

Notes on general-relativistic continuum electromagneto-thermo-mechanics

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Personal notes on topics in general-relativistic continuum electromagneto-thermo-mechanics.

1 Bases of multivector spaces

Take an ordered coordinate system (t, x, y, z) , which also defines an orientation. We shall usually assume that t has dimensions of time and x, y, z of length, and usually that they are orthogonal if a metric is defined.

The associated bases for inner-oriented multivector fields are

$$dt \quad dx \quad dy \quad dz \quad (1)$$

$$d^2tx \quad d^2ty \quad d^2tz \quad d^2yz \quad d^2zx \quad d^2xy \quad (2)$$

$$d^3xyz \quad -d^3tyz \quad -d^3tzx \quad -d^3txy \quad (3)$$

$$d^4txyz \quad (4)$$

and analogously for inner-oriented multivector fields. The particular choice of ordering and sign is such that these bases are related by volume duality, see below.

The outer-oriented unit scalar is $\tilde{1}$, with outer orientation $txyz$; note that it's only defined on a coordinate patch. A twisted or outer 3-covector such as $d^3\tilde{x}yz$ has an associated outer direction, in this case positive t . We adopt this shorter notation for the outer-oriented versions of the bases above:

$$-d_{xyz} \quad d_{tyz} \quad d_{tzx} \quad d_{txy} \quad (5)$$

$$d_{yz}^2 \quad d_{zx}^2 \quad d_{xy}^2 \quad d_{tx}^2 \quad d_{ty}^2 \quad d_{tz}^2 \quad (6)$$

$$d_t^3 \quad d_x^3 \quad d_y^3 \quad d_z^3 \quad (7)$$

$$d^4 \quad (8)$$

so that $-d_{xyz} := d\tilde{t}$ and so on. Similar notation is used for outer-oriented multivector fields; for instance $-\partial_{xyz} := \partial_{\tilde{t}}$.

Here is an example of an outer-oriented 3-covector written in terms of the bases and notation above. Note the position of the super- and sub-scripts:

$$\begin{aligned} & n_{xyz} d^3 \tilde{x} y z - n_{tyz} d^3 \tilde{t} y z - n_{tzz} d^3 \tilde{t} z x - n_{txy} d^3 \tilde{t} x y \\ & \equiv n^t d_t^3 + n^x d_x^3 + n^y d_y^3 + n^z d_z^3 \end{aligned} \quad (9)$$

with $n^t \equiv n_{xyz}$, $n^x \equiv n_{tyz}$, and so on.

2 Linear transformations of multivector bases

Often we must consider multiplication of the basis of vectors or covectors by a square matrix, say

$$M_{\alpha\beta} du^\beta,$$

leading to an object in a vector space of the same dimension. The two most important examples for us are:

- coordinate transformations, for example

$$d\alpha' = \frac{\partial \alpha'}{\partial \alpha} d\alpha$$

- raising or lowering of indices with the metric, for example

$$d\alpha \mapsto g^{\alpha\beta} \partial_\beta.$$

The corresponding operations for multivectors involve compound matrices of the original transformation matrix¹. For the spaces of 3-vectors and 3-covectors we have simplified formulae²:

$$\begin{aligned} d_{\alpha'}^3 &= \det\left(\frac{\partial \alpha'}{\partial \alpha}\right) \frac{\partial \alpha}{\partial \alpha'} d_\alpha^3 \\ \partial^3 \alpha' &= \det\left(\frac{\partial \alpha}{\partial \alpha'}\right) \frac{\partial \alpha'}{\partial \alpha} \partial^3 \alpha \end{aligned} \quad (10)$$

note how the 3-covectors almost transforms as 1-vectors and 3-vectors and 1-covectors, apart from a determinant factor.

In an analogous way

$$\begin{aligned} d_\alpha^3 &\mapsto \frac{1}{|g|} g_{\alpha\beta} \partial^3 \beta \\ \partial^3 \alpha &\mapsto |g| g^{\alpha\beta} d_\beta^3 \end{aligned} \quad (11)$$

¹ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199. ² Gantmacher 2000 § I.4 eq. (33).

3 Metric

We take the metric g to have signature $(-, +, +, +)$ and dimensions of area. The square root of its negative determinant is denoted shortly

$$\sqrt{g} := \sqrt{-\det g} . \quad (12)$$

The volume element induced by the metric g has dimensions of volume-time and is denoted (note the boldface)

$$\gamma := \frac{1}{c} \sqrt{g} \, d^4 \tilde{t} x y z \equiv \frac{\sqrt{g}}{c} \, d^4 \quad (13)$$

and its corresponding inverse, a twisted 4-vector:

$$\gamma^{-1} := \frac{c}{\sqrt{g}} \, \partial^4 \quad (14)$$

Contraction with the volume element or its inverse establishes a “volume duality” between outer n -covectors and inner $(4 - n)$ -vectors:

$$\left(\begin{array}{cccc} \partial_{xyz}^3 & \partial_{tyz}^3 & \partial_{tzx}^3 & \partial_{txy}^3 \\ \partial_{tx}^2 & \partial_{ty}^2 & \partial_{tz}^2 & \partial_{yz}^2 & \partial_{zx}^2 & \partial_{xy}^2 \\ \partial_t & \partial_x & \partial_y & \partial_z \\ \partial_{txyz}^4 \end{array} \right) \begin{array}{c} \xleftarrow{\frac{\sqrt{g}}{c} \gamma^{-1}} \\ \xrightarrow{\gamma \frac{c}{\sqrt{g}}} \end{array} \left(\begin{array}{cccc} d_{xyz} & d_{tyz} & d_{tzx} & d_{txy} \\ d_{yz}^2 & d_{zx}^2 & d_{xy}^2 & d_{tx}^2 & d_{ty}^2 & d_{tz}^2 \\ d_t^3 & d_x^3 & d_y^3 & d_z^3 \\ d^4 \end{array} \right) \quad (15)$$

This is the reason why in older literature an outer-oriented n -covector is treated as a $(4 - n)$ -“vector density”, that is, a vector divided by the square root of the volume element.

If the coordinates are orthonormal at some point, then the metric at that point has components

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad (16)$$

and the volume element is simply d^4 .

4 Four-stress

The stress-energy-momentum tensor, or simply 4-stress, is a covector-valued 3-covector field, the 3-covector being outer-oriented. It has the dimensions of an action, and can be decomposed as

$$\mathbf{T} = \epsilon \mathbf{d}_t^3 \otimes dt + q^i \mathbf{d}_i^3 \otimes dt + p_j \mathbf{d}_t^3 \otimes dx^j + \pi_j^i \mathbf{d}_i^3 \otimes dx^j \quad (17)$$

the indices i, j running over x, y, z , and where

$$\begin{aligned} \epsilon &= \text{volumic energy} & q^i &= \text{aeric energy flux} \\ p_i &= \text{volumic momentum} & \pi_j^i &= \text{3-stress} \end{aligned} \quad (18)$$

measured in the coordinate system $txyz$. The energy ϵ is a density per unit *coordinate* volume xyz , and possibly includes a conversion factor for the time unit. The component q^i is an energy per coordinate area and coordinate time, and possibly includes a conversion factor for the time unit. The component p_i is a momentum density per unit coordinate volume, and includes a conversion factor for the length x^i . The components π_j^i are forces per unit coordinate area, possibly including conversion factors for the time unit and the length x^j .

Suppose the coordinates $txyz$ are orthogonal and the metric at a point has diagonal components

$$\begin{bmatrix} -c^2 g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \quad (19)$$

The diagonal elements g_{tt}, \dots include a dimensions or unit transformation factor. For instance, if x has dimensions of angle, then g_{xx} has dimensions of area per squared angle.

Contracting the 4-stress with the volume element and the inverse metric we obtain:

$$\begin{aligned} \gamma^{-1} \cdot \mathbf{T} \cdot \mathbf{g}^{-1} &= -\frac{1}{c g_{tt} \sqrt{g}} \epsilon \partial_t \otimes \partial_t - \frac{1}{c g_{tt} \sqrt{g}} q^i \partial_i \otimes \partial_t + \\ &\quad \sum_j \frac{c}{g_{jj} \sqrt{g}} p_j \partial_t \otimes \partial_j + \sum_j \frac{c}{g_{jj} \sqrt{g}} \pi_j^i \partial_i \otimes \partial_j \end{aligned} \quad (20)$$

Appendices

A Matrix representation

Column-vectors, row-vectors, and matrices are useful to encode the components of geometric objects, but they are too few to faithfully encode the difference between co- and contra-variant objects and multivectors of different orders. Here are some useful conventions and procedures.

1-vectors represented by column-matrices

1-covectors represented by row-matrices

3-vectors represented by row-matrices

3-covectors represented by column-matrices

objects such as metrics, covector-valued 3-covectors, tensor products of 1-vector and 1-covector, and similar must be represented by matrices.

Here are the matrix representations of several operations. Consider these objects:

- u^\bullet is a 1-vector, represented by the column-matrix u .
- Similarly for v^\bullet .
- ω_\bullet is a 1-covector, represented by the row-matrix ω .
- $g_{\bullet\bullet}$ is a co-covector, represented by the matrix g . The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $g^{-1\bullet\bullet}$ is a contra-contravector, inverse of g , that is: $g \cdot g^1 = \text{id}_\bullet$. The rows of the matrix represent the left side of the tensor product; and the columns, the right side.
- $Q_{|...|}$ is a 3-covector, represented by the column-matrix Q .
- $T_{|...|}$ is a 1-covector-valued 3-covector, represented by the matrix T . The rows represent the 3-covector components; the columns, the 1-covector components.
- $\gamma_{|...|}$ is a 4-covector, represented by the number γ .
- $\gamma^{-1|.....|}$ is a 4-vector, represented by the number γ^{-1} .
- The Jacobian matrix $\frac{\partial x'}{\partial x}$ from “old” coordinates x to “new” coordinates x' is represented by the matrix J . The rows correspond to the new coordinates x' ; the columns, to the old x .

- The inverse-Jacobian matrix $\frac{\partial x}{\partial x'}$ from new coordinates x' to old coordinates x is represented by the matrix J^{-1} . The rows correspond to the old coordinates x ; the columns, to the new x' .

Then – note that the order on the right side is important:

- Contractions, index raising and lowering

$$\text{(object)} \qquad \qquad \text{(matrix repr)} \qquad \qquad (21)$$

$$\omega \cdot u \qquad \qquad \omega u \equiv u^\top \omega^\top \quad \text{(number)} \qquad (22)$$

$$v \cdot g \cdot u \qquad \qquad v^\top g u \quad \text{(number)} \qquad (23)$$

$$g \cdot u \qquad \qquad u^\top g^\top \quad \text{(row-matrix)} \qquad (24)$$

$$\omega \cdot g^{-1} \qquad \qquad g^{-\top} \omega^\top \quad \text{(column-matrix)} \qquad (25)$$

$$\gamma^{-1} \cdot Q \qquad \qquad \gamma^{-1} Q \quad \text{(column-matrix)} \qquad (26)$$

$$\gamma^{-1} \cdot T \qquad \qquad \gamma^{-1} T \quad \text{(column-matrix)} \qquad (27)$$

$$T \cdot u \qquad \qquad T u \quad \text{(column-matrix)} \qquad (28)$$

$$g \cdot (\gamma^{-1} \cdot T) \cdot u \qquad \qquad \gamma^{-1} g T u \quad \text{(matrix)} \qquad (29)$$

- Transformations

$$\text{(old coords)} \quad \mapsto \quad \text{(new coords)} \qquad (30)$$

$$u \mapsto J u \qquad (31)$$

$$\omega \mapsto \omega J^{-1} \qquad (32)$$

$$g \mapsto J^{-\top} g J^{-1} \qquad (33)$$

$$Q \mapsto \frac{1}{\det J} J Q \qquad (34)$$

$$\gamma \mapsto \frac{1}{\det J} \gamma \qquad (35)$$

$$\gamma^{-1} \mapsto \det J \gamma^{-1} \qquad (36)$$

$$T \mapsto \frac{1}{\det J} J T J^{-1} \qquad (37)$$

B Checks about optimal representation of 4-stress

First a lemma about the exterior derivative of some outer-oriented 3-covectors:

$$\begin{aligned} d(f d_t^3) &= d\left(f d^3 \tilde{x} y z\right) = \partial_t f dt \wedge d^3 \tilde{x} y z + 0 = \partial_t f d^4 \tilde{x} y z \\ d(f d_x^3) &= -d\left(f d^3 \tilde{t} y z\right) = -\partial_x f dx \wedge d^3 \tilde{t} y z + 0 = \partial_x f d^4 \tilde{t} x y z \quad (38) \\ d(f d_i^3) &= \partial_i f d^4 \tilde{t} x y z \end{aligned}$$

And a lemma about the covariant derivative of some inner-oriented 1-covectors:

$$\begin{aligned} \nabla(dt) &= -\Gamma_{tt}^t dt \otimes dt - \Gamma_{tj}^t dt \otimes dx^j - \Gamma_{it}^t dx^i \otimes dt - \Gamma_{ij}^t dx^i \otimes dx^j \quad (39) \\ \nabla(dx^k) &= -\Gamma_{tt}^k dt \otimes dt - \Gamma_{tj}^k dt \otimes dx^j - \Gamma_{it}^k dx^i \otimes dt - \Gamma_{ij}^k dx^i \otimes dx^j \end{aligned}$$

Then

$$0 = D\mathbf{T}$$

$$\begin{aligned} &= D(e d_t^3 \otimes dt + q^i d_i^3 \otimes dt + p_j d_t^3 \otimes dx^j + \pi_j^i d_i^3 \otimes dx^j) \\ &= \partial_t e d^4 \otimes dt + \partial_i q^i d^4 \otimes dt + \partial_t p_j d^4 \otimes dx^j + \partial_i \pi_j^i d^4 \otimes dx^j - \\ &\quad \left[-e \Gamma_{tt}^t d_t^3 \wedge dt \otimes dt - e \Gamma_{tj}^t d_t^3 \wedge dt \otimes dx^j + 0 + \right. \\ &\quad - q^i \Gamma_{it}^t d_i^3 \wedge dx^i \otimes dt - q^i \Gamma_{ij}^t d_i^3 \wedge dx^i \otimes dx^j + 0 + \\ &\quad - p_j \Gamma_{tt}^j d_t^3 \wedge dt \otimes dt - p_k \Gamma_{tj}^k d_t^3 \wedge dt \otimes dx^j + 0 + \\ &\quad \left. - \pi_k^i \Gamma_{it}^k d_i^3 \wedge dx^i \otimes dt - \pi_k^i \Gamma_{ij}^k d_i^3 \wedge dx^i \otimes dx^j \right] \quad (40) \\ &= d^4 \otimes \left[\begin{aligned} &\partial_t e dt - e \Gamma_{tt}^t dt - e \Gamma_{tj}^t dx^j + \\ &\partial_i q^i dt - q^i \Gamma_{it}^t dt - q^i \Gamma_{ij}^t dx^j + \\ &\partial_t p_j dx^j - p_j \Gamma_{tt}^j dt - p_k \Gamma_{tj}^k dx^j + \\ &\partial_i \pi_j^i dx^j - \pi_k^i \Gamma_{it}^k dt - \pi_k^i \Gamma_{ij}^k dx^j \end{aligned} \right] \end{aligned}$$

This corresponds to the four balance equations

$$\partial_t e + \partial_i q^i = e \Gamma_{tt}^t + q^i \Gamma_{it}^t + p_j \Gamma_{tt}^j + \pi_k^i \Gamma_{it}^k \quad (41)$$

$$\partial_t p_j + \partial_i \pi_j^i = e \Gamma_{tj}^t + q^i \Gamma_{ij}^t + p_k \Gamma_{tj}^k + \pi_k^i \Gamma_{ij}^k \quad (42)$$

Radial case

$$\partial_t e + \partial_r q^r = e \Gamma_{tt}^t + q^r \Gamma_{rt}^t + p_r \Gamma_{tt}^r + \pi_r^r \Gamma_{rt}^r \quad (43)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \Gamma_{tr}^t + q^r \Gamma_{rr}^t + p_r \Gamma_{tr}^r + \pi_r^r \Gamma_{rr}^r \quad (44)$$

$$\partial_t e + \partial_r q^r = q^r \frac{g}{c^2} + p_r g \quad (45)$$

$$\partial_t p_r + \partial_r \pi_r^r = e \frac{g}{c^2} \quad (46)$$

Let's consider a Cartesian coordinate system over a small neighbourhood on the Earth's surface, with z pointing upwards. In the Newtonian approximation we have³

$$\Gamma_{jt}^t = \Gamma_{tj}^t = \frac{GM}{c^2} \frac{x^j}{r^3} \approx \frac{g}{c^2} \quad \Gamma_{tt}^j = GM \frac{x^j}{r^3} \approx g \quad (47)$$

where g is the standard acceleration, considered positive. Take also $p_j \approx mv_j$ and $e \approx mc^2 + \frac{1}{2}mv^2$.

The balances above become

$$\partial_t e + \partial_i q^i = q^z \frac{g}{c^2} + p_z g \quad (48)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} \quad (49)$$

Also,

$$\begin{aligned} \Gamma_{tt}^t &\approx -2 \frac{g}{c^2} v(t) & \Gamma_{jt}^t &= \Gamma_{tj}^t \approx \frac{g}{c^2} \\ \Gamma_{tt}^j &\approx g - 2 \frac{g}{c^2} v(t)^2 - \dot{v}(t) & \Gamma_{jt}^j &= \Gamma_{tj}^j \approx \frac{g}{c^2} v(t) \end{aligned} \quad (50)$$

$$\partial_t e + \partial_i q^i = -e 2 \frac{g}{c^2} v(t) + q^z \frac{g}{c^2} + p_j \left(g - 2 \frac{g}{c^2} v(t)^2 - \dot{v}(t) \right) + \pi_z^z \frac{g}{c^2} v(t) \quad (51)$$

$$\partial_t p_z + \partial_i \pi_z^i = e \frac{g}{c^2} + p_z \frac{g}{c^2} v(t) \quad (52)$$

³ Poisson & Will 2014 §5.2.3.

C Works with useful content

- Eq. (21) in⁴: fluid with heat conduction.
- Factor 1/2 in kinetic energy: § 6.3.3 in⁵

For transformation or raising:⁶.

$$\gamma' \cdot (B \wedge A) = (-1)^{\deg A \deg B} \gamma' \cdot (A \wedge B) \quad (53)$$

with $\deg(B) = n - \deg(A)$ ⁷

Compound matrices:⁸

Bibliography

(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

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⁴ Maugin 1974. ⁵ Gourgoulhon 2012. ⁶ Gantmacher 2000 § I.4 eq. (33). ⁷ Barnabei et al. 1985 prop. 4.1. ⁸ Choquet-Bruhat et al. 1996 § IV.A.1 p. 199, Problem 1 p. 270.