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Dimensional analysis in relativity and in differential geometry

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This note provides a short guide to dimensional analysis in Lorentzian and general relativity and in differential geometry. It tries to revive Dorgelo and Schouten's notion of 'intrinsic' or 'absolute' dimension of a tensorial quantity. The intrinsic dimension is independent of the dimensions of the coordinates and expresses the physical and operational meaning of a tensor. The dimensional analysis of several important tensors and tensor operations is summarized. In particular it is shown that the components of a tensor need not have all the same dimension, and that the Riemann (once contravariant and thrice covariant), Ricci (twice covariant), and Einstein (twice covariant) curvature tensors are dimensionless. The relation between dimension and operational meaning for the metric and stress-energy-momentum tensors, and the dimension of Einstein's constant are also discussed.

per la piccola Emma

1 Introduction

From the point of view of dimensional analysis, do all components of a tensor need to have the same dimension? What happens to these components if we choose coordinates that don't have dimensions of length or time? And if the components of a tensor have different dimensions, then does it make sense to speak of "the dimension of the tensor"? What are the dimensions of the metric and of the curvature tensors? What is the dimension of the constant in the Einstein equations?

A sense of insecurity often gets hold of many students (and possibly of some researchers) in relativity, when they have to discuss and answer this kind of questions. This is evident in many question & answer websites and wiki pages, where several incorrect or unfounded statements about dimensional analysis in relativity are in circulation. (For the record, the answers to the questions above are: No – See eq. (12) below – Yes – They are dimensionless – Either $\mathsf{M}^{-1}\mathsf{L}$ or $\mathsf{M}^{-1}\mathsf{L}^{-1}\mathsf{T}^{2}$, depending on the dimension you assign the metric tensor, see §§ 9–11.)

Several factors contribute to these misconceptions and insecurity. Modern texts in Lorentzian and general relativity commonly use geometrized units. They say that, for finding the dimension of some constant in a tensorial equation, it's sufficient to compare the dimensions of the terms in the equation. But the application of this procedure is sometimes not so immediate, because some tensors don't have universally agreed dimensions – prime example the metric tensor. Older texts often use four coordinates with dimension of length, and base their dimensional analyses on that specific choice. They even multiply[1] mcvittie tensorial components having dimension of time by powers of c. Such common practices therefore give students the impression that coordinates ought to always be lengths, and that all components of a tensor ought to have the same dimension. Yet, students cannot find such rules explicitly stated anywhere. We'll see shortly that no such rules in fact exist, nor are they necessary.

Dimensional analysis is thus not very self-evident in relativity and in differential geometry. The present note wants to provide a short but exhaustive guide to it. Some important dimensional-analysis questions in general relativity are also consistently settled in this note; for example the dimension of the Riemann curvature tensor, or the effect of the covariant or Lie derivatives on dimensions.

The application of dimensional analysis in relativity is most straightforward and self-evident if we rely on the coordinate-free or intrinsic approach to differential geometry, briefly recalled below, and if we adopt the perhaps overlooked notion of *intrinsic dimension* of a tensor. The intrinsic dimension of a tensor was introduced under the name ' absolute dimension ' by Schouten and Dorgelo[2] and used in Truesdell & Toupin[3]. As its name implies, this dimension is independent of the choice and dimensions of coordinate functions. It is distinct from the dimensions of the tensor's *components*, which instead depend on the dimensions of the coordinates. The intrinsic dimension of a tensor is determined by the latter's physical and operational[4] meaning. It is therefore a natural notion for dimensional analysis in relativity.

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1 e.g. Tolman 1949 p. 71 eq. (37.1); Landau & Lifshitz 1996 p. 80 eq. (32.15); Adler et al. 1975 p. 332 eq. (10.15); Penfield & Haus 1967 § 5.7; for exception see e.g. Truesdell & Toupin 1960 § F.III.280; Kitano 2013 § X. **2** Dorgelo & Schouten 1946; Schouten 1989 ch. VI. **3** Truesdell & Toupin 1960 Appendix II. **4** Bridgman 1958.

Here is a synopsis of the rest of this note. A brief reminder of the intrinsic approach to differential geometry, with references, is given in the next section, together with some special notation necessary to our discussion. Section 3 gives a simple example of dimensional analysis in a two-dimensional spacetime; I believe that this example will actually be enough for most readers to get the basic way of reasoning; such readers can work out the rest for themselves whenever they need and don't need to read the rest of this note. Sections 4–7 offer a more systematic discussion and a synopsis of dimensional analysis for the main tensorial operations. The notion of intrinsic dimension is explained in § 5. The intrinsic dimensions of the various curvature tensors, of the metric tensor, and of the stress-energy-momentum tensor are separately discussed in §§ 8–10. The (contravariant and thrice covariant) Riemann and (fully covariant) Ricci tensors, in particular, are found to have intrinsic dimension 1, that is, to be dimensionless. The operational motivation of the two standard choices for the dimension of the metric tensor are also discussed. The dimension of the constant in the Einstein equations is finally derived in § 11.

This note obviously assumes familiarity with basic tensor calculus and related notions, for example of co- and contra-variance, tensor product, contraction. Some passages assume familiarity with the exterior calculus of differential forms. The general ideas, however, should be understandable even without such familiarity.

Finally, quoting Truesdell & Toupin[5], "dimensional analysis remains a controversial and somewhat obscure subject. We do not attempt a complete presentation here". References about recent developments in this subject are given in the summary of § 12.

2 Intrinsic view of differential-geometric objects: brief reminder and notation

From the intrinsic point of view, a tensor is defined by its geometric properties. For example, a vector $\boldsymbol{v}$ is an object that operates on functions defined on the (spacetime) manifold, yielding new functions, with the properties $\boldsymbol{v}(af + bg) = a\boldsymbol{v}(f) + b\boldsymbol{v}(g)$ and $\boldsymbol{v}(fg) = \boldsymbol{v}(f)g + f\boldsymbol{v}(g)$ for all functions f, g and reals a, b. A covector $\boldsymbol{\omega}$ is an object that operates

[5] Truesdell & Toupin 1960 Appendix § 7 footnote 4.

on vector fields, yielding functions, with the property $\boldsymbol{\omega}(f\boldsymbol{u} + g\boldsymbol{v}) = f\,\boldsymbol{\omega}(\boldsymbol{u}) + g\,\boldsymbol{\omega}(\boldsymbol{v})$ for all vector fields $\boldsymbol{u}, \boldsymbol{v}$ and functions f, g. The sum of vector or covector fields (the latter also called 1-forms), and their products by functions are defined in an obvious way. Tensors are constructed from these objects; see also the end of this section for a slightly different point of view.

A system of coordinates (x^i) is just a set of linearly independent functions. This set gives rise to a set of vectors fields $\bigl(\frac{\partial}{\partial x^i}\bigr)$ and to a set of covector fields $(\mathrm{d}x^i)$ by the obvious requirements that $\frac{\partial}{\partial x^i}(x^j) = \delta_i^{\ j}$ and $\mathrm{d}x^i\bigl(\frac{\partial}{\partial x^j}\bigr) = \delta^i_{\ j}$. These two sets can be used as bases to express all other vectors and covectors as linear combinations. A vector field $\boldsymbol{v}$ can thus be written as

$$\boldsymbol{v} \equiv \sum_i v^i \frac{\partial}{\partial x^i} \equiv v^i \frac{\partial}{\partial x^i}\;, \tag{1}$$

where the *functions* $v^i := \boldsymbol{v}(x^i)$ are its components with respect to the basis $\bigl(\frac{\partial}{\partial x^i}\bigr)$. Analogously for a covector field.

For the presentation of the intrinsic view I recommend the excellent texts by Choquet-Bruhat et al. (1996), Boothby (2003), Abraham et al. (1988), and more on the general-relativity side Misner et al. (1973 ch. 9), Gourgoulhon (2012 ch. 2), Penrose & Rindler (2003 ch. 4).

For the notation in dimensional analysis I use ISO conventions:[6] $\dim(\boldsymbol{A})$ is the dimension of the quantity $\boldsymbol{A}$, and among the base quantities are mass M, length L, time T, temperature Θ, electric current I. Note that I don't discuss units – it doesn't matter here whether the unit for length is the metre or the centimetre, for example.

Throughout this note c denotes the speed of light, with $\dim(c) = \mathsf{L}\mathsf{T}^{-1}$. Its numerical quantity value $\{c\}$ depends on the chosen units of length and time.

The number, ordering, and symmetries of a tensor's covariant and contravariant "slots"[7] will often be important in our discussion. The traditional coordinate-free notation '$\boldsymbol{A}$' omits this information. We thus need a coordinate-free notation that makes it explicit. Penrose & Rindler[8] propose an abstract-index notation where '$A_i^{\ jk}$', for example, denotes

[6] ISO 2009 § 5. [7] Misner et al. 1973 § 3.2. [8] Penrose & Rindler 2003 § 2.2.

a tensor covariant in its first slot and contravariant in its second and third slots. Every index in this notation is “a *label* whose sole purpose is to keep track of the type of tensor under discussion”[9]. So this notation doesn’t stand for the set of *components* of the tensor. For the latter set, Penrose & Rindler use **bold** indices instead: ‘ $A_{\boldsymbol{i}}{}^{\boldsymbol{jk}}$ ’. But in our discussion the difference between a tensor and its set of components is crucial, and Penrose & Rindler’s abstract-index notation unfortunately lends itself to conceptual and typographic misunderstanding.

I shall therefore use a notation such as $\boldsymbol{A}_{\bullet}{}^{\bullet\bullet}$ to indicate that $\boldsymbol{A}$ is covariant in its first slot and contravariant in its second and third slots. Its components would thus be $(A_i{}^{jk})$. For brevity I’ll call this a ‘ co-contra-contra-variant ’ tensor, with an obvious naming generalization for other tensor types. A set of completely antisymmetric slots will be put within bars: thus the notation $\boldsymbol{A}^{\bullet}{}_{|\bullet\bullet|}$ means that $\boldsymbol{A}$ is completely antisymmetric in its last two covariant slots. Finally, in accord with convenient modern terminology, a completely antisymmetric contravariant tensor of rank k will be called ‘ k-vector ’; and a completely antisymmetric covariant tensor of rank k, a ‘ k-covector ’. The terms ‘ multi-vector ’ and ‘ multi-covector ’ are used when k isn’t specified.

The only weak points of the notation just explained are the operations of transposition and contraction, which literal indices depict so well instead. Considering that transposition is a generalization of matrix transposition, and contraction a generalization of trace, I’ll use the following notation:

- $\boldsymbol{A}^{\intercal\alpha\beta}$ is the transposition (swapping) of the αth and βth slots. Its coordinate-free definition is

$$(\boldsymbol{A}^{\intercal\alpha\beta})(\dots,\overset{\alpha\text{th slot}}{\zeta},\dots,\underset{\beta\text{th slot}}{\boldsymbol{\eta}},\dots) \coloneqq \boldsymbol{A}(\dots,\overset{\alpha\text{th slot}}{\boldsymbol{\eta}},\dots,\underset{\beta\text{th slot}}{\zeta},\dots) \tag{2}$$

for all $\zeta, \boldsymbol{\eta}$ of appropriate variance type.

- $\operatorname{tr}_{\alpha\beta} \boldsymbol{A}$ is the contraction of the αth and βth slots, which must have opposite variant types; note that we may have $\beta < \alpha$. Its coordinate-free

[9] Penrose & Rindler 2003 p. 75.

definition is

$$(\operatorname{tr}_{\alpha\beta} \boldsymbol{A})(\dots,\dots,\dots) := \sum_i \boldsymbol{A}(\dots, \overset{\alpha\text{th slot}}{\boldsymbol{u}_i}, \dots, \underset{\beta\text{th slot}}{\boldsymbol{\omega}^i}, \dots) \tag{3}$$

for any arbitrary complete and linearly independent sets $\{\boldsymbol{u}_i\}$, $\{\boldsymbol{\omega}^j\}$ such that $\boldsymbol{\omega}^j(\boldsymbol{u}_i) = \delta^j{}_i$.

In index notation the two operations above are the familiar

$$A_{\dots}{}^{\overset{\alpha\text{th slot}}{i}}{}_{\dots\underset{\beta\text{th slot}}{j}\dots} \mapsto A_{\dots\underset{\alpha\text{th slot}}{j}\dots}{}^{\overset{\beta\text{th slot}}{i}}{}_{\dots} \qquad \text{and} \qquad A_{\dots}{}^{\overset{\alpha\text{th slot}}{i}}{}_{\dots\underset{\beta\text{th slot}}{i}\dots} .$$

For the sake of notation economy I'll denote the contraction of the closest slots of two tensor-multiplied tensors by simple juxtaposition. For example, if $\boldsymbol{A} \equiv \boldsymbol{A}_{\bullet}{}^{\bullet}$, $\boldsymbol{B} \equiv \boldsymbol{B}_{\bullet\bullet}$, and v is a vector, then

$$\boldsymbol{AB} := \operatorname{tr}_{23}(\boldsymbol{A}_{\bullet}{}^{\bullet} \otimes \boldsymbol{B}_{\bullet\bullet}) , \qquad \boldsymbol{B}v := \operatorname{tr}_{23}(\boldsymbol{B}_{\bullet\bullet} \otimes v^{\bullet}) . \tag{4}$$

This notation makes sense considering tensors as linear operators.

Contraction and transposition will be discussed only sparsely, so I hope you won't find the notation above too uncomfortable.

It is possible to build the tensor-product architecture not on vectors and covectors, but on multi-vectors and multi-covectors, with their straight and twisted (also called 'odd' or 'axial') orientations. This elegant and powerful geometric point of view leads to deeper physical insight and is gaining popularity in the literature. For its presentation I recommend the texts of Bossavit (1991 especially ch. 3), Burke (1983; 1987; 1980; 1995), de Rham (1984 ch. 2), Schouten (1954), Deschamps (1970; 1981) and Lindell (2004).

In the notation above, the bars identify k-vectors and -covectors for $k > 1$. Thus $\boldsymbol{A}^{\bullet}{}_{|\bullet\bullet|}$ indicates that $\boldsymbol{A}$ belongs to the tensor product of 1-vectors and 2-covectors; it's also called a 1-vector-valued 2-covector. To avoid burdening the notation I won't add symbols denoting straight or twisted orientation, but I'll explicitly say in the text when any object has a twisted orientation.

3 An introductory two-dimensional example

Let me first present a simple example of dimensional analysis in a two-dimensional spacetime. I provide very little explanation, letting the analysis speak for itself. The next sections will give a longer discussion of the general point of view, of the assumptions, and of cases with more elaborate geometric objects.

In a region of a two-dimensional spacetime we use coordinates (x, y). These coordinates allow us to uniquely label every event in the region (otherwise they wouldn't be coordinates). Let us say that coordinate x has dimension of temperature, and y of specific entropy:

$$\dim(x) = \Theta\,, \quad \dim(y) = \mathrm{s} := \mathsf{L}^2\mathsf{T}^{-2}\Theta^{-1}\,. \tag{5}$$

This choice may be possible for several reasons. For example, the region could be occupied by a heat-conducting material; in a specific spacetime foliation, its temperature increases along each 1-dimensional spacelike slice, and its entropy density is uniform on each slice but increases from slice to slice.[10] Owing to this kind of monotonic behaviour for these quantities, if we are given a pair of temperature & specific-entropy values we can identify a unique event associated to them in this spacetime region. They can thus be used as a coordinate system. The point here is that coordinates can have any dimensions owing to physical reasons. In atmospheric and ocean dynamics, for example, pressure or mass density are sometimes used as coordinates for depth[11].

From these coordinates we construct two covector fields $(\mathrm{d}x, \mathrm{d}y)$, and two vector fields $\bigl(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\bigr)$ that serve as bases for the spaces of tangent covectors, vectors, and tensors. Their dimensions are

$$\begin{aligned} &\dim(\mathrm{d}x) = \Theta && \dim(\mathrm{d}y) = \mathrm{s}\,,\\ &\dim\Bigl(\frac{\partial}{\partial x}\Bigr) = \Theta^{-1} && \dim\Bigl(\frac{\partial}{\partial y}\Bigr) = \mathrm{s}^{-1}\,. \end{aligned} \tag{6}$$

Consider a contra-co-variant tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$ in this region. Using the basis fields above it can be written as

$$\boldsymbol{A} = A^x{}_x \frac{\partial}{\partial x} \otimes \mathrm{d}x + A^x{}_y \frac{\partial}{\partial x} \otimes \mathrm{d}y + A^y{}_x \frac{\partial}{\partial y} \otimes \mathrm{d}x + A^y{}_y \frac{\partial}{\partial y} \otimes \mathrm{d}y\,, \tag{7}$$

where $A^x{}_x := \boldsymbol{A}\bigl(\mathrm{d}x, \frac{\partial}{\partial x}\bigr)$ and so on are the components of the tensor in the coordinate system (x, y).

[10] For general-relativistic thermomechanics see e.g. Eckart 1940; Maugin 1974; 1978a,b,c,d; Muschik & von Borzeszkowski 2014. [11] Griffies 2004 ch. 6; Vallis 2006 § 2.6.2.

By the rules of dimensional analysis, the two sides of the expansion above must have the same dimension. The same holds for the four summands on the right side. Denoting $\mathsf{A} \coloneqq \dim(\boldsymbol{A})$, we thus have the four equations

$$\begin{aligned} \mathsf{A} &= \dim({A^x}_x) & \qquad \mathsf{A} &= \dim({A^x}_y)\,\Theta^{-1}\,\mathrm{s} \\ \mathsf{A} &= \dim({A^y}_x)\,\Theta\,\mathrm{s}^{-1} & \qquad \mathsf{A} &= \dim({A^y}_y)\;, \end{aligned}$$

or

$$\begin{aligned} \dim({A^x}_x) &= \mathsf{A} & \qquad \dim({A^x}_y) &= \mathsf{A}\,\Theta\,\mathrm{s}^{-1} \equiv \mathsf{A}\,\mathsf{L}^{-2}\mathsf{T}^2\Theta^2 \\ \dim({A^y}_x) &= \mathsf{A}\,\Theta^{-1}\,\mathrm{s} & \qquad \dim({A^y}_y) &= \mathsf{A}\;. \end{aligned} \tag{8}$$

The intrinsic dimension of the tensor $\boldsymbol{A}$ is A. The expansion (7) shows that this dimension is independent of the coordinate system, by construction – such expansion could be done in any other coordinate system, and the left side would be the same. The effect of coordinate transformations is examined more in detail in § 5. The intrinsic dimension A is determined by the physical and operational meaning of the tensor; see §§ 9, 10 for concrete examples. Together with the dimensions of the coordinates it determines the dimensions of the components, eq. (8), which need not be all equal.

This simple example should have disclosed the main points of dimensional analysis on manifolds, which will now be discussed in more generality. In the derivation above we silently adopted a couple of natural conventions; for example, that the tensor product behaves similarly to multiplication with regard to dimensions. Such conventions are briefly discussed in § 12.

4 Coordinates

From a physical point of view, a coordinate is just a function that associates values of some physical quantity with the events in a region (the domain of the coordinate chart) of spacetime. Together with the other coordinates, such function allows us to uniquely identify every event within that region. Any physical quantity will do: the distance from something, the time elapsed since something, an angle, an energy density, the strength of a magnetic flux, a temperature, and so on. A coordinate can thus have any dimension: length L, time T, angle 1, temperature Θ, magnetic flux $\Phi \coloneqq \mathsf{ML^2T^{-2}I^{-1}}$, and so on.

The functional relation between two sets of coordinates must of course be dimensionally consistent. For example, if $\dim(x^0) = \mathsf{T}$ and $\dim(x^1) = \mathsf{L}$, and we introduce a coordinate $\xi(x^0, x^1)$ with dimension 1, additive in the previous two, then we must have $\xi = ax^0 + bx^1$ with $\dim(a) = \mathsf{T}^{-1}$ and $\dim(b) = \mathsf{L}^{-1}$.

5 Tensors: intrinsic dimension and components' dimensions

Consider a system of coordinates (x^i) with dimensions (X_i), and the ensuing sets of covector fields (1-forms) $\mathrm{d}x^i$ and of vector fields $\bigl(\frac{\partial}{\partial x^i}\bigr)$, bases for the cotangent and tangent spaces. Their tensor products are bases for the tangent spaces of higher tensor types.

The differential $\mathrm{d}x^i$ traditionally has the same dimension as x^i: $\dim(\mathrm{d}x^i) = \mathsf{X}_i$, and the vector $\frac{\partial}{\partial x^i}$ traditionally has the inverse dimension: $\dim\bigl(\frac{\partial}{\partial x^i}\bigr) = {\mathsf{X}_i}^{-1}$.

For our discussion let's take a concrete example: a contra-co-variant tensor field $\boldsymbol{A} \equiv \boldsymbol{A}^{\bullet}{}_{\bullet}$. The discussion generalizes to tensors of other types in an obvious way.

The tensor $\boldsymbol{A}$ can be expanded in terms of the basis vectors and covectors, as in § 2 and in the example of § 3:

$$\boldsymbol{A} = A^i{}_j \, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \equiv A^0{}_0 \, \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^0 + A^0{}_1 \, \frac{\partial}{\partial x^0} \otimes \mathrm{d}x^1 + \cdots . \tag{9}$$

Each function

$$A^i{}_j \coloneqq \boldsymbol{A}\Bigl(\mathrm{d}x^i, \frac{\partial}{\partial x^j}\Bigr) \tag{10}$$

is a component of the tensor in this coordinate system.

To make dimensional sense, all terms in the sum (9) must have the same dimension. This is possible only if the generic component $A^i{}_j$ has dimension

$$\dim\bigl(A^i{}_j\bigr) = \mathsf{A}\, \mathsf{X}_i \, {\mathsf{X}_j}^{-1}, \tag{11}$$

where A is common to all components. In fact, the $\mathsf{X}_i \, {\mathsf{X}_j}^{-1}$ term cancels the ${\mathsf{X}_i}^{-1}\, \mathsf{X}_j$ term coming from $\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ in the sum (9), and each summand therefore has dimension A.

The generalization of the formula above to tensors of other types is obvious:

$$\dim\bigl(A^{ij\dots}{}_{kl\dots}\bigr) = \mathsf{A}\,\mathsf{X}_i\,\mathsf{X}_j \cdots \mathsf{X}_k{}^{-1}\,\mathsf{X}_l{}^{-1} \cdots \qquad \text{where the ordering of the indices doesn't matter.} \tag{12}$$

Clearly the components can have different dimensions[12]. But this isn't an issue. What matters is that the sum (9) be dimensionally consistent.

The dimension A, which is also the dimension of the sum (9), I'll call the *intrinsic dimension* of the tensor $\boldsymbol{A}$, and we write

$$\dim(\boldsymbol{A}) = \mathsf{A}. \tag{13}$$

This dimension is independent of any coordinate system. It reflects the physical or operational[13] meaning of the tensor. We shall see an example of such an operational analysis in §§ 9 and 10 for the metric and stress-energy-momentum tensors.

The notion of intrinsic dimension was introduced by Dorgelo and Schouten[14] under the name 'absolute dimension'. I find the adjective 'intrinsic' more congruous to modern terminology and less prone to suggest spurious connections with absolute values.

Different coordinate systems lead to different dimensions of the *components* of a tensor $\boldsymbol{A}$, but the intrinsic dimension of the tensor remains the same. Formula (12) for the dimensions of the components is consistent under changes of coordinates. For example, in new coordinates $(\bar{x}^k)$ with dimensions $(\bar{\mathsf{X}}_k)$, the new components of $\boldsymbol{A}$ are

$$\bar{A}^k{}_l = A^i{}_j\,\frac{\partial \bar{x}^k}{\partial x^i}\,\frac{\partial x^j}{\partial \bar{x}^l} \tag{14}$$

and a quick check shows that $\dim(\bar{A}^k{}_l) = \mathsf{A}\,\bar{\mathsf{X}}_k\,\bar{\mathsf{X}}_l^{-1}$, consistently with the general formula (12).

In the following I'll drop the adjective 'intrinsic' when it is clear from the context.

If in eq. (14), relating intrinsic and component dimensions, we have $\mathsf{X}_1 = \mathsf{X}_2 = \ldots = \mathsf{X}$, then all components also have equal dimensions. So *if*

12 see Fokker 1965 § VII.1 p. 88 for an example of metric tensor with components having different dimensions.
13 Bridgman 1958; see also Synge 1960a § A.2; Truesdell & Toupin 1960 §§ A.3–4.
14 Dorgelo & Schouten 1946; Schouten 1989 ch. VI.

we use a system of coordinates having equal dimensions, the components of any tensor must also have equal dimensions. This justifies the common practice in the literature.

Choosing coordinates of different dimensions, however, has several advantages. First, it allows us to use dimensional analysis as a heuristic tool to determine the variance type of a tensor; we'll see an example in § 10. Second, it can lead to components with familiar dimensions. For example, if we use a timelike coordinate of dimension T and spacelike coordinates of dimension L (without c factors), then the timelike and spacelike components of the charge-current form have dimensions $\mathsf{I}\mathsf{L}^{-3}\mathsf{T}$ and $\mathsf{I}\mathsf{L}^{-2}$ – the familiar charge density and surface current density.

6 Tensor operations

By the reasoning of the previous section, which simply applies standard dimensional considerations to the basis expansion (9), it's easy to find the resulting intrinsic dimension of various operations and operators on tensors and tensor fields.

Here is a summary of the dimensional rules for the main differential-geometric operations and operators, except for the covariant derivative, the metric, and related tensors, discussed more in depth in §§ 8–9 below. Some of these rules are actually definition or conventions, as briefly discussed in their description. The others can be proved; I only give proof for some of them, leaving the other proofs as an exercise. For reference, in brackets I give the section of Choquet-Bruhat et al. (1996) where these operations are defined.

- The *tensor product* [III.B.5] multiplies dimensions:

$$\dim(\boldsymbol{A} \otimes \boldsymbol{B}) = \dim(\boldsymbol{A}) \dim(\boldsymbol{B}). \tag{15}$$

This is actually a definition or convention. We tacitly used this rule already in the example of § 3 and in § 5 for the coordinate expansion (9). It is a natural definition, because for tensors of order 0 (functions) the tensor product is just the ordinary product, and the dimension of a product is the product of the dimensions. This definition doesn't lead to inconsistencies.

• The *contraction* [III.B.5] or trace of the αth and βth slots of a tensor has the same dimension as the tensor:

$$\dim(\operatorname{tr}_{\alpha\beta} \boldsymbol{A}) = \dim(\boldsymbol{A}). \tag{16}$$

Note that the formula above only holds *without raising or lowering indices*; see § 9 for those operations.

This operation can be traced back to the duality of vectors and covectors mentioned in § 2: a covector field $\boldsymbol{\omega}$ operates linearly on a vector field $\boldsymbol{v}$ to yield a function $f = \boldsymbol{\omega}(\boldsymbol{v})$. Also in this case we have that $\dim(f) = \dim(\boldsymbol{\omega})\dim(\boldsymbol{v})$ by definition or convention, and the rule (16) follows from this convention. Also in this case this convention seems very natural, owing to the linearity properties of the trace, and doesn't lead to inconsistencies.

• The *transposition*[15] of the αth and βth slots of a tensor has the same dimension as the tensor:

$$\dim(\boldsymbol{A}^{\top\alpha\beta}) = \dim(\boldsymbol{A}). \tag{17}$$

• The *Lie bracket* [III.B.3] of two vectors has the product of their dimensions:

$$\dim([\boldsymbol{u}, \boldsymbol{v}]) = \dim(\boldsymbol{u})\dim(\boldsymbol{v}). \tag{18}$$

In fact, in coordinates (x^i) the bracket can be expressed as

$$[\boldsymbol{u}, \boldsymbol{v}] = \left(u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j}\right) \frac{\partial}{\partial x^i}, \tag{19}$$

and equating the dimensions of the left and right sides, considering that

$$\dim(u^i) = \dim(\boldsymbol{u})\,\mathsf{X}_i, \quad \dim(v^i) = \dim(\boldsymbol{v})\,\mathsf{X}_i, \tag{20}$$

we find again that all X terms cancel out, leaving the result (18).

• The *pull-back* [III.A.2], *tangent map* [III.B.1] and *push-forward* of a map F between manifolds don't change the dimensions of the tensors they map. The reason, evident from their definitions, is that they all rest on the pull-back of functions: $F^*(f) \coloneqq f \circ F$. Being a composition, the pull-back of a function has the same dimension of the function.

15 called "building an isomer" by Schouten 1954 § I.3 p. 13; 1989 § II.4 p. 20.

• The *Lie derivative* [III.C.2] of a tensor with respect to a vector field has the product of the dimensions of the tensor and of the vector:

$$\dim(\mathsf{L}_{v}\boldsymbol{A}) = \dim(v)\dim(\boldsymbol{A}). \tag{21}$$

Regarding operations and operators on differential forms:

• The *exterior product* [IV.A.1] of two differential forms multiplies their dimensions:

$$\dim(\boldsymbol{\omega}\wedge\boldsymbol{\tau}) = \dim(\boldsymbol{\omega})\dim(\boldsymbol{\tau}). \tag{22}$$

• The *interior product* [IV.A.4] (also called 'dual' or 'dot' product) of a vector and a form multiplies their dimensions:

$$\dim(v\,\lrcorner\,\boldsymbol{\omega}) = \dim(v)\dim(\boldsymbol{\omega}). \tag{23}$$

This equation also holds for the generalized inner product[16] of a multi-vector v and a multi-covector $\boldsymbol{\omega}$.

• The *exterior derivative* [IV.A.2] of a form has the same dimension of the form:

$$\dim(\mathrm{d}\boldsymbol{\omega}) = \dim(\boldsymbol{\omega}). \tag{24}$$

This can be proven using the identity $\mathrm{d}\circ(v\lrcorner) + (v\lrcorner)\circ\mathrm{d} = \mathsf{L}_{v}$ or similar identities[17] together with eqs (21) and (23).

• The *integral* [IV.B.1] of a form over a submanifold (or more generally a chain) M has the same dimension as the form:

$$\dim\Bigl(\int_{M}\boldsymbol{\omega}\Bigr) = \dim(\boldsymbol{\omega}). \tag{25}$$

The reason is that the integral of a form over a submanifold or chain ultimately rests on the standard definition of integration on the real line[18], which satisfies the dimensional rule above. In fact, the integral is invariant with respect to reparameterizations of the chain; it depends only on its image (some texts[19] even define chains as equivalence classes determined by their image).

All rules above extend in obvious ways to tensor densities, and apply regardless whether the objects have straight or twisted orientation.

16 Deschamps 1970; 1981 Appendices; Lindell 2004; Truesdell & Toupin 1960 § F.I.267; Misner et al. 1973 Box 4.1, p. 92, item 4; see also Porta Mana 2019a. **17** Curtis & Miller 1985 ch. 9 p. 180 Theorem 9.78; Abraham et al. 1988 § 6.4 Theorem 6.4.8. **18** e.g. Choquet-Bruhat et al. 1996 §§ IV.B.1–2; de Rham 1984 § 5 p. 21, § 6 p. 24; Abraham et al. 1988 § 7.1; Boothby 2003 § VI.2. **19** e.g. Martin 2004 § 10.4 p. 297; Fecko 2006 § 7.3.

7 Curves and integral curves, 4-velocity

Consider a curve into spacetime, $C\colon s \mapsto P(s)$, with the parameter s having dimension $\dim(s) = \mathsf{S}$.

If we consider the events of the spacetime manifold as dimensionless quantities, then the dimension of the tangent or velocity vector $\dot{C}$ to the curve is

$$\dim(\dot{C}) = \mathsf{S}^{-1}\ , \tag{26}$$

owing to the definition[20]

$$\dot{C} \coloneqq \frac{\partial(x^i \circ C)}{\partial s}\,\frac{\partial}{\partial x^i}\ . \tag{27}$$

This has a quirky interesting consequence. Given a vector field $\boldsymbol{v}$ we say that C is an integral curve for it if

$$\boldsymbol{v} = \dot{C} \tag{28}$$

at all events $C(s)$ in the image of the curve (or more precisely $\boldsymbol{v}_{C(s)} = \dot{C}_{C(s)}$ in usual differential-geometric notation[21]). From the point of view of dimensional analysis this definition can only be valid if $\boldsymbol{v}$ has dimension S^{-1}. If $\boldsymbol{v}$ and s^{-1} have different dimensions – a case which could happen for physical reasons – the condition (27) must be modified into $\boldsymbol{v} = k\dot{C}$, where k is a possibly dimensionful constant. This is equivalent to considering an affine and dimensional reparameterization of C.

The worldline of a small body is a curve into spacetime parameterized by some time t (usually but not necessarily the proper time; see § 9), with $\dim(t) = \mathsf{T}$. The 4-velocity of the body at a given spacetime event is the tangent vector to the worldline thus parameterized. From eq. (26) it follows that *the 4-velocity has intrinsic dimension* T^{-1}.

You might find this result counter-intuitive. Probably the reason is that you're thinking in terms of components. There is no contradiction: if we use a system of coordinates with dimensions L and T, then the components of the 4-velocity have dimensions L/T and 1. But the fact that the intrinsic dimension is T^{-1}, with no lengths involved, is quite sensible. We have equipped the body with some kind of clock, so we can say how much time has passed between two spacetime events intersected by the

[20] Choquet-Bruhat et al. 1996 § III.B.1; Boothby 2003 § IV.(1.9). [21] Choquet-Bruhat et al. 1996 § III.B.1.

body's worldline. But we cannot say what their distance is, because no metric has been introduced yet.

If we introduce a metric tensor, The *norm* of the 4-velocity, calculated using the metric tensor, has dimension LT^{-1} or 1, depending on the dimension chosen for the metric tensor; see § 9.

8 Connection, covariant derivative, curvature tensors

Consider an arbitrary connection[22] with covariant derivative ∇. For the moment we don't assume the presence of any metric structure.

The covariant derivative of the product $f\boldsymbol{v}$ of a function and a vector satisfies[23]

$$\nabla(f\boldsymbol{v}) = \mathrm{d}f \otimes \boldsymbol{v} + f\nabla\boldsymbol{v}. \tag{29}$$

The first summand, from formulae (24) and (15), has dimension $\dim(f)\dim(\boldsymbol{v})$; for dimensional consistency this must also be the dimension of the second summand. Thus

$$\dim(\nabla\boldsymbol{v}) = \dim(\boldsymbol{v}). \tag{30}$$

It follows that the *directional* covariant derivative $\nabla_{\boldsymbol{u}}$ has dimension

$$\dim(\nabla_{\boldsymbol{u}}\boldsymbol{v}) = \dim(\boldsymbol{u})\dim(\boldsymbol{v}), \tag{31}$$

and by its derivation properties[24] we see that formula (30) extends from vectors to tensors of arbitrary type.

In the coordinate system (x^i), the action of the covariant derivative is carried by the *connection coefficients* or Christoffel symbols $(\mathit{\Gamma}^i_{\ jk})$ defined by

$$\nabla\frac{\partial}{\partial x^k} = \mathit{\Gamma}^i_{\ jk}\ \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i}\ . \tag{32}$$

From this equation and eqs (15), (30) it follows that these coefficients have dimensions

$$\dim(\mathit{\Gamma}^i_{\ jk}) = \mathsf{X}_i\,\mathsf{X}_j{}^{-1}\,\mathsf{X}_k{}^{-1}\ . \tag{33}$$

[22] Choquet-Bruhat et al. 1996 § V.B.
[23] Choquet-Bruhat et al. 1996 § V.B.1.
[24] Choquet-Bruhat et al. 1996 § V.B.1 p. 303.

The *torsion* $\boldsymbol{\tau}^{\bullet}{}_{|\bullet\bullet|}$, *Riemann curvature* $\boldsymbol{Rie}^{\bullet}{}_{\bullet|\bullet\bullet|}$, and *Ricci curvature* $\boldsymbol{Ric}_{\bullet\bullet}$ tensors are defined by[25]

$$\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}) \coloneqq \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}]\,, \tag{34}$$

$$\boldsymbol{Rie}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) \coloneqq \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}\,, \tag{35}$$

$$\boldsymbol{Ric}_{\bullet\bullet} \coloneqq \operatorname{tr}_{13}\boldsymbol{Rie}^{\bullet}{}_{\bullet|\bullet\bullet|}\,. \tag{36}$$

From these definitions and the results of § 6 we find the dimensional requirements

$$\dim\bigl(\boldsymbol{\tau}^{\bullet}{}_{|\bullet\bullet|}\bigr)\dim(\boldsymbol{u})\dim(\boldsymbol{v}) = \dim(\boldsymbol{u})\dim(\boldsymbol{v})\,, \tag{37}$$

$$\dim\bigl(\boldsymbol{Rie}^{\bullet}{}_{\bullet|\bullet\bullet|}\bigr)\dim(\boldsymbol{w})\dim(\boldsymbol{u})\dim(\boldsymbol{v}) = \dim(\boldsymbol{w})\dim(\boldsymbol{u})\dim(\boldsymbol{v})\,, \tag{38}$$

$$\dim(\boldsymbol{Ric}_{\bullet\bullet}) = \dim\bigl(\boldsymbol{Rie}^{\bullet}{}_{\bullet|\bullet\bullet|}\bigr)\,, \tag{39}$$

which imply that *torsion, Riemann curvature, and Ricci curvature tensors are dimensionless*:

$$\dim\bigl(\boldsymbol{\tau}^{\bullet}{}_{|\bullet\bullet|}\bigr) = \dim\bigl(\boldsymbol{Rie}^{\bullet}{}_{\bullet|\bullet\bullet|}\bigr) = \dim(\boldsymbol{Ric}_{\bullet\bullet}) = 1\,. \tag{40}$$

This result is sensible, because the notion of local parallelism, which these tensors express, doesn't involve any notion of distance or angle[26]. The exact contra- and co-variant primitive type of these tensors is very important in the equations above. If a metric tensor is also introduced and used to raise or lower any indices of these tensors, the resulting tensors will have different dimensions; see next section, especially eq. (48).

Misner et al.[27] say that "curvature", by which they seem to mean the Riemann tensor, has dimension L^{-2}. This statement is seemingly at variance with the dimensionless results (40). But I believe that Misner et al. refer to the *components* of the Riemann tensor in specific coordinates of dimension L and using geometrized units. In such specific coordinates every *component* $R^{i}{}_{jkl}$ does indeed have dimension L^{-2}, according to the general formula (12), if and only if the intrinsic dimension of $\boldsymbol{Rie}$ is unity, $\dim(\boldsymbol{Rie}) = 1$. So I believe that Misner et al.'s statement actually agrees with the results (40). This possible misunderstanding shows the importance of distinguishing between the intrinsic dimension, which doesn't depend on any specific coordinate choice, and component dimensions, which do.

25 Choquet-Bruhat et al. 1996 § V.B.1. **26** cf. Porta Mana 2019b. **27** Misner et al. 1973 p. 35.

The formulae above are also valid if a metric is defined and the connection is compatible with it. The connection coefficients in this case are defined in terms of the metric tensor, but using the results of § 9 it's easy to see that eqs (30), (31), (33), (40) still hold.

9 Metric and related tensors and operations

Let us now consider a metric tensor $\boldsymbol{g}_{\bullet\bullet}$. What is its intrinsic dimension $\dim(\boldsymbol{g})$? There seem to be two choices in the literature; both can be derived from the operational meaning of the metric.

Consider a (timelike) worldline $s \mapsto C(s)$, $s \in [a, b]$, between events $C(a)$ and $C(b)$. The metric tells us the *proper time* $\triangle t$ elapsed for an observer having that worldline, according to the formula

$$\triangle t = \int_a^b \sqrt{\left|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]\right|}\, \mathrm{d}s \;. \tag{41}$$

From the results of § 6 this formula implies that $\mathsf{T} \equiv \dim(\triangle t) = \sqrt{\dim(\boldsymbol{g}_{\bullet\bullet})}$, independently of the dimension of s, and therefore

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{T}^2. \tag{42}$$

Most authors[28], however, prefer to include a dimensional factor $1/c$ in the definition (41):

$$\triangle t = \int_a^b \frac{1}{c}\sqrt{\left|\boldsymbol{g}[\dot{C}(s), \dot{C}(s)]\right|}\, \mathrm{d}s \;, \tag{43}$$

thus obtaining

$$\dim(\boldsymbol{g}_{\bullet\bullet}) = \mathsf{L}^2. \tag{44}$$

The choice (44) seems also supported by the traditional expression for the "line element ds^2" as it appears in many works:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \;, \tag{45}$$

possibly with opposite signature; for an exception with dimension T^2 see Kilmister[29]. If the coordinates (t, x, y, z) have the dimensions suggested

28 e.g. Fock 1964 § V.62 eq. (62.02); Curtis & Miller 1985 ch. 11 eq. (11.21); Rindler 1986 § 5.3 eq. (5.6); Hartle 2003 ch. 6 eq. (6.24). **29** Kilmister 1973 ch. II p. 25.

by their symbols, this formula has dimension L^2, so that if we interpret "ds^2" as $\boldsymbol{g}$ we find $\dim(\boldsymbol{g}) = \mathsf{L}^2$. The line-element expression above often has an ambiguous differential-geometric meaning, however, because it may also represent the metric applied to some *unspecified* vector, that is, $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v})$, where $\boldsymbol{v}$ is left unspecified[30]. In this case we have

$$\mathsf{L}^2 = \dim(\boldsymbol{g})\ \dim(\boldsymbol{v})^2$$

and the dimension of $\boldsymbol{g}$ is ambiguous or undefined, because the vector $\boldsymbol{v}$ could have any dimension.

The standard choices for $\dim(\boldsymbol{g})$ are thus L^2 or T^2. The second choice has some merit for reasons discussed by Synge and Bressan[31]. Synge gives a vivid summary:[32]

> We are now launched on the task of giving physical meaning to the Riemannian geometry [...]. It is indeed a Riemannian *chronometry* rather than *geometry*, and the word *geometry*, with its dangerous suggestion that we should go about measuring *lengths* with *yardsticks*, might well be abandoned altogether in the present connection

In fact, to measure the proper time $\triangle t$ defined above we only need to ensure that a clock has the worldline C, and then take the difference between the clock's final and initial times. On the other hand, consider the case when the curve C is *spacelike*. Its proper length is still defined by the integral (41) apart from a dimensional constant. Its measurement, however, is more involved than in the timelike case. It requires dividing the curve into very short pieces, and having specially-chosen observers (orthogonal to the pieces) measure each piece. But the measurement of each piece actually relies on the measurement of *proper time*: each observer uses radar distance[33], sending a lightlike signal which bounces back at the end of the piece, and measuring the time it takes to come back. Even if rigid rods are used, their calibration still relies on a measurement of time – this is also reflected in the current definition of the standard metre[34].

The first choice, $\dim(\boldsymbol{g}) = \mathsf{L}^2$, seems by far the most common, however. It has the merit that the projection of the metric onto a spacelike hypersurface also has dimension L^2, which is sensible from a Newtonian point

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30 cf. Misner et al. 1973 Box 3.2 D p. 77. **31** Synge 1960b §§ III.2–4; Bressan 1978 §§ 15, 18. **32** Synge 1960b § III.3 pp. 108–109. **33** Frankel 1979 ch. 2; Landau & Lifshitz 1996 § 84. **34** BIPM 1983 p. 98; Giacomo 1984 p. 25.

of view; and such projections are at the heart of the 3 + 1 formulations of general relativity[35]. In the following we shall see how both choices affect some dimensional results.

The metric $\boldsymbol{g}$ can be considered as an operator mapping vectors $\boldsymbol{v}$ to covectors $\boldsymbol{\omega}$, which we can compactly write as $\boldsymbol{\omega} = \boldsymbol{g}\boldsymbol{v}$ instead of the cumbersome $\boldsymbol{\omega} = \mathrm{tr}_{23}(\boldsymbol{g} \otimes \boldsymbol{v})$. The *inverse metric tensor* $\boldsymbol{g}^{-1\bullet\bullet}$ is then defined by the formula

$$\boldsymbol{g}\,\boldsymbol{g}^{-1} = \mathbf{id}_{\bullet}{}^{\bullet}\,, \tag{46}$$

where $\mathbf{id}_{\bullet}{}^{\bullet} \colon \boldsymbol{\omega} \mapsto \boldsymbol{\omega}$ is the dimensionless identity operator (also a tensor) on the cotangent space. Hence

$$\dim(\boldsymbol{g}^{-1}) = \dim(\boldsymbol{g})^{-1}\,. \tag{47}$$

The operation of *raising or lowering an index* of a tensor represents a contraction of the tensor product of that tensor with the metric or the metric inverse, for example $\underline{\boldsymbol{A}}_{\bullet\bullet} \coloneqq \boldsymbol{g}\boldsymbol{A} \equiv \mathrm{tr}_{23}(\boldsymbol{g}_{\bullet\bullet} \otimes \boldsymbol{A}^{\bullet}{}_{\bullet})$ from the mixed tensor $\boldsymbol{A}^{\bullet}{}_{\bullet}$, and similarly for tensors of other types. Therefore every lowering of a tensor's index multiplies its dimension by $\dim(\boldsymbol{g})$, and every rising divides it by $\dim(\boldsymbol{g})$:

$$\begin{aligned} \dim(\underline{\boldsymbol{A}}_{\dots\bullet\dots}) &= \dim(\boldsymbol{A}_{\dots}{}^{\bullet}{}_{\dots})\,\dim(\boldsymbol{g}) \\ \dim(\overline{\boldsymbol{B}}_{\dots}{}^{\bullet}{}_{\dots}) &= \dim(\boldsymbol{B}_{\dots\bullet\dots})\,\dim(\boldsymbol{g})^{-1}\,. \end{aligned} \tag{48}$$

A metric tensor on a four-dimensional spacetime determines a unique twisted 4-covector field or 4-form, called the volume element[36]. It can be seen as a completely antisymmetric tensor of rank 4 (whose coordinate transformation includes the sign of the Jacobian), with only one non-zero component equal to the square root of the determinant of the components (g_{ij}) of the metric:

$$\sqrt{|\det(g_{ij})|}\;\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,,$$

where $\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ actually has a twisted orientation. Note a non-vanishing volume element can be defined on orientable and non-orientable manifolds alike. From the results of § 6 it can be shown that

[35] Gourgoulhon 2012; Alcubierre 2008; Misner et al. 1973 ch. 21. I thank I. Bengtsson for this remark. [36] De Rham 1984 § V.24; Choquet-Bruhat et al. 1996 § V.A.4; Abraham et al. 1988 § 6.2.

the 4-form above has intrinsic dimension $\dim(\boldsymbol{g})^2$ (in an n-dimensional spacetime it has dimension $\dim(\boldsymbol{g})^{n/2}$). It's convenient to multiply it by a power of c, and to define the *proper volume element* $\boldsymbol{\gamma}_{|\bullet\bullet\bullet\bullet|}$ as follows:

$$\boldsymbol{\gamma} := \begin{cases} \frac{1}{c}\sqrt{|\det(g_{ij})|}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ c^3\sqrt{|\det(g_{ij})|}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \end{cases} \quad \text{if } \dim(\boldsymbol{g}) := \begin{cases} \mathsf{L}^2 \\ \mathsf{T}^2 \end{cases}. \tag{49}$$

As a consequence we have

$$\dim\bigl(\boldsymbol{\gamma}_{|\bullet\bullet\bullet\bullet|}\bigr) = \mathsf{L}^3\mathsf{T} \tag{50}$$

independently of whether $\dim(\boldsymbol{g})$ equals L^2 or T^2. This convention has several advantages; it moreover implies that the hypervolume of a four-dimensional region, given by the integral of $\boldsymbol{\gamma}$ (see § 6), also has dimension $\mathsf{L}^3\mathsf{T}$ – which is a reasonable result for a *spacetime* region.

In general the metric $\boldsymbol{g}$ induces volume, area, and line elements on three-, two-, and one-dimensional regions. It is convenient to multiply these elements by appropriate powers of c so that the region's volume has intuitive dimensions; such as L^3 for a spacelike three-dimensional region and $\mathsf{L}^2\mathsf{T}$ for a timelike one. Indeed the definition of proper time (43) is doing exactly this, including a factor $1/c$ in the induced line element on a timelike curve.

The proper volume element appears in the definition of the *star operator* on covectors and forms. This operator acts by first rising all indices of a covector and then taking the generalized inner product (see § 6) with the proper volume element. For example, for a 2-covector $\omega \equiv \omega_{\bullet\bullet}$,

$$*\,\omega := (\boldsymbol{g}^{-1}\omega\boldsymbol{g}^{-1}) \rfloor \boldsymbol{\gamma}\,. \tag{51}$$

From this definition it is clear that the star operator's effect on the dimension depends on the degree of the form it operates on[37].

The *inverse proper volume element* is the 4-vector field $\boldsymbol{\gamma}^{-1}$, with twisted orientation, having unit generalized inner product with the proper volume element: $\boldsymbol{\gamma}^{-1} \rfloor \boldsymbol{\gamma} = 1$. Its intrinsic dimension is therefore

$$\dim\bigl(\boldsymbol{\gamma}^{-1|\bullet\bullet\bullet\bullet|}\bigr) = \dim(\boldsymbol{\gamma})^{-1} \equiv \mathsf{L}^{-3}\mathsf{T}^{-1} \tag{52}$$

again independently of whether $\dim(\boldsymbol{g})$ equals L^2 or T^2.

37 I personally prefer to avoid the star operator and explicitly use the inner product with the proper volume element; cf. Bossavit 1991 §§ 4.1–2.

Note that the inverse proper volume element is dimensionally and numerically different from the tensor $\overline{\boldsymbol{\gamma}}$ obtained by raising all indices of $\boldsymbol{\gamma}$. It can be proven, for example using an orthonormal basis, that the relation between the two is

$$\overline{\boldsymbol{\gamma}} = -\frac{1}{c^2}\boldsymbol{\gamma}^{-1} . \tag{53}$$

More generally the negative factor is replaced with $(-1)^s$, where s is the number of negative eigenvalues of the metric tensor.

In a rectangular Cartesian coordinate system (x^0, x^1, x^2, x^3) with dimensions $(\mathsf{T}, \mathsf{L}, \mathsf{L}, \mathsf{L})$, adapted to an inertial (freely falling) observer at a specific spacetime event, the components of the metric tensor $\boldsymbol{g}$ with $\dim(\boldsymbol{g}) = \mathsf{L}^2$ and of the proper volume element $\boldsymbol{\gamma}$ at that event are

$$(g_{ij}) = \begin{pmatrix} -c^2 & 0\,\mathrm{ms}^{-1} & 0\,\mathrm{ms}^{-1} & 0\,\mathrm{ms}^{-1} \\ 0\,\mathrm{ms}^{-1} & 1 & 0 & 0 \\ 0\,\mathrm{ms}^{-1} & 0 & 1 & 0 \\ 0\,\mathrm{ms}^{-1} & 0 & 0 & 1 \end{pmatrix} , \qquad \gamma_{0123} = 1 . \tag{54}$$

the units of the zero components above are arbitrary of course, as long as they have dimension $\mathsf{L}\mathsf{T}^{-1}$. Many authors use a metric with opposite sign.

The formulae for the covariant derivative (30), connection coefficients (33), and curvature tensors (40) remain valid for a connection compatible with the metric. In this case the connection coefficients can be obtained from the metric by the formulae[38]

$$\Gamma^i_{\ jk} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} g_{jl} + \frac{\partial}{\partial x^j} g_{kl} - \frac{\partial}{\partial x^l} g_{jk}\right) g^{li} , \tag{55}$$

and it's easily verified that the dimensions of these coefficients given in eq. (33) still hold, as do The results for the curvature tensors (40).

The *scalar curvature* R and the co-co-variant *Einstein tensor* $\boldsymbol{G}_{\bullet\bullet}$

$$R \coloneqq \operatorname{tr}(\boldsymbol{Ric}\,\boldsymbol{g}^{-1}) , \qquad \boldsymbol{G}_{\bullet\bullet} \coloneqq \boldsymbol{Ric} - \tfrac{1}{2} R\,\boldsymbol{g} \tag{56}$$

38 Choquet-Bruhat et al. 1996 § V.B.2.

have dimensions

$$\dim(R) = \dim(\boldsymbol{g})^{-1} \equiv \begin{cases} \mathsf{T}^{-2} \\ \mathsf{L}^{-2} \end{cases} \qquad \text{if } \dim(\boldsymbol{g}) \coloneqq \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases} \quad , \tag{57}$$

$$\dim(\boldsymbol{G}_{\bullet\bullet}) = 1 \ , \tag{58}$$

that is, *the twice covariant Einstein tensor is dimensionless, independently of the dimension of the metric tensor*.

10 Stress-energy-momentum tensor and four-momentum

The operational meaning of the stress-energy-momentum tensor is still surrounded by some mystery and will be briefly discussed later. Let's first try to find the dimension of this tensor with a heuristic approach, which also shows the usefulness of intrinsic dimensional analysis on differential manifolds.

The stress-energy-momentum tensor for a material continuum at a spacetime event embodies the energy density (including rest energy) ϵ, surface energy-flux density q_r (comprising convected energy density, heating, and working), momentum density p_r, and stress (including convected momentum density) σ_{sr} of the material at that event. Here the indices $r, s \in \{x, y, z\}$, and the vertical positions of the indices do *not* denote any variance type. These quantities are measured by an inertial observer at that event using a system of one timelike and three spacelike coordinates (t, x, y, z). If these coordinates have dimensions $(\mathsf{T}, \mathsf{L}, \mathsf{L}, \mathsf{L})$, then the dimensions of the quantities above are

$$\begin{aligned} \dim(\epsilon) &= \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2} \ , \qquad & \dim(q_r) &= \mathsf{E}\mathsf{L}^{-2}\mathsf{T}^{-1} \equiv \mathsf{M}\mathsf{T}^{-3} \ , \\ \dim(p_r) &= \mathsf{E}\mathsf{L}^{-4}\mathsf{T} \equiv \mathsf{M}\mathsf{L}^{-2}\mathsf{T}^{-1} \ , & \dim(\sigma_{rs}) &= \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2} \ . \end{aligned} \tag{59}$$

Suppose we want to construct a spacetime tensor containing these 16 independent quantities as components (I'm not assuming a priori the symmetry of this tensor: it can be seen as a consequence of the Einstein equations, and it only needs to hold for the sum of the stress-energy-momentum tensors from all kinds of matter). What should the variance type and the intrinsic dimension of such a tensor be?

Since we have 16 components, this tensor should belong to the tensor product of two tangent spaces, each spanned by four basis elements.

There are four such spaces: vectors, covectors, 3-vectors, and 3-covectors. Let's use shorthands such as $\partial_{tzy} \coloneqq \partial_t \wedge \partial_z \wedge \partial_y$ and $\mathrm{d}tzy \coloneqq \mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}y$. These four spaces then have the following coordinate-induced bases and corresponding dimensions:

$$(\partial_t, \partial_x, \partial_y, \partial_z): \qquad (\mathsf{T}^{-1}, \mathsf{L}^{-1}, \mathsf{L}^{-1}, \mathsf{L}^{-1}), \tag{60a}$$

$$(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z): \qquad (\mathsf{T}, \mathsf{L}, \mathsf{L}, \mathsf{L}), \tag{60b}$$

$$(\partial_{xyz}, \partial_{tzy}, \partial_{txz}, \partial_{tyx}): \qquad (\mathsf{L}^{-3}, \mathsf{L}^{-2}\mathsf{T}^{-1}, \mathsf{L}^{-2}\mathsf{T}^{-1}, \mathsf{L}^{-2}\mathsf{T}^{-1}), \tag{60c}$$

$$(\mathrm{d}xyz, \mathrm{d}tzy, \mathrm{d}txz, \mathrm{d}tyx): \qquad (\mathsf{L}^{3}, \mathsf{L}^{2}\mathsf{T}, \mathsf{L}^{2}\mathsf{T}, \mathsf{L}^{2}\mathsf{T}) \tag{60d}$$

(the orderings are chosen to minimize the minus signs appearing from inner products with a volume element). There are therefore 4×4 possible tensor-product spaces, each constructed by the product of two of the four spaces above; and thus 16 possible alternatives to represent our stress-energy-momentum tensor. Energy density is intuitively associated with the purely timelike component of this tensor, stress with the purely spacelike ones, and surface energy-flux density and momentum density with the mixed timelike-spacelike components.

Consider the following first alternative, obtained from the tensor product of the space (60a) with itself; omit y- and z-terms for brevity:

$$\boldsymbol{T} \overset{?}{=} \epsilon\ \partial_t \otimes \partial_t + q_x\ \partial_t \otimes \partial_x + p_x\ \partial_x \otimes \partial_t + \sigma_{xx}\ \partial_x \otimes \partial_x + \cdots$$

The first and third summands of this expression have incompatible intrinsic dimensions $\mathsf{E}\mathsf{L}^{-3}\mathsf{T}^{-2}$ and $\mathsf{E}\mathsf{L}^{-5}$. This alternative is therefore rejected because dimensionally inconsistent. Similar dimensional analyses on the remaining fifteen alternatives show that only four are dimensionally consistent:

$$\boldsymbol{T}_{\bullet}{}^{\bullet} = \epsilon\ \mathrm{d}t \otimes \partial_t + q_x\ \mathrm{d}t \otimes \partial_x + p_x\ \mathrm{d}x \otimes \partial_t + \sigma_{xx}\ \mathrm{d}x \otimes \partial_x + \cdots \tag{61a}$$

$$\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|} = \epsilon\ \mathrm{d}t \otimes \mathrm{d}xyz + q_x\ \mathrm{d}t \otimes \mathrm{d}tzy + \\ p_x\ \mathrm{d}x \otimes \mathrm{d}xyz + \sigma_{xx}\ \mathrm{d}x \otimes \mathrm{d}tzy + \cdots \tag{61b}$$

$$\boldsymbol{T}^{|\bullet\bullet\bullet|\bullet} = \epsilon\ \partial_{xyz} \otimes \partial_t + q_x\ \partial_{xyz} \otimes \partial_x + \\ p_x\ \partial_{tzy} \otimes \partial_t + \sigma_{xx}\ \partial_{tzy} \otimes \partial_x + \cdots \tag{61c}$$

$$\boldsymbol{T}^{|\bullet\bullet\bullet|}{}_{|\bullet\bullet\bullet|} = \epsilon\ \partial_{xyz} \otimes \mathrm{d}xyz + q_x\ \partial_{xyz} \otimes \mathrm{d}tzy + \\ p_x\ \partial_{tzy} \otimes \mathrm{d}yxz + \sigma_{xx}\ \partial_{tzy} \otimes \mathrm{d}tzy + \cdots \tag{61d}$$

Our analysis of the intrinsic dimensions therefore restricts the stress-energy-momentum tensor to be one of the four alternatives above. Note that their kind of orientation, straight or twisted, is also still undetermined.

To further restrict the possibilities let's consider a couple of additional, interrelated heuristic arguments. First, the notions of energy and momentum density, surface energy-flux density, and stress imply some kind of integration over three-dimensional spacelike or timelike regions. Such integration needs a 3-form and thus excludes alternatives (61a) and (61c). Second, the total energy measured by a specific observer within a *topologically* specified three-dimensional spatial region is considered to be independent of the metric extent of that region (just like charge, although the latter is also independent of the observer's motion). The energy *density* therefore does depend on the metric properties and must change accordingly. Similar arguments hold for the surface energy flux, and for the momentum density and stress, provided we consider mass as a spatial topological invariant (for a given observer). Only the second tensor alternative (67) above is consistent with these requirements. Finally, the value of the energy density should not change under a change in the orientation of the spacelike coordinates. This means that the 3-covector slot in alternative (61b) should have a twisted orientation.

A heuristic application of intrinsic dimensional analysis, combined with a integration, rescaling, and reorientation arguments, thus tells us that the stress-energy-momentum tensor has variance type $\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|}$, or equivalently that it's a covector-valued 3-covector, or a four-times-covariant tensor completely antisymmetric in three slots. The 3-covector or antisymmetric slots have a twisted orientation. This tensor has the dimension of an *action*:

$$\dim(\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|}) = \mathsf{ET} \equiv \mathsf{ML^2T^{-1}} \,. \tag{62}$$

Note that we couldn't have obtained this result by dimensional analysis without considering *intrinsic* dimensions. Traditionally we would have multiplied the components by appropriate powers of c to ensure they all had the same dimension; but this procedure would have left the variance type undetermined. This shows the value of intrinsic-dimensional analysis, as opposed to dimensional analysis focused on components only.

This form of the stress-energy-momentum tensor appears for example in Einstein's original work[39], Cartan's[40], Brillouin's[41], Truesdell & Toupin's[42], who try to find an expression universally valid in Newtonian, Lorentzian, and general-relativistic mechanics. It also appears in more recent works[43] which approach its definition from diverse points of view. This is also the form obtained with variational principles from an action Lagrangean[44], from which it is easily seen that this tensor has the intrinsic dimension of an action.

Note that some of the works just cited speak of a tensor of variance type $\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|}$, others of a once covariant and once contravariant "V(olume)-tensor" or "tensor density", which has variance type $\boldsymbol{T}_{\bullet}{}^{\bullet}{}_{|\bullet\bullet\bullet\bullet|}$. But these two objects are geometrically equivalent, in the sense that their independent components have the same transformation law under changes of coordinates[45]. They are related by a contraction with $\omega \otimes \omega^{-1}$, where ω is an arbitrary non-vanishing 4-form (this object has only one independent component, which transforms as the scalar 1).

To obtain a twice covariant tensor to be used in the Einstein equations, we can first take the inner product of the antisymmetric part by the inverse proper volume element, obtaining a co-contravariant tensor equivalent to (61a) above, and then lower the new contravariant slot using the metric tensor:

$$\bar{\boldsymbol{T}}_{\bullet\bullet} = (\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|} \lfloor \gamma^{-1})\boldsymbol{g} \tag{63}$$

with intrinsic dimension

$$\begin{aligned}\dim(\bar{\boldsymbol{T}}_{\bullet\bullet}) &= \dim\big(\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|}\big)\dim(\gamma)^{-1}\ \dim(\boldsymbol{g})\\ &= \begin{cases}\mathsf{EL}^{-1} \equiv \mathsf{MLT}^{-2}\\ \mathsf{EL}^{-3}\mathsf{T}^{2} \equiv \mathsf{ML}^{-1}\end{cases} \qquad \text{if } \dim(\boldsymbol{g}) := \begin{cases}\mathsf{L}^2\\ \mathsf{T}^2\end{cases},\end{aligned} \tag{64}$$

that is, it has the dimension of a *force* if the metric has dimension L^2.

"$^*\boldsymbol{T}$" by Misner et al. , the works by Segev[46], the discussion by Burke

[39] Einstein 1914 § C.9. [40] Cartan 1923 § 13. [41] cf. Brillouin 1924 § 7. [42] Truesdell & Toupin 1960 § F.IV.288. [43] Misner et al. 1973 ch. 14 Exercise 14.18, ch. 15; Hehl & McCrea 1986; Gronwald & Hehl 1997; cf. also Segev 1986; Segev & Rodnay 1999; Segev 2000a,b; 2002; Kanso et al. 2007. [44] Misner et al. 1973 § 21.3; Hawking & Ellis 1994 § 3.3; Pauli 1958 § IV.55; Gotay & Marsden 1992 see also. [45] Schouten 1989 § II.8 p. 30. [46] Segev 2002; 1986; Segev & Rodnay 1999; Segev 2000a,b.

According to the discussion of intrinsic dimension in § 5, the first alternative is dimensionally inconsistent: its first The second and third alternatives are similarly inconsistent. Only the last alternative is intrinsic-dimensionally consistent. It is co-contra-variant and has intrinsic dimension of volumic energy, EL^{-3}.

The heuristic dimensional analysis above therefore suggests a co-contra-variant stress-energy-momentum tensor $\boldsymbol{T}_{\bullet}^{\bullet}$ with intrinsic dimension

$$\dim(\boldsymbol{T}_{\bullet}^{\bullet}) = \mathsf{EL}^{-3} \equiv \mathsf{ML}^{-1}\mathsf{T}^{-2} \,. \tag{65}$$

The analysis above doesn't exclude other possibilities. For example, instead of the space vectors as the left-side term of the tensor product, we could use the four-dimensional space of 3-covectors. This would lead to the dimensionally consistent expression

$$\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|} = \epsilon \,\mathrm{d}t \otimes (\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) + q_x \,\mathrm{d}t \otimes (\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z) + \cdots \tag{66}$$

with intrinsic dimension

$$\dim\bigl(\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|}\bigr) = \mathsf{ET} \equiv \mathsf{ML}^{2}\mathsf{T}^{-1} \,. \tag{67}$$

We could also use 4-covector-valued co-vectors as the right side of the tensor product, obtaining the dimensionally consistent expression

$$\boldsymbol{T}_{|\bullet\bullet\bullet|\bullet}{}^{\bullet} = \epsilon \,\mathrm{d}t \otimes (\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) + q_x \,\mathrm{d}t \otimes (\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z) + \cdots \tag{68}$$

with intrinsic dimension

$$\dim\bigl(\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|}\bigr) = \mathsf{ET} \equiv \mathsf{ML}^{2}\mathsf{T}^{-1} \,. \tag{69}$$

Note that we couldn't have obtained this result without considering intrinsic dimensions. Traditionally we would have multiplied the components by appropriate powers of c to ensure they all had the same dimension; but this procedure would have left the variance type undetermined. This shows the value of intrinsic-dimensional analysis, as opposed to dimensional analysis focused on components only.

determine all geometric properties of the tensor. Any tensor operations that preserve the intrinsic dimensionality lead to other dimensionally consistent alternatives. For example, the tensor above could have its covariant or contravariant slot with a twisted orientation. We could also take the inner product of either slot with the proper volume element or its inverse; this would be equivalent to choosing the four-dimensional spaces of 3-vectors or 3-covectors as components of the tensor product, in the analysis above. Finally we could also tensor-multiply the tensor by the proper volume element

or multiply the tensor by a obtaining the following variants:

$$\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|} \coloneqq \boldsymbol{T}_{\bullet}^{\ \bullet} \rfloor \gamma \tag{70}$$

$$\boldsymbol{T}_{\bullet|\bullet\bullet\bullet|} \coloneqq \boldsymbol{T}_{\bullet}^{\ \bullet} \rfloor \gamma \tag{71}$$

From the components (76), the dimensions (74), and the relation between intrinsic and component dimensions (12) we can thus find the intrinsic dimension of the stress-energy-momentum tensor. To determine the variance type of the components (T_{ij}) consider the four alternatives in which it might be written in terms of basis vectors and covectors: You can check that the first alternative is dimensionally inconsistent: its first two summands have incompatible dimensions $\mathsf{EL^{-3}T^{2}}$ and $\mathsf{EL^{-1}}$. The second and third alternatives are similarly inconsistent. Only the last alternative is dimensionally consistent. It is co-contra-variant and has a dimension of volumic energy, $\mathsf{EL^{-3}}$.

$$\boldsymbol{T} \overset{?}{=} \epsilon\ \partial_t \otimes \partial_t + q_x\ \partial_t \otimes \partial_x + p_x\ \partial_x \otimes \partial_t + \sigma_{x\,x}\ \partial_x \otimes \partial_x + \cdots\ .$$

The expression above, however, is dimensionally inconsistent: its first two summands have incompatible intrinsic dimensions $\mathsf{EL^{-3}T^{2}}$ and $\mathsf{EL^{-1}}$.

with basis $(\partial_t, \partial_x, \partial_y, \partial_z)$, of covectors $(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, of 3-vectors

Consider the balances of energy and of momentum (or of force) for a material continuum in Newtonian mechanics, in the absence of volumic forces and volumic heating[47]. Choose an inertial frame in which the material continuum is instantaneously at rest at a specific spacetime point. Use rectangular Cartesian coordinates $(t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$, with standard dimensions of time and length, adapted to such frame. Let the indices r, s run only through spatial coordinates. The two balances then assume, at the specified point, the following component expressions:

$$\partial_t \epsilon + \sum_{r=1}^{3} \partial_r q_r = 0\,, \tag{72}$$

$$\partial_t p_s + \sum_{r=1}^{3} \partial_r \sigma_{sr} = 0\,, \quad s \in \{1, 2, 3\} \tag{73}$$

[47] Truesdell & Toupin 1960; Truesdell 1991; Samohýl & Pekař 2014; Bird et al. 2002.

where the indices' levels do *not* denote any variance type yet. Here ϵ is the volumic internal energy, q_i are the components of the areic energy flux, p_i are the components of the volumic momentum, and σ_{ij} are the components of the stress tensor, a force per unit area[48]. σ also contains macroscopic momentum transport; this makes the derivation less neat Their dimensions are

$$\dim(\epsilon) = \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}\,, \qquad \dim(q_i) = \mathsf{E}\mathsf{L}^{-2}\mathsf{T}^{-1} \equiv \mathsf{M}\mathsf{T}^{-3}\,, \tag{74a}$$

$$\dim(p_i) = \mathsf{M}\mathsf{L}\mathsf{T}^{-1}\,, \qquad \dim(\sigma_{ij}) = \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}\,. \tag{74b}$$

The two balances can be suggestively combined into a four-dimensional equation:

$$\textstyle\sum_j \partial_j T_{ij} = 0 \tag{75}$$

with

$$T_{ij} \coloneqq \begin{pmatrix} \epsilon & q_r \\ p_s & \sigma_{sr} \end{pmatrix}\,, \tag{76}$$

where indices i, j run over all four coordinates, The combined balance resembles a four-dimensional divergence. Cartan[49] in fact showed that Newtonian mechanics can be formulated in fully covariant form in terms of a special connection ∇, and the terms T_{ij} are indeed four-dimensional components of a stress-energy-momentum tensor $\boldsymbol{T}$. This reformulation is usually called Newton-Cartan theory; for details see especially Truesdell & Toupin[50]. Cartan's connection can even express Newtonian gravitation, for which see Misner et al.[51].

The balances of energy and momentum (72), (73) in Newtonian mechanics are thus covariantly expressed in coordinate-free form as

$$\mathrm{tr}_{13}\, \nabla_{\bullet} \boldsymbol{T}_{\bullet}{}^{\bullet} = 0\,, \quad \text{commonly written } \nabla \cdot \boldsymbol{T} = 0\,, \tag{77}$$

with components $\nabla_j \boldsymbol{T}_i{}^j = 0$. This is the same equation that can be obtained from the Einstein equations in general relativity from the fact that the Einstein tensor has zero divergence. Indeed this is the starting point to define the stress part of the tensor $\boldsymbol{T}$ in relativity: it is assumed

48 considered as pressure rather than tension, hence its sign; cf. Bird et al. 2002 § 1.2 p. 19.
49 Cartan 1923; 1924. **50** Truesdell & Toupin 1960 §§ B.II.152–154, D.II.203–205, D.V.238, F.IV.285–289; see also Marsden & Hughes 1994 § 2.4. **51** Misner et al. 1973 ch. 12; also Trautman 1964; Dixon 1975.

to be locally equal to the Newtonian one in an inertial (free-fall) frame in which the material continuum is momentarily at rest. note on momentum of heat, refer to[52]

From the components (76), the dimensions (74), and the relation between intrinsic and component dimensions (12) we can thus find the intrinsic dimension of the stress-energy-momentum tensor. To determine the variance type of the components (T_{ij}) consider the four alternatives in which it might be written in terms of basis vectors and covectors: You can check that the first alternative is dimensionally inconsistent: its first two summands have incompatible dimensions $\mathsf{E}\mathsf{L}^{-3}\mathsf{T}^{2}$ and $\mathsf{E}\mathsf{L}^{-1}$. The second and third alternatives are similarly inconsistent. Only the last alternative is dimensionally consistent. It is co-contra-variant and has a dimension of volumic energy, $\mathsf{E}\mathsf{L}^{-3}$.

We thus find

$$\dim(\boldsymbol{T}_{\bullet}^{\bullet}) = \mathsf{E}\mathsf{L}^{-3} \equiv \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2}\,, \tag{78}$$

$$\dim(\boldsymbol{T}_{\bullet\bullet}) = \dim(\boldsymbol{T}_{\bullet}^{\bullet})\dim(\boldsymbol{g}) = \begin{cases}\mathsf{M}\mathsf{L}^{-1}\\ \mathsf{M}\mathsf{L}\mathsf{T}^{-2} \equiv \mathsf{E}\mathsf{L}^{-1}\end{cases} \quad \text{if } \dim(\boldsymbol{g}) := \begin{cases}\mathsf{T}^{2}\\ \mathsf{L}^{2}\end{cases} \tag{79}$$

[53]

Here ϵ is the volumic internal energy, q_i are the components of the areic energy flux, p_i are the components of the volumic momentum, and σ_{ij} are the components of the stress tensor, a force per unit area. Hence
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To find the dimension of the stress-energy-momentum $\boldsymbol{T}$, or '4-stress' for short, let's start with the analysis of the (3-)stress σ in Newtonian mechanics. The stress σ is the projection of the 4-stress $\boldsymbol{T}$ onto a spacelike tangent plane with respect to some observer[54]. If we assume that such spatial projection preserves the intrinsic dimension, then the 4-stress and the stress have the same intrinsic dimension.

In Newtonian mechanics the stress σ is an object that, integrated over the boundary of a body, gives the total surface force acting on the

[52] Eckart 1940 p. 923. [53] Tolman 1949 §§ III.35–37. [54] Gourgoulhon 2012 § 3.4.1; Smarr & York 1978; York 1979; Smarr et al. 1980; Wilson & Mathews 2007 § 1.3; the projection doesn't need to be orthogonal: Marsden & Hughes 1994 § 2.4; Hehl & Obukhov 2003 § B.1.4.

body[55] (such integration requires a flat connection). This means that it must be represented by a "force-valued" 2-form. Force, in turn, can be interpreted as an object that, integrated over a (spacelike) trajectory, gives an energy – the work done by the force along the trajectory. It's therefore a 1-form. Putting these two requirements together, the stress turns out to be a covector-valued 2-form, equivalent to a tensor $\sigma_{\bullet\,\bullet\bullet}$ antisymmetric in its last two indices. Integrated over a surface, and then over a trajectory, it yields an energy. From § 6, integration of a form does not change the dimension of the form. Therefore

$$\dim(\sigma_{\bullet\,\bullet\bullet}) = \mathsf{E} \equiv \mathsf{ML}^2\mathsf{T}^{-2} \,. \tag{80}$$

But usually the stress is represented by a co-contra-variant tensor $\sigma_{\bullet}{}^{\bullet}$. The latter is obtained by contracting the last two slots of $\sigma_{\bullet\,\bullet\bullet}$ with the inverse of the volume element of the 3-metric – this is the duality[56] between k-vectors and $(n-k)$-covectors induced by the metric (and an orientation choice), where n is the geometric dimension of the manifold. If we assume the Newtonian 3-metric to have dimension L^2, it can be shown similarly to § 9 that its volume element has dimension L^3, and the inverse volume element has dimension L^{-3}. Thus we obtain

$$\dim(\sigma_{\bullet}{}^{\bullet}) = \mathsf{EL}^{-3} \equiv \mathsf{ML}^{-1}\mathsf{T}^{-2} \,, \tag{81}$$

an energy density (or 'volumic energy' according to ISO[57]).

Since the stress $\sigma_{\bullet}{}^{\bullet}$ is the projection of $\boldsymbol{T}_{\bullet}{}^{\bullet}$ and the projection preserves the intrinsic dimension, we finally find that $\boldsymbol{T}_{\bullet}{}^{\bullet}$ also has the dimension of an energy density:

$$\dim(\boldsymbol{T}_{\bullet}{}^{\bullet}) = \mathsf{EL}^{-3} \equiv \mathsf{ML}^{-1}\mathsf{T}^{-2} \,. \tag{82}$$

Note that other co- or contra-variant versions of the 4-stress have different intrinsic dimension, because they're obtained by lowering or raising indices. For example, $\dim(\boldsymbol{T}_{\bullet\bullet}) = \dim(\boldsymbol{T}_{\bullet}{}^{\bullet})\dim(\boldsymbol{g}) = \mathsf{ML}^{-1}$ if $\dim(\boldsymbol{g}) \coloneqq \mathsf{T}^2$.

Let me add a passing remark. Even though in most texts the 4-stress is represented by a tensor of order 2, as above, its most fitting geometrical nature is still shrouded in mystery from the kinematic and the dynamical points of view. There are indications that it could be more properly

[55] Truesdell 1991 ch. III. [56] Bossavit 1991 § 4.1.2. [57] ISO 2009 item A.6.2.

represented by a covector-valued 3-form (equivalent to a tensor $\boldsymbol{T}_{\bullet\,|\bullet\bullet\bullet|}$ antisymmetric in the last three slots), or by a 3-vector-valued 3-form (equivalent to a tensor $\boldsymbol{T}^{\bullet\bullet\bullet}{}_{\bullet\bullet\bullet}$ antisymmetric in the first three and last three slots), for reasons connected with integration, similar to those mentioned above for the stress $\sigma_{\bullet\,\bullet\bullet}$. See for example the discussion about "$^*\boldsymbol{T}$" by Misner et al.[58], the works by Segev[59], the discussion by Burke[60]. add[61]

11 The constant in the Einstein equations

We finally arrive at the Einstein equations,

$$\boldsymbol{G} = \kappa \boldsymbol{T} \tag{83}$$

where κ (sometimes seen with a minus[62] depending on the signature of the metric or on the orientation of the stress) is Einstein's constant. For the dimension of κ we thus find

$$\dim(\kappa) = \dim(\boldsymbol{G}_{\bullet}{}^{\bullet})\ \dim(\boldsymbol{T}_{\bullet}{}^{\bullet})^{-1} \equiv \begin{cases} \mathsf{M}^{-1}\mathsf{L} \\ \mathsf{M}^{-1}\mathsf{L}^{-1}\mathsf{T}^{2} \end{cases} \qquad \text{if } \dim(\boldsymbol{g}) \coloneqq \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases} \tag{84}$$

This constant can be obtained from the dimensions of Newton's gravitational constant $\dim(G) = \mathsf{M}^{-1}\mathsf{L}^{3}\mathsf{T}^{-2}$ (this is not the Einstein tensor $\boldsymbol{G}$!) and of the speed of light $\dim(c) = \mathsf{L}\mathsf{T}^{-1}$ only in the following ways, with an 8π factor coming from the Newtonian limit:

$$\kappa = \begin{cases} 8\pi G/c^{2} \\ 8\pi G/c^{4} \end{cases} \qquad \text{if } \dim(\boldsymbol{g}) \coloneqq \begin{cases} \mathsf{T}^{2} \\ \mathsf{L}^{2} \end{cases} \quad . \tag{85}$$

The second choice is by far the most common, consistently with the most common choice of $\dim(\boldsymbol{g}) = \mathsf{L}^2$ discussed before. The first choice appears for example in Fock[63] and Adler et al.[64].

12 Summary and conclusions

We have seen that dimensional analysis, with its familiar rules, can be seamlessly performed in Lorentzian and general relativity and in

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58 Misner et al. 1973 ch. 15. **59** Segev 2002; 1986; Segev & Rodnay 1999; Segev 2000a,b. **60** Burke 1987 § 41. **61** Kanso et al. 2007. **62** e.g. Tolman 1949 § 78 eq. (78.3); Fock 1964 § 52 eq. (52.06); Rindler 2006 § 14.2 eq. (14.8). **63** Fock 1964 § 55 eqs (55.15) and (52.06). **64** Adler et al. 1975 § 10.5 eq. (10.98).

differential geometry if we adopt the coordinate-free approach typical of modern texts. In this approach each tensor has an *intrinsic* dimension (a notion introduced by Schouten and Dorgelo). This dimension doesn't depend on the dimensions of the coordinates, and is determined by the physical and operational meaning of the tensor. It is therefore generally more profitable to focus on the intrinsic dimension of a tensor rather than on the dimensions of its components. The dimension of each specific component is easily found by formula (12): it's the product of the intrinsic dimension by the dimension of the ith coordinate function for each contravariant index i, by the inverse of the dimension of the jth coordinate function for each covariant index j.

add note on usefulness of coordinate-free way to settle questions as in § 10?

Dimensional analysis in differential geometry seems to rest on two main conventions: the tensor product and the action of covectors on vectors behave analogously to usual multiplication for the purposes of dimensional analysis. Alternative, equivalent sets of conventions can also be considered, for example involving the exterior derivative.

We found or re-derived some essential results for general relativity, in particular that the Riemann $\boldsymbol{Rie}^{\bullet}{}_{\bullet\bullet\bullet}$ and Ricci $\boldsymbol{Ric}_{\bullet\bullet}$ curvature tensors are dimensionless, and that the Einstein tensor $\boldsymbol{G}_{\bullet}{}^{\bullet}$ has the inverse dimension of the metric tensor. Maybe these results can be of importance for some current research involving scales and conformal factors[65]. We also discussed the operational reasons behind two common choices of dimension for the metric tensor.

Since the dimensions of the components are usually different from the intrinsic dimension and depend on the coordinates, I recommend to avoid statements such as "the tensor $A_i{}^{jk}$ has dimension X", which leave it unclear whether "$A_i{}^{jk}$" is meant to represent the tensor in general (as in Penrose & Rindler's notation), or to represent its set of components, or to represent just a specific component.

Dimensional analysis remains a controversial, obscure, but fascinating subject still today, 60 years from Truesdell & Toupin's remark quoted in the Introduction. For an overview of some recent and creative approaches to it, going beyond Bridgman's text[66] (whose point of view

65 e.g. Röhr & Uggla 2005; Cadoni & Tuveri 2019. **66** Bridgman 1963.

is in many respects at variance with modern developments: see the following references), I recommend for example the works by Mari et al.[67], Domotor and Batitsky[68], Kitano[69], the extensive analysis by Dybkaer[70], the historical review by de Boer[71], and references therein.

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Note: this paper was rejected by arXiv without any kind of scientific motivation. It's sad and alarming that the possibility of pre-print feedback is curtailed like that.

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("de *X*" is listed under D, "van *X*" under V, and so on, regardless of national conventions.)

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[67] Mari & Giordani 2012; Frigerio et al. 2010. [68] Domotor 2017; Domotor & Batitsky 2016; Domotor 2012. [69] Kitano 2013. [70] Dybkær 2010. [71] De Boer 1995.

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