Notes on multivector algebra on differential manifolds

P.G.L. Porta Mana
<pgl@portamana.org>

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1 Multivector tensor algebra

The idea is to build tensors not from the two spaces of vectors and covectors, but from the $2N^2$ spaces of multivectors and multicovectors with their possible straight and twisted orientations.

The exterior algebra is an algebra independent of the tensor one, and it expresses very intuitive geometric relations[1].

The fact that it is independent of the tensor algebra is clear from the fact that we can establish several inequivalent relations between the tensor and exterior products, none of them being canonical.

Antisymmetrizer A (a projection):

$$AT \coloneqq \frac{1}{(\deg T)!} \sum_{\pi} \operatorname{sgn}(\pi)\, T \circ \pi \tag{1}$$

Abraham et al. (1988), Choquet-Bruhat et al. (1996), Bossavit (1991) use this relation:

$$\begin{aligned} \alpha \wedge \beta &\equiv \frac{(\deg \alpha + \deg \beta)!}{(\deg \alpha)!\,(\deg \beta)!}\, A(\alpha \otimes \beta) \\ &\equiv \frac{1}{(\deg \alpha)!\,(\deg \beta)!} \sum_{\pi} \operatorname{sgn}(\pi)\, (\alpha \otimes \beta) \circ \pi\,, \end{aligned} \tag{2}$$

but relations with different multiplicative factors are also possible.

It's best to define the exterior product intrinsically, with its multilinear, associative, and graded-commutative properties.

[1] cf. Deschamps 1970; 1981.

2 Inner or dual or dot product

For a vector u and covector ω with $\deg u \leqslant \deg \omega$ it's defined as

$$\begin{aligned} u \rfloor \omega &\coloneqq \omega(u) \quad \text{if } \deg u = \deg \omega \\ (u \rfloor \omega)(v) &\coloneqq (u \wedge v) \rfloor \omega \equiv \omega(u \wedge v) \quad \text{if } \deg u < \deg \omega \end{aligned} \tag{3}$$

It's possible to define an inner product from the right side, but it gives the same result as above except for a sign:

$$\begin{aligned} (\omega \lfloor u)(v) \coloneqq (v \wedge u) \rfloor \omega &\equiv (-1)^{\deg u\ \deg v}\, (u \wedge v) \rfloor \omega \\ &\equiv (-1)^{\deg u\ \deg v}\, (u \rfloor \omega)(v) \end{aligned} \tag{4}$$

$$\implies \omega \lfloor u \equiv (-1)^{\deg u\ (\deg \omega - \deg u)}\, u \rfloor \omega \equiv (-1)^{\deg u\ (\deg \omega - 1)}\, u \rfloor \omega$$

For 1-vectors in particular:

$$\omega \lfloor u \equiv (-1)^{\deg \omega - 1}\, u \rfloor \omega \qquad \text{if } \deg u = 1 \tag{5}$$

Also

$$\begin{aligned} u \lfloor \omega &\coloneqq \omega(u) \equiv u \rfloor \omega \quad \text{if } \deg u = \deg \omega \\ (u \lfloor \omega)(\xi) &\coloneqq u \lfloor (\xi \wedge \omega) \equiv (\xi \wedge \omega)(u) \quad \text{if } \deg u > \deg \omega \end{aligned} \tag{6}$$

and

$$\begin{aligned} (\omega \rfloor u)(\xi) \coloneqq u \rfloor (\omega \wedge \xi) &\equiv (-1)^{\deg \omega\ \deg \xi}\, u \rfloor (\xi \wedge \omega) \\ &\equiv (-1)^{\deg \omega\ \deg \xi}\, (u \lfloor \omega)(\xi) \end{aligned} \tag{7}$$

$$\implies \omega \rfloor u \equiv (-1)^{\deg \omega\ (\deg v - 1)}\, u \lfloor \omega$$

The lower hook in "$\rfloor$" and "$\lfloor$" is useful to denote the object with lower degree, to know how to apply the sign in the graded-commutativity property and to know what kind of object – multivector or multicovector – one obtains.

If we define the degree of vectors to be negative, we can say that $\alpha \rfloor \beta$ yields an object of degree $\deg(\alpha \rfloor \beta) = \deg \alpha + \deg \beta$, no matter whether α is a vector and β a covector or vice versa. With this convention we could use the more compact dot-notation[2]

$$\begin{gathered} \alpha \cdot \beta \\ \text{with} \qquad \deg(\alpha \cdot \beta) = \deg \alpha + \deg \beta \\ \beta \cdot \alpha = (-1)^{\min\{|\deg \alpha|, |\deg \beta|\}\ (\deg \alpha + \deg \beta)}\, \alpha \cdot \beta\ . \end{gathered} \tag{8}$$

[2] cf. Truesdell & Toupin 1960 § F.I.267.

But it doesn't make much sense to use a unique symbol, because it would not represent an associative operation (unlike the wedge).

The inner product with a 1-vector or a 1-covector is a graded derivation.

3 Tensor products and equivalent objects

When we take tensor products of exterior objects, some special objects and some canonical correspondences between different classes of objects appear.

Take for example a non-zero N-covector γ. The tensor product $\gamma \otimes \gamma^{-1}$ is an N-vector-valued N-covector. This object is independent of the specific γ we chose. It has only one non-zero component, of value 1, which is invariant under basis or coordinate changes[3]. It has properties similar to those of a scalar, and the tensor space of N-vector-valued N-covectors is similar to those of scalars (Schouten (1989) § II.8 p. 29: "This is not a new geometric conception, only a new notation enabling us to get rid of a lot of indices").

The inner product of a p-vector u with the object above is an N-vector-valued $(N-p)$-covector:

$$u \cdot \gamma \otimes \gamma^{-1} = \omega \otimes \gamma^{-1} \qquad \text{with } \omega = u \cdot \gamma\,. \tag{9}$$

The space of p-vectors is therefore equivalent, in a canonical way, to the space of N-vector-valued $(N-p)$-covectors. The independent components transform in the same way under a change of basis/coordinates; see the example in Schouten (1989) § II.8 p. 30 bottom.

As Schouten[4] says, "Hence the geometrical meanings of corresponding quantities do not differ. There is only a difference in notation. [...] The use of Δ-densities is sometimes convenient; [...] the formulae contain less indices".

With the coordinate-free (and index-free) approach the use of such quantities offers no advantages.

4 Inner product with N-covector

We can use several possible conventions in defining an inner product of multivectors and multicovectors. One basic requirement is that the

3 Schouten 1989 § II.8 p. 29 bottom. **4** Schouten 1989 § II.8 p. 30.

usual inner product, which has the recursive property

$$u \cdot (\omega \wedge \xi) = (u \cdot \omega) \wedge \xi \tag{10}$$

for every 1-vector u, be respected. This property says that the first slots of the covector on the right are combined first. Another basic requirement is that

$$(u \wedge v \wedge \cdots) \cdot (\omega \wedge \xi \wedge \dots) = (u \cdot \omega)\,(v \cdot \xi) \cdots \tag{11}$$

for the same number of 1-vectors and 1-covectors.

There are two main decisions in the extension: what to do if the vector and covector are swapped, and what to do if the vector has higher order than the covector. Reasonable alternatives for the first are

$$(\omega \wedge \xi) \cdot u \coloneqq \omega \wedge (\xi \cdot u) \quad \text{recursively} \tag{12a}$$

or

$$(\omega \wedge \xi) \cdot u \coloneqq u \cdot (\omega \wedge \xi) = (u \cdot \omega) \wedge \xi\ , \tag{12b}$$

for every 1-vector u. Reasonable alternatives for the second are

$$(u \wedge v) \cdot \omega = u \wedge (v \cdot \omega) \quad \text{recursively} \tag{13a}$$

or

$$(u \wedge v) \cdot \omega \coloneqq (u \cdot \omega) \wedge v \tag{13b}$$

for every 1-covector ω.

In other words, we must choose among

(I1) *the adjacent maximal sets of slots of the two terms of the inner product should always be combined* and *the two terms of the inner product should be left in the order they are;*

(I2) *the adjacent maximal sets of slots of the two terms of the inner product should always be combined* and *the multivector should always be put on the left of the multicovector as a first thing*;

(I3) *the first maximal sets of slots of the two terms of the inner product should always be combined.*

Let's examine the consequences of such choices in particular cases:

$$\partial_x \cdot (\mathrm{d}x \wedge \mathrm{d}y) = \begin{cases} \mathrm{d}y & \text{I1} \\ \mathrm{d}y & \text{I2} \\ \mathrm{d}y & \text{I3} \end{cases} \tag{14}$$

$$(\mathrm{d}x \wedge \mathrm{d}y) \cdot \partial_x = \begin{cases} -\mathrm{d}y & \text{I1} \\ \mathrm{d}y & \text{I2} \\ \mathrm{d}y & \text{I3} \end{cases} \tag{15}$$

$$(\partial_x \wedge \partial_y) \cdot \mathrm{d}y = \begin{cases} \partial_x & \text{I1} \\ \partial_x & \text{I2} \\ -\partial_x & \text{I3} \end{cases} \tag{16}$$

$$\mathrm{d}y \cdot (\partial_x \wedge \partial_y) = \begin{cases} -\partial_x & \text{I1} \\ \partial_x & \text{I2} \\ -\partial_x & \text{I3} \end{cases} \tag{17}$$

$$\partial_x \cdot (\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = \begin{cases} \mathrm{d}y \wedge \mathrm{d}z & \text{I1} \\ \mathrm{d}y \wedge \mathrm{d}z & \text{I2} \\ \mathrm{d}y \wedge \mathrm{d}z & \text{I3} \end{cases} \tag{18}$$

$$(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) \cdot \partial_x = \begin{cases} \mathrm{d}y \wedge \mathrm{d}z & \text{I1} \\ \mathrm{d}y \wedge \mathrm{d}z & \text{I2} \\ \mathrm{d}y \wedge \mathrm{d}z & \text{I3} \end{cases} \tag{19}$$

$$(\partial_x \wedge \partial_y \wedge \partial_z) \cdot (\mathrm{d}y \wedge \mathrm{d}z) = \begin{cases} \partial_x & \text{I1} \\ \partial_x & \text{I2} \\ \partial_x & \text{I3} \end{cases} \tag{20}$$

$$(\mathrm{d}y \wedge \mathrm{d}z) \cdot (\partial_x \wedge \partial_y \wedge \partial_z) = \begin{cases} \partial_x & \text{I1} \\ \partial_x & \text{I2} \\ \partial_x & \text{I3} \end{cases} \tag{21}$$

Let's explore some consequences of the three alternatives.

Under the third alternative (I3), in even dimensions the inner products with the straight volume element γ and then with its inverse γ^{-1} becomes an anti-involution, see eqs (14) and (17); whereas in odd dimensions it's an involution. This may be confusing and cumbersome when one is studying a space with an unspecified dimension.

Under the first alternative (I1), in even dimensions the inner product with the straight volume element and then its inverse may be an involution, provided that the product happens from opposite sides. For a tensor product this would lead to the equalities involving transposition, for example

$$u \otimes v = \{\gamma^{-1} \cdot [\gamma \cdot (u \otimes v) \cdot \gamma]^{\intercal} \cdot \gamma^{-1}\}^{\intercal} \qquad \text{(I1)} \tag{22}$$

or

$$u \otimes v = \{\gamma^{-1} \cdot [(u \otimes v) \cdot \gamma]^{\intercal}\}^{\intercal} \qquad \text{(I1)}, \tag{23}$$

whereas in odd dimensions such transposition wouldn't be necessary. This could perhaps be obviated by using two pairs of symbols such as "$\rfloor$" and "$\lfloor$" with $a \lfloor b \coloneqq b \rfloor a$. But again in odd dimensions such distinction would often be unnecessary.

The second alternative (I2) seems to lead to the least cumbersome consequences, valid in even and odd dimensions alike.

So we can define the inner product starting from 1-vectors and 1-covectors as

$$\begin{gathered} u \cdot \omega \equiv \omega \cdot u \coloneqq \omega(u) \\ (u \wedge v) \cdot (\omega \wedge \xi \wedge \zeta) \equiv (\omega \wedge \xi \wedge \zeta) \cdot (u \wedge v) \coloneqq (u \cdot \omega)\,(v \cdot \xi)\,\zeta \\ (u \wedge v \wedge w) \cdot (\omega \wedge \xi) \equiv (\omega \wedge \xi) \cdot (u \wedge v \wedge w) \coloneqq u\,(v \cdot \omega)\,(w \cdot \xi) \end{gathered} \tag{24}$$

and generalizing.

5 Orientation

One way to define inner orientation is inductively as follows.

The orientation of a point, a 0-flat, is + or −.

The orientation of an n-flat bounded by pairs of parallel $(n-1)$-flats is determined by giving an orientation of the $(n-1)$-flats in such a way that parallel pairs have opposite orientations, and the orientations of the common $(n-2)$-flats cancel each other.

So the orientation of a line is determined by assigning + and − signs to its boundary points. The orientation of a parallelogram is determined by assigning opposite orientations to its opposite sides. The orientation of the interior is effectively given in terms of that of its boundary. This is similar, but not equivalent, to the definition in Tonti[5]. The orientation in terms of circulation is then equivalent to the orientation of outward direction at a point of the boundary, followed by the orientation of the boundary at that point.

Because of the latter correspondence this definition of orientation seems to give a more straightforward understanding of the connection between inner and outer orientations.

5 Tonti 2013 ch. 3.

Below: old text

If γ is a non-zero N-covector (hypervolume covector), so that γ^{-1} is its dual N-vector, we have

$$\gamma^{-1} \cdot (u \cdot \gamma) = (\gamma \cdot u) \cdot \gamma^{-1} = u \qquad \gamma \cdot (\omega \cdot \gamma^{-1}) = (\gamma^{-1} \cdot \omega) \cdot \gamma = \omega \tag{25}$$

for any multivector u and multicovector ω[6]. According to (4),

$$\gamma \cdot u = (-1)^{\deg(u)\,(N-1)}\, u \cdot \gamma \tag{26}$$

and analogously for the dual case.

Therefore $\gamma^{-1} \cdot (\gamma \cdot u) = (-1)^{\deg(u)\,(N-1)}\, u \neq u = (\gamma^{-1} \cdot \gamma) \cdot u$, which shows that the inner product is non-associative in general.

6 Star operator

This section has mistakes

Comparison of definitions of star operator:

Denote with $\overline{\omega}$ the p-vector obtained by rising all slots of ω with a metric g, and with $\underline{\mathrm{v}}$ the reverse operation. Let γ be the volume element induced by the metric.

Choquet-Bruhat et al. (1996 § V.A.4):

$$*\,\omega \coloneqq \overline{\omega} \cdot \gamma \tag{27}$$

Applying twice:

$$\begin{aligned} **\omega = \overline{\overline{\omega} \cdot \gamma} \cdot \gamma = (\omega \cdot \gamma^{-1}) \cdot \gamma = \\ (-1)^{\deg(\omega)\,(N-1)}(\gamma^{-1} \cdot \omega) \cdot \gamma = (-1)^{\deg(\omega)\,(N-1)}\omega \end{aligned} \tag{28}$$

So $*^{-1} = (-1)^{\deg(\omega)\,(N-1)}\, *$. Compare with Bossavit (1991 § 4.2 Ex. 71).

[6] cf. Schouten 1989 § II.7 p. 28.

7 Twisted scalars, vectors, covectors

A twisted scalar is a positive number with an associated outer orientation. We can specify such orientation locally for example by giving an ordered list of coordinate functions. Denote such a unit twisted scalar by

$$\underset{txyz}{1} \tag{29}$$

It satisfies

$$\begin{gathered}\underset{txyz}{1} \cdot \underset{txyz}{1} = 1 \\ a \cdot \underset{txyz}{1} = \underset{txyz}{a} \quad \text{for any scalar } a \\ \underset{txyz}{-1} = \underset{xtyz}{1} \quad \text{or any other odd permutation}\end{gathered} \tag{30}$$

$\underset{txyz}{1}$

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(“de *X*” is listed under D, “van *X*” under V, and so on, regardless of national conventions.)

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