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Kinematics and dynamics from a modern perspective

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The kinematics of mechanical, thermodynamic, and electromagnetic phenomena is developed in such a way as to be used for Newtonian, Lorentzian, and general relativity. The same is done, as much as possible, for their dynamics as well

1 Scribbles and memos

1.1 Twisted objects

Denote the wedge product by juxtaposition: $\mathrm{d}t \wedge \mathrm{d}x =: \mathrm{d}t\,\mathrm{d}x$ and so on; abbreviate $\mathrm{d}t\,\mathrm{d}x =: \mathrm{d}tx$ and so on; and denote twisted differential forms by $\underset{\sim}{\mathrm{d}}$.

Where an ordered set of coordinate functions (t, x, y, z) is chosen, the twisted unit $\underline{1}$ is defined. It has unit magnitude and outer-orientation $txyz$, and the property $\underline{1} \cdot \underline{1} = 1$. In general it is only defined in chart domain, where coordinate functions can be defined, but not globally. We obtain twisted vectors and covectors by multiplying their non-twisted counterparts by the twisted unit. For a vector or covector ω we have that the orientation of its twisted counterpart is such that (see Burke 1983)

$$\big\{\{\underline{\omega}\}, \{\omega\}\big\} = \big\{\underline{1}\big\} \,. \tag{1}$$

Said otherwise, in the product $\underline{1} \cdot \omega$ the *right* side of the orientation of $\underline{1}$ cancels out with the orientation of ω. This rule must be respected even if we invert the product order, so $\underline{1} \cdot \omega \equiv \omega \cdot \underline{1}$.

Multiplying a twisted vector or covector by the twisted unit, we obtain its non-twisted counterpart. The resulting orientation is obtained by cancelling out the orientation of the twisted object and the *left* side of the orientation of $\underline{1}$.

We have:

$$\{\underline{1}\} = txyz\ ; \tag{2}$$

$$\{\underset{\sim}{\mathrm{d}}t\} = -xyz\ , \quad \{\underset{\sim}{\mathrm{d}}x\} = tyz\ , \quad \{\underset{\sim}{\mathrm{d}}y\} = tzx\ , \quad \{\underset{\sim}{\mathrm{d}}z\} = txy\ ; \tag{3}$$

$$\{\underset{\sim}{\mathrm{d}}tx\} = yz\ , \quad \{\underset{\sim}{\mathrm{d}}ty\} = zx\ , \quad \{\underset{\sim}{\mathrm{d}}tz\} = xy\ , \tag{4}$$

$$\{\underset{\sim}{\mathrm{d}}xy\} = tz\ , \quad \{\underset{\sim}{\mathrm{d}}yz\} = tx\ , \quad \{\underset{\sim}{\mathrm{d}}zx\} = ty\ ; \tag{5}$$

$$\{\underset{\sim}{\mathrm{d}}tyz\} = -x\ , \quad \{\underset{\sim}{\mathrm{d}}tzx\} = -y\ , \quad \{\underset{\sim}{\mathrm{d}}txy\} = -z\ , \quad \{\underset{\sim}{\mathrm{d}}xyz\} = t\ ; \tag{6}$$

$$\{\underset{\sim}{\mathrm{d}}txyz\} = +1\ . \tag{7}$$

The minus signs appear in the odd ranks when we have t and an even number of other coordinates after the "$\underset{\sim}{\mathrm{d}}$". These minus signs flip if we keep t always to the right, with orientation $xyzt$.

Note that considering, say, the *function* x, we have

$$\{\underline{x}\} = \begin{cases} \{txyz\} & \text{if } x > 0\ , \\ -\{txyz\} & \text{if } x < 0\ . \end{cases} \tag{8}$$

1.2 Charge and current densities

We can represent charge density and current density in one geometrical entity. Consider an ordered coordinate system (t, x, y, z). The charge-current density is

$$\begin{aligned} &\rho\ \underset{\sim}{\mathrm{d}}xyz - j_x\ \underset{\sim}{\mathrm{d}}tyz - j_y\ \underset{\sim}{\mathrm{d}}tzx - j_z\ \underset{\sim}{\mathrm{d}}txy \equiv \\ &\qquad\qquad \rho\ \underset{\sim}{\mathrm{d}}xyz - \mathrm{d}t\ \big(j_x\ \underset{\sim}{\mathrm{d}}yz + j_y\ \underset{\sim}{\mathrm{d}}zx + j_z\ \underset{\sim}{\mathrm{d}}xy\big)\ . \end{aligned} \tag{9}$$

It has the dimensions of charge (current · time). The minus signs appear so that the x-component of the current, for example, is positive when $j_x > 0$ and so on.

This object automatically give us net volume charge when integrated over a three-dimensional region at constant time, or the net flux of charge when integrated over a two-dimensional surface – possibly even moving –

over a lapse of time. Consider for example a 3-volume V at constant time, having *outer* orientation in the positive t direction and parameterized by

$$(u, v, w) \mapsto (t_0, u, v, w)\,. \tag{10}$$

On it, the basis twisted 1-forms map to

$$\begin{gathered}\underset{-xyz}{\mathrm{d}t}\Big|_V = 0\,, \quad \underset{tyz}{\mathrm{d}x}\Big|_V = \underset{vw}{\mathrm{d}u}\,, \quad \underset{tzx}{\mathrm{d}y}\Big|_V = \underset{wu}{\mathrm{d}v}\,, \quad \underset{txy}{\mathrm{d}z}\Big|_V = \underset{uv}{\mathrm{d}w}\,,\\ \underset{t}{\mathrm{d}}xyz\Big|_V = \underset{+}{\mathrm{d}}uvw\,.\end{gathered} \tag{11}$$

Then we have

$$\bigl(\rho\,\underset{\sim}{\mathrm{d}}xyz - j_x\,\underset{\sim}{\mathrm{d}}tyz - j_y\,\underset{\sim}{\mathrm{d}}tzx - j_z\,\underset{\sim}{\mathrm{d}}txy\bigr)\Big|_V = \rho\,\underset{+}{\mathrm{d}}uvw \tag{12}$$

and the current density gives no contribution.

Bibliography

("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

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