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Kinematics and dynamics from a modern perspective

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The kinematics of mechanical, thermodynamic, and electromagnetic phenomena is developed in such a way as to be used for Newtonian, Lorentzian, and general relativity. The same is done, as much as possible, for their dynamics as well

1 Scribbles and memos

1.1 Twisted objects

Denote the wedge product by juxtaposition: $\mathrm{d}t \wedge \mathrm{d}x =: \mathrm{d}t\,\mathrm{d}x$ and so on; abbreviate $\mathrm{d}t\,\mathrm{d}x =: \mathrm{d}tx$ and so on; and denote twisted differential forms by $\utilde{\mathrm{d}}$.

Where an ordered set of coordinate functions (t, x, y, z) is chosen, the twisted unit $\utilde{1}$ is defined. It has unit magnitude and outer-orientation $txyz$, and the property $\utilde{1} \cdot \utilde{1} = 1$. In general it is only defined in chart domain, where coordinate functions can be defined, but not globally. We obtain twisted vectors and covectors by multiplying their non-twisted counterparts by the twisted unit. For a vector or covector ω we have that the orientation of its twisted counterpart is such that (see Burke 1983)

$$\bigl\{\{\utilde{\omega}\}, \{\omega\}\bigr\} = \bigl\{\utilde{1}\bigr\} . \tag{1}$$

Said otherwise, in the product $\utilde{1} \cdot \omega$ the *right* side of the orientation of $\utilde{1}$ cancels out with the orientation of ω. This rule must be respected even if we invert the product order, so $\utilde{1} \cdot \omega \equiv \omega \cdot \utilde{1}$.

Multiplying a twisted vector or covector by the twisted unit, we obtain its non-twisted counterpart. The resulting orientation is obtained by cancelling out the orientation of the twisted object and the *left* side of the orientation of $\utilde{1}$.

We have:

$$\{\underset{\sim}{1}\} = txyz\ ; \tag{2}$$

$$\{\underset{\sim}{\mathrm{d}}t\} = -xyz\ , \quad \{\underset{\sim}{\mathrm{d}}x\} = tyz\ , \quad \{\underset{\sim}{\mathrm{d}}y\} = tzx\ , \quad \{\underset{\sim}{\mathrm{d}}z\} = txy\ ; \tag{3}$$

$$\{\underset{\sim}{\mathrm{d}}tx\} = yz\ , \quad \{\underset{\sim}{\mathrm{d}}ty\} = zx\ , \quad \{\underset{\sim}{\mathrm{d}}tz\} = xy\ , \tag{4}$$

$$\{\underset{\sim}{\mathrm{d}}xy\} = tz\ , \quad \{\underset{\sim}{\mathrm{d}}yz\} = tx\ , \quad \{\underset{\sim}{\mathrm{d}}zx\} = ty\ ; \tag{5}$$

$$\{\underset{\sim}{\mathrm{d}}tyz\} = -x\ , \quad \{\underset{\sim}{\mathrm{d}}tzx\} = -y\ , \quad \{\underset{\sim}{\mathrm{d}}txy\} = -z\ , \quad \{\underset{\sim}{\mathrm{d}}xyz\} = t\ ; \tag{6}$$

$$\{\underset{\sim}{\mathrm{d}}txyz\} = +1\ . \tag{7}$$

The minus signs appear in the odd ranks when we have t and an even number of other coordinates after the "$\underset{\sim}{\mathrm{d}}$". These minus signs flip if we keep t always to the right, with orientation $xyzt$.

Note that considering, say, the *function* x, we have

$$\{\underset{\sim}{x}\} = \begin{cases} \{txyz\} & \text{if } x > 0\ , \\ -\{txyz\} & \text{if } x < 0\ . \end{cases} \tag{8}$$

1.2 Charge and current densities

We can represent charge density and current density in one geometrical entity. Consider an ordered coordinate system (t, x, y, z). The charge-current density is

$$\begin{aligned} \boldsymbol{Q} &\coloneqq \rho\ \underset{\sim}{\mathrm{d}}xyz - j_x\ \underset{\sim}{\mathrm{d}}tyz - j_y\ \underset{\sim}{\mathrm{d}}tzx - j_z\ \underset{\sim}{\mathrm{d}}txy \\ &\equiv \rho\ \underset{\sim}{\mathrm{d}}xyz - \mathrm{d}t\ \left(j_x\ \underset{\sim}{\mathrm{d}}yz + j_y\ \underset{\sim}{\mathrm{d}}zx + j_z\ \underset{\sim}{\mathrm{d}}xy\right)\ . \end{aligned} \tag{9}$$

It has the dimensions of charge (current · time). The minus signs appear so that the x-component of the current, for example, is positive when $j_x > 0$ and so on.

This object automatically give us net volume charge when integrated over a three-dimensional region at constant time, or the net flux of charge when integrated over a two-dimensional surface – possibly even moving – over a lapse of time. Consider for example a 3-volume V at constant time, having *outer* orientation in the positive t direction and parameterized by

$$(u, v, w) \mapsto (t_0, u, v, w)\ . \tag{10}$$

On it, the basis twisted 1-forms map to

$$\begin{gathered}
\underset{-xyz}{\mathrm{d}t}\big|_V = 0\;, \quad \underset{tyz}{\mathrm{d}x}\big|_V = \underset{vw}{\mathrm{d}u}\;, \quad \underset{tzx}{\mathrm{d}y}\big|_V = \underset{wu}{\mathrm{d}v}\;, \quad \underset{txy}{\mathrm{d}z}\big|_V = \underset{uv}{\mathrm{d}w}\;, \\
\underset{t}{\mathrm{d}}xyz\big|_V = \underset{+}{\mathrm{d}}uvw\;.
\end{gathered} \tag{11}$$

Then we have

$$\big(\rho\,\underset{\sim}{\mathrm{d}}xyz - j_x\,\underset{\sim}{\mathrm{d}}tyz - j_y\,\underset{\sim}{\mathrm{d}}tzx - j_z\,\underset{\sim}{\mathrm{d}}txy\big)\big|_V = \rho\,\underset{+}{\mathrm{d}}uvw \tag{12}$$

and the current density gives no contribution.

The charge-current density also correctly transform under coordinate changes. Consider for example

$$(t', x', y', z') = (t, x - vt, y, z)\;, \tag{13}$$

for which

$$\begin{gathered}
\mathrm{d}t' = \mathrm{d}t\;, \quad \mathrm{d}y' = \mathrm{d}y\;, \quad \mathrm{d}z' = \mathrm{d}z\;, \\
\mathrm{d}x' = \mathrm{d}x - v\,\mathrm{d}t\;, \qquad \mathrm{d}x = \mathrm{d}x' + v\,\mathrm{d}t'\;;
\end{gathered} \tag{14}$$

$$\begin{aligned}
\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= (\mathrm{d}x' + v\,\mathrm{d}t')\,\mathrm{d}y'\,\mathrm{d}z' = \mathrm{d}x'y'z' + v\,\mathrm{d}t'y'z'\;, \\
\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}x &= \mathrm{d}t'\,\mathrm{d}z'\,(\mathrm{d}x' + v\,\mathrm{d}t') = \mathrm{d}t'\,\mathrm{d}z'\,\mathrm{d}x'\;, \\
\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y &= \mathrm{d}t'\,(\mathrm{d}x' + v\,\mathrm{d}t')\,\mathrm{d}y' = \mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}y'\;.
\end{aligned} \tag{15}$$

The charge-current density can then be rewritten as

$$\begin{aligned}
\boldsymbol{Q} &= \rho\,\underset{\sim}{\mathrm{d}}xyz - j_x\,\underset{\sim}{\mathrm{d}}tyz - j_y\,\underset{\sim}{\mathrm{d}}tzx - j_z\,\underset{\sim}{\mathrm{d}}txy \\
&= \rho\,\underset{\sim}{\mathrm{d}}x'y'z' - (j_x - \rho\,v)\,\underset{\sim}{\mathrm{d}}t'y'z' - j_y\,\underset{\sim}{\mathrm{d}}t'z'x' - j_z\,\underset{\sim}{\mathrm{d}}t'x'y'\;,
\end{aligned} \tag{16}$$

which is indeed the correct transformation for the charge density and the x-component of the current density (Kovetz 2000 eq. (5.8)). It is important to note that we did not make any assumptions regarding spacetime symmetries and metric. The transformation (13) is a Galilei boost between Galileian inertial frames, if we assume Newtonian relativity, and a non-symmetry preserving coordinate transformation in Lorentzian or general relativity. So there is no contradiction with any of these theories. If we assume that Lorentzian relativity holds and (t, x, y, z) is a Lorentzian inertial frame, then a metric-preserving transformation would instead be

$$\begin{gathered}
(t', x', y', z') = \Big((t - x\,v/c^2)/\gamma,\ (x - vt)/\gamma,\ y,\ z\Big)\;, \\
\gamma \coloneqq \sqrt{1 - v^2/c^2}\;,
\end{gathered} \tag{17}$$

and a calculation similar to the previous one shows that the components of the charge-density would again transform as expected (Kovetz 2000 eqs (12.17)–(12.18)).

The law of charge conservation is simply expressed by

$$\mathrm{d}\boldsymbol{Q} = 0\,, \tag{18}$$

which leads to, considering permutations and antisymmetry,

$$\begin{gathered}\mathrm{d}\boldsymbol{Q} = \partial_t\rho\ \underset{\sim}{\mathrm{d}}txyz - \partial_x j_x\ \underset{\sim}{\mathrm{d}}xtyz - \partial_y j_y\ \underset{\sim}{\mathrm{d}}ytzx - \partial_x j_x\ \underset{\sim}{\mathrm{d}}ztxy = \\ \bigl(\partial_t\rho + \partial_x j_x + \partial_y j_y + \partial_z j_z\bigr)\ \underset{\sim}{\mathrm{d}}txyz = 0\,,\end{gathered} \tag{19}$$

implying the familiar (Kovetz 2000 eq. (1.14))

$$\partial_t\rho + \partial_x j_x + \partial_y j_y + \partial_z j_z = 0\,, \tag{20}$$

but now shown to be *valid in any coordinate system*.

Bibliography

("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

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