial_{xyz}^3 \quad \partial_{tyz}^3 \quad \partial_{tzx}^3 \quad \partial_{txy}^3 \\ \partial_{tx}^2 \quad \partial_{ty}^2 \quad \partial_{tz}^2 \quad \partial_{yz}^2 \quad \partial_{zx}^2 \quad \partial_{xy}^2 \\ \partial_t \quad \partial_x \quad \partial_y \quad \partial_z \\ 1 \end{pmatrix} \begin{matrix} \xleftarrow{\partial^4 \cdot} \\ \xrightarrow{\cdot d^4} \end{matrix} \begin{pmatrix} \tilde{1} \\ d_{xyz} \quad d_{tyz} \quad d_{tzx} \quad d_{txy} \\ d_{yz}^2 \quad d_{zx}^2 \quad d_{xy}^2 \quad d_{tx}^2 \quad d_{ty}^2 \quad d_{tz}^2 \\ d_t^3 \quad d_x^3 \quad d_y^3 \quad d_z^3 \\ d^4 \end{pmatrix} \quad (9)$$

These duals have special properties. For instance, for any 3-covector \mathbf{N} , we have

$$\mathbf{N} \cdot (\partial^4 \cdot \mathbf{N}) \cdot \mathbf{N} = 0 \quad (10)$$

that is, the dual of \mathbf{N} is a vector belonging in the kernel of \mathbf{N} .

If $\boldsymbol{\gamma}$ is a non-zero 4-covector and $\boldsymbol{\gamma}^{-1}$ the inverse 4-vector, that is, $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{\gamma} = 1$, and if \mathbf{N} is a 3-covector and $\boldsymbol{\phi}$ a 1-covector, we have the useful identity

$$\mathbf{N} \wedge \boldsymbol{\phi} = (\mathbf{N} \cdot \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{\phi}) \boldsymbol{\gamma} , \quad (11)$$

which also holds as long as the degrees of \mathbf{N} and $\boldsymbol{\phi}$ sum up to 4.

We often consider vector- or covector-valued covectors, written as sums of tensor products. They can be outer- or inner-oriented. For instance

$$d_t^3 \otimes dx \equiv d^3 \tilde{x} y z \otimes dx , \quad d_x^3 \otimes \partial_y . \quad (12)$$

The operation \mathbf{A} between a vector-valued covector and a covector-valued covector is the contraction of their vector- and covector-valued parts and the exterior product of their covector parts. For instance, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are covectors, $\boldsymbol{\omega}$ is a covector, and \mathbf{u} is a vector, then

$$(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \mathbf{A} (\boldsymbol{\psi} \otimes \mathbf{u}) := (\boldsymbol{\phi} \wedge \boldsymbol{\psi}) \otimes (\boldsymbol{\omega} \cdot \mathbf{u}) \quad (13)$$

As another example,

$$(d_t^3 \otimes dx) \mathbf{A} (d_t \otimes \partial_x) = (d_t^3 \wedge d_t) (dx \cdot \partial_x) = -d^4 . \quad (14)$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta} du^\beta ,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$d\alpha' = \frac{\partial \alpha'}{\partial \alpha} d\alpha$$

- raising or lowering of indices with the metric, for example

$$d\alpha \mapsto g^{\alpha\beta} \partial_\beta .$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix¹. For the spaces of 3-vectors and 3-covectors we have simplified formulae²:

$$\begin{aligned} d_{\alpha'}^3 &= \det\left(\frac{\partial \alpha'}{\partial \alpha}\right) \frac{\partial \alpha}{\partial \alpha'} d_{\alpha}^3 \\ \partial^3 \alpha' &= \det\left(\frac{\partial \alpha}{\partial \alpha'}\right) \frac{\partial \alpha'}{\partial \alpha} \partial^3 \alpha \end{aligned} \quad (15)$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} d_{\alpha}^3 &\mapsto \frac{1}{|g|} g_{\alpha\beta} \partial^3 \beta \\ \partial^3 \alpha &\mapsto |g| g^{\alpha\beta} d_{\beta}^3 \end{aligned} \quad (16)$$

3 Metric

We take the metric \mathbf{g} to have signature $(-, +, +, +)$ and dimensions of area. The determinant and the negative of its square root are denoted shortly

$$g := \det \mathbf{g} \quad \sqrt{g} := \sqrt{-\det \mathbf{g}}. \quad (17)$$

The volume element induced by the metric \mathbf{g} has dimensions of volume-time and is denoted (note the boldface)

$$\boldsymbol{\gamma} := \frac{\sqrt{g}}{c} d^4 \tilde{x} y z \equiv \frac{\sqrt{g}}{c} d^4 \quad (18)$$

and its corresponding inverse, a twisted 4-vector:

$$\boldsymbol{\gamma}^{-1} := \frac{c}{\sqrt{g}} \partial^4. \quad (19)$$

¹ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199. ² Gantmacher 2000 § I.4 eq. (33).

Contraction with the volume element or its inverse establishes a “volume duality” between outer n -covectors and inner $(4 - n)$ -vectors:

$$\begin{pmatrix} \partial^4 \\ \partial^3_{xyz} & \partial^3_{tyz} & \partial^3_{tzx} & \partial^3_{txy} \\ \partial^2_{tx} & \partial^2_{ty} & \partial^2_{tz} & \partial^2_{yz} & \partial^2_{zx} & \partial^2_{xy} \\ \partial_t & \partial_x & \partial_y & \partial_z \\ 1 \end{pmatrix} \begin{matrix} \xleftarrow{\frac{\sqrt{g}}{c} \boldsymbol{\gamma}^{-1} \cdot} \\ \xrightarrow{\cdot \boldsymbol{\gamma} \frac{c}{\sqrt{g}}} \end{matrix} \begin{pmatrix} \tilde{1} \\ d_{xyz} & d_{tyz} & d_{tzx} & d_{txy} \\ d^2_{yz} & d^2_{zx} & d^2_{xy} & d^2_{tx} & d^2_{ty} & d^2_{tz} \\ d^3_t & d^3_x & d^3_y & d^3_z \\ d^4 \end{pmatrix} \quad (20)$$

This is the reason why in older literature an outer-oriented n -covector is treated as a $(4 - n)$ -“vector density”, that is, a vector divided by the square root of the volume element.

The metric on the space of 1-vectors also induces metrics on the spaces of 1-covectors, 2-vectors and 2-covectors, and so on. In particular, the metrics $\overset{3}{g}^{-1}$ on the space of 3-covectors and $\overset{3}{g}$ on the space of 3-vectors can be written in coordinates as

$$\overset{3}{g}^{-1} = \frac{g^{\mu\nu}}{g} \partial^3_\mu \otimes \partial^3_\nu \quad \text{with dimensions length}^{-6} \quad (21)$$

$$\overset{3}{g} = g g^{\mu\nu} d^3_\mu \otimes d^3_\nu \quad \text{with dimensions length}^6. \quad (22)$$

With these we can define squared norms $\|\cdot\|^2$ on all those spaces. Note in particular the following identity:

$$\|\boldsymbol{\gamma}^{-1} \cdot \mathbf{N}\|^2 = -c^2 \|\mathbf{N}\| \quad \text{for every 3-covector } \mathbf{N}. \quad (23)$$

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (24)$$

and the volume element is simply d^4 .

4 Matter current

The amount-of-matter current \mathbf{N} is an outer-oriented 3-covector

$$\mathbf{N} = N d_t^3 + J^i d_i^3 \quad (25)$$

of dimensions “amount of matter”, typically measured in moles, where

- N is the volumic amount of matter, measured per unit coordinate volume.
- J^i is the aeric flux of amount of matter (equivalent to a molar flux), measured per unit coordinate area and unit coordinate time.

The current for every particular kind of matter satisfies the conservation law

$$d\mathbf{N} = 0 \quad \text{or} \quad \partial_t N + \partial_i J^i = 0 \quad (26)$$

independent of any metric.

The common contravariant form of the matter current, “ N^μ ”, is obtained by contracting the matter current with the inverse volume element:

$${}'N^\mu \triangleq \boldsymbol{\gamma}^{-1} \cdot \mathbf{N} = \frac{c}{\sqrt{g}} N \partial_t + \frac{c}{\sqrt{g}} J^i \partial_i. \quad (27)$$

If a metric is present, a four-velocity \mathbf{U} can be associated with the matter current \mathbf{N} , defined by the following properties and identity:

$$\mathbf{U} \cdot \mathbf{N} = 0 \quad \|\mathbf{U}\|^2 = -c^2 \quad (28)$$

$$\mathbf{U} = \frac{1}{\|\mathbf{N}\|} \boldsymbol{\gamma}^{-1} \cdot \mathbf{N} \quad (29)$$

which also implies (for normal matter)

$$\mathbf{N} = \|\mathbf{N}\| \mathbf{U} \cdot \boldsymbol{\gamma} \quad (30)$$

For normal matter (as opposed to antimatter) $\|\mathbf{N}\|^2 \geq 0$.

5 Four-stress

The stress-energy-momentum tensor, or simply four-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\begin{aligned} \mathbf{T} &= T^\mu{}_\nu \, d^3_\mu \otimes dx^\nu \\ &= -\epsilon \, d^3_t \otimes dt - q^i \, d^3_i \otimes dt + p_j \, d^3_i \otimes dx^j + \pi^i_j \, d^3_i \otimes dx^j \end{aligned} \quad (31)$$

the indices i, j running over x, y, z , and where:

- The energy ϵ is a density per unit *coordinate* volume xyz , and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this energy comprises “rest” energy, kinetic energy, potential gravitational energy, and centrifugal potential energy.
- The energy flux q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this flux term includes transport of the energy term above and also heat flux.
- The momentum p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i .
- The compressive three-stress π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j . If the point at which the four-stress is considered has a matter-current, then this stress comprises momentum transport and internal stresses.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \quad (32)$$

The diagonal elements g_{tt}, \dots include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

The common contra-contra-variant form of the stress, “ $T^{\mu\nu}$ ”, is obtained by contracting the four-stress with the inverse volume element and the inverse metric:

$$\begin{aligned} {}'T^{\mu\nu} \triangleq \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1} &= -\frac{c}{\sqrt{g}} \frac{g^{tt}}{\sqrt{g}} \epsilon \partial_t \otimes \partial_t - \frac{c}{\sqrt{g}} \frac{g^{tt}}{\sqrt{g}} q^i \partial_i \otimes \partial_t \\ &\quad + \sum_j \frac{c}{\sqrt{g}} \frac{g^{jj}}{\sqrt{g}} p_j \partial_t \otimes \partial_j + \sum_j \frac{c}{\sqrt{g}} \frac{g^{jj}}{\sqrt{g}} \pi_j^i \partial_i \otimes \partial_j . \end{aligned} \quad (33)$$

One important detail in finding the Newtonian approximation of “energy density” is that *one takes different zeros of energy density in different coordinate systems*: the zero is taken as the molar mass times the molar density in the current coordinate system. By ‘zero’ I mean the arbitrary separation between “mass” and “energy”.

The *total* four-stress satisfies the balance equation

$$D\boldsymbol{T} = 0 \quad (34)$$

which is equivalent to the four balance equations

$$\begin{aligned} \partial_t e + \partial_i q^i &= e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k \\ \partial_t p_j + \partial_i \pi_j^i &= -e \Gamma_{tj}^t - q^i \Gamma_{ij}^t + p_k \Gamma_{tj}^k + \pi_k^i \Gamma_{ij}^k \end{aligned} \quad (35)$$

In general relativity the *total* four-stress also satisfies

$$(\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1})^\top - \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1} = 0 . \quad (36)$$

The four-stress determines an association between any 1-vector \boldsymbol{V} field and an outer-oriented 3-covector field, interpreted as a current:

$$\boldsymbol{V} \mapsto \boldsymbol{T} \cdot \boldsymbol{V} . \quad (37)$$

This current satisfies the balance equation

$$d(\boldsymbol{T} \cdot \boldsymbol{V}) = -\boldsymbol{T} \wedge \nabla \boldsymbol{V} = \text{tr}(\boldsymbol{T}^\top \cdot \boldsymbol{\gamma}^{-1} \cdot \nabla \boldsymbol{V}) \boldsymbol{\gamma} \quad (38)$$

which is a conservation law if \boldsymbol{V} is a Killing vector.

For the special case $\mathbf{V} = \partial_\alpha$ the formula above becomes

$$d(T^\mu_\alpha d^3_\mu) = T^\mu_\nu \Gamma^\nu_{\mu\alpha} d^4 \iff \partial_\mu T^\mu_\alpha = T^\mu_\nu \Gamma^\nu_{\mu\alpha} \quad (39)$$

Consider a region where there is a non-vanishing matter current \mathbf{N} with associated four-velocity \mathbf{U} , and define

$$\bar{\mathbf{U}} = -\frac{1}{c} \mathbf{g} \cdot \mathbf{U} \quad (40)$$

which satisfies

$$\bar{\mathbf{U}} \cdot \mathbf{U} = 1, \quad \nabla \mathbf{U} \cdot \bar{\mathbf{U}} = 0. \quad (41)$$

The last equality can be proved from $\nabla \mathbf{g} = 0$ and

$$0 = -\nabla(c^2) = \nabla(\mathbf{U} \cdot \mathbf{g} \cdot \mathbf{U}) = 2(\nabla \mathbf{U}) \cdot \mathbf{g} \cdot \mathbf{U}. \quad (42)$$

We can associate with the matter a four-stress \mathbf{T} which can be decomposed as follows:

$$\begin{aligned} \mathbf{T} &= -\epsilon \mathbf{N} \otimes \bar{\mathbf{U}} + \mathbf{N} \otimes \mathbf{P} - (\bar{\mathbf{U}} \wedge \mathbf{Q}) \otimes \bar{\mathbf{U}} + \bar{\mathbf{U}} \wedge \mathbf{S} \\ \text{with } \mathbf{P} \cdot \mathbf{U} &= 0 \quad \mathbf{Q} \cdot \mathbf{U} = 0 \quad \mathbf{U} \cdot \mathbf{S} = 0 \quad \mathbf{S} \cdot \mathbf{U} = 0 \end{aligned} \quad (43)$$

where

- ϵ is a scalar, the molar energy density.
- \mathbf{P} is a 1-covector, the molar momentum density.
- \mathbf{Q} is a 2-covector, the areic energy-flux density.
- \mathbf{S} is a 1-covector-valued 2-covector, the three-stress related to the Cauchy stress.

Using the four-velocity \mathbf{U} associated with the matter current we obtain what could be called the “internal-energy current”:

$$\mathbf{T} \cdot \mathbf{U} \equiv -\epsilon \mathbf{N} - \bar{\mathbf{U}} \mathbf{Q} \quad (44)$$

which, from eqs (38), (40), (26), satisfies the balance law

$$d(\mathbf{T} \cdot \mathbf{U}) = -\mathbf{T} \mathbin{\wedge} \nabla \mathbf{U} \quad (45)$$

or

$$\begin{aligned} d(-\epsilon \mathbf{N} - \bar{\mathbf{U}} \mathbf{Q}) &= (\epsilon \mathbf{N} \otimes \bar{\mathbf{U}}) \cdot \nabla \mathbf{U} - (\mathbf{N} \otimes \mathbf{P}) \cdot \nabla \mathbf{U} \\ &\quad + (\bar{\mathbf{U}} \mathbf{Q} \otimes \bar{\mathbf{U}}) \cdot \nabla \mathbf{U} - \bar{\mathbf{U}} \mathbf{S} \cdot \nabla \mathbf{U} \end{aligned} \quad (46)$$

or simply

$$\mathbf{N}d\epsilon + \bar{\mathbf{U}}d\mathbf{Q} - \mathbf{Q}d\bar{\mathbf{U}} = -\mathbf{N}\nabla\mathbf{U} \cdot \mathbf{P} - (\bar{\mathbf{U}}\mathbf{S}) \cdot \nabla\mathbf{U}. \quad (47)$$

The last balance equation corresponds to eq. (28) in Eckart (1940), except for the fact that here we are keeping the momentum \mathbf{P} and heat flux \mathbf{Q} distinct.

When \mathbf{P} , \mathbf{S} , \mathbf{Q} are zero, one speaks of the four-stress of “dust”. It is noteworthy that in this case the four-stress is essentially proportional to the matter 3-covector, multiplied by a vector collinear with the matter four-velocity:

$$\mathbf{T} = -\mathbf{N} \otimes (\epsilon \bar{\mathbf{U}}). \quad (48)$$

If we take the four-velocity itself as reference, given that $\bar{\mathbf{U}}\mathbf{U} = -c^2$, the energy 3-covector we obtain is proportional to the matter 3-covector:

$$-c^2\epsilon \mathbf{N} \quad (49)$$

For this reason ϵ is called the “proper internal energy”. If we take other vector fields as reference, then this energy will pick up further multiplicative terms consisting in the projection of the matter four-velocity \mathbf{U} onto the reference vector field. In particular taking ∂_t or ∂_i as reference we’ll have the projections of the four-velocity onto the coordinate covectors as multiplicative factors.

6 Electromagnetic field

The electromagnetic field, represented by the Faraday 2-covector, is typically decomposed as³

$$\begin{aligned} \mathbf{F} &= \mathbf{E} dt + \mathbf{B} \\ &\equiv E_x d^2xt + E_y d^2yt + E_z d^2zt + B^x d^2yz + B^y d^2zx + B^z d^2xy. \end{aligned} \quad (50)$$

Given a system of coordinates, this decomposition is unique: the 2-covector \mathbf{B} does not have dt components, that is, it has ∂_t in its kernel; and so does the 1-covector \mathbf{E} .

³ Frankel 1979 ch. 9.

The conservation of magnetic flux is expressed by

$$d\mathbf{F} = 0 \quad (51)$$

or equivalently

$$\begin{aligned} \partial_i B^i &= 0 & (d^3xyz \text{ component}) \\ \partial_t B^x + \partial_y E_z - \partial_z E_y &= 0 & (d^3tyz) \\ \partial_t B^y + \partial_z E_x - \partial_x E_z &= 0 & (d^3txz) \\ \partial_t B^z + \partial_x E_y - \partial_y E_x &= 0 & (d^3txy) \end{aligned} \quad (52)$$

It should be noted that all components of $d\mathbf{E}$ containing dt also disappear, because of the exterior product $\mathbf{E} dt$. The differential d therefore operates on \mathbf{E} as if this 1-covector belonged to a three-dimensional manifold with coordinates (x, y, z) . Let's denote this operation by \mathbf{d} ; it is connected with the "curl" operator. Then the equations above can be written

$$d\mathbf{B} = 0 \quad \partial_t \mathbf{B} + d\mathbf{E} = 0. \quad (53)$$

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- \mathbf{u}^\bullet is a 1-vector, represented by the column-matrix \mathbf{u} .

- Similarly for \boldsymbol{v}^* .
- $\boldsymbol{\omega}_\bullet$ is a 1-covector, represented by the row-matrix $\boldsymbol{\omega}$.
- $\boldsymbol{g}_{\bullet\bullet}$ is a covector-valued covector, represented by the matrix \boldsymbol{g} . The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\boldsymbol{g}^{-1\bullet\bullet}$ is a vector-valued vector, inverse of \boldsymbol{g} , that is: $\boldsymbol{g} \cdot \boldsymbol{g}^1 = \text{id}_\bullet^\bullet$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\boldsymbol{Q}_{|\bullet\bullet\bullet|}$ is a 3-covector, represented by the column-matrix \boldsymbol{Q} .
- $\boldsymbol{T}_{|\bullet\bullet\bullet|}$ is a covector-valued 3-covector, represented by the matrix \boldsymbol{T} . The rows represent the 3-covector components; the columns, the covector components.
- $\boldsymbol{\gamma}_{|\bullet\bullet\bullet|}$ is a 4-covector, represented by the number $\boldsymbol{\gamma}$.
- $\boldsymbol{\gamma}^{-1|\bullet\bullet\bullet|}$ is a 4-vector, represented by the number $\boldsymbol{\gamma}^{-1}$.
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from “old” coordinates x to “new” coordinates x' is represented by the matrix \boldsymbol{J} . The rows correspond to the new coordinates x' ; the columns, to the old x .
- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x is represented by the matrix \boldsymbol{J}^{-1} . The rows correspond to the old coordinates x ; the columns, to the new x' .

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

$$\text{(object)} \qquad \qquad \qquad \text{(matrix repr)} \qquad \qquad \qquad (54)$$

$$\boldsymbol{\omega} \cdot \boldsymbol{u} \qquad \qquad \qquad \boldsymbol{\omega} \boldsymbol{u} \equiv \boldsymbol{u}^\top \boldsymbol{\omega}^\top \quad \text{(number)} \qquad (55)$$

$$\boldsymbol{v} \cdot \boldsymbol{g} \cdot \boldsymbol{u} \qquad \qquad \qquad \boldsymbol{v}^\top \boldsymbol{g} \boldsymbol{u} \quad \text{(number)} \qquad (56)$$

$$\boldsymbol{g} \cdot \boldsymbol{u} \qquad \qquad \qquad \boldsymbol{u}^\top \boldsymbol{g}^\top \quad \text{(row-matrix)} \qquad (57)$$

$$\boldsymbol{\omega} \cdot \boldsymbol{g}^{-1} \qquad \qquad \qquad \boldsymbol{g}^{-\top} \boldsymbol{\omega}^\top \quad \text{(column-matrix)} \qquad (58)$$

$$\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{Q} \qquad \qquad \qquad \boldsymbol{\gamma}^{-1} \boldsymbol{Q} \quad \text{(column-matrix)} \qquad (59)$$

$$\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \qquad \qquad \qquad \boldsymbol{\gamma}^{-1} \boldsymbol{T} \quad \text{(column-matrix)} \qquad (60)$$

$$\boldsymbol{T} \cdot \boldsymbol{u} \qquad \qquad \qquad \boldsymbol{T} \boldsymbol{u} \quad \text{(column-matrix)} \qquad (61)$$

$$\boldsymbol{g} \cdot (\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T}) \cdot \boldsymbol{u} \qquad \qquad \qquad \boldsymbol{\gamma}^{-1} \boldsymbol{g} \boldsymbol{T} \boldsymbol{u} \quad \text{(matrix)} \qquad (62)$$

- Transformations

$$(\text{old coords}) \mapsto (\text{new coords}) \quad (63)$$

$$\mathbf{u} \mapsto \mathbf{J}\mathbf{u} \quad (64)$$

$$\boldsymbol{\omega} \mapsto \boldsymbol{\omega}\mathbf{J}^{-1} \quad (65)$$

$$\mathbf{g} \mapsto \mathbf{J}^{-\top}\mathbf{g}\mathbf{J}^{-1} \quad (66)$$

$$\mathbf{Q} \mapsto \frac{1}{\det \mathbf{J}} \mathbf{J}\mathbf{Q} \quad (67)$$

$$\boldsymbol{\gamma} \mapsto \frac{1}{\det \mathbf{J}} \boldsymbol{\gamma} \quad (68)$$

$$\boldsymbol{\gamma}^{-1} \mapsto \det \mathbf{J} \boldsymbol{\gamma}^{-1} \quad (69)$$

$$\mathbf{T} \mapsto \frac{1}{\det \mathbf{J}} \mathbf{J}\mathbf{T}\mathbf{J}^{-1} \quad (70)$$

B Checks about optimal representation of four-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$\begin{aligned} d(f d_t^3) &= d(f d^3 \tilde{x} y z) = \partial_t f dt \wedge d^3 \tilde{x} y z + 0 = \partial_t f d^4 \tilde{x} y z \\ d(f d_x^3) &= -d(f d^3 \tilde{t} y z) = -\partial_x f dx \wedge d^3 \tilde{t} y z + 0 = \partial_x f d^4 \tilde{t} x y z \\ d(f d_i^3) &= \partial_i f d^4 \tilde{t} x y z \end{aligned} \quad (71)$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned} \nabla(dt) &= -\Gamma_{tt}^t dt \otimes dt - \Gamma_{tj}^t dt \otimes dx^j - \Gamma_{it}^t dx^i \otimes dt - \Gamma_{ij}^t dx^i \otimes dx^j \\ \nabla(dx^k) &= -\Gamma_{tt}^k dt \otimes dt - \Gamma_{tj}^k dt \otimes dx^j - \Gamma_{it}^k dx^i \otimes dt - \Gamma_{ij}^k dx^i \otimes dx^j \end{aligned} \quad (72)$$

If $\boldsymbol{\phi}$ is a 3-covector, $\boldsymbol{\omega}$ a 1-covector, and D the exterior covariant derivative, then

$$D(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) = (d\boldsymbol{\phi}) \otimes \boldsymbol{\omega} - \boldsymbol{\phi} \wedge \nabla \boldsymbol{\omega} \quad (73)$$

and in particular

$$D(\boldsymbol{\phi} \otimes dx^\alpha) = (d\boldsymbol{\phi}) \otimes dx^\alpha + \Gamma_{\mu\nu}^\alpha (\boldsymbol{\phi} \wedge dx^\mu) \otimes dx^\nu. \quad (74)$$

Let's also consider the contraction with a 1-vector \mathbf{u} :

$$\begin{aligned}
 D(\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \mathbf{u}) &\equiv d[\boldsymbol{\phi}(\boldsymbol{\omega} \cdot \mathbf{u})] \\
 &= (d\boldsymbol{\phi})(\boldsymbol{\omega} \cdot \mathbf{u}) - \boldsymbol{\phi} \wedge d(\boldsymbol{\omega} \cdot \mathbf{u}) \\
 &= d\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \mathbf{u} - \boldsymbol{\phi} \wedge \nabla \boldsymbol{\omega} \cdot \mathbf{u} - \boldsymbol{\phi} \wedge \nabla \mathbf{u} \cdot \boldsymbol{\omega} \\
 &\equiv (d\boldsymbol{\phi} \otimes \boldsymbol{\omega} - \boldsymbol{\phi} \wedge \nabla \boldsymbol{\omega}) \cdot \mathbf{u} - \boldsymbol{\phi} \wedge \nabla \mathbf{u} \cdot \boldsymbol{\omega} \\
 &\equiv D(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \mathbf{u} - (\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \mathbin{\wedge} \nabla \mathbf{u} .
 \end{aligned} \tag{75}$$

and in particular

$$D(\boldsymbol{\phi} \otimes dx^\alpha \cdot \mathbf{u}) = d\boldsymbol{\phi} u^\alpha - \boldsymbol{\phi} \wedge dx^\beta \partial_\beta u^\alpha \tag{76}$$

A balance equation with the exterior covariant derivative then becomes

$$D\mathbf{T} = 0 \quad \implies \quad d(\mathbf{T} \cdot \mathbf{u}) = -\mathbf{T} \mathbin{\wedge} \nabla \mathbf{u} \tag{77}$$

Then

$$0 = D\mathbf{T}$$

$$\begin{aligned}
 &= D(-e d_t^3 \otimes dt - q^i d_i^3 \otimes dt + p_j d_t^3 \otimes dx^j + \pi_j^i d_i^3 \otimes dx^j) \\
 &= -\partial_t e d^4 \otimes dt - \partial_i q^i d^4 \otimes dt + \partial_t p_j d^4 \otimes dx^j + \partial_i \pi_j^i d^4 \otimes dx^j - \\
 &\quad \left[e \Gamma_{tt}^t d_t^3 \wedge dt \otimes dt + e \Gamma_{tj}^t d_t^3 \wedge dt \otimes dx^j + 0 \right. \\
 &\quad q^i \Gamma_{it}^t d_i^3 \wedge dx^i \otimes dt + q^i \Gamma_{ij}^t d_i^3 \wedge dx^i \otimes dx^j + 0 \\
 &\quad - p_j \Gamma_{tt}^j d_t^3 \wedge dt \otimes dt - p_k \Gamma_{tj}^k d_t^3 \wedge dt \otimes dx^j + 0 \\
 &\quad \left. - \pi_k^i \Gamma_{it}^k d_i^3 \wedge dx^i \otimes dt - \pi_k^i \Gamma_{ij}^k d_i^3 \wedge dx^i \otimes dx^j \right] \\
 &= d^4 \otimes [\\
 &\quad - \partial_t e dt + e \Gamma_{tt}^t dt + e \Gamma_{tj}^t dx^j \\
 &\quad - \partial_i q^i dt + q^i \Gamma_{it}^t dt + q^i \Gamma_{ij}^t dx^j \\
 &\quad + \partial_t p_j dx^j - p_j \Gamma_{tt}^j dt - p_k \Gamma_{tj}^k dx^j \\
 &\quad + \partial_i \pi_j^i dx^j - \pi_k^i \Gamma_{it}^k dt - \pi_k^i \Gamma_{ij}^k dx^j \\
 &\quad] \\
 \end{aligned} \tag{78}$$

The equation above corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k \tag{79}$$

$$\partial_t p_j + \partial_i \pi_j^i = -e \Gamma_{tj}^t - q^i \Gamma_{ij}^t + p_k \Gamma_{tj}^k + \pi_k^i \Gamma_{ij}^k \tag{80}$$

For $\mathbf{T} \cdot \mathbf{u}$ we find first

$$\mathbf{T} \cdot \mathbf{u} = (-e u^t + p_j u^j) d_t^3 + (-q^i u^t + \pi_j^i u^j) d_i^3 \quad (81)$$

$$\begin{aligned} \mathbf{T} \wedge \nabla \mathbf{u} = & (-e \partial_t u^t + p_j \partial_t u^j) d_t^3 \wedge dt + (-q^i \partial_i u^t + \pi_j^i \partial_i u^j) d_i^3 \wedge dx^i \\ & + \Gamma \text{ terms} \end{aligned} \quad (82)$$

and therefore

$$\begin{aligned} \partial_t(-e u^t + p_j u^j) + \partial_i(-q^i u^t + \pi_j^i u^j) = \\ -e \partial_t u^t + p_j \partial_t u^j - q^i \partial_i u^t + \pi_j^i \partial_i u^j \\ - (e \Gamma_{tt}^t + q^i \Gamma_{it}^t - p_j \Gamma_{tt}^j - \pi_k^i \Gamma_{it}^k) u^t \\ - (e \Gamma_{tj}^t + q^i \Gamma_{ij}^t - p_k \Gamma_{tj}^k - \pi_k^i \Gamma_{ij}^k) u^j \end{aligned} \quad (83)$$

Radial case

$$\partial_t e + \partial_r q^r = e \Gamma_{tt}^t + q^r \Gamma_{rt}^t + p_r \Gamma_{tt}^r + \pi_r^r \Gamma_{rt}^r \quad (84)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \Gamma_{tr}^t + q^r \Gamma_{rr}^t + p_r \Gamma_{tr}^r + \pi_r^r \Gamma_{rr}^r \quad (85)$$

$$\partial_t e + \partial_r q^r = q^r \frac{g}{c^2} + p_r g \quad (86)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \frac{g}{c^2} \quad (87)$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have⁴

$$\Gamma_{jt}^t = \Gamma_{ij}^t = \frac{GM}{c^2} \frac{x^j}{r^3} \approx \frac{g}{c^2} \quad \Gamma_{tt}^j = GM \frac{x^j}{r^3} \approx g \quad (88)$$

where g is the standard acceleration, considered positive. Take also $p_j \approx mv_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

⁴ Poisson & Will 2014 §5.2.3.

The balances above become

$$\partial_t e + \partial_i q^i = q^z \frac{g}{c^2} + p_z g \quad (89)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} \quad (90)$$

Also,

$$\begin{aligned} \Gamma_{tt}^t &\approx -2 \frac{g}{c^2} v(t) & \Gamma_{jt}^t &= \Gamma_{tj}^t \approx \frac{g}{c^2} \\ \Gamma_{tt}^j &\approx g - 2 \frac{g}{c^2} v(t)^2 - \dot{v}(t) & \Gamma_{jt}^j &= \Gamma_{tj}^j \approx \frac{g}{c^2} v(t) \end{aligned} \quad (91)$$

$$\partial_t e + \partial_i q^i = -e 2 \frac{g}{c^2} v(t) + q^z \frac{g}{c^2} + p_j \left(g - 2 \frac{g}{c^2} v(t)^2 - \dot{v}(t) \right) + \pi_z^z \frac{g}{c^2} v(t) \quad (92)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} + p_z \frac{g}{c^2} v(t) \quad (93)$$

C Works with useful content

- Eq. (21) in⁵: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in⁶

For transformation or raising:⁷.

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B} \gamma \cdot (A \wedge B) \quad (94)$$

with $\deg(B) = n - \deg(A)$ ⁸

Compound matrices:⁹

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(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

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⁵ Maugin 1974. ⁶ Gourgoulhon 2012. ⁷ Gantmacher 2000 § I.4 eq. (33). ⁸ Barnabei et al. 1985 prop. 4.1. ⁹ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199, Problem 1 p. 270.

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