Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z), which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

There are 18 basic types of geometric objects, which represent physical quantities of different types. We have inner-oriented scalars, inner-oriented 1-vectors up to 4-vectors, inner-oriented 1-covectors up to 4-covectors, and all their outer-oriented counterparts.

The exterior product '$\wedge$' of vectors and covectors is usually represented just by juxtaposition, and a shortened notation is used for exterior products of several exterior derivatives 'd'. For instance

$$\mathrm{d}^2xy \coloneqq \mathrm{d}x \wedge \mathrm{d}y \qquad \partial^2_{xy} \coloneqq \partial_x \wedge \partial_y \ . \tag{1}$$

The associated bases for inner-oriented covector fields are

$$\mathrm{d}t \quad \mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z \tag{2}$$
$$\mathrm{d}^2tx \quad \mathrm{d}^2ty \quad \mathrm{d}^2tz \quad \mathrm{d}^2yz \quad \mathrm{d}^2zx \quad \mathrm{d}^2xy \tag{3}$$
$$\mathrm{d}^3xyz \quad -\mathrm{d}^3tyz \quad -\mathrm{d}^3tzx \quad -\mathrm{d}^3txy \tag{4}$$
$$\mathrm{d}^4txyz \tag{5}$$

and analogously for inner-oriented vector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it's only defined on a coordinate patch. It is idempotent: $\tilde{1}\tilde{1} = 1$.

A twisted or outer-oriented 3-covector such as $\mathrm{d}^3\tilde{x}yz$ has an associated outer direction, in this case positive t. We adopt this shorter notation for the outer-oriented versions of the bases above (analogous to the notation in Gotay & Marsden 1992 § 2 p. 371):

$$-\mathrm{d}_{xyz} \quad \mathrm{d}_{tyz} \quad \mathrm{d}_{tzx} \quad \mathrm{d}_{txy} \tag{6a}$$

$$\mathrm{d}^2_{yz} \quad \mathrm{d}^2_{zx} \quad \mathrm{d}^2_{xy} \quad \mathrm{d}^2_{tx} \quad \mathrm{d}^2_{ty} \quad \mathrm{d}^2_{tz} \tag{6b}$$

$$\mathrm{d}^3_t \quad \mathrm{d}^3_x \quad \mathrm{d}^3_y \quad \mathrm{d}^3_z \tag{6c}$$

$$\mathrm{d}^4 \tag{6d}$$

so that $-\mathrm{d}_{xyz} \coloneqq \mathrm{d}\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial xyz \coloneqq \partial_{\tilde{t}}$. Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} &n_{xyz}\,\mathrm{d}^3\tilde{x}yz - n_{tyz}\,\mathrm{d}^3\tilde{t}yz - n_{tzx}\,\mathrm{d}^3\tilde{t}zx - n_{txy}\,\mathrm{d}^3\tilde{t}xy \\ &\equiv n^t\,\mathrm{d}^3_t + n^x\,\mathrm{d}^3_x + n^y\,\mathrm{d}^3_y + n^z\,\mathrm{d}^3_z \end{aligned} \tag{7}$$

$$\text{with } n^t \equiv n_{xyz}\,,\ n^x \equiv n_{tyz}\,,\ \text{ and so on.}$$

Contraction or dot-product of vectors and covectors is denoted by '$\cdot$', and always contracts the maximal possible amount of contiguous elements. For instance

$$\mathrm{d}^3xyz \cdot \partial^2_{yz} = \mathrm{d}x \qquad -\partial^3_{txy} \cdot \mathrm{d}x = \partial^2_{ty}\,. \tag{8}$$

Contractions with the 4-vector ∂^4 and 4-covector d^4 establish a duality between outer n-covectors and inner $(4-n)$-vectors:

$$\left.\begin{matrix} \partial^4 \\ \partial^3_{xyz}\ \partial^3_{tyz}\ \partial^3_{tzx}\ \partial^3_{txy} \\ \partial^2_{tx}\ \partial^2_{ty}\ \partial^2_{tz}\ \partial^2_{yz}\ \partial^2_{zx}\ \partial^2_{xy} \\ \partial_t\ \partial_x\ \partial_y\ \partial_z \\ 1 \end{matrix}\right) \begin{matrix} \xleftarrow{\ \partial^4\,\cdot\ } \\ \xrightarrow{\ \cdot\,\mathrm{d}^4\ } \end{matrix} \left(\begin{matrix} \tilde{1} \\ \mathrm{d}_{xyz}\ \mathrm{d}_{tyz}\ \mathrm{d}_{tzx}\ \mathrm{d}_{txy} \\ \mathrm{d}^2_{yz}\ \mathrm{d}^2_{zx}\ \mathrm{d}^2_{xy}\ \mathrm{d}^2_{tx}\ \mathrm{d}^2_{ty}\ \mathrm{d}^2_{tz} \\ \mathrm{d}^3_t\ \mathrm{d}^3_x\ \mathrm{d}^3_y\ \mathrm{d}^3_z \\ \mathrm{d}^4 \end{matrix}\right. \tag{9}$$

These duals have special properties. For instance, for any 3-covector $\boldsymbol{N}$, we have

$$\boldsymbol{N} \cdot (\partial^4 \cdot \boldsymbol{N}) \cdot \boldsymbol{N} = 0 \tag{10}$$

that is, the dual of $\boldsymbol{N}$ is a vector belonging in the kernel of $\boldsymbol{N}$.

If $\boldsymbol{\gamma}$ is a non-zero 4-covector and $\boldsymbol{\gamma}^{-1}$ the inverse 4-vector, that is, $\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma}^{-1}\cdot\boldsymbol{\gamma} = 1$, and if $\boldsymbol{N}$ is a 3-covector and $\boldsymbol{\phi}$ a 1-covector, we have the useful identity

$$\boldsymbol{N} \wedge \boldsymbol{\phi} = (\boldsymbol{N}\cdot\boldsymbol{\gamma}^{-1}\cdot\boldsymbol{\phi})\,\boldsymbol{\gamma}\ , \tag{11}$$

which also holds as long as the degrees of $\boldsymbol{N}$ and $\boldsymbol{\phi}$ sum up to 4.

We often consider vector- or covector-valued covectors, written as sums of tensor products. They can be outer- or inner-oriented. For instance

$$\mathrm{d}^3_t \otimes \mathrm{d}x \equiv \mathrm{d}^3\tilde{x}yz \otimes \mathrm{d}x\ , \qquad \mathrm{d}^3_x \otimes \partial_y\ . \tag{12}$$

The operation $\curlywedge$ between a vector-valued covector and a covector-valued covector is the contraction of their vector- and covector-valued parts and the exterior product of their covector parts. For instance, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are covectors, $\boldsymbol{\omega}$ is a covector, and $\boldsymbol{u}$ is a vector, then

$$(\boldsymbol{\phi}\otimes\boldsymbol{\omega}) \curlywedge (\boldsymbol{\psi}\otimes\boldsymbol{u}) \coloneqq (\boldsymbol{\phi}\wedge\boldsymbol{\psi})\otimes(\boldsymbol{\omega}\cdot\boldsymbol{u}) \tag{13}$$

As another example,

$$(\mathrm{d}^3_t \otimes \mathrm{d}x) \curlywedge (\mathrm{d}_t \otimes \partial_x) = (\mathrm{d}^3_t \wedge \mathrm{d}_t)\,(\mathrm{d}x\cdot\partial_x) = -\mathrm{d}^4\ . \tag{14}$$

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta}\,\mathrm{d}u^\beta\ ,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$\mathrm{d}\alpha' = \frac{\partial\alpha'}{\partial\alpha}\,\mathrm{d}\alpha$$

- raising or lowering of indices with the metric, for example

$$\mathrm{d}\alpha \mapsto g^{\alpha\beta}\,\partial_\beta\ .$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix[1]. For the spaces of 3-vectors and 3-covectors we have simplified formulae[2]:

$$\begin{aligned} \mathrm{d}^3_{\alpha'} &= \det\left(\tfrac{\partial\alpha'}{\partial\alpha}\right)\,\frac{\partial\alpha}{\partial\alpha'}\,\mathrm{d}^3_{\alpha} \\ \partial^3\alpha' &= \det\left(\tfrac{\partial\alpha}{\partial\alpha'}\right)\,\frac{\partial\alpha'}{\partial\alpha}\,\partial^3\alpha \end{aligned} \tag{15}$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} \mathrm{d}^3_{\alpha} &\mapsto \frac{1}{|g|}\,g_{\alpha\beta}\,\partial^3\beta \\ \partial^3\alpha &\mapsto |g|\,g^{\alpha\beta}\,\mathrm{d}^3_{\beta} \end{aligned} \tag{16}$$

3 Metric

We take the metric $\boldsymbol{g}$ to have signature $(-,+,+,+)$ and dimensions of area. The determinant and the negative of its square root are denoted shortly

$$g \coloneqq \det\boldsymbol{g} \qquad \sqrt{g} \coloneqq \sqrt{-\det\boldsymbol{g}}\,. \tag{17}$$

The volume element induced by the metric $\boldsymbol{g}$ has dimensions of volume-time and is denoted (note the boldface)

$$\boldsymbol{\gamma} \coloneqq \frac{\sqrt{g}}{c}\mathrm{d}^4\tilde{t}xyz \equiv \frac{\sqrt{g}}{c}\,\mathrm{d}^4 \tag{18}$$

and its corresponding inverse, a twisted 4-vector:

$$\boldsymbol{\gamma}^{-1} \coloneqq \frac{c}{\sqrt{g}}\,\partial^4\,. \tag{19}$$

[1] Choquet-Bruhat et al. 1996 § IV.A.1 p. 199. [2] Gantmacher 2000 § I.4 eq. (33).

Contraction with the volume element or its inverse establishes a "volume duality" between outer n-covectors and inner $(4-n)$-vectors:

$$\left.\begin{matrix} \partial^4 \\ \partial^3_{xyz}\ \ \partial^3_{tyz}\ \ \partial^3_{tzx}\ \ \partial^3_{txy} \\ \partial^2_{tx}\ \ \partial^2_{ty}\ \ \partial^2_{tz}\ \ \partial^2_{yz}\ \ \partial^2_{zx}\ \ \partial^2_{xy} \\ \partial_t\ \ \partial_x\ \ \partial_y\ \ \partial_z \\ 1 \end{matrix}\right) \quad \begin{matrix} \xleftarrow{\ \frac{\sqrt{g}}{c}\,\boldsymbol{\gamma}^{-1}\,\cdot\ } \\[2ex] \xrightarrow{\ \cdot\,\boldsymbol{\gamma}\,\frac{c}{\sqrt{g}}\ } \end{matrix} \quad \left(\begin{matrix} \tilde{1} \\ \mathrm{d}_{xyz}\ \ \mathrm{d}_{tyz}\ \ \mathrm{d}_{tzx}\ \ \mathrm{d}_{txy} \\ \mathrm{d}^2_{yz}\ \ \mathrm{d}^2_{zx}\ \ \mathrm{d}^2_{xy}\ \ \mathrm{d}^2_{tx}\ \ \mathrm{d}^2_{ty}\ \ \mathrm{d}^2_{tz} \\ \mathrm{d}^3_t\ \ \mathrm{d}^3_x\ \ \mathrm{d}^3_y\ \ \mathrm{d}^3_z \\ \mathrm{d}^4 \end{matrix}\right. \tag{20}$$

This is the reason why in older literature an outer-oriented n-covector is treated as a $(4-n)$-"vector density", that is, a vector divided by the square root of the volume element.

The metric on the space of 1-vectors also induces metrics on the spaces of 1-covectors, 2-vectors and 2-covectors, and so on. In particular, the metrics $\overset{3}{\boldsymbol{g}}{}^{-1}$ on the space of 3-covectors and $\overset{3}{\boldsymbol{g}}$ on the space of 3-vectors can be written in coordinates as

$$\overset{3}{\boldsymbol{g}}{}^{-1} = \frac{g_{\mu\nu}}{g}\,\partial^3\mu \otimes \partial^3\nu \qquad \text{with dimensions length}^{-6} \tag{21}$$

$$\overset{3}{\boldsymbol{g}} = g\, g^{\mu\nu}\,\mathrm{d}^3_\mu \otimes \mathrm{d}^3_\nu \qquad \text{with dimensions length}^{6}\ . \tag{22}$$

With these we can define squared norms $\|\boldsymbol{.}\|^2$ on all those spaces. Note in particular the following identity:

$$\|\boldsymbol{\gamma}^{-1}\cdot\boldsymbol{N}\|^2 = -c^2\,\|\boldsymbol{N}\| \qquad \text{for every 3-covector } \boldsymbol{N}. \tag{23}$$

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{24}$$

and the volume element is simply d^4.

4 Matter current

The amount-of-matter current $\boldsymbol{N}$ is an outer-oriented 3-covector

$$\boldsymbol{N} = N\,\mathrm{d}^3_t + J^i\,\mathrm{d}^3_i \tag{25}$$

of dimensions "amount of matter", typically measured in moles, where

- N is the volumic amount of matter, measured per unit coordinate volume.
- J^i is the aeric flux of amount of matter (equivalent to a molar flux), measured per unit coordinate area and unit coordinate time.

The current for every particular kind of matter satisfies the conservation law

$$\mathrm{d}\boldsymbol{N} = 0 \quad \text{or} \quad \partial_t N + \partial_i J^i = 0 \tag{26}$$

independent of any metric.

The common contravariant form of the matter current, "N^μ", is obtained by contracting the matter current with the inverse volume element:

$$`\,N^\mu\,' \mathrel{\hat{=}} \boldsymbol{\gamma}^{-1}\cdot\boldsymbol{N} = \frac{c}{\sqrt{g}}\,N\,\partial_t + \frac{c}{\sqrt{g}}\,J^i\,\partial_t\ . \tag{27}$$

If a metric is present, a four-velocity $\boldsymbol{U}$ can be associated with the matter current $\boldsymbol{N}$, defined by the following properties and identity:

$$\boldsymbol{U}\cdot\boldsymbol{N} = 0 \qquad \|\boldsymbol{U}\|^2 = -c^2 \tag{28}$$

$$\boldsymbol{U} = \frac{1}{|\|\boldsymbol{N}\||}\,\boldsymbol{\gamma}^{-1}\cdot\boldsymbol{N} \tag{29}$$

which also implies (for normal matter)

$$\boldsymbol{N} = \|\boldsymbol{N}\|\,\boldsymbol{U}\cdot\boldsymbol{\gamma} \tag{30}$$

For normal matter (as opposed to antimatter) $\|\boldsymbol{N}\|^2 \geqslant 0$.

5 Four-stress

The stress-energy-momentum tensor, or simply four-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\begin{aligned} \boldsymbol{T} &= T^{\mu}_{\ \nu}\ \mathrm{d}^3_{\mu} \otimes \mathrm{d}x^{\nu} \\ &= -\epsilon\, \mathrm{d}^3_t \otimes \mathrm{d}t - q^i\, \mathrm{d}^3_i \otimes \mathrm{d}t + p_j\, \mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\, \mathrm{d}^3_i \otimes \mathrm{d}x^j \end{aligned} \tag{31}$$

the indices i, j running over x, y, z, and where:

- The energy ϵ is a density per unit *coordinate* volume xyz, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this energy comprises "rest" energy, kinetic energy, potential gravitational energy, and centrifugal potential energy.
- The energy flux q^j is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. If the point at which the four-stress is considered has a matter-current, then this flux term includes transport of the energy term above and also heat flux.
- The momentum p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i.
- The compressive three-stress π^i_j are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j. If the point at which the four-stress is considered has a matter-current, then this stress comprises momentum transport and internal stresses.

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} -|g_{tt}| & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \tag{32}$$

The diagonal elements $g_{tt}, \ldots$ include a transformation factor for dimensions or units. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

The common contra-contra-variant form of the stress, “$T^{\mu\nu}$”, is obtained by contracting the four-stress with the inverse volume element and the inverse metric:

$$
\begin{aligned}
{}^{\backprime} T^{\mu\nu\,\prime} \equiv \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1} = \frac{c\,\lvert g^{tt} \rvert}{\sqrt{g}}\, \epsilon\, \partial_t \otimes \partial_t + \frac{c\,\lvert g^{tt} \rvert}{\sqrt{g}}\, q^i\, \partial_i \otimes \partial_t \\
+ \frac{c\, g^{ij}}{\sqrt{g}}\, p_i\, \partial_t \otimes \partial_j + \frac{c\, g^{kj}}{\sqrt{g}}\, \pi^i_k\, \partial_i \otimes \partial_j \;.
\end{aligned}
\tag{33}
$$

One important detail in finding the Newtonian approximation of “energy density” is that *one takes different zeros of energy density in different coordinate systems*: the zero is taken as the molar mass times the molar density *in the current coordinate system*. By ‘zero’ I mean the arbitrary separation between “mass” and “energy”.

The *total* four-stress satisfies the balance equation

$$
\mathrm{D}\boldsymbol{T} = 0 \tag{34}
$$

which is equivalent to the four balance equations

$$
\begin{aligned}
\partial_t \epsilon + \partial_i q^i &= \epsilon\, \Gamma^t_{tt} + q^i\, \Gamma^t_{it} - p_j\, \Gamma^j_{tt} - \pi^i_k\, \Gamma^k_{it} \\
\partial_t p_j + \partial_i \pi^i_j &= -\epsilon\, \Gamma^t_{tj} - q^i\, \Gamma^t_{ij} + p_k\, \Gamma^k_{tj} + \pi^i_k\, \Gamma^k_{ij}
\end{aligned}
\tag{35}
$$

In general relativity the *total* four-stress also satisfies

$$
\left(\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1}\right)^{\intercal} - \boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{g}^{-1} = 0 \;. \tag{36}
$$

The four-stress determines an association between any 1-vector $\boldsymbol{V}$ field and an outer-oriented 3-covector field, interpreted as a current:

$$
\boldsymbol{V} \mapsto \boldsymbol{T} \cdot \boldsymbol{V} \;. \tag{37}
$$

The equation $\mathrm{D}\boldsymbol{T} = 0$ is equivalent to the condition

$$
\mathrm{d}(\boldsymbol{T} \cdot \boldsymbol{V}) = -\boldsymbol{T} \curlywedge \nabla \boldsymbol{V} = \mathrm{tr}(\boldsymbol{T}^{\intercal} \cdot \boldsymbol{\gamma}^{-1} \cdot \nabla \boldsymbol{V})\, \boldsymbol{\gamma} \qquad \text{for any } \boldsymbol{V} \tag{38}
$$

which is a conservation law if $\boldsymbol{V}$ is a Killing vector. We only need to require this for four independent vector fields, which we can choose to be $\{\partial_\alpha\}$. For the special case $\boldsymbol{V} = \partial_\alpha$ the formula above becomes, thanks to equations (75) and (76),

$$\begin{aligned} \mathrm{d}(\boldsymbol{T}\cdot\partial_\alpha) &= -\boldsymbol{T} \curlywedge \nabla\partial_\alpha & \iff \\ \mathrm{d}(T^\mu{}_\alpha\, \mathrm{d}^3_\mu) &= T^\mu{}_\nu\, \Gamma^\nu_{\mu\alpha}\, \mathrm{d}^4 & \iff \\ \partial_\mu T^\mu{}_\alpha &= T^\mu{}_\nu\, \Gamma^\nu_{\mu\alpha} \end{aligned} \tag{39}$$

which is the common expression of $\mathrm{D}\boldsymbol{T} = 0$, where $\boldsymbol{T}$ is a tensor density.

Consider a region where there is a non-vanishing matter current $\boldsymbol{N}$ with associated four-velocity $\boldsymbol{U}$, and define

$$\bar{\boldsymbol{U}} = -\frac{1}{c}\,\boldsymbol{g}\cdot\boldsymbol{U} \tag{40}$$

which statisfies

$$\bar{\boldsymbol{U}}\cdot\boldsymbol{U} = 1\,, \qquad \nabla\boldsymbol{U}\cdot\bar{\boldsymbol{U}} = 0\,. \tag{41}$$

The last equality can be proved from $\nabla\boldsymbol{g} = 0$ and

$$0 = -\nabla(c^2) = \nabla(\boldsymbol{U}\cdot\boldsymbol{g}\cdot\boldsymbol{U}) = 2\,(\nabla\boldsymbol{U})\cdot\boldsymbol{g}\cdot\boldsymbol{U}\,. \tag{42}$$

We can associate with the matter a four-stress $\boldsymbol{T}$ which can be decomposed as follows:

$$\begin{gathered} \boldsymbol{T} = -\epsilon\,\boldsymbol{N}\otimes\bar{\boldsymbol{U}} + \boldsymbol{N}\otimes\boldsymbol{P} - (\bar{\boldsymbol{U}}\wedge\boldsymbol{Q})\otimes\bar{\boldsymbol{U}} + \bar{\boldsymbol{U}}\wedge\boldsymbol{S} \\ \text{with}\quad \boldsymbol{P}\cdot\boldsymbol{U} = 0 \quad \boldsymbol{Q}\cdot\boldsymbol{U} = 0 \quad \boldsymbol{U}\cdot\boldsymbol{S} = 0 \quad \boldsymbol{S}\cdot\boldsymbol{U} = 0 \end{gathered} \tag{43}$$

where

- ϵ is a scalar, the molar energy density.
- $\boldsymbol{P}$ is a 1-covector, the molar momentum density.
- $\boldsymbol{Q}$ is a 2-covector, the areic energy-flux density.
- $\boldsymbol{S}$ is a 1-covector-valued 2-covector, the three-stress related to the Cauchy stress.

Using the four-velocity $\boldsymbol{U}$ associated with the matter current we obtain what could be called the "internal-energy current":

$$\boldsymbol{T} \cdot \boldsymbol{U} \equiv -\epsilon\, \boldsymbol{N} - \bar{\boldsymbol{U}}\, \boldsymbol{Q} \tag{44}$$

which, from eqs (38), (40), (26), satisfies the balance law

$$\mathrm{d}\big(\boldsymbol{T} \cdot \boldsymbol{U}\big) = -\boldsymbol{T} \curlywedge \nabla\boldsymbol{U} \tag{45}$$

or

$$\begin{aligned} \mathrm{d}\big(-\epsilon\, \boldsymbol{N} - \bar{\boldsymbol{U}}\, \boldsymbol{Q}\big) = \big(\epsilon\, \boldsymbol{N} \otimes \bar{\boldsymbol{U}}\big) \cdot \nabla\boldsymbol{U} - \big(\boldsymbol{N} \otimes \boldsymbol{P}\big) \cdot \nabla\boldsymbol{U} \\ + \big(\bar{\boldsymbol{U}}\, \boldsymbol{Q} \otimes \bar{\boldsymbol{U}}\big) \cdot \nabla\boldsymbol{U} - \bar{\boldsymbol{U}}\, \boldsymbol{S} \cdot \nabla\boldsymbol{U} \end{aligned} \tag{46}$$

or simply

$$\boldsymbol{N}\mathrm{d}\epsilon + \bar{\boldsymbol{U}}\mathrm{d}\boldsymbol{Q} - \boldsymbol{Q}\,\mathrm{d}\bar{\boldsymbol{U}} = -\boldsymbol{N}\,\nabla\boldsymbol{U} \cdot \boldsymbol{P} - (\bar{\boldsymbol{U}}\,\boldsymbol{S}) \cdot \nabla\boldsymbol{U}\ . \tag{47}$$

The last balance equation corresponds to eq. (28) in Eckart (1940), except for the fact that here we are keeping the momentum $\boldsymbol{P}$ and heat flux $\boldsymbol{Q}$ distinct.

When $\boldsymbol{P}$, $\boldsymbol{S}$, $\boldsymbol{Q}$ are zero, one speaks of the four-stress of "dust". It is noteworthy that in this case the four-stress is essentially proportional to the matter 3-covector, multiplied by a vector collinear with the matter four-velocity:

$$\boldsymbol{T} = -\boldsymbol{N} \otimes (\epsilon\, \bar{\boldsymbol{U}})\ . \tag{48}$$

If we take the four-velocity itself as reference, given that $\bar{\boldsymbol{U}}\boldsymbol{U} = -c^2$, the energy 3-covector we obtain is proportional to the matter 3-covector:

$$-c^2 \epsilon\, \boldsymbol{N} \tag{49}$$

For this reason ϵ is called the "proper internal energy". If we take other vector fields as reference, then this energy will pick up further multiplicative terms consisting in the projection of the matter four-velocity $\boldsymbol{U}$ onto the reference vector field. In particular taking ∂_t or ∂_i as reference we'll have the projections of the four-velocity onto the coordinate covectors as multiplicative factors.

6 Balance laws

Given a coordinate system (t, x^i), we can use formula (38) with a special set of ten vector fields, obtaining a set of ten balance laws[3]. The vector fields are, separated into four groups of one, three, three, and three,

$$\partial_t \tag{50}$$

$$\partial_{x^i} \tag{51}$$

$$\boldsymbol{g}^{-1} \cdot (t\,\mathrm{d}x^i - x^i\,\mathrm{d}t) \tag{52}$$

$$\boldsymbol{g}^{-1} \cdot (x^i\,\mathrm{d}x^j - x^j\,\mathrm{d}x^i)\,. \tag{53}$$

Note that the last two groups of fields are dimensionally consistent no matter what the dimensions of the coordinates are. Also note that when the metric has its standard diagonal form, the third set has a plus sign instead of a minus when written explicitly in vector form.

7 Electromagnetic field

The electromagnetic field, represented by the Faraday 2-covector, is typically decomposed as[4]

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{E}\,\mathrm{d}t + \boldsymbol{B}\\ &\equiv E_x\,\mathrm{d}^2xt + E_y\,\mathrm{d}^2yt + E_z\,\mathrm{d}^2zt + B^x\,\mathrm{d}^2yz + B^y\,\mathrm{d}^2zx + B^z\,\mathrm{d}^2xy\,.\end{aligned} \tag{54}$$

Given a system of coordinates, this decomposition is unique: the 2-covector $\boldsymbol{B}$ does not have $\mathrm{d}t$ components, that is, it has ∂_t in its kernel; and so does the 1-covector $\boldsymbol{E}$.

The conservation of magnetic flux is expressed by

$$\mathrm{d}\boldsymbol{F} = 0 \tag{55}$$

or equivalently

$$\begin{aligned}
&\partial_i B^i = 0 && (\mathrm{d}^3xyz \text{ component})\\
&\partial_t B^x + \partial_y E_z - \partial_z E_y = 0 && (\mathrm{d}^3tyz)\\
&\partial_t B^y + \partial_z E_x - \partial_x E_z = 0 && (\mathrm{d}^3tzx)\\
&\partial_t B^z + \partial_x E_y - \partial_y E_x = 0 && (\mathrm{d}^3txy)
\end{aligned} \tag{56}$$

[3] Hawking & Ellis 1994 § 3.2. [4] Frankel 1979 ch. 9.

It should be noted that all components of $\mathrm{d}\boldsymbol{E}$ containing $\mathrm{d}t$ also disappear, because of the exterior product $\boldsymbol{E}\,\mathrm{d}t$. The differential d therefore operates on $\boldsymbol{E}$ as if this 1-covector belonged to a three-dimensional manifolds with coordinates (x, y, z). Let's denote this operation by $\mathbf{d}$; it is connected with the "curl" operator. Then the equations above can be written

$$\mathbf{d}\boldsymbol{B} = 0 \qquad \partial_t \boldsymbol{B} + \mathbf{d}\boldsymbol{E} = 0\ . \tag{57}$$

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- $\boldsymbol{u}^{\bullet}$ is a 1-vector, represented by the column-matrix $\boldsymbol{u}$.
- Similarly for $\boldsymbol{v}^{\bullet}$.
- $\boldsymbol{\omega}_{\bullet}$ is a 1-covector, represented by the row-matrix $\boldsymbol{\omega}$.
- $\boldsymbol{g}_{\bullet\bullet}$ is a covector-valued covector, represented by the matrix $\boldsymbol{g}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $\boldsymbol{g}^{-1\bullet\bullet}$ is a vector-valued vector, inverse of $\boldsymbol{g}$, that is: $\boldsymbol{g} \cdot \boldsymbol{g}^{1} = \mathbf{id}_{\bullet}^{\bullet}$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.

- $\boldsymbol{Q}_{|\bullet\bullet\bullet|}$ is a 3-covector, represented by the column-matrix $\boldsymbol{Q}$.
- $\boldsymbol{T}_{|\bullet\bullet\bullet|\bullet}$ is a covector-valued 3-covector, represented by the matrix $\boldsymbol{T}$. The rows represent the 3-covector components; the columns, the covector components.
- $\gamma_{|\bullet\bullet\bullet\bullet|}$ is a 4-covector, represented by the number $\boldsymbol{\gamma}$.
- $\boldsymbol{\gamma}^{-1\,|\bullet\bullet\bullet\bullet|}$ is a 4-vector, represented by the number $\boldsymbol{\gamma}^{-1}$
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from "old" coordinates x to "new" coordinates x' is represented by the matrix $\boldsymbol{J}$. The rows correspond to the new coordinates x'; the columns, to the old x.
- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x' is represented by the matrix $\boldsymbol{J}^{-1}$. The rows correspond to the old coordinates x; the columns, to the new x'.

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

(object)	(matrix repr)	(58)
$\boldsymbol{\omega} \cdot \boldsymbol{u}$	$\boldsymbol{\omega u} \equiv \boldsymbol{u}^\intercal \boldsymbol{\omega}^\intercal$ (number)	(59)
$\boldsymbol{v} \cdot \boldsymbol{g} \cdot \boldsymbol{u}$	$\boldsymbol{v}^\intercal \boldsymbol{g u}$ (number)	(60)
$\boldsymbol{g} \cdot \boldsymbol{u}$	$\boldsymbol{u}^\intercal \boldsymbol{g}^\intercal$ (row-matrix)	(61)
$\boldsymbol{\omega} \cdot \boldsymbol{g}^{-1}$	$\boldsymbol{g}^{-\intercal} \boldsymbol{\omega}^\intercal$ (column-matrix)	(62)
$\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{Q}$	$\boldsymbol{\gamma}^{-1} \boldsymbol{Q}$ (column-matrix)	(63)
$\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T}$	$\boldsymbol{\gamma}^{-1} \boldsymbol{T}$ (column-matrix)	(64)
$\boldsymbol{T} \cdot \boldsymbol{u}$	$\boldsymbol{T u}$ (column-matrix)	(65)
$\boldsymbol{g} \cdot (\boldsymbol{\gamma}^{-1} \cdot \boldsymbol{T}) \cdot \boldsymbol{u}$	$\boldsymbol{\gamma}^{-1}\, \boldsymbol{g T u}$ (matrix)	(66)

- Transformations

$$\begin{aligned}
\text{(old coords)} &\mapsto \text{(new coords)} && (67)\\
\boldsymbol{u} &\mapsto \boldsymbol{J}\boldsymbol{u} && (68)\\
\boldsymbol{\omega} &\mapsto \boldsymbol{\omega}\boldsymbol{J}^{-1} && (69)\\
\boldsymbol{g} &\mapsto \boldsymbol{J}^{-\intercal}\boldsymbol{g}\boldsymbol{J}^{-1} && (70)\\
\boldsymbol{Q} &\mapsto \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{J}\boldsymbol{Q} && (71)\\
\boldsymbol{\gamma} &\mapsto \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{\gamma} && (72)\\
\boldsymbol{\gamma}^{-1} &\mapsto \det \boldsymbol{J}\,\boldsymbol{\gamma}^{-1} && (73)\\
\boldsymbol{T} &\mapsto \frac{1}{\det \boldsymbol{J}}\,\boldsymbol{J}\boldsymbol{T}\boldsymbol{J}^{-1} && (74)
\end{aligned}$$

B Checks about optimal representation of four-stress

A lemma about the exterior derivative of some outer-oriented 3-covectors:

$$\begin{gathered}
\mathrm{d}\big(f\,\mathrm{d}^3_t\big) = \mathrm{d}\Big(f\,\mathrm{d}^3\widetilde{xyz}\Big) = \partial_t f \mathrm{d}t \wedge \mathrm{d}^3\widetilde{xyz} + 0 = \partial_t f\ \mathrm{d}^4\widetilde{txyz}\\
\mathrm{d}\big(f\,\mathrm{d}^3_x\big) = -\mathrm{d}\Big(f\,\mathrm{d}^3\widetilde{tyz}\Big) = -\partial_x f \mathrm{d}x \wedge \mathrm{d}^3\widetilde{tyz} + 0 = \partial_x f\ \mathrm{d}^4\widetilde{txyz}\\
\mathrm{d}\big(f\,\mathrm{d}^3_\alpha\big) = \partial_\alpha f\ \mathrm{d}^4\widetilde{txyz}
\end{gathered} \tag{75}$$

Another lemma about the exterior product of 3- and 1-covectors:

$$\begin{gathered}
\mathrm{d}^3_t \wedge \mathrm{d}t = -\mathrm{d}^4 \qquad \mathrm{d}^3_x \wedge \mathrm{d}x = -\mathrm{d}^4 \qquad \dots\\
\mathrm{d}^3_\alpha \wedge \mathrm{d}x^\beta = -\delta^\beta_\alpha \mathrm{d}^4
\end{gathered} \tag{76}$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned}
\boldsymbol{\nabla}(\mathrm{d}t) &= -\Gamma^t_{tt}\,\mathrm{d}t\otimes\mathrm{d}t - \Gamma^t_{tj}\,\mathrm{d}t\otimes\mathrm{d}x^j - \Gamma^t_{it}\,\mathrm{d}x^i\otimes\mathrm{d}t - \Gamma^t_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j\\
\boldsymbol{\nabla}\big(\mathrm{d}x^k\big) &= -\Gamma^k_{tt}\,\mathrm{d}t\otimes\mathrm{d}t - \Gamma^k_{tj}\,\mathrm{d}t\otimes\mathrm{d}x^j - \Gamma^k_{it}\,\mathrm{d}x^i\otimes\mathrm{d}t - \Gamma^k_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j
\end{aligned} \tag{77}$$

If $\boldsymbol{\phi}$ is a 3-covector, $\boldsymbol{\omega}$ a 1-covector, and D the exterior covariant derivative, then

$$\mathrm{D}(\boldsymbol{\phi}\otimes\boldsymbol{\omega}) = (\mathrm{d}\boldsymbol{\phi})\otimes\boldsymbol{\omega} - \boldsymbol{\phi}\wedge\nabla\boldsymbol{\omega} \tag{78}$$

and in particular

$$\mathrm{D}\big(\boldsymbol{\phi} \otimes \mathrm{d}x^{\alpha}\big) = (\mathrm{d}\boldsymbol{\phi}) \otimes \mathrm{d}x^{\alpha} + \mathit{\Gamma}^{\alpha}_{\mu\nu} \big(\boldsymbol{\phi} \wedge \mathrm{d}x^{\mu}\big) \otimes \mathrm{d}x^{\nu} \;. \tag{79}$$

Let's also consider the contraction with a 1-vector $\boldsymbol{u}$:

$$\begin{aligned} \mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u}) &\equiv \mathrm{d}[\boldsymbol{\phi}\,(\boldsymbol{\omega} \cdot \boldsymbol{u})] \\ &= (\mathrm{d}\boldsymbol{\phi})\,(\boldsymbol{\omega} \cdot \boldsymbol{u}) - \boldsymbol{\phi} \wedge \mathrm{d}(\boldsymbol{\omega} \cdot \boldsymbol{u}) \\ &= \mathrm{d}\boldsymbol{\phi} \otimes \boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega} \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega} \\ &\equiv (\mathrm{d}\boldsymbol{\phi} \otimes \boldsymbol{\omega} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{\omega}) \cdot \boldsymbol{u} - \boldsymbol{\phi} \wedge \nabla\boldsymbol{u} \cdot \boldsymbol{\omega} \\ &\equiv \mathrm{D}(\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \cdot \boldsymbol{u} - (\boldsymbol{\phi} \otimes \boldsymbol{\omega}) \curlywedge \nabla\boldsymbol{u} \;. \end{aligned} \tag{80}$$

and in particular

$$\mathrm{D}(\boldsymbol{\phi} \otimes \mathrm{d}x^{\alpha} \cdot \boldsymbol{u}) = \mathrm{d}\boldsymbol{\phi}\, u^{\alpha} - \boldsymbol{\phi} \wedge \mathrm{d}x^{\beta}\, \partial_{\beta} u^{\alpha} \tag{81}$$

A balance equation with the exterior covariant derivative then becomes

$$\mathrm{D}\boldsymbol{T} = 0 \qquad \Longrightarrow \qquad \mathrm{d}(\boldsymbol{T} \cdot \boldsymbol{u}) = -\boldsymbol{T} \curlywedge \nabla\boldsymbol{u} \tag{82}$$

Then

$$\begin{aligned} 0 &= \mathrm{D}\boldsymbol{T} \\ &= \mathrm{D}\big(-e\, \mathrm{d}^3_t \otimes \mathrm{d}t - q^i\, \mathrm{d}^3_i \otimes \mathrm{d}t + p_j\, \mathrm{d}^3_t \otimes \mathrm{d}x^j + \pi^i_j\, \mathrm{d}^3_i \otimes \mathrm{d}x^j\big) \\ &= -\partial_t e\; \mathrm{d}^4 \otimes \mathrm{d}t - \partial_i q^i\; \mathrm{d}^4 \otimes \mathrm{d}t + \partial_t p_j\; \mathrm{d}^4 \otimes \mathrm{d}x^j + \partial_i \pi^i_j\; \mathrm{d}^4 \otimes \mathrm{d}x^j - \\ &\quad \Big[e\, \mathit{\Gamma}^t_{tt}\, \mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t + e\, \mathit{\Gamma}^t_{tj}\, \mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0 \\ &\quad q^i\, \mathit{\Gamma}^t_{it}\, \mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t + q^i\, \mathit{\Gamma}^t_{ij}\, \mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j + 0 \\ &\quad - p_j\, \mathit{\Gamma}^j_{tt}\, \mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}t - p_k\, \mathit{\Gamma}^k_{tj}\, \mathrm{d}^3_t \wedge \mathrm{d}t \otimes \mathrm{d}x^j + 0 \\ &\quad - \pi^i_k\, \mathit{\Gamma}^k_{it}\, \mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}t - \pi^i_k\, \mathit{\Gamma}^k_{ij}\, \mathrm{d}^3_i \wedge \mathrm{d}x^i \otimes \mathrm{d}x^j\Big] \\ &= \mathrm{d}^4 \otimes \Big[\\ &\quad - \partial_t e\; \mathrm{d}t + e\, \mathit{\Gamma}^t_{tt}\, \mathrm{d}t + e\, \mathit{\Gamma}^t_{tj}\, \mathrm{d}x^j \\ &\quad - \partial_i q^i\, \mathrm{d}t + q^i\, \mathit{\Gamma}^t_{it}\, \mathrm{d}t + q^i\, \mathit{\Gamma}^t_{ij}\, \mathrm{d}x^j \\ &\quad + \partial_t p_j\, \mathrm{d}x^j - p_j\, \mathit{\Gamma}^j_{tt}\, \mathrm{d}t - p_k\, \mathit{\Gamma}^k_{tj}\, \mathrm{d}x^j \\ &\quad + \partial_i \pi^i_j\, \mathrm{d}x^j - \pi^i_k\, \mathit{\Gamma}^k_{it}\, \mathrm{d}t - \pi^i_k\, \mathit{\Gamma}^k_{ij}\, \mathrm{d}x^j \\ &\quad \Big] \end{aligned} \tag{83}$$

The equation above corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} - p_j\,\Gamma^j_{tt} - \pi^i_k\,\Gamma^k_{it} \tag{84}$$

$$\partial_t p_j + \partial_i \pi^i_j = -e\,\Gamma^t_{tj} - q^i\,\Gamma^t_{ij} + p_k\,\Gamma^k_{tj} + \pi^i_k\,\Gamma^k_{ij} \tag{85}$$

For $\boldsymbol{T}\cdot\boldsymbol{u}$ we find first

$$\boldsymbol{T}\cdot\boldsymbol{u} = (-e\,u^t + p_j\,u^j)\,\mathrm{d}^3_t + (-q^i\,u^t + \pi^i_j\,u^j)\,\mathrm{d}^3_i \tag{86}$$

$$\begin{aligned}\boldsymbol{T} \curlywedge \nabla\boldsymbol{u} = {} & (-e\,\partial_t u^t + p_j\,\partial_t u^j)\,\mathrm{d}^3_t \wedge \mathrm{d}t + (-q^i\,\partial_i u^t + \pi^i_j\,\partial_i u^j)\,\mathrm{d}^3_i \wedge \mathrm{d}x^i \\ & + \Gamma \text{ terms}\end{aligned} \tag{87}$$

and therefore

$$\begin{aligned}\partial_t(-e\,u^t + p_j\,u^j) + \partial_i(-q^i\,u^t + \pi^i_j\,u^j) = {} & \\ & - e\,\partial_t u^t + p_j\,\partial_t u^j - q^i\,\partial_i u^t + \pi^i_j\,\partial_i u^j \\ & - (e\,\Gamma^t_{tt} + q^i\,\Gamma^t_{it} - p_j\,\Gamma^j_{tt} - \pi^i_k\,\Gamma^k_{it})\,u^t \\ & - (e\,\Gamma^t_{tj} + q^i\,\Gamma^t_{ij} - p_k\,\Gamma^k_{tj} - \pi^i_k\,\Gamma^k_{ij})\,u^j\end{aligned} \tag{88}$$

Radial case

$$\partial_t e + \partial_r q^r = e\,\Gamma^t_{tt} + q^r\,\Gamma^t_{rt} + p_r\,\Gamma^r_{tt} + \pi^r_r\,\Gamma^r_{rt} \tag{89}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\Gamma^t_{tr} + q^r\,\Gamma^t_{rr} + p_r\,\Gamma^r_{tr} + \pi^r_r\,\Gamma^r_{rr} \tag{90}$$

$$\partial_t e + \partial_r q^r = q^r\,\frac{g}{c^2} + p_r\,g \tag{91}$$

$$\partial_t p_r + \partial_r \pi^r_r = e\,\frac{g}{c^2} \tag{92}$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have[5]

$$\Gamma^t_{jt} = \Gamma^t_{tj} = \frac{GM}{c^2}\frac{x^j}{r^3} \approx \frac{g}{c^2} \qquad \Gamma^j_{tt} = GM\frac{x^j}{r^3} \approx g \tag{93}$$

where g is the standard acceleration, considered positive. Take also $p_j \approx m v_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

[5] Poisson & Will 2014 § 5.2.3.

The balances above become

$$\partial_t e + \partial_i q^i = q^z \frac{g}{c^2} + p_z\, g \tag{94}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\, \frac{g}{c^2} \tag{95}$$

Also,

$$\begin{aligned} \Gamma^t_{tt} &\approx -2\frac{g}{c^2}v(t) \qquad & \Gamma^t_{jt} = \Gamma^t_{tj} &\approx \frac{g}{c^2} \\ \Gamma^j_{tt} &\approx g - 2\frac{g}{c^2}\, v(t)^2 - \dot{v}(t) \qquad & \Gamma^j_{jt} = \Gamma^j_{tj} &\approx \frac{g}{c^2}v(t) \end{aligned} \tag{96}$$

$$\partial_t e + \partial_i q^i = -e\, 2\frac{g}{c^2}v(t) + q^z\, \frac{g}{c^2} + p_j \left(g - 2\frac{g}{c^2}\, v(t)^2 - \dot{v}(t)\right) + \pi^z_z\, \frac{g}{c^2}v(t) \tag{97}$$

$$\partial_t p_z + \partial_i \pi^i_z = e\, \frac{g}{c^2} + p_z\, \frac{g}{c^2}v(t) \tag{98}$$

C Works with useful content

- Eq. (21) in Maugin 1974: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in Gourgoulhon 2012

For transformation or raising: Gantmacher 2000 § I.4 eq. (33).

$$\gamma \cdot (B \wedge A) = (-1)^{\deg A \deg B}\gamma \cdot (A \wedge B) \tag{99}$$

with $\deg(B) = n - \deg(A)$ Barnabei et al. 1985 prop. 4.1

Compound matrices: Choquet-Bruhat et al. 1996 § IV.A.1 p. 199, Problem 1 p. 270

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(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

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