Force, inertia, metric
in Newtonian relativity and general relativity

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This note is an exploration of some notions around kinematics, dynamics, matter, inertia, and force in Newtonian- and general-relativistic continuum mechanics. The conceptual pivot of this exploration is this: Neither in Newtonian relativity or in general relativity should there be any distinction between inertial and non-inertial motion, or between forced and unforced motion.

> More than ever hour after
> hour work is never over
> (Daft Punk 2001a)

1 A bundle

Tangled bundles are fun and stressful to untangle. The most tangled ones I know of are: the bundle of quantum theory and its physical meaning, the bundle of the brain's workings, and the bundle of general relativity and Newtonian mechanics.

The last 'tangled bundle' is the most deserving of this metaphorical name, because it unswervingly has that characteristic: you pick and pull a piece of thread that sticks out, as it seems that'll loosen the bundle; and instead you notice that a piece of thread at the other end is tightening up. Its thread, moreover, has several different colours; and you cannot hope to untangle the bundle by focusing on one colour only.

In this note I present some of the pieces of thread that sticks out in the bundle of general relativity and Newtonian mechanics, explaining why it seems that pulling some of them should loosen it, and showing how other threads get tightened in doing so instead. The thread is coloured by different disciplines, including continuum thermomechanics and the theory of inner- and outer-oriented differential forms. Some readers may be unfamiliar with one or another of these disciplines, but I hope they'll bear with me and when needed consult the reference I give. For reasons

that will become clear in the exposition, I will speak of 'Newtonian relativity' rather than 'mechanics'.

My presentation won't be linear, owing to the nature of the problem: we'll need to pull the thread in some part of the bundle, then look at some other part, sometimes diametrically opposite, to see what happened there.

The main notions that we'll deal with are: mass, body, force, inertia, stress, momentum, inner- and outer-oriented differential forms, metric, spacetime, slicings, frames, energy, and also temperature. The main pieces of thread that we'll pull are these:

- The idea of a principle: *neither in Newtonian relativity or in general relativity should there be any distinction between inertial and non-inertial motion, or between forced and unforced motion.*

- The possibility of rephrasing general relativity in terms of *force* rather than momentum.

- To consider inertia as a force, in Newtonian and in general relativity.

- Newton's second law mixes frame-dependent and frame independent quantities.

- Newtonian thermomechanics can be restated as a theory of *statics* in some reference frame, provided inertia is included among the forces. This static point of view is even more natural in general relativity, from a four-dimensional point of view.

- In general relativity, the notion of geodesic is overestimated. Almost no objects of our everyday experience follow geodesics.

- Mass and its conservation are purely differential-geometric notions; they don't need a metric.

- Many other electro-thermo-mechanical notions seem to be metric-free. It should be useful to keep their purely differential-geometric nature well in sight, without mixing the metric in them. 'Indeed, the notion of metric is a very complicated one: it requires measurements with clocks and scales. generally with *rigid* bodies, which are themselves things of extreme complexity. Hence it seems undesirable to take the notion of ametrie as a fundament, also of phenomena which are much simpler and independent of it' (van Dantzig 1934a § 1, pp. 521–522).

- There is a difference between momentum and '$m\mathbf{n}$'.

- Can forces exist where there is no mass? Force and

- The idea that an electromagnetic field can also sustain a force (Page 1958 § 158, pp. 534–535; Ericksen 2007).

2   Prerequisites

I will freely use the differential-geometric notions of inner- and outer-oriented forms (also called 'straight' and 'twisted'), explained by Burke (1995; 1987 ch. IV; 1983), Bossavit (1991 chs 2, 3; 1998), Hehl et al. (2003 ch. A), and going back to Schouten & van Dantzig (1940) at the latest.

I will assume some familiarity with the '3 + 1' decomposition of Einstein's equations (Smarr et al. 1978; York 1979; Smarr et al. 1980) and with the basic principles of Newtonian continuum thermomechanics (Truesdell 1991; Samohýl et al. 2014; Truesdell et al. 1960).

3   Slicings and frames

We consider the four-dimensional manifold formed by all physical events – spacetime – as a primitive concept. Speaking of these events we usually adopt notions of *simultaneity* and of *identity of location*. These notions are usually determined by physical phenomena; e.g. we perceive as simultaneous such events as send light rays that reach our eyes at the same time. But we could also decree by convention that a set of events is simultaneous or occurs at the same place, as long as such convention has some differentiability.

A choice of simultaneity is determined by a *slicing* of spacetime, represented by a function t, or equivalently by an exact 1-form $\mathrm{d}t$ (York 1979). We call each 3-surface with constant t a *slice*, and its points *places* at time t (Smarr et al. 1978; York 1979; Smarr et al. 1980; Marsden et al. 1994 § 2.4). The choice of a slicing is, for the moment, completely arbitrary.

The 'time' t above is a convention: it does not represent the physical time beaten by a periodic phenomenon, a clock. For a small spacetime displacement V of such a clock, the corresponding small lapse in physical time is given in Newtonian relativity by contraction $V \cdot \mathrm{d}t_a$ with a specific 1-form $\mathrm{d}t_a$, or equivalently by the square root of the contraction with the degenerate quadratic form $\mathrm{d}t_a \otimes \mathrm{d}t_a$; in general relativity the quadratic form given by the metric γ is used instead.

The notion of motion also implies a convention about the correspondence of places on slices at different times, so that we can say 'this place

today is the same as that place yesterday'. Such a correspondence can be made with a vector field U such that

$$U \cdot dt = 1. \tag{1}$$

The flow of this vector field puts points on different slices into mutual correspondence. It's sometimes called an observer, or a field of observers, or a *frame*, the term we use here.

A slicing t and a frame U on an n-dimensional manifold induce a natural pair of projection operators: $dt \otimes U$ projects along the frame, and $I - dt \otimes U$ onto the slices. With these projections every m-vector v can be represented on each $(n-1)$-slice as the pair $(\hat{v}, \bar{v})$ of an $(m-1)$-vector and an m-vector

$$\hat{v} \coloneqq v \cdot dt, \qquad \bar{v} \coloneqq v - \hat{v} \wedge U. \tag{2}$$

Every m-form α as the pair $(\hat{\alpha}, \bar{\alpha})$ of an $(m-1)$-form and an m-form

$$\hat{\alpha} \coloneqq U \cdot \alpha, \qquad \bar{\alpha} \coloneqq \alpha - dt \wedge \hat{\alpha}. \tag{3}$$

All of the projected components have a well-defined meaning on each slice. The wedge-product, exterior differentiation, and contraction of these objects in the n-manifold can be rewritten (Hehl et al. 2003 § B.1.4) as pairs of operations on their slice representatives:

$$(\hat{\alpha}, \bar{\alpha}) \wedge (\hat{\beta}, \bar{\beta}) = (\hat{\alpha} \wedge \bar{\beta} + (-1)^m \bar{\alpha} \wedge \hat{\beta}, \bar{\alpha} \wedge \bar{\beta}) \tag{4}$$

$$d(\hat{\alpha}, \bar{\alpha}) = (L_U \bar{\alpha} - d\hat{\alpha}, d\bar{\alpha}), \tag{5}$$

$$(\hat{v}, \bar{v}) \cdot (\hat{\alpha}, \bar{\alpha}) = (-\bar{v} \cdot \hat{\alpha}, \bar{v} \cdot \bar{\alpha} + \hat{v} \cdot \hat{\alpha}) \tag{6}$$

4   Mass

Mass is the first piece of thread that we'll try to pull.

Newtonian continuum mechanics is based on the notions of *body* and *mass*, traditionally introduced as separate mathematical objects.

Bodies are represented as abstract three-dimensional manifolds, of which we can consider submanifolds, i.e. sub-bodies. These bodies are immersed and move in space with time. This representation is typical in both Newtonian (Truesdell 1991 §§ I.2–3; Noll 1959; 1973) and Lorentzian or general (Grot et al. 1966a; Carter et al. 1972) relativity. This mathematical representation reflects our experience with 'bodies' we deal with in

everyday life, for example a tree lulled by the wind and its individual branches, or a leaf dragged by a water stream, or pieces of differently coloured silly putty getting mixed together.
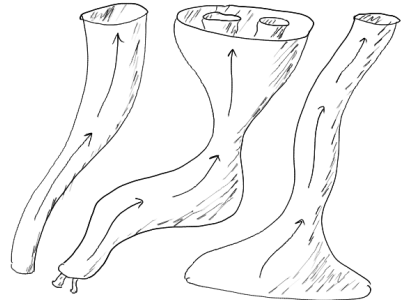
Mass is a property of a body, represented as a measure on the respective manifold, with its usual conservation property. In both Newtonian (Truesdell 1991 §§ I.4; Noll 1959; 1973) and relativistic (Grot et al. 1966a; Rezzolla et al. 2013 ch. 3) continuum mechanics mass is typically introduced intertwined with the metric (Carter et al. 1972 being an exception).

But it turns out that mass and body can be introduced at a single stroke (cf. Carter et al. 1972), as two aspects of the same entity, and without a metric. The idea is analogous to that of charge (Burke 1983; Bossavit 1998; Hehl et al. 2000; 2003 § B.1).

This beautiful representation is by a closed, outer-oriented 3-form M, the *matter 3-form*, satisfying

$$\mathrm{d}M = 0. \tag{7}$$

This object has a simple and beautiful geometrical meaning: it can be visualized in spacetime as a set of uninterrupted tubes possessing a definite orientation along them. The tubes are chosen by us: if we select a closed two-dimensional surface on a spacetime slice, where $M \neq 0$, the matter 3-form selects a unique tube with 3-dimensional section containing that 2-surface. This represents our ability to mark a small

Two-dimensional representation of tubes of a closed outer-oriented 3-form M

body of matter – say, by drawing a '◯' sign on an body, or throwing some leaves in flowing water – and then to follow it as it moves and deforms in space as time passes, with respect to some slicing and frame. A closed outer-oriented 3-form is the perfect mathematical representation of this.

Note that by 'matter' I mean something which possesses mass, and I'm thus excluding the electromagnetic field, for example, which is often considered as matter in general relativity.

If we were to include a theory of mixtures (e.g. Truesdell 1984 app. 5B), we would represent different substances by different matter 3-forms, which could also coexist in the same spacetime region.

The matter 3-form M provides a frame V wherever it doesn't vanish. It is called a Lagrangean or *material* frame (Smarr et al. 1978; 1980) and defined by

$$V \cdot M = 0, \qquad V \cdot dt = 1. \tag{8}$$

It represents the correspondence of places at different times defined by simply saying that each part of that body is at rest. Note that V is not the 4-velocity of matter; the latter is given by the condition $V \cdot dt_a = 1$ in Newtonian relativity and $|V \cdot \gamma \cdot V| = 1$ in general relativity. A matter 3-form cannot be used to define time lapse or distance, owing to its purely differential-geometric nature.

The matter 3-form of a body can also be used to define a volume element on each slice, by decreeing that each unit of mass of that body occupies a unit volume.

Once we have a slicing t and a frame U, we can project the matter 3-form:

$$p := -\hat{M} \equiv -U \cdot M, \qquad m := \overline{M} \equiv M - dt \wedge (U \cdot M). \tag{9}$$

We can speak of the amount of matter contained at time t within a 3-dimensional region on a slice: it's the integral of m (or M) over that region. We can speak of the rate at which matter is flowing through a stationary outer-oriented 2-dimensional surface on a slice: it's the integral of the outer-oriented 2-form p over that surface. We can also speak of the spacelike velocity of matter with respect to the frame U at time t: it's the vector v, tangent to the slice, satisfying

$$v \cdot dt = 0, \qquad v \cdot M = -U \cdot M \quad \text{or} \quad v \cdot m = p, \tag{10a}$$

or equivalently, in terms of the Eulerian frame (8),

$$v := V - U; \tag{10b}$$

i.e. it's the difference between the frame U and the Eulerian frame V. This is a velocity in the sense of our convention of slicing & frame, not in the metric sense. For the latter, we would have to replace the left eq. (8) with a normalization condition.

The outer-oriented 3-form m is a density when it's restricted to a slice. It corresponds to '$\rho d\tau$', where 'ρ' is mass density (mass/volume) and $d\tau$ a volume element. Thus its unit of measure is mass. The outer-oriented 2-form p is related – but doesn't correspond – to 'ρv', where v is the traditional velocity (length/time). This product is the traditional

momentum, which is identical with mass flux in continuum mechanics. But, as we'll see, we have to keep these two notions distinct. The 2-form p is mass-flux, but not momentum.

With the projections (9) and the expression for the differential (5), the mass balance (7) can be expressed as

$$\mathrm{L}_U m + \mathrm{d}p = 0, \tag{11}$$

since $\mathrm{d}m \equiv 0$ on a slice by construction. This simple, metric-free formula should be compared with the one involving the covariant derivative or several metric volume elements when matter flow is expressed by a 1-vector or a 1-form instead (e.g. Rezzolla et al. 2013 eq. (7.205), § 7.3, p. 361).

By specifying a slicing, a frame, and a matter 3-form, we are specifying a differential-geometric kinematics. We're only short of a time lapse and a distance for a kinematics in the traditional sense. The major aim of Newtonian relativity is to specify the causes that lead to a particular kinematic.

We have here pulled the following piece of thread: mathematically defining mass as a three-form, with its conservation (7) or (11). This seems to be most natural and convenient: metric-free and valid in both Newtonian and general relativity. The tangled bundle loosens here.

But alas, it gets more tangled elsewhere. The projection p of the matter 3-form on the frame, eq. (9)$_{\mathrm{left}}$, is a 2-form, and traditionally it would be considered as 'momentum'. As momentum it would be part of the energy-momentum-stress tensor T, which has order 2. This only works if we represent it as a *1*-form or a vector. Should we change our representation of the energy-momentum-stress tensor T then, to something of a different order? We'll see in §** that there are other reasons to do so indeed. But in Einstein's equations $G = 8\pi T$ this tensor is equated to the Einstein tensor G; thus we would need to find a different representation of the latter, too.

5   Force in Newtonian relativity

The second piece of thread we're going to pull is *force*. This is actually a smaller bundle in itself, and we'll examine it from several angles.

***why no need to speak of conservation

***dynamics comes from stating that the total force and total torque vanish

***affine structure needed, related to Newton's 3rd law. But if we assume only contact forces, it can be expressed locally without parallel transport. Exception: add spacelike surfaces from a 4D point of view

***that would suffice. But 1st law of thermodynamics, needed for thermomechanics, mixes force in energy equation as work.

***can there be force where there's no mass? And where there's no EM field?

***when the constitutive equation for a force 'visibly' depends on space or time variables, as is the case for elastic or inertial forces, the amount of work done by the force can be 'read' in a change of that variable. This change is usually called 'energy'. Typical examples are elastic force $-kx$ and elastic energy, the integral of $-kx\dot{x}$ being equal to $-\frac{1}{2}kx^2$ no matter how x depends on time; and inertial force $-m\ddot{x}$ and kinetic energy, the integral of $-m\ddot{x}\dot{x}$ being equal to $-\frac{1}{2}m\dot{x}^2$ no matter how x depends on time. For viscous forces like $-\eta\dot{x}$ we usually don't speak of 'viscous energy' probably because the integral of $-\eta\dot{x}^2$ does not have a closed form. This fact is obviously related to the difference between conservative and non-conservative forces and to the existence of potentials.

5.1 Constitutive equations

5.2 Geometric representation

In most presentations of the Newtonian mechanics of point masses, force is represented as a vector bound to a point mass. If we take the line integral of this vector along the trajectory of the point mass, we obtain the work made by the force on the mass. In continuum mechanics this picture becomes more complex. We deal with extended bodies, and the total force on a body or on part thereof is obtained by integrating two kinds of forces: *bulk forces*, also called body forces, which act on a mass or volume element and must be integrated over a three-dimensional region, and *surface forces*, also called contact forces, which act on an area element of the body's surface and must be integrated over a two-dimensional region.

Let's see what kind of geometric representation these operations suggest.

The fact that a force is to be integrated over a line to obtain work suggests that it's better viewed as a *covector-valued* object, ready to be

integrated over a line without the need of a scalar product (Burke 1995 § 12; Schouten 1951 § VII.2; van Dantzig 1954 § 2; Burke 1987 chs VI–VII; Bamberg et al. 1990 ch. 7). Let's adopt this representation for the moment, although we'll see later that a different one may be more appropriate.

If a bulk force is to be integrated over a volume, then it should be a covector-valued, outer-oriented 3-form. If we think of it as operating on a mass element (outer-oriented 3-form) instead, then it should be a covector-valued, outer-oriented 3-vector. Here we see the core of one of the questions of § 1: are we to interpret forces as 'acting' on matter? or as entities that can exist where there's no matter? General relativity suggests the latter interpretation. But we'll see that we can bypass the question of this representation in general relativity, because the notion of bulk force disappears there, leaving only that of surface force.

Surface forces, being integrated over a surface, should be covector-valued, outer-oriented 2-forms.

The integration of bulk or surface forces, which are covector-valued object, is possible in Newtonian mechanics thanks to the affine structure of its three-dimensional space. This kind of integration is impossible or at best ambiguous with the path-dependent parallel transport of general relativity; this impossibility is related to the absence of a 'conservation law' for 4-momentum (Pauli 1958 § 21; Eddington 1930 § 59; Landau et al. 1994 § 96; also Alderman et al. 1970).

**1-form also suggested by principle of virtual work

5.3 Frame indifference

5.4 Work and frames

5.5 Inertia as force and affine structure

*** only affine structure is necessary to transport and sum forces
    *** static formulation (cf. van Dantzig 1934b § 6, p. 652)

5.6 Bulk and surface forces

5.7 Four-dimensional generalizations

*** 4th component, spacelike 3-surface

5.8   Newtonian relativity

The main aim of continuum mechanics is to describe how bodies move and deform in spacetime in response to forces.

In the history of science we see two main kinds of scientific theory.

One kind of theory is based on the 'stuff' that is believed to be ultimate and fundamental at the time of conception, although usually proven not to be ultimate some decades later. Examples are vortex theories from the Baroque and contemporary high-energy particle theories.

The other kind of theory, humbler and more grandiose at the same time, tries to make allowance for any new phenomena that may be discovered.

Newtonian thermomechanics belongs to the second kind. Its aim is not to describe this or that particular particle or field, or this or that fundamental interaction. Its purpose is the description, within particular ranges and resolutions of space and time, of any kind of material and substance that humans use now or will discover in the future. For this purpose it uses some concepts devised, with their generality, to be able to accommodate as many new phenomena as possible. The main one is the concept of force.

Our exploration cannot only be based on mathematical manipulation of the Einstein equations, but also requires a clear view of the way continuum thermomechanics is used.

Newtonian continuum thermomechanics, which I'm here calling 'Newtonian relativity', has been refined under more than 300 years and has today a crystal-clear, yet far from finished, method and structure:

• Its purpose is the description, within particular ranges and resolutions of space, time, energy, of any kind of material or substance that Man uses now or will discover in the future. This must be contrasted with contemporary particle high-energy physics, whose goal is the description of *only* those forces and kind of matter believed to be 'ultimate'. The scope of Newtonian continuum thermomechanics is from this point of view wider than that of particle physics.

• For this description it uses five fundamental balance laws and a set of nine physical quantities:

– balances: mass, force, torque, energy, entropy;

– quantities: mass, displacement, temperature, stress, internal energy, heating flux, entropy, body force, body heating.

The quantities and balances above are postulated to apply to any kind of matter or substance. Mass, displacement, temperature are independent quantities. Body force and body heating must be given in any particular problem and express the effect of the surroundings The remaining are dependent quantities.

• Each particular kind of matter, with its different behaviours and responses, is characterized by a set of *constitutive equations*. These equations express the particular dependence of the dependent quantities on the independent ones, and together with the balances and initial and boundary conditions yield a well-posed system of integro-differential equations. Specifically, the constitutive equations give stress, internal energy, heating flux, and entropy as functionals of mass, displacement, temperature. The difference between al kinds of liquids, gases, solid, plasmas resides in the constitutive equations.

Sometimes internal energy is taken and independent, and temperature as dependent with an appropriate constitutive equation.

The separation of balance laws and constitutive equations is fundamental and reflects the aim of continuum mechanics to describe *any* kind of force and matter that can be discovered, not just those that our century believes to be the only existing ones. We can make a parallel with Hamiltonian mechanics: Hamilton's general equations correspons to the balance laws; particular Hamiltonians correspond to particular constitutive equations.

6　Newtonian-relativistic continuum mechanics: kinematics

7　Newtonian-relativistic continuum mechanics: dynamics

7.1　Dynamics with absolute time

Traditional presentations of Newtonian-relativistic dynamics and of Newton-Cartan theory starts with the notion of *absolute time t_a*, which is a unique slicing of spacetime. We'll presently see that this notion, together with a geometry and a connection on the slices, is mathematically necessary when the notion of force is represented by objects living on 3-surfaces. But later we'll consider the possibility of avoiding absolute time. In fact, the notion of simultaneity in Newtonian-relativistic mechanics is inspired by what we simultaneously *see*, together with the assumption that the velocity of light is practically infinite. But we could also define

simultaneity in terms of what we simultaneously *hear*, for example. How could Newtonian mechanics be formulated in that case? We'll give a possible answer later.

A force on a body at time t_a in Newtonian mechanics has two important characteristics: (1) it is meant to be integrated along the trajectory of every small region of the body, to yield the work done on that region; (2) it is meant to be obtained from the sum of infinitesimal body forces acting over each part of the region, or surface forces over its boundary.

Owing to the first characteristic, many geometers (van Dantzig 1954; Burke 1980; 1987; 1995; Bossavit 1991; 1998; Hehl et al. 1999; 2003) have advocated the representation of force as a covector. Their arguments are valid and geometrically beautiful, and therefore we assume for the moment that forces are covector-valued objects.

Note, however, that the covector-valued conception of force is geometrically unsatisfying, because velocity is a frame-dependent object. Later on I will use a frame-indifferent notion more primitive than velocity.

The second characteristic says that a body force must be a covector-valued outer-oriented 3-form, while a surface force a covector-valued outer-oriented 2-form.

The total force on a body is the integral of the body forces over its volume and of the surface forces over its boundary. Since forces are vector-valued objects, this integration is only possible if the slices have an *affine* structure, that is, a parallel transport with vanishing curvature. From this point of view, the affine structure is demanded by the dynamics, rather than by the kinematics.

Some forces between bodies, like the gravitational force or a purely elastic stress, are moreover defined to depend on a metric, which must therefore be introduced besides the affine structure. It would be interesting to explore whether metric and affine structure could be introduced in terms of these forces instead.

So in the present section let's assume that we have a unique slicing $t \equiv t_a$, representing absolute time, and whose slices have a Euclidean structure: metric $\boldsymbol{g}_t$ and flat parallel transport ∇_t; the index t will be omitted, but it is important to note that there is no specific relation between metrics and parallel transport on different slices. We are not assuming the existence of a Newton-Cartan connection. A frame $\boldsymbol{U}$ that satisfies $L_{\boldsymbol{U}}\boldsymbol{g} = 0$ is called a *rigid frame*.

Indicating the sum of all body forces acting on a body by f, and that of the surface forces by τ, the *balance of forces* states that

$$\int_V f + \oint_{\partial V} \tau = 0 \qquad \forall V \tag{12}$$

for each outer-oriented closed 3-volume V within a slice. Note that the result of the integration is a 1-form, not a scalar. The local version of the balance is

$$f + \nabla \tau = 0, \tag{13}$$

where ∇ is the *exterior covariant derivative* (Choquet-Bruhat et al. 1996 § Vbis.A.4; Misner et al. 2003 § 14.5; Frankel 2012 § 9.3; see also Segev 2000a,b; Segev et al. 2000; Segev 2002; Segev et al. 2012; Kanso et al. 2007).

The fact that we have to use the *exterior* covariant derivative, rather than the covariant derivative, may have important consequences later.

7.2 Interlude: forces and motion – is there a balance of momentum?

How does the balance of forces relate to motion?

The vast majority of literature and textbook do not speak of balance of forces; they follow a different path that sounds more or less as follows:

There is a law of 'conservation of momentum', which states that the rate of change of momentum of a body region is proportional to the forces exerted on it:

$$\text{``}\, d_t \int_V mv = \int_V f + \oint_{\partial V} \tau \,\text{''}, \tag{14a}$$

or

$$\text{``}\, \partial_t(mv) + \text{div}(mv \otimes v) = \text{div}\,\tau + f \,\text{''}. \tag{14b}$$

This equation directly connects forces and motion, and looks like a conservation equation with fluxes and sources. This conservation law is only valid in an inertial frame. We can make it valid in every rigid frame provided we add some 'fictitious forces' looking like

$$\text{``}\, m\left[-\partial_t^2 x^\star + 2\Omega^\star \cdot \partial_t(x - x^\star) + (\partial_t \Omega^\star - \Omega^{\star 2}) \cdot (x - x^\star)\right] \,\text{''}, \tag{15}$$

in that non-inertial rigid frame.

The presentation above is geometrically unsatisfying, however.

The right-hand side of eq. (14) is frame-indifferent; that is, the definition and measurement of τ and f do not depend on a choice of frame, inertial or otherwise. For example, the force exerted on a pointlike object attached to a Hookean spring with constant k and elongation l has intensity kl and direction along the elongation. This is a complete description of the force, and the notion of frame is completely irrelevant to it. Similarly for the gravitational force: it is given by $m \otimes \gamma$, where γ is a frame-indifferent covector field. This field is in turn determined by the mass distribution via $\mathrm{d}\gamma = -4\pi Gm$, again a frame-indifferent definition.

The momentum density mv on left-hand side of eq. (14) and the 'fictitious forces' (15) require the specification of a frame instead.

Therefore eq. (14), with or without fictitious forces, mixes up frame-indifferent and frame-dependent quantities. This doesn't make sense geometrically.

We find a geometrically more satisfying formulation by using some ideas that can possibly be found in a work by Jacob Bernoulli (1703) (see Truesdell 1968 p. 104) (Noll 1963; Truesdell 1991 § I.13). Consider a small enough body region, so that deformation and torques can be neglected. In a problem of *statics* in an inertial frame, the balance of forces (12) holds. Now consider a fact and two assumptions:

- a small body region can always be considered at rest in a suitably chosen frame;
- the laws of statics should hold in any frame in which the body region is at rest;
- an inertial frame is a rigid frame at rest on average with respect to the total distribution of matter in the universe, which is mostly associated with the fixed stars.

The consequence of these statements is that *the balance of forces* (12) *is valid in any frame; among these forces we must account for one related to the relative motion of the body region with respect to the total distribution of matter in the universe, approximately identified with the 'fixed stars'*.

This force is the inertial force. Let's see how it is expressed in an arbitrary rigid frame U; remember that what we call 'time derivative' is the Lie derivative $\mathrm{L}_U =: \partial_t$. In this frame let the position of the body region be $x(t)$, and let the total distribution of matter in the universe have centre of mass at $x^\star(t)$ and average instantaneous rotation given by the 2-form $\Omega^\star(t)$; in § A I explain how they are determined by the mass distribution through a sort of Mach's principle. Then the inertial force is

the covector-valued 3-form

$$f^\star := -m \otimes \left(\Omega^\star \cdot g^{-1} \cdot \Omega^\star - \partial_t \Omega^\star - 2\Omega^\star \partial_t + g{\partial_t}^2\right) \cdot (x - x^\star). \quad (16)$$

The definition of this force requires a metric on each slice. The definition of its source, see § A, also requires a parallel transport on each slice. But it does not require a 4-dimensional connection on the whole spacetime. If we choose a rigid frame in which the total matter of the universe does not rotate on average, $\Omega^\star = 0$, and moves with uniform velocity, $\partial_t x^\star = \text{const}$, then we recover the familiar expression $f^\star = -mg \cdot {\partial_t}^2 x$ – the g is necessary to transform a vector into a covector. This kind of rigid frame is called inertial. The use of a *rigid* frame is not necessary; a frame-independent expression of the inertial force is given below.

Including the inertial force in the sum of all other body forces f, the balance of forces (12) includes the familiar Newtonian equations of motion. It is the inertial force that in most physical situations links kinematics and dynamics, specifically by linking forces upon a body to its motion with respect to the rest of matter in the universe. But there are also situations in which inertia is inessential. For example, consider a small body of negligible mass attached to a Hookean spring inside a fluid, the other end of the spring being at rest with respect to the fluid. The forces acting on the body are the Hookean and the viscous ones. In a suitable frame of reference the balance of forces is $-kx - \eta\partial_t x = 0$, leading to a specific motion $x(t) = e^{-kt/\eta}x(0)$ with respect to the fluid. In this example the inertial force plays no role and it is not important whether the frame is inertial or not.

As regards 'conservation of momentum', we see that the time-derivative of momentum in Newtonian mechanics is not a quantity with a general geometric meaning: it is the simplified form assumed by a particular force in a particular frame. From this point of view conservation of momentum is not a general geometric notion in Newtonian-relativistic mechanics.

The inertial force on the mass M can be written in a frame-free way as $-g^{-1} \cdot \mathrm{L}_{U^\star}(U^\star \cdot M)$, where $U^\star$ is a vector field transverse to the slices, defined in § A. Some forces, like inertial, viscous, and electromagnetic ones, require objects transverse to the slicing for their definition, whereas others, like elastic or gravitational ones, don't. This suggests that the presentation in terms of an absolute-time slicing can be geometrically improved upon.

7.3 Balance of energy

To each matter element on each slice is also associated an internal energy density ϵm, represented by an outer-oriented 3-form proportional to the mass 3-form m. The balance of energy states that the change in the total energy in a 3-volume V on a slice must come from the energy transported thither or thence, from the work done by the forces, and from body and surface heating, represented by outer-oriented 3-form and 2-form:

$$\partial_t(\epsilon m) + \mathrm{d}(v \cdot \epsilon m) = v \cdot f + \mathrm{d}(v \cdot \tau) + h + \mathrm{d}q, \tag{17}$$

with $\partial_t(\epsilon m) \equiv \mathrm{L}_u(\epsilon m)$. The rate of change of kinetic energy is simply the work done by the inertial force, and is therefore included in the term $v \cdot f$. The traditional expression with '$\partial_t + v \cdot \nabla$' appears if we write the 3-form ϵm as a multiple of the volume element $\sqrt{g}$ – one more example of a unnecessary use of metric.

　　If we assume that body forces and body heating vanish, then we can rewrite this equation as

$$\partial_t(\epsilon m) + \mathrm{d}(v \cdot \epsilon m - v \cdot \tau - q) = 0, \tag{18}$$

which suggests that energy can also be considered like an outer-oriented closed 3-form, with a flux 2-form

$$j \equiv -v \cdot \epsilon m + v \cdot \tau + q. \tag{19}$$

7.4 Constitutive equations

✠ This section is disconnected from the rest. To be rewritten

　　The balances of mass, force, energy give a system of 1+3+1 real-valued equations. They involve at least 4+3+6+5 real-valued components: mass density and flux, body force and stress, energy density and flux, and body supply of energy. The independent variables in these equations are usually assumed to be the four components of the mass 3-form – that is, mass density and mass flux or velocity – and the temperature. Sometimes the energy density is exchanged with the temperature in the role of independent variable.

　　The balance laws alone are therefore not sufficient to determine the evolution of all fields. Moreover, although the balances are assumed to hold for all forms of matter, different kinds of matter behave in different ways.

Constitutive equations are equations that gives the dependent fields as functions or functionals of the independent ones. They express the different behaviours of different kinds of matter, and together with the balance laws yield a closed system of differential or integro-differential equations.

The most important constitutive equation in our discussion is that for the 4-stress. All 4-components of the stress τ are in the simplest cases functionals of the mass 3-form M, the internal energy ϵ or temperature θ, and the 3-metrics $\boldsymbol{g}$:

$$\boldsymbol{T} = \boldsymbol{T}(M, \theta, \boldsymbol{g}), \tag{20a}$$

or, in a given slicing and frame,

$$\boldsymbol{T} = \begin{pmatrix} \epsilon(m, \boldsymbol{p}, \theta, \boldsymbol{g}) & \boldsymbol{q}(m, \boldsymbol{p}, \theta, \boldsymbol{g}) \\ \boldsymbol{j}(m, \boldsymbol{p}, \theta, \boldsymbol{g}) & \boldsymbol{\tau}(m, \boldsymbol{p}, \theta, \boldsymbol{g}) \end{pmatrix}. \tag{20b}$$

The dependence on the metrics can be complicated: for example, the stress at a point can depend on the difference between the 3-metric there and the metric from previous 3-surfaces, transported by the velocity $\boldsymbol{u}$ associated with the mass flux(Grot et al. 1966a; Carter et al. 1972). The difference can be non-vanishing even if the 3-metric is Euclidean on all 3-surfaces. The only difference between Newtonian and general relativistic mechanics is that all 3-metrics are flat in the former but not in the latter.

7.5   Dynamics without absolute time

The formulation of dynamics of § 7.1 has the advantage of being frame-indifferent: inertial frames don't have a special status, and inertia is just one force among many others. But that formulation still has several interrelated unpleasant features:

- a special absolute-time slicing is used,
- forces, energy, energy fluxes are only defined on the slices of that slicing,
- forces are meant to act on velocities, which are frame-dependent,
- a flat parallel transport is needed to integrate the forces.

A first step in lifting forces from their plain existence on slices to a fourfilling existence in spacetime, and in unchaining them from frame-dependent velocities, is to assume that they act directly on the four-dimensional mass flux expressed by the matter 3-form. This requires that

they have *outer-oriented 3-vector* values. It also means that their physical dimension acquires a $[\text{mass}^{-1}]$ factor.

A 3-vector has four components. The additional component of a force is interpreted, both in Newtonian (Truesdell et al. 1960 §§ 152–154, 288–289; Grot et al. 1966a § 2.3) and Lorentzian (Eckart 1940; Grot et al. 1966a § 2.3; Maugin 1978) relativity, as the combination of power and heating. Thus we arrive at the notion of *heating-force* tensor, which corresponds to the 'four-force' of Lorentzian relativity.

In this new point of view a surface force – 4-stress – is a outer-oriented-3-vector-valued outer-oriented 3-form. Contracted with a matter 3-form it yields another outer-oriented 3-form, which could be interpreted as the spacetime flux of some quantity. From general relativity we foresee that this quantity is the energy flux.

✠ To be added: formulation with 2-point energy-momentum-stress tensor (Truesdell et al. 1960 § 288)

7.6   Decoupling force from body

In Newtonian relativity, force is something experienced by a body, rather than something in spacetime. But there are signs that forces need to be decoupled also in Newtonian relativity.

7.7   An interesting development?

This leads to some interesting developments. Interpret a 4-stress $\boldsymbol{T}$ as an object which contracted with a matter 3-form $\boldsymbol{M}$ yields its corresponding energy-flux outer-oriented 3-form $\boldsymbol{E}$:

$$\boldsymbol{E} = \boldsymbol{T} \cdot \boldsymbol{M}. \tag{21}$$

This is similar to the '$^*\boldsymbol{T}$' of Misner et al. (Misner et al. 2003 ch. 15), with the difference that $\boldsymbol{T}$ is 3-covector valued.

Now postulate that energy flux must always be conserved,

$$\mathrm{d}\boldsymbol{E} = 0. \tag{22}$$

We obtain a set of two balances and a relation between them:

$$\mathrm{d}\boldsymbol{M} = 0, \qquad \mathrm{d}\boldsymbol{E} = 0, \qquad \boldsymbol{E} = \boldsymbol{T} \cdot \boldsymbol{M}. \tag{23}$$

Question: does this system *imply* Einstein's equations? The point is that we can now introduce a four-dimensional exterior covariant derivative

∇ and combine the equations above as follows:

$$d\boldsymbol{E} = d(\boldsymbol{T} \cdot \boldsymbol{M}) = \nabla\boldsymbol{T} \cdot \boldsymbol{M} - \boldsymbol{T} \cdot d\boldsymbol{M} = \nabla\boldsymbol{T} \cdot \boldsymbol{M} = 0 \qquad (24)$$

and *asking this to hold for every $\boldsymbol{M}$*, we have

$$\nabla\boldsymbol{T} = 0, \qquad (25)$$

which corresponds to Misner et al.'s '$\boldsymbol{d^*T} = 0$' (Misner et al. 2003 ch. 15).

This point of view seems to say that *any exterior covariant derivative ∇ will do* – possibly with additional restrictions coming from restrictions on positivity of mass and energy fluxes. Such a possibility is not so strange when we consider that Einstein's equations can be rewritten, in a completely equivalent way, in terms of a flat connection with non-vanishing torsion, or a mixture of the two (de Andrade et al. 2000; Arcos et al. 2004; Aldrovandi et al. 2013; Pereira 2014; Cai et al. 2016). It seems that what's fundamental here is the use of a derivation operator with the properties of an *exterior* covariant derivative, rather than the use of a covariant derivative or parallel transport per se.

Finally, Einstein's equations $\boldsymbol{G} = \boldsymbol{T}$ could simply be interpreted as *the most general integral of the equation $\nabla\boldsymbol{T} = 0$*. This possibility is related to an interesting remark by Truesdell:

> When the field theories were discovered in the eighteenth century, solutions in arbitrary functions such as those presented in this subchapter were sought earnestly, but, for the most part, sought in vain. In the nineteenth century, researches on partial differential equations turned away from such general solutions so as to concentrate upon boundary-value problems. When, in the twentieth century, the general solutions were at last obtained, scarce attention was paid to them, and to this day they remain virtually unknown. Though so far they have been used but rarely, they might turn out to be illuminating in studies of underdetermined systems, where the conventional viewpoint of partial differential equations has gained little. (Truesdell et al. 1960 p. 594)

✠ To be added: refs to Finzi, Beltrami

✠ To be added: discussion about pp. 18–19 of (Weatherall 2017): use differential forms for Gauss-Green theorem, instead of covariant derivative

8   General relativity

8.1   Aims

General relativity gives us the equation $\boldsymbol{G} = \kappa\boldsymbol{T}$; for some time after its proposal it was not clear which were independent, dependent, and boundary-condition quantities; a distinction that is necessary to set up a well-posed boundary-value problem. Misner et al. (2003 ch. 21) discuss this question at length.

The Einstein equations are clearly suited for a continuum description of matter, not a particulate one: because they involve the notion of stress, which is an exquisitely continuum notion. And since stress is the staple of constitutive equations in Newtonian mechanics, and does not commit to specific forms of matter or forces, then the Einstein equations do not commit to specific forms of matter or forces either, unlike particle-physics theories. From this point of view the Einstein equations appear as a harmonization of a part of the Newtonian balance laws and of geometry. They – especially the stress appearing in them – still need constitutive equations.

The Einstein equations don't include the balances of mass and entropy (Eckart 1940; Misner et al. 2003 ch. 22). This fact is very clear in numerical relativity (Disconzi 2014 p. 1918; Wilson et al. 2007 § 2.2; Gourgoulhon 2012 § 6.3.2; Baumgarte et al. 2010 ch. 5; Rezzolla et al. 2013 § 2.3).

8.2   3 + 1 formulation

Spacetime can be sliced and a frame introduced just like in the Newtonian case of § 3.

The Einstein equations can be decomposed by projection along the slices t and the frame $\boldsymbol{V}$, giving rise to a part parallel to the surfaces with 3×3 real components, a part parallel to the vector field with 1 component, and a mixed part with 3 components. Our freedom in choosing the slicing t and the frame $\boldsymbol{V}$ appears in the decomposed equations as the presence of and undetermined 'lapse' function and an undetermined 'shift' vector field. When a metric is present we can choose the frame $\boldsymbol{V}$ to be the unit normal to the 3-surfaces. This choice corresponds to a constant unit lapse function and a vanishing shift vector (Smarr et al. 1978; 1980). Other choices of lapse and shift are more convenient for the initial-value problem, for example because they can correspond to a slicing that covers most of spacetime while avoiding singularities (Smarr et al. 1978). But our present

discussion is mainly conceptual; we therefore use the simplest lapse and shift.

The decomposed Einstein equations can be found compactly written in a number of texts (Wilson et al. 2007 § 1.3; Gourgoulhon 2007 § 4.3.2; Alcubierre 2008 ch. 2; Rezzolla et al. 2013 § 7.2.2). The decomposition of the Einstein tensor $\boldsymbol{G}$ yields the 3-metric $\boldsymbol{g} = (g_{ab})$ of the 3-surfaces, having inverse $\boldsymbol{g}^{-1} = (g^{ab})$, with its compatible connection ∇ having Ricci curvature $\boldsymbol{R} = (R_{ab})$, and the extrinsic curvature $\boldsymbol{K} = (K_a{}^b)$ of the 3-surfaces. The decomposition of the energy-stress tensor $\boldsymbol{T}$ yields the internal energy ϵ, energy flux $\boldsymbol{j}$, and spatial stress $\boldsymbol{\tau} = (\tau_a{}^b)$:

$$\boldsymbol{T} = \epsilon \, \mathrm{d}t \otimes V + \boldsymbol{j} \otimes V + \mathrm{d}t \otimes (\boldsymbol{j} \cdot \boldsymbol{g}^{-1}) + \boldsymbol{\tau}. \tag{26}$$

The Einstein equations $\boldsymbol{G} = 8\pi\boldsymbol{T}$ are decomposed into the evolution equations

$$\partial_t \boldsymbol{g} = -2\boldsymbol{K} \cdot \boldsymbol{g}, \tag{27a}$$

$$\partial_t \boldsymbol{K} = \boldsymbol{K} \operatorname{tr} \boldsymbol{K} + \boldsymbol{R} \cdot \boldsymbol{g}^{-1} + 4\pi (\operatorname{tr} \boldsymbol{\tau} - \epsilon) - 8\pi\boldsymbol{\tau}, \tag{27b}$$

and the constraint equations

$$16\pi\epsilon = (\operatorname{tr} \boldsymbol{K})^2 - \boldsymbol{K} : \boldsymbol{K} + \boldsymbol{R} : \boldsymbol{g}^{-1}, \tag{28a}$$

$$8\pi\boldsymbol{j} = \nabla \cdot (\boldsymbol{K}^\top - \mathsf{I} \operatorname{tr} \boldsymbol{K}). \tag{28b}$$

Note that

$$\boldsymbol{K} \cdot \boldsymbol{g} = -\frac{1}{2}\partial_t \boldsymbol{g} \equiv -\frac{1}{2}\mathrm{L}_V \boldsymbol{g} \equiv -\frac{1}{\sqrt{\boldsymbol{g}}}\mathrm{L}_V \sqrt{\boldsymbol{g}} \equiv -\nabla \cdot *** \tag{29}$$

It is interesting to note that '3-momentum' in relativity appears as the divergence of a 3-tensor – just like the force from a stress.

To these equations we must add the mass balance, equivalent to eq. (11):

$$\partial_t m = -\mathrm{d}\boldsymbol{p}. \tag{30}$$

In these expressions $\partial_t \equiv \mathrm{L}_V$.

This gives us a total of $6 + 6 + 1 + 3 + 1$ real-component equations for a total of $6 + 6 + 6 + 1 + 3 + 1 + 3 + 1$ real components of the fields $\boldsymbol{g}$, $\boldsymbol{K}$, $\boldsymbol{\tau}$, ϵ, $\boldsymbol{j}$, m, $\boldsymbol{p}$, and temperature θ. Then we need $6 + 1 + 3$ equations, which turn out to be, as in the Newtonian case, the constitutive equations (46)

for the energy-stress:

$$\epsilon = \epsilon(m, \boldsymbol{p}, \theta, \boldsymbol{g}, \boldsymbol{K}), \tag{31a}$$

$$j = j(m, \boldsymbol{p}, \theta, \boldsymbol{g}, \boldsymbol{K}), \tag{31b}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}(m, \boldsymbol{p}, \theta, \boldsymbol{g}, \boldsymbol{K}). \tag{31c}$$

8.3   Parentetical remark: divergence of energy-stress

A strange point appears in most general-relativity texts I've seen, especially for numerical relativity. They remark that the balance equations $\bar{\nabla} \cdot \boldsymbol{T} = 0$ are a consequence of the Einstein equations $\boldsymbol{G} = 8\pi\boldsymbol{T}$, and therefore solving the latter automatically yields the former. But to find the numerical solutions they actually use the former!

The balance $\bar{\nabla} \cdot \boldsymbol{T} = 0$ is indeed contained in eqs (27)–(28). For example, combining the time derivative of the constraint equation (28b) and the 3-divergence of the evolution equation (27b) we obtain the spatial part of $\bar{\nabla} \cdot \boldsymbol{T} = 0$: the balance of momentum. A quick check of such combination of (28b) and (27b) shows in fact the appearance of the time-derivative of j and the divergence of the 3-stress. A similar combination yields the balance of energy:

$$\partial_t \epsilon = \epsilon \operatorname{tr} \boldsymbol{K} - \boldsymbol{g}^{-1} : \nabla j + \boldsymbol{K} : \boldsymbol{\tau} \tag{32a}$$

$$\partial_t j = j \operatorname{tr} \boldsymbol{K} - \nabla \cdot \boldsymbol{\tau}^{\mathsf{T}}, \tag{32b}$$

The reason for using the divergence equation $\bar{\nabla} \cdot \boldsymbol{T} = 0$ is clearly explained by Frittelli (1997) (the only work I've found that explains this). The constraint equations (28) should be enforced on *every* 3-surface, i.e. for all times t, while the metric and extrinsic curvature are evolved by eqs (27). If we did so there would be no need to use $\bar{\nabla} \cdot \boldsymbol{T} = 0$ in the solution.

But it would be more convenient if the constraints (28) could be enforced on one 3-surface only, for example the initial one, and be automatically satisfied on all other 3-surfaces, i.e. at all subsequent times. It turns out that the divergence equation $\bar{\nabla} \cdot \boldsymbol{T} = 0$ does exactly this: it makes the constraints automatically satisfied at all times if they are satisfied at the initial time.

Loosely speaking, the system

$$\begin{cases} \text{evolution equations (27)} \\ \text{constraint equations (28) for all } t \end{cases}$$

is equivalent to the system

$$\begin{cases} \text{evolution equations (27)} \\ \text{divergence equation } \bar{\nabla} \cdot \boldsymbol{T} = 0 \\ \text{constraint equations (28) for initial } t \text{ only} \end{cases}$$

and the latter is more convenient to solve.

8.4 How to solve the decomposed equations

Let us consider how equations (27)–(31) could be solved.

Imagining a rough numerical timestepping scheme, in which for example eq. (27a) becomes

$$\boldsymbol{g}(t + \triangle t) \approx \boldsymbol{g}(t) - 2\boldsymbol{K}(t) \cdot \boldsymbol{g}(t)\,\triangle t, \tag{33}$$

it seems that the equations (27)–(31) can be solved this way:

1. Choose all fields such as to satisfy the constraints (28) and constitutive equations (31) on an initial 3-surface at time t.

2. Calculate metric $\boldsymbol{g}$, extrinsic curvature $\boldsymbol{K}$, mass density m at time $t + \triangle t$ from the evolution eqs (27) and balance (30):

$$\partial_t \boldsymbol{g} = -2\boldsymbol{K} \cdot \boldsymbol{g},$$
$$\partial_t \boldsymbol{K} = \boldsymbol{K}\operatorname{tr}\boldsymbol{K} + \boldsymbol{R} \cdot \boldsymbol{g}^{-1} + 4\pi(\operatorname{tr}\boldsymbol{\tau} - \epsilon) - 8\pi\boldsymbol{\tau},$$
$$\partial_t m = -\mathrm{d}\boldsymbol{p}.$$

Note that these evolution equations require, in particular, the mass flux $\boldsymbol{p}$, energy ϵ, stress $\boldsymbol{\tau}$ at time t.

3. Calculate the mass flux $\boldsymbol{p}$, temperature θ, energy ϵ, energy flux $\boldsymbol{j}$, stress $\boldsymbol{\tau}$ at time $t + \triangle t$ from the system of constraints (28) and constitutive eqs (31), given the newly found $\boldsymbol{g}, \boldsymbol{K}, m$:

$$16\pi\epsilon = (\operatorname{tr}\boldsymbol{K})^2 - \boldsymbol{K} : \boldsymbol{K} + \boldsymbol{R} : \boldsymbol{g}^{-1}, \qquad 8\pi\boldsymbol{j} = \nabla \cdot (\boldsymbol{K}^{\mathsf{T}} - \boldsymbol{I}\operatorname{tr}\boldsymbol{K}),$$
$$\epsilon = \epsilon(m, \boldsymbol{p}, \theta, \boldsymbol{g}, \boldsymbol{K}), \qquad\qquad \boldsymbol{j} = \boldsymbol{j}(m, \boldsymbol{p}, \theta, \boldsymbol{g}, \boldsymbol{K}),$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}(m, \boldsymbol{p}, \theta, \boldsymbol{g}, \boldsymbol{K}).$$

Note that this system *implicitly* yields, in particular, the mass flux $\boldsymbol{p}$, energy ϵ, stress $\boldsymbol{\tau}$ needed in the next timestep; see step above.

Now we have all fields at time $t + \triangle t$.

4. Set $t + \triangle t \to t$, go to step 2.

It's important to remark that the motion of matter, expressed by the mass flux $\boldsymbol{p}$ in this particular reference frame and slicing, is determined in an *implicit* way in the scheme above. It would be worth investigating how geodesic equations (Geroch et al. 1975; Weatherall 2011) peep out from the present scheme.

This scheme can be generalized to more complex temporal dependences in the constitutive equations, like dependence on derivatives, memory, etc..

8.5   A possible reduction and reinterpretation

An interesting aspect of the equations above is that energy density and energy flux seem to have an intermediate role. We could think of solving the constraint and constitutive equations of step 3 for mass density m and mass flux $\boldsymbol{p}$ in terms of the 4-metric $\boldsymbol{g}$, $\boldsymbol{K}$ and the temperature θ; this dependence could involve integro-differential operators. The mass-balance $\partial_t m = -\mathrm{d}\boldsymbol{p}$ would then become an evolution equation for the temperature. The constitutive equation for the stress would also be reduced.

We would be left with the system

$$\boldsymbol{K} := -\tfrac{1}{2}\partial_t \boldsymbol{g} \cdot \boldsymbol{g}^{-1}, \tag{34a}$$

$$\partial_t \boldsymbol{K} = \boldsymbol{K}\operatorname{tr}\boldsymbol{K} - \tfrac{1}{4}(\operatorname{tr}\boldsymbol{K})^2 + \tfrac{1}{4}\boldsymbol{K}:\boldsymbol{K} + \tfrac{3}{4}\boldsymbol{R}\cdot\boldsymbol{g}^{-1} + 4\pi\operatorname{tr}\boldsymbol{\tau} - 8\pi\boldsymbol{\tau}, \tag{34b}$$

$$\partial_t m(\boldsymbol{g}, \boldsymbol{K}, \theta) = -\mathrm{d}\boldsymbol{p}(\boldsymbol{g}, \boldsymbol{K}, \theta), \tag{34c}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{g}, \boldsymbol{K}, \theta), \tag{34d}$$

a set of first-order differential equations for the 3-metric $\boldsymbol{g}$, extrinsic curvature $\boldsymbol{K}$, and the temperature θ; or second-order for the 3-metric $\boldsymbol{g}$ and the temperature θ. This system seems to say that

- mass density, mass flux, energy density, and energy flux are temperature-dependent manifestations of the 4-metric;

- the 4-metric can interact with itself in different ways, expressed by particular temperature-dependent constitutive relations.

Would this satisfy Einstein?

Appendices

A Addendum: inertial force from mass

⚓ Refer to Weatherall (2017).

Inertial force is presented here as the force exerted from the total matter in the universe. It is proportional to the change, per unit time, in the momentum seen from a frame representing a sort of average position of the total matter: the frame of a rigid body with the same mass centre and second moment, expressed by the Euler tensor.

Suppose that the 3-form $m^\star$ representing the total matter density in the universe is well-defined. Now consider the slice E_t at time t with its affine structure. The mass centre of the matter distribution at each time t is

$$P^\star(t) := \int_{E_t} P \frac{m^\star(P)}{\int_{E_t} m^\star}; \tag{35}$$

this integration is a convex combination and makes sense with the affine structure.

A vector field can be defined at each place P by $P \mapsto P - P^\star(t)$ thanks to the affine structure. The Euler tensor (Truesdell 1991 § I.10) of the mass distribution with respect to its mass centre is

$$e(t) := \int_{E_t} \big(P - P^\star(t)\big) \otimes \big(P - P^\star(t)\big) \otimes m^\star(P). \tag{36}$$

This tensor-valued integration is possible thanks to the affine structure of the space slice.

The Euler tensor is symmetric and, if the total mass distribution is enough well-behaved, it is also positive. It has three principal axes along vectors $a_i(t)$ with respect to the metric g, defined by the generalized eigenvalue equation

$$[e(t) - \lambda_i(t)g] \cdot a_i(t) = 0. \tag{37}$$

We define the 'frame of the fixed stars' $U^\star$ as that frame that maps the principal axes on all slices into themselves, and in which the mass centre is stationary. These conditions are expressed by

$$\mathrm{L}_{U^\star}[e(t) - \lambda_i(t)g] = 0, \qquad U^\star|_{P^\star(t)} = \dot{P}^\star. \tag{38}$$

The inertial force field upon a mass density M is then defined as the opposite of the change in momentum as seen from the fixed-star frame:

$$-g^{-1} \cdot \mathrm{L}_{U^\star}(M \cdot U^\star) \equiv -g^{-1} \cdot (\mathrm{L}_{U^\star}M) \cdot U^\star. \tag{39}$$

B   Bits and pieces

We have seen that a velocity v can be assigned to a mass 3-form M by eq. (8), if a slicing t and a frame U are provided. Can we relate power to force and to the mass-flux 2-form p directly? thus avoiding the use of a slicing? In this case force has to be contracted with an outer-oriented 3-covector, so it must be an object with *outer-oriented 3-vector* values. If a volume element is given, there is a unique correspondence between covectors and outer-oriented 3-vectors. In mechanics there's always the metric volume element $\sqrt{g}$ sneaking around, so it's quite possible that the covector-valued conception of force is tacitly using a metric volume element.

   **must be 3-vector-valued to act directly on mass 3-form rather than a vector-velocity derived from it. Stress must be integrated over 3-regions.

   **Once the 4-stress acts on the mass 3-form, it yields an energy 3-form.

   **If we say that the total energy flux must vanish, independently on the mass 3-form? then we can say that the total stress must vanish? and interpret Einstein's equations as $G - T = 0$? Then G would be the stress associated with the gravito-inertial force? Compare (Misner et al. 2003 ch. 15), with the difference that we need to consider T (their '$^*T$') as 3-vector-valued, rather than simply 1-vector-valued. Then

$$0 = \mathrm{d}(M \cdot T)$$
$$= (\mathrm{d}M) \cdot T + M \cdot \nabla T \qquad (40)$$
$$= 0 + M \cdot \nabla T$$

and asking this to hold for every M, we have $\nabla T = 0$ (their '$d^*T = 0$').

   In Newtonian mechanics we can choose frames that preserve the Euclidean metric of the slices: $\mathrm{L}_U g = 0$, called rigid frames. In general relativity it is generally impossible to choose a vector field this way, but we can choose one normal to the slices with respect to a spacetime 4-metric; this is called a *Eulerian* frame (Smarr et al. 1978; 1980). An Eulerian frame can also be chosen in such a way that $\mathrm{L}_U \sqrt{g} = 0$, a condition called 'maximal slicing' (Smarr et al. 1978 § III.B)[also Lichnerowicz (1944)].

   The integral over a 3-surface is related to the number of tubes that intersect it.

   We can take the tubes associated with M as small as we please: in the limit they represent the trajectories in spacetime of 'particles' of the continuum.

C   Einstein: matter as geometry

One of Einstein's goals was to represent matter as a manifestation of the gravitational field (see Havas et al. 1962; Havas 1967). He tried to achieve this by representing matter particles as singularities of the field (Einstein et al. 1938), but this programme wasn't very successful.

The present note explores the idea of realizing Einstein's wish in a slightly different fashion: by interpreting mass, energy, and their fluxes as manifestations of the gravitational field. These manifestations depend on the temperature. Moreover, the gravitational field can interact with itself in a variety of ways, also dependent on the temperature; these ways are expressed in the stress. This idea appears when we try to combine Newtonian continuum thermomechanics and general relativity.

To explore this idea I first give in § D a summary of the aim of Newtonian continuum thermomechanics and of its basic equations, stressing the importance of constitutive equations. It is necessary to discuss the 'aim' because, independently of the different invariance groups and geometrical structures of Newtonian mechanics and general relativity, the two theories seem to have different perspectives regarding the modelling of matter and fields. Newtonian continuum thermomechanics is a theory of crystal conceptual and methodological clarity, thanks to its 300 years of history. Some of its concepts and methods may be brought into general relativity and offer new interpretations of the latter.

I apologize for § D, which is actually very confusing, because I tried to keep it short and to give the broadest possible view at the same time. To correct this I give a summary of its key points at the end of it.

Then in § 8 I give a summary of the '3 + 1' (or 'Arnowitt-Deser-Misner', or 'initial-value') formulation of the Einstein equations, showing in particular how the energy-stress-momentum balance equations arise from them. I then merge the 3 + 1 formulation with some of the continuum-mechanical concepts previously summarized, also discussing some points that aren't clear to me.

I conclude with a 3 + 1 spacetime presentation of Newtonian continuum mechanics, slightly different from Newton-Cartan theory. This presentation shows that the Newtonian geometrical structure is not so different from the general-relativistic one, and also reveals that the interpretation of some core concepts becomes actually foggier in general relativity.

This note excludes quantum theory and particle physics; concepts like particle, point-mass, baryon number never enter the discussion – I'm not familiar with baryon conservation. Our point of view and primitives are based on mass, force, energy, equations of balance and motion, and similar. Lagrangeans, Hamiltonians, actions, extremum principles, and similar notions are also avoided because I don't like them.

D   Continuum thermomechanics

D.1   Spacetime formulation

The kinematic structures of Newtonian and general-relativistic mechanics are very similar if not identical. We start from a four-dimensional differential manifold representing spacetime. No metric or connection are given yet. On this manifold we choose a slicing determined by a function t, or equivalently by an exact 1-form dt.

We assume that each 3-surface, or slice, of this slicing is equipped with a 3-metric $\boldsymbol{g}_t$ and a metric-compatible connection ∇.

In Newtonian mechanics we postulate the existence of an 'absolute time', represented by a particular slicing. Its 3-metrics are postulated to be Euclidean, and therefore their connections are flat. These postulates, however, are of a dynamic, not kinematic, nature; they are related to inertial and gravitational forces. Nothing forbids us to use a different slicing to describe the flow of matter.

Although each slice has a connection, no connection is defined on the whole spacetime. This means that there is no canonical correspondence between points on two different slices: we cannot say 'this place, now, is the *same* as that place, yesterday'. Such a correspondence is made via a vector field $\boldsymbol{U}$ such that

$$\boldsymbol{U} \cdot dt = 1. \tag{41}$$

The flow of this vector field puts points on different slices into mutual correspondence. It's variously called an 'observer', or a field of observers, or a frame. In Newtonian mechanics we can choose frames that preserve the Euclidean metric of the slices: $L_{\boldsymbol{U}}\boldsymbol{g} = 0$, called rigid frames. In general relativity it is generally impossible to choose a vector field this way, but we can choose one normal to the slices with respect to a spacetime 4-metric; this is called a *Eulerian* frame (Smarr et al. 1978; 1980). An Eulerian frame can also be chosen in such a way that $L_{\boldsymbol{U}}\sqrt{g} = 0$, a condition called 'maximal slicing' (Smarr et al. 1978 § III.B)[also Lichnerowicz (1944)].

The tensors $\mathrm{d}t \otimes U$ and $\mathbf{I} - \mathrm{d}t \otimes U$ induce a projective decomposition of any geometric object into parts parallel to the vector field U, called timelike, and to parts parallel the slices of the slicing (Hehl et al. 2003 § B.1.4), called spacelike. These projections do not depend on the metric or the connection.

The geometric structure just sketched is common to Newtonian mechanics and general relativity. In particular, also Newtonian mechanics has a 'lapse' and 'shift' in its equations, but they are chosen so as to make the gravito-inertial force assume the simplest possible form.

D.2 Energy-force

Mystery still surrounds the concepts of force and energy in Newtonian and general-relativistic mechanics and their differences in the two theories. In both theories the energy-force seems to be a covector-valued object, because it is meant to be integrated along a segment of line of matter flow u to yield the work done by force on matter along that segment.

In Newtonian mechanics, force is traditionally an object that also has to be integrated on a closed 2-surface or a 3-volume to give the total force acting on matter within that volume. Forces τ to be integrated over 2-surfaces are called stresses; forces f to be integrated over 3-volumes are called body forces. Such integration of a covector-valued object would require a flat connection: integrating means parallel-transporting the force covectors to a common point to sum them; but the result would depend on the path along which they are transported if the connection weren't flat (Truesdell et al. 1960 § 238). The balance of force, however, stating that the total force on every 3-volume vanishes, can be expressed locally using a divergence, and this can be generalized to non-flat connections.

Very important is the *gravito-inertial body force* exerted on matter at some spacetime point, defined by

$$f^{\star} := m\gamma - U^{\star} \cdot \mathrm{L}_{U^{\star}} M \cdot g^{-1} \tag{42}$$

and discussed in the previous section. Its definition in Newtonian mechanics requires a specific slicing, whose slices have Euclidean metrics g and a flat connection, and inertial and gravitational fields $U^{\star}$, γ. But besides this fact it can be treated similarly in both Newtonian and general-relativistic mechanics.

Most texts say that the 'time derivative of momentum', ma, is an effect of force. They also call 'inertial forces' some expressions appearing

in non-inertial frames, and often say that these are 'fictitious'. But the spacetime covariant perspective shows that the quantity $-ma$, where a is *the acceleration with respect to the fixed stars* has to be considered as a force. This force is the internal pressure we feel whenever we accelerate with respect to the fixed stars. It's rightly called *inertial force*. The gravitational force is given by the familiar expression $-m\gamma$, where γ is the gravitational field.

In Newtonian mechanics the combined gravito-inertial force $f^\star$ depends locally on three quantities:

(1) the mass form M,
(2) a particular frame $U^\star$, called the fixed-stars frame,
(3) the 3-metric g.

Its exact expression $f^\star(M, U^\star, g)$ in terms of these quantities is not important right now. It is something like $-m\,(L_{U^\star}v + \gamma)$, where v is a velocity covector obtained with the help of the metric g from the mass flux $U^\star \cdot M$. The gravitational force γ is in turn determined locally by the mass 3-form M via $d\gamma = -4\pi Gm$ – no metric or connection required. The fixed-stars frame $U^\star$ can also be thought as determined, non-locally, by the mass form M and the 3-metric g, or possibly just the 3-connection. I discuss this in §**; it is a form of Mach's principle.

The difference between Newtonian and general-relativistic mechanics is only in the dependence just explained, which is replaced by a local dependence in general relativity.

The fourth component of the gravito-inertial force is the change in kinetic and gravitational potential energies. The fact that kinetic energy is just the work per unit time done by the inertial force is clear from the relation $\partial_t\left(\frac{1}{2}mv^2\right) = (m\partial_t v) \cdot v$.

The discussion just given about the inertial force is very imprecise, but this lack of precision does not affect the discussion about the Einstein equations to be made later.

The second and third fundamental balances of continuum mechanics are those of force and energy, which says that the total energy-force on matter vanishes:

$$f + \nabla \cdot T = 0. \tag{43}$$

The gravito-inertial force is counted within the total body force f, and in a rigid frame with adapted coordinates it gives the familiar 'time derivative of momentum' term. In a rigid frame with adapted coordinates

the above balances become

$$\partial_t \epsilon = \nabla \cdot \boldsymbol{q} + h, \tag{44}$$

$$\partial_t \boldsymbol{j} = \nabla \cdot \tau + \boldsymbol{b}, \tag{45}$$

with $\partial_t = L_u$. The heat flux $\boldsymbol{q}$ and the momentum $\boldsymbol{q}$ are separate entities in Newtonian mechanics, but in general relativity their distinction is not clear-cut: 'heat carries momentum' and 'momentum carries energy' (Eckart 1940).

D.3 Torque

The concept of torque and its balance would require a rather long discussion. For the present note it is sufficient to remember that its balance traditionally leads to a symmetric stress tensor τ.

D.4 Constitutive equations

The balances of mass, force, energy give a system of 1+3+1 real-valued equations. They involve at least 4+3+6+5 real-valued components: mass density and flux, body force and stress, energy density and flux, and body supply of energy. The independent variables in these equations are usually assumed to be the four components of the mass 3-form – that is, mass density and mass flux or velocity – and the temperature. Sometimes the energy density is exchanged with the temperature in the role of independent variable.

The balance laws alone are therefore not sufficient to determine the evolution of all fields. Moreover, although the balances are assumed to hold for all forms of matter, different kinds of matter behave in different ways.

Constitutive equations are equations that gives the dependent fields as functions or functionals of the independent ones. They express the different behaviours of different kinds of matter, and together with the balance laws yield a closed system of differential or integro-differential equations.

The most important constitutive equation in our discussion is that for the 4-stress. All 4-components of the stress τ are in the simplest cases functionals of the mass 3-form M, the internal energy ϵ or temperature θ, and the 3-metrics $\boldsymbol{g}$:

$$\boldsymbol{T} = \boldsymbol{T}(M, \theta, \boldsymbol{g}), \tag{46a}$$

or, in a given slicing and frame,

$$T = \begin{pmatrix} \epsilon(m, \boldsymbol{p}, \theta, \boldsymbol{g}) & q(m, \boldsymbol{p}, \theta, \boldsymbol{g}) \\ j(m, \boldsymbol{p}, \theta, \boldsymbol{g}) & \tau(m, \boldsymbol{p}, \theta, \boldsymbol{g}) \end{pmatrix}. \tag{46b}$$

The dependence on the metrics can be complicated: for example, the stress at a point can depend on the difference between the 3-metric there and the metric from previous 3-surfaces, transported by the velocity $\boldsymbol{u}$ associated with the mass flux (Grot et al. 1966a; Carter et al. 1972). The difference can be non-vanishing even if the 3-metric is Euclidean on all 3-surfaces. The only difference between Newtonian and general relativistic mechanics is that all 3-metrics are flat in the former but not in the latter.

D.5   Remaining conceptual problems

We've seen that mass has a beautiful, conceptually intuitive, and metric-free geometric representation independent of the Newtonian or general-relativistic setting. The appropriate geometric representation of force, energy, and their balance is still unclear and discussed today.

We could conjecture that metric and connection are only related to the definition of the gravito-inertial force, and the other kinds of forces do not need such structure. In that case the equation of energy-force balance (43) is unsatisfactory because it involves a covariant derivative and therefore a connection.

Relativity theory seems to suggest that we can speak about 'surface forces' in a generalized sense, where spacelike surfaces can also be considered, and the gravito-inertial force would be such a 'spacelike surface force'. The integration of such forces is still problematic, however, because spacetime is not flat. A way out of this could be to consider force – a covector – to be something that must first be contracted with a vector field – yielding a scalar-valued form – and then integrated. This vector field could be a field of 'virtual displacements', its contraction with the force yielding 'virtual work'; this is indeed another way to conceive force in Newtonian mechanics (Truesdell et al. 1960 § 238). Some studies (Segev 2002; Hehl et al. 2003; Segev 1986; Segev et al. 1999; Segev 2000a,b) suggest that stress could be considered as a map transforming a 3-form like matter or charge into an energy 3-form – thus with 4 ×4 real components, as required. This transformation would be metric-free.

Finally, an opposite conjecture is that force and energy could just be manifestations of the metric.

D.6   Summary

I imagine that this section on Newtonian mechanics was poorly written and very confusing. Unfortunately I haven't managed to find a good balance between shortness and broadness. I summarize here its 'take-home messages' necessary for the next section:

- Mass and its balance rely only on the differential-geometric structure. They are the same in Newtonian and general-relativistic mechanics.

- The kinematic of mass relies on the concepts of slicing and frame, chosen arbitrarily, without metrics or connections. It is the same in Newtonian and general-relativistic mechanics.

- Force and energy, and therefore the dynamic of mass, rely on the introduction of metrics and connections. They do so in similarl ways in Newtonian and general-relativistic mechanics.

- An important difference between Newtonian and general-relativistic mechanics is in the form of the gravito-inertial force and its dependence on mass, metric, and a special slicing.

- The independent fields in Newtonian mechanics are mass 3-density, mass flux, temperature, 3-geometry. The energy-force tensor is given as a functional of them in a *constitutive equation*, which depends on the kind of matter considered. Mass balance, energy-force balance, and the constitutive equation yield a well-posed system of equations for the evolution of the independent fields.

D.7   Doubts and remarks

I believe that some points in the most common treatments of Newtonian and general-relativistic mechanics are an obstacle to a clearer understanding of the relationships between geometry, matter, and gravitation:

- The metric tensor is nonchalantly spread everywhere in the mathematical formulae: to raise or lower indices, introduce volume elements, and similar operations. But many physical quantities *and their balance equations* can actually be fully expressed in differential-geometrical, non-metric terms. This was shown, for example, by Kottler (1922a,b) and van Dantzig (1934c,a,d,b,e; 1937; 1940; 1954).

  Mass density, mass flux, and the balance of mass are one example, as we saw in § **??**. Most literature instead (e.g., Smarr et al. 1980 p. 171) writes the

balance of mass introducing a metric volume element $\sqrt{|g|}$ (cf. Choquet-Bruhat et al. 1996 § V.B.4, pp. 317–318), but the metric should play no role here.

• The metric-compatible connection is also nonchalantly introduced everywhere from the start. Since metric and connection represent the gravito-inertial force, their introduction represents the appearance of dynamics before the kinematic is completed.

Studies by Segev (2002) indicate that the energy-momentum-stress tensor might also be defined in metric-free terms; compare the balance of momentum rewritten with the metric volume element in Smarr et al. (1980 p. 171).

Thanks

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('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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