Notes on multivector algebra on differential manifolds

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1 Multivector tensor algebra

The idea is to build tensors not from the two spaces of vectors and covectors, but from the $2N^2$ spaces of multivectors and multicovectors with their possible straight and twisted orientations.

The exterior algebra is an algebra independent of the tensor one, and it expresses very intuitive geometric relations[1].

The fact that it is independent of the tensor algebra is clear from the fact that we can establish several inequivalent relations between the tensor and exterior products, none of them being canonical.

Antisymmetrizer A (a projection):

$$AT \coloneqq \frac{1}{(\deg T)!} \sum_{\pi} \operatorname{sgn}(\pi)\, T \circ \pi \tag{1}$$

Abraham et al. (1988), Choquet-Bruhat et al. (1996), Bossavit (1991) use this relation:

$$\begin{aligned} \alpha \wedge \beta &\equiv \frac{(\deg \alpha + \deg \beta)!}{(\deg \alpha)!\,(\deg \beta)!}\, A(\alpha \otimes \beta) \\ &\equiv \frac{1}{(\deg \alpha)!\,(\deg \beta)!} \sum_{\pi} \operatorname{sgn}(\pi)\, (\alpha \otimes \beta) \circ \pi\,, \end{aligned} \tag{2}$$

but relations with different multiplicative factors are also possible.

It's best to define the exterior product intrinsically, with its multilinear, associative, and graded-commutative properties.

[1] cf. Deschamps 1970; 1981.

2 Inner or dual or dot product

For a vector u and covector ω with $\deg u \leqslant \deg \omega$ it's defined as

$$
\begin{gathered}
u \rfloor \omega \coloneqq \omega(u) \quad \text{if } \deg u = \deg \omega \\
(u \rfloor \omega)(v) \coloneqq (u \wedge v) \rfloor \omega \equiv \omega(u \wedge v) \quad \text{if } \deg u < \deg \omega
\end{gathered}
\tag{3}
$$

It's possible to define an inner product from the right side, but it gives the same result as above except for a sign:

$$
\begin{aligned}
(\omega \lfloor u)(v) \coloneqq (v \wedge u) \rfloor \omega &\equiv (-1)^{\deg u\ \deg v}\, (u \wedge v) \rfloor \omega \\
&\equiv (-1)^{\deg u\ \deg v}\, (u \rfloor \omega)(v)
\end{aligned}
\tag{4}
$$
$$
\implies \omega \lfloor u \equiv (-1)^{\deg u\ (\deg \omega - \deg u)}\, u \rfloor \omega \equiv (-1)^{\deg u\ (\deg \omega - 1)}\, u \rfloor \omega
$$

For 1-vectors in particular:

$$
\omega \lfloor u \equiv (-1)^{\deg \omega - 1}\, u \rfloor \omega \qquad \text{if } \deg u = 1 \tag{5}
$$

Also

$$
\begin{gathered}
u \lfloor \omega \coloneqq \omega(u) \equiv u \rfloor \omega \quad \text{if } \deg u = \deg \omega \\
(u \lfloor \omega)(\xi) \coloneqq u \lfloor (\xi \wedge \omega) \equiv (\xi \wedge \omega)(u) \quad \text{if } \deg u > \deg \omega
\end{gathered}
\tag{6}
$$

and

$$
\begin{aligned}
(\omega \rfloor u)(\xi) \coloneqq u \rfloor (\omega \wedge \xi) &\equiv (-1)^{\deg \omega\ \deg \xi}\, u \rfloor (\xi \wedge \omega) \\
&\equiv (-1)^{\deg \omega\ \deg \xi}\, (u \lfloor \omega)(\xi)
\end{aligned}
\tag{7}
$$
$$
\implies \omega \rfloor u \equiv (-1)^{\deg \omega\ (\deg v - 1)}\, u \lfloor \omega
$$

The lower hook in "$\rfloor$" and "$\lfloor$" is useful to denote the object with lower degree, to know how to apply the sign in the graded-commutativity property and to know what kind of object – multivector or multicovector – one obtains.

If we define the degree of vectors to be negative, we can say that $\alpha \rfloor \beta$ yields an object of degree $\deg(\alpha \rfloor \beta) = \deg \alpha + \deg \beta$, no matter whether α is a vector and β a covector or vice versa. With this convention we could use the more compact dot-notation[2]

$$
\begin{gathered}
\alpha \cdot \beta \\
\text{with} \qquad \deg(\alpha \cdot \beta) = \deg \alpha + \deg \beta \\
\beta \cdot \alpha = (-1)^{\min\{|\deg \alpha|, |\deg \beta|\}\ (\deg \alpha + \deg \beta)}\, \alpha \cdot \beta\ .
\end{gathered}
\tag{8}
$$

[2] cf. Truesdell & Toupin 1960 § F.I.267.

But it doesn't make much sense to use a unique symbol, because it would not represent an associative operation (unlike the wedge).

The inner product with a 1-vector or a 1-covector is a graded derivation.

3 Inner product with N-covector

If γ is an N-covector (hypervolume covector), so that γ^{-1} is its dual N-vector, we have

$$\gamma^{-1} \cdot (u \cdot \gamma) = (\gamma \cdot u) \cdot \gamma^{-1} = u \qquad \gamma \cdot (\omega \cdot \gamma^{-1}) = (\gamma^{-1} \cdot \omega) \cdot \gamma = \omega \tag{9}$$

for any vector u and form ω. According to (4),

$$\gamma \cdot u = (-1)^{\deg(u)\ (N-1)}\, u \cdot \gamma \tag{10}$$

and analogously for the dual case.

Therefore $\gamma^{-1} \cdot (\gamma \cdot u) = (-1)^{\deg(u)\ (N-1)}\, u \neq u = (\gamma^{-1} \cdot \gamma) \cdot u$, which shows that the inner product is non-associative in general.

4 Tensor products and equivalent objects

Bibliography

("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

Abraham, R., Marsden, J. E., Ratiu, T. (1988): *Manifolds, Tensor Analysis, and Applications*, 2nd ed. (Springer, New York). First publ. 1983. DOI:10.1007/978-1-4612-1029-0.

Bossavit, A. (1991): *Differential Geometry: for the student of numerical methods in electromagnetism*. https://www.researchgate.net/publication/200018385_Differential_Geometry_for_the_student_of_numerical_methods_in_Electromagnetism.

Choquet-Bruhat, Y., DeWitt-Morette, C., Dillard-Bleick, M. (1996): *Analysis, Manifolds and Physics. Part I: Basics*, rev. ed. (Elsevier, Amsterdam). First publ. 1977.

Deschamps, G. A. (1970): *Exterior differential forms*. In: Deschamps, de Jager, John, Lions, Moisseev, Sommer, Tihonov, Tikhomirov, et al. (1970): p. III:111–161.

— (1981): *Electromagnetics and differential forms*. Proc. IEEE **69**^{6}, 676–696.

Deschamps, G. A., de Jager, E. M., John, F., Lions, J. L., Moisseev, N., Sommer, F., Tihonov, A. N., Tikhomirov, V., et al. (1970): *Mathematics applied to physics*. (Springer, Berlin). Ed. by É. Roubine.

Flügge, S., ed. (1960): *Handbuch der Physik: Band III/1: Prinzipien der klassischen Mechanik und Feldtheorie [Encyclopedia of Physics: Vol. III/1: Principles of Classical Mechanics and Field Theory]*. (Springer, Berlin). doi:10.1007/978-3-642-45943-6.

Truesdell III, C. A., Toupin, R. A. (1960): *The Classical Field Theories*. In: Flügge (1960): I–VII, 226–902. With an appendix on invariants by Jerald LaVerne Ericksen. doi:10.1007/978-3-642-45943-6_2.