

The beauty of Graßmann spaces

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Draft of 4 February 2019 (first drafted 18 April 2017)

A new view on Graßmann spaces

Note: Dear Reader & Peer, this manuscript is being peer-reviewed by you. Thank you.

1 An algebra of geometric affine? objects

The idea of ‘a system of operations on geometric entities, and their consequences, analogous to those that algebra has on the numbers’ (Peano 2000 Preface) is quite old. Peano traces it back to Leibniz. Its first consistent formulation is universally attributed to Graßmann (1878; 1862). His work, though, has been tragically undervalued to the present day, as lamented by Barnabei, Brini, Rota (1985 § 1).

In this essay (in the etymological sense of this term, implying tentativeness and a want of finish) we present an algebra of geometric objects in the spirit of Graßmann and Peano, but with additional features coming from the calculus of exterior and interior objects (Veblen et al. 1932; Schouten et al. 1940; Schouten 1989; Burke 1983; 1987; 1995; Bossavit 2002; 2003).

OLD TEXT

✚ IMPORTANT: several statements below are *false* and will be corrected later. This happens because they concern notions at the ‘boundaries’ of the references given in the first section, which aren’t developed in any of those works. For example, statements about the scalar multiplication of outer-weighted objects. I’m working this out.

✚ Most important todo:

- clarify if and how Whitney forms and ‘elements’ fit into Graßmann algebra.

✚ Other refs to check: Barnabei et al. 1985; Crapo 2009; Brini et al. 2011; Bossavit 2004; Whitney 1957; Bossavit 2005; Arnold et al. 2006; Stern et al. 2015

2 Why

These notes present Graßmann spaces by combining Graßmann’s ideas as summarized by Peano (1888), ideas from geometric algebra Dorst et al. 2007; Li 2008, the concept of outer-oriented or ‘twisted’ geometric objects Veblen et al. 1932; Schouten et al. 1940; Schouten 1989; Burke 1983; 1987; 1995; Bossavit 2002; 2003, the theory of Whitney elements introduced by Bossavit 2005; 2003; Whitney 1957; Gawlik et al. 2010; Brini et al. 2011; Arnold et al. 2010 and insights from Barnabei, Brini, Rota’s (2011; 1985), Goldman’s (2002; 2000), and Crapo’s (2009) brilliant presentations.

✚ add also Bamberg et al. 1990; 1992; Frankel 2012

These notes do not contain anything new, except maybe the glue joining the works above. We will rest content if we manage to spark your curiosity and to spur you to take a look at them.

Graßmann spaces are beautiful. They share many important properties with projective spaces, yet are as intuitive as affine spaces. They represent notions like point, vector, functional, form on an equal geometric level. Their geometric operations have a beautiful algebra and include a simplified version of the operations of integration, differentiation, boundary and the de Rham theory of chains and cochains on manifolds. Their geometric notions can be given intuitive physical interpretations and be computationally used for the numerical solution of partial integro-differential equations. ✚ Maybe add something about their relation to Cayley algebras

The primitives of Graßmann spaces are the same as for affine spaces: point, straight line, plane, and so on; and the notion of parallelism. But each of these geometric objects has also a *weight density* and an *orientation*. Weight and orientation can be of *inner* kind, that is, defined on the geometric object itself; or of *outer* kind, that is, defined on the complement subspace of the object.

3 Basic geometric objects and properties

3.1 Basic geometric objects


The basic geometric objects of a Graßmann space are those of an affine space: points, straight lines, planes, and so on. We assume these are well-known to you. Let's agree to use the terms *point*, *line*, *plane* with their usual 0-, 1-, 2-dimensional meanings. We call *r-plane* their *r*-dimensional generalization, a 0-plane being a point, and so on. In a space of dimension N we call *hyperplane* an $(N - 1)$ -plane; there is only one N -plane, the space itself.

The notion of parallelism is very important, and we assume that it is also well-known to you. Just like in an affine space, there is no notion of angle or orthogonality, nor is there an absolute measure of length, area, and so on. We want to remind that the notion of parallelism, though, allows us to meaningfully speak of the ratio of lengths of segments lying on parallel lines, the ratio of areas lying on parallel planes, and so on.

Important: a weight density, to be introduced shortly, is *not* the same as a relative length, area, etc., although these notions are related.

3.2 Complements

The notion of *complement* is essential. Let's explain it in detail, starting with an example.

Consider a line in a 3-dimensional space. We can consider the set of this line and all lines parallel to it, see fig. . The 'points' of this set are the parallel lines of our original space. This set is 2-dimensional and has an affine structure and a notion of parallelism. For example, two parallel lines in this set correspond to two parallel planes containing some of the parallel lines our set is made of. This set is called the *complement* of our line.


If we select a plane intersecting our line in only one point, then there is a one-one correspondence between it and the complement. For example, a point on the plane corresponds to the unique line parallel to our initial one and passing through this point, and vice versa; and this line is a ‘point’ in the complement. We can therefore speak of objects, properties, constructions on the complement of our line by referring to a specific plane intersecting this line – remembering, though, that such properties are not specific to that plane but shared by all other planes intersecting this line.

Generalizing to an r -plane in an N -dimensional space, its complement is $(N - r)$ -dimensional and is in one-one correspondence with each $(N - r)$ -plane intersecting the original r -plane in only one point.

Two parallel r -planes have the same complement. The complement of a point is isomorphic to the whole space, and the complement of the whole space is in one-one correspondence with each point.

3.3 Weight densities

With each r -plane we can associate a *weight density*, or simply ‘weight’, represented by a non-negative real number. Of a point we simply say that it has a weight.

Assigning a weight to an r -plane is different from assigning a unit length to it, although the two notions are related. Consider for example the two parallel lines of fig. . The first has a total weight w associated with the segment AB , the second a total weight $2w$ associated with the segment CD , which has twice the length of AB . The two parallel lines have therefore the same weight density.

It is therefore only meaningful to compare the weight densities of parallel r -planes.

3.4 Orientations

It is easy to familiarize with the notion of orientation of a line, plane, volume. We can imagine to traverse a line with two points A, B by going from A to B or from B to A . On a plane we can imagine to ‘spin’ clockwise or counter-clockwise. In a volume we can identify two cork-screw senses. In each case there are two possible orientations. In the 0-dimensional case of a point we convene to have the two symbolic orientations ‘+’ and ‘-’.

Mathematically orientation is usually defined in terms of some equivalence class. This shows that, as with all notions defined in terms of equivalence classes, the mathematical formalism is still too primitive to capture it well. In this note we would like to consider orientation as intuitive and primitive, and will try not to use equivalence classes to define it.

The choice of an orientation on a line determines a unique ordering of every two distinct points on it. The choice of an orientation on a plane determines a unique ordering of every three non-collinear points on it, modulo an even number of permutations in the ordering. And so on for r -planes.

If a particular orientation is understood on a set of r -planes, we can understand a negative weight as indicating a reversed orientation. This corresponds to multiplying the positive weight by -1 , as explained below.


3.5 Inner and outer properties


Weight and orientation were defined above as ‘lying’ within an r -plane. For this reason they are called *inner*, and we say that an r -plane has an inner weight or is inner-oriented.

Now consider the complement of an r -plane in an N -dimensional space. This is an $(N - r)$ -dimensional affine space, and also the unique $(N - r)$ -plane within this space. We can associate with it a weight and an orientation. As it happens with all properties defined on a complement, they can be mapped one-to-one onto every $(N - r)$ -plane intersecting the initial r -plane in our original space; each such $(N - r)$ -plane therefore acquires a weight and an orientation. Vice versa we can speak of the weight and orientation of a complement by referring to any $(N - r)$ -plane intersecting our initial r -plane.

When we assign a weight w to the complement of an r -plane, we say that the latter has an *outer* weight $1/w$ – note the reciprocal. When we assign an orientation to the complement, we say that the r -plane has an *outer* orientation.


Inner and outer weights and orientations can be assigned independently of each other. Given an r -plane we have therefore four possibilities: we can assign to it an inner weight and inner orientation, an outer weight and outer orientation, an inner weight and outer orientation, an outer weight and inner orientation. 🧩 Can we also assign all four?

Let's see the special case of points in 2-dimensional space, i.e. a plane; see fig. . A point with inner weight and inner orientation simply has an associated real number, positive or negative. A point with outer weight m and outer orientation assigns a weight surface density $1/m$ and a circulation sense to the whole plane. A point with inner weight and outer orientation has an associated positive real number and a circulation for the plane; a negative real number indicates that the opposite circulation must be taken. Finally, a point with outer weight m and inner orientation assigns a weight surface density $1/m$ to the whole plane and has an associated $+$ or $-$ sign; a negative density indicates that the opposite sign must be taken.

The case of a line in 3-dimensional space is illustrated in fig. .


4 Scalar multiplication and sum

We indicate weighted and oriented r -planes simply by their weights, if this does not cause confusion. A negative weight means an opposite orientation with respect to a tacitly understood one.

Each r -plane with a weight and orientation, either inner or outer, can be multiplied by a real number. The result is the same r -plane with its weight multiplied by the absolute value of the number, and the same or a reversed orientation depending on whether the number is positive or negative. See fig. .

Two r -planes, both having inner or outer properties, can also be summed.

Let's start with the sum of weighted points, m_1, m_2, \dots . This is an interesting operation because it generalizes affine combination and leads to properties alike those of projective space.

First assume that the sum of the weights does not vanish: $\sum_j m_j \neq 0$. The result of the sum of the weighted points is a point given by the usual affine combination of the points with normalized coefficients $m_i / \sum_j m_j$, and with an associated weight $\sum_j m_j$.  **Add definition or example of affine combination.** The resulting weight and orientation are inner or outer depending on whether those of all the summand points are.

Now consider the case of vanishing total weight, and for simplicity consider just two points: $m_1 + m_2 = 0$. By writing $m_2 = \epsilon - m_1$ and considering smaller and smaller values of ϵ , we see that the resulting

point is ‘at infinity’ and has a vanishing weight; its orientation is therefore also undetermined.

Points of this kind are called *vectors*. They have indeed all properties of usual ‘free vectors’. Consider for example two points A, B with unit weights and positive orientation, and the vector $v := B - A$. By summing this vector to the point A we obtain the point B : $A + v = B$. Thus v really behaves as a vector from A to B , translating the point A to B when summed to the former.

Vectors also give Graßmann spaces properties typical of projective spaces. Consider for example the unit-weight points A, B on the line a , and two unit-weight points C, D on the parallel line b and having the same distance as A, B . The vectors $B - A$ and $D - C$ are the same. This can be seen by rewriting the equality $B - A = D - C$ as $A + D = B + C$, which is indeed satisfied because the result of both $A + D$ and $B + C$ is the point at the centre of their parallelogram, with a weight of 2. This means that the vector $B - A$ is a point belonging to both parallel lines a and b , which therefore ‘meet at infinity’. A vector therefore characterizes the common direction, or *attitude*, and inner orientation of a set of parallel lines.

But the presence of weights leads to a much richer and interesting structure than in affine and projective spaces. Consider for example the vector $v := B - A$, and sum it to the point A having a large weight m , which we denote mA . The result is the point $B + (m - 1)A$, located at relative distances $(m - 1)/m$ from B and $1/m$ from A . As the weight m increases, this resultant point gets closer to A . Thus the amount by which a weighted point is translated by a vector depends on the point’s weight, a ‘heavy’ point being translated less than a ‘light’ one.

For an exploration of these fascinating properties and their use we recommend Goldman’s brilliant articles Goldman 2000; 2002.

✚ Explore and add more curiosities of this kind

Above we spoke of the total sum of the weights of some points, implicitly assuming that these weights can somehow be summed independently of the points they belong to. This is possible because all points have the same complement – the whole space – or equivalently because they are all ‘parallel’ to one another.

The weights of parallel r -planes can similarly be added, taking care of their orientations, independently of the r -planes they belong to. The

reason is exactly the same as for the comparison of lengths, surfaces, etc., of parallel objects in an affine space.

Sums of parallel r -planes with total zero weight are objects called r -vectors, analogous to vectors. They can be imagined as r -planes ‘at infinity’ with vanishing weight. An r -vector is ‘common’ to a set of parallel $(r + 1)$ -planes, and therefore characterizes their r -direction or attitude and inner orientation. Again this gives many projective-like properties to a Graßmann space.

✚ The following example needs the introduction of wedge first Let’s consider the two-dimensional case of *bivectors*. Consider the antiparallel lines $A \wedge B$ and $-C \wedge D$, the distances AB and CD being equal. The total weight of these lines is zero. The sum $A \wedge B - C \wedge D$ is a bivector. It can be considered as a ‘line’ that ‘goes around’ the plane containing $A \wedge B$ and $C \wedge D$, with a circulation sense agreeing with those of AB and CD . See fig. ✚ .

The segment CD is parallel to AB , hence there is a vector w such that $C = A + w$ and $D = B + w$. The bivector $\alpha := A \wedge B - C \wedge D$ is therefore equal to $(B - A) \wedge w$, or $\alpha = v \wedge w$, with $v := B - A$. A bivector is therefore equal to the product of two vectors.

Very interesting is the case of r -planes that are skew, that is, do not lie on a common $(r + 1)$ -plane. Their weights cannot meaningfully be summed. ***

✚ Explain sum of lines and planes with examples

The literature uses the term *extensors* to indicate points, lines, etc. that can be both of the usual kind and ‘at infinity’, reserving ‘point’ etc. only for those not at infinity. ✚ Shall we also use it?

5 Join, meet, contraction

Besides sum and scalar multiplication, Graßmann spaces have three kinds of multiplicative operations: join, meet, and contraction.

The join operates on inner-weighted objects, giving a new inner-weighted object of higher dimension. The meet operates on outer-weighted objects, giving a new outer-weighted object of lower dimension, hence higher codimension.

5.1 Wedge or join

The common notion of vector is usually associated with a line, but we have seen that vectors in Graßmann spaces are special kinds of points, and indeed they have properties more similar to points than lines.

A line with inner weight and orientation also has many similarities to a vector. The main difference is that it is not a free vector, because it is bound to that particular line; but it is not a bound vector either, because it has no definite initial point within that line.

✚ Not sure whether the paragraphs above are understandable

Graßmann spaces have also another operation, the *wedge product*, which yields *inner-weighted* objects of higher dimension from lower-dimensional ones.

Let's start with two inner-weighted, inner-oriented points m_1, m_2 . The result of their wedge product $m_1 \wedge m_2$ is the line passing through them, having weight $m_1 m_2$ and inner orientation going from the first to the second; if the total weight is negative then the opposite orientation must be taken. See fig. ✚.

✚ Clarify the difference between 'wedge' and 'join' in the literature

The wedge product of *outer-weighted* objects yields a lower-dimensional object instead. ✚ That's why Barnabei, Brini, Rota Barnabei et al. 1985; Brini et al. 2011 indicate the wedge of inner-weighted objects with \vee and that of outer-weighted ones with \wedge , reminding of \cup and \cap .

Consider the exterior algebra of a 4-dimensional vector space, with four linearly independent vectors e_1, e_2, e_3, e_4 . The products $e_1 \wedge e_2$ and $e_3 \wedge e_4$ can be associated with two planes containing the respective vectors. These planes intersect at one point only, the origin. It is for this reason that the sum $e_1 \wedge e_2 + e_3 \wedge e_4$ cannot be written as $a \wedge b$.

6 Contraction or meet

✚ There are two main approaches around:

- The 'geometric-algebra school' introduce a metric and a dual space, and define the meet in terms of contractions with duals and scalar products. Whitney seems to do likewise, at least at the beginning of his book.
- The 'Rota school' introduce a volume element (N -covector) and define the meet in terms of the latter. Their approach is to view a

Peano space in terms of invariance under the special linear group – that is, the preservation of volumes.

Neither school introduce a distinction between inner- and outer-weighted objects. Barnabei, Brini, & Rota make one valid observation:

Elie Cartan found the regressive product to be superfluous and awkward. By vector space duality, a pairing of the two exterior algebras of V and V^* could easily be made with only one kind of product, the one that came to be called the wedge. [...] [However, T]he dual space V^* of a vector space V plays no role in such a calculus: a hyperplane is an object living in V , and its identification with a linear functional is a step backwards in clarity.

I think that their geometric viewpoint would be even more elegant by distinguishing between inner and outer weights. Covectors are like vectors, but characterized by an outer weight. From this point of view, moreover, we note that the traditional wedge of covectors yields their *intersection*, whereas the wedge of vectors give their *union*. This different behaviour is a direct consequence of the only ways in which inner and outer weights can be meaningfully be combined – and it doesn't require the notion of contraction or a special N -covector; both may be introduced later.

7 Bases

8 Boundary and differential

✚ The following sections would aim to embed the calculus of simplicial r -chains, presented in Bossavit (2005 esp. §§ 23.2–3) into the algebra of Graßmann spaces. It's not clear whether this is meaningful or possible yet, but some very interesting mathematical curiosities suggest that those two theories are part of one common framework.

One curiosity is that the boundary of an r -chain always seems to correspond to an $(r - 1)$ -vector, i.e. an $(r - 1)$ -plane 'at infinity' with vanishing weight. Consider for example the segment AB . Its boundary is $\partial(AB) = B - A$, which interpreted within Graßmann-space algebra is a vector, a point at infinity with vanishing weight. Same goes for the triangle ABC : its boundary is $\partial(ABC) = AB + BC + CA$. If we interpret juxtaposition as wedge product, we see that the result is a bivector, a line

at infinity with vanishing weight. So one wonders whether the calculus of simplicial r -chains can be somehow interpreted as a sub-calculus of r -vectors in Graßmann space. If this is true, then what is the relation between the boundary operator ∂ and the Graßmann operations?

A related question appears considering for example a 0-chain like $\sum_i m_i A_i$. In the chain calculus this is just a formal expression; in Graßmann space it is equivalent to a specific point B with a specific weight. Does this mean that, when we 'integrate' a 0-form on this chain, the result is the same as evaluating it on the resulting point B ? It's indeed a possibility, given the affine structure in which both are embedded.

In the theory of chains and differential forms on manifolds, a simplicial r -chain is defined as a *formal* sum $\sum_i w_i C_i$ of simplicial r -submanifolds C_i with weights w_i . Differential r -forms can then be integrated on such chains.

In a Graßmann space we can interpret the formal sum above as a new, specific r -plane C' with an associated weight and, owing to the affine properties of the space and its algebra, the 'integral' of an r -form on this r -plane is equal to its integral on the initial chain. This is the basis for discretization and interpolation schemes to numerically solve partial integro-differential equations on manifolds.

✚ Whitney forms = basis forms in the spaces of r -planes derived from $N + 1$ basis points?

9 Whitney, de Rham, computation

10 Representing physical quantities

Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area. [. . .]

There is another distinction between different kinds of directed quantities [. . .]. This is the distinction between longitudinal and rotational properties. *Maxwell (1881 §§ 13, 15)*

Physical quantities are associated with a point, like temperature; with a line, like electromotive force; with a surface, like mass or energy flux or current; with a volume, like charge. And we suppose them to vary with a particular kind of continuity with respect to variations of these

geometric extensions. For this reason they are represented by inner- or outer-oriented differential forms, defined on a manifold and meant to be evaluated at a point or integrated over a line, surface, volume. The laws that govern them are expressed as particular mathematical relations among such integrals.

Approximate solutions to particular problems may be obtained by selecting a discrete and finite set of points, lines, surfaces, volumes on the manifold, requiring the laws to be satisfied within this set, and then interpolating the values of the fields on geometric extensions outside this set.

For example, we can associate a temperature with each point in such discrete set, and an electromotive force with each line, a current with each surface, a charge with each volume therein. A Graßmann space comes handy in such a discretization: the amount of a physical quantity over a particular point, line, and so on is represented by a single geometric entity, like a weighted point and so on. The interpolation is then automatically achieved by the sum of such geometric entities in the Graßmann space.

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