Open Science Framework DOI:10.31219/osf.io/***

Kinematics, dynamics, force, inertia, metric in Newtonian relativity and in general relativity [draft]

P.G.L. Porta Mana
<piero.mana@ntnu.no>

16 October 2016; updated 9 December 2019

This note is an exploration of some notions around kinematics, dynamics, matter, inertia, and force in Newtonian- and general-relativistic continuum mechanics. The conceptual pivots of this exploration are these: (i) kinematics should not allow us to distinguish between Newtonian relativity and general relativity; (ii) in neither theory should there be any distinction between inertial and non-inertial motion, or between forced and unforced motion.
Note: Dear Reader & Peer, this manuscript is being peer-reviewed by you. Thank you.

This presentation is mainly discursive. For formulae see Porta Mana (2016).

1 Kinematics

1.1 Spacetime and matter

Our first primitive is the notion of spacetime, as the seats of all events. It's represented by a 4D differential manifold. In the following I misuse 'topological' as a shorthand for 'differential-geometric', to denote objects that only require a differential structure – so no connections, metrics, volume elements, or similar – for their definition. In spacetime there's no predefined notion of simultaneity nor of 'identity through time'.

Our second primitive is the notion of matter, of bodies. How to represent it? Matter is characterized by our ability to mark a small part of it – as when we draw a '◯' sign on an body, or throw some leaves in flowing water – and then to follow it as it moves. It relates to the notions of identity, individuality, permanence through time. Two topological objects can represent these notions: an inner-oriented vector field or an outer-oriented 3-covector field[1] (or '3-form'). The first can be depicted as a collection of 1D curves, the second as a collection of tubes with 3D cross-sections[2]. Either has the property that, given one event in spacetime, a unique 1D curve through it is selected. Either is in

[1] Schouten 1989. [2] Misner et al. 1973; Schouten 1989.

turn determined by such curves but only up to a multiplicative scalar function.

For the vector field, different functions determine different vectors tangent to the curves; they could represent a 4-velocity. But we know from Lorentzian relativity that such freedom is irrelevant: 4-velocities are by convention normalized to unity. For the 3-covector field, on the other hand, different scalar functions determine different cross-sections of the tubes.

The latter fact is connected with an additional feature of such a field: it can be integrated over an outer-oriented 3D surface (count the 3D tubes which cross that surface). Later on, such a surface could be characterized as spacelike or timelike. This 3-covector field thus allows us to speak of the 'quantity' of matter within a 3D region at some time, or the quantity of matter that has flowed through a 2D surface during a given time. It is not possible to decompose this flux into a 'density' and a 'velocity', though.

If such a field is closed (its exterior derivative, which is a topological operation, vanishes), then the matter it represents satisfies a conservation law: the total quantity of matter flowing through a closed 2D surface in a given time must be equal to the increase of matter within that surface during the same time. This property is useful in the theory of mixtures[3], where we model different kinds of matter that can transform into each other, their total amount being conserved. This situation can be represented by several non-closed 3-covector fields, their sum being closed. In general relativity, conservation of matter is more precisely conservation of baryonic-number[4]. In Newtonian relativity traditionally it's conservation of mass, but it's easy to show that this theory could be formulated with baryon conservation instead; a similar reformulation in fact takes place within the theory of mixtures mentioned above.

So we represent matter as an outer-oriented 3-covector field, called the 'matter field'.

A spacetime and one or more matter fields give us a topological kinematics. Such kinematics doesn't allow us to say that some parts of matter are at rest or in motion, or approaching or diverging. But it allows us to follow their trajectories through spacetime. It is remarkable that the conservation of matter turns out to be a kinematic and topological

[3] Samohýl 1975; Bowen 1976; Samohýl et al. 2014; Faria 2003. [4] Eckart 1940; Misner et al. 1973 § 22.2; Gourgoulhon 2012; Alcubierre 2008.

property. It is also remarkable that this is a true 'field' (an association with a geometric object to each point in spacetime) and yet it represents particles and their trajectories. This shows that there is no dualism between field and matter.

The notions of identity, individuality, permanence through time can also be generalized to non-pointlike objects, such as 1D curves or 2D surfaces. The idea, for example, is that we can *identify* and *follow* a given curve (or tubular region) through time, although the points on it cannot be followed because they keep no identity through time. These kinds of objects can also be represented by m-covector fields. The most important example is the electromagnetic field, an inner-oriented 2-covector. These objects can also satisfy conservation laws ('the number of curves cannot increase'), expressed as the closure of their covector fields.

1.2 Simultaneity and rest; frames

The intuitive notion of simultaneity comes to us mainly from our sense of vision; our intuition would probably be different if it came from sound. I take the point of view, suggested by general relativity, that there is no natural spacetime-extended definition of simultaneity. We can make an arbitrary foliation of spacetime into 3D surfaces, which we can call 'instants', and even label each instant with a number, but this construction and labels need not have any physical meaning.

Even when instants are introduced, spacetime doesn't offer any canonical isomorphism among them. That is, there's no way of saying 'this event at this instant and that event at that instant occur at the *same* place'. It is possible to establish an arbitrary isomorphism among instants through a vector field not tangent to any of them. This vector field can also be chosen to have unit product with the 1-covector associated with the foliation. This construction needs not have any physical meaning. This vector field allows us to compare field quantities at different instants via the Lie derivative, which therefore acts as a 'time derivative at fixed position'.

The combination of a foliation and a vector field as described above is called a *frame* or an *observer*[5]. Once a frame is introduced we can speak of motion or rest of matter with respect to that frame, and even define a velocity and a space-density of matter[6]. But again, these are arbitrary

[5] Taub 1978; Smarr et al. 1978; York 1979; Smarr et al. 1980; Hehl et al. 2003 § B.1.4.
[6] Porta Mana 2016 § 3.2.

constructions. A frame is a scaffolding that allows us to formulate kinematics in terms of 'rest' and topological versions of 'velocity' and 'density'. But it is clear that it doesn't add any physical feature. The rest of the development of the theory can be done without it. Frames may be useful for our intuition and to set up initial-value problems in numerical relativity.

It is possible to choose frames with particular physical meaning. For example we can choose the vector field of the frame to be tangent to the tubes of a matter field. That matter will then be at rest in that frame, by decree. This frame is called *Lagrangean* or *comoving*[7]. Other choices become possible when a metric or a connection or a 4-stress tensor are introduced, and we have *Eulerian* frames, *inertial* or *free-fall* frames, and several others.

2 Metric

The kinematics just discussed is purely topological, even if we can introduce synthetic notions of time-lapse and rest into it. The question is whether metric and chronometric notions should be part of kinematics or dynamics. The answer depends on what we want 'kinematics' and 'dynamics' to mean, and has room for subjective preferences.

My preference is that kinematics should allow us to describe *all imaginable* motions – it therefore has a counterfactual character. It should place no constraints on any motion. Note that the law of conservation of matter doesn't place any constraints on the motion of matter. On the other hand, dynamics is what places such constraints. It should do so by introducing new geometrical objects (such as force) that satisfy general laws. These objects should be functionals (constitutive relations) of the kinematic objects; thus their laws indirectly place constraints on the kinematic objects.

I therefore see four alternatives for the metric:

(i) it's a kinematic object – a form of matter, a field (the Ether![8]);

(ii) it isn't a form of matter, but it's part of the kinematic infrastructure, like spacetime;

(iii) it's a dynamic object, like force;

(iv) it isn't a dynamic object, but it's part of the dynamic infrastructure.

[7] Taub 1978; Smarr et al. 1978; York 1979; Smarr et al. 1980. [8] Dirac 1951.

Let's see what Newtonian and general relativity suggest about these alternatives.

In Newtonian relativity, kinematics traditionally also includes chronometric and metric notions: a distinguished time foliation and a distinguished flat 3-metric at each instant. These foliation and 3-metrics are fixed. From this point of view they could be classified as 'kinematic infrastructure', alternative (ii) above.

Foliation and 3-metrics, however, do *not* determine any natural isomorphism among instants – 'this place at this instant and that place at that instant are the *same*'. Newtonian relativity traditionally includes also such an isomorphism, which can be represented by an affine connection, leading to Newton-Cartan theory[9]. Foliation, 3-metrics, and affine connection together represent the 'absolute space' which 'in its own nature, without relation to anything external, remains always similar and immovable'[10].

We see that these objects together do introduce dynamical notions in the form of special motions: absolute rest, straight lines, constant speed. From this point of view they seem to belong to dynamics rather than kinematics; the affine connection does, at the very least.

In general relativity the 4-metric and its canonical affine connection are not fixed: there are several possible ones. In a $3+1$ formulation of the equations of motions, the 4-metric evolves, just like a kinematic quantity. Thus it could be classified as (i) above.

In both Newtonian and general relativity the metric also appear in peculiar places. For example, the stress of many materials, like elastic ones, depends on the strain, which in both relativity theories turns out to be the Lie derivative of the 3-metric with respect to the vector field induced by the matter field with respect to the time foliation[11]. Most important, the traditional dynamic equations can be made, in both theories, to take the form

$$\nabla \cdot T = 0, \tag{1}$$

where T is the stress-energy-momentum tensor, which I call '4-stress', and ∇ is the covariant derivative with respect to the 4-metric. The metric thus directly enters the dynamic equations. The same is true if we start from the Einstein equation $G = T$ in general relativity. The metric's

[9] Cartan 1923; 1924; 1925; Trautman 1964; 1966; Ellis 1971; Künzle 1972; Earman et al. 1973; Ehlers 1973. [10] Newton 1974a *Definitions*, *Scholium*. [11] Rayner 1963; Carter et al. 1972; Carter 1973; Barrabes 1975; Maugin 1978.

appearance in such places makes a choice between the four alternatives above very difficult.

The discussion above becomes even more complicated owing to two additional considerations and questions. First, some equations that seem to involve the metric can be reformulated in a metric-free way; matter conservation is an example. Second, the 4-metric in the standard formulation of general relativity allows us to speak of the 'size of the universe' in an absolute way. This doesn't make sense if we view spacetime and distance in a relational way[12]. Is it possible that a more 'relational' formulation should eliminate the motional degrees of freedom that the metric enjoys in general relativity? But we must also remember that this theory is not only meant to describe the universe as a whole: it can also be applied to parts of it. From this point of view it does makes sense that the metric should give us the absolute size of the region under study. Third, is it possible that metric could be *the* dynamical object, all others being just defined in terms of it? Or vice versa that metric could be fully defined in terms of the 4-stress?

I believe that much work has still to be done to disentangle the metric from other geometric objects, to arrive at an appealing and conceptually simple representation of physical ideas.

3 Dynamics

[12] Barbour et al. 1982; Anderson et al. 2003; Barbour 2003a,b.

Bibliography

(‘de *X*’ is listed under D, ‘van *X*’ under V, and so on, regardless of national conventions.)

Alcubierre, M. (2008): *Introduction to 3 + 1 Numerical Relativity*. (Oxford University Press, Oxford).

Anderson, E., Barbour, J. B., Foster, B. Z., Murchadha, N. Ó. (2003): *Scale-invariant gravity: geometrodynamics*. Class. Quantum Grav. **20**^{8}, 1571–1604. See also Barbour (2003a).

Barbour, J. B., Bertotti, B. (1982): *Mach's principle and the structure of dynamical theories*. Proc. R. Soc. Lond. A **382**^{1783}, 295–306.

Barbour, J. B. (2003a): *Scale-invariant gravity: particle dynamics*. Class. Quantum Grav. **20**^{8}, 1543–1570. See also Anderson, Barbour, Foster, Murchadha (2003).

— (2003b): *Dynamics of pure shape, relativity, and the problem of time*. In: *Decoherence and entropy in complex systems*. Ed. by H.-T. Elze (Springer, New York), 15–35.

Barrabes, C. (1975): *Elastic and thermoelastic media in general relativity*. Nuovo Cimento B **28**^{2}, 377–394.

Bowen, R. M. (1976): *Theory of mixtures*. In: Eringen (1976), 1–127.

Cartan, É. (1923): *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)*. Annales scientifiques de l'École Normale Supérieure **40**, 325–412. See also Cartan (1924; 1925); transl. in Cartan (1986).

— (1924): *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie) (Suite)*. Annales scientifiques de l'École Normale Supérieure **41**, 1–25. See also Cartan (1923; 1925); transl. in Cartan (1986).

— (1925): *Sur les variétés à connexion affine et la théorie de la relativité généralisée (deuxième partie)*. Annales scientifiques de l'École Normale Supérieure **42**, 17–88. See also Cartan (1923; 1924); transl. in Cartan (1986).

— (1986): *On Manifolds with an Affine Connection and the Theory of General Relativity*. (Bibliopolis, Napoli). Transl. of Cartan (1923; 1924; 1925) by Anne Magnon and Abhay Ashtekar.

Carter, B. (1973): *Elastic perturbation theory in general relativity and a variation principle for a rotating solid star*. Commun. Math. Phys. **30**^{4}, 261–286.

Carter, B., Quintana, H. (1972): *Foundations of general relativistic high-pressure elasticity theory*. Proc. R. Soc. Lond. A **331**^{1584}, 57–83.

Dirac, P. A. M. (1951): *Is there an æther?* Nature **168**^{4282}, 906–907.

Earman, J., Friedman, M. (1973): *The meaning and status of Newton's law of inertia and the nature of gravitational forces*. Philos. Sci. **40**^{3}, 329–359.

Eckart, C. (1940): *The thermodynamics of irreversible processes. III. Relativistic theory of the simple fluid*. Phys. Rev. **58**^{10}, 919–924.

Ehlers, J. (1973): *Survey of general relativity theory*. In: Israel (1973), 1–125.

Ellis, G. F. R. (1971): *Relativistic cosmology*. In: Sachs (1971), 104–182.

Eringen, A. C., ed. (1976): *Continuum Physics. Vol. III – Mixtures and EM Field Theories*. (Academic Press, New York).

Faria, S. H. (2003): *Mechanics and thermodynamics of mixtures with continuous diversity: from complex media to ice sheets*. PhD thesis. (Technische Universität Darmstadt, Darmstadt). `http://elib.tu-darmstadt.de/diss/000307/`.

Gourgoulhon, É. (2012): *3+1 Formalism in General Relativity: Bases of Numerical Relativity*. (Springer, Heidelberg). First publ. 2007 as `arXiv:gr-qc/0703035`.

Hehl, F. W., Obukhov, Y. N. (2003): *Foundations of Classical Electrodynamics: Charge, Flux, and Metric*. (Birkhäuser, Boston).

Hoffmann, B., ed. (1966): *Perspectives in Geometry and Relativity: Essays in Honor of Václav Hlavatý*. (Indiana University Press, Bloomington and London).

Israel, W., ed. (1973): *Relativity, Astrophysics and Cosmology*. (Reidel, Dordrecht).

Künzle, H. P. (1972): *Galilei and lorentz structures on space-time: comparison of the corresponding geometry and physics*. Ann. Inst. Henri Poincaré Phys. th. **17**^{4}, 337–362.

Maugin, G. A. (1978): *Elasticity and electro-magneto-elasticity of general-relativistic systems*. Gen. Relat. Gravit. **9**^{6}, 541–549.

Misner, C. W., Thorne, K. S., Wheeler, J. A. (1973): *Gravitation*, repr. (W. H. Freeman and Company, New York). First publ. 1970.

Newton, I. (1726): *Philosophiæ naturalis principia mathematica*, 'Tertia aucta & emendata' ed. (Guil. & Joh. Innys, London). http://www.archive.org/details/principiareprint00newtuoft, http://www.archive.org/details/principia00newtuoft. First publ. 1687; transl. in Newton (1846; 1974a,b).

— (1846): *Newton's Principia: The Mathematical Principles of Natural Philosophy*, first American, 'carefully rev. and corr.' ed. (Daniel Adee, New York). Transl. of Newton (1726) by Andrew Motte. With Newton's system of the world and a life of the author by N. W. Chittenden.

— (1974a): *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his system of the world. Vol. One: The Motion of Bodies*. (University of California Press, Berkeley). Transl. of Newton (1726) by Andrew Motte, rev. and supplied with an historical and explanatory appendix by Florian Cajori.

— (1974b): *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his system of the world. Vol. Two: The System of the World*. (University of California Press, Berkeley). Transl. of Newton (1726) by Andrew Motte, rev. and supplied with an historical and explanatory appendix by Florian Cajori.

Porta Mana, P. G. L. (2016): *Force, inertia, metric in Newtonian relativity and general relativity*. In preparation; draft available at Open Science Framework doi:10.17605/osf.io/rvcxs.

Rayner, C. B. (1963): *Elasticity in general relativity*. Proc. R. Soc. Lond. A **272**^{1348}, 44–53.

Sachs, R. K., ed. (1971): *General Relativity and Cosmology*. (Academic Press, New York).

Samohýl, I. (1975): *Comparison of classical and rational thermodynamics of reacting fluid mixtures with linear transport properties*. Collection Czechoslov. Chem. Commun. **40**, 3421–3435. Transl. by K. Micka.

— (1987): *Thermodynamics of Irreversible Processes in Fluid Mixtures (Approached by Rational Thermodynamics)*. (Teubner, Leipzig). Partially rewritten 2014 as Samohýl, Pekař (2014).

Samohýl, I., Pekař, M. (2014): *The Thermodynamics of Linear Fluids and Fluid Mixtures*. (Springer, Cham). First published 1987 as Samohýl (1987).

Schouten, J. A. (1989): *Tensor Analysis for Physicists*, corr. second ed. (Dover, New York). First publ. 1951.

Smarr, L. L., ed. (1979): *Sources of Gravitational Radiation*. (Cambridge University Press, Cambridge).

Smarr, L., Taubes, C., Wilson, J. R. (1980): *General relativistic hydrodynamics: the comoving, Eulerian, and velocity potential formalisms*. In: Tipler (1980), ch. 11, 157–183.

Smarr, L., York Jr., J. W. (1978): *Kinematical conditions in the construction of spacetime*. Phys. Rev. D **17**^{10}, 2529–2551.

Taub, A. H. (1978): *Relativistic fluid mechanics*. Ann. Rev. Fluid Mech. **10**, 301–332.

Tipler, F. J., ed. (1980): *Essays in General Relativity: A Festschrift for Abraham Taub*. (Academic Press, New York).

Trautman, A. (1964): *Foundations and current problems of general relativity*. In: Trautman, Pirani, Bondi (1964), 1–248. Notes by G. Dixon, P. Florides, G. Lemmer.

— (1966): *Comparison of Newtonian and relativistic theories of space-time*. In: Hoffmann (1966), ch. 42, 413–425.

Trautman, A., Pirani, F. A. E., Bondi, H. (1964): *Lectures on General Relativity*. (Prentice-Hall, Englewood Cliffs, USA). Notes by G. Dixon, P. Florides, G. Lemmer, J. J. J. Marek, J. Hyde; ed. by S. Deser and K. W. Ford.

York Jr., J. W. (1979): *Kinematics and dynamics of general relativity*. In: Smarr (1979), 83–126.