The beauty of Grassmann spaces

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line art by I. Bengtsson

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A new view on Grassmann spaces

Note: Dear Reader & Peer, this manuscript is being peer-reviewed by you. Thank you.

Synopsis:

- Intro
- Overview
- Primitives (flats, weights, orientations)
- Sum of points with inner weight
- Sum of points with outer weight
- Sum of lines and higher-dim objects, both kinds of weight
- Product (join) of objects with inner weights
- Product (meet) of objects with outer weights
- Product between objects with inner and outer weights
- Combination of all products into a non-associative one
- Further discussion: non-necessity of forms etc

1 An algebra of geometric objects

The idea of a calculus analogous to 'algebraic calculus, but in which the entities on which the calculations are carried out are geometric objects, instead of numbers' (Peano 1888 ch. I, our transl.) is quite old. Peano traced it back to Leibniz. He also expressed the opinion that 'before long this geometric calculus, or something analogous, will replace methods currently adopted in higher education' (Peano 1888 Prefazione, our transl.).

The first consistent formulation of such a calculus is universally attributed to Grassmann (1878; 1862). Grassmann's work was undervalued and suffered a convoluted history, an outlook of which is given in the translator's notes of that work (Grassmann 1995; 2000) and in the introductory parts of several other works (Peano 1888; Barnabei et al. 1985; Crapo 2009; Browne 2012; Vargas 2016). His work is surely still undervalued today. We believe part of the reason to be the beautiful but also complicated and partially antagonist ways in which it has been developed and presented in the past fifty years (examples are Hestenes

1968; Hestenes et al. 1987; Barnabei et al. 1985; Dorst et al. 2002; Li 2008; Crapo 2009; Brini et al. 2011; Dorst et al. 2011; Gunn 2011; Browne 2012; González Calvet 2016; Vargas 2016). These modern presentations would probably be too difficult to digest for high-school students.

Yet the basic ideas behind Grassmann's work are quite intuitive and easy to visualize. They seem within the reach of a high-school student. And they are very powerful, allowing a student to state in a couple of lines geometric properties that would require more cumbersome formulae in analytic geometry.

In this essay – in the etymological sense of this term, implying tentativeness and a want of finish – we present an algebra of geometric objects in the spirit of Grassmann and Peano. It differs from theirs and from modern treatments in some mathematical respects, such as the avoidance of metric notions and the inclusion of 'exterior' and 'interior' geometric properties (Veblen et al. 1932; Schouten et al. 1940; Schouten 1989; Burke 1983; 1987; 1995; Bossavit 2002; 2003). It also differs from modern treatments in the presentation style, which is mainly visual, not formal, and tries to avoid advanced concepts (duals, adjoints, matroids, and so on).

2 Overview

3 Primitives

Our arena is three-dimensional space with its notion of *parallelism*. We won't use vectorial or metric notions such as origin, distance, or angle; technically speaking we're considering an affine space. Denote the dimension of space by $N = 3$.

Our attention will focus on points, straight lines, planes, and the whole space. By 'line' we'll always mean 'straight line'. When we need to consider any one of these kinds of geometric objects, we'll call it a *flat* (for obvious reasons). When we need to keep in mind the dimension of a flat, but the dimension is arbitrary, say r, we'll speak of an r*-flat*. Thus a point is a 0-flat, a line a 1-flat, and so on.

The calculus to be introduced allows us to express geometric statements about flats in a compact symbolic way. Some are very familiar statements of solid geometry. For example, that a line passes through two distinct points, or that a line is the intersection of two incident planes. Others are statements about relations that are not encountered in pure solid geometry, but that we encounter in physics. For example, the statement that a point is the centre of mass of two point-masses located at two particular points. This second kind of statements makes our space richer and has consequences for the first, more familiar kind of statements.

To make the generalized kind of statements we need to enrich the geometric objects we're considering. To each geometric object of our four basic kinds we attribute two characteristics: a *weight* and a *sense*, and each of these can be of two possible types: *inner* or *outer*. For example, a point can have an inner weight and an inner sense, or an outer weight and an outer sense; or an inner weight and an outer sense; or, finally, an outer weight and an inner sense. A line can also comes in those four varieties; and so on. Weight comes in continuous degrees, whereas sense has only two possible values.

3.1 Inner weight

A point with an inner weight is very akin to a point-mass in physics. The inner weight can indeed be thought of as the mass (or charge, when combined with an inner sense as explained below) of the point and leads to similar statements and effects. We can freely choose a unit of mass, and it is possible to compare the mass of any two points. The weight of a point can thus be represented by a non-negative real number.

The inner weight of a line can be thought of as a fixed length on the line itself. Picture Unlike the case of the points, a unit weight is not defined, but it is possible to compare two possible weights of the same line or of two parallel lines, but not of two lines that are not parallel (incident or skew). With such comparison we can determine the ratio of the two weights; it is a non-negative real number. Possibly add discussion of how such weights are compared Picture

The inner weight of a plane can be thought of as a fixed area on the plane; but, if we visualize this area as the region within a closed curve on the plane, we must keep in mind that the shape of such region is unimportant. Picture A unit weight is not defined, but it is possible to compare and determine the ratio of the inner weights for the same plane or for two parallel planes, but not for two incident planes.

The inner weight of the whole space can be thought of as a fixed volume; but, as in the case of the plane, if we visualize this volume as the region within a closed surface we must keep in mind that the shape of such surface is unimportant. Also in this case a unit weight is undefined but we can compare and determine the ratio of two possible weights. Picture

3.2 Inner sense

The inner sense of a point is simply a sign, '+' or '−'. The combination of inner weight and inner sense can thus be represented by a real number.

The inner sense of a line can be visualized as the direction of motion by which we travel on the line, or of the direction of a flow happening on the line, both familiar concepts in physics. As for inner weight, it is possible to compare the senses of parallel lines, but not of non-parallel ones.

The inner sense of a plane can be visualized as the direction of a circular motion taking place on the plane. We can compare the inner senses of parallel planes but not of non-parallel ones.

The inner sense of space (sounds funny) can be visualized as the direction of a screw motion happening in space, again a familiar concept in physics. We usually speak of a right-handed or dextrogyre sense, '↻', and of a left-handed or laevogyre sense, '↺'.

Pictures

3.3 Outer properties

Before introducing outer weights and senses it may be useful to help intuition about what we mean with 'outer'. Outer properties can be visualized in several ways; we discuss two, the first more pictorial, the second more technical.

Consider a point, and imagine to displace it a little. The freedom we have in this displacement is three-dimensional. We can thus imagine a three-dimensional region surrounding the point, representing its possible displacements.

Consider a line, and imagine to displace it a little keeping it parallel to itself. The freedom we have in this displacement is effectively two-dimensional: two displacements that can be obtained from each other by a displacement parallel to the line don't lead to observable differences. We can thus imagine a two-dimensional region surrounding the line, representing its possible effective displacements.

For a plane, the same process lead us to visualize a one dimensional region extending on both sides of the plane.

For the space there is no effective displacement: the region to visualize would be zero-dimensional.

The 'outer region' of an r-flat has therefore dimension $N - r$.

A more technical way of visualizing outer properties is in terms of equivalence classes. For this, let's first agree to extend the meaning of 'parallel' by saying that every point is by definition parallel to every other point.

Consider a point. The set of points parallel to (or coincident with) it is the whole space. Let's call this the *complementary space* of the point.

Consider a line, and consider the set of lines parallel to it. This set is called the complementary space of the line. Each element of this set is a line. The complementary space is two-dimensional. We can in fact set up a one-to-one association between each element (a line) of this set with a point on a plane: select a plane that intersects the original line (without containing it in full). If we select a point on this plane, there's a unique line parallel to the original one that passes through the chosen point. Vice versa, if we select a line parallel to the original one, this line passes through a unique point of the plane. Each point of this plane is thus in one-to-one correspondence with an element of the complementary space. Note that in this way we can associate the complementary space with every plane that intersect the original line, so none of such planes is preferable to the others. We can thus visualize the complementary space of a line as a plane intersecting it, keeping in mind that the exact position and inclination of the plane don't matter.

In an analogous way we can construct and visualize the complementary space of a plane: it has one dimension, its elements are the planes parallel to the original one, and it can be visualized as a line intersecting it, keeping in mind that the exact position and inclination of such line don't matter.

In the case of the whole space, the complementary set has just one element: the space itself. It is therefore zero-dimensional.

The complementary space of an r-flat has therefore dimension $N - r$.

3.4 Outer weights

Outer weights are analogous to inner weights, with the essential difference that they are defined *on the complementary space* of the point, line, plane, or space under consideration. Since the complementary space of an r-flat is $(N - r)$-dimensional, the outer weights of point, line, plane, space are akin to the inner weights of space, plane, line, point.

A point with an outer weight can be visualized as a point with an associated volume. We can imagine such volume as enveloping the point or being beside it, keeping in mind that the shape and location of such volume aren't important.

A line with an outer weight can be visualized as a line crossed by a flat surface of given area; or as a tube, containing the original line, made of parallel lines and having that cross-sectional area. The cross-sectional shape of the tube doesn't matter.

A plane with an outer weight can be visualized as a plane crossed by a segment with given length; or as two parallel planes, one on each side of the original plane, or one of which is the original plane.

Finally, space with an outer orientation can be visualized as the whole space with an associated non-negative number.

Pictures

3.5 Outer senses

Outer senses are senses chosen on the complementary space of the flat under consideration. As in the case of outer weights this leads to properties for points, lines, planes, space that are reversed in dimensionality with respect to inner senses.

A point with outer sense can be visualized as a point with a screw motion of the space around it.

A line with outer sense can be visualized as a line with a rotatory motion around it.

A plane with outer sense can be visualized as a plane with an associated crossing direction from one side to the other.

Finally, outer sense for space is simply a sign, '+' or '−'. The combination of outer sense and outer weight for space is thus represented by a real number.

Pictures

3.6 Summary: zoology of geometric objects

Each point, line, plane, and space comes thus in four varieties: inner-weighted and inner-oriented, outer-weighted and outer-oriented, inner-

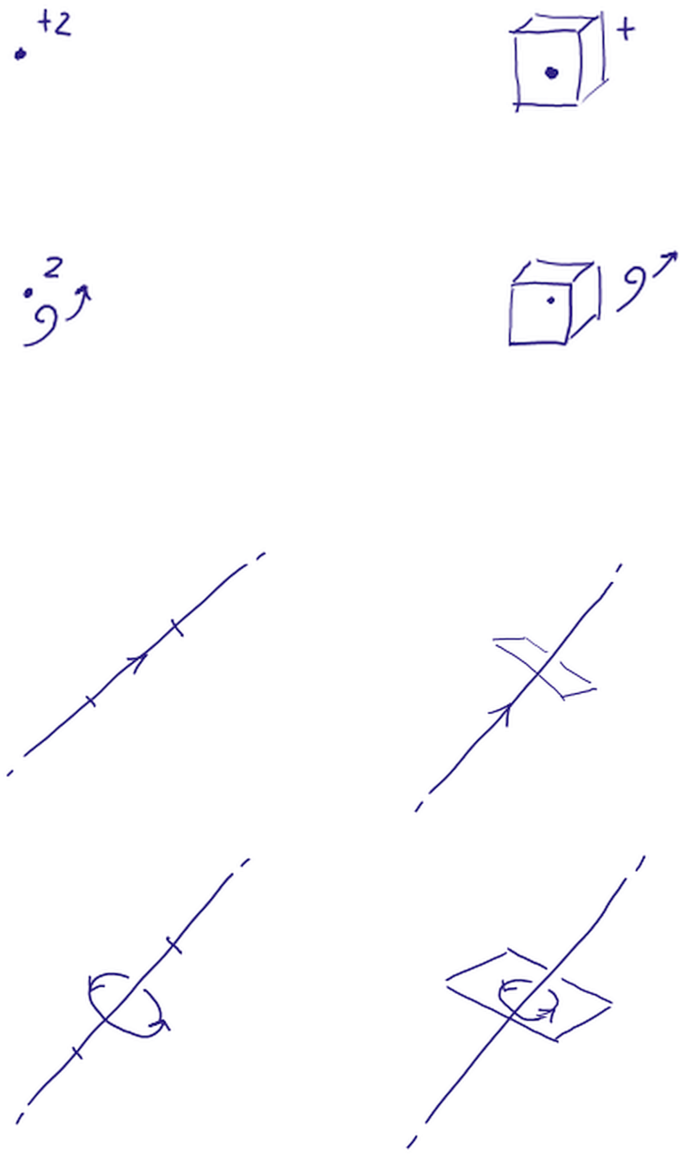

Figure 1 The four kinds of points and lines

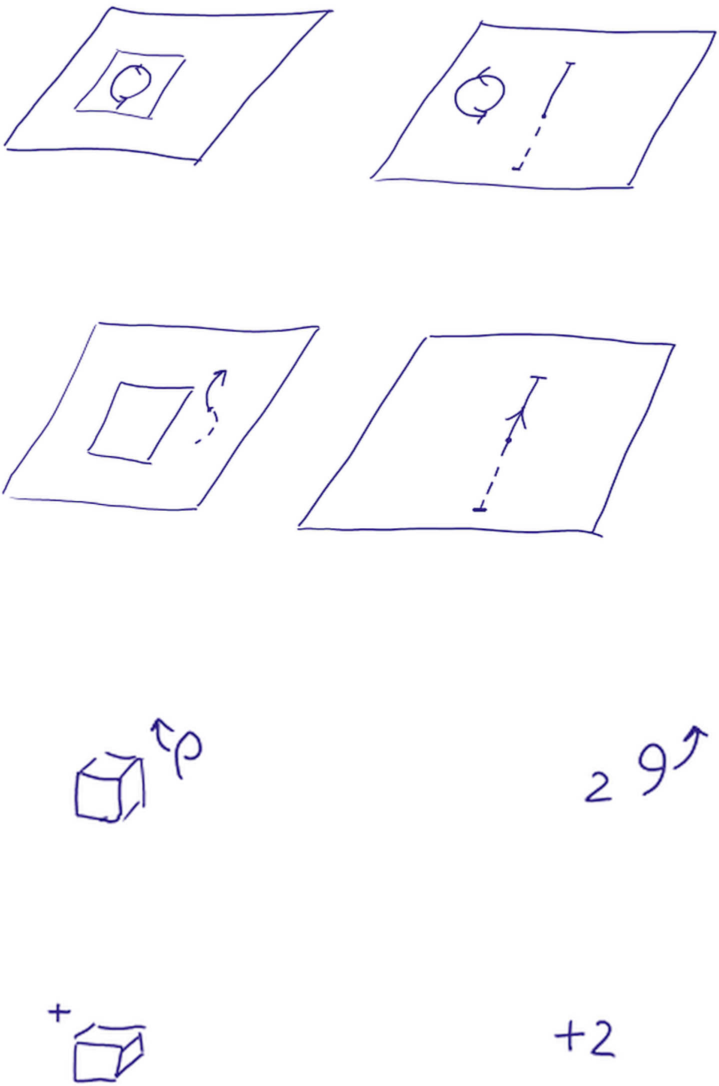

Figure 2 The four kinds of planes and space

weighted and outer-oriented, and outer-weighted and inner-oriented. We are thus dealing with sixteen different kinds of geometrical objects. Their visual representation is shown in figs 1–2.

We have mentioned that the inner weight and inner sense of a point can be jointly represented by a real number: its absolute value represents the weight, its sign the sense. The same is true for the outer weight and outer sense of space.

In the cases of a point with inner weight and outer sense, and of space with outer weight and inner sense, the combination of weight and sense is represented by a non-negative number with a screw sense. It is convenient to consider negative numbers also in these cases, however: the presence of a negative sign simply indicates that the opposite screw sense should be considered; for example, $\circlearrowleft(-2) \equiv \circlearrowright 2$.

The four kinds of space are especially important. Space with outer weight and outer sense is just represented by a real number. We'll see that it has all the properties of a *scalar*, and therefore we call it that way. Space with outer weight and inner sense behaves like a scalar with a screw sense (cf. Bossavit 1991 § 3.2.1); we call it an *axial scalar*. Space with inner weight and outer sense behaves like a 'volume element', and we call it a *pseudoscalar*. Finally, we call space with inner weight and inner sense an *axial pseudoscalar*. Note that scalars and axial scalars have a well defined unit value, whereas pseudoscalars of either kind don't.

We can indicate the weight of a flat independently of its sense by an absolute-value notation, '$|\cdot|$'.

Note on Maxwell's discussion of such objects

> Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area. [...]
>
> There is another distinction between different kinds of directed quantities [...]. This is the distinction between longitudinal and rotational properties. *Maxwell (1881 §§ 13, 15)*

- mass and charge density: space with inner weight and outer sense
- current density: line with outer weight and inner sense
- momentum density: line with outer weight and inner sense?
- electric field: plane with outer weight and outer sense
- magnetic field: line with outer weight and outer sense

Is it best to introduce products first??

4 Sum of kindred objects

We next show how objects in each of these sixteen categories can be combined together to obtain an object in the same category. We assume that the reader already has an intuitive idea of how to sum or subtract two lengths on the same line or on two parallel lines, two areas on the same plane or on two parallel planes, and two volumes. Remember that such sums and subtractions do not require the notions of distance or angle.

4.1 Sum of points

The sum of points with inner weight and sense generalizes the calculation of a centre of mass and the operation of affine combination. It's associative and commutative: it can be done in an arbitrary order.

The sum of the point A with inner weight and sense a and the point B with inner weight and sense b is a point C of inner weight and sense $a+b$, located on the line determined by A and B, and such that the segment with endpoints C, A is $|b/(a+b)|$ times the segment with endpoints A, B, and the segment with endpoints B, C is $|a/(a+b)|$ times the segment with endpoints A, B. A moment's thought shows that these two requirements determine a unique location for C. If the ratio $b/(a+b)$ is negative, then C and B lie on opposite sides of A; analogously for the ratio $a/(a+b)$.
Pictures

For the sums of three non-collinear points and four mutually non-coplanar points similar constructions exist, with the ratios referring to areas and volumes.

5 Products

Let's indicate an r-flat with inner weight with $(\dot{r}, N-r)$ and an r-flat with outer weight with $(r, N \dot{-} r)$: this redundant notation help us remember the dimensions of the flat and of its complementary space, and of where the weight resides.

This table gives the type of flat found by multiplying flats with inner weights – the *progressive product*:

$\vee$	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$
$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	
$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$		
$(\dot{2},1)$	$(\dot{3},0)$			
$(\dot{3},0)$				

the empty entries being zero.

This table gives the type of flat found by multiplying flats with outer weights – the *regressive product*:

$\wedge$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$
$(0,\dot{3})$				$(0,\dot{3})$
$(1,\dot{2})$			$(0,\dot{3})$	$(1,\dot{2})$
$(2,\dot{1})$		$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$
$(3,\dot{0})$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$

the empty entries being zero.

This table gives the type of flat found by multiplying flats with inner weights and flats with outer weights – the *contraction*:

$\cdot$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$
$(\dot{0},3)$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$	$(\dot{0},3)$
$(\dot{1},2)$	$(2,\dot{1})$	$(3,\dot{0})$	$(\dot{0},3)$	$(\dot{1},2)$
$(\dot{2},1)$	$(3,\dot{0})$	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$
$(\dot{3},0)$	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$

From the last two tables we see that a space with outer weight, type $(3,\dot{0})$, behaves like a scalar. Moreover, the general structure of the three tables suggest that we could complete the first two in the following way:

	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$
$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$
$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$
$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$
$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$

	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$
$(0,\dot{3})$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$
$(1,\dot{2})$	$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$
$(2,\dot{1})$	$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$
$(3,\dot{0})$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$

On the other hand we can consider the table for the contraction as really an extension of progressive (above the anti-diagonal) and regressive (anti-diagonal and below) products. Likewise, the new parts in the last two tables extend the regressive and progressive products. This way we are left with two products with the following tables:

$\vee$	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$
$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$		$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$	
$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$			$(2,\dot{1})$	$(3,\dot{0})$		
$(\dot{2},1)$	$(\dot{3},0)$				$(3,\dot{0})$			
$(\dot{3},0)$								
$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$		$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	
$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$			$(\dot{2},1)$	$(\dot{3},0)$		
$(2,\dot{1})$	$(3,\dot{0})$				$(\dot{3},0)$			
$(3,\dot{0})$								

empty entries being zero.

$\wedge$	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$
$(\dot{0},3)$				$(0,\dot{3})$				$(\dot{0},3)$
$(\dot{1},2)$			$(0,\dot{3})$	$(1,\dot{2})$			$(\dot{0},3)$	$(\dot{1},2)$
$(\dot{2},1)$		$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$		$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$
$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$
$(0,\dot{3})$				$(\dot{0},3)$				$(0,\dot{3})$
$(1,\dot{2})$			$(\dot{0},3)$	$(\dot{1},2)$			$(0,\dot{3})$	$(1,\dot{2})$
$(2,\dot{1})$		$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$		$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$
$(3,\dot{0})$	$(\dot{0},3)$	$(\dot{1},2)$	$(\dot{2},1)$	$(\dot{3},0)$	$(0,\dot{3})$	$(1,\dot{2})$	$(2,\dot{1})$	$(3,\dot{0})$

empty entries being zero.

Note the following relations regarding weights, similar to the multiplication of signs:

$$
\begin{aligned}
&\text{inner} \vee \text{inner} = \text{inner} &\qquad& \text{inner} \wedge \text{inner} = \text{outer} \\
&\text{outer} \vee \text{outer} = \text{inner} && \text{outer} \wedge \text{outer} = \text{outer} \\
&\text{inner} \vee \text{outer} = \text{outer} && \text{inner} \wedge \text{outer} = \text{inner}
\end{aligned}
\tag{1}
$$

The question whether we can replace the zero entries of the first two tables pivots around this problem:

Consider five points with inner weight: $A_1, \dots, A_5$. The fact that all of them lie in the same space can be expressed with the progressive product as $A_1 \vee \cdots \vee A_5 = 0$. The regressive product of the space $A_1 \vee \cdots \vee A_4$ and the point A_5 is instead non-zero: $(A_1 \vee \cdots \vee A_4) \wedge A_5 \neq 0$. There is no way to make allowance for both these statements with one product only, even if non-associative.

OLD TEXT

5.1 r-flats

r-dimensional *flats* or r-flats: a 1-flat is a straight line, a 2-flat a plane, the 3-flat is the whole space, and a 0-flat is a point. Two flats can be parallel, incident, or skew example pictures. Every r-flat is itself an affine space of r dimensions.

5.2 r-coflats

An r-dimensional *co-flat* or r-coflat, is the set of r-flats parallel to a given r-flat. For example, given a straight line, the set of lines parallel to it is a 1-flat. Every r-coflat is itself an affine space of $N - r$ dimensions. Every r-flat determines a unique r-coflat, but not vice versa.

We can always set up a non-canonical identification between an r-coflat and an $(N - r)$-flat. Consider for example a line. The set of lines parallel to it constitutes a 1-coflat or co-line. Select a plane that intersects this line without containing it. Each line parallel to the given one passes through one point of this plane, in a one-one correspondence. The points of this plane thus correspond to the points of the 1-coflat determined by the given line. Every plane that intersect the original line has this isomorphism with the 1-coflat; this is why this identification is non-canonical. In a similar fashion, if we select a plane, the set of planes parallel to it constitute a 2-coflat or co-plane. This set is an affine space of 1 dimension. Choose a line intersecting the original plane in one point. Each point on this line selects a unique plane parallel to the original one, and thus selects one point on the 2-coflat.

5.3 r-extensions

Given an r-flat, we can select a r-dimensional region of it (not necessarily simply connected) delimited by a closed $(r - 1)$-dimensional manifold. For example, a 2-dimensional region bounded by closed curve on a plane add example pictures. The absence of a metric doesn't allow us to speak of the exact extension of this region – its length, or area, or volume. But parallelism allows us to give a numerical value to the *ratio* of the extensions of two different regions on the same r-flat or on parallel r-flats. In particular, it allows us to say whether two such regions have the same extension. An r-extension is the equivalence class of the regions having the same extension. It can be visualized as a region on an r-flat, but the location and shape of this region don't matter, as long as we preserve its extension.

We define a 0-extension as a positive real number.

Pictures

5.4 r-coextensions

Now consider an r-coflat. It's an affine space, so we can consider an $(N-r)$-dimensional region on it and an $(N-r)$-extension. This is called an r-coextension.

Coextensions can be visualized in two ways. Consider a 1-coextension. We select a line, and this determines a co-line. We can visualize this co-line as a plane intersecting the original line. A 1-coextension is visualized as a closed 2-dimensional region on this plane, with the understanding that its location and shape don't matter, as long as its area is the same. Thus a 1-coextension can be identified with a 2-dimensional flat region intersecting a given line picture. On the other hand, consider the abstract 2-dimensional region in the co-line. This region has a boundary; the points of this boundary represent lines parallel to the original one. We can thus visualize a 1-coextension as a tube formed by parallel lines, with the understanding that the cross-sectional shape of the tube doesn't matter, as long as its cross-sectional area with respect to any plane cutting it, remains the same.

Proceeding in a similar way, we can visualize:

- a 2-coextension as a segment on a line intersecting a given plane, or as two parallel planes;
- a 0-coextension as a 3-dimensional region of space, with the understanding that its shape and location don't matter, as long as its volume is the same;
- a 3-coextension is defined as a positive real number.

Pictures

5.5 Orientations

Our primitive objects are point, line, plane, space. But we equip each of these with two characteristics: a *weight* and an *orientation*. Weight and orientation can each be of two kinds: *inner*, that is, defined on the geometric object itself; or of *outer* kind, that is, defined on the complement subspace of the object.

OLD TEXT

IMPORTANT: several statements below are *false* and will be corrected later. This happens because they concern notions at the 'boundaries' of the references given in the first section, which aren't developed in any of those works. For example, statements about the scalar multiplication of outer-weighted objects. I'm working this out.

Most important todo:

- clarify if and how Whitney forms and 'elements' fit into Grassmann algebra.

Other refs to check: Barnabei et al. 1985; Crapo 2009; Brini et al. 2011; Bossavit 2004; Whitney 1957; Bossavit 2005; Arnold et al. 2006; Stern et al. 2015

6 Why

These notes present Grassmann spaces by combining Grassmann's ideas as summarized by Peano (1888), ideas from geometric algebra Dorst et al. 2007; Li 2008, the concept of outer-oriented or 'twisted' geometric objects Veblen et al. 1932; Schouten et al. 1940; Schouten 1989; Burke 1983; 1987; 1995; Bossavit 2002; 2003, the theory of Whitney elements introduced by Bossavit Bossavit 2005; 2003; Whitney 1957; Gawlik et al. 2010; Brini et al. 2011; Arnold et al. 2010 and insights from Barnabei, Brini, Rota's (2011; 1985), Goldman's (2002; 2000), and Crapo's (2009) brilliant presentations. add also Bamberg et al. 1990; 1992; Frankel 2012

These notes do not contain anything new, except maybe the glue joining the works above. We will rest content if we manage to spark your curiosity and to spur you to take a look at them.

Grassmann spaces are beautiful. They share many important properties with projective spaces, yet are as intuitive as affine spaces. They represent notions like point, vector, functional, form on an equal geometric level. Their geometric operations have a beautiful algebra and include a simplified version of the operations of of integration, differentiation, boundary and the de Rham theory of chains and cochains on manifolds. Their geometric notions can be given intuitive physical interpretations and be computationally used for the numerical solution of partial integro-differential equations. Maybe add something about their relation to Cayley algebras

The primitives of Grassmann spaces are the same as for affine spaces: point, straight line, plane, and so on; and the notion of parallelism. But each of these geometric objects has also a *weight density* and an *orientation*. Weight and orientation can be of *inner* kind, that is, defined on the geometric object itself; or of *outer* kind, that is, defined on the complement subspace of the object.

7 Basic geometric objects and properties

7.1 Basic geometric objects

The basic geometric objects of a Grassmann space are those of an affine space: points, straight lines, planes, and so on. We assume these are well-known to you. Let's agree to use the terms *point*, *line*, *plane* with their usual 0-, 1-, 2-dimensional meanings. We call *r-plane* their r-dimensional generalization, a 0-plane being a point, and so on. In a space of dimension N we call *hyperplane* an $(N-1)$-plane; there is only one N-plane, the space itself.

The notion of parallelism is very important, and we assume that it is also well-known to you. Just like in an affine space, there is no notion of angle or orthogonality, nor is there an absolute measure of length, area, and so on. We want to remind that the notion of parallelism, though, allows us to meaningfully speak of the ratio of lengths of segments lying on parallel lines, the ratio of areas lying on parallel planes, and so on.

Important: a weight density, to be introduced shortly, is *not* the same as a relative length, area, etc., although these notions are related.

7.2 Complements

The notion of *complement* is essential. Let's explain it in detail, starting with an example.

Consider a line in a 3-dimensional space. We can consider the set of this line and all lines parallel to it, see fig. . The 'points' of this set are the parallel lines of our original space. This set is 2-dimensional and has an affine structure and a notion of parallelism. For example, two parallel lines in this set correspond to two parallel planes containing some of the parallel lines our set is made of. This set is called the *complement* of our line.

If we select a plane intersecting our line in only one point, then there is a one-one correspondence between it and the complement. For example, a point on the plane corresponds to the unique line parallel to our initial one and passing through this point, and vice versa; and this line is a 'point' in the complement. We can therefore speak of objects, properties, constructions on the complement of our line by referring to a specific plane intersecting this line – remembering, though, that such properties are not specific to that plane but shared by all other planes intersecting this line.

Generalizing to an r-plane in an N-dimensional space, its complement is $(N-r)$-dimensional and is in one-one correspondence with each $(N-r)$-plane intersecting the original r-plane in only one point.

Two parallel r-planes have the same complement. The complement of a point is isomorphic to the whole space, and the complement of the whole space is in one-one correspondence with each point.

7.3 Weight densities

With each r-plane we can associate a *weight density*, or simply 'weight', represented by a non-negative real number. Of a point we simply say that it has a weight.

Assigning a weight to an r-plane is different from assigning a unit length to it, although the two notions are related. Consider for example the two parallel lines of fig. . The first has a total weight w associated with the segment AB, the second a total weight $2w$ associated with the segment CD, which has twice the length of AB. The two parallel lines have therefore the same weight density.

It is therefore only meaningful to compare the weight densities of parallel r-planes.

7.4 Orientations

It is easy to familiarize with the notion of orientation of a line, plane, volume. We can imagine to traverse a line with two points A, B by going from A to B or from B to A. On a plane we can imagine to 'spin' clockwise or counter-clockwise. In a volume we can identify two cork-screw senses. In each case there are two possible orientations. In the 0-dimensional case of a point we convene to have the two symbolic orientations '+' and '−'.

Mathematically orientation is usually defined in terms of some equivalence class. This shows that, as with all notions defined in terms of equivalence classes, the mathematical formalism is still too primitive to capture it well. In this note we would like to consider orientation as intuitive and primitive, and will try not to use equivalence classes to define it.

The choice of an orientation on a line determines a unique ordering of every two distinct points on it. The choice of an orientation on a plane determines a unique ordering of every three non-collinear points on it, modulo an even number of permutations in the ordering. And so on for r-planes.

If a particular orientation is understood on a set of r-planes, we can understand a negative weight as indicating a reversed orientation. This corresponds to multiplying the positive weight by -1, as explained below.

7.5 Inner and outer properties

Weight and orientation were defined above as 'lying' within an r-plane. For this reason they are called *inner*, and we say that an r-plane has an inner weight or is inner-oriented.

Now consider the complement of an r-plane in an N-dimensional space. This is an $(N-r)$-dimensional affine space, and also the unique $(N-r)$-plane within this space. We can associate with it a weight and an orientation. As it happens with all properties defined on a complement, they can be mapped one-to-one onto every $(N-r)$-plane intersecting the initial r-plane in our original space; each such $(N-r)$-plane therefore acquires a weight and an orientation. Vice versa we can speak of the weight and orientation of a complement by referring to any $(N-r)$-plane intersecting our initial r-plane.

When we assign a weight w to the complement of an r-plane, we say that the latter has an *outer* weight $1/w$ – note the reciprocal. When we assign an orientation to the complement, we say that the r-plane has an *outer* orientation.

Inner and outer weights and orientations can be assigned independently of each other. Given an r-plane we have therefore four possibilities: we can assign to it an inner weight and inner orientation, an outer weight and outer orientation, an inner weight and outer orientation, an outer weight and inner orientation. Can we also assign all four?

Let's see the special case of points in 2-dimensional space, i.e. a plane; see fig. . A point with inner weight and inner orientation simply has an associated real number, positive or negative. A point with outer weight m and outer orientation assigns a weight surface density $1/m$ and a circulation sense to the whole plane. A point with inner weight and outer orientation has an associated positive real number and a circulation for the plane; a negative real number indicates that the opposite circulation must be taken. Finally, a point with outer weight m and inner orientation assigns a weight surface density $1/m$ to the whole plane and has an associated + or − sign; a negative density indicates that the opposite sign must be taken.

The case of a line in 3-dimensional space is illustrated in fig. .

8 Scalar multiplication and sum

We indicate weighted and oriented r-planes simply by their weights, if this does not cause confusion. A negative weight means an opposite orientation with respect to a tacitly understood one.

Each r-plane with a weight and orientation, either inner or outer, can be multiplied by a real number. The result is the same r-plane with its weight multiplied by the absolute value of the number, and the same or a reversed orientation depending on whether the number is positive or negative. See fig.

Two r-planes, both having inner or outer properties, can also be summed.

Let's start with the sum of weighted points, $m_1, m_2, \dots$. This is an interesting operation because it generalizes affine combination and leads to properties alike those of projective space.

First assume that the sum of the weights does not vanish: $\sum_j m_j \neq 0$. The result of the sum of the weighted points is a point given by the usual affine combination of the points with normalized coefficients $m_i / \sum_j m_j$, and with an associated weight $\sum_j m_j$. Add definition or example of affine combination. The resulting weight and orientation are inner or outer depending on whether those of all the summand points are.

Now consider the case of vanishing total weight, and for simplicity consider just two points: $m_1 + m_2 = 0$. By writing $m_2 = \epsilon - m_1$ and considering smaller and smaller values of ϵ, we see that the resulting point is ‘at infinity’ and has a vanishing weight; its orientation is therefore also undetermined.

Points of this kind are called *vectors*. They have indeed all properties of usual ‘free vectors’. Consider for example two points A, B with unit weights and positive orientation, and the vector $v \coloneqq B - A$. By summing this vector to the point A we obtain the point B: $A + v = B$. Thus v really behaves as a vector from A to B, translating the point A to B when summed to the former.

Vectors also give Grassmann spaces properties typical of projective spaces. Consider for example the unit-weight points A, B on the line a, and two unit-weight points C, D on the parallel line b and having the same distance as A, B. The vectors $B - A$ and $D - C$ are the same. This can be seen by rewriting the equality $B - A = D - C$ as $A + D = B + C$, which is indeed satisfied because the result of both $A + D$ and $B + C$ is the point at the centre of their parallelogram, with a weight of 2. This means that the vector $B - A$ is a point belonging to both parallel lines a and b, which therefore ‘meet at infinity’. A vector therefore characterizes the common direction, or *attitude*, and inner orientation of a set of parallel lines.

But the presence of weights leads to a much richer and interesting structure than in affine and projective spaces. Consider for example the vector $v \coloneqq B - A$, and sum it to the point A having a large weight m, which we denote mA. The result is the point $B + (m - 1)A$, located at relative distances $(m - 1)/m$ from B and $1/m$ from A. As the weight m increases, this resultant point gets closer to A. Thus the amount by which a weighted point is translated by a vector depends on the point’s weight, a ‘heavy’ point being translated less than a ‘light’ one.

For an exploration of these fascinating properties and their use we recommend Goldman’s brilliant articles Goldman 2000; 2002.

Explore and add more curiosities of this kind

Above we spoke of the total sum of the weights of some points, implicitly assuming that these weights can somehow be summed independently of the points they belong to. This is possible because all points have the same complement – the whole space – or equivalently because they are all ‘parallel’ to one another.

The weights of parallel r-planes can similarly be added, taking care of their orientations, independently of the r-planes they belong to. The reason is exactly the same as for the comparison of lengths, surfaces, etc., of parallel objects in an affine space.

Sums of parallel r-planes with total zero weight are objects called r-vectors, analogous to vectors. They can be imagined as r-planes 'at infinity' with vanishing weight. An r-vector is 'common' to a set of parallel $(r+1)$-planes, and therefore characterizes their r-direction or attitude and inner orientation. Again this gives many projective-like properties to a Grassmann space.

The following example needs the introduction of wedge first Let's consider the two-dimensional case of *bivectors*. Consider the antiparallel lines $A \wedge B$ and $-C \wedge D$, the distances AB and CD being equal. The total weight of these lines is zero. The sum $A \wedge B - C \wedge D$ is a bivector. It can be considered as a 'line' that 'goes around' the plane containing $A \wedge B$ and $C \wedge D$, with a circulation sense agreeing with those of AB and CD. See fig. .

The segment CD is parallel to AB, hence there is a vector w such that $C = A + w$ and $D = B + w$. The bivector $\alpha \coloneqq A \wedge B - C \wedge D$ is therefore equal to $(B - A) \wedge w$, or $\alpha = v \wedge w$, with $v \coloneqq B - A$. A bivector is therefore equal to the product of two vectors.

Very interesting is the case of r-planes that are skew, that is, do not lie on a common $(r+1)$-plane. Their weights cannot meaningfully be summed. ***

Explain sum of lines and planes with examples

The literature uses the term *extensors* to indicate points, lines, etc. that can be both of the usual kind and 'at infinity', reserving 'point' etc. only for those not at infinity. Shall we also use it?

9 Join, meet, contraction

Besides sum and scalar multiplication, Grassmann spaces have three kinds of multiplicative operations: join, meet, and contraction.

The join operates on inner-weighted objects, giving a new inner-weighted object of higher dimension. The meet operates on outer-weighted objects, giving a new outer-weighted object of lower dimension, hence higher codimension.

9.1 Wedge or join

The common notion of vector is usually associated with a line, but we have seen that vectors in Grassmann spaces are special kinds of points, and indeed they have properties more similar to points than lines.

A line with inner weight and orientation also has many similarities to a vector. The main difference is that it is not a free vector, because it is bound to that particular line; but it is not a bound vector either, because it has no definite initial point within that line.

Not sure whether the paragraphs above are understandable

Grassmann spaces have also another operation, the *wedge product*, which yields *inner-weighted* objects of higher dimension from lower-dimensional ones.

Let's start with two inner-weighted, inner-oriented points m_1, m_2. The result of their wedge product $m_1 \wedge m_2$ is the line passing through them, having weight $m_1 m_2$ and inner orientation going from the first to the second; if the total weight is negative then the opposite orientation must be taken. See fig. .

Clarify the difference between 'wedge' and 'join' in the literature

The wedge product of *outer-weighted* objects yields a lower-dimensional object instead. That's why Barnabei, Brini, Rota Barnabei et al. 1985; Brini et al. 2011 indicate the wedge of inner-weighted objects with $\vee$ and that of outer-weighted ones with $\wedge$, reminding of $\cup$ and $\cap$.

Consider the exterior algebra of a 4-dimensional vector space, with four linearly independent vectors e_1, e_2, e_3, e_4. The products $e_1 \wedge e_2$ and $e_3 \wedge e_4$ can be associated with two planes containing the respective vectors. These planes intersect at one point only, the origin. It is for this reason that the sum $e_1 \wedge e_2 + e_3 \wedge e_4$ cannot be written as $a \wedge b$.

10 Contraction or meet

There are two main approaches around:

- The 'geometric-algebra school' introduce a metric and a dual space, and define the meet in terms of contractions with duals and scalar products. Whitney seems to do likewise, at least at the beginning of his book.
- The 'Rota school' introduce a volume element (N-covector) and define the meet in terms of the latter. Their approach is to view a

Peano space in terms of invariance under the special linear group – that is, the preservation of volumes.

Neither school introduce a distinction between inner- and outer-weighted objects. Barnabei, Brini, & Rota make one valid observation:

> Elie Cartan found the regressive product to be superfluous and awkward. By vector space duality, a pairing of the two exterior algebras of V and V^* could easily be made with only one kind of product, the one that came to be called the wedge. [...] [However, T]he dual space V^* of a vector space V plays no role in such a calculus: a hyperplane is an object living in V, and its identification with a linear functional is a step backwards in clarity.

I think that their geometric viewpoint would be even more elegant by distinguishing between inner and outer weights. Covectors are like vectors, but characterized by an outer weight. From this point of view, moreover, we note that the traditional wedge of covectors yields their *intersection*, whereas the wedge of vectors give their *union*. This different behaviour is a direct consequence of the only ways in which inner and outer weights can be meaningfully be combined – and it doesn't require the notion of contraction or a special N-covector; both may be introduced later.

11 Bases

12 Boundary and differential

The following sections would aim to embed the calculus of simplicial r-chains, presented in Bossavit (2005 esp. §§ 23.2–3) into the algebra of Grassmann spaces. It's not clear whether this is meaningful or possible yet, but some very interesting mathematical curiosities suggest that those two theories are part of one common framework.

One curiosity is that the boundary of an r-chain always seems to correspond to an $(r-1)$-vector, i.e. an $(r-1)$-plane 'at infinity' with vanishing weight. Consider for example the segment AB. Its boundary is $\partial(AB) = B - A$, which interpreted within Grassmann-space algebra is a vector, a point at infinity with vanishing weight. Same goes for the triangle ABC: its boundary is $\partial(ABC) = AB + BC + CA$. If we interpret juxtaposition as wedge product, we see that the result is a bivector, a line at infinity with vanishing weight. So one wonders whether the calculus of simplicial r-chains can be somehow interpreted as a sub-calculus of r-vectors in Grassmann space. If this is true, then what is the relation between the boundary operator ∂ and the Grassmann operations?

A related question appears considering for example a 0-chain like $\sum_i m_i A_i$. In the chain calculus this is just a formal expression; in Grassmann space it is equivalent to a specific point B with a specific weight. Does this mean that, when we 'integrate' a 0-form on this chain, the result is the same as evaluating it on the resulting point B? It's indeed a possibility, given the affine structure in which both are embedded.

In the theory of chains and differential forms on manifolds, a simplicial r-chain is defined as a *formal* sum $\sum_i w_i C_i$ of simplicial r-submanifolds C_i with weights w_i. Differential r-forms can then be integrated on such chains.

In a Grassmann space we can interpret the formal sum above as a new, specific r-plane C' with an associated weight and, owing to the affine properties of the space and its algebra, the 'integral' of an r-form on this r-plane is equal to its integral on the initial chain. This is the basis for discretization and interpolation schemes to numerically solve partial integro-differential equations on manifolds.

Whitney forms = basis forms in the spaces of r-planes derived from $N + 1$ basis points?

13 Whitney, de Rham, computation

14 Representing physical quantities

> Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area. [...]
>
> There is another distinction between different kinds of directed quantities [...]. This is the distinction between longitudinal and rotational properties. *Maxwell (1881 §§ 13, 15)*

Physical quantities are associated with a point, like temperature; with a line, like electromotive force; with a surface, like mass or energy flux or current; with a volume, like charge. And we suppose them to vary with a particular kind of continuity with respect to variations of these geometric extensions. For this reason they are represented by inner- or outer-oriented differential forms, defined on a manifold and meant to be evaluated at a point or integrated over a line, surface, volume. The laws that govern them are expressed as particular mathematical relations among such integrals.

Approximate solutions to particular problems may be obtained by selecting a discrete and finite set of points, lines, surfaces, volumes on the manifold, requiring the laws to be satisfied within this set, and then interpolating the values of the fields on geometric extensions outside this set.

For example, we can associate a temperature with each point in such discrete set, and an electromotive force with each line, a current with each surface, a charge with each volume therein. A Grassmann space comes handy in such a discretization: the amount of a physical quantity over a particular point, line, and so on is represented by a single geometric entity, like a weighted point and so on. The interpolation is then automatically achieved by the sum of such geometric entities in the Grassmann space.

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('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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