Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z), which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

There are 18 basic types of geometric objects, which represent physical quantities of different types. We have inner-oriented scalars, inner-oriented 1-vectors up to 4-vectors, inner-oriented 1-covectors up to 4-covectors, and all their outer-oriented counterparts.

The exterior product '$\wedge$' of vectors and covectors is usually represented just by juxtaposition, and a shortened notation is used for exterior products of several exterior derivatives 'd'. For instance

$$\mathrm{d}^2 xy \coloneqq \mathrm{d}x \wedge \mathrm{d}y \qquad \partial^2_{xy} \coloneqq \partial_x \wedge \partial_y \ . \tag{1}$$

The associated bases for inner-oriented covector fields are

$$\mathrm{d}t \quad \mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z \tag{2}$$

$$\mathrm{d}^2 tx \quad \mathrm{d}^2 ty \quad \mathrm{d}^2 tz \quad \mathrm{d}^2 yz \quad \mathrm{d}^2 zx \quad \mathrm{d}^2 xy \tag{3}$$

$$\mathrm{d}^3 xyz \quad -\mathrm{d}^3 tyz \quad -\mathrm{d}^3 tzx \quad -\mathrm{d}^3 txy \tag{4}$$

$$\mathrm{d}^4 txyz \tag{5}$$

and analogously for inner-oriented vector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it's only defined on a coordinate patch. It is idempotent: $\tilde{1}\tilde{1} = 1$.

A twisted or outer-oriented 3-covector such as $\mathrm{d}^3\tilde{x}yz$ has an associated outer direction, in this case positive t. We adopt this shorter notation for the outer-oriented versions of the bases above (analogous to the notation in Gotay & Marsden 1992 § 2 p. 371):

$$-\mathrm{d}_{xyz} \quad \mathrm{d}_{tyz} \quad \mathrm{d}_{tzx} \quad \mathrm{d}_{txy} \tag{6a}$$

$$\mathrm{d}^2_{yz} \quad \mathrm{d}^2_{zx} \quad \mathrm{d}^2_{xy} \quad \mathrm{d}^2_{tx} \quad \mathrm{d}^2_{ty} \quad \mathrm{d}^2_{tz} \tag{6b}$$

$$\mathrm{d}^3_{t} \quad \mathrm{d}^3_{x} \quad \mathrm{d}^3_{y} \quad \mathrm{d}^3_{z} \tag{6c}$$

$$\mathrm{d}^4 \tag{6d}$$

so that $-\mathrm{d}_{xyz} \coloneqq \mathrm{d}\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz \coloneqq \partial_{\tilde{t}}$. Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} &n_{xyz}\,\mathrm{d}^3\tilde{x}yz - n_{tyz}\,\mathrm{d}^3\tilde{t}yz - n_{tzx}\,\mathrm{d}^3\tilde{t}zx - n_{txy}\,\mathrm{d}^3\tilde{t}xy \\ &\equiv n^t\,\mathrm{d}^3_t + n^x\,\mathrm{d}^3_x + n^y\,\mathrm{d}^3_y + n^z\,\mathrm{d}^3_z \\ &\qquad \text{with } \; n^t \equiv n_{xyz}\,,\; n^x \equiv n_{tyz}\,,\; \text{ and so on.} \end{aligned} \tag{7}$$

Contraction or dot-product of vectors and covectors is denoted by '$\cdot$', and always contracts the maximal possible amount of contiguous elements. For instance

$$\mathrm{d}^3xyz \cdot \partial^2_{yz} = \mathrm{d}x \qquad -\partial^3_{txy} \cdot \mathrm{d}x = \partial^2_{ty}\;. \tag{8}$$

Contractions with the 4-vector ∂^4 and 4-covector d^4 establish a duality between outer n-covectors and inner $(4-n)$-vectors:

$$\left.\begin{matrix} \partial^4 \\ \partial^3_{xyz}\;\partial^3_{tyz}\;\partial^3_{tzx}\;\partial^3_{txy} \\ \partial^2_{tx}\;\partial^2_{ty}\;\partial^2_{tz}\;\partial^2_{yz}\;\partial^2_{zx}\;\partial^2_{xy} \\ \partial_t\;\partial_x\;\partial_y\;\partial_z \\ 1 \end{matrix}\right) \;\begin{matrix} \xleftarrow{\;\partial^4\,\cdot\;} \\ \xrightarrow{\;\cdot\,\mathrm{d}^4\;} \end{matrix}\; \left(\begin{matrix} \tilde{1} \\ \mathrm{d}_{xyz}\;\mathrm{d}_{tyz}\;\mathrm{d}_{tzx}\;\mathrm{d}_{txy} \\ \mathrm{d}^2_{yz}\;\mathrm{d}^2_{zx}\;\mathrm{d}^2_{xy}\;\mathrm{d}^2_{tx}\;\mathrm{d}^2_{ty}\;\mathrm{d}^2_{tz} \\ \mathrm{d}^3_t\;\mathrm{d}^3_x\;\mathrm{d}^3_y\;\mathrm{d}^3_z \\ \mathrm{d}^4 \end{matrix}\right. \tag{9}$$

These duals have special properties. For instance, for any 3-covector $\boldsymbol{N}$, we have

$$\boldsymbol{N} \cdot (\partial^4 \cdot \boldsymbol{N}) \cdot \boldsymbol{N} = 0 \tag{10}$$

that is, the dual of $\boldsymbol{N}$ is a vector belonging in the kernel of $\boldsymbol{N}$.

If γ is a non-zero 4-covector and γ^{-1} the inverse 4-vector, that is, $\gamma \cdot \gamma^{-1} = \gamma^{-1} \cdot \gamma = 1$, and if $\boldsymbol{N}$ is a 3-covector and $\boldsymbol{\phi}$ a 1-covector, we have the useful identity

$$\boldsymbol{N} \wedge \boldsymbol{\phi} = (\boldsymbol{N} \cdot \gamma^{-1} \cdot \boldsymbol{\phi})\, \gamma \ , \tag{11}$$

which also holds as long as the degrees of $\boldsymbol{N}$ and $\boldsymbol{\phi}$ sum up to 4.

We often consider vector- or covector-valued covectors, written as sums of tensor products. They can be outer- or inner-oriented. For instance

$$\mathrm{d}^3_t \otimes \mathrm{d}x \equiv \mathrm{d}^3\tilde{x}yz \otimes \mathrm{d}x \ , \qquad \mathrm{d}^3_x \otimes \partial_y \ . \tag{12}$$

The operation $\curlywedge$ between a vector-valued covector and a covector-valued covector is the contraction of their vector- and covector-valued parts and the exterior product of their covector parts. For instance, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are covectors, $\boldsymbol{\omega}$ is a covector, and $\boldsymbol{u}$ is a vector, then

$$(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \curlywedge (\boldsymbol{\psi} \otimes \boldsymbol{u}) \coloneqq (\boldsymbol{\phi} \wedge \boldsymbol{\psi}) \otimes (\boldsymbol{\omega} \cdot \boldsymbol{u}) \tag{13}$$

As another example,

$$(\mathrm{d}^3_t \otimes \mathrm{d}x) \curlywedge (\mathrm{d}_t \otimes \partial_x) = (\mathrm{d}^3_t \wedge \mathrm{d}_t)\,(\mathrm{d}x \cdot \partial_x) = -\mathrm{d}^4 \ . \tag{14}$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta}\, \mathrm{d}u^\beta \ ,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$\mathrm{d}\alpha' = \frac{\partial \alpha'}{\partial \alpha}\, \mathrm{d}\alpha$$

- raising or lowering of indices with the metric, for example

$$\mathrm{d}\alpha \mapsto g^{\alpha\beta}\, \partial_\beta \ .$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix[1]. For the spaces of 3-vectors and 3-covectors we have simplified formulae[2]:

$$\begin{aligned} \mathrm{d}^3_{\alpha'} &= \det\bigl(\tfrac{\partial\alpha'}{\partial\alpha}\bigr)\, \frac{\partial\alpha}{\partial\alpha'}\, \mathrm{d}^3_{\alpha} \\ \partial^3\alpha' &= \det\bigl(\tfrac{\partial\alpha}{\partial\alpha'}\bigr)\, \frac{\partial\alpha'}{\partial\alpha}\, \partial^3\alpha \end{aligned} \tag{15}$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} \mathrm{d}^3_{\alpha} &\mapsto \frac{1}{|g|}\, g_{\alpha\beta}\, \partial^3\beta \\ \partial^3\alpha &\mapsto |g|\, g^{\alpha\beta}\, \mathrm{d}^3_{\beta} \end{aligned} \tag{16}$$

3 Metric

We take the metric $\boldsymbol{g}$ to have signature $(-,+,+,+)$ and dimensions of area. The determinant and the negative of its square root are denoted shortly

$$g \coloneqq \det \boldsymbol{g} \qquad \sqrt{g} \coloneqq \sqrt{-\det \boldsymbol{g}}\,. \tag{17}$$

The volume element induced by the metric $\boldsymbol{g}$ has dimensions of volume-time and is denoted (note the boldface)

$$\boldsymbol{\gamma} \coloneqq \frac{\sqrt{g}}{c} \mathrm{d}^4 \tilde{t}xyz \equiv \frac{\sqrt{g}}{c}\, \mathrm{d}^4 \tag{18}$$

and its corresponding inverse, a twisted 4-vector:

$$\boldsymbol{\gamma}^{-1} \coloneqq \frac{c}{\sqrt{g}}\, \partial^4\,. \tag{19}$$

[1] Choquet-Bruhat et al. 1996 § IV.A.1 p. 199. [2] Gantmacher 2000 § I.4 eq. (33).

Contraction with the volume element or its inverse establishes a "volume duality" between outer n-covectors and inner $(4-n)$-vectors:

$$\left.\begin{matrix} \partial^4 \\ \partial^3_{xyz}\ \ \partial^3_{tyz}\ \ \partial^3_{tzx}\ \ \partial^3_{txy} \\ \partial^2_{tx}\ \ \partial^2_{ty}\ \ \partial^2_{tz}\ \ \partial^2_{yz}\ \ \partial^2_{zx}\ \ \partial^2_{xy} \\ \partial_t\ \ \partial_x\ \ \partial_y\ \ \partial_z \\ 1 \end{matrix}\right) \begin{matrix} \xleftarrow{\frac{\sqrt{g}}{c}\,\gamma^{-1}\,\cdot} \\ \xrightarrow{\cdot\,\gamma\,\frac{c}{\sqrt{g}}} \end{matrix} \left(\begin{matrix} \tilde{1} \\ \mathrm{d}_{xyz}\ \ \mathrm{d}_{tyz}\ \ \mathrm{d}_{tzx}\ \ \mathrm{d}_{txy} \\ \mathrm{d}^2_{yz}\ \ \mathrm{d}^2_{zx}\ \ \mathrm{d}^2_{xy}\ \ \mathrm{d}^2_{tx}\ \ \mathrm{d}^2_{ty}\ \ \mathrm{d}^2_{tz} \\ \mathrm{d}^3_t\ \ \mathrm{d}^3_x\ \ \mathrm{d}^3_y\ \ \mathrm{d}^3_z \\ \mathrm{d}^4 \end{matrix}\right. \tag{20}$$

This is the reason why in older literature an outer-oriented n-covector is treated as a $(4-n)$-"vector density", that is, a vector divided by the square root of the volume element.

The metric on the space of 1-vectors also induces metrics on the spaces of 1-covectors, 2-vectors and 2-covectors, and so on. In particular, the metrics $\overset{3}{g}{}^{-1}$ on the space of 3-covectors and $\overset{3}{g}$ on the space of 3-vectors can be written in coordinates as

$$\overset{3}{g}{}^{-1} = \frac{g_{\mu\nu}}{g}\,\partial^3\mu \otimes \partial^3\nu \qquad \text{with dimensions length}^{-6} \tag{21}$$

$$\overset{3}{g} = g\,g^{\mu\nu}\,\mathrm{d}^3_\mu \otimes \mathrm{d}^3_\nu \qquad \text{with dimensions length}^{6}\ . \tag{22}$$

With these we can define squared norms $\|.\|^2$ on all those spaces. Note in particular the following identity:

$$\|\gamma^{-1}\cdot \boldsymbol{N}\|^2 = -c^2\,\|\boldsymbol{N}\| \qquad \text{for every 3-covector } \boldsymbol{N}. \tag{23}$$

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{24}$$

and the volume element is simply d^4.

4 Matter current

The amount-of-matter current $\boldsymbol{N}$ is an outer-oriented 3-covector

$$\boldsymbol{N} = N\,\mathrm{d}^3_t + J^i\,\mathrm{d}^3_i \tag{25}$$

of dimensions "amount of matter", typically measured in moles, where

- N is the volumic amount of matter, measured per unit coordinate volume.
- J^i is the aeric flux of amount of matter (equivalent to a molar flux), measured per unit coordinate area and unit coordinate time.

The current for every particular kind of matter satisfies the conservation law

$$\mathrm{d}\boldsymbol{N} = 0 \quad \text{or} \quad \partial_t N + \partial_i J^i = 0 \tag{26}$$

independent of any metric.

The common contravariant form of the matter current, "N^μ", is obtained by contracting the matter current with the inverse volume element:

$$`N^\mu\text{'} \mathrel{\hat{=}} \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{N} = \frac{c}{\sqrt{g}}\,N\,\partial_t + \frac{c}{\sqrt{g}}\,J^i\,\partial_t\ . \tag{27}$$

If a metric is present, a four-velocity $\boldsymbol{U}$ can be associated with the matter current $\boldsymbol{N}$, defined by the following properties and identity:

$$\boldsymbol{U} \cdot \boldsymbol{N} = 0 \qquad \|\boldsymbol{U}\|^2 = -c^2 \tag{28}$$

$$\boldsymbol{U} = \frac{1}{|\|\boldsymbol{N}\||}\,\gamma^{-1} \cdot \boldsymbol{N} \tag{29}$$

which also implies (for normal matter)

$$\boldsymbol{N} = \|\boldsymbol{N}\|\,\boldsymbol{U} \cdot \gamma \tag{30}$$

For normal matter (as opposed to antimatter) $\|\boldsymbol{N}\|^2 \geqslant 0$.

5 Four-stress

The stress-energy-momentum tensor, or simply four-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\begin{aligned} \boldsymbol{T} &= T^{\mu}_{\ \nu}\ \mathrm{d}^3_{\mu} \otimes \mathrm{d}x^{\nu} \\ &= -\epsilon\, \mathrm{d}^3_t \otimes \mathrm{d}t - q^i\, \mathrm{d}^3_i \otimes \mathrm{d}t + p_j\, \mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\, \mathrm{d}^3_i \otimes \mathrm{d}x^j \end{aligned} \tag{31}$$

the indices i, j running over x, y, z, and where:

- The energy ϵ is a density per unit *coordinate* volume xyz, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this energy comprises "rest" energy, kinetic energy, potential gravitational energy, and centrifugal potential energy.
- The energy flux q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this flux term includes transport of the energy term above and also heat flux.
- The momentum p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i.
- The compressive three-stress π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j. If the point at which the four-stress is considered has a matter-current, then this stress comprises momentum transport and internal stresses.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \tag{32}$$

The diagonal elements $g_{tt}, \dots$ include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

The common contra-contra-variant form of the stress, "$T^{\mu\nu}$", is obtained by contracting the four-stress with the inverse volume element and the inverse metric:

$$
\begin{aligned}
`T^{\mu\nu}\,' \mathrel{\hat{=}} \boldsymbol{\gamma}^{-1}\cdot\boldsymbol{T}\cdot\boldsymbol{g}^{-1} = -\frac{c\,g^{tt}}{\sqrt{g}}\,\epsilon\,\partial_t\otimes\partial_t - \frac{c\,g^{tt}}{\sqrt{g}}\,q^i\,\partial_i\otimes\partial_t \\
+\sum_j \frac{c\,g^{jj}}{\sqrt{g}}\,p_j\,\partial_t\otimes\partial_j + \sum_j \frac{c\,g^{jj}}{\sqrt{g}}\,\pi^i_j\,\partial_i\otimes\partial_j\;.
\end{aligned}
\tag{33}
$$

One important detail in finding the Newtonian approximation of "energy density" is that *one takes different zeros of energy density in different coordinate systems*: the zero is taken as the molar mass times the molar density *in the current coordinate system*. By 'zero' I mean the arbitrary separation between "mass" and "energy".

The *total* four-stress satisfies the balance equation

$$
\mathrm{D}\boldsymbol{T} = 0 \tag{34}
$$

which is equivalent to the four balance equations

$$
\begin{aligned}
\partial_t e + \partial_i q^i &= e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} - p_j\,\Gamma^j_{tt} - \pi^i_k\,\Gamma^k_{it} \\
\partial_t p_j + \partial_i \pi^i_j &= -e\,\Gamma^t_{tj} - q^i\,\Gamma^t_{ij} + p_k\,\Gamma^k_{tj} + \pi^i_k\,\Gamma^k_{ij}
\end{aligned}
\tag{35}
$$

In general relativity the *total* four-stress also satisfies

$$
\left(\boldsymbol{\gamma}^{-1}\cdot\boldsymbol{T}\cdot\boldsymbol{g}^{-1}\right)^{\intercal} - \boldsymbol{\gamma}^{-1}\cdot\boldsymbol{T}\cdot\boldsymbol{g}^{-1} = 0\;. \tag{36}
$$

The four-stress determines an association between any 1-vector $\boldsymbol{V}$ field and an outer-oriented 3-covector field, interpreted as a current:

$$
\boldsymbol{V} \mapsto \boldsymbol{T}\cdot\boldsymbol{V}\;. \tag{37}
$$

This current satisfies the balance equation

$$
\mathrm{d}(\boldsymbol{T}\cdot\boldsymbol{V}) = -\boldsymbol{T} \curlywedge \nabla\boldsymbol{V} = \operatorname{tr}(\boldsymbol{T}^{\intercal}\cdot\boldsymbol{\gamma}^{-1}\cdot\nabla\boldsymbol{V})\,\boldsymbol{\gamma} \tag{38}
$$

which is a conservation law if $\boldsymbol{V}$ is a Killing vector.

For the special case $\boldsymbol{V} = \partial_\alpha$ the formula above becomes

$$\mathrm{d}(T^{\mu}{}_{\alpha}\ \mathrm{d}^3_{\mu}) = T^{\mu}{}_{\nu}\ \Gamma^{\nu}_{\mu\alpha}\,\mathrm{d}^4 \qquad \Longleftrightarrow \qquad \partial_\mu T^{\mu}{}_{\alpha} = T^{\mu}{}_{\nu}\ \Gamma^{\nu}_{\mu\alpha} \tag{39}$$

Consider a region where there is a non-vanishing matter current $\boldsymbol{N}$ with associated four-velocity $\boldsymbol{U}$, and define

$$\bar{\boldsymbol{U}} = -\frac{1}{c}\,\boldsymbol{g}\cdot\boldsymbol{U} \tag{40}$$

which statisfies

$$\bar{\boldsymbol{U}}\cdot\boldsymbol{U} = 1\ , \qquad \nabla\boldsymbol{U}\cdot\bar{\boldsymbol{U}} = 0\ . \tag{41}$$

The last equality can be proved from $\nabla\boldsymbol{g} = 0$ and

$$0 = -\nabla(c^2) = \nabla(\boldsymbol{U}\cdot\boldsymbol{g}\cdot\boldsymbol{U}) = 2\,(\nabla\boldsymbol{U})\cdot\boldsymbol{g}\cdot\boldsymbol{U}\ . \tag{42}$$

We can associate with the matter a four-stress $\boldsymbol{T}$ which can be decomposed as follows:

$$\begin{aligned} &\boldsymbol{T} = -\epsilon\,\boldsymbol{N}\otimes\bar{\boldsymbol{U}} + \boldsymbol{N}\otimes\boldsymbol{P} - (\bar{\boldsymbol{U}}\wedge\boldsymbol{Q})\otimes\bar{\boldsymbol{U}} + \bar{\boldsymbol{U}}\wedge\boldsymbol{S} \\ &\text{with}\quad \boldsymbol{P}\cdot\boldsymbol{U} = 0 \quad \boldsymbol{Q}\cdot\boldsymbol{U} = 0 \quad \boldsymbol{U}\cdot\boldsymbol{S} = 0 \quad \boldsymbol{S}\cdot\boldsymbol{U} = 0 \end{aligned} \tag{43}$$

where

- ϵ is a scalar, the molar energy density.
- $\boldsymbol{P}$ is a 1-covector, the molar momentum density.
- $\boldsymbol{Q}$ is a 2-covector, the areic energy-flux density.
- $\boldsymbol{S}$ is a 1-covector-valued 2-covector, the three-stress related to the Cauchy stress.

Using the four-velocity $\boldsymbol{U}$ associated with the matter current we obtain what could be called the "internal-energy current":

$$\boldsymbol{T}\cdot\boldsymbol{U} \equiv -\epsilon\,\boldsymbol{N} - \bar{\boldsymbol{U}}\,\boldsymbol{Q} \tag{44}$$

which, from eqs (38), (40), (26), satisfies the balance law

$$\mathrm{d}\big(\boldsymbol{T}\cdot\boldsymbol{U}\big) = -\boldsymbol{T}\curlywedge\nabla\boldsymbol{U} \tag{45}$$

or

$$\begin{aligned} \mathrm{d}\big(-\epsilon\,\boldsymbol{N} - \bar{\boldsymbol{U}}\,\boldsymbol{Q}\big) = \big(\epsilon\,\boldsymbol{N}\otimes\bar{\boldsymbol{U}}\big)\cdot\nabla\boldsymbol{U} - \big(\boldsymbol{N}\otimes\boldsymbol{P}\big)\cdot\nabla\boldsymbol{U} \\ + \big(\bar{\boldsymbol{U}}\,\boldsymbol{Q}\otimes\bar{\boldsymbol{U}}\big)\cdot\nabla\boldsymbol{U} - \bar{\boldsymbol{U}}\,\boldsymbol{S}\cdot\nabla\boldsymbol{U} \end{aligned} \tag{46}$$

or simply

$$N\,\mathrm{d}\epsilon + \bar{U}\,\mathrm{d}Q - Q\,\mathrm{d}\bar{U} = -N\,\nabla U \cdot P - (\bar{U}\,S) \cdot \nabla U\ . \tag{47}$$

The last balance equation corresponds to eq. (28) in Eckart (1940), except for the fact that here we are keeping the momentum P and heat flux Q distinct.

6 Electromagnetic field

The electromagnetic field, represented by the Faraday 2-covector, is typically decomposed as[3]

$$\begin{aligned} \boldsymbol{F} &= E\,\mathrm{d}t + \boldsymbol{B} \\ &\equiv E_x\,\mathrm{d}^2xt + E_y\,\mathrm{d}^2yt + E_z\,\mathrm{d}^2zt + B^x\,\mathrm{d}^2yz + B^y\,\mathrm{d}^2zx + B^z\,\mathrm{d}^2xy \end{aligned} \tag{48}$$

The conservation of magnetic flux is expressed by

$$\mathrm{d}\boldsymbol{F} = 0 \tag{49}$$

or equivalently

$$\begin{aligned} &\partial_i B^i = 0 && (\mathrm{d}^3xyz \text{ component}) \\ &\partial_t B^x + \partial_y E_z - \partial_z E_y = 0 && (\mathrm{d}^3tyz) \\ &\partial_t B^y + \partial_z E_x - \partial_x E_z = 0 && (\mathrm{d}^3tzx) \\ &\partial_t B^z + \partial_x E_y - \partial_y E_x = 0 && (\mathrm{d}^3txy) \end{aligned} \tag{50}$$

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

[3] Frankel 1979 ch. 9.

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- $\boldsymbol{u}^{\bullet}$ is a 1-vector, represented by the column-matrix $\boldsymbol{u}$.
- Similarly for $\boldsymbol{v}^{\bullet}$.
- $\boldsymbol{\omega}_{\bullet}$ is a 1-covector, represented by the row-matrix $\boldsymbol{\omega}$.
- $\boldsymbol{g}_{\bullet\bullet}$ is a co-covector, represented by the matrix $\boldsymbol{g}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\boldsymbol{g}^{-1^{\bullet\bullet}}$ is a contra-contravector, inverse of $\boldsymbol{g}$, that is: $\boldsymbol{g} \cdot \boldsymbol{g}^{1} = \mathbf{id}_{\bullet}^{\bullet}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\boldsymbol{Q}_{|\bullet\bullet\bullet|}$ is a 3-covector, represented by the column-matrix $\boldsymbol{Q}$.
- $\boldsymbol{T}_{|\bullet\bullet\bullet|\bullet}$ is a 1-covector-valued 3-covector, represented by the matrix $\boldsymbol{T}$. The rows represent the 3-covector components; the columns, the 1-covector components.
- $\gamma_{|\bullet\bullet\bullet\bullet|}$ is a 4-covector, represented by the number γ.
- $\gamma^{-1^{|\bullet\bullet\bullet\bullet|}}$ is a 4-vector, represented by the number γ^{-1}
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from "old" coordinates x to "new" coordinates x' is represented by the matrix $\boldsymbol{J}$. The rows correspond to the new coordinates x'; the columns, to the old x.
- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x' is represented by the matrix $\boldsymbol{J}^{-1}$. The rows correspond to the old coordinates x; the columns, to the new x'.

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

$$
\begin{array}{lll}
\text{(object)} & \text{(matrix repr)} & (51)\\
\boldsymbol{\omega}\cdot\boldsymbol{u} & \boldsymbol{\omega}\boldsymbol{u} \equiv \boldsymbol{u}^{\intercal}\boldsymbol{\omega}^{\intercal} \quad \text{(number)} & (52)\\
\boldsymbol{v}\cdot\boldsymbol{g}\cdot\boldsymbol{u} & \boldsymbol{v}^{\intercal}\boldsymbol{g}\boldsymbol{u} \quad \text{(number)} & (53)\\
\boldsymbol{g}\cdot\boldsymbol{u} & \boldsymbol{u}^{\intercal}\boldsymbol{g}^{\intercal} \quad \text{(row-matrix)} & (54)\\
\boldsymbol{\omega}\cdot\boldsymbol{g}^{-1} & \boldsymbol{g}^{-\intercal}\boldsymbol{\omega}^{\intercal} \quad \text{(column-matrix)} & (55)\\
\gamma^{-1}\cdot\boldsymbol{Q} & \gamma^{-1}\boldsymbol{Q} \quad \text{(column-matrix)} & (56)\\
\gamma^{-1}\cdot\boldsymbol{T} & \gamma^{-1}\boldsymbol{T} \quad \text{(column-matrix)} & (57)\\
\boldsymbol{T}\cdot\boldsymbol{u} & \boldsymbol{T}\boldsymbol{u} \quad \text{(column-matrix)} & (58)\\
\boldsymbol{g}\cdot(\gamma^{-1}\cdot\boldsymbol{T})\cdot\boldsymbol{u} & \gamma^{-1}\,\boldsymbol{g}\boldsymbol{T}\boldsymbol{u} \quad \text{(matrix)} & (59)
\end{array}
$$

- Transformations

$$
\begin{array}{rcll}
\text{(old coords)} & \mapsto & \text{(new coords)} & (60)\\
\boldsymbol{u} & \mapsto & \boldsymbol{J}\boldsymbol{u} & (61)\\
\boldsymbol{\omega} & \mapsto & \boldsymbol{\omega}\boldsymbol{J}^{-1} & (62)\\
\boldsymbol{g} & \mapsto & \boldsymbol{J}^{-\intercal}\boldsymbol{g}\boldsymbol{J}^{-1} & (63)\\
\boldsymbol{Q} & \mapsto & \dfrac{1}{\det\boldsymbol{J}}\,\boldsymbol{J}\boldsymbol{Q} & (64)\\
\gamma & \mapsto & \dfrac{1}{\det\boldsymbol{J}}\,\gamma & (65)\\
\gamma^{-1} & \mapsto & \det\boldsymbol{J}\,\gamma^{-1} & (66)\\
\boldsymbol{T} & \mapsto & \dfrac{1}{\det\boldsymbol{J}}\,\boldsymbol{J}\boldsymbol{T}\boldsymbol{J}^{-1} & (67)
\end{array}
$$

B Checks about optimal representation of four-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$
\begin{gathered}
\mathrm{d}\big(f\,\mathrm{d}^3_t\big) = \mathrm{d}\Big(f\,\mathrm{d}^3\tilde{x}yz\Big) = \partial_t f\,\mathrm{d}t\wedge\mathrm{d}^3\tilde{x}yz + 0 = \partial_t f\;\mathrm{d}^4 t\tilde{x}yz\\
\mathrm{d}\big(f\,\mathrm{d}^3_x\big) = -\mathrm{d}\Big(f\,\mathrm{d}^3\tilde{t}yz\Big) = -\partial_x f\,\mathrm{d}x\wedge\mathrm{d}^3\tilde{t}yz + 0 = \partial_x f\;\mathrm{d}^4 t\tilde{x}yz\\
\mathrm{d}\big(f\,\mathrm{d}^3_i\big) = \partial_i f\;\mathrm{d}^4 t\tilde{x}yz
\end{gathered}
\qquad (68)
$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned}\boldsymbol{\nabla}(\mathrm{d}t) &= -\Gamma^t_{tt}\,\mathrm{d}t \otimes \mathrm{d}t - \Gamma^t_{tj}\,\mathrm{d}t \otimes \mathrm{d}x^j - \Gamma^t_{it}\,\mathrm{d}x^i \otimes \mathrm{d}t - \Gamma^t_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \\ \boldsymbol{\nabla}\big(\mathrm{d}x^k\big) &= -\Gamma^k_{tt}\,\mathrm{d}t \otimes \mathrm{d}t - \Gamma^k_{tj}\,\mathrm{d}t \otimes \mathrm{d}x^j - \Gamma^k_{it}\,\mathrm{d}x^i \otimes \mathrm{d}t - \Gamma^k_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j\end{aligned} \tag{69}$$

If $\boldsymbol{\omega}$ is a 3-covector, $\boldsymbol{\phi}$ a 1-covector, and D the exterior covariant derivative, then

$$\mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) = (\mathrm{d}\boldsymbol{\phi}) \otimes \boldsymbol{\omega} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega} \tag{70}$$

and in particular

$$\mathrm{D}\big(\boldsymbol{\phi} \otimes \mathrm{d}x^\alpha\big) = (\mathrm{d}\boldsymbol{\phi}) \otimes \mathrm{d}x^\alpha + \Gamma^\alpha_{\mu\nu}\,\big(\boldsymbol{\phi} \wedge \mathrm{d}x^\mu\big) \otimes \mathrm{d}x^\nu\ . \tag{71}$$

Let's also consider the contraction with a 1-vector $\boldsymbol{u}$:

$$\begin{aligned}\mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u}) &= \mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \boldsymbol{u} - (\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \curlywedge \nabla\boldsymbol{u} \\ &\equiv \mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega} \\ &= \mathrm{d}\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega}\ .\end{aligned} \tag{72}$$

and in particular

$$\mathrm{D}(\boldsymbol{\phi} \otimes \mathrm{d}x^\alpha \cdot \boldsymbol{u}) = \mathrm{d}\boldsymbol{\phi}\, u^\alpha - \boldsymbol{\phi} \wedge \mathrm{d}x^\beta\, \partial_\beta u^\alpha \tag{73}$$

A balance equation with the exterior covariant derivative then becomes

$$\mathrm{D}\boldsymbol{T} = 0 \qquad \Longrightarrow \qquad \mathrm{d}(\boldsymbol{T} \cdot \boldsymbol{u}) = -\boldsymbol{T} \curlywedge \nabla\boldsymbol{u} \tag{74}$$

Then

$$
\begin{aligned}
0 &= \mathrm{D}\boldsymbol{T}\\
&= \mathrm{D}\bigl(-e\,\mathrm{d}^3_t \otimes \mathrm{d}t - q^i\,\mathrm{d}^3_i \otimes \mathrm{d}t + p_j\,\mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\,\mathrm{d}^3_i \otimes \mathrm{d}x^j\bigr)\\
&= -\partial_t e\ \mathrm{d}^4 \otimes \mathrm{d}t - \partial_i q^i\,\mathrm{d}^4 \otimes \mathrm{d}t + \partial_t p_j\,\mathrm{d}^4 \otimes \mathrm{d}x^j + \partial_i \pi^i_j\,\mathrm{d}^4 \otimes \mathrm{d}x^j -\\
&\quad \Bigl[e\,\Gamma^t_{tt}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t + e\,\Gamma^t_{tj}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0\\
&\quad q^i\,\Gamma^t_{it}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t + q^i\,\Gamma^t_{ij}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j + 0\\
&\quad - p_j\,\Gamma^j_{tt}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t - p_k\,\Gamma^k_{tj}\,\mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0\\
&\quad - \pi^i_k\,\Gamma^k_{it}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t - \pi^i_k\,\Gamma^k_{ij}\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j\Bigr]\\
&= \mathrm{d}^4 \otimes \bigl[\\
&\quad - \partial_t e\ \mathrm{d}t + e\,\Gamma^t_{tt}\,\mathrm{d}t + e\,\Gamma^t_{tj}\,\mathrm{d}x^j\\
&\quad - \partial_i q^i\,\mathrm{d}t + q^i\,\Gamma^t_{it}\,\mathrm{d}t + q^i\,\Gamma^t_{ij}\,\mathrm{d}x^j\\
&\quad + \partial_t p_j\,\mathrm{d}x^j - p_j\,\Gamma^j_{tt}\,\mathrm{d}t - p_k\,\Gamma^k_{tj}\,\mathrm{d}x^j\\
&\quad + \partial_i \pi^i_j\,\mathrm{d}x^j - \pi^i_k\,\Gamma^k_{it}\,\mathrm{d}t - \pi^i_k\,\Gamma^k_{ij}\,\mathrm{d}x^j\\
&\quad \bigr]
\end{aligned}
\tag{75}
$$

The equation above corresponds to the four balance equations

$$
\partial_t e + \partial_i q^i = e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} - p_j\,\Gamma^j_{tt} - \pi^i_k\,\Gamma^k_{it} \tag{76}
$$

$$
\partial_t p_j + \partial_i \pi^i_j = -e\,\Gamma^t_{tj} - q^i\,\Gamma^t_{ij} + p_k\,\Gamma^k_{tj} + \pi^i_k\,\Gamma^k_{ij} \tag{77}
$$

For $\boldsymbol{T} \cdot \boldsymbol{u}$ we find first

$$
\boldsymbol{T} \cdot \boldsymbol{u} = (-e\,u^t + p_j\,u^j)\,\mathrm{d}^3_t + (-q^i\,u^t + \pi^i_j\,u^j)\,\mathrm{d}^3_i \tag{78}
$$

$$
\begin{aligned}
\boldsymbol{T} \curlywedge \nabla\boldsymbol{u} &= (-e\,\partial_t u^t + p_j\,\partial_t u^j)\,\mathrm{d}^3_t \wedge \mathrm{d}t + (-q^i\,\partial_i u^t + \pi^i_j\,\partial_i u^j)\,\mathrm{d}^3_i \wedge \mathrm{d}x^i\\
&\qquad + \Gamma \text{ terms}
\end{aligned}
\tag{79}
$$

and therefore

$$\begin{aligned}
\partial_t(-e\,u^t + p_j\,u^j) + \partial_i(-q^i\,u^t + \pi^i_j\,u^j) = {}&\\
&-e\,\partial_t u^t + p_j\,\partial_t u^j - q^i\,\partial_i u^t + \pi^i_j\,\partial_i u^j\\
&-(e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} - p_j\,\Gamma^j_{tt} - \pi^i_k\,\Gamma^k_{it})\,u^t\\
&-(e\,\Gamma^t_{tj} + q^i\,\Gamma^t_{ij} - p_k\,\Gamma^k_{tj} - \pi^i_k\,\Gamma^k_{ij})\,u^j
\end{aligned} \tag{80}$$

Radial case

$$\partial_t e + \partial_r q^r = e\,\Gamma^t_{tt} + q^r\,\Gamma^t_{rt} + p_r\,\Gamma^r_{tt} + \pi^r_r\,\Gamma^r_{rt} \tag{81}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\Gamma^t_{tr} + q^r\,\Gamma^t_{rr} + p_r\,\Gamma^r_{tr} + \pi^r_r\,\Gamma^r_{rr} \tag{82}$$

$$\partial_t e + \partial_r q^r = q^r\,\frac{g}{c^2} + p_r\,g \tag{83}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\frac{g}{c^2} \tag{84}$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have[4]

$$\Gamma^t_{jt} = \Gamma^t_{tj} = \frac{GM}{c^2}\frac{x^j}{r^3} \approx \frac{g}{c^2} \qquad \Gamma^j_{tt} = GM\frac{x^j}{r^3} \approx g \tag{85}$$

where g is the standard acceleration, considered positive. Take also $p_j \approx m v_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

The balances above become

$$\partial_t e + \partial_i q^i = q^z\,\frac{g}{c^2} + p_z\,g \tag{86}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\,\frac{g}{c^2} \tag{87}$$

Also,

[4] Poisson & Will 2014 § 5.2.3.

$$\Gamma^{t}_{tt} \approx -2\frac{g}{c^2}v(t) \qquad \Gamma^{t}_{jt} = \Gamma^{t}_{tj} \approx \frac{g}{c^2}$$
$$\Gamma^{j}_{tt} \approx g - 2\frac{g}{c^2}\,v(t)^2 - \dot{v}(t) \qquad \Gamma^{j}_{jt} = \Gamma^{j}_{tj} \approx \frac{g}{c^2}v(t) \tag{88}$$

$$\partial_t e + \partial_i q^i = -e\,2\frac{g}{c^2}v(t) + q^z\,\frac{g}{c^2} + p_j\left(g - 2\frac{g}{c^2}\,v(t)^2 - \dot{v}(t)\right) + \pi^z_z\,\frac{g}{c^2}v(t) \tag{89}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\,\frac{g}{c^2} + p_z\,\frac{g}{c^2}v(t) \tag{90}$$

C Works with useful content

- Eq. (21) in[5]: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in[6]

For transformation or raising:[7].

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B}\gamma \cdot (A \wedge B) \tag{91}$$

with $\deg(B) = n - \deg(A)$[8]

Compound matrices:[9]

Bibliography

(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

Barnabei, M., Brini, A., Rota, G.-C. (1985): *On the exterior calculus of invariant theory*. J. Algebra **96**[1], 120–160. doi:10.1016/0021-8693(85)90043-2.

Choquet-Bruhat, Y., DeWitt-Morette, C., Dillard-Bleick, M. (1996): *Analysis, Manifolds and Physics. Part I: Basics*, rev. ed. (Elsevier, Amsterdam). First publ. 1977.

Eckart, C. (1940): *The thermodynamics of irreversible processes. III. Relativistic theory of the simple fluid*. Phys. Rev. **58**[10], 919–924. doi:10.1103/PhysRev.58.919.

Frankel, T. (1979): *Gravitational Curvature: An Introduction to Einstein's Theory*. (W. H. Freeman and Company, San Francisco).

Gantmacher, F. R. (2000): *The Theory of Matrices. Vol. 1*, repr. (American Mathematical Society, Providence, USA). https://archive.org/details/gantmacher-the-theory-of-matrices-vol-1-1959. First publ. in Russian 1959. Transl. by K. A. Hirsch.

Gotay, M. J., Marsden, J. E. (1992): *Stress-energy-momentum tensors and the Belinfante-Rosenfeld formula*. Contemp. Math. **132**, 367–392. https://www.cds.caltech.edu/~marsden/bib/1992/05-GoMa1992/, doi:https://doi.org/10.1090/conm/132.

[5] Maugin 1974. [6] Gourgoulhon 2012. [7] Gantmacher 2000 § I.4 eq. (33). [8] Barnabei et al. 1985 prop. 4.1. [9] Choquet-Bruhat et al. 1996 § IV.A.1 p. 199, Problem 1 p. 270.

Gourgoulhon, É. (2012): *3+1 Formalism in General Relativity: Bases of Numerical Relativity*. (Springer, Heidelberg). First publ. 2007 as arXiv doi:10.48550/arXiv.gr-qc/0703035. doi:10.1007/978-3-642-24525-1.

Maugin, G. A. (1974): *Constitutive equations for heat conduction in general relativity*. J. Phys. A **7**4, 465–484. doi:10.1088/0305-4470/7/4/010.

Poisson, E., Will, C. M. (2014): *Gravity: Newtonian, Post-Newtonian, Relativistic*. (Cambridge University Press, Cambridge). doi:10.1017/CBO9781139507486.