Notes on multivector algebra on differential manifolds

P.G.L. Porta Mana
<pgl@portamana.org>

28 October 2020; updated 11 November 2020

1 Multivector tensor algebra

The idea is to build tensors not from the two spaces of vectors and covectors, but from the $2N^2$ spaces of multivectors and multicovectors with their possible straight and twisted orientations.

The exterior algebra is an algebra independent of the tensor one, and it expresses very intuitive geometric relations[1].

The fact that it is independent of the tensor algebra is clear from the fact that we can establish several inequivalent relations between the tensor and exterior products, none of them being canonical.

Antisymmetrizer A (a projection):

$$AT \coloneqq \frac{1}{(\deg T)!} \sum_{\pi} \operatorname{sgn}(\pi)\, T \circ \pi \tag{1}$$

Abraham et al. (1988), Choquet-Bruhat et al. (1996), Bossavit (1991) use this relation:

$$\begin{aligned} \alpha \wedge \beta &\equiv \frac{(\deg\alpha + \deg\beta)!}{(\deg\alpha)!\,(\deg\beta)!}\, A(\alpha\otimes\beta) \\ &\equiv \frac{1}{(\deg\alpha)!\,(\deg\beta)!} \sum_{\pi} \operatorname{sgn}(\pi)\,(\alpha\otimes\beta)\circ\pi\,, \end{aligned} \tag{2}$$

but relations with different multiplicative factors are also possible.

It's best to define the exterior product intrinsically, with its multilinear, associative, and graded-commutative properties.

[1] cf. Deschamps 1970; 1981.

2 Inner or dual or dot product

For a vector u and covector ω with $\deg u \leqslant \deg \omega$ it's defined as

$$
\begin{gathered}
u \rfloor \omega \coloneqq \omega(u) \quad \text{if } \deg u = \deg \omega \\
(u \rfloor \omega)(v) \coloneqq (u \wedge v) \rfloor \omega \equiv \omega(u \wedge v) \quad \text{if } \deg u < \deg \omega
\end{gathered}
\tag{3}
$$

It's possible to define an inner product from the right side, but it gives the same result as above except for a sign:

$$
\begin{aligned}
(\omega \lfloor u)(v) \coloneqq (v \wedge u) \rfloor \omega &\equiv (-1)^{\deg u \;\deg v}\, (u \wedge v) \rfloor \omega \\
&\equiv (-1)^{\deg u \;\deg v}\, (u \rfloor \omega)(v)
\end{aligned}
\tag{4}
$$
$$
\implies \omega \lfloor u \equiv (-1)^{\deg u \;(\deg \omega - \deg u)}\, u \rfloor \omega \equiv (-1)^{\deg u \;(\deg \omega - 1)}\, u \rfloor \omega
$$

For 1-vectors in particular:

$$
\omega \lfloor u \equiv (-1)^{\deg \omega - 1}\, u \rfloor \omega \qquad \text{if } \deg u = 1 \tag{5}
$$

Also

$$
\begin{gathered}
u \lfloor \omega \coloneqq \omega(u) \equiv u \rfloor \omega \quad \text{if } \deg u = \deg \omega \\
(u \lfloor \omega)(\xi) \coloneqq u \lfloor (\xi \wedge \omega) \equiv (\xi \wedge \omega)(u) \quad \text{if } \deg u > \deg \omega
\end{gathered}
\tag{6}
$$

and

$$
\begin{aligned}
(\omega \rfloor u)(\xi) \coloneqq u \rfloor (\omega \wedge \xi) &\equiv (-1)^{\deg \omega \;\deg \xi}\, u \rfloor (\xi \wedge \omega) \\
&\equiv (-1)^{\deg \omega \;\deg \xi}\, (u \lfloor \omega)(\xi)
\end{aligned}
\tag{7}
$$
$$
\implies \omega \rfloor u \equiv (-1)^{\deg \omega \;(\deg v - 1)}\, u \lfloor \omega
$$

The lower hook in "$\rfloor$" and "$\lfloor$" is useful to denote the object with lower degree, to know how to apply the sign in the graded-commutativity property and to know what kind of object – multivector or multicovector – one obtains.

If we define the degree of vectors to be negative, we can say that $\alpha \rfloor \beta$ yields an object of degree $\deg(\alpha \rfloor \beta) = \deg \alpha + \deg \beta$, no matter whether α is a vector and β a covector or vice versa. With this convention we could use the more compact dot-notation[2]

$$
\begin{gathered}
\alpha \cdot \beta \\
\text{with} \qquad \deg(\alpha \cdot \beta) = \deg \alpha + \deg \beta \\
\beta \cdot \alpha = (-1)^{\min\{|\deg \alpha|, |\deg \beta|\}\;(\deg \alpha + \deg \beta)}\, \alpha \cdot \beta \;.
\end{gathered}
\tag{8}
$$

[2] cf. Truesdell & Toupin 1960 § F.I.267.

But it doesn't make much sense to use a unique symbol, because it would not represent an associative operation (unlike the wedge).

The inner product with a 1-vector or a 1-covector is a graded derivation.

3 Tensor products and equivalent objects

When we take tensor products of exterior objects, some special objects and some canonical correspondences between different classes of objects appear.

Take for example a non-zero N-covector γ. The tensor product $\gamma \otimes \gamma^{-1}$ is an N-vector-valued N-covector. This object is independent of the specific γ we chose. It has only one non-zero component, of value 1, which is invariant under basis or coordinate changes[3]. It has properties similar to those of a scalar, and the tensor space of N-vector-valued N-covectors is similar to those of scalars (Schouten (1989) § II.8 p. 29: "This is not a new geometric conception, only a new notation enabling us to get rid of a lot of indices").

The inner product of a p-vector u with the object above is an N-vector-valued $(N-p)$-covector:

$$u \cdot \gamma \otimes \gamma^{-1} = \omega \otimes \gamma^{-1} \qquad \text{with } \omega = u \cdot \gamma\,. \tag{9}$$

The space of p-vectors is therefore equivalent, in a canonical way, to the space of N-vector-valued $(N-p)$-covectors. The independent components transform in the same way under a change of basis/coordinates; see the example in Schouten (1989) § II.8 p. 30 bottom.

As Schouten[4] says, "Hence the geometrical meanings of corresponding quantities do not differ. There is only a difference in notation. [...] The use of Δ-densities is sometimes convenient; [...] the formulae contain less indices".

With the coordinate-free (and index-free) approach the use of such quantities offers no advantages.

3 Schouten 1989 § II.8 p. 29 bottom. **4** Schouten 1989 § II.8 p. 30.

4 Inner product with N-covector

If γ is a non-zero N-covector (hypervolume covector), so that γ^{-1} is its dual N-vector, we have

$$\gamma^{-1} \cdot (u \cdot \gamma) = (\gamma \cdot u) \cdot \gamma^{-1} = u \qquad \gamma \cdot (\omega \cdot \gamma^{-1}) = (\gamma^{-1} \cdot \omega) \cdot \gamma = \omega \tag{10}$$

for any multivector u and multicovector ω[5]. According to (4),

$$\gamma \cdot u = (-1)^{\deg(u)\,(N-1)}\, u \cdot \gamma \tag{11}$$

and analogously for the dual case.

Therefore $\gamma^{-1} \cdot (\gamma \cdot u) = (-1)^{\deg(u)\,(N-1)}\, u \neq u = (\gamma^{-1} \cdot \gamma) \cdot u$, which shows that the inner product is non-associative in general.

5 Star operator

This section has mistakes

Comparison of definitions of star operator:

Denote with $\overline{\omega}$ the p-vector obtained by rising all slots of ω with a metric g, and with $\underline{v}$ the reverse operation. Let γ be the volume element induced by the metric.

Choquet-Bruhat et al. (1996 § V.A.4):

$$*\,\omega \coloneqq \overline{\omega} \cdot \gamma \tag{12}$$

Applying twice:

$$\begin{aligned} **\omega = \overline{\overline{\omega} \cdot \gamma} \cdot \gamma = (\omega \cdot \gamma^{-1}) \cdot \gamma = \\ (-1)^{\deg(\omega)\,(N-1)} (\gamma^{-1} \cdot \omega) \cdot \gamma = (-1)^{\deg(\omega)\,(N-1)} \omega \end{aligned} \tag{13}$$

So $*^{-1} = (-1)^{\deg(\omega)\,(N-1)}\,*$. Compare with Bossavit (1991 § 4.2 Ex. 71).

[5] cf. Schouten 1989 § II.7 p. 28.

6 Twisted scalars, vectors, covectors

A twisted scalar is a positive number with an associated outer orientation. We can specify such orientation locally for example by giving an ordered list of coordinate functions. Denote such a unit twisted scalar by

$$\underset{txyz}{1} \tag{14}$$

It satisfies

$$\begin{gathered}\underset{txyz}{1} \cdot \underset{txyz}{1} = 1 \\ \underset{txyz}{-1} = \underset{xtyz}{1} \quad \text{or any other odd permutation} \\ a \cdot \underset{txyz}{1} = \underset{txyz}{a} \quad \text{for any scalar } a\end{gathered} \tag{15}$$

$\underset{txyz}{1}$

Bibliography

("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

Abraham, R., Marsden, J. E., Ratiu, T. (1988): *Manifolds, Tensor Analysis, and Applications*, 2nd ed. (Springer, New York). First publ. 1983. DOI:10.1007/978-1-4612-1029-0.

Bossavit, A. (1991): *Differential Geometry: for the student of numerical methods in electromagnetism*. https://www.researchgate.net/publication/200018385_Differential_Geometry_for_the_student_of_numerical_methods_in_Electromagnetism.

Choquet-Bruhat, Y., DeWitt-Morette, C., Dillard-Bleick, M. (1996): *Analysis, Manifolds and Physics. Part I: Basics*, rev. ed. (Elsevier, Amsterdam). First publ. 1977.

Deschamps, G. A. (1970): *Exterior differential forms*. In: Deschamps, de Jager, John, Lions, Moisseev, Sommer, Tihonov, Tikhomirov, et al. (1970): p. III:111–161.

— (1981): *Electromagnetics and differential forms*. Proc. IEEE **69**6, 676–696.

Deschamps, G. A., de Jager, E. M., John, F., Lions, J. L., Moisseev, N., Sommer, F., Tihonov, A. N., Tikhomirov, V., et al. (1970): *Mathematics applied to physics*. (Springer, Berlin). Ed. by É. Roubine.

Flügge, S., ed. (1960): *Handbuch der Physik: Band III/1: Prinzipien der klassischen Mechanik und Feldtheorie [Encyclopedia of Physics: Vol. III/1: Principles of Classical Mechanics and Field Theory]*. (Springer, Berlin). DOI:10.1007/978-3-642-45943-6.

Schouten, J. A. (1989): *Tensor Analysis for Physicists*, corr. 2nd ed. (Dover, New York). First publ. 1951.

Truesdell III, C. A., Toupin, R. A. (1960): *The Classical Field Theories*. In: Flügge (1960): I–VII, 226–902. With an appendix on invariants by Jerald LaVerne Ericksen. DOI:10.1007/978-3-642-45943-6_2.